COMPLEX REPRESENTATIONS OF FINITE GENERAL LINEAR GROUPS

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ABSTRACT

COMPLEX REPRESENTATIONS OF FINITE GENERAL LINEAR GROUPS

In this thesis, we determine complex irreducible representations of GL(2,K), the group of 2 by 2 invertible matrices over a finite field K. Actually, this is done by Ilya Piatetski-Shapiro in 1983. In his article [1], Shapiro classifies the irreducible representations of the group GL(2,K) by using the definition of induced module depends as a space of functions. The aim of this thesis is to rewrite the article using the induction module definition constructed by a tensor product. We start the thesis by reminding some basic definitions and theorems related to our topic. Then we determine the commutator subgroup of GL(2,K) and introduce some special subgroups of GL(2,K). The number of irreducible representations of a finite group is equal to the number of conjugacy classes of that group. Hence we calculate the conjugacy classes of GL(2,K). We determine irreducible representations of GL(2,K) through irreducible representations of the subgroups of it and quotient groups.

ÖZET

SONLU LİNEER GRUPLARIN KARMAŞIK TEMSİLLERİ

Bu savda, sonlu bir K cismi üzerine olan ikiye iki tersinir matrisler grubu GL(2,K) in karmaşık indirgenemez temsillerini belirleyeceğiz. Aslında bu daha önce Ilya Piatetski-Shapiro tarafından 1983 yılında yapıldı. Makalesinde ([1]), Shapiro GL(2,K) in indirgenemez temsillerini indüklenmiş modülün fonksiyonlar uzayına bağlı tanımını kullanarak sınıflandırıyor. Bu savın amacı makaleyi tensör çarpımı üzerine kurulu indüklenmiş modül tanımı kullanarak yeniden yazmaktır. Makaleye konumuzla alakalı temel tanım ve teoremleri hatırlatarak başlayacağız. Daha sonra GL(2,K) in değişeç alt grubunu belirleyeceğiz ve GL(2,K) in bazı özel alt gruplarını tanıtacağız. Sonlu bir grubun indirgenemez temsillerinin sayısı eşlenik sınıflarının sayısına eşittir. Bu sebepten GL(2,K) in eşlenik sınıflarını hesaplayacağız. Bölüm grupları ve GL(2,K) in indirgenemez temsilleri aracılığıyla GL(2,K) in indirgenemez temsillerini belirleyeceğiz.

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LIST OF SYMBOLS

$\mathbb{C}$	Complex numbers
$\mathrm{Def}^{G}_{G/N}$	deflation from G to G/N
$\|G\|$	order of the group G
G'	commutator subgroup of G
$\|G : H\|$	index of H in G
GL(2,K)	2×2 invertible matrices with entries from K
$H\backslash G/K$	(H, K)-double cosets in G
$[H\backslash G/K]$	a set of representatives of (H, K)-double cosets in G
Id_n	$n \times n$ identity matrix
Ind^{G}_{H}	induction from H to G
$\mathrm{Inf}^{G}_{G/N}$	inflation from G/N to G
$\mathrm{Irr}(G)$	the set of irreducible representations of G
$\mathrm{Iso}(\phi)$	isogation by ϕ
$J(V_\rho)$	Jacquet module of a representation ρ of G
K	field with q elements where $q > 2$
$K^\times$	the set of invertible elements of K
$\Bbbk[G]$	group algebra of the group G over the field $\Bbbk$
ker	Kernel
Res^{G}_{H}	restriction from G to H
SL(2,K)	2×2 K-matrices with determinant equal to 1

1. INTRODUCTION

In this chapter we introduce basic definitions and theorems related to our topic. The definitions and theorems in this section can be found in [2], [3] and [4].

Throughout the thesis, all vector spaces we considered are finite dimensional and over $\mathbb{C}$, the field of complex numbers.

Definition 1.1. *Let G be a group and V be a vector space. A group homomorphism*

$$\rho\colon G \to \mathrm{GL}(V)$$

is called a ($\mathbb{C}$-linear) representation of G on V. The vector space V is called the representation space of ρ and is also denoted by V_ρ.

Let ρ be a representation of G on V. Then it defines a G-action on V by

$$g \cdot v = \rho(g)(v) \quad \text{for any } g \in G,\, v \in V.$$

Definition 1.2. *Let ρ be a representation of G on V. The subgroup*

$$\{g \in G \mid \ \rho(g) = \mathrm{Id}_n\}$$

of G where $n = \dim(V)$ is called the kernel of ρ and denoted by $\ker(\rho)$.

Remark 1.3. *A representation ρ of G on V is called the trivial representation if $\rho(g) = \mathrm{Id}_n$ for all $g \in G$ where $n = \dim(V)$, equivalently, if $\ker(\rho) = G$.*

Remark 1.4. *The dimension (or degree) of ρ is defined as the dimension of V.*

Definition 1.5. *Let $(M,+)$ be an abelian group. Let R be a ring with unity. We call M a (left) R-module if there is a function $R \times M \to M$, written $(r,m) \mapsto rm$ which satisfies*

- $r(m_1 + m_2) = rm_1 + rm_2$
- $(r_1 + r_2)m = r_1 m + r_2 m$
- $(r_1 r_2)m = r_1(r_2 m)$
- $1_R m = m$

for all $m, m_1, m_2 \in M$ and for all $r, r_1, r_2 \in R$.

Definition 1.6. *Let $\Bbbk$ be a field. An algebra over $\Bbbk$ or a $\Bbbk$-algebra is a ring M which is also a $\Bbbk$-vector space such that*

$$k(xy) = (kx)y = x(ky) \quad \text{for all } x, y \in M,\ k \in \Bbbk.$$

Definition 1.7. *Let G be a finite group and $\Bbbk$ be a field. Consider the set of formal sums*

$$\left\{ \sum_{g \in G} r_g g \;\middle|\; r_g \in \Bbbk \right\}$$

and define multiplication and addition operations and scalar multiplication on it as follows.

- $\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g) g,$
- $\left(\sum_{g \in G} r_g g \right) \cdot \left(\sum_{h \in G} s_h h \right) = \sum_{g,h \in G} (r_g . s_h) gh,$
- $r \left(\sum_{g \in G} r_g g \right) = \sum_{g \in G} r r_g g$

where $r_g, s_g, s_h, r \in \Bbbk$.

The above structure defines an algebra, called the group algebra of G over $\Bbbk$ and denoted by $\Bbbk[G]$.

From now on we work with $\mathbb{C}[G]$-modules. The following proposition shows that $\mathbb{C}[G]$-modules and $\mathbb{C}$-linear representations of G are actually the same structures.

Proposition 1.8. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation, we can regard V as a $\mathbb{C}[G]$-module where the multiplication of G on V is given by*

$$g \cdot v = \rho(g)(v) \quad \text{for } g \in G \text{ and } v \in V.$$

For the other side, let V be a $\mathbb{C}[G]$-module. Let $\mathfrak{B}$ be any basis of V. Then

$$\begin{aligned} \rho\colon G &\to \mathrm{GL}(V) \\ g &\mapsto [g]_{\mathfrak{B}} \end{aligned}$$

defines a representation of G, where $[g]_{\mathfrak{B}}$ is the matrix representation of the linear transformation

$$\begin{aligned} V &\to V \\ v &\mapsto g \cdot v \end{aligned}$$

in the basis $\mathfrak{B}$.

From now on we use the terms 'representation of G over $\mathbb{C}$' and '$\mathbb{C}[G]$-module' to mean the same structure, under the above correspondence.

Definition 1.9. *A non-zero representation $\rho\colon G \to \mathrm{GL}(V)$ of G is said to be irreducible if it has no proper nonzero $\mathbb{C}[G]$-submodules.*

Notation. *The set of irreducible representations of G is denoted by $\mathrm{Irr}(G)$.*

Lemma 1.10. *Let G be a finite group. Then number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Lemma 1.11. *Let G be a finite group. We have*

$$\sum_{\rho \in \mathrm{Irr}(G)} (\mathrm{dim}\rho)^2 = |G|.$$

Corollary 1.12. *Let G be a finite abelian group and ρ be an irreducible representation of G. Then* $\mathrm{dim}\rho = 1$.

Now we introduce methods to produce new representations from given representations.

Definition 1.13. *Let $\rho\colon G \to \mathrm{GL}(V)$ and $\mu\colon H \to \mathrm{GL}(W)$ be representations of finite groups G and H, respectively. Then we can define a representation $\rho \times \mu$ of $G \times H$ on $V \otimes W$ by*

$$(\rho \times \mu)(g,h)(v \otimes w) = \rho(g)(v) \otimes \mu(h)(w) \quad \textit{for } g \in G, h \in H, v \in V, w \in W.$$

Lemma 1.14. *Let G and H be finite groups. Suppose $\{\rho_i \mid \ 1 \leq i \leq n\}$ and $\{\mu_i \mid \ 1 \leq i \leq m\}$ are the set of irreducible representations of G and H, respectively. Then $\{\rho_i \times \mu_j \mid \ 1 \leq i \leq n, 1 \leq j \leq m\}$ is the set of irreducible representations of $G \times H$.*

Definition 1.15. *Let G be a finite group and ρ and ϕ be representations of G on U and V, respectively. Then we define a representation $\rho \oplus \phi$ of G on $U \oplus V$ by*

$$(\rho \oplus \phi)(g)(u \oplus v) = \rho(g)u \oplus \phi(g)w \ \ \textit{for } g \in G,\ u \in U \textit{ and } v \in V.$$

Definition 1.16. *Let G be a finite group, H be a subgroup of G and ρ be a representation of G. The restriction of ρ from G to H, denoted by $\mathrm{Res}^G_H\rho$, is the representation* *of H given by*

$$\mathrm{Res}_H^G \rho(h) = \rho(h) \quad \textit{for all } h \in H.$$

Definition 1.17. *Let G be a finite group, N be a normal subgroup of G and ρ be a representation of the group G/N. We define a representation $\mathrm{Inf}_{G/N}^G \rho$ of G called inflation of ρ from G/N to G by*

$$\mathrm{Inf}_{G/N}^G \rho(g) = \rho(gN) \quad \textit{for all } g \in G.$$

Remark 1.18. *(i) Inflation of an irreducible representation is irreducible.*

(ii) Let G be a finite group and N be a normal subgroup of G. Let ρ_1 and ρ_2 be representations of G/N. Then

$$\mathrm{Inf}_{G/N}^G(\rho_1 \oplus \rho_2) = \mathrm{Inf}_{G/N}^G \rho_1 \oplus \mathrm{Inf}_{G/N}^G \rho_2.$$

Remark 1.19. *Let G be a finite group and N be a normal subgroup of G. If ρ_1 and ρ_2 are two nonisomorphic representations of G/N, then $\mathrm{Inf}_{G/N}^G \rho_1 \ncong \mathrm{Inf}_{G/N}^G \rho_2$.*

Definition 1.20. *Let G be a finite group and N be a normal subgroup of G. Let ρ be an irreducible representation of G. We define a representation $\mathrm{Def}_{G/N}^G \rho$ of G/N called deflation from G to G/N by*

$$\mathrm{Def}_{G/N}^G \rho(gN) = \begin{cases} \rho(g), & \textit{if } N \subseteq \ker(\rho), \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, for an arbitrary representation ρ of G, we define

$$\mathrm{Def}_{G/N}^G \rho = \bigoplus_{\rho_i \in \mathrm{Irr}(G)} c_i \mathrm{Def}_{G/N}^G \rho_i. \quad \textit{where } \rho = \bigoplus_{\rho_i \in \mathrm{Irr}(G)} c_i \rho_i.$$

Definition 1.21. *Let G and H be finite isomorphic groups. Let $\varphi\colon G \to H$ be an isomorphism between them. Let ρ be a representation of G. Then $\rho \circ \varphi^{-1}$ gives a representation of H, denoted by $\operatorname{Iso}(\varphi)\rho$ and called isogation by ϕ.*

Definition 1.22. *Let G be a finite group, $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G and $\mathfrak{B}$ be a basis for V. Then the character χ of ρ is defined by*

$$\begin{aligned}\chi\colon G &\to \mathbb{C}\\ g &\mapsto \operatorname{tr}([g]_{\mathfrak{B}})\end{aligned}$$

where $\operatorname{tr}([g]_{\mathfrak{B}})$ is trace of the matrix $[g]_{\mathfrak{B}}$.

Remark 1.23. *(i) Trace of a matrix of a linear transformation is independent of the chosen basis. Thus the above map is well defined.*

(ii) We have $\chi(1) = \dim(V)$. We call $\chi(1)$ the dimension (or the degree) of χ.

Note that we can identify 1-dimensional representations with 1-dimensional characters. From now on, we consider them as the same.

Notation. *The set of one-dimensional characters of a group G is denoted by $\widehat{G}$.*

The following Lemma can be found in [5].

Lemma 1.24. *Let G be a finite abelian group. Then $\widehat{G}$ forms a group under multiplication and $\widehat{G}$ is isomorphic to G.*

Definition 1.25. *Let G be a finite group. For any character χ of G, the function*

$$\begin{aligned}\bar{\chi}\colon G &\to \mathbb{C}\\ g &\mapsto \overline{\chi(g)}\end{aligned}$$

where $\overline{\chi(g)}$ is the complex conjugate of $\chi(g)$, defines a character of G and called the conjugate character of χ.

Remark 1.26. *Let G be a finite group, 1 be the trivial character and χ be any character of G. Then*

- $\bar{\chi}(g) = \chi(g^{-1})$ *for all* $g \in G$,
- $\bar{\chi}\chi = \chi\bar{\chi} = 1$ *if* χ *is one dimensional.*

Definition 1.27. *Let G be a group. The subgroup*

$$< ghg^{-1}h^{-1} \ \ | \ \ g, h \in G >$$

is defined as the commutator subgroup of G and denoted by G'.

Remark 1.28. *The group G/G' is the largest abelian quotient.*

Lemma 1.29. *The commutator subgroup G' of the group G acts trivially on any 1-dimensional representation of G.*

Lemma 1.30. *Let G be a finite group. For any irreducible representation ρ of G/G', the representation $\mathrm{Inf}_{G/G'}^{G}\rho$ is a one-dimensional representation of G. Moreover, all one-dimensional representations of G can be obtained in this way. In particular, there are $|G : G'|$ many one-dimensional representations of G.*

The definition of induced module in [1] is as follows:

Definition 1.31. *Let G be a finite group and H be a subgroup of G. Let ρ be a representation of H on W. Let*

$$V = \{\psi \colon G \to W \ \ | \ \ \psi(hg) = \rho(h)(\psi(g)) \text{ for all } h \in H, \ g \in G\}.$$

Note that the set V is a $\mathbb{C}[G]$-module where the G-action on V is as follows:

$$(g\psi)(g') = \psi(g'g) \ \text{ for } g, g' \in G \text{ and } \psi \in V.$$

The $\mathbb{C}[G]$-module V is called the induced module of the $\mathbb{C}[H]$-module W.

On the other hand, in [4] there is an equivalent definition of induced module given as follows:

Definition 1.32. *Let G be a finite group and H be a subgroup of G. Assume W is a left $\mathbb{C}[H]$-module. Note that we can regard $\mathbb{C}[G]$ as a $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule via left and right multiplication. Then the tensor product*

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

is a left $\mathbb{C}[G]$-module. It is called the induced module of the $\mathbb{C}[H]$-module W and is denoted by $\mathrm{Ind}_H^G W$.

Let V be as in Definition 1.31. Then the following $\mathbb{C}[G]$-module isomorphism provides that the definitions of induced representation of W, given in Definiton 1.31 and Definition 1.32 coincide.

$$\begin{aligned} &\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V \\ &\left(\sum_{g\in G} s_g g\right) \otimes w \mapsto \left(\psi : x \mapsto \sum_{h\in H} s_{x^{-1}h} hw\right) \end{aligned}$$

where $s_g \in \mathbb{C},\ x \in G,\ w \in W$.

The above map can be found in [6] where the case is extended from $\mathbb{C}$ to any field of characteristic zero.

Remark 1.33. *We have*

$$\dim(\mathrm{Ind}_H^G W) = |G : H|(\dim W).$$

The above maps between representations satisfy the following compatibility relations, see [7].

Let G be a finite group.

(i) *Transitivity Relations*:

a. Assume K and H are subgroups of G such that $K \leq H \leq G$. Let ρ and ϕ be representations of G and K, respectively. Then

$$\mathrm{Res}^H_K \mathrm{Res}^G_H \rho = \mathrm{Res}^G_K \rho, \qquad \mathrm{Ind}^G_H \mathrm{Ind}^H_K \phi = \mathrm{Ind}^G_K \phi.$$

b. Assume $\varphi\colon G \to H$ and $\psi\colon H \to K$ are group isomorphisms and ρ is a representation of G. Then

$$\mathrm{Iso}(\psi)\mathrm{Iso}(\varphi)\rho = \mathrm{Iso}(\psi\varphi)\rho.$$

c. Assume N and M are normal subgroups of G such that $N \leq M$. Let ρ and ϕ be representations of G/M and G, respectively. Then

$$\mathrm{Inf}^G_{G/N} \mathrm{Inf}^{G/N}_{G/M} \rho = \mathrm{Inf}^G_{G/M} \rho, \qquad \mathrm{Def}^{G/N}_{G/M} \mathrm{Def}^G_{G/N} \phi = \mathrm{Def}^G_{G/M} \phi.$$

(ii) *Commutation Relations*: **a.** Assume $\psi\colon G \to H$ is a group isomorphism, and K is a subgroup of G. Let ρ and ϕ be representations of G and K, respectively. Then

$$\mathrm{Iso}(\psi')\mathrm{Res}^G_K \rho = \mathrm{Res}^H_{\psi(K)} \mathrm{Iso}(\psi)\rho, \qquad \mathrm{Iso}(\psi)\mathrm{Ind}^G_K \phi = \mathrm{Ind}^H_{\psi(K)} \mathrm{Iso}(\psi')\phi,$$

where $\psi'\colon K \to \psi(K)$ is the restriction of ψ.

b. Assume $\psi\colon G \to H$ is a group isomorphism, and N is a normal subgroup of G. Let ρ and ϕ be representations of G and G/N, respectively. Then

$$\mathrm{Iso}(\psi'')\mathrm{Def}^G_{G/N} \rho = \mathrm{Def}^H_{H/\psi(N)} \mathrm{Iso}(\psi)\rho, \qquad \mathrm{Iso}(\psi)\mathrm{Inf}^G_{G/N} \phi = \mathrm{Inf}^H_{H/\psi(N)} \mathrm{Iso}(\psi'')\phi,$$

where $\psi'' \colon G/N \to H/\psi(N)$ is the group isomorphism induced by ψ.

c. (*Mackey Formula*) Assume H and K are subgroups of G. Let ρ be a representation of K. Then

$$\mathrm{Res}^G_H \mathrm{Ind}^G_K \rho = \bigoplus_{x \in [H\backslash G/K]} \mathrm{Ind}^H_{H\cap {}^xK} \mathrm{Iso}(\gamma_x) \mathrm{Res}^K_{H^x \cap K} \rho$$

where $[H\backslash G/K]$ is a set of representatives of (H, K)-double cosets in G, and where $H^x = \{x^{-1}hx \mid h \in H\}$, ${}^xK = \{xkx^{-1} \mid k \in K\}$ and $\gamma_x \colon H^x \cap K \to H \cap {}^xK$ is the group isomorphism obtained by conjugation by x i.e. $\gamma_x(a) = xax^{-1}$.

d. Assume N and M are normal subgroups of G. Let ρ be a representation of G/M. Then

$$\mathrm{Def}^G_{G/N} \mathrm{Inf}^G_{G/M} \rho = \mathrm{Inf}^{G/N}_{G/NM} \mathrm{Def}^{G/M}_{G/NM} \rho.$$

Remark 1.34. *Observe that when $N = M$,*

$$\mathrm{Def}^G_{G/N} \mathrm{Inf}^G_{G/N} \rho = \mathrm{Inf}^{G/N}_{G/N} \mathrm{Def}^{G/N}_{G/N} \rho = \rho.$$

e. Assume H is a subgroup of G, and N is a normal subgroup of G. Let ρ and ψ be representations of H and G/N, respectively. Then

$$\mathrm{Def}^G_{G/N} \mathrm{Ind}^G_H \rho = \mathrm{Ind}^{G/N}_{HN/N} \mathrm{Iso}(\varphi) \mathrm{Def}^H_{H/H\cap N} \rho,$$
$$\mathrm{Res}^G_H \mathrm{Inf}^G_{G/N} \psi = \mathrm{Inf}^H_{H/H\cap N} \mathrm{Iso}(\varphi^{-1}) \mathrm{Res}^{G/N}_{HN/N} \psi$$

where $\varphi \colon H/H \cap N \to HN/N$ is the canonical group isomorphism.

f. Assume H is a subgroup of G and N is a normal subgroup of G such that $N \leq H$. Let ρ and ψ be representations of G and H/N, respectively. Then

$$\mathrm{Res}^{G/N}_{H/N} \mathrm{Def}^G_{G/N} \rho = \mathrm{Def}^H_{H/N} \mathrm{Res}^G_H \rho, \qquad \mathrm{Ind}^G_H \mathrm{Inf}^H_{H/N} \psi = \mathrm{Inf}^G_{G/N} \mathrm{Ind}^{G/N}_{H/N} \psi.$$

Definition 1.35. *Let G be a finite group. A function $\psi\colon G \to \mathbb{C}$ is called a class function on G if it is constant on conjugacy classes of G.*

Remark 1.36. *Let G be a finite group. The set of class functions on G form a $\mathbb{C}$-vector space under function addition and usual scalar multiplication on functions. Moreover, the set of irreducible characters of G form a basis for that vector space.*

Definition 1.37. *Let G be a finite group. For any two functions $\psi, \zeta\colon G \to \mathbb{C}$ define*

$$\langle \psi, \zeta \rangle := \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\zeta(g)}.$$

The definition above gives an inner product on the vector space of class functions.

Remark 1.38. *Let ψ, ζ be two irreducible characters of G. Then*

$$\langle \psi, \zeta \rangle = \begin{cases} 0, & \text{if } \psi \neq \zeta, \\ 1, & \text{if } \psi = \zeta. \end{cases}$$

Theorem 1.39 (Frobenius Reciprocity Theorem)**.** *Let G be finite group, H be a subgroup of G, and let ζ and ψ be representations of G and H, respectively. Then*

$$\left\langle \psi, \mathrm{Res}_H^G \zeta \right\rangle = \left\langle \mathrm{Ind}_H^G \psi, \zeta \right\rangle.$$

Theorem 1.40. *Let G be a finite group and N be a normal subgroup of it, and let ζ and ψ be representations of G and G/N, respectively. Then*

$$\left\langle \zeta, \mathrm{Inf}_{G/N}^G \psi \right\rangle = \left\langle \mathrm{Def}_{G/N}^G \zeta, \psi \right\rangle.$$

2. THE GROUP GL(2,K)

Let K be a field with q elements where $q > 2$. Denote by GL(2,K) the set of invertible matrices with entries coming from the field K. We will examine the complex irreducible representations of GL(2,K). From now on, we use the letter G instead of GL(2,K) for simplicity.

2.1. Commutator Subgroup of GL(2,K)

In this section we prove that $G' = \mathrm{SL}(2,K)$ where SL(2,K) is the subgroup of GL(2,K) consisting of matrices whose determinants are equal to $1 \in K$. Observe that for any $g, h \in G$ the commutator of g and h satisfies

$$\det(ghg^{-1}h^{-1}) = 1.$$

Moreover, the determinant of any element of G' generated by these elements is equal to 1. Hence we conclude that $G' \subseteq \mathrm{SL}(2,K)$.

For the converse, let $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,K)$ be arbitrary. Then we have $ad - bc = 1$ i.e. $d = a^{-1} + a^{-1}bc$. By standard computations from group theory, we write

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}c & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

Now we will show that each of the matrices in the decomposition above belongs to the set of commutators of G.

- Let $k \in K$ be arbitrary. Choose $x \in K^\times$ such that $x+1 \neq 0$ (such an x exists since $q > 2$). Now let $g = \begin{pmatrix} x+1 & k \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} x & k \\ 0 & 1 \end{pmatrix}$ be elements in GL(2,K). Observe that

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

  Thus, for all $k \in K$ we have $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in G'$.

- Let $k \in K$ be arbitrary. Let $g' = (h^t)^{-1}$ and $h' = (g^t)^{-1}$ where g, h are as above and g^t and h^t are the transposes of h and g, respectively. Then,

$$g'h'g'^{-1}h'^{-1} = (h^t)^{-1}(g^t)^{-1}h^tg^t = (ghg^{-1}h^{-1})^t = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

  Thus, for all $k \in K$ we have $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in G'$.

- Let $k \in K^\times$ be arbitrary. Now, suppose that $g = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \in$ GL(2,K) and $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in$ GL(2,K). Observe that $ghg^{-1}h^{-1} = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$. Thus, for all $k \in K$ we have $\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \in G'$.

Hence, we conclude that $G' =$ SL(2,K).

Remark 2.1. *The function $f \colon G/G' \to K^\times$ defined by $f(gG') = \det(g)$ for $g \in G$ is a group isomorphism.*

Now, we introduce some particular subgroups of GL(2,K) and assign some letters to them for the rest of the thesis.

(i) The subgroup of upper triangular matrices,

$$B := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha,\delta \in K^{\times};\ \beta \in K \right\},$$

is called the Borel subgroup of $\mathrm{GL}(2,K)$. It has $q(q-1)^2$ many elements. Observe that for any $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} \in B$,

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha' & \delta^{-1}(-\alpha'\beta + \alpha\beta' + \beta\delta') \\ 0 & \delta' \end{pmatrix}. \tag{2.2}$$

Claim 2.2. *The set* $\Gamma := \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mid \gamma \in K \right\}$ *and* $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *form a set of representatives for the right (and also left) cosets of* B *in* $\mathrm{GL}(2,K)$.

Proof. For $\gamma_1, \gamma_2, \gamma \in K$ such that $\gamma_1 \neq \gamma_2$, we have

$$\begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma_2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma_1 - \gamma_2 & 1 \end{pmatrix} \notin B$$

and

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -\gamma \end{pmatrix} \notin B.$$

The calculations above show that they form distinct coset representatives of B in G. Also to see that they are complete set of representatives of B in G, take

$s := \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in \mathrm{GL}(2,K)$;

- If $s_{21} = 0$, then $s \in B$.

- If $s_{21} \neq 0$ and $s_{22} = 0$ (which implies $s_{12} \neq 0$ since $s \in \mathrm{GL}(2,K)$), then

$$s = \begin{pmatrix} s_{12} & s_{11} \\ 0 & s_{21} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Bw.$$

- If $s_{21} \neq 0$ and $s_{22} \neq 0$ (which implies s_{22}^{-1} exists), then

$$s = \begin{pmatrix} s_{11} - s_{12}s_{22}^{-1}s_{21} & s_{12} \\ 0 & s_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{22}^{-1}s_{21} & 1 \end{pmatrix} \in B \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

where $\gamma = s_{22}^{-1}s_{21}$. Observe that $s_{11} - s_{12}s_{22}^{-1}s_{21} \neq 0$ since $\det(s) \neq 0$.

□

Hence B has $q+1$ many cosets in G. Then $|G| = |B||G : B| = (q-1)^2q(q+1)$.

(ii) The abelian subgroup of all unipotent upper triangular matrices

$$U := \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in K \right\}$$

has q elements and isomorphic to the group $(K, +)$ via the group isomorphism from U to K defined by $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mapsto \beta$. Here $(K, +)$ corresponds to the additive group structure of the field K. In (2.2), if we take $\alpha' = \delta' = 1$, we see that U is a normal subgroup of B. Also, to obtain the commutator elements of B, if we multiply (2.2) by $\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}^{-1}$ from the right, we get $\begin{pmatrix} 1 & \beta'\delta'^{-1}(\alpha\delta^{-1} - 1) - \beta\delta^{-1}(\alpha'\delta'^{-1} - 1) \\ 0 & 1 \end{pmatrix}$ which is an element of U. Thus, B' is a subgroup of U. For the converse, as the elements g, h that are defined in (2.1) are in B, (2.1) yields that any element of U is a commutator of some elements from B. Therefore, $U = B'$.

(iii) The subgroup of diagonal matrices is the group

$$D := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha,\delta \in K^\times \right\}.$$

It is isomorphic to $K^\times \times K^\times$ via the group isomorphism $\nu\colon K^\times \times K^\times \to D$ by $\nu\big((\alpha,\delta)\big) = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$. It is also an abelian group. Observe that $U, D \subseteq B$, $U \cap D = \{1\}$ and $|UD| = \dfrac{|U| \cdot |D|}{|U \cap D|} = \dfrac{q(q-1)^2}{1} = |B|$. Hence, we conclude that $B = UD$. Furthermore, B is the semi-direct product of U by D. Also, one can easily check that $\theta\colon D \to B/U$ by $\theta(d) = dU$ is an isomorphism. From now on, we will use letter κ for the isomorphism $\kappa\colon K^\times \times K^\times \to B/U$ defined by the composition $\kappa = \theta \circ \nu$.

(iv) Another normal subgroup of B,

$$P := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha \in K^\times,\, \beta \in K \right\},$$

has $(q-1)q$ elements. Similarly, by letting $\delta' = 1$ in (2.2), one can easily see that P is a normal subgroup of B. We have shown that $B' = U$. The same argument used to prove it can be applied to get $P' = U$.

(v) The center of G,

$$Z := \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in K^\times \right\},$$

is a subgroup of B with $q-1$ elements. It is isomorphic to $K^\times$ via the group isomorphism $\iota\colon K^\times \to Z$ defined by $\iota(\delta) = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$. Notice that $Z \cap P = \{1\}$. Also, $ZP = B$ since $|ZP| = \dfrac{|Z| \cdot |P|}{|Z \cap P|} = \dfrac{(q-1)q(q-1)}{1} = |B|$ and $Z, P \subset B$.

Therefore, B is the direct product of Z and P as they are normal in B.

(vi) The subgroup of P,

$$A := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in K^\times \right\},$$

has $(q-1)$ elements and is isomorphic to $K^\times$. We assign letter η to the isomorphism $\eta \colon K^\times \to A$ defined by $\eta(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$. Observe that U, A are subgroups of P such that $U \cap A = \{1\}$ and so $|UA| = \dfrac{|U| \cdot |A|}{|U \cap A|} = \dfrac{q(q-1)}{1} = |P|$. Hence, we conclude that $P = UA$. Furthermore, P is the semi-direct product of U by A. From now on, we will use the letter ξ for the isomorphism $\xi \colon A \to P/U$ given by $\xi(a) = aU$. Also observe that $|AZ| = \dfrac{|A| \cdot |Z|}{|A \cap Z|} = \dfrac{(q-1)(q-1)}{1} = |D|$. Hence D is the direct product of A and Z as D is abelian.

We summarize what is explained above in the following graph:

$$\begin{tikzcd}
& G \arrow[d, dash] & \\
& B \arrow[dl, equal] \arrow[dr, dash] & \\
P \arrow[d, equal] \arrow[dr, dash] & & D \arrow[dl, equal] \arrow[d, equal] \\
U \arrow[dr, dash] & A \arrow[d, dash] & Z \arrow[dl, dash] \\
& \{1\} &
\end{tikzcd}$$

where we have equalities $B = P \times Z = U \rtimes D$ and $U = B' = P'$.

2.2. Bruhat's Decomposition

The following theorem gives us a decomposition of G, where we use it to calculate some double coset representatives.

Lemma 2.3 ([1]). *We have a decomposition of G, called Bruhat's decomposition*

$$G = B \sqcup BwU$$

where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \setminus B$. So $ad - bc \neq 0$ and $c \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b - ac^{-1}d & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} \in BwU.$$

□

Corollary 2.4. *We have* $[B \setminus G/B] = \{1, w\}$.

2.3. The conjugacy classes of GL(2,K)

We know that the number of irreducible representations of a group is equal to the number of conjugacy classes of it. Hence, we want to classify the conjugacy classes of G. An element M of G has two eigenvalues.

Case 1: One of the eigenvalues of M belongs to K. Since characteristic polynomial of M is degree 2, if one of the roots of the polynomial is in K, so does the other. If both eigenvalues of M are equal, say to α, then the Jordan form of M is one of the followings

$$c_1(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\ c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

If M has two distinct eigenvalues α and β in K, then M is diagonalizable and the Jordan form of M is $c_3(\alpha,\beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

If we let α and β vary, we get different conjugacy classes. Thus, there are $(q-1)$ matrices in each of the forms $c_1(\alpha)$ and $c_2(\alpha)$. Also, there are $\frac{1}{2}\ (q-1)(q-2)$ many conjugacy classes of the form $c_3(\alpha,\beta)$.

Case 2: Both eigenvalues of M do not belong to K. Let $p(x)$ be the characteristic polynomial of M and $\alpha, \bar{\alpha}$ be the roots of $p(x)$ i.e. $p(x) = x^2 - (\alpha+\bar{\alpha})x + \alpha\bar{\alpha}$. Then they belong to the quadratic extension $L := K[\alpha]$. Let v be any nonzero element of the K-vector space K^2.

Claim. *The vectors v and Mv form a basis for K^2.*

Proof. By contradiction, assume they are linearly dependent. Then there exists $\lambda \in K$ such that Mv=λv , which implies that M has an eigenvalue in K, contrary to our hypothesis. □

Now we want to write the matrix M with respect to the basis $\{v, Mv\}$. Observe that M sends v and Mv to Mv and M^2v, respectively. By Cayley-Hamilton Theorem, we have $p(M) = 0$. Thus, we obtain $M^2 - (\alpha+\bar{\alpha})M + \alpha\bar{\alpha} = 0$. Hence,

$$M(Mv) = M^2v = (\alpha+\bar{\alpha})Mv - \alpha\bar{\alpha}v.$$

Thus, the corresponding matrix to M in the basis $\{v, Mv\}$ is

$$c_4(\alpha) = \begin{pmatrix} 0 & -\alpha\bar{\alpha} \\ 1 & \alpha + \bar{\alpha} \end{pmatrix}.$$

Observe that $c_4(\alpha) = c_4(\bar{\alpha})$. Also, if α and β are different elements which are not conjugate in K (i.e. which are not roots of the same quadratic polynomial in K), then $c_4(\alpha)$ is not conjugate to $c_4(\beta)$ since they have distinct eigenvalues. Thus, the number of conjugacy classes of this form is equal to the half of the number of the elements in $L \setminus K$. L has q^2 elements as a quadratic extension of K which has q elements. Thus there are $\frac{1}{2}(q^2 - q)$ conjugacy classes of the form $c_4(\alpha)$. Thus, we proved the following.

Theorem 2.5 ([1]). *Conjugacy classes of G can be classified as below:*

(i) $(q-1)$ *classes of the form* $c_1(\alpha)$ *where* $\alpha \in K$ *and* $c_1(\alpha)$ *is the set of diagonalizable matrices in* G *whose both eigenvalues equal to* α*.*

(ii) $(q-1)$ *classes of the form* $c_2(\alpha)$ *where* $\alpha \in K$ *and* $c_2(\alpha)$ *is the set of non-diagonalizable matrices in* G *whose both eigenvalues equal to* α*.*

(iii) $\frac{1}{2}(q-1)(q-2)$ *classes of the form* $c_3(\alpha, \beta)$ *where* $\alpha, \beta \in K$ *,* $\alpha \neq \beta$ *and* $c_3(\alpha, \beta)$ *is the set of matrices in* G *with eigenvalues* α *and* β*.*

(iv) $\frac{1}{2}(q^2 - q)$ *classes of the form* $c_4(\alpha)$ *where* $\alpha \in L \setminus K$ *and* $c_4(\alpha)$ *is the set of matrices in* K *whose one of the eigenvalues equal to* α*.*

3. THE REPRESENTATIONS OF GL(2,K)

3.1. One-Dimensional Representations of G

Let $\mu_1, \mu_2, \ldots, \mu_{q-1}$ be all irreducible characters of the group $K^\times$. Then, all characters of G/G' can be formed by $\mathrm{Iso}(f^{-1})\mu_i$ where f is as defined in Remark 2.1.

Lemma 3.1 ([1]). *There are $(q-1)$ many one-dimensional irreducible representation of G, which are $\mu_i \circ \det$ for $i = 1, 2, \ldots, q-1$.*

Proof. By Lemma 1.30 there are $|G/G'| = |K^\times| = q-1$ many one-dimensional representations of G which are obtained by inflating characters from G/G' to G. Observe that for each character $\mathrm{Iso}(f^{-1})\mu_i$ of G/G' where $i = 1, 2, \ldots, q-1$, for each $g \in G$

$$\mathrm{Inf}_{G/G'}^{G}(\mathrm{Iso}(f^{-1})\mu_i)(g) = (\mathrm{Iso}(f^{-1})\mu_i)(gG') = \mu_i(f(gG') = \mu_i(\mathrm{det} g) = \mu_i \circ \det(g).$$

Hence we get the desired result by Remark 1.19. □

3.2. Representations of P

For the rest of the thesis, we fix a non-trivial character ψ of $(K, +)$. Now, for every $a \in A$, we can form a non-trivial character ψ_a of U by

$$\psi_a(u) := \psi(a_{11}u_{12}), \text{ for } u \in U$$

where $a = \begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix}$. Now, one can easily check that $\psi_a(uu') = \psi_a(u).\psi_a(u')$, which shows that ψ_a is a one-dimensional character of U.

Also, observe that

$$
\begin{aligned}
\psi_a = \psi_{a'} &\Rightarrow \psi_a(u) = \psi_{a'}(u) \text{ for all } u \in U \\
&\Rightarrow \psi(a_{11}k) = \psi(a'_{11}k) \text{ for all } k \in K \\
&\Rightarrow \psi((a_{11} - a'_{11})k) = 0 \text{ for all } k \in K \\
&\Rightarrow a_{11} - a'_{11} = 0 \\
&\Rightarrow a_{11} = a'_{11} \\
&\Rightarrow a = a'.
\end{aligned}
$$

Hence, each different choice of $a \in A$ gives us a different character of U. Thus, by this way, we get $(q-1)$ characters. These characters and the trivial character of U are all the characters of U since U is abelian and has q elements.

Claim 3.2 ([1])**.** $\mathrm{Res}_U^P \mathrm{Ind}_U^P \psi_1 = \bigoplus_{a \in A} \psi_a$.

Proof. We have equalities $[U \backslash P/U] = [U \backslash P] = A$, since U is a normal subgroup of P and since $P = UA$, respectively. Then, by Mackey Formula

$$
\begin{aligned}
\mathrm{Res}_U^P \mathrm{Ind}_U^P \psi_1 &= \bigoplus_{a \in [U \backslash P/U] = A} \mathrm{Ind}_{U \cap {}^aU}^U \mathrm{Iso}(\gamma_a) \mathrm{Res}_{U^a \cap U}^U \psi_1 \\
&\overset{(*)}{=} \bigoplus_{a \in A} \mathrm{Ind}_U^U \mathrm{Iso}(\gamma_a) \mathrm{Res}_U^U \psi_1 \\
&= \bigoplus_{a \in A} \mathrm{Iso}(\gamma_a) \psi_1 = \bigoplus_{a \in A} \psi_{a^{-1}} = \bigoplus_{a \in A} \psi_a
\end{aligned}
$$

where Equality (*) is the result of that $a \in P$ and $U \trianglelefteq P$ implies $U^a = {}^aU = U$. $\square$

Theorem 3.3 ([1])**.** *Irreducible representations of the group P consists of*

(a) $(q-1)$ many one-dimensional characters obtained by $\mathrm{Inf}_{P/U}^P \mathrm{Iso}(\xi)\chi$ where χ is a one-dimensional character of A, ξ is the isomorphism defined in Section 2.1 and

(b)A $(q-1)$-dimensional irreducible representation that is $\pi = \mathrm{Ind}_U^P \psi_1$.

Proof. (a) We have $P' = U$. By Lemma 1.30, every one dimensional character of P is inflated from an irreducible (one-dimensional) character of the quotient group P/U. Also since $A \cong P/U$, for each irreducible character χ of A, $\mathrm{Inf}_{P/U}^{P}\mathrm{Iso}(\xi)\chi$ gives us an (distinct) irreducible character of P. As A has $(q-1)$ many one-dimensional characters, by this way, we get $(q-1)$ many one-dimensional characters of P.

(b) By Remark 1.33, we have $\dim(\pi) = \dim(\mathrm{Ind}_U^P\psi_1) = \frac{|P|}{|U|}\dim(\psi_1) = q-1$. Also using Frobenius Reciprocity Theorem and by Claim 3.2, we get

$$\langle \pi, \pi \rangle = \left\langle \mathrm{Ind}_U^P\psi_1, \mathrm{Ind}_U^P\psi_1 \right\rangle = \left\langle \psi_1, \mathrm{Res}_U^P\mathrm{Ind}_U^P\psi_1 \right\rangle = \left\langle \psi_1, \bigoplus_{a\in A}\psi_a \right\rangle = 1$$

Thus, π is an irreducible representation of P, having dimension $q-1$. We have the equality:

$$(\dim\pi)^2 + \sum_{\chi\in\mathrm{Irr}(A)} (\dim\mathrm{Inf}_{P/U}^{P}\mathrm{Iso}(\xi)\chi)^2 = (q-1)^2 + (q-1)\cdot 1^2 = (q-1)q = |P|.$$

Thus, there is no additional representation of P by Lemma 1.11. □

3.3. Representations of B

We have seen that $B = P \times Z$. Therefore, by Lemma 1.14, all the irreducible representations of B can be written as a product of an irreducible representation of P and an irreducible representation of Z. The product of a one-dimensional character of P and a one-dimensional character of Z gives us a one-dimensional character of B. There are $(q-1)^2$ of them. The product of the $(q-1)$-dimensional representation π of P and a one-dimensional representation of Z gives us a $(q-1)$-dimensional representation of B. There are $q-1$ of them. These representations are all distinct and gives us all the irreducible representations of B.

There is another way to see one-dimensional characters of B. Remember that κ is an isomorphism from $K^\times \times K^\times$ to B/U. For every character $(\mu_i \times \mu_j)$ of $K^\times \times K^\times$, we get a distinct one-dimensional character of B via $\mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ and denote it by $\mu_{i,j}$ for simplicity. Hence, by this way, we get all $(q-1)^2$ many one-dimensional characters of B.

We obtained the following theorem:

Theorem 3.4 ([1]). *The group B has*

(i) $(q-1)^2$ *many one-dimensional characters obtained by* $\mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$*, and denoted by* $\mu_{i,j}$ *where* $(\mu_i \times \mu_j)$ *is a character of* $K^\times \times K^\times$*.*

(ii) $(q-1)$ *many* $(q-1)$*-dimensional irreducible representations obtained by* $\pi \times \chi_i$ *for an irreducible representation* χ_i *of* Z*.*

Note that for any $(q-1)$-dimensional irreducible representation $\pi \times \chi_i$ of B, we obtain $\mathrm{Res}_P^B(\pi \times \chi_i) = \pi$.

Now, we make some observations about one-dimensional representations of B. If $\mu_{i,j}$ is a one-dimensional character given by $\mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ as above, then

$$\begin{aligned}
\mu_{i,j}\left(\begin{pmatrix} \alpha & * \\ 0 & \delta \end{pmatrix}\right) &= \mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)\left(\begin{pmatrix} \alpha & * \\ 0 & \delta \end{pmatrix}\right) \\
&= \mathrm{Iso}(\kappa)(\mu_i \times \mu_j)\left(\begin{pmatrix} \alpha & * \\ 0 & \delta \end{pmatrix} U\right) \\
&= (\mu_i \times \mu_j)\left(\kappa^{-1}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} U\right)\right) \\
&= (\mu_i \times \mu_j)(\alpha, \delta) = \mu_i(\alpha) \cdot \mu_j(\delta).
\end{aligned}$$

Observe the relation between $\mu_{i,j}$ and $\mu_{j,i}$ on D:

$$\begin{aligned}
\mu_{j,i}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) &= \mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_j \times \mu_i)\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) \\
&= \mu_j(\alpha)\cdot\mu_i(\delta) \\
&= \mu_{i,j}\left(\begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}\right) \\
&= \mu_{i,j}\left(w\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}w^{-1}\right),
\end{aligned}$$

i.e. $\mu_{j,i}(d) = \mu_{i,j}(wdw^{-1})$ for all $d \in D$ where w is as described in Lemma 2.3.

Lemma 3.5. *For one-dimensional representations $\mu_{i,j}$ and $\mu_{k,l}$ of B, we have $\mathrm{Res}_D^B\mu_{i,j} = \mathrm{Res}_D^B\mu_{k,l}$ if and only if $\mu_{i,j} = \mu_{k,l}$.*

Proof. By transitivity and commutation relations in Section 1, we have

$$\begin{aligned}
\mathrm{Res}_D^B\mu_{i,j} &= \mathrm{Res}_D^B\mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \\
&= \mathrm{Inf}_{D/D\cap U}^{D}\mathrm{Iso}(\theta^{-1})\mathrm{Res}_{DU/U}^{B/U}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \\
&= \mathrm{Inf}_{D}^{D}\mathrm{Iso}(\theta^{-1})\mathrm{Res}_{B/U}^{B/U}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \\
&= \mathrm{Iso}(\theta^{-1}\kappa)(\mu_i \times \mu_j). \\
&= \mathrm{Iso}(\nu)(\mu_i \times \mu_j).
\end{aligned}$$

Thus,

$$\begin{gathered}
\mathrm{Res}_D^B\mu_{i,j} = \mathrm{Res}_D^B\mu_{k,l} \text{ if and only if} \\
\mathrm{Iso}(\nu)(\mu_i \times \mu_j) = \mathrm{Iso}(\nu)(\mu_k \times \mu_l) \text{ if and only if} \\
(\mu_i \times \mu_j) = (\mu_k \times \mu_l) \text{ if and only if} \\
\mu_{i,j} = \mu_{k,l}.
\end{gathered}$$

□

Lemma 3.6 ([1]). *We have $\mu_{i,j} = \mu_{j,i}$ if and only if $\mu_i = \mu_j$.*

Proof.

$$\begin{aligned} \mu_{i,j} = \mu_{j,i} &\text{ if and only if} \\ \mathrm{Inf}^B_{B/U}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) = \mathrm{Inf}^B_{B/U}\mathrm{Iso}(\kappa)(\mu_j \times \mu_i) &\text{ if and only if} \\ (\mu_i \times \mu_j) = (\mu_j \times \mu_i) &\text{ if and only if} \\ \mu_i = \mu_j. \end{aligned}$$

□

3.4. Inducing One-Dimensional Characters from B to G

In this section, we find irreducible components of representations of G obtained by inducing one-dimensional characters of B. First, the article [1] defines the Jacquet Module of a representation ρ of G as

$$J(V_\rho) = \{v \in V_\rho \mid \rho(u)v = v \text{ for every } u \in U\}$$

where V_ρ is the representation space of ρ and shows that it is a B-space. Let V' be the B complement of $J(V_\rho)$ in V_ρ. Then no element of V' is fixed by U. Now observe that

$$\mathrm{Inf}^B_{B/U}\mathrm{Def}^B_{B/U}\mathrm{Res}^G_B V_\rho = \mathrm{Inf}^B_{B/U}\mathrm{Def}^B_{B/U}(J(V_\rho) \oplus V') = \mathrm{Inf}^B_{B/U}\mathrm{Def}^B_{B/U} J(V_\rho) = J(V_\rho).$$

Here, since $U \subseteq \ker(\rho)$ on $J(V_\rho)$ (also for all irreducible components of it) we have the last equality by definition of deflation and inflation. Thus, from now on, we will use the equality $J(V_\rho) = \mathrm{Inf}^B_{B/U}\mathrm{Def}^B_{B/U}\mathrm{Res}^G_B V_\rho$ as the definition. The representation of $J(V_\rho)$ is $\mathrm{Inf}^B_{B/U}\mathrm{Def}^B_{B/U}\mathrm{Res}^G_B \rho$. For simplicity, we will use the notation $J(\rho)$ instead

$\mathrm{Inf}_{B/U}^{B}\mathrm{Def}_{B/U}^{B}\mathrm{Res}_{B}^{G}\rho$. Note that for any one-dimensional character $\mu_{i,j}$ of B,

$$\dim(\mathrm{Ind}_{B}^{G}\mu_{i,j}) = |G : B| = q + 1.$$

Lemma 3.7 ([1]). *If $\mu_{i,j}$ is a one-dimensional representation of B, then we have $\dim(J(\mathrm{Ind}_{B}^{G}\mu_{i,j})) = 2$ and $J(\mathrm{Ind}_{B}^{G}\mu_{i,j})$ has two irreducible components $\mu_{i,j}$ and $\mu_{j,i}$.*

Proof. By Mackey Formula, we have

$$\begin{aligned}
\mathrm{Res}_{B}^{G}\mathrm{Ind}_{B}^{G}\mu_{i,j} &= \bigoplus_{x\in[B\backslash G/B]} \mathrm{Ind}_{B\cap {}^{x}B}^{B}\mathrm{Iso}(\gamma_x)\mathrm{Res}_{B^{x}\cap B}^{B}\mu_{i,j} \\
&\overset{(*)}{=} \mathrm{Ind}_{B\cap B}^{B}\mathrm{Res}_{B}^{B}\mu_{i,j} \oplus \mathrm{Ind}_{B\cap {}^{w}B}^{B}\mathrm{Iso}(\gamma_w)\mathrm{Res}_{B^{w}\cap B}^{B}\mu_{i,j} \\
&= \mu_{i,j} \oplus \mathrm{Ind}_{D}^{B}\mathrm{Iso}(\gamma_w)\mathrm{Res}_{D}^{B}\mu_{i,j} \\
&= \mu_{i,j} \oplus \mathrm{Ind}_{D}^{B}\mathrm{Res}_{D}^{B}\mu_{j,i}
\end{aligned}$$

where Equality (*) comes from Corollary 2.4. Thus, we have

$$\mathrm{Res}_{B}^{G}\mathrm{Ind}_{B}^{G}\mu_{i,j} = \mu_{i,j} \ \oplus \ \mathrm{Ind}_{D}^{B}\mathrm{Res}_{D}^{B}\mu_{j,i}. \tag{3.1}$$

Then by commutation relations in Section 1 and Remark 1.34, we have the following equations.

$$
\begin{aligned}
& J(\mathrm{Ind}_B^G \mu_{i,j}) \\
& = \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B \mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} \\
& \overset{(*)}{=} \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B (\mu_{i,j} \oplus \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}) \\
& = \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B \mu_{i,j} \oplus \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i} \\
& = \mathrm{Inf}_{B/U}^B \cancel{\mathrm{Def}_{B/U}^B \mathrm{Inf}_{B/U}^B} \mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \\
& \qquad\qquad \oplus \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B \mathrm{Ind}_D^B \mathrm{Res}_D^B \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_j \times \mu_i) \\
& = \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \oplus \mathrm{Inf}_{B/U}^B \mathrm{Def}_{B/U}^B \mathrm{Ind}_D^B \mathrm{Res}_D^B \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_j \times \mu_i) \\
& = \mu_{i,j} \oplus \mathrm{Inf}_{B/U}^B \mathrm{Ind}_{DU/U}^{B/U} \mathrm{Iso}\theta \cancel{\mathrm{Def}_{D/D\cap U}^D \mathrm{Inf}_{D/D\cap U}^D} \mathrm{Iso}\theta^{-1} \mathrm{Res}_{DU/U}^{B/U} \mathrm{Iso}(\kappa)(\mu_j \times \mu_i) \\
& = \mu_{i,j} \oplus \mathrm{Inf}_{B/U}^B \cancel{\mathrm{Ind}_{B/U}^{B/U}} \cancel{\mathrm{Iso}\theta \mathrm{Iso}\theta^{-1}} \cancel{\mathrm{Res}_{B/U}^{B/U}} \mathrm{Iso}(\kappa)(\mu_j \times \mu_i) \\
& = \mu_{i,j} \oplus \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_j \times \mu_i) \\
& = \mu_{i,j} \oplus \mu_{j,i}
\end{aligned}
$$

where Equality (*) comes from (3.1). Since each $\mu_{i,j}$ and $\mu_{j,i}$ is one-dimensional, $J(\mathrm{Ind}_B^G \mu_{i,j})$ has dimension 2. Thus we have the conclusion :

$$
J(\mathrm{Ind}_B^G \mu_{i,j}) = \mu_{i,j} \oplus \mu_{j,i}. \tag{3.2}
$$

□

Lemma 3.8 ([1]). *Let $\mu_{i,j}$ be a one-dimensional representation of B. Then*

$$
\left\langle \mathrm{Res}_P^G \mathrm{Ind}_B^G \mu_{i,j}, \pi \right\rangle = 1.
$$

Proof. We have, by Frobenius Reciprocity Theorem,

$$\begin{aligned}
\left\langle \mathrm{Res}_P^G \mathrm{Ind}_B^G \mu_{i,j}, \pi \right\rangle &= \left\langle \mathrm{Res}_P^B \mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j}, \pi \right\rangle \\
&\overset{(*)}{=} \left\langle \mathrm{Res}_P^B (\mu_{i,j} \ \oplus \ \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}), \pi \right\rangle \\
&= \left\langle \mathrm{Res}_P^B \mu_{i,j}, \pi \right\rangle + \left\langle \mathrm{Res}_P^B \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}, \pi \right\rangle \\
&= 0 + \left\langle \mathrm{Res}_D^B \mu_{j,i}, \mathrm{Res}_D^B \mathrm{Ind}_P^B \pi \right\rangle \\
&= \left\langle \mathrm{Res}_D^B \mu_{j,i}, \mathrm{Res}_D^B \mathrm{Ind}_U^B \psi_1 \right\rangle \\
&\overset{(**)}{=} \left\langle \mathrm{Res}_D^B \mu_{j,i}, \mathrm{Ind}_{\{1\}}^D \mathrm{Res}_{\{1\}}^U \psi_1 \right\rangle \\
&= \left\langle \mathrm{Res}_{\{1\}}^B \mu_{j,i}, \mathrm{Res}_{\{1\}}^U \psi_1 \right\rangle = 1.
\end{aligned}$$

Equation 3.1 gives the Equality (*). Mackey formula and $[D\backslash B/U] = \{1\}$ implies that $\mathrm{Res}_D^B \mathrm{Ind}_U^B \psi_1 = \mathrm{Ind}_{D\cap U}^D \mathrm{Res}_{U\cap D}^U \psi_1$. Here $U \cap D = \{1\}$ which gives Equality (**). $\square$

Lemma 3.9. *If $\mu_{i,j}$ is a one-dimensional character of B, then*

$$\mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} = J(\mathrm{Ind}_B^G \mu_{i,j}) \oplus \lambda = \mu_{i,j} \oplus \mu_{j,i} \oplus \lambda$$

where λ is a $(q-1)$-dimensional irreducible representation of B.

Proof. We have $\mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} = \mu_{i,j} \oplus \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}$ by (3.1). Observe that the representation $\mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}$ is a q-dimensional representation of B. All irreducible representations of B are described in Theorem 3.4. Hence, we conclude that $\mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}$ is either a sum of q many one-dimensional characters of B or it is a sum of a one-dimensional character and a $(q-1)$-dimensional irreducible representation of B. However it can not be a sum of one-dimensional characters of B since in this case $\mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j}$ would be a sum of one-dimensional characters of B and then $\mathrm{Res}_P^G \mathrm{Ind}_B^G \mu_{i,j}$ would be a sum of one-dimensional characters of P which contradicts with Lemma 3.8. Therefore $\mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}$ is a sum of a one-dimensional character and a $(q-1)$-dimensional irreducible representation of B. By Frobenius Reciprocity Theorem, we have

$$\left\langle \mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i}, \mu_{j,i} \right\rangle = \left\langle \mathrm{Res}_D^B \mu_{j,i}, \mathrm{Res}_D^B \mu_{j,i} \right\rangle = 1.$$

Thus,

$$\mathrm{Ind}_D^B \mathrm{Res}_D^B \mu_{j,i} = \mu_{j,i} \oplus \lambda$$

for some $(q-1)$-dimensional irreducible representation λ of B. Thus, we obtain

$$\mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} = \mu_{i,j} \oplus \mu_{j,i} \oplus \lambda.$$

Hence, by using Equality 3.2, we conclude that

$$\mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} = J(\mathrm{Ind}_B^G \mu_{i,j}) \oplus \lambda.$$

$\square$

Lemma 3.10 ([1]). *If $\mu_{i,j}$ is a one-dimensional character of B, then $\mathrm{Ind}_B^G \mu_{i,j}$ has at most two irreducible components. Moreover, $\mathrm{Ind}_B^G \mu_{i,j}$ is irreducible if $\mu_{i,j} \neq \mu_{j,i}$ and reducible otherwise.*

Proof. We have, by Frobenius Reciprocity Theorem,

$$\begin{aligned}
\left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{i,j} \right\rangle &= \left\langle \mu_{i,j}, \mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j} \right\rangle \\
&\overset{(*)}{=} \langle \mu_{i,j}, \mu_{i,j} \oplus \mu_{j,i} \oplus \lambda \rangle \\
&= \langle \mu_{i,j}, \mu_{i,j} \rangle + \langle \mu_{i,j}, \mu_{j,i} \rangle + \langle \mu_{i,j}, \lambda \rangle \\
&= 1 + \langle \mu_{i,j}, \mu_{j,i} \rangle + 0 \\
&\overset{(**)}{=} \begin{cases} 1 & \text{if} \quad \mu_{i,j} \neq \mu_{j,i} \\ 2 & \text{if} \quad \mu_{i,j} = \mu_{j,i} \end{cases}
\end{aligned}$$

where (*) comes from Lemma 3.9, here λ is a $(q-1)$-dimensional irreducible representation of B and (**) comes from Equation 3.2. □

Lemma 3.11 ([1]). *If $\mu_{i,j}$ is a one-dimensional character of B and $\mu_{i,j} = \mu_{j,i}$ then $\mathrm{Ind}_B^G \mu_{i,j}$ has a one-dimensional component.*

Proof. Let $\mu_{i,j} = \mu_{j,i}$. Then by Lemma 3.10, there are exactly two irreducible components of $\mathrm{Ind}_B^G \mu_{i,j}$ i.e. $\mathrm{Ind}_B^G \mu_{i,j} = \chi_1 \oplus \chi_2$ for some irreducible representations χ_1, χ_2 of G. Then restricting it to B and using Lemma 3.9 we get

$$\mu_{i,j} \oplus \mu_{i,j} \oplus \lambda = \mathrm{Res}_B^G \chi_1 \oplus \mathrm{Res}_B^G \chi_2.$$

Also, $\left\langle \mathrm{Res}_B^G \chi_i, \mu_{i,j} \right\rangle = \left\langle \chi_i, \mathrm{Ind}_B^G \mu_{i,j} \right\rangle = 1$ for each i. Without loss of generality, we may assume that $\mathrm{Res}_B^G \chi_1 = \mu_{i,j}$ which implies that $\dim \chi_1 = 1$. Hence, we conclude that $\mathrm{Ind}_B^G \mu_{i,j}$ has a one-dimensional component. □

Corollary 3.12 ([1]). *If $\mu_{i,j}$ is a one-dimensional character of B and $\mathrm{Ind}_B^G \mu_{i,j}$ is reducible, then $\mathrm{Ind}_B^G \mu_{i,j}$ has a one-dimensional component.*

Proof. The result follows from Lemma 3.10 and Lemma 3.11. □

Lemma 3.13 ([1]). *Assume $\mu_{i,j}$ and $\mu_{k,l}$ are two distinct one-dimensional characters of B. Then $\left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{k,l} \right\rangle \neq 0$ if and only if $\mu_{k,l} = \mu_{j,i}$*

Proof. By using Frobenius Reciprocity Theorem, we have

$$\begin{aligned} \left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{k,l} \right\rangle &= \left\langle \mathrm{Res}_B^G \mathrm{Ind}_B^G \mu_{i,j}, \mu_{k,l} \right\rangle \\ &\overset{(*)}{=} \langle \mu_{i,j} \oplus \mu_{j,i} \oplus \lambda, \mu_{k,l} \rangle \\ &= \langle \mu_{i,j}, \mu_{k,l} \rangle + \langle \mu_{j,i}, \mu_{k,l} \rangle + \langle \lambda, \mu_{k,l} \rangle \\ &= 0 + \langle \mu_{j,i}, \mu_{k,l} \rangle + 0 \\ &= \langle \mu_{j,i}, \mu_{k,l} \rangle \end{aligned}$$

where (*) comes from Lemma 3.9. Therefore $\left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{k,l} \right\rangle \neq 0$ if and only if $\mu_{j,i} = \mu_{k,l}$. □

Lemma 3.14 ([1]). *Assume $\mu_{i,j}$ and $\mu_{k,l}$ are two distinct one-dimensional characters of B. Then $\mathrm{Ind}_B^G \mu_{i,j} = \mathrm{Ind}_B^G \mu_{k,l}$ if and only if $\mu_{k,l} = \mu_{j,i}$, or in the open form, $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_k \times \mu_l) = \mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ if and only if $\mu_k = \mu_j$ and $\mu_l = \mu_i$.*

Proof. Assume $\mu_{i,j}$ and $\mu_{k,l}$ are two distinct one-dimensional characters of B. If $\mathrm{Ind}_B^G \mu_{i,j} = \mathrm{Ind}_B^G \mu_{k,l}$, then $\left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{k,l} \right\rangle \neq 0$. Hence, by Lemma 3.13 we have $\mu_{k,l} = \mu_{j,i}$.

For the other side, assume $\mu_{k,l} = \mu_{j,i}$. Since $\mu_{i,j}$ and $\mu_{k,l}$ are distinct characters, we get that $\mu_{i,j} \neq \mu_{j,i}$ and $\mu_{k,l} \neq \mu_{l,k}$. Thus by Lemma 3.10, $\mathrm{Ind}_B^G \mu_{i,j}$ and $\mathrm{Ind}_B^G \mu_{k,l}$ are irreducible and $\left\langle \mathrm{Ind}_B^G \mu_{i,j}, \mathrm{Ind}_B^G \mu_{k,l} \right\rangle \neq 0$ by Lemma 3.13. Hence we obtain $\mathrm{Ind}_B^G \mu_{i,j} = \mathrm{Ind}_B^G \mu_{k,l}$. □

As a summary of this section, we have the following result.

Theorem 3.15 ([1]). *If $\mu_{i,j}$ and $\mu_{k,l}$ are one-dimensional characters of B, then*

(i) $\dim(\mathrm{Ind}_B^G \mu_{i,j}) = q+1$;

(ii) $\mathrm{Ind}_B^G \mu_{i,j}$ *is irreducible if and only if* $\mu_{i,j} \neq \mu_{j,i}$;

(iii) $\mathrm{Ind}_B^G \mu_{i,j}$ *is a direct sum of a* 1*-dimensional and a* q*-dimensional irreducible representation if it is reducible;*

(iv) $\mathrm{Ind}_B^G \mu_{k,l} = \mathrm{Ind}_B^G \mu_{i,j}$ *if and only if* $\mu_{k,l} = \mu_{i,j}$ *or* $\mu_{k,l} = \mu_{j,i}$.

Now, based on this theorem, we will classify irreducible representations of G, obtained by induction from B. Let $\mu_{i,j} = \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ be a one-dimensional character of B.

Case 1: If $\mu_i = \mu_j$, which is equivalent to say $\mu_{i,j} = \mu_{j,i}$, then by (ii) and (iii) of Theorem 3.15 we get $\mathrm{Ind}_B^G \mu_{i,j}$ is the direct sum of a one-dimensional representation, which we denote it by $\rho'_{(\mu_i,\mu_i)}$, and a q-dimensional irreducible representation, which we denote it by $\rho_{(\mu_i,\mu_i)}$. Since $K^\times$ has $q-1$ characters, by this way, we get $(q-1)$ many one dimensional characters of G and $q-1$ many q-dimensional irreducible representations of G. Lemma 3.13 ensures that they are all distinct. Also observe that

$$\mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_i)\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right) = \mu_i(\alpha)\cdot\mu_i(\delta) = \mu_i \circ \det\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right).$$

Thus,

$$\left\langle \mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_i), \mu_i \circ \det \right\rangle = \left\langle \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_i), \mathrm{Res}_B^G \mu_i \circ \det \right\rangle = 1$$

i.e. $\rho'_{(\mu_i,\mu_i)} = \mu_i \circ \det$.

Case 2: If $\mu_i \neq \mu_j$, which is equivalent to say $\mu_{i,j} \neq \mu_{j,i}$, then $\mathrm{Ind}_B^G \mu_{i,j}$ is a $(q+1)$-dimensional irreducible representation of G by (ii) of Theorem 3.15. We denote it by $\rho_{(\mu_i,\mu_j)}$. All distinct irreducible representations of B induce distinct representation of G except the pairs $\mu_{i,j}$ and $\mu_{j,i}$ by (iv) of Theorem 3.15. Then there are $\dfrac{(q-1)(q-2)}{2}$ many $(q+1)$-dimensional irreducible representations of G obtained by this way.

Thus, we have the following theorem:

Theorem 3.16 ([1]). *The irreducible representations of G which are summands of the induced representations $\mathrm{Ind}_B^G \mu_{i,j}$ where $\mu_{i,j} = \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ is a one-dimensional character of B consist of the followings:*

(i) $q-1$ *many one-dimensional characters denoted by* $\rho'_{(\mu_i,\mu_i)}$ *and equal to* $\mu_i \circ \det$ *where they are all the one-dimensional characters of* G*,*

(ii) $q-1$ *many* q*-dimensional irreducible representations denoted by* $\rho_{(\mu_i,\mu_i)}$*,*

(iii) $\dfrac{(q-1)(q-2)}{2}$ *many* $(q+1)$*-dimensional irreducible representations denoted by* $\rho_{(\mu_i,\mu_j)}$*.*

3.5. Cuspidal Representations of G

Irreducible representations of G that are not components of $\mathrm{Ind}_B^G \mu_{i,j}$, where $\mu_{i,j}$ is one-dimensional character of B, are said to be cuspidal.

Lemma 3.17. *There are $\frac{1}{2}(q^2 - q)$ many cuspidal representations of G.*

Proof. By Lemma 1.10, Theorem 2.5 and Theorem 3.16. □

Lemma 3.18 ([1]). *Let ρ be a representation of G. Then $J(\rho) \neq 0$ if and only if there exists a one-dimensional character $\mu_{i,j}$ of B such that $\langle \rho, \mathrm{Ind}_B^G \mu_{i,j} \rangle \neq 0$.*

Proof. Let ρ be a representation of G and $\mu_{i,j}$ be any one-dimensional character of B. Then, by using Theorem 1.40 and Frobenius Reciprocity Theorem,

$$\begin{aligned}\langle J(\rho), \mu_{i,j}\rangle &= \langle \mathrm{Inf}_{B/U}^{B}\mathrm{Def}_{B/U}^{B}\mathrm{Res}_{B}^{G}\rho, \mathrm{Inf}_{B/U}^{B}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)\rangle \\ &= \left\langle \rho, \mathrm{Ind}_{B}^{G}\mathrm{Inf}_{B/U}^{B}\,\cancel{\mathrm{Def}_{B/U}^{B}\mathrm{Inf}_{B/U}^{B}}\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)\right\rangle \\ &= \left\langle \rho, \mathrm{Ind}_{B}^{G}\mu_{i,j}\right\rangle\end{aligned}$$

where the cancellation comes from Remark 1.34.

Let $J(\rho) \neq 0$. Since B/U is abelian, by Corollary 1.12 any representation of B/U is a sum of some one-dimensional representations of B/U. Then by (ii) of Remark 1.18 and the fact that inflating a representation does not change the dimension, we conclude that $J(\rho)$ is sum of some one-dimensional representations of B. Thus $\langle J(\rho), \mu_{i,j}\rangle \neq 0$ for some one-dimensional character $\mu_{i,j}$ of B. Hence, by the equations above, for that character $\mu_{i,j}$ of B, we have $\left\langle \rho, \mathrm{Ind}_{B}^{G}\mu_{i,j}\right\rangle \neq 0$.

For the other side, assume $\left\langle \rho, \mathrm{Ind}_{B}^{G}\mu_{i,j}\right\rangle \neq 0$ for some one-dimensional character $\mu_{i,j}$ of B. Then $\langle J(\rho), \mu_{i,j}\rangle \neq 0$ for some one-dimensional character $\mu_{i,j}$ of B which implies that $J(\rho) \neq 0$. □

Lemma 3.18 shows us that an irreducible representation ρ of G is cuspidal if and only if $J(\rho) = 0$.

Lemma 3.19 ([1]). *Let ρ be a cuspidal representation of G. Then $\mathrm{Res}_{P}^{G}\rho = r\pi$ for some positive integer r. In particular $\dim\rho = r.(q-1)$ is a multiple of $q-1$.*

Proof. Let ρ be any irreducible representation of G. Assume that $\mathrm{Res}_{P}^{G}\rho$ contains a one-dimensional character of P as a summand. By Theorem 3.3 they are of the form $\mathrm{Inf}_{P/U}^{P}\mathrm{Iso}(\xi)\chi$ for some irreducible representation χ of A. Then for that χ, by

Frobenius Rciprocity Theorem and commutation relations in Section 1, we get

$$\begin{aligned} 0 &\neq \left\langle \mathrm{Res}_P^G \rho, \mathrm{Inf}_{P/U}^P \mathrm{Iso}(\xi)\chi \right\rangle \\ &= \left\langle \mathrm{Res}_B^G \rho, \mathrm{Ind}_P^B \mathrm{Inf}_{P/U}^P \mathrm{Iso}(\xi)\chi \right\rangle \\ &= \left\langle \mathrm{Res}_B^G \rho, \mathrm{Inf}_{B/U}^B \mathrm{Ind}_{P/U}^{B/U} \mathrm{Iso}(\xi)\chi \right\rangle . \end{aligned}$$

By similar argument as in proof of Lemma 3.18, we say that $\mathrm{Inf}_{B/U}^B \mathrm{Ind}_{P/U}^{B/U} \mathrm{Iso}(\xi)\chi$ is a sum of some one-dimensional characters of B. Then by the inequality above, we conclude that $\mathrm{Res}_B^G \rho$ contains a one-dimensional representation of B which implies that ρ is not cuspidal. Hence for any cuspidal representation ρ of G, there is no one-dimensional component of $\mathrm{Res}_P^G \rho$. Therefore, $\mathrm{Res}_P^G \rho = r\pi$ for some positive integer r by Theorem 3.3. □

Lemma 3.20 ([1]). *(i) If ρ is a cuspidal representation of G, then $\mathrm{Res}_P^G \rho = \pi$ and $\mathrm{dim}\rho = (q-1)$.*

(ii) Conversely, if ρ is a representation of G such that $\mathrm{Res}_P^G \rho = \pi$, then ρ is cuspidal.

Proof. (i) By Lemma 3.19, for any cuspidal representation ρ of G $\mathrm{dim}\rho = r(q-1)$. By Lemma 1.11, Theorem 3.16 and Lemma 3.17 we get

$$\begin{aligned} |G| &= \sum_{\sigma \in \mathrm{Irr}(G)} (\mathrm{dim}\sigma)^2 \\ &= (q-1)1^2 + (q-1)q^2 + \frac{1}{2}(q-1)(q-2)(q+1)^2 + \frac{1}{2}(q^2-q)r^2(q-1)^2. \end{aligned}$$

Also $|G| = (q-1)^2 q(q+1)$ implies that $r = 1$. Therefore, $\mathrm{Res}_P^G \rho = \pi$ and $\mathrm{dim}\rho = (q-1)$.

(ii) Let ρ be a representation of G such that $\mathrm{Res}_P^G \rho = \pi$. We see that ρ is irreducible since its restriction to P is irreducible. By contradiction assume that ρ is not cuspidal. Then for some one-dimensional character $\mu_{i,j}$ of B, we get

$$0 \neq \left\langle \rho, \mathrm{Ind}_B^G \mu_{i,j} \right\rangle = \left\langle \mathrm{Res}_B^G \rho, \mu_{i,j} \right\rangle$$

by Frobenius Reciprocity Theorem. Thus we obtain that $\mu_{i,j}$ is a summand of $\mathrm{Res}_B^G \rho$. Then if we restrict it to P we obtain that $\mathrm{Res}_P^B \mu_{i,j}$ is a summand of $\mathrm{Res}_P^G \rho$ which is equal to π by assumption. It gives us a contradiction. Hence $\mathrm{Res}_P^G \rho = \pi$ implies that ρ is cuspidal.

□

3.6. Inducing $(q-1)$-Dimensional Representations from B to G

In this section, we find irreducible components of representations of G obtained by inducing $(q-1)$-dimensional irreducible representations of B. We know that $Z \cong K^\times$ and $A \cong K^\times$. Then any character of Z can be seen of the form $\mathrm{Iso}(\iota)\mu_i$ and any character of A can be seen of the form $\mathrm{Iso}(\eta)\mu_i$ where ι, η are the isomorphisms defined in Section 2.1 and μ_i is a character of $K^\times$. For simplicity, we assign letters $\dot{\mu}_i$ and $\ddot{\mu}_i$ for the representations $\mathrm{Iso}(\iota)\mu_i$ and $\mathrm{Iso}(\eta)\mu_i$, respectively.

Let $\dot{\mu}_r$ be a fixed character of Z. Then $\pi \times \dot{\mu}_r$ is a $(q-1)$-dimensional irreducible representation of B by Theorem 3.4. In this section we will decompose the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.

Lemma 3.21. *We have* $\mathrm{Res}_A^P \pi = \bigoplus_{k=1}^{q-1} \ddot{\mu}_k$.

Proof. Observe that

$$\mathrm{Res}_A^P \pi = \mathrm{Res}_A^P \mathrm{Ind}_U^P \psi_1 \overset{(*)}{=} \mathrm{Ind}_{A\cap U}^A \mathrm{Res}_{A\cap U}^U \psi_1 = \mathrm{Ind}_{\{1\}}^A \mathrm{Res}_{\{1\}}^U \psi_1 \overset{(**)}{=} \bigoplus_{k=1}^{q-1} \ddot{\mu}_k$$

where $(*)$ comes from Mackey Formula and the fact that $P = AU$ and $(**)$ can be seen by Frobenius Reciprocity Theorem. $\square$

We have seen that D is the direct product of A and Z. Therefore any irreducible character of D can be seen as the direct product of the irreducible characters of A and Z by Lemma 1.14. In view of this information, we have the following Lemma.

Lemma 3.22. *Let $\pi \times \dot{\mu}_i$ be a $(q-1)$-dimensional irreducible representation of B. Then*

$$\mathrm{Res}^B_D(\pi \times \dot{\mu}_i) = \bigoplus_{k=1}^{q-1}(\ddot{\mu}_k \times \dot{\mu}_i).$$

Proof. Let $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ be an arbitrary element of D. Then

$$\begin{aligned}
\pi \times \dot{\mu}_i \left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right) &= \pi \times \dot{\mu}_i \left(\begin{pmatrix} \alpha\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \right) \\
&= \pi \left(\begin{pmatrix} \alpha\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \dot{\mu}_i \left(\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \right) \\
&= \mathrm{Res}^P_A \pi \left(\begin{pmatrix} \alpha\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \dot{\mu}_i \left(\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \right) \\
&= (\mathrm{Res}^P_A \pi) \times \dot{\mu}_i \left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right).
\end{aligned}$$

Thus, by Lemma 3.21 we have the result

$$\mathrm{Res}^B_D(\pi \times \dot{\mu}_i) \overset{(*)}{=} (\mathrm{Res}^P_A \pi) \times \dot{\mu}_i = \left(\bigoplus_{k=1}^{q-1} \ddot{\mu}_k \right) \times \dot{\mu}_i = \bigoplus_{k=1}^{q-1} (\ddot{\mu}_k \times \dot{\mu}_i).$$

$\square$

First, we decompose the representation $\mathrm{Res}_B^G \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$, then decompose the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ with the help of it. By Mackey Formula we have

$$\begin{aligned}
\mathrm{Res}_B^G \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r) &= \bigoplus_{\substack{x \in [B \backslash G / B] \\ = \{1, w\}}} \mathrm{Ind}_{B \cap {}^x B}^B \mathrm{Iso}(\gamma_x) \mathrm{Res}_{B^x \cap B}^B (\pi \times \dot{\mu}_r) \\
&= \mathrm{Ind}_{B \cap B}^B \mathrm{Res}_B^B (\pi \times \dot{\mu}_r) \oplus \mathrm{Ind}_{B \cap {}^w B}^B \mathrm{Iso}(\gamma_w) \mathrm{Res}_{B^w \cap B}^B (\pi \times \dot{\mu}_r) \\
&= (\pi \times \dot{\mu}_r) \oplus \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w) \mathrm{Res}_D^B (\pi \times \dot{\mu}_r) \\
&\overset{(*)}{=} (\pi \times \dot{\mu}_r) \oplus \left[\mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w) \left(\bigoplus_{k=1}^{q-1} (\ddot{\mu}_k \times \dot{\mu}_r) \right) \right] \\
&\overset{(**)}{=} (\pi \times \dot{\mu}_r) \oplus \left(\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w) (\ddot{\mu}_k \times \dot{\mu}_r) \right)
\end{aligned}$$

where $(*)$ is by Lemma 3.22. Thus we have

$$\mathrm{Res}_B^G \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r) = (\pi \times \dot{\mu}_r) \oplus \left(\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w) (\ddot{\mu}_k \times \dot{\mu}_r) \right) \tag{3.3}$$

Now, the next aim is to find the irreducible components of $\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r)$. Below, we describe two ways to obtain irreducible characters of D, one is through the characters of $K^\times \times K^\times$ and the other is through the characters of $A \times Z$. Observe that the diagram

$$\begin{tikzcd}
 & \begin{pmatrix} \alpha\delta & 0 \\ 0 & \delta \end{pmatrix} & \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \arrow[l, mapsto] & \\
\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} & D & A \times Z \arrow[l, "\gamma_w"'] & \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \\
 & K^\times \times K^\times \arrow[u, "\nu"] \arrow[ur, "\iota \times \eta"'] & & \\
 & (\alpha, \delta) \arrow[uul, mapsto] \arrow[uurr, mapsto] & &
\end{tikzcd}$$

does not commute. To compare the characters:

$$\begin{tikzcd}
& \mathrm{Iso}(\gamma_w)(\ddot{\mu}_i \times \dot{\mu}_j) & & \ddot{\mu}_i \times \dot{\mu}_j \arrow[ll, mapsto] & \\
\mathrm{Iso}(\nu)(\mu_i \times \mu_j) & D & & A \times Z \arrow[ll, "\gamma_w"'] & \ddot{\mu}_i \times \dot{\mu}_j \\
& & K^\times \times K^\times \arrow[ul, "\nu"] \arrow[ur, "\iota \times \eta"'] & & \\
& & \mu_i \times \mu_j \arrow[uull, mapsto] \arrow[uurr, mapsto] & &
\end{tikzcd}$$

The diagram does not commute also. The claim below, describe the conditions on characters of $K^\times \times K^\times$ and $A \times Z$ which makes the corresponding representations in D equal.

Claim 3.23. *Let* $\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_l)$ *and* $\mathrm{Iso}(\nu)(\mu_i \times \mu_j)$ *be two irreducible characters of* D *where* μ_k, μ_l, μ_i, μ_j *are irreducible characters of* $K^\times$ *and* ν *is the isomorphism defined in Section* 2.1. *Then*

$$\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_l) = \mathrm{Iso}(\nu)(\mu_i \times \mu_j) \text{ if and only if } k = j \text{ and } \overline{\mu_k}\mu_l = \mu_i$$

where $\bar{\mu}_k$ *is the conjugate character of* μ_k.

Proof. Observe that

$$
\begin{aligned}
&\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_l) = \mathrm{Iso}(\nu)(\mu_i \times \mu_j) \\
\Longleftrightarrow\ &\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_l)\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) = \mathrm{Iso}(\nu)(\mu_i \times \mu_j)\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\ddot{\mu}_k \times \dot{\mu}_l \left(\begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}\right) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\ddot{\mu}_k \times \dot{\mu}_l \left(\begin{pmatrix} \delta\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\ddot{\mu}_k \left(\begin{pmatrix} \delta\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \dot{\mu}_l \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\mu_k(\delta\alpha^{-1})\mu_l(\alpha) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\mu_k(\delta)\mu_k(\alpha^{-1})\mu_l(\alpha) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\overset{(*)}{\Longleftrightarrow}\ &\mu_k(\delta)\overline{\mu_k}(\alpha)\mu_l(\alpha) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\mu_k(\delta)\overline{\mu_k}\mu_l(\alpha) = \mu_i(\alpha)\mu_j(\delta) \quad \forall \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\mu_k = \mu_j \text{ and } \overline{\mu_k}\mu_l = \mu_i \\
\Longleftrightarrow\ &k = j \text{ and } \overline{\mu_k}\mu_l = \mu_i
\end{aligned}
$$

where $(*)$ is by Remark 1.26. □

The following Lemma calculates one-dimensional irreducible components of the representation $\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r)$.

Lemma 3.24. *Let $\mu_{i,j} = \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ be a one dimensional character of B. Then*

$$
\left\langle \bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mu_{i,j} \right\rangle = \begin{cases} 1, & \text{if} \quad \mu_i = \bar{\mu}_j \mu_r, \\ 0, & \text{otherwise.} \end{cases}
$$

Proof. By Frobenius Reciprocity Theorem, we have

$$\begin{aligned}
\left\langle \bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mu_{i,j} \right\rangle &= \sum_{k=1}^{q-1} \left\langle \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mu_{i,j} \right\rangle \\
&= \sum_{k=1}^{q-1} \left\langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mathrm{Res}_D^B \mu_{i,j} \right\rangle \\
&\overset{(*)}{=} \sum_{k=1}^{q-1} \langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mathrm{Iso}(\nu)(\mu_i \times \mu_j) \rangle \\
&\overset{(**)}{=} \langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_j \times \dot{\mu}_r), \mathrm{Iso}(\nu)(\mu_i \times \mu_j) \rangle \\
&\overset{(***)}{=} \begin{cases} 1, & \text{if} \quad \mu_i = \bar{\mu}_j \mu_r, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Here $(*)$ comes from the proof of the Lemma 3.5, $(**)$ and $(***)$ comes from Claim 3.23. □

Corollary 3.25. *The representation* $\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r)$ *has* $(q-1)$*-many one-dimensional components of the form* $\mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\bar{\mu}_j \mu_r \times \mu_j)$ *for* $j = 1, 2, \ldots, q-1$.

The following Lemma calculates $(q-1)$-dimensional irreducible components of the representation $\bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r)$.

Lemma 3.26. *Let* $\pi \times \dot{\mu}_j$ *be a* $(q-1)$*-dimensional irreducible representation of* B*. Then*

$$\left\langle \bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \pi \times \dot{\mu}_j \right\rangle = \begin{cases} q-1, & \textit{if} \quad j = r, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. First observe that

$$
\begin{aligned}
&\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r) = \ddot{\mu}_l \times \dot{\mu}_j \\
\Longleftrightarrow\ &\mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r)\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) = \ddot{\mu}_l \times \dot{\mu}_j \left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) \text{ for all } \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\ddot{\mu}_k \times \dot{\mu}_r \left(\begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}\right) = \ddot{\mu}_l \times \dot{\mu}_j \left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right) \text{ for all } \alpha, \delta \in K^\times \\
\overset{(*)}{\Longleftrightarrow}\ &\mu_k(\delta)\bar{\mu}_k\mu_r(\alpha) = \mu_l(\alpha)\bar{\mu}_l\mu_j(\beta) \text{ for all } \alpha, \delta \in K^\times \\
\Longleftrightarrow\ &\mu_k = \bar{\mu}_l\mu_j \text{ and } \bar{\mu}_k\mu_r = \mu_l \\
\Longleftrightarrow\ &\mu_k = \bar{\mu}_l\mu_j \text{ and } \bar{\mu}_l\bar{\mu}_k\mu_r = \bar{\mu}_l\mu_l \\
\overset{(**)}{\Longleftrightarrow}\ &\mu_k = \bar{\mu}_l\mu_j \text{ and } \bar{\mu}_j\mu_r = 1 \\
\Longleftrightarrow\ &\mu_k = \bar{\mu}_l\mu_j \text{ and } j = r
\end{aligned}
$$

where $(*)$ is calculated before in the proof of Claim 3.23 and $(**)$ is by Remark 1.26. If we fix $j = r$ then there are $(q-1)$ many pairs of (k, l) satisfying the above equality. By Frobenius Reciprocity Theorem, we have

$$
\begin{aligned}
\left\langle \bigoplus_{k=1}^{q-1} \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \pi \times \dot{\mu}_j \right\rangle &= \sum_{k=1}^{q-1} \left\langle \mathrm{Ind}_D^B \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \pi \times \dot{\mu}_j \right\rangle \\
&= \sum_{k=1}^{q-1} \left\langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \mathrm{Res}_D^B(\pi \times \dot{\mu}_j) \right\rangle \\
&\overset{(**)}{=} \sum_{k=1}^{q-1} \left\langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), \bigoplus_{l=1}^{q-1} (\ddot{\mu}_l \times \dot{\mu}_j) \right\rangle \\
&= \sum_{k=1}^{q-1} \sum_{l=1}^{q-1} \left\langle \mathrm{Iso}(\gamma_w)(\ddot{\mu}_k \times \dot{\mu}_r), (\ddot{\mu}_l \times \dot{\mu}_j) \right\rangle \\
&\overset{(***)}{=} \begin{cases} q-1, & \text{if} \quad j = r, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

where $(**)$ is by Lemma 3.22 and $(***)$ is by the above observation. $\square$

Corollary 3.27. *The decomposition of* $\mathrm{Res}_B^G\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ *to irreducible components is given by*

$$\mathrm{Res}_B^G\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r) = q(\pi \times \dot{\mu}_r) \oplus \bigoplus_{j=1}^{q-1} \mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_j\mu_r \times \mu_j)$$

Proof. Proof comes directly from (3.3), Corollary 3.25 and Lemma 3.26. □

Corollary 3.27 gives the decomposition of $\mathrm{Res}_B^G\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ to irreducible components. Now, we are ready to decompose $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.

Corollary 3.28. *Let* $\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_j)$ *be a one dimensional character of* B. *Then*

$$\left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \right\rangle = \begin{cases} 1, & \textit{if} \quad \mu_i = \bar{\mu}_j\mu_r, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By Frobenius Reciprocity Theorem and by Corollary 3.27, we have

$$\begin{aligned} &\left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \right\rangle \\ &\qquad = \left\langle \mathrm{Res}_B^G\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_j) \right\rangle = \begin{cases} 1, & \text{if} \quad \mu_i = \bar{\mu}_j\mu_r, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

□

The following lemma proves that the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has no one-dimensional component.

Lemma 3.29. *Let* χ *be a one dimensional character of* G. *Then*

$$\left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \chi \right\rangle = 0.$$

Proof. By Frobenius Reciprocity Theorem, we have,

$$\left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \chi \right\rangle = \left\langle \pi \times \dot{\mu}_r, \mathrm{Res}_B^G \chi \right\rangle = 0$$

since they are irreducible representations of G having different degree. $\square$

Now we examine the character μ_r and then continue from where we left off.

Definition 3.30. *We call μ_r a square character of $K^\times$ if $\mu_r = \mu_i\mu_i$ for some $i \in \{1, 2, \ldots, q-1\}$.*

Remark 3.31. *We know that any polynomial of degree n over a field has at most n roots. Thus, for an element $k \in K^\times$ the polynomial $x^2 - k$ has at most 2 roots in K, indeed in $K^\times$. Also, since $x^2 - k$ is a polynomial of degree 2, it has either two roots in $K^\times$ or has no root in $K^\times$. Therefore, if μ_r is a square character of $K^\times$, then there are exactly two irreducible characters μ_i and μ_j of $K^\times$ which are solutions of $x^2 = \mu_r$. This is because the character group of $K^\times$, denoted by $\widehat{K^\times}$, is isomorphic to the group $K^\times$ as $K^\times$ is abelian.*

Observe that $\{\mu_1^2, \mu_2^2, \ldots, \mu_{q-1}^2\} \subset \{\mu_1, \mu_2, \ldots, \mu_{q-1}\}$ where in the left side each element stand twice by Remark 3.31. Hence there are $\frac{(q-1)}{2}$ many square characters and $\frac{(q-1)}{2}$ many non-square characters of $K^\times$.

Remark 3.32. *By Remark 3.31, if μ_r is a square character of $K^\times$ then there exists $i, j \in \{1, 2 \ldots, q-1\}$ such that $\mu_r = \mu_i\mu_i = \mu_j\mu_j$ which is equivalent to say $\bar{\mu}_i\mu_r = \mu_i$ and $\bar{\mu}_j\mu_r = \mu_j$. For remaining $k \in \{1, 2 \ldots, q-1\} \setminus \{i, j\}$ we have $\mu_r \neq \mu_k\mu_k$ which is equivalent to say $\bar{\mu}_k\mu_r \neq \mu_k$. By Lemma 3.10 this implies that the representations $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_i\mu_r \times \mu_i)$ and $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_j\mu_r \times \mu_j)$ of G (or simply $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_i)$ and $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_j \times \mu_j)$) are reducible and for remaining $k \in \{1, 2 \ldots, q-1\} \setminus \{i, j\}$ $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_k\mu_r \times \mu_k)$ is irreducible.*

Following Lemma investigates $(q+1)$-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ for a square character μ_r of $K^\times$.

Lemma 3.33. *Let μ_r be a square character of $K^\times$ such that $\mu_r = \mu_i\mu_i = \mu_j\mu_j$ for some $i, j \in \{1, 2 \dots, q-1\}$. Then there are $\frac{(q-3)}{2}$ many $(q+1)$-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ of the form $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_k\mu_r \times \mu_k)$ for $k \in \{1, 2 \dots, q-1\} \setminus \{i, j\}$. We divided it by 2 since each counted as twice although they appear in once.*

Proof. We know by Theorem 3.16 and Lemma 3.20 that any $(q+1)$-dimensional irreducible representation of G is obtained by inducing one-dimensional characters of B. Observe that, by Corollary 3.28, we have

$$\left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_k \times \bar{\mu}_k\mu_r)\right\rangle = 1$$

since $\overline{\bar{\mu}_k\mu_r}\mu_r = \mu_k$. However, by Lemma 3.14

$$\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_k \times \bar{\mu}_k\mu_r) = \mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_k\mu_r \times \mu_k).$$

Then by Remark 3.32 for $k \in \{1, 2 \dots, q-1\} \setminus \{i, j\}$, all $(q+1)$-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ are given by $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\bar{\mu}_k\mu_r \times \mu_k)$ with each has multiplicity 1. $\square$

Following Lemma investigates q-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ for a square character μ_r of $K^\times$.

Lemma 3.34. *Let μ_r be a square character of $K^\times$ such that $\mu_r = \mu_i\mu_i = \mu_j\mu_j$ for some $i, j \in \{1, 2 \dots, q-1\}$. Then all q-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ are $\rho_{(\mu_i,\mu_i)}$ and $\rho_{(\mu_j,\mu_j)}$ where they are q-dimensional components of the $(q+1)$-dimensional representations $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_i \times \mu_i)$ and $\mathrm{Ind}_B^G\mathrm{Inf}_{B/U}^B\mathrm{Iso}(\kappa)(\mu_j \times \mu_j)$, respectively as described in Theorem 3.15*

Proof. We know by Theorem 3.16 and Lemma 3.20 that any q-dimensional irreducible representation of G is a component of a reducible representation obtained by inducing one-dimensional representation of B. By (iii) of Theorem 3.15 a reducible representation obtained by inducing one-dimensional representation of B is the sum of a one-dimensional character and a q-dimensional irreducible representation. Corollary 3.28, Lemma 3.29 and Remark 3.32 implies that $\rho_{(\mu_i,\mu_i)}$ and $\rho_{(\mu_j,\mu_j)}$ are all q-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ where they are q-dimensional components of $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_i \times \mu_i)$ and $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\mu_j \times \mu_j)$, respectively. □

Up to here we have found that, for a square character μ_r, $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has no one dimensional component, $\frac{(q-3)}{2}$ many $(q+1)$-dimensional irreducible components and 2 many q-dimensional components. Following Lemma gives the number of q-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.

Lemma 3.35. *Let μ_r be a square character. Then there are $\frac{q-1}{2}$ many $(q-1)$-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.*

Proof. Let n be the number of $(q-1)$-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$. Then, we get $n = \frac{q-1}{2}$ by the equality

$$(q-1)(q+1) = \mathrm{dim}\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r) = 2q + \frac{q-3}{2}(q+1) + n(q-1).$$

□

We proved the folllowing:

Proposition 3.36. *Let μ_r be a square character such that $\mu_r = \mu_i\mu_i = \mu_j\mu_j$ for some $i, j \in \{1, 2 \ldots, q-1\}$. Then irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ consist of the following:*

- $\frac{(q-1)}{2}$ *many $(q-1)$-dimensional irreducible components.*

- *2 many q-dimensional irreducible components, $\rho_{(\mu_i,\mu_i)}$ and $\rho_{(\mu_j,\mu_j)}$ where they are described as in Theorem* 3.15
- $\frac{(q-3)}{2}$ *many $(q+1)$-dimensional irreducible components of the form* $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\bar{\mu}_k\mu_r \times \mu_k)$ *for $k \in \{1, 2 \ldots, q-1\} \setminus \{i, j\}$ where of those that the same counted at once.*

Now we continue with the case that μ_r is not a square.

Remark 3.37. *Let μ_r be not a square character. Then, by definition $\mu_r \neq \mu_i\mu_i$ for all $i \in \{1, 2, \ldots, q-1\}$ which is equivalent to say $\bar{\mu}_i\mu_r \neq \mu_i$. Then the representation* $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\bar{\mu}_i\mu_r \times \mu_i)$ *is irreducible for all $i \in \{1, 2, \ldots, q-1\}$ by Lemma* 3.10*.*

Following three Lemmas investigate $(q+1)$-dimensional, q-dimensional and $(q-1)$-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ for a non-square character μ_r of $K^\times$.

Lemma 3.38. *Let μ_r be not a square. Then there are $\frac{(q-1)}{2}$ many $(q+1)$-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ where they are of the form $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\bar{\mu}_i\mu_r \times \mu_i)$. We divided it by 2 since each counted as twice although they appear in once.*

Proof. The proof is almost same as the proof of Lemma 3.33. We have the result by Corollary 3.28, by Lemma 3.14 and by Remark 3.37. □

Lemma 3.39. *Let μ_r be not a square. Then there is no q-dimensional irreducible component of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.*

Proof. We know, by Theorem 3.16 and Lemma 3.20 that any q-dimensional irreducible representation of G appear as a summand of a reducible representation obtained by inducing one-dimensional representation of B. Then the result is followed by Corollary 3.28 and Remark 3.37. □

Up to here we have found that, for a square character μ_r, $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has no one dimensional component and no q-dimensional component by Lemma 3.29 and Lemma 3.39, has $\frac{(q-1)}{2}$ many $(q+1)$-dimensional irreducible components by Lemma 3.38. Following Lemma gives the number of q-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.

Lemma 3.40. *Let μ_r be not a square character. Then there are $\frac{(q+1)}{2}$ many $(q-1)$-dimensional irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$.*

Proof. Let n be the number of $(q-1)$-dimensional irreducible components of the representation $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$. Then, we get $n = \frac{q+1}{2}$ by the equality

$$(q-1)(q+1) = \dim(\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)) = \frac{(q-1)}{2}(q+1) + n(q-1).$$

□

We proved the folllowing:

Proposition 3.41. *Let μ_r be a non-square character. Then irreducible components of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ consist of the following:*

- *$\frac{(q-1)}{2}$ many $(q-1)$-dimensional irreducible components.*
- *$\frac{(q-1)}{2}$ many $(q+1)$-dimensional irreducible components of the form $\mathrm{Ind}_B^G \mathrm{Inf}_{B/U}^B \mathrm{Iso}(\kappa)(\bar{\mu}_i \mu_r \times \mu_i)$ for $i \in \{1, 2 \ldots, q-1\}$ where of those that the same counted at once.*

Lemma 3.42. *Let μ_r be a representation of $K^\times$ which is either square or not a square. Then each $(q-1)$ dimensional component of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has multiplicity 1.*

Proof. Let ρ be a $(q-1)$-dimensional irreducible summand of $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$. By Frobenius Reciprocity Theorem, we have

$$1 \leq \left\langle \mathrm{Ind}_B^G(\pi \times \dot{\mu}_r), \rho \right\rangle = \left\langle \pi \times \dot{\mu}_r, \mathrm{Res}_B^G \rho \right\rangle$$

where the right hand side is smalller or equal to 1 since both $\pi \times \dot{\mu}_r$ and $\mathrm{Res}_B^G \rho$ has dimension $(q-1)$ and $\pi \times \dot{\mu}_r$ is irreducible. Hence we have the result. □

To sum up what we have done in this subsection, we have the following theorem

Proposition 3.43. *Degree $q-1$ irreducible representations of G can be classified as follows:*

(i) If μ_r is a square character of $K^\times$, then $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has $\frac{(q-1)}{2}$ many $(q-1)$-dimensional irreducible representations as summands,

(ii) If μ_r is not a square character of $K^\times$, then $\mathrm{Ind}_B^G(\pi \times \dot{\mu}_r)$ has $\frac{(q+1)}{2}$ many $(q-1)$-dimensional irreducible representations as summands

where each is distinct.

The proposition covers

$$\frac{(q-1)}{2}\frac{(q-1)}{2} + \frac{(q-1)}{2}\frac{(q+1)}{2} = \frac{q(q-1)}{2}$$

many $(q-1)$-dimensional irreducible representations of G. They are all by Lemma 3.17.

4. CONCLUSION

In this thesis, we study the irreducible representations of the group GL(2,K), the group of invertible matrices over a finite field K. In Section 3.4, we found the irreducible representations of G coming through induction of one-dimensional characters of B and in Section 3.6 we classified irreducible representations of G coming through induction of $(q-1)$-dimensional representations of B.

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