T.R.
YUZUNCU YIL UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

DIFFERENTIAL QUADRATURE METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

M.Sc. THESIS

PRESENTED BY: Sagvan Kareem MOHAMMEDALİ
SUPERVISOR: Asst. Prof. Dr. Nagehan ALSOY-AKGÜN

VAN-2017

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ACCEPTANCE and APPROVAL PAGE

This thesis entitled “ Differential quadrature method for partial differential equations” presented by Sagvan Kareem MOHAMMEDALİ under supervision of Asst. Prof. Dr. Nagehan Alsoy-Akgün in the Department of Mathematics has been accepted as a M. Sc. Thesis according to the rules of Higher Education Instiution of Republic of Turkey on 06/ 01/ 2017 with unanimity of the member of jury.

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THESIS STATEMENT

All information presented in the thesis obtained in the frame of ethical behavior and academic rules. In addition all kinds of information that does not belong to me have been cited appropriately in the thesis prepared by the thesis writing rules.

Signature

Sagvan Kareem MOHAMMEDALİ

ÖZET

KISMİ TÜREVLİ DENKLEMLER İÇİN DİFERANSİYEL KARELEME YÖNTEMİ

MOHAMMEDALİ, Sagvan Kareem
Yüksek Lisans Tezi, Matematik Anabilim Dalı
Tez Danışmanı : Yrd. Doç. Dr. Nagehan ALSOY-AKGÜN
Ocak 2017, 49 sayfa

Bu tez çalışmasında, kısmi türevli denklemler ile tanımlanmış problemlerin diferansiyel kareleme yöntemiyle çözümleri verilmiştir. İlk üç problemin denklemleri teorik çözümleri mevcut olan Poisson, Helmholtz ve conveksiyon-difüzyon-reaksiyon denklemleri olup Dirichlet tipindeki sınır koşullarına sahiptirler. Elde edilen sonuçlar grafikler ve tablolar yardımıyla teorik çözümler ile karşılaştırmalı olarak verilmiştir.

Sonraki iki problemde sırasıyla zamana bağlı difüzyon ve konveksiyon-difüzyon denklemleri yine diferansiyel kareleme yöntemi ile çözülmüştür. Bu denklemler orijinal halleri ile çözülmek yerine homojen olmayan modifiye edilmiş Helmholtz denklemlerine dönüştürülmüş ve sonra diferansiyel kareleme yöntemi çözüm prosedürü uygulanmıştır. Homojen olmayan modifiye edilmiş Helmholtz denklemlerini elde etmek için önce denklemin zaman türevleri ileri sonlu farklar yöntemi kullanılarak iki zaman düzeyinde açılmıştır. Ayrıca Laplace terimleri içinde bulunan bilinmeyen fonksiyon için bir parametre yardımıyla yeni bir açılım yapılmıştır. Bunlar denklem içerisinde yerine konulup denklemler yeniden yazıldığında iteratif formda homojen olmayan modifiye edilmiş Helmholtz denklemleri elde edilmiştir. Böylece zaman türevi için farklı bir yöntem kullanmaya gerek kalmamış ve dolayısıyla sayısal kararlılık analizi yapma ihtiyacı ortadan kalkmıştır.

Ayrıca bu problemlerde sınır koşulları Dirichlet ve Neumann tipinde olup Neumann tipindeki sınır koşullarının diferansiyel kareleme yöntemindeki uygulaması detaylı olarak verilmiştir.

Anahtar kelimeler: Diferansiyel kareleme yöntemi, Difüzyon denklemi, Helmholtz tipindeki denklemler, Kısmi türevli denklemler, Konveksiyon-difüzyon, Konveksiyon-difüzyon-reaksiyon denklemi.

ABSTRACT

DIFFERANTIAL QUADRATURE METHOD FOR PARTIAL DIFFERENTIAL EQUATION

MOHAMMEDALİ, Sagvan Kareem
M. Sc. Thesis, Mathematics
Supervisor : Asst. Prof. Dr. Nagehan ALSOY-AKGÜN
January 2017, 49 sayfa

In this thesis, partial differential equations are solved by using differential quadrature method. First three problems are Poisson, Helmholtz and convection-diffusion-reaction equations with the Dirichlet type boundary conditions which have the exact solutions. Obtained results are given using graphs and tables, and are compared with the exact solutions.

Next two problems are time dependet diffusion and convection-diffusion equations, respectively, and these are again solved by using differential quadrature method. For these equations differential quadrature solution procedure performed after transforming the give equations into the modified Helmholtz equation. In order to obtain modified Helmholtz equation, first, time derivatives are approximated using forward difference approximation at two time levesl. Also, unknown function located in the Laplace term is approximated using a relaxation parameter. These approximations are inserted into the equations. Nonhomogeneous modified Helmholtz equations in an iterative form are obtained by rearranging the equations. Therefore, the need of another time integration scheme is eliminated, and stability problems are diminished.

Also, in these problems, the boundary conditions are taken as both Dirichlet and Neumann types and the procedure for Neumann type boundary condition is explained in details.

Keywords: Convection-diffusion equation, Convection-diffusion-reaction equation, Differential quadrature method, Diffusion equation, Helmholtz-type equations, Partial differential equation.

ACKNOWLEDGMENTS

I would like to thank my supervisor Asst. Prof. Nagehan ALSOY-AKGÜN, for her guidance, advice, support and her helpful suggestion during my work to complete this thesis. I am grateful to my family for their immolation, subsidization and guidance. Finally, I wish to thank all of my teacher in the department of Mathematics in the Yüzüncü Yıl University for their encouragement and for supporting me in my studying.

2017
Sagvan Kareem MOHAMMEDALİ

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1. INTRODUCTION

Partial Differential Equations (PDE's) together with the suitable boundary conditions represent many mathematical model of engineering or physical problems. For example, acoustic and microwaves can be modeled by Helmholtz equation. Usually, it is not possible to find a closed-form solution for the PDE's. Due to its importance in many research area, it is necessary to develop an approximate solution for these equations.

There are many numerical solution techniques and each of them has different advantages to the other. In the solution procedure of any numerical method, a finite set of number which are stored in a computer memory is used for representing a continuous function in a differential equation. So, choosing a computationally efficient numerical method in terms of computer memory is an important step for the solution of PDE. Among currently used solution procedure, finite difference method (FDM), finite element method (FEM), differential quadrature method (DQM), boundary element method (BEM) and dual reciprocity boundary element method (DRBEM) are the most commonly used methods. When these methods are compared it is understood that the differences of the methods are the type of approximation of the variables and domain discretization where the problems are defined.

DQM is a numerical solution technique which was proposed by Bellman in the early 70s (Bellman et al., 1971; Bellman et al., 1972). Then, the method was used for the solution of many problems in engineering and physical sciences. The important point for the method to determine the weighting coefficients and for the first order derivative two different methods are suggested by Bellman et al. (Bellman et al., 1972). In the engineering, most early applications of DQM are used Bellman's method for computing the weighting coefficients (Bellman et al., 1971; Bellman et al., 1972; Bellman et al., 1974; Bellman et al., 1975a,b; Hu and Hu, 1974; Mingle, 1977; Wang ,1982; Civan and Sliepcevich, 1983a,b; Civan and Sliepcevich, 1984a,b; Naadimuthu et al., 1984; Bert et al., 1988; Bert et al., 1988; Bert et al., 1989; Jang et al., 1989). Lagrange interpolation polynomials are used as a test function to compute weighting coefficients (Quan and Chang, 1986a,b). Also, explicit formulations to the weighting coefficients for first and second order derivatives are obtained in the same studies. Generalization of these formulas for higher order derivatives using the higher order polynomial approximation and a recurrence relationship for the weighting coefficients using the simple algebraic formulation of the weighting coefficients for the first order derivative are given by Sue and Richards (Sue and Richards 1990; Sue 1991). A numerical study is presented using DQM for one dimensional inverse heat equation by Repaci in (Repaci, 1991). It is called inverse problem due to its missing boundary condition and this problem is eliminated by using a measurement of the temperature in an inner point of the space domain as a missing boundary condition. The DQM is extended by Lam for the solution of two-dimensional partial differential equations to encompass problems with arbitrary geometry (Lam, 1993). The results of thermal and torsional problems showed reasonably good accuracy when they are compared with the other solutions.

In the literature, there are many applications of DQM such as fluid mechanics, static and dynamic structural mechanics, lubrication mechanics, static aeroelasticity and biosciences. DQM solution of a model of an isothermal reactor with axial mixing is presented by (Civan, 1994). In this study, DQ method alleviates the numerical difficulties encountered in finite difference and quadrature solutions while satisfying the boundary conditions accurately. DQM is accepted as a good alternative to the conventional numerical solution techniques such as the finite difference and finite element methods. A state-of-the-art review of the differential quadrature method is presented in (Berta and Malik, 1996).

Wu and Liu claim that boundary-value and initial-value differential equations with a linear or nonlinear nature can be solved using Differential Quadrature Method (Wu and Liu, 1999). The difference between the classical DQM is the function values and some derivatives wherever necessary are chosen as independent variables. Differential Quadrature Element Method (DQEM) is proposed by Chang to solve steady-state heat conduction problems (Chang, 1999). He used the irregular elements and the numerical results are presented by demonstrating the developed DQEM steady-state heat conduction analysis model.

The first comprehensive work as a text book for the DQM is presented by Shu (Shu, 2000). First order initial value problems was solved by Fung using the DQM (Fung, 2001). In the solution procedure, the time derivative is taken at a sampling grid point as a weighted linear sum of the given initial condition and the function values which gives an unconditionally stable algorithm. The roots of Legendre Polynomials are taken the sampling grid points. In the second part of the study, this algorithm was extended for the solution of second order initial value problems. In (Tanaka and Chen, 2001), a numerical application of dual reciprocity

BEM (DRBEM) and differential quadrature (DQM) for the time-dependent diffusion problems is presented by Tanaka and Chen. The spatial partial derivatives and the time derivative are discretized by using DRBEM and DQM, respectively. Another study was presented by Fung for the imposition of boundary conditions containing higher order derivatives (Fung, 2003). In this study, the weighting coefficient matrices in the DQM is modified which overcome the limitation of the previous solution procedure for the imposition of boundary conditions.

Ece and Büyük solved steady natural convection flow under a magnetic field in an inclined rectangular enclosure heated and cooled on adjacent walls with various Grashof and Hartmann numbers by using DQM (Ece and Büyük, 2006). Lo et al. used DQM as solution technique for the solution of the benchmark problem of 2D unsteady natural convection flow in a cavity in (Lo et al., 2007). In their study, they used second order finite difference approximation for the time derivative.

A study for unsteady natural convection in a cavity under a magnetic field by presented by Alsoy-Akgün and Tezer-Sezgin in (Alsoy-Akgün and Tezer-Sezgin, 2013) using DQM and DRBEM. In the study, the vorticity transport and energy equations in the governing equations are transformed to the modified Helmholtz equation and the results obtained from DRBEM and DQM are compared in terms of accuracy and computational cost. Alsoy-Akgün extended this study to the natural convection flow of water-based nanofluid in the study (Alsoy-Akgün, 2016).

In this thesis, some partial differential equations such as Poisson equation, Helmholtz equation, modified Helmholtz equation, diffusion equation, convection-diffusion equation are solved by using DQM. Time dependent equations are transformed to the modified Helmholtz equations by approximating time derivative terms using forward difference approximation. The need of another time integration scheme for time derivatives is eliminated by solving obtained nonhomogeneous modified Helmholtz equations. In the solution procedure, Chebyshev-Gauss-Lobatto grid points which are located near the end point are used for all problem. Numerical results are given as tables and graphs, and are discussed by comparing with the exact solution.

2. DIFFERENTIAL QUADRATURE METHOD

Differential Quadrature Method (DQM) is a numerical discretization procedure for the solution of differential equations. First, it was developed by R. Bellman and his associates in the early 1970's. Bellman introduced that an accurate solution can be obtained using considerably small number of mesh points with DQM in (Bellman et al, 1971; Bellman et al, 1972). In the DQM, all the derivatives of any order can be expressed as a linear summation of all the function values along a mesh line. In the solution procedure, the weighting coefficients are determined by using the mesh information. So, it is easy to obtain a system of algebraic equations for any differential equation.

In this thesis, the physical problems governed by partial differential equations are solved with polynomial-based differential quadrature method by using non-uniform mesh point distribution. In this chapter, first, polynomial-based differential quadrature method will be described for one-dimensional problem. Non-uniform mesh point distribution is explained in the next section. Then, the DQM discretization of the governing partial differential equations together with the implementation of the of boundary conditions for both Dirichlet and Neumann type boundary conditions are described.

2.1 Polynomial-based Differential Quadrature Method

The aim of the any numerical method is to obtain a solution for the initial or boundary value problems by transforming the governing equation into the algebraic equations in terms of the discrete values of the function at discrete points of the solution domain. Differential Quadrature Method (DQM) is based on the approximation of the first order derivative of sufficiently smooth function with respect to coordinate direction at any mesh point using a linear sum of the values of function at all the points in one direction (Shu, 2000). In this section, DQM is explained for one-dimensional problem.

First order derivative for one dimensional problem can be approximated as

$$\frac{\partial f}{\partial x}(x_i) = \sum_{j=1}^{N} a_{ij} f(x_j), \qquad \text{for} \qquad i = 1, 2, \ldots, N \tag{2.1.1}$$

where a_{ij} is the weighting coefficients and N is the number of the mesh points in the domain. The weighting coefficients can be changed depending on the location of the x_i and determination of the weighting coefficient is the first step of the DQM.

Mathematical theories claim that a well-posed PDE have a solution function but, in general, this solution may not be written in a closed form. On the other hand, this solution function can be written in approximated form using the higher order polynomials.

Weierstrass' first theorem: Let $f(x)$ be a real valued continuous function defined in a closed interval $[a,b]$. Then there exists a sequence of polynomials $P_n(x)$ which converges to $f(x)$ uniformly as n goes to infinity or for every $\epsilon > 0$, there exists a polynomial $P_n(x)$ of degree $n = n(\epsilon)$ such that the inequality

$$|f(x) - P_n(x)| \leq \epsilon \tag{2.1.2}$$

holds through the interval $[a,b]$.

So, a solution function of a PDE can be approximated as

$$f(x) \approx P_N(x) = \sum_{k=0}^{N-1} c_k x^k \tag{2.1.3}$$

where $P_N(x)$ is a polynomial of degree less than $N-1$, c_k is a constant coefficient. A polynomial of degree less than $N-1$ sets up a N dimensional linear vector space V_N together with the operation of vector addition and scalar multiplication.

The set of $\{1, x, x^2, \dots, x^{N-1}\}$ is linearly independent in the vector space V_N. Therefore,

$$S_k(x) = x^{k-1}, \qquad k = 1, 2, \dots, N$$

is a basis of this vector space.

In order to obtained a numerical solution of a PDE, the solution domain must be discretized and than discreet values of solution function are found out at these discrete points. So, for one dimensional problem, a closed interval $[a,b]$ is divided into $N-1$ parts using N mesh points with the coordinates $a = x_1, x_2, \dots, x_n = b$. Evaluating $f(x_i)$ at mesh point x_i using Equation (2.1.1), we can obtain the following the system of equations

$$\begin{cases} c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_{N-1} x_1^{N-1} = f(x_1) \\ c_0 + c_1 x_2 + c_2 x_2^2 + \ldots + c_{N-1} x_2^{N-1} = f(x_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_{N-1} x_N^{N-1} = f(x_N). \end{cases} \tag{2.1.4}$$

The coefficient matrix of this system is of `Vandermonde` form. Since it is nonsingular matrix, this system has a unique solution. So, after solving the system we can obtain the coefficients c_0, $c_1, \ldots, c_{N-1}$ and approximated matrix $P_N(x)$. But, if N increases, the dimension of the coefficient matrix increases and it becomes ill-conditioned. So, the inverse of the coefficient matrix cannot be obtained easily.

This problem can be eliminated using the Lagrange interpolation polynomial

$$f(x) = \sum_{k=1}^{N} l_k(x) f(x_k) \tag{2.1.5}$$

where

$$\begin{aligned} l_k(x) &= \frac{M(x)}{M^{(1)}(x_k)(x-x_k)} \\ M(x) &= (x-x_1)(x-x_2)\cdots(x-x_N) = N(x,x_k)(x-x_k) \\ M^{(1)}(x_k) &= (x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x-x_N) = \prod_{k=1,k\neq i}^{N}(x_k-x_i) \\ N(x_k,x_j) &= M^{(1)}(x_k)\delta_{kj} \end{aligned} \tag{2.1.6}$$

Therefore, the approximated polynomial can be obtained using the functional values at the mesh points. Here, $l_k(x)$ is the k-th degree Lagrange polynomial and it also can be written as

$$l_k(x) = \frac{N(x,x_k)}{M^{(1)}(x_k)} \tag{2.1.7}$$

where it has the property

$$l_k(x_i) = \begin{cases} 1, & \text{when} \quad k = i \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.8}$$

First and second order derivatives of $f(x)$ can be obtained by differentiating Equation (2.1.5) as

$$f^{(1)}(x) = \sum_{k=1}^{N} l_k^{(1)}(x) f(x_k), \tag{2.1.9}$$

$$f^{(2)}(x) = \sum_{k=1}^{N} l_k^{(2)}(x) f(x_k) \tag{2.1.10}$$

and the first and second derivatives of Lagrange function are

$$l_k^{(1)}(x) = \frac{N^{(1)}(x, x_k)}{M^{(1)}(x_k)}, \tag{2.1.11}$$

$$l_k^{(2)}(x) = \frac{N^{(2)}(x, x_k)}{M^{(1)}(x_k)}. \tag{2.1.12}$$

For the computation of the weighting coefficients a practical notation is used by Shu (Shu, 2000) as

$$f^{(1)}(x_i) = \sum_{k=1}^{N} a_{ik} f(x_k), \tag{2.1.13}$$

$$f^{(2)}(x_i) = \sum_{k=1}^{N} b_{ik} f(x_k) \tag{2.1.14}$$

where

$$a_{ik} = \frac{N^{(1)}(x_i, x_k)}{M^{(1)}(x_k)}, \tag{2.1.15}$$

$$b_{ik} = \frac{N^{(2)}(x_i, x_k)}{M^{(1)}(x_k)} \tag{2.1.16}$$

where $N^{(1)}(x, x_k)$ and $N^{(2)}(x, x_k)$ are the first and second order derivatives of $N(x, x_k)$ with respect to x. A recurrence relation formulation can obtained for the higher order derivatives of $M(x)$ by successive differentiating with respect to x as

$$M^{(m)}(x) = N^{(m)}(x, x_k)(x - x_k) + mN^{(m-1)}(x, x_k) \qquad m = 1, 2, \cdots, N-1 \tag{2.1.17}$$

where $M^{(m)}(x)$ and $N^{(m)}(x, x_k)$ are the m-th order derivative of $M(x)$ and $N(x, x_k)$, respectively. Using Equation (2.1.17) we get

$$\begin{aligned} N^{(1)}(x_i, x_k) &= \frac{M^{(1)}(x_i)}{x_i - x_k}, \qquad i \neq k \\ N^{(1)}(x_i, x_i) &= \frac{M^{(2)}(x_i)}{2} \end{aligned} \tag{2.1.18}$$

and

$$N^{(2)}(x_i, x_k) = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_k)}{x_i - x_k}, \qquad i \neq k$$

$$N^{(2)}(x_i, x_i) = \frac{M^{(3)}(x_i)}{3}. \tag{2.1.19}$$

Substituting equations (2.1.18) and (2.1.19) into equations (2.1.15) and (2.1.16), respectively, we achieved

$$a_{ik} = \frac{M^{(1)}(x_i)}{(x_i - x_k)M^{(1)}(x_k)}, \qquad i \neq k,$$

$$a_{ii} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)} \tag{2.1.20}$$

and

$$b_{ik} = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_k)}{(x_i - x_k)M^{(1)}(x_k)}, \qquad i \neq k$$

$$b_{ii} = \frac{M^{(3)}(x_i)}{3M^{(1)}(x_i)}. \tag{2.1.21}$$

Thus, the weighting coefficients can be computed easily using the mesh point x_i for $i \neq k$. On the other hand, for the computation of the weighting coefficients a_{ii} and b_{ii}, we need to compute $M^{(2)}(x)$ and $M^{(3)}(x)$, and their computations are very difficult. But, using the property of linear vector space, if one set of base polynomials satisfies a linear operator so does another set of polynomials (Shu, 2000), this problem can be eliminated. As it mentioned before, x^{k-1}, $k = 1, \ldots, N$ is another set of base polynomial and when $k = 1$ it satisfies the property

$$\sum_{k=1}^{N} a_{ik} = 0 \qquad \texttt{or} \qquad a_{ii} = -\sum_{k=1, k \neq i}^{N} a_{ik} \tag{2.1.22}$$

and

$$\sum_{k=1}^{N} b_{ik} = 0 \qquad \texttt{or} \qquad b_{ii} = -\sum_{k=1, k \neq i}^{N} b_{ik}. \tag{2.1.23}$$

So, equations (2.1.22) and (2.1.23) are good alternative for the computation of the weighting coefficients a_{ii} and b_{ii}, respectively. Also, using the equations (2.1.20) and (2.1.21) we can obtain

$$b_{ik} = 2a_{ik}\left(a_{ii} - \frac{1}{x_i - x_k}\right), \qquad i \neq k. \tag{2.1.24}$$

A general formulation for the higher order derivatives is

$$
\begin{aligned}
w_{ij}^{(m)} &= m\left(a_{ij}w_{ii}^{(m-1)} - \frac{w_{ij}^{(m-1)}}{x_i - x_j}\right), \qquad i \neq j \\
w_{ii}^{(m)} &= -\sum_{j=1, i\neq j}^{N} w_{ij}^{(m)}, \qquad i = j
\end{aligned}
\tag{2.1.25}
$$

where $m = 2, 3, ..., N-1$, $i, j = 1, 2, ..., N$ and a_{ij} are the weighting coefficients of the first order derivative. This formulation is called Shu's recurrence relation formulation for high order derivatives.

2.2 The Type of Grid Point Distribution

Generally, numerical solution methods prefer to use uniform grid point distribution due to its simplicity. It is also possible for the differential quadrature method. However, nonuniform grid point distributions give more stable results than the uniform grid point distributions. In this thesis, Chebyshev-Gauss-Lobatto (CGL) points which enable one to take grid points close to the boundary points, are used for all problems.

The definition of $i-$th degree Chebyshev polynomial is

$$
T_i(x) = \cos i\theta \qquad \theta = \arccos x \tag{2.2.1}
$$

and the roots of $|T_N(x)| = 1$ in the interval $[1, -1]$ are taken as the Chebyshev-Gauss-Lobatto points which are given by

$$
x_i = \cos\left(\frac{i\pi}{N}\right), \qquad i = 0, 1, ..., N. \tag{2.2.2}
$$

For any physical domain $[a, b]$, the following coordinate transformation

$$
x = \frac{b-a}{2}(1-\xi) + a \tag{2.2.3}
$$

which maps the interval $[a, b]$ in the x-domain onto the interval $[1, -1]$ in the ξ-domain.

DQM is going to be applied to some problems which have exact solution. Numerical results are given in terms of tables and graphics which involves the comparison with the exact solution.

3. APPLICATION OF DQM

In this chapter, some applications of Differential Quadrature Method are presented. There are five different problems with the exact solution. All problems are defined in a rectangular domain. First three equations are Poisson, convection-diffusion-reaction and Helmholtz equations and all of them have the Dirichlet type boundary conditions. Next two problems are diffusion and convection-diffusion equations and they have both Dirichlet and Neumann type boundary conditions.

Solutions for all problems are presented graphically and numerical solutions are compared with the exact solutions using the contourlines on the same graphics. In the computations, maximum absolute error is defined as

$$\texttt{Max. Abs. Error} = \max_{i,j} |U_{num}(x_i, y_j) - U_{ex}(x_i, y_j)| \qquad i, j = 1, \ldots, N$$

where N is the number of mesh points in one direction. Computer programs are written using FORTRAN Language and all the graphs are drawn using MATLAB.

3.1 Problem 1

Consider the Dirichlet problem (Bialecki and Karageorghis, 2004)

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega = [-1,1]\times[-1,1] \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.1.1}$$

with

$$f(x, y) = 32\pi^2 \sin(4\pi x)\sin(4\pi y)$$

where the exact solution is

$$u(x, y) = \sin(4\pi x)\sin(4\pi y).$$

DQM discretization of the Equation (3.1.1) at the point (x_i, y_j) is

$$-\left(\sum_{k=1}^{N} b_{ik}u_{kj} + \sum_{k=1}^{N} \overline{b}_{jk}u_{ik}\right) = f(x_i, y_j) \tag{3.1.2}$$

where b_{ik} and $\overline{b}_{jk}$ are the weighted coefficients of second order derivatives of u with respect to x and y, respectively. Since the all value of u are known at the boundary, DQM discretization can be written as

$$-\left(\sum_{k=2}^{N-1} b_{ik}u_{kj} + \sum_{k=2}^{N-1} \overline{b}_{jk}u_{ik}\right) = \widetilde{f}_{i,j} \qquad (3.1.3)$$

for $i = 2,...,N-1$ and $j = 2,...,N-1$ and $\widetilde{f}_{i,j}$ is

$$\widetilde{f}_{i,j} = f(x_i, y_j) - (b_{i1}u_{1j} + b_{iN}u_{Nj} + \overline{b}_{j1}u_{i1} + \overline{b}_{jN}u_{iN}).$$

This problem is solved using different number of mesh points and the results are compared with the exact solution. Computations are carrying using N = 16, 20, 24, 28 and 32 in one direction and the discretization of mesh points are taken as

$$x_i = -\cos\left(\frac{i-1}{N-1}\right)\pi \qquad i = 1,...,N$$

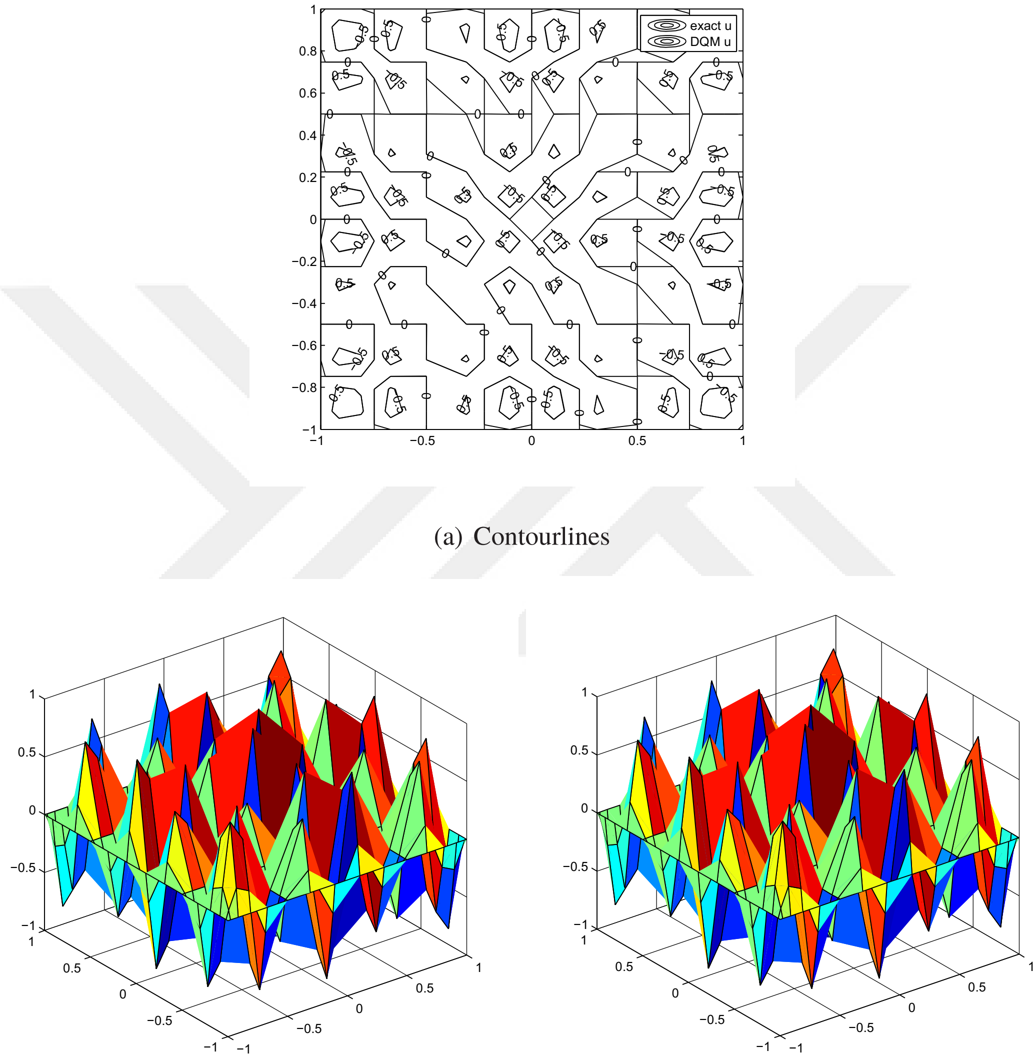
$$y_j = -\cos\left(\frac{j-1}{N-1}\right)\pi \qquad j = 1,...,N.$$

Table 3.1 Maximum absolute errors for Problem 1 with different mesh points.

N	Max. Abs. Error
16	1.91×10^{-2}
20	2.40×10^{-4}
24	1.25×10^{-6}
28	5.90×10^{-7}
32	6.00×10^{-7}

The results for the problem 1 are given in terms of maximum absolute errors in Table (3.1). From the table, the minimum error can be obtained for higher values of mesh points. When the number of mesh points N is increased the contours and graphs become smooth as can be seen from Figures (3.1.1)-(3.1.3). For this problem, there is no need to increase N since the solution obtained with N = 24 is already very accurate. Because, when N increases, the size of the coefficient matrix increases, thus, it causes extra computational effort.

For the next problems, the results are going to be given for suitable number of mesh points for obtaining accurate results and for drawing the contours.

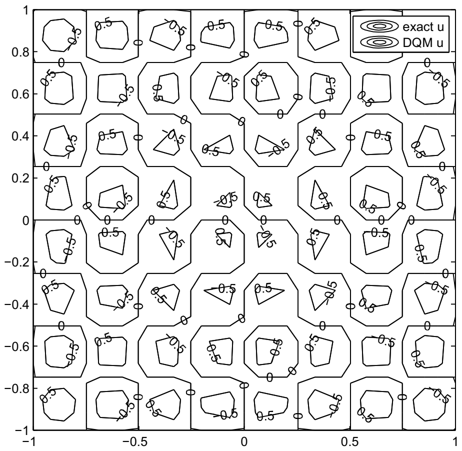


(a) Contourlines

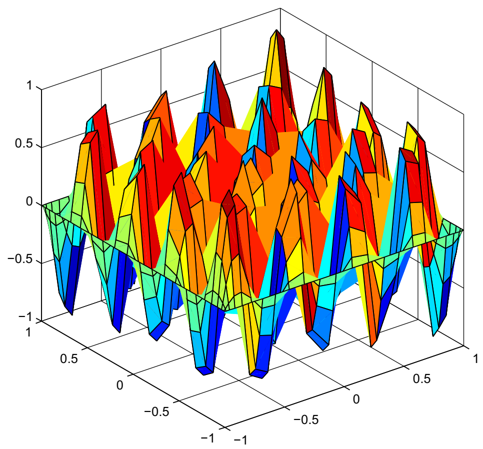
(b) Exact Solution

(c) DQM Solution

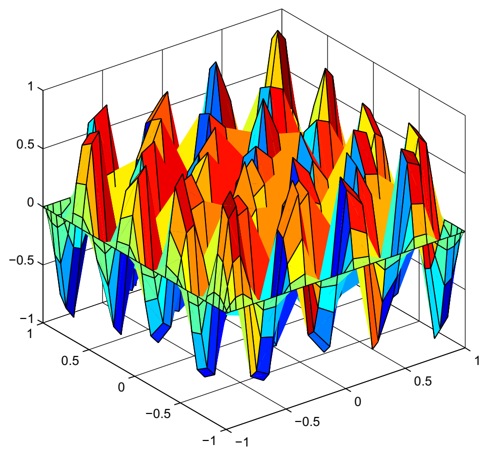
Figure 3.1.1 DQM Solution of Problem 1 for $N = 16$.



(a) Contourlines



(b) Exact Solution



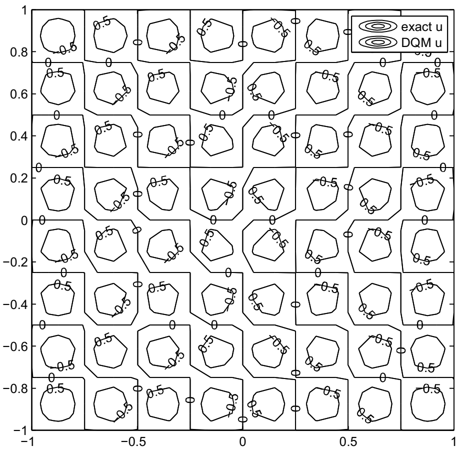
(c) DQM Solution

Figure 3.1.2 DQM Solution of Problem 1 for $N = 24$.

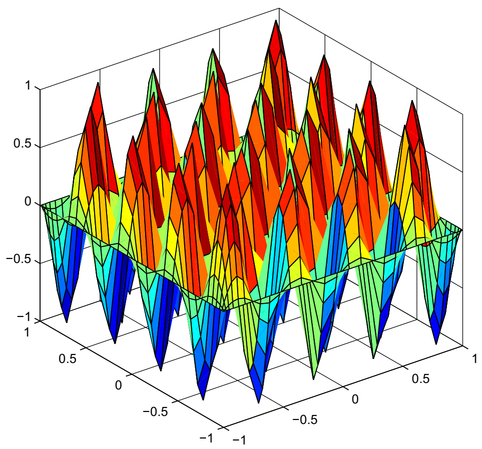
3.2 Problem 2

The next Dirichlet problem is defined as (Bialecki and Karageorghis, 2004)

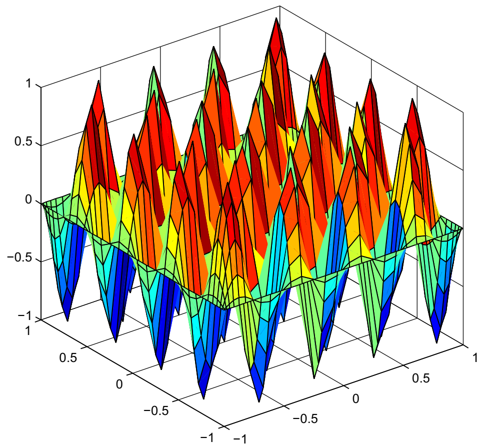
$$\begin{aligned} -\nabla(p(x,y)\nabla u)+u &= f \qquad \texttt{in} \qquad \Omega=[-1,1]\times[-1,1] \\ u &= 0 \qquad \texttt{on} \qquad \partial\Omega \end{aligned} \tag{3.2.1}$$



(a) Contourlines



(b) Exact Solution



(c) DQM Solution

Figure 3.1.3 DQM Solution of Problem 1 for $N = 32$.

where

$$p(x,y) = 1 + x^2 y^2$$

and the exact solution is

$$u(x,y) = \sin^2(\pi x)\sin^2(\pi y)e^{x+y}.$$

Using the exact solution inhomogeneous term is obtained as

$$\begin{aligned} f(x,y) = \; & -2xy^2e^{x+y}\sin^2(\pi y)\left[\sin^2(\pi x)+\pi\sin^2(2\pi x)\right] \\ & -2x^2ye^{x+y}\sin^2(\pi x)\left[\sin^2(\pi y)+\pi\sin^2(2\pi y)\right] \\ & -(1+x^2y^2)e^{x+y}\Big(\sin^2(\pi y)\left[\sin^2(\pi x)+2\pi\sin(2\pi x)+2\pi^2\cos(2\pi x)\right] \\ & +\sin^2(\pi x)\left[\sin^2(\pi y)+2\pi\sin(2\pi y)+2\pi^2\cos(2\pi y)\right]\Big)+\sin^2(\pi x)\sin^2(\pi y)e^{x+y}. \end{aligned}$$

After using DQM, the discretized form of the Equation (3.2.1) is

$$-\left(P_x(x_i,y_j)\sum_{k=1}^{N}a_{ik}u_{kj}+P_y(x_i,y_j)\sum_{k=1}^{N}\overline{a}_{jk}u_{ik}\right)-\left(P(x_i,y_j)\sum_{k=1}^{N}b_{ik}u_{kj}+P(x_i,y_j)\sum_{k=1}^{N}\overline{b}_{jk}u_{ik}\right)+u_{ij}=f_{ij} \qquad (3.2.2)$$

where a_{ik} and $\overline{a}_{jk}$ are the weighted coefficients of first order derivatives of u, and b_{ik} and $\overline{b}_{jk}$ are the weighted coefficients of second order derivatives of u with respect to x and y, respectively. Imposing the Dirichlet boundary conditions to the Equation (3.2.2) we get,

$$-\left(P_x(x_i,y_j)\sum_{k=2}^{N-1}a_{ik}u_{kj}+P_y(x_i,y_j)\sum_{k=2}^{N-1}\overline{a}_{jk}u_{ik}\right)-\left(P(x_i,y_j)\sum_{k=2}^{N-1}b_{ik}u_{kj}+P(x_i,y_j)\sum_{k=2}^{N-1}\overline{b}_{jk}u_{ik}\right)+u_{ij}=\tilde{f}_{ij} \qquad (3.2.3)$$

where

$$\begin{aligned} \widetilde{f}_{i,j} = \; & f_{i,j}-\{-\left[P_x(x_i,y_j)a_{i1}u_{1j}+P_y(x_i,y_j)\overline{a}_{j1}u_{i1}\right]-\left[P(x_i,y_j)b_{i1}u_{1j}+P(x_i,y_j)\overline{b}_{j1}u_{i1}\right] \\ & -\left[P_x(x_i,y_j)a_{iN}u_{Nj}+P_y(x_i,y_j)\overline{a}_{jN}u_{iN}\right]-\left[P(x_i,y_j)b_{iN}u_{1j}+P(x_i,y_j)\overline{b}_{jN}u_{iN}\right]\} \end{aligned}$$

In the computations, the same mesh points are used with the previous problem and the results are compared with the exact solution. The maximum absolute errors are computed for N = 12, 16, 20, 24 and 26 and results are listed in Table (3.2). In Figures (3.2.1)-(3.2.3), exact and numerical results are given numerically.

From the table and figures, N = 20 is enough to obtain accurate solutions and there is no need to increase the number of mesh points. As compared with the Problem 1 the method require less mesh points to obtain numerical results with the higher of the accuracy for the problem.

Table 3.2 Maximum absolute errors for Problem 2 with different mesh points.

N	Max. Abs. Error
12	6.96×10^{-4}
16	1.19×10^{-6}
20	4.00×10^{-8}
24	2.00×10^{-8}
26	1.00×10^{-8}

3.3 Problem 3

Consider the problem

$$\begin{aligned} \nabla^2 u + 0.5u &= f \qquad \texttt{in} \qquad \Omega = [0,1]\times[0,1] \\ u &= 0 \qquad \texttt{on} \qquad \partial\Omega \end{aligned} \tag{3.3.1}$$

with

$$f(x,y) = (-2\pi^2 + 0.5)\sin(\pi x)\sin(\pi y)$$

where the exact solution is

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Mesh points which are used at the discretization are

$$\begin{aligned} x_i &= \frac{1}{2}\left(1-\cos\left(\frac{i-1}{N-1}\right)\pi\right) \qquad i = 1,\ldots,N \\ y_j &= \frac{1}{2}\left(1-\cos\left(\frac{j-1}{N-1}\right)\pi\right) \qquad j = 1,\ldots,N. \end{aligned}$$

At any point (x_i, y_j) Equation (3.3.1) can be discretized as

$$\sum_{k=1}^{N} b_{ik}u_{kj} + \sum_{k=1}^{N} \overline{b}_{jk}u_{ik} + 0.5u_{ij} = f(x_i, y_j) \tag{3.3.2}$$

where b_{ik} and $\overline{b}_{jk}$ are the weighted coefficients of second order derivatives of u with respect to x and y, respectively. Since the all value of u are known at the boundary, DQM discretization can be written as

$$\sum_{k=2}^{N-1} b_{ik}u_{kj} + \sum_{k=2}^{N-1} \overline{b}_{jk}u_{ik} + 0.5u_{ij} = \widetilde{f_{i,j}} \tag{3.3.3}$$

for $i = 2, \ldots, N-1$ and $j = 2, \ldots, N-1$ and $\widetilde{f_{i,j}}$ is

$$\widetilde{f_{i,j}} = f(x_i, y_j) - (b_{i1}u_{1j} + b_{iN}u_{Nj} + \overline{b}_{j1}u_{i1} + \overline{b}_{jN}u_{iN}).$$

Table 3.3 Maximum absolute errors for Problem 3 with different mesh points.

N	Max. Abs. Error
3	2.412×10^{-1}
5	5.511×10^{-3}
7	2.125×10^{-5}
9	1.520×10^{-7}
11	1.510×10^{-7}
13	1.510×10^{-7}

Computations are carried out using the different number of mesh points and the maximum absolute errors are given in Table (3.3). Also, the results are given in Figures(3.3.1)-(3.3.3) graphically. From the table and figures it can be seen that the best accuracy comparing with the exact solution of the problem is obtained when the number of mesh points increases, as in previous examples.

3.4 Problem 4

The usual form of the diffusion equation is

$$\begin{aligned} &\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \texttt{in} \qquad \Omega = [0, L] \times [0, L] \\ &u(0, y, t) = 0, \qquad u(x, 0, t) = 0, \\ &q(L, y, t) = 0, \qquad q(x, L, t) = 0 \end{aligned} \tag{3.4.1}$$

where k is the diffusivity coefficient. The square plate initially at the temperature u_0 and the exact solution of the problem is give in (Patridge et al., 1992),

$$u(x, y, t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} A_{lm} \sin\left(\frac{l\pi x}{L}\right) \sin\left(\frac{l\pi y}{L}\right) \exp\left[-\left(\frac{kl^2\pi^2}{L^2} + \frac{km^2\pi^2}{L^2}\right)t\right]$$

where

$$A_{lm} = \frac{4u_0}{lm\pi^2}[(-1)^l - 1][(-1)^m - 1].$$

In the computation, the parameters are taken as $L = 1.5$, $k = 1.25$ and $u_0 = 30$.

The diffusion equation can be written in the form of a modified Helmholtz equation. For this aim, first, the time derivative is approximated using the forward difference approximation as

$$\nabla^2 u = \frac{1}{k}\Big(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\Big) \tag{3.4.2}$$

where Δt is the time step, $u^{(n)}$ and $u^{(n+1)}$ represent the value of u at current and advance time level, respectively. Approximating the u located in the Laplace term using the relaxation parameter $0 < \theta < 1$, Equation (3.4.2) can be written as

$$\nabla^2\left(\theta u^{(n+1)} + (1-\theta)u^{(n)}\right) = \frac{1}{k}\Big(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\Big). \tag{3.4.3}$$

Rewriting the Equation (3.4.3) a modified Helmholtz equation can be obtained as

$$\nabla^2 u^{(n+1)} - \lambda^2 u^{(n+1)} = -\Big(\frac{1-\theta}{\theta}\Big)\nabla^2 u^{(n)} - \lambda^2 u^{(n)} \tag{3.4.4}$$

where $\lambda^2 = \dfrac{1}{k\Delta t\theta}$. Equation (3.4.4) is the inhomogeneous modified Helmholtz equation since the right hand side of the equation can be computed using know value of $u^{(n)}$. Now, this equation can be solved using DQM.

In the DQM solution procedure, the Neumann and Dirichlet type boundary conditions are imposed separately. Discretization points are taken as

$$\begin{aligned} x_i &= \frac{3}{4}\left(1 - \cos\left(\frac{i-1}{N-1}\right)\pi\right) \qquad i = 1, \ldots, N \\ y_j &= \frac{3}{4}\left(1 - \cos\left(\frac{j-1}{N-1}\right)\pi\right) \qquad j = 1, \ldots, N. \end{aligned}$$

After discretizing Equation (3.4.4) at the domain with N mesh points and imposing the Dirichlet type boundary conditions we get

$$\sum_{k=2}^{N} b_{ik} u_{kj}^{(n+1)} + \sum_{k=2}^{N} \overline{b}_{jk} u_{ik}^{(n+1)} - \lambda^2 u_{ij}^{(n+1)} = s_{ij} \tag{3.4.5}$$

where

$$s_{ij} = -\Big(\frac{1-\theta}{\theta}\Big)\Big[\sum_{k=1}^{N} b_{ik} u_{kj}^{(n)} + \sum_{k=1}^{N} \overline{b}_{jk} u_{ik}^{(n)}\Big] - \lambda^2 u_{ij}^{(n)} - (b_{i1}u_{1j} + \overline{b}_{j1}u_{i1}). \tag{3.4.6}$$

Neumann type boundary conditions are defined at the top and the right walls as

$$\begin{aligned} \sum_{k=1}^{N} \overline{a}_{Nk} u_{ik}^{(n+1)} &= 0, \qquad i = 2, \ldots, N \\ \sum_{k=1}^{N} a_{Nk} u_{kj}^{(n+1)} &= 0, \qquad j = 2, \ldots, N-1. \end{aligned} \tag{3.4.7}$$

Thus, Equations (3.4.5) and (3.4.7) give a set of algebraic equations and it can be solved iteratively.

In the solution procedure, Crank-Nicolson ($\theta = 1/2$) and Galerkin ($\theta = 2/3$) schemes are used as in (Patridge and Sensale, 2000). $N = 24$ mesh points are used in one direction and all the results are given to show that the effects of time increment Δt and relaxation parameter θ.

Table 3.4 Maximum absolute errors for Problem 4 at $t = 1.2$ with $N = 24$.

	Relaxation Parameter	Iteration	Max. Abs. Err.
Case 1	$\theta = 1/2$	96	2.27177
$\Delta t = 0,01250$	$\theta = 2/3$	96	0.04716
Case 2	$\theta = 1/2$	192	1.74417
$\Delta t = 0,00625$	$\theta = 2/3$	192	0.02359
Case 3	$\theta = 1/2$	384	0.03492
$\Delta t = 0,003125$	$\theta = 2/3$	384	0.01180
Case 4	$\theta = 1/2$	768	0.04125
$\Delta t = 0,001575$	$\theta = 2/3$	768	0.01065

From the Table (3.4) it can be concluded that the Galerkin Scheme with the smallest time increment (case 4) shows the very well agreement with the exact solution of the problem. In Figures (3.4.1) and (3.4.2), we give the results for case 4 since the smaller Δt gives better accuracy for both, Galerkin and Crank-Nicolson schemes. Since the Galerkin scheme gives very good agreement with the exact solution of the problem, it will be used in the rest of the analysis of the problem.

In the problem, results are also obtained and compared with the exact solution at different time levels. Computations are carried out to show that a comparison of the time variation at $t = 0.5$, 1.0, 1.5, and 2.0, and results are given in the Figures (3.4.3), (3.4.4), (3.4.5) and (3.4.6), respectively. From the figures it is observe that contourlines show a circular behavior and it takes the maximum value at the right bottom corner of the cavity. Also, the value of the solutions decrease at time advances and the steady state results are obtained around with $t = 2.25$.

A comparison between the exact and numerical solutions at the right bottom corner for increasing time levels are given in Figure (3.4.7). From the figure, the largest error occur at the beginning of the solution procedure and then the error starts to decrease. This behavior is expected for the thermal shock problem. Because the shock is applied to the mathematical model suddenly but to the computational model linearly.

3.5 Problem 5

The next problem of the convection diffusion equation is modeled as in (Patridge and Sensale, 2000)

$$\nabla^2 u = \frac{1}{K}\frac{\partial u}{\partial t} + c_x\frac{\partial u}{\partial x} + c_y\frac{\partial u}{\partial y} + du \tag{3.5.1}$$

with the boundary conditions in a square region $[0,1]\times[0,0.7]$

$$u(0,y,t) = 300, \qquad q(x,0,t) = 0,$$
$$u(1,y,t) = 10, \qquad q(x,0.7,t) = 0.$$

Here, K is the dispersion coefficient, c_x and c_y are the velocity components and d is the coefficient of the chemical reactor. In the computations these coefficients are taken as

$$K = 1,$$
$$c_x = dx + \log\frac{10}{300} - \frac{d}{2}$$
$$c_y = 0$$

The exact solution of the problem is (Patridge and Sensale, 2000)

$$u(x,y,t) = 300\exp\left(\frac{d}{2}x^2 + \frac{10}{300}x - \frac{d}{2}x\right).$$

A modified Helmholtz equation can be obtained for the convection diffusion equation using the same procedure with the previous problem. So, first, the time derivative is expanded using the forward difference approximation as

$$\frac{\partial u}{\partial t} = \frac{u^{(n+1)} - u^{(n)}}{\Delta t}. \tag{3.5.2}$$

Then, using the relaxation parameter θ the solution term u in the Laplace term is approximated as

$$u^{(n+1)} = \theta u^{(n+1)} + (1-\theta)u^{(n)}.$$

After substituting these approximations into the Equation (3.5.1) and taking all the other terms at n-th time level new form of the equation is written as

$$\nabla^2 u^{(n+1)} - \lambda^2 u^{(n+1)} = -\left(\frac{1-\theta}{\theta}\right)\nabla^2 u^{(n)} - \lambda^2 u^{(n)} + \frac{1}{K\theta}\left(c_x\frac{\partial u^{(n)}}{\partial x} + c_y\frac{\partial u^{(n)}}{\partial y} + du^{(n)}\right) \tag{3.5.3}$$

where $\lambda^2 = \dfrac{1}{K\theta\Delta t}$.

Now, the iterative form of the convection diffusion equation can solve using DQM. Equation (3.5.3) is discretized at the domain using N mesh points in one direction. In the discretization the mesh points are taken as

$$\begin{aligned} x_i &= \frac{1}{2}\left(1-\cos\left(\frac{i-1}{N-1}\right)\pi\right) \qquad i = 1,\ldots,N \\ y_j &= \frac{0.3}{2}\left(1-\cos\left(\frac{j-1}{N-1}\right)\pi\right) \qquad j = 1,\ldots,N. \end{aligned}$$

For the unknown value of u at the interior points of the domain, using the boundary conditions, Equation (3.5.3) takes to form

$$\sum_{k=2}^{N-1} b_{ik}u_{kj}^{(n+1)} + \sum_{k=1}^{N} \overline{b}_{jk}u_{ik}^{(n+1)} - \lambda^2 u_{ij}^{(n+1)} = s_{ij} \tag{3.5.4}$$

where

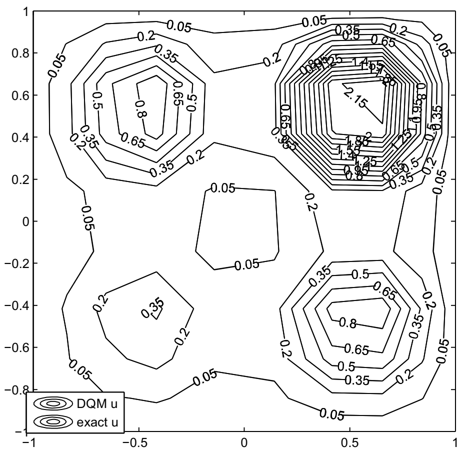
$$\begin{aligned} s_{ij} = {} & -\left(\frac{1-\theta}{\theta}\right)\left[\sum_{k=1}^{N} b_{ik}u_{kj}^{(n)} + \sum_{k=1}^{N}\overline{b}_{jk}u_{ik}^{(n)}\right] - \lambda^2 u_{ij}^{(n)} + \frac{1}{K\theta}\left(c_x\sum_{k=1}^{N} a_{ik}u_{kj}^{(n)} + c_y\sum_{k=1}^{N}\overline{a}_{jk}u_{ik}^{(n)} + du_{ij}^{(n)}\right) \\ & -(b_{i1}u_{1j} + b_{iN}u_{Nj}) \end{aligned} \tag{3.5.5}$$

Due to the Neumann type boundary conditions, the equations are obtained for the bottom and top wall using the DQM as

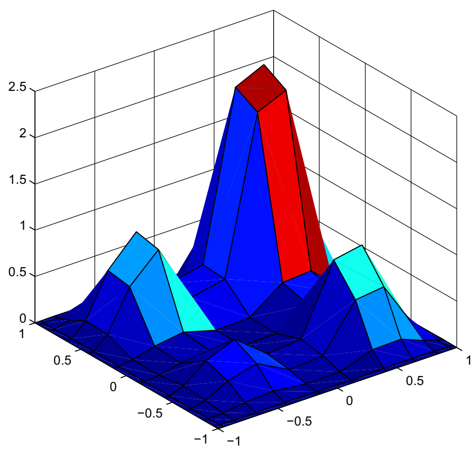
$$\begin{aligned} \sum_{k=1}^{N} \bar{a}_{1k} u_{ik}^{(n+1)} &= 0, \qquad i = 2, \ldots, N-1 \\ \sum_{k=1}^{N} \bar{a}_{Nk} u_{ik}^{(n+1)} &= 0, \qquad i = 2, \ldots, N-1. \end{aligned} \tag{3.5.6}$$

Thus, together with the Equations (3.5.4) we have $N \times (N-2)$ equation for $N \times (N-2)$ unknowns and obtained system can be solved iteratively.

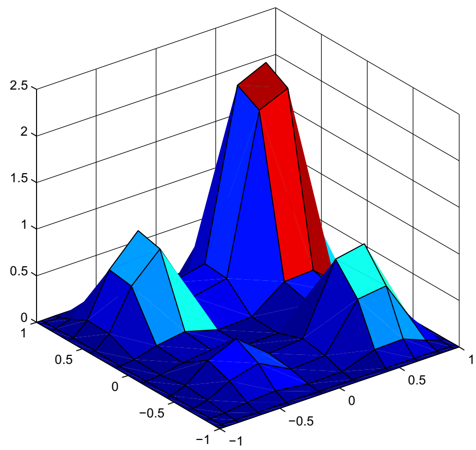
In the solution procedure of the problem for all analysis, $\theta = 2/3$ is used as a relaxation parameter and $N = 24$ mesh points are used in one direction. In the first analysis, time variation of u at $y = 0.6$ is given for $d = 1$. When time increases, the DQM solution converges to the steady state exact solution and the behavior of the solution at different time level is given in Figure 3.5.1. The numerical results at the steady state time level $t = 0.1$ are also given for the values of $d = 1, 5, 20$ and 40 in Figures 3.5.2, 3.5.3, 3.5.4 and 3.5.5, respectively. From the figures for all values of d, obtained results are good agreement with the steady state exact solutions.



(a) Contourlines

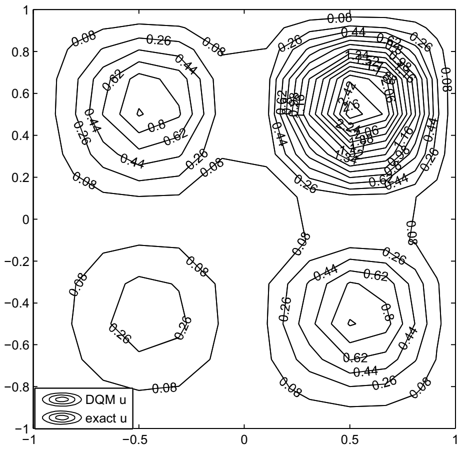


(b) Exact Solution

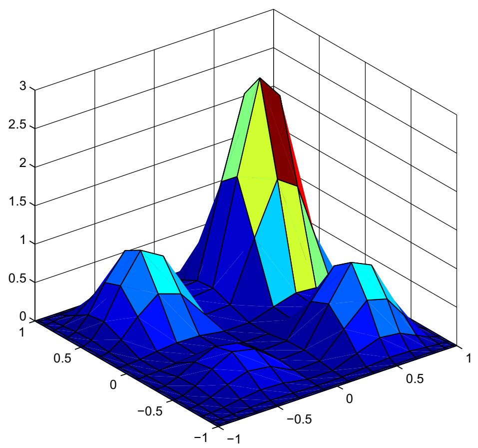


(c) DQM Solution

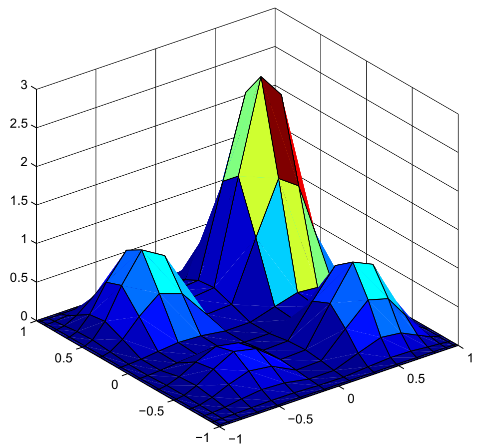
Figure 3.2.1 DQM Solution of Problem 2 for $N = 12$.



(a) Contourlines

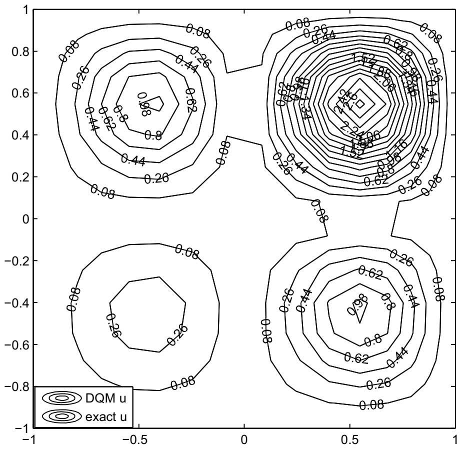


(b) Exact Solution

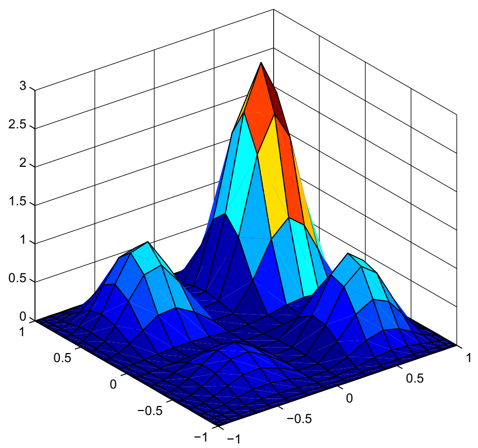


(c) DQM Solution

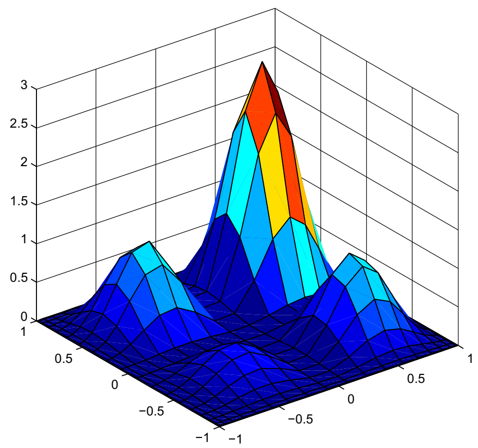
Figure 3.2.2 DQM Solution of Problem 2 for $N = 16$.



(a) Contourlines

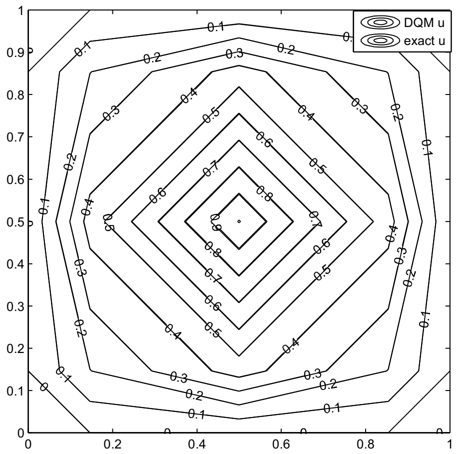


(b) Exact Solution

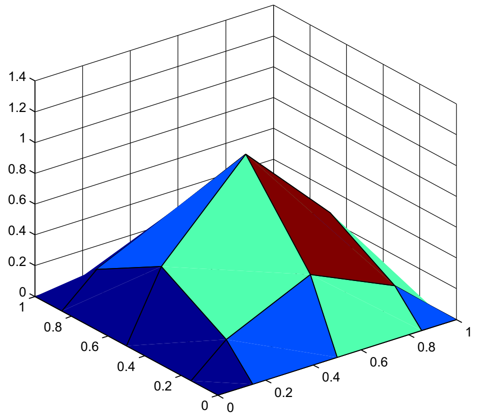


(c) DQM Solution

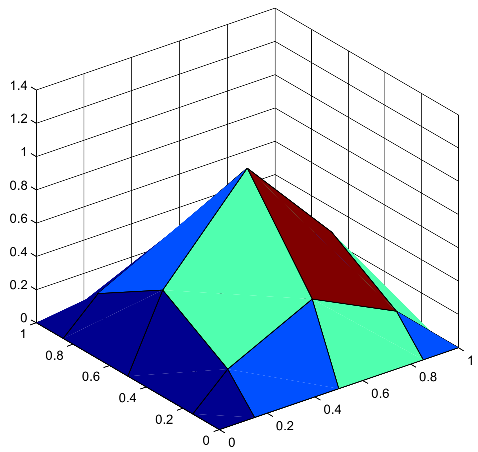
Figure 3.2.3 DQM Solution of Problem 2 for $N = 20$.



(a) Contourlines

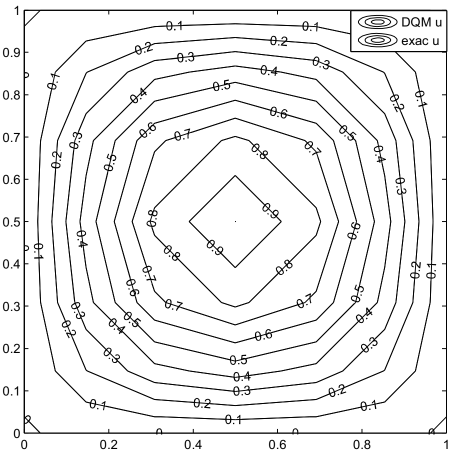


(b) Exact Solution

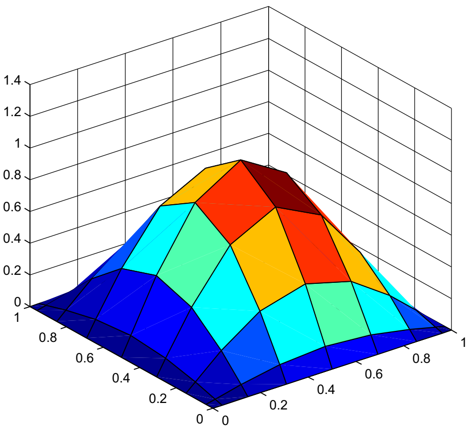


(c) DQM Solution

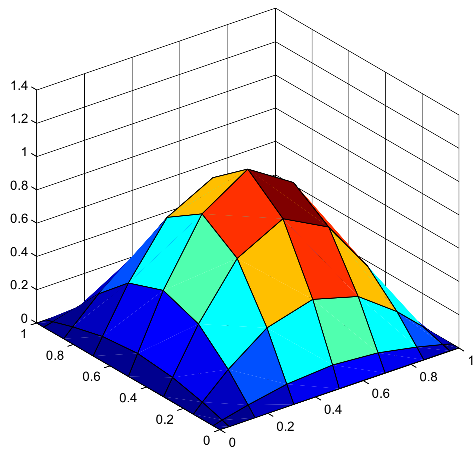
Figure 3.3.1 DQM Solution of Problem 3 for $N = 5$.



(a) Contourlines

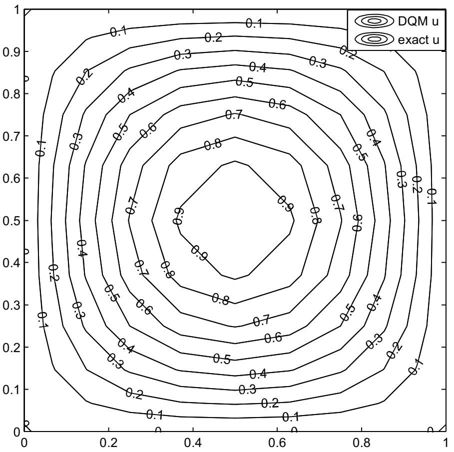


(b) Exact Solution

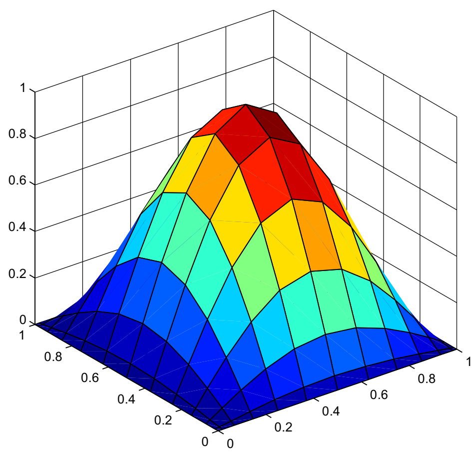


(c) DQM Solution

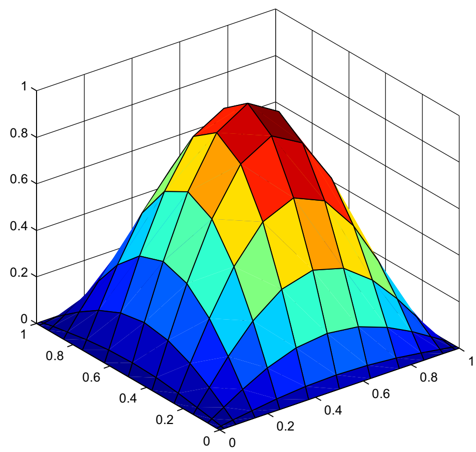
Figure 3.3.2 DQM Solution of Problem 3 for $N = 9$.



(a) Contourlines

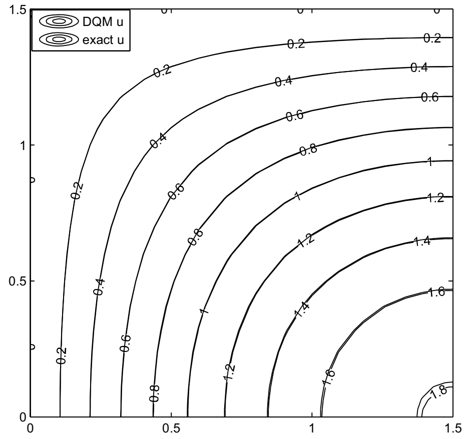


(b) Exact Solution

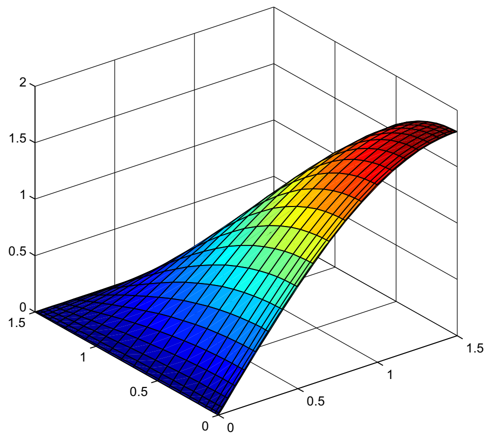


(c) DQM Solution

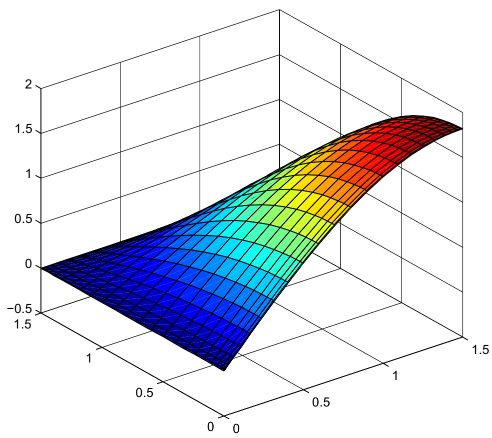
Figure 3.3.3 DQM Solution of Problem 3 for $N = 13$.



(a) Contourlines

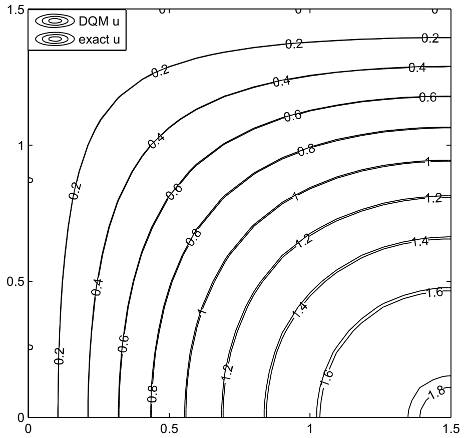


(b) Exact Solution

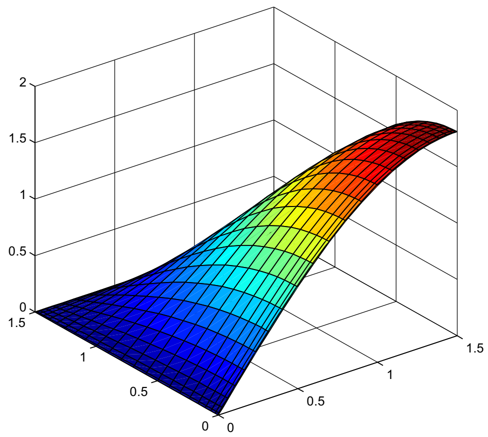


(c) DQM Solution

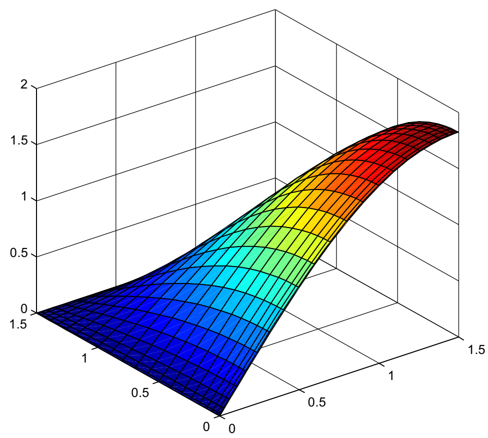
Figure 3.4.1 DQM Solution of Problem 4 for $\theta = 1/2$, $N = 24$ and $t = 1.2$.



(a) Contourlines

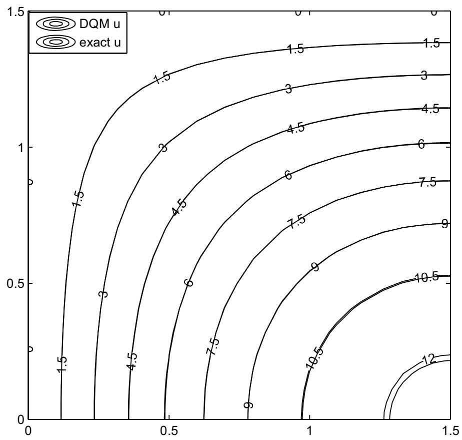


(b) Exact Solution

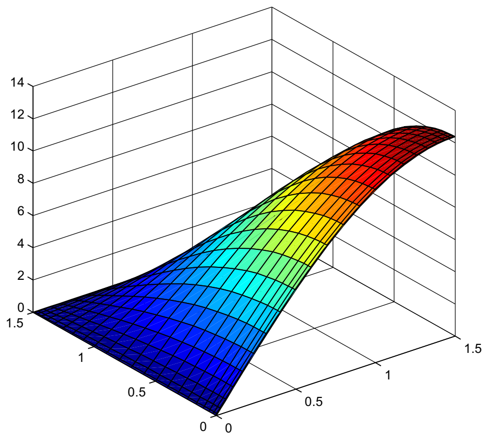


(c) DQM Solution

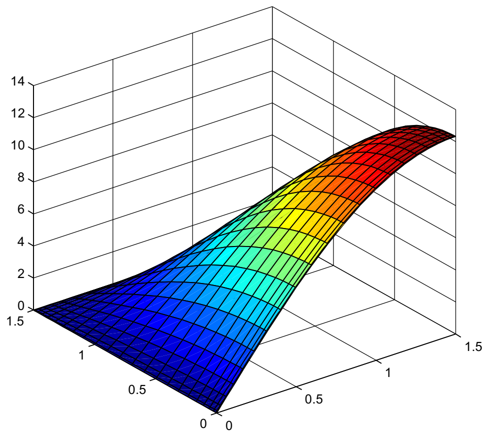
Figure 3.4.2 DQM Solution of Problem 4 for $\theta = 2/3$, $N = 24$ and $t = 1.2$.



(a) Contourlines

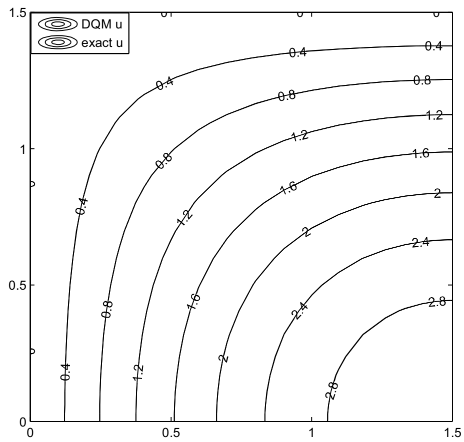


(b) Exact Solution

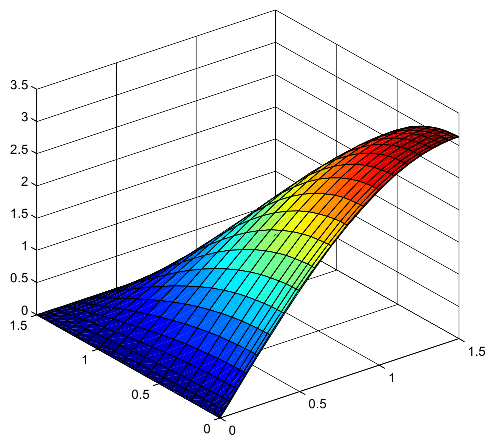


(c) DQM Solution

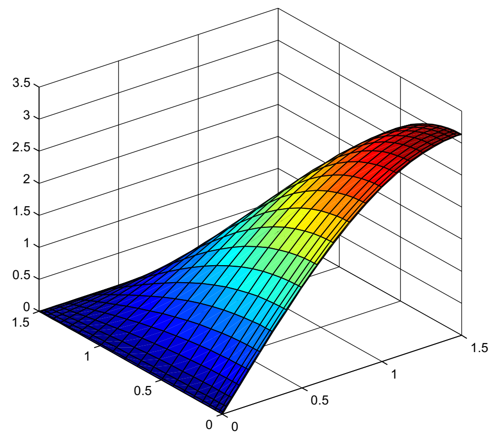
Figure 3.4.3 DQM Solution of Problem 4 for $\theta = 2/3$, $N = 24$ and $t = 0.5$.



(a) Contourlines

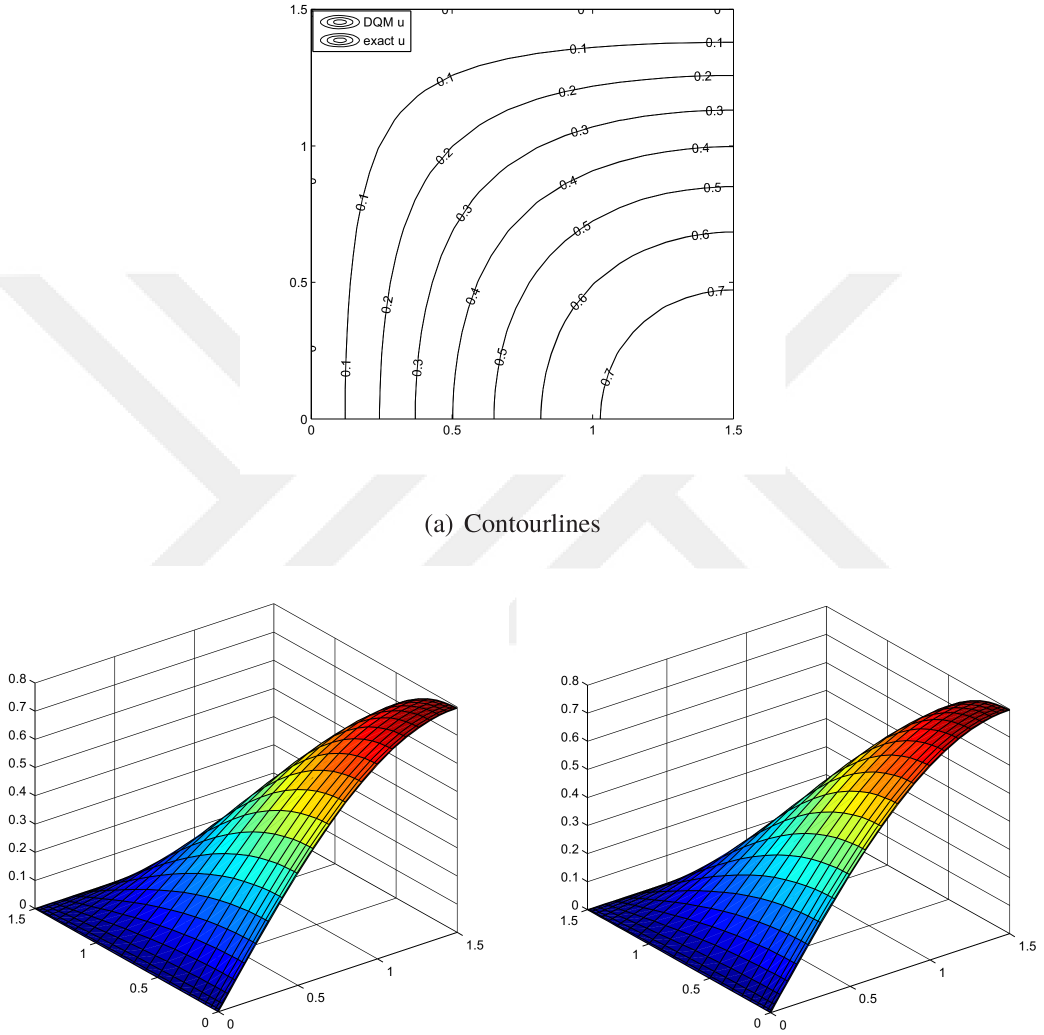


(b) Exact Solution



(c) DQM Solution

Figure 3.4.4 DQM Solution of Problem 4 for $\theta = 2/3$, $N = 24$ and $t = 1.0$.

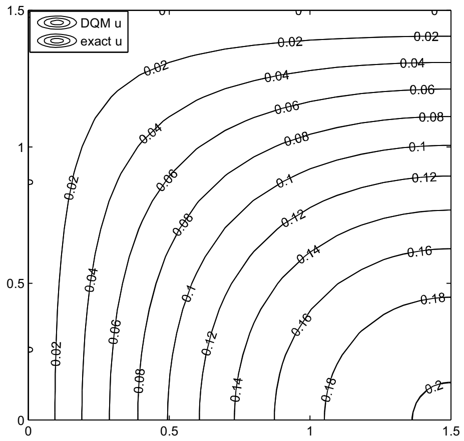


(a) Contourlines

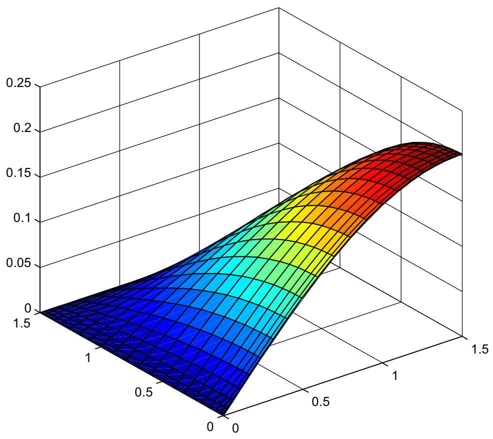
(b) Exact Solution

(c) DQM Solution

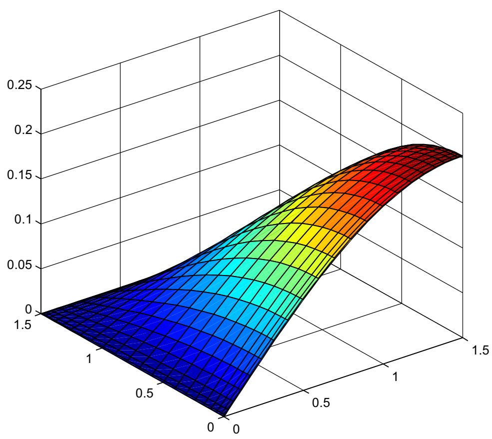
Figure 3.4.5 DQM Solution of Problem 4 for $\theta = 2/3$, $N = 24$ and $t = 1.5$.



(a) Contourlines



(b) Exact Solution



(c) DQM Solution

Figure 3.4.6 DQM Solution of Problem 4 for $\theta = 2/3$, $N = 24$ and $t = 2.0$.

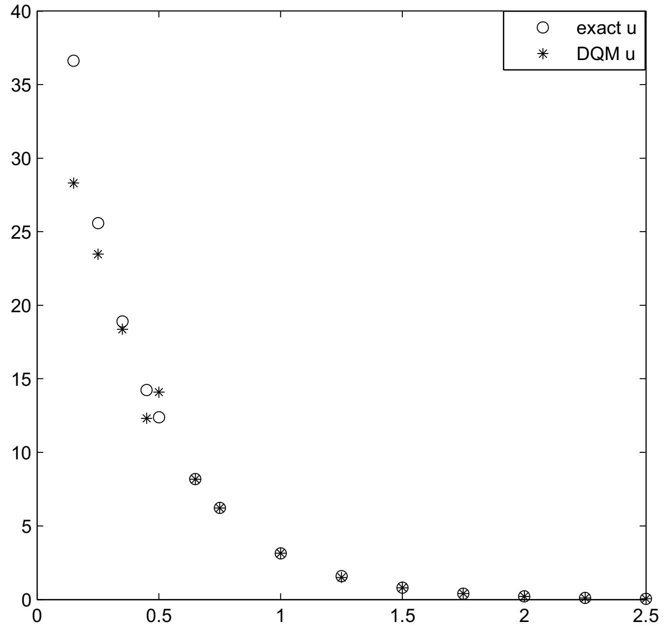


Figure 3.4.7 Time variation of u at right bottom corner.

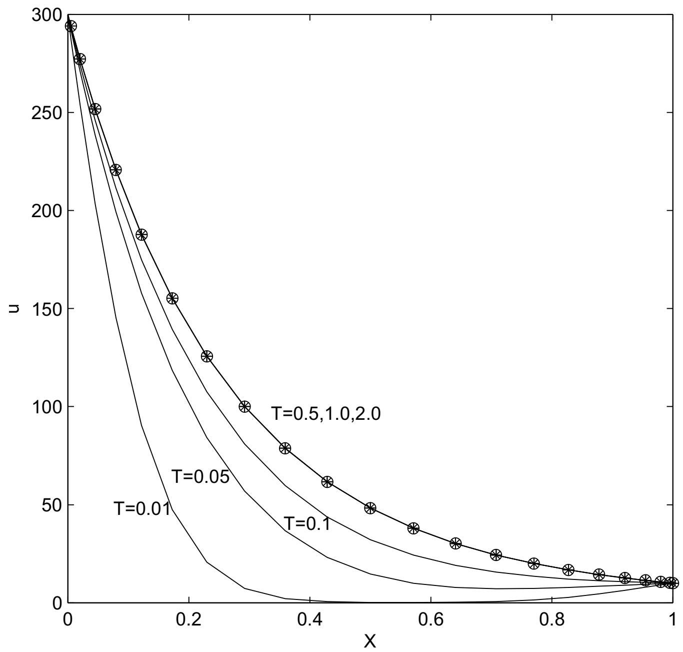
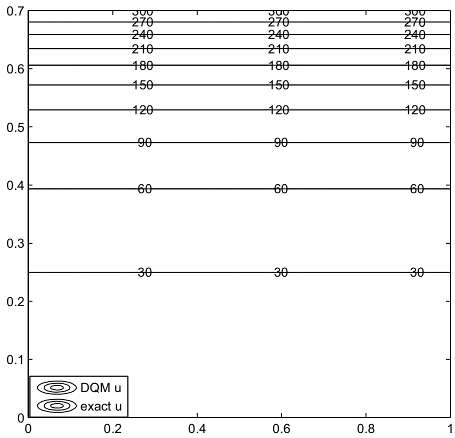
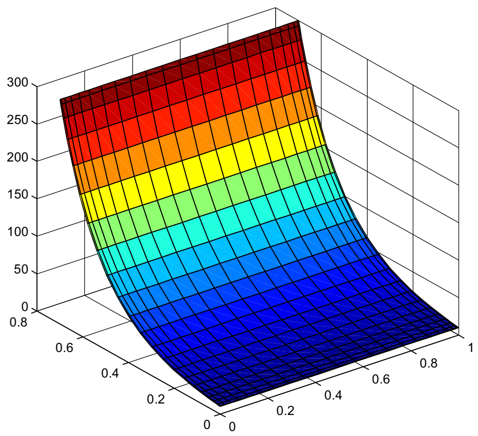


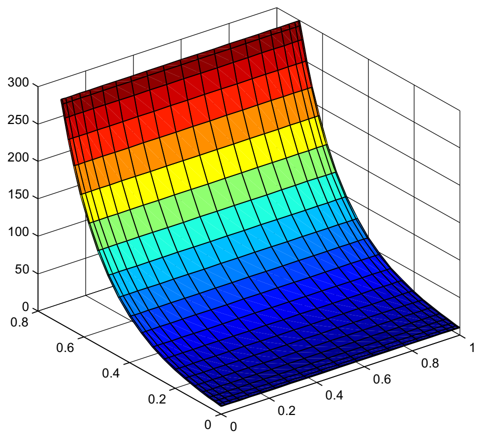
Figure 3.5.1 Time variation of u at $y = 0.6$.



(a) Contourlines

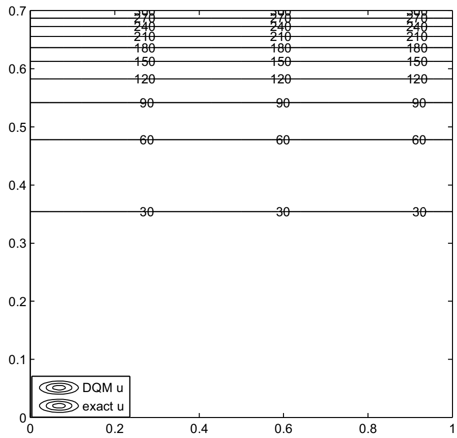


(b) Exact Solution

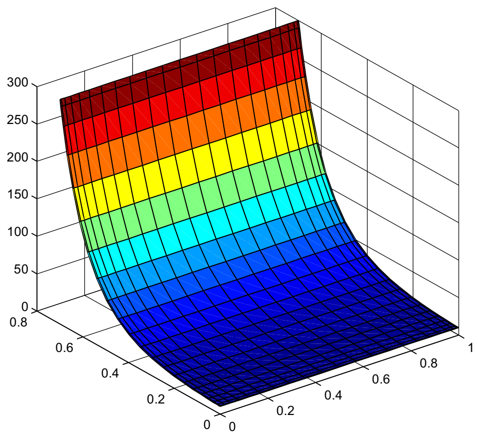


(c) DQM Solution

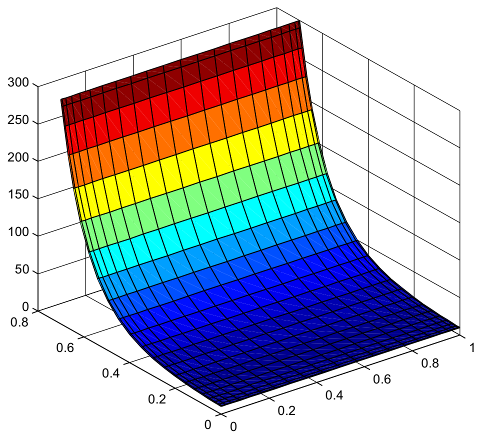
Figure 3.5.2 DQM Solution of Problem 5 for $\theta = 2/3$, $N = 24$, $T = 1.0$ and $d = 1.0$.



(a) Contourlines

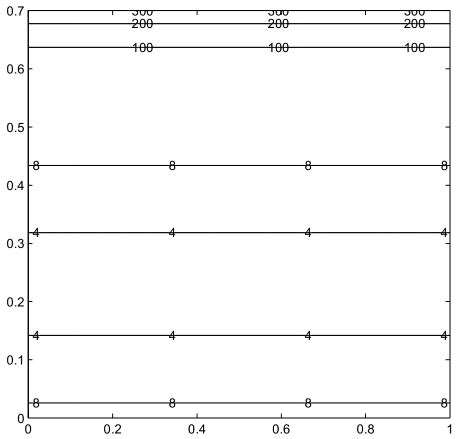


(b) Exact Solution

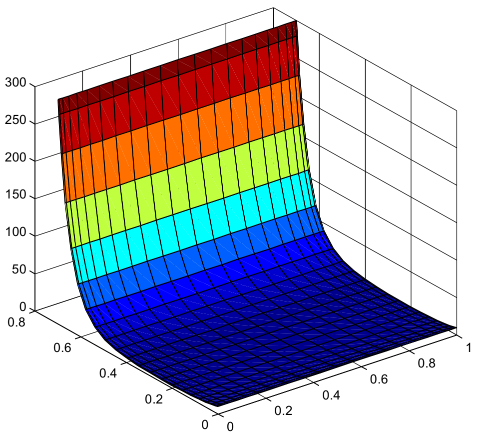


(c) DQM Solution

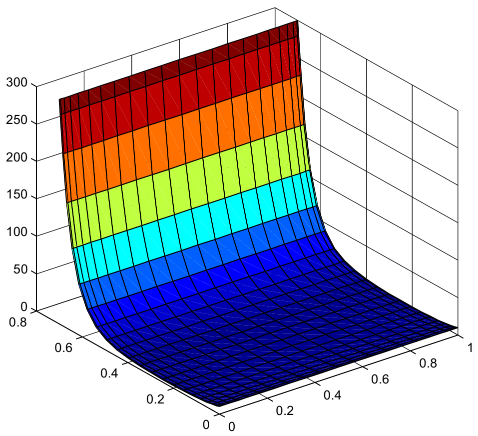
Figure 3.5.3 DQM Solution of Problem 5 for $\theta = 2/3$, $N = 24$, $T = 1.0$ and $d = 5.0$.



(a) Contourlines

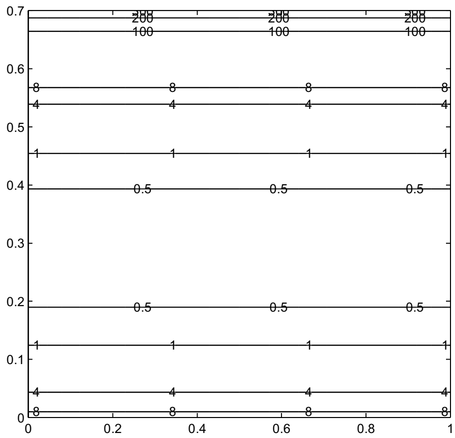


(b) Exact Solution

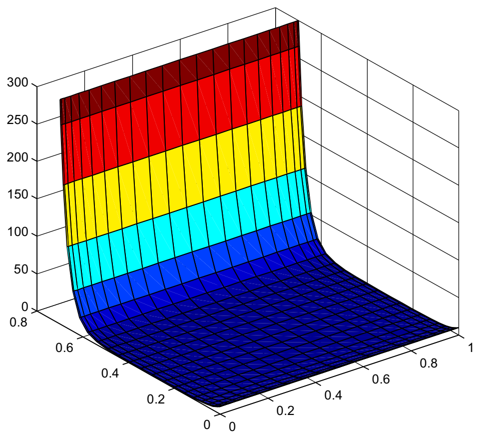


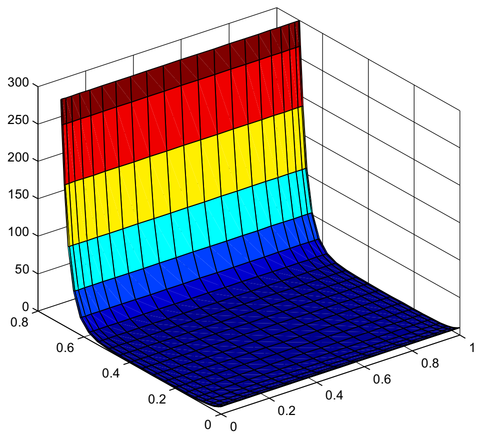
(c) DQM Solution

Figure 3.5.4 DQM Solution of Problem 5 for $\theta = 2/3$, $N = 24$, $T = 1.0$ and $d = 20.0$.



(a) Contourlines





(b) Exact Solution

(c) DQM Solution

Figure 3.5.5 DQM Solution of Problem 5 for $\theta = 2/3$, $N = 24$, $T = 1.0$ and $d = 40.0$.

4. CONCLUSION

This thesis is devoted to the differential quadrature solution of partial differential equations. First, polynomial-based differential quadrature method is explained for one-dimensional problem. The implementation of boundary conditions is described for both Dirichlet and Neumann type boundary conditions. In this section also nonuniform grid point distribution is explained by giving Chebyshev-Gauss-Lobatto grid points.

Then DQM is expanded to the two-dimensional problem. As test problems Poisson, Helmholtz and modified Helmholtz equations are used and implementation of Dirichlet boundary conditions are given for these problems. DQM solution of inhomogeneous Helmholtz-type equations which are the transformed form of time-dependent are given at the next problems. In order to obtain nonhomogeneous modified Helmholtz equation, the forward finite difference discretization is used for the time derivative and relaxation parameter is used for the unknown function which is located in the Laplace terms. Therefore we do not need to use any time integration scheme for the time derivative and eliminate the stability problems. Two different relaxation parameters are used in the computations which are Crank-Nicolson ($\theta = 1/2$) and Galerkin ($\theta = 2/3$) schemes.

DQM is a domain discretization method but very accurate results can be obtained using considerably small number of the mesh points. Behind this, DQM is quite simple since it is based on interpolation of solution and its derivatives by polynomials and at the end of the solution procedure the system of ordinary differential equations in time is constructed.

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APPENDIX

GENİŞLETİLMİŞ TÜRKÇE ÖZET

EXTENDED TURKISH SUMMARY

Bu tez çalışmasında, diferansiyel kareleme yöntemi kullanılarak kısmi türevli denklemler ile tanımlanmış problemlerin nümerik çözümleri grafikler ve tablolar yardımıyla tam çözümler ile karşılaştırmalı olarak verilmiştir. Diferansiyel kareleme yönteminde bir fonksiyonun her hangi bir mertebeden türevleri, bir doğru boyunca ki şebeke noktalarındaki fonksiyonun değerlerinin lineer toplamı şeklinde ifade edilebilir. Az sayıda şebeke noktası kullanılarak çok iyi sonuçlar elde edilebildiğinde oldukça kullanışlı bir nümerik çözüm yöntemidir. Metodun uygulanışında ki en önemli nokta ağırlıklandırılmış katsayıların elde edilmesidir ve bu katsayılar şebeke noktalarının koordinatları kullanılarak hesaplanabilmektedir. Böylece, bir diferansiyel denklem için cebirsel bir denklem sistemi elde etmek oldukça kolaydır.

Bu çalışmada kısmi türevli diferansiyel denklemlerin diferansiyel kareleme yöntemi ile elde edilmiş çözümleri düzgün olmayan şebeke dağılımları kullanılarak elde edilmiştir. İkinci bölümde, önce bir boyutlu problemler için yöntem açıklanmıştır. Birinci ve ikinci mertebeden türevler için katsayıların hesabı verildikten sonra birinci mertebeden türevlerde kullanılan katsayılar yardımıyla yüksek mertebeden türevlerin katsayılarının hesabı için formüller verilmiştir. Bu çalışmada düzgün olmayan şebeke olarak Chebyshev-Gauss-Lobatto noktaları kullanılmış ve bu noktaların tanımları yine bu bölümde verilmiştir.

Üçüncü bölümde diferansiyel kareleme yönteminin uygulama problemleri verilmiştir. Bütün problemler iki boyutlu olup yöntem iki boyutlu problemler için genişletilmiştir. Çözülen problemlerin tam çözümleri mevcut olduğu için elde edilen nümerik sonuçlar karï£¡laştï£¡rma yapılarak verilmiştir. İlk problem

$$\begin{array}{rlll} -\triangle u & = f & \quad \texttt{in} \quad & \Omega = [-1,1]\times[-1,1] \\ u & = 0 & \quad \texttt{on} \quad & \partial\Omega \end{array} \tag{4.0.1}$$

şeklinde bir Poisson denklemidir. Sağ taraf fonksiyonu

$$f(x,y) = 32\pi^2 \sin(4\pi x)\sin(4\pi y)$$

ve problemin tam çözümü

$$u(x,y) = \sin(4\pi x)\sin(4\pi y)$$

şeklindedir.

İkinci problem

$$\begin{aligned} -\nabla(p(x,y)\nabla u) + u &= f \qquad \texttt{in} \qquad \Omega = [-1,1]\times[-1,1] \\ u &= 0 \qquad \texttt{on} \qquad \partial\Omega \end{aligned} \tag{4.0.2}$$

öyleki

$$p(x,y) = 1 + x^2 y^2$$

şeklinde modifiye edilmiş Helmholtz denklemidir ve tam çözümü

$$u(x,y) = \sin^2(\pi x)\sin^2(\pi y)e^{x+y}.$$

şeklindedir. Tam çözümü kullanılarak sağ taraf fonksiyonu

$$\begin{aligned} f(x,y) = \; & -2xy^2 e^{x+y}\sin^2(\pi y)\left[\sin^2(\pi x) + \pi\sin^2(2\pi x)\right] \\ & -2x^2 y e^{x+y}\sin^2(\pi x)\left[\sin^2(\pi y) + \pi\sin^2(2\pi y)\right] \\ & -(1+x^2y^2)e^{x+y}\Big(\sin^2(\pi y)\left[\sin^2(\pi x) + 2\pi\sin(2\pi x) + 2\pi^2\cos(2\pi x)\right] \\ & +\sin^2(\pi x)\left[\sin^2(\pi y) + 2\pi\sin(2\pi y) + 2\pi^2\cos(2\pi y)\right]\Big) + \sin^2(\pi x)\sin^2(\pi y)e^{x+y}. \end{aligned}$$

şeklinde elde edilir.

Üçüncü problem

$$\begin{aligned} \nabla^2 u + 0.5u &= f \qquad \texttt{in} \qquad \Omega = [0,1]\times[0,1] \\ u &= 0 \qquad \texttt{on} \qquad \partial\Omega \end{aligned} \tag{4.0.3}$$

şeklinde bir Helmholtz denklemidir ve nonhomejen terim

$$f(x,y) = (-2\pi^2 + 0.5)\sin(\pi x)\sin(\pi y)$$

şeklindedir. Problemin tam çözümü

$$u(x,y) = \sin(\pi x)\sin(\pi y)$$

olarak verilmiştir.

Sonraki iki problemde sırasıyla zamana bağlı difüzyon ve konveksiyon-difüzyon denklemleri yine diferansiyel kareleme yöntemi ile çözülmüştür. Bu denklemler orijinal halleri ile çözülmek yerine homojen olmayan modifiye edilmiş Helmholtz denklemlerine dönüştürülmüş ve sonra diferansiyel kareleme yöntemi çözüm prosedürü uygulanmıştır. Homojen olmayan modifiye edilmiş Helmholtz denklemlerini elde etmek için önce denklemin zaman türevleri ileri sonlu farklar yöntemi kullanılarak iki zaman düzeyinde açılmıştır. Ayrıca Laplace terimleri içinde bulunan bilinmeyen fonksiyon için bir parametre yardımıyla yeni bir açılım yapılmıştır.Bunlar denklem içerisinde yerine konulup denklemler yeniden yazıldığında iteratif formda homojen olmayan modifiye Helmholtz denklemleri elde edilmiştir. Böylece zaman türevi için farklı bir yöntem kullanmaya gerek kalmamış ve dolayısıyla sayısal kararlılık analizi yapma ihtiyacı ortadan kalkmıştır.

Dördüncü problem difüzyon denklemidir

$$\begin{aligned} &\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \texttt{in} \qquad \Omega = [0,L]\times[0,L] \\ &u(0,y,t) = 0, \qquad u(x,0,t) = 0, \\ &q(L,y,t) = 0, \qquad q(x,L,t) = 0 \end{aligned} \tag{4.0.4}$$

şeklinde tanımlanır. Burada k diffüsiviti katsayısıdır. Problemin tam çözümü

$$u(x,y,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} A_{lm}\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{l\pi y}{L}\right)\exp\left[-\left(\frac{kl^2\pi^2}{L^2}+\frac{km^2\pi^2}{L^2}\right)t\right]$$

öyleki

$$A_{lm} = \frac{4u_0}{lm\pi^2}[(-1)^l - 1][(-1)^m - 1].$$

şeklinde verilmiştir. Modifiye edilmiş Helmholtz denklemi formundaki difüzyon denklemi ise

$$\nabla^2 u^{(n+1)} - \lambda^2 u^{(n+1)} = -\left(\frac{1-\theta}{\theta}\right)\nabla^2 u^{(n)} - \lambda^2 u^{(n)} \tag{4.0.5}$$

öyleki $\lambda^2 = \dfrac{1}{k\Delta t\theta}$. Modifiye edilmiş Helmholtz denklemi elde eldilirken zamana bağlı türev için

$$\frac{\partial u}{\partial t} = \frac{u^{(n+1)} - u^{(n)}}{\Delta t} \tag{4.0.6}$$

ve Laplace terimi içindeki bilinmeyen fonksiyon u için

$$u = \theta u^{(n+1)} + (1-\theta)u^{(n)} \tag{4.0.7}$$

yaklaşımları kullanılmıştır. Burada Δt zaman artırımı, $u^{(n)}$ ve $u^{(n+1)}$ u'nun sırası ile şimdiki ve ileri zamandaki değerlerini temsil eder.

Son olarak konveksiyon-difüzyon denklemi

$$\nabla^2 u = \frac{1}{K}\frac{\partial u}{\partial t} + c_x\frac{\partial u}{\partial x} + c_y\frac{\partial u}{\partial y} + du \tag{4.0.8}$$

ve $[0,1]\times[0,1]$ karesel bölgedeki sınır koşulları

$$\begin{aligned} u(0,y,t) &= 300, & q(x,0,t) &= 0, \\ u(1,y,t) &= 10, & q(x,0.7,t) &= 0. \end{aligned}$$

şeklinde tanımlanmıştır. Hesaplamalarda katsayılar

$$\begin{aligned} K &= 1, \\ c_x &= dx + \log\frac{10}{300} - \frac{d}{2} \\ c_y &= 0 \end{aligned}$$

şeklinde alınmıştır. Problemin tam çözümü ise

$$u(x,y,t) = 300\exp\left(\frac{d}{2}x^2 + \frac{10}{300}x - \frac{d}{2}x\right).$$

şeklindedir. Difüzyon denkleminde kullanılan yaklaşımlar bu denklem için de kullanıldığında

$$\nabla^2 u^{(n+1)} - \lambda^2 u^{(n+1)} = -\left(\frac{1-\theta}{\theta}\right)\nabla^2 u^{(n)} - \lambda^2 u^{(n)} + \frac{1}{K\theta}\left(c_x\frac{\partial u^{(n)}}{\partial x} + c_y\frac{\partial u^{(n)}}{\partial y} + du^{(n)}\right) \tag{4.0.9}$$

öyleki $\lambda^2 = \dfrac{1}{K\theta\Delta t}$ şeklinde modifiye edilmiş Helmholtz denklemi elde edilir.

Bütün problemler için elde edilen diferansiyel kareleme yöntemi çözümleri değerlendirildiğinde, kullanım olarak oldukça basit bir yöntem olmasına karşın yöntem gayet iyi sonuçlar elde edilebilmiştir.

CURRICULUM VITAE

He was born in Duhok from Iraq, in 1986. He went to Duhok University between 2006 and 2010. He work as a primary school teacher in Duhok. He started his master study in Yüzüncü Yıl University at the September of 2014.

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