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MASTER THESIS

GAME COLORING OF GRAPHS

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GAME COLORING OF GRAPHS

YEMİN METNİ

Yüksek Lisans Tezi olarak sunduğum "Game Coloring of Graphs" adlı çalışmanın, tarafımdan bilimsel ahlak ve geleneklere aykırı düşecek bir yardıma başvurmaksızın yazılığını ve yararlandığım eserlerin bibliyografyada gösterilenlerden oluştuğunu, bunlara atıf yapılarak yararlanılmış olduğunu belirtir ve bunu onurumla doğrularım.

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MELEK ÇELİK

ABSTRACT

Master Thesis

GAME COLORING OF GRAPHS

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Mathematics

This thesis consists of three chapters. Firstly, an introductory approach is given. In the first chapter, basic notions of graph theory are given. In chapter two, edge game coloring of graphs is introduced, then game chromatic index of some graphs are mentioned. In chapter three, game coloring of graphs is explained and game chromatic number of some type of graphs are given. Finally, some variations of game chromatic number are examined.

Key words: Game Coloring Of Graphs, Game Chromatic Number, Game Chromatic Index.

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INTRODUCTION

Games can be used to model some conflicting interests or to model the worst type of incorrect behaviour of a system. It is assumed that the incorrect system uses an intelligent strategy to try to prevent us from reaching our target.

Various games on graphs have been considered among the most intensively studied is the two–player coloring game. In 1991, Bodlaender (Bodlaender, 1991) introduced the game–coloring of graphs. Let G be a graph and X = {1, 2, . . ., k} be a set of colors. In the coloring game, Player 1 and Player 2 make moves alternatively with Player 1 moving first. Each feasible move consists of choosing an uncolored vertex, and coloring it with a color from X, so that adjacent vertices get distinct colors. The game ends as soon as one of the two players can no longer execute any feasible move. Player 1 wins if all the vertices of G are colored, otherwise Player 2 wins.

The edge version of the game coloring of graphs is defined similarly and has first been studied by Cai and Zhu (Cai and Zhu, 2001).

This thesis aims to review literature about the game coloring and edge game coloring of graphs. Game chromatic number and game chromatic index of graphs are taken into consideration and various theorems in the literature are explained.

CHAPTER 1

PRELIMINARIES

In this chapter, we provide the necessary background and motivation for this study on the game–coloring of graphs. We start in Section 1.1 by giving some definitions of standard graph-theoretical terms used throughout the remainder of the thesis. We next define common families of graph in graph theory in Section 1.2 Then, in Section 1.3 we introduce the graph operations.

1.1 Graphs

A **graph** $G = (V, E)$ consists of two sets: a non-empty finite set V and a finite set E. The elements of V are called vertices (or points or nodes) and the elements of E are called edges (or lines). Each edge is identified with a pair of vertices. The set $V(G)$ is called the **vertex set** of G, and the set $E(G)$ is called the **edge set** of $E(G)$ (Balakrishnan and Ranganathan, 2000). Each edge is identified with a pair of vertices. If the edges of a graph G are identified with ordered pairs of vertices, then G is called a **directed** graph. Otherwise G is called an **undirected** graph (Wilson, 1996).

The cardinality of the vertex set of a graph G is called the **order** of G and is commonly denoted by $n(G)$, or more simply by n when the graph under consideration is clear; while the cardinality of its edge set is the **size** of G and is often denoted by $m(G)$ or m. A graph with no edges is called an **empty** graph. A graph with no vertices (and hence no edges) is called a **null** graph (Wilson, 1996).

Graphs are finite or infinite according to their order; however the graphs we consider are all finite. If a graph allows more than one edge (but yet a finite number) between the same pair of vertices in a graph, the resulting structure is a **multi-graph**. Such edges are called **parallel** or **multiple edges**. An edge that joins a single endpoint to itself is known as a **loop**. Graphs that allow parallel edges and loops are called **pseudographs**. A **simple graph** is a graph with no parallel edges and loops (Balakrishnan and Ranganathan, 2000). A graph G is **planar** if there exists a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G (Wilson, 1996).

An edge is said to be **incident** on its end vertices. Two vertices are **adjacent** if they are the end vertices of an edge. The neighborhood of *v*, *N*(*v*), is the set of vertices adjacent to *v*. If two edges have a common end vertex, then these edges are said to be **adjacent** (Wilson, 1996).

Let *G* be a graph and $v \in V$. The number of edges incident at *v* is called the **degree** of the vertex *v* in *G* and denoted by *deg*(*v*). A loop at *v* is to be counted twice in computing the degree of *v*. A vertex of degree 0 is called an **isolated** vertex. A graph *G* is **regular** if all the vertices of *G* are of equal degree. If every vertex of *G* has degree *r*, then *G* is called *r-regular*.

$\delta(G) = \min\{\deg(v) \mid v \in V\}$ denotes the **minimum degree** of *G*. Similarly,

$\Delta(G) = \max\{\deg(v) \mid v \in V\}$ denotes the **maximum degree** of *G* (Balakrishnan and Ranganathan, 2000).

A graph *H* is called a **subgraph** of *G* if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph *H* of *G* is said to be an **induced subgraph** of *G* if each edge of *G* having its endpoints in *V* (*H*) is also an edge of *H*. A subgraph *H* of *G* is a **spanning subgraph** of *G*, if $V(H) = V(G)$ (Balakrishnan and Ranganathan, 2000).

1.2 Common Families of Graphs

A simple graph *G* is said to be **complete** if every pair of distinct vertices of *G* are adjacent in *G*. It is denoted by K_n (Balakrishnan and Ranganathan, 2000).

A **bipartite** graph *G* is a graph whose vertex–set *V* can be partitioned into two subsets *U* and *W*, such that each edge of *G* has one endpoint in *U* and one endpoint in *W*. The pair *U*, *W* is called a vertex bipartition of *G*, and *U* and *W* are called the bipartition subsets (Wilson, 1996).

A **complete bipartite** graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has m vertices in one of its bipartition subsets and *n* vertices in the other is denoted by $K_{m,n}$ (Wilson, 1996). A complete bipartite graph of the form $K_{1,n}$ is called a **star** (Balakrishnan and Ranganathan, 2000). A **walk** in a simple graph *G* is a sequence $v_0\, e_1\, v_1 \ldots v_{k-1}\, e_n\, v_n$ of vertices and distinct edges such that consecutive vertices in the sequence are adjacent. The walk is closed if $v_0 = v_n$ and is open otherwise. A **path** is a walk with no repeated vertex. P_n denotes a path on *n* vertices. The **length** of a walk or path is its number of edges.

A **cycle** is a closed walk of length at least three in which the vertices are distinct except the first and the last. C_n denotes a cycle on n vertices. A cycle is odd or even according as its length is odd or even. The graph obtained from C_n by joining each vertex to a new vertex v is called **wheel**. W_n denotes a wheel on $n + 1$ vertices.

A graph is said to be **acyclic** if it has no cycles. A **tree** is a connected acyclic graph (Balakrishnan and Ranganathan, 2000).

1.3 Graph Operations

Operations on graphs produce new graphs from old ones. They may be separated into the following major categories.

1.3.1 Unary operations

Unary operations create a new graph from the old one.

1.3.1.1 Elementary operations

These are sometimes called "editing operations" on graphs. They create a new graph from the original one by a simple, local change, such as addition or deletion of a vertex or an edge, merging and splitting of vertices, edge contraction, etc.

Vertex Removal: If v_i is a vertex of a graph $G = (V, E)$, then $G - v_i$ is the induced subgraph of G on the vertex set $V - v_i$; that is, $G - v_i$ is the graph obtained after removing from G the vertex v_i and all the edges incident on v_i (Wilson, 1996).

Edge Removal: If e_i is an edge of a graph $G = (V, E)$, then $G - e_i$ is the subgraph of G that results after removing from G the edge e_i. Note that the end vertices of e_i are not removed from G (Wilson, 1996).

1.3.1.2 Advanced operations

The **complement** $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(\overline{G})$ (Wilson, 1996).

Power of graph: The k-th power G^k of a graph G is a supergraph formed by adding an edge between all pairs of vertices of G with distance at most k. The *second power* of a graph is also called its square (http://en.wikipedia.org/ wiki/ Graph_ operations).

Line graph: Given a graph *G*, its line graph *L*(*G*) is a graph such that each vertex of *L*(*G*) represents an edge of *G*; and two vertices of *L*(*G*) are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in *G* (http://en.wikipedia.org/wiki/Graph_operations).

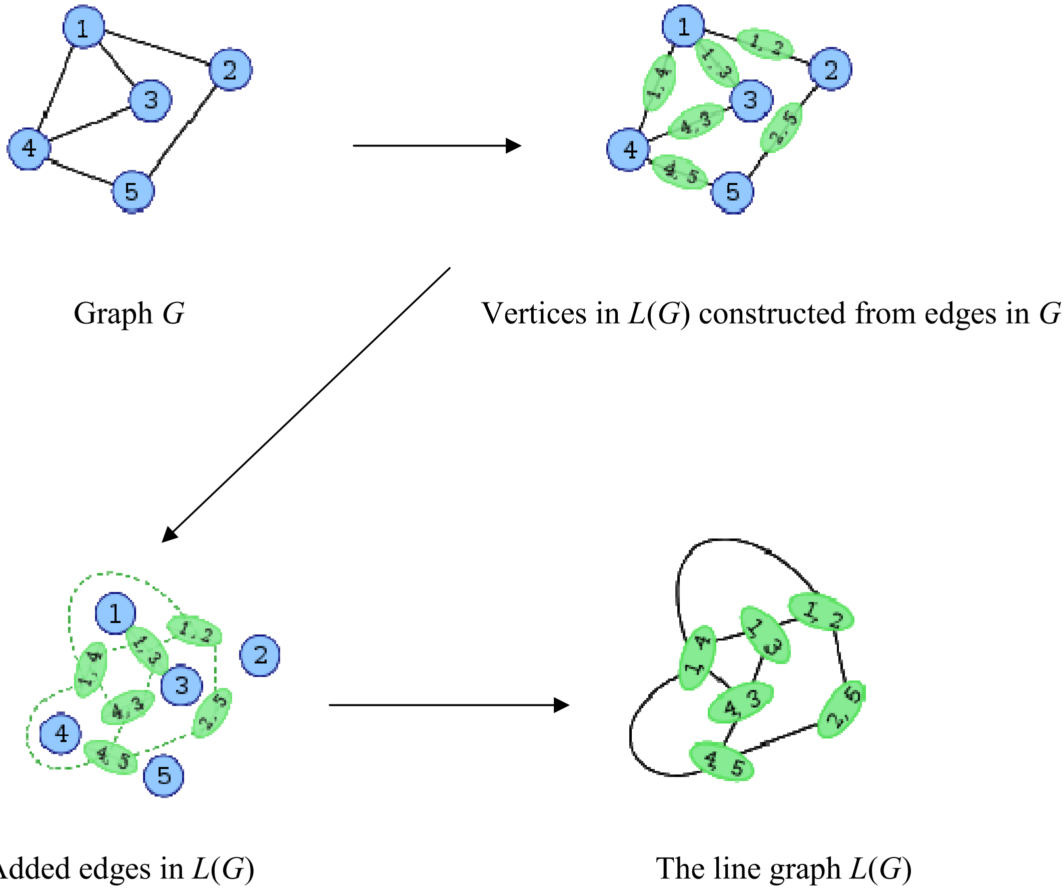

Figure 1.1: The line graph *L* (*G*)

1.3.2 Binary operations

Binary operations create a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$:

The **union** of two graphs is formed by taking the union of the vertices and edges of the graphs. Thus the union of two graphs is always disconnected (Balakrishnan and Ranganathan, 2000).

The **join** $G + H$ of the graph G and H is obtained from the graph union $G + H$ by adding an edge between each vertex of G and each vertex of H (Balakrishnan and Ranganathan, 2000).

The **Cartesian product** $G = G_1 \times G_2$ has $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = u_2$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. A convenient way of drawing $G_1 \times G_2$ is first to place a copy of G_2 at each vertex of G_1 and then to join corresponding vertices of G_2 in those copies of G_2 placed at adjacent vertices of G_1 (Wilson, 1996).

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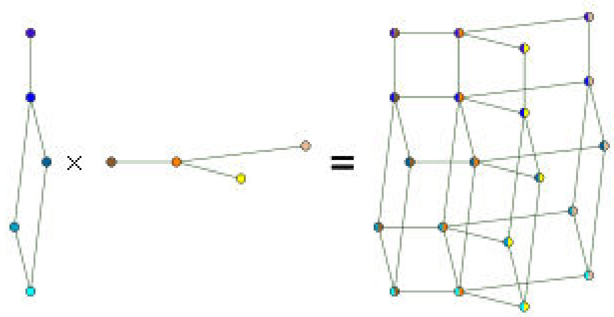

Figure 1.2: The Cartesian product of graphs

CHAPTER 2

THE GAME CHROMATIC INDEX

The edge version of the game coloring of graphs is defined similarly and has first been studied by Cai and Zhu (Cai and Zhu, 2001), Alice (A) and Bob (B), move alternately. A move consists in coloring an uncolored edge of *G* with a color from *X* so that adjacent edges are not colored with the same color. The game ends when no more move is possible. Alice wins if every edge is colored at the end of the game, otherwise Bob wins (Andres, 2005).

A graph *G* is called *k*–game-colorable if Alice has a winning strategy for $|X| = k$ and the smallest number k such that Alice has a winning strategy with *k* colors in edge game coloring *G* is called the *game chromatic index* of *G*, denoted by $\chi_g^{'}$(G) (Lam, Shiu and Xu, 1991).

To be precise, the game chromatic index may depend on who is to make the first move. It is generally assumed that Alice moves first, but this requirement is irrelevant for the trees with $\Delta \geq 6$ (Dinski and Zhu, 1999).

Theorem 2.1 (Lam, Shiu and Xu, 1991): Chromatic index of a graph *G* is a lower bound of the game chromatic index of *G*.

$$\chi^{'}(G) \leq \chi_g^{'}(G)$$

Theorem 2.2 (Lam, Shiu and Xu, 1991): Each edge is adjacent to at most $2\Delta(G) - 2$ distinct edges, where $\Delta(G)$ denotes the maximum degree of G. So $2\Delta(G) - 1$ is a trivial upper bound of $\chi_g^{'}$(G).

$$\Delta(G) \leq \chi_g^{'}(G) \leq 2\Delta(G) - 1$$

2.1 Game Chromatic Index Of Some Graphs

Theorem 2.1.1 (Lam, Shiu and Xu, 1991): If P_n is a path then $\chi_g'(P_n) = 3$ when $n \geq 5$.

Theorem 2.1.2 (Lam, Shiu and Xu, 1991): If C_n is a cycle then $\chi_g'(C_n) = 3$.

Theorem 2.1.3 (Lam, Shiu and Xu, 1991): If $K_{1,n}$ is a star then $\chi_g'(K_{1,n}) = n$.

Theorem 2.1.4 (Lam, Shiu and Xu, 1991): If W_n is a wheel then $\chi_g'(W_3) = 5$, $\chi_g'(W_n) = n + 1$ when $n \geq 4$.

Proof: $\chi_g'(W_3) = 5$ can be verified directly.

Suppose $n \geq 4$. We shall call an edge joining the center to a vertex of the circle a spoke. The end vertex of a spoke lying on C_n is called its end. An edge joining the ends of two spokes is called a *rim*. A bare spoke is uncolored and have no colored rim incident with it.

It is enough to show that Player 1 can color all spokes with $n + 1$ colors when $n \geq 4$. Any rim is incident with two spokes and two other rims. Hence $n + 1 \geq 5$ assumes that there will be feasible color for any rim after all spokes are colored. Let c_k denote the k-th color introduced during the game. Player 1 plans to color r spokes with r distinct colors, for $1 \leq r \leq n - 2$. Initially, he colors an arbitrarily chosen spoke with color c_1. Suppose $r - 1$ spokes have been colored with $r - 1$ colors when it is Player 2's turn, where $2 \leq r \leq n - 2$, so there are at least 3 uncolored spokes.

If Player 2 colors a spoke, he would be helping Player 1 to accomplish his goal. If Player 2 colors a rim, he can at his best prevent at most 2 of the uncolored spokes from being colored with c_r by Player 1, but there are at least 3 uncolored spokes.

When Player 1 colors the $(n - 2)$ the spoke, he should ensure that no bare spokes are next to each other afterwards. This can be done by choosing to color a bare spoke if there are 2 next to each other, and by choosing to color the middle one if there are three of them next to each other. With this precaution, at best, Player 2 can color the rim incident with one of the 2 remaining uncolored spokes with c_{n-1}. Nevertheless, Player 1 can color the other one with c_{n-1}. So no matter what Player 2 does next, Player 1 can color the last uncolored spoke with c_n or c_{n+1}.

If Player 2 chooses to color the $(n-2)$ – th spoke with c_{n-2}, Player 1 can simply color one of the 2 uncolored spokes with c_{n-1}. Any move by Player 2 cannot prevent. Player 1 from coloring the last uncolored spoke with c_n or c_{n+1}. ■

Theorem 2.1.5 (Lam, Shiu and Xu, 1991):

$\chi_g'(T) \leq \Delta(T) + 2$ for each tree T, if $\Delta \geq 3$.

Proof: We give a winning strategy for Player 1 using $\Delta(T) + 2$ colors.

Initially, Player 1 chooses an arbitrary edge $e = v_0 v_1$ of T, where $\deg_T(v_0) = 1$, and assigns a color to it. Let $T^* = \{e\}$. Henceforth, T is regarded as a diagraph with v_0 as its root.

Suppose that Player 2 has just moved by coloring an edge e_1. Let P the directed path from v_0 to e_1 in T, and let e^* be the last edge P has in common with T^*. We update T^* to $T^* \cup$ P, i.e., $T^* := T^* \cup \{P\}$.

If e^* is uncolored, then we assign a feasible color to e^*. If e^* is colored and T^* contains an uncolored edge f, then assign a feasible color to f. Otherwise, we color any edge f adjacent to same edges of T^* and let $T^* := T^* \cup \{f\}$.

Suppose $f = \overrightarrow{uv}$ is the last edge of the directed path in T from v_0 to v. If at most one arc out of v has been colored, then the total number of colored arcs incident with f is at most $\Delta(G)$. As soon as a second outgoing arc of v has been colored, Player 1 will color f unless it has already been colored beforehand. At this moment, at most $\Delta(G) + 2$ available colors. Player 1 can always find a feasible color for f. ■

Theorem 2.1.6 (Erdös, Faigle and Kern, 1993): For any $\Delta \geq 2$ there exists a tree with game chromatic index equal to $\Delta + 1$.

Proof: It suffices to exhibit a tree $T = T_\Delta$ such that Alice has no winning strategy for T if the number of colors is $k = \Delta$. For $\Delta = 2$ this is trivial: Take any sufficiently long path. (If B is move first, a path with at least 5 edges is needed)

For $\Delta \geq 3$ we let $T = T_{\Delta}$ be the unique rooted tree of height 2 with $\Delta + 1$ nodes of degree Δ and $\Delta(\Delta - 1)$ leaves. Thus the root v is incident with Δ "base edges" and each base edge in turn is adjacent to $\Delta - 1$ leaf edges.

For $\Delta = 3$ the claim is straightforward to check. (B can create a "critical edge" after two moves of A, no matter who starts). The case $\Delta \geq 4$ can be solved similarly as follows. In his first moves B colors base edges in such a way that (after his move) each of the remaining uncolored base edges has only uncolored adjacent leaf edges. He proceeds this way as long as (after his move) there are still (at least) two such uncolored base edges left. Then A will have last in two further steps as in case $\Delta = 3$.

■

Theorem 2.1.7 (Cai and Zhu, 2001): If F is a forest of maximum degree 3, then $\chi'_g(F) \leq 4$.

Proof: Let F be a forest of maximum degree 3. For convenience, we shall assume that each vertex of F either has degree 1 or has degree 3. The strategy we give here for Alice can be easily adopted to apply to those forests with degree 2 vertices. We shall prove that if the color set is $\{1, 2, 3, 4\}$, then Alice has a winning strategy.

We need to define a few terms. Let F be a forest with some edges (properly)–colored by colors from $\{1, 2, 3, 4\}$. We say an edge e of H is safe if either e is colored, or at least three of the edges incident to e are colored, and two of them are colored with the same color. We define a c-block B of H to be a maximal subtree of F such that each safe edge of B is leaf-edge of B. Intuitively, the c-blocks of a partially colored forest could be obtained by cutting, each connected component becomes a c-block. Note that each safe edge belongs to two c-blocks, and each other edge belongs to one c-block.

In the process of the game, the edges of F are successively colored. At each stage, the forest F is partially colored forest. For that partially colored forest, we have a family of c-blocks. When another edge is colored the Family of c-blocks is changed. It is obvious that each time a new edge is colored, some old c-block may break into two or three new c-blocks. So the c-blocks become smaller and smaller, and eventually, when all the edges are colored, each c-blocks is a star.

An important observation is that when an edge e is being colored, it only affect the edges in the c-block at the previous stage that contains e. In other words, we may regard different c-blocks of the forest *F* as disjoint subtrees, and consider each of the c-blocks separately.

Another observation is that if an uncolored edge is a safe edge, then it belongs to two c-blocks. However, one of the c-blocks is a star $K_{1,3}$.

We shall prove by induction that Alice has a strategy for the game so that at any stage, after her move (and before Bob takes his next move), each c-block *B* of *F* has the following property:

(∗): *B* contains at most 3 colored edges, and that no uncolored edge is incident to three edges colored with distinct colors.

Initially, this is certainly true.

Suppose this is the case at a certain stage, i.e., after Alice finished her move, the partially colored forest has property (∗). Then Bob color an edge, say *e*, with color *j*. We shall show that Alice can choose an uncolored edge and a suitable color for that edge so that after she colors the chosen edge with that color, each c-block of the resulting partially colored forest has property (∗). We shall only describe Alice's strategy, and leave to the readers to verify that the resulting partially colored forest does have the required property.

If *e* was not a safe edge before Bob colors it, then *e* belonged to a single c-block. If the edge *e* was a safe edge before Bob colors it, then *e* belonged to two c-blocks, however, one of the c-block is a star, and hence Alice need not to *B*, that contains the edge e. If *B* contains only one colored edge. Then after Bob colors *e*, *B* breaks into two c-blocks, and each of the two c-blocks has at most two colored edges. In this case, Alice can easily find a suitable edge and color it with a suitable color, so that the resulting new c-blocks have property (∗). We omit the details.

Assume that B has two colored edges. After Bob colors *e*, B breaks into two or three new c-blocks (B breaks into three new c-blocks only if an uncolored edge becomes a safe edge), say $B_1', B_2', \ldots$. It is easy to see that at most one new c-block contains three colored edges.

If each new c-block contains at most two colored edges, then again it is easy for Alice to find a suitable edge and color it with a suitable color, so that the resulting new c-blocks have property (∗).

Assume there is one c-block, say B_1', contains three colored edges. If the three colored edges have a common end vertex, then it is again trivial. If two of the colored edges have a common end vertex, then Alice colors the third edge incident to that vertex with any legal color. Thus we assume that no two of the three colored edges have a common end vertex. If the three colored edges are as shown in Fig. 2. 1(a), where the thick edges denote the colored edges, and x, y denote colors, then Alice color edge e' with color y. In case $x = y$, then Alice color e' with any legal color. Otherwise the three colored edges are as shown in Fig. 2. 1 (b), where e_3 is not adjacent to e', and either e_1 or e_2 (or both) is not adjacent to e'. In this case, Alice color the indicated edge e' with the color of e_3, or e_1 if the color of e_3 is not legal for e'.

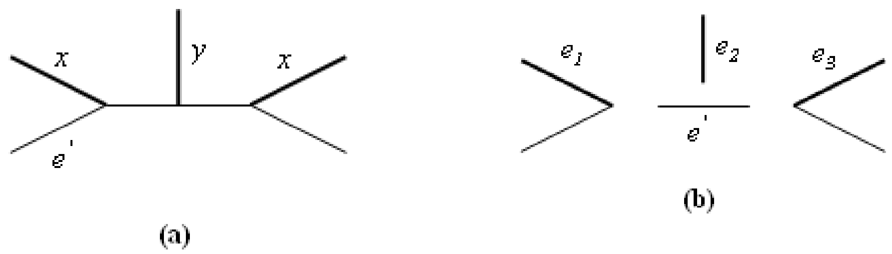

Figure 2.1:

Finally, we consider the case that B contains three colored edge. After Bob colors the edge *e*, B breaks into two or three new c-blocks. Again, at most one new c-block contains more than two colored edges. If each of the new blocks contains at most three colored edges, then Alice use the same strategy as described in the previous case. Assume now that one of the new c-blocks contains four colored edges. If there is one colored edge which has distance ≥ 2 to each of the other three colored edges, then Alice simply consider the other three colored edges, and use the rules described in the previous paragraph. However, in Fig.2.1(b), there are three edges which could be the edge e'. By carefully choosing that edge (among the three possible choices), Alice can make sure that in the resulting partially colored forest, each new c-block has at most three colored edges.

Assume that none of the four colored edges has distance ≥ 2 to every other colored edges in B_1'.

If there is an edge which has distance ≥ 1 to each of the other three colored edges, then it must be as depicted in Fig. 2.2 (a). If the three edges e_3, e_4 and e^* have a common end vertex and e_2, e_3, e_4 are colored with three distinct colors, then Alice color the edge e'' with the color of e_3 or e_4, whichever is legal. In this case, after coloring e'', the edge e^* becomes a safe edge, and will seperate the colored edges into two different blocks. Otherwise, Alice color the edge e' with the color of e_2 (or in case e_1 and e_2 are colored the same color, then color e' with any legal color.) In this case, after coloring e', the edge between e_1 and e_2 become a safe edge, and will seperate the colored edges into two different blocks.

Assume now that each of the colored edges is adjacent to another colored edge. Then the colored edges are as depicted in Fig. 2.2 (b) or Fig. 2.2(c). In Fig.2.2(c), Alice colors the indicated edge e' with color *x*. In Fig. 2(b), we note that the indicated edge e' is not adjacent to e_3 and e_4, because if so, before Bob's move, e' is adjacent to three colored edges, and by the induction hypothesis, two of the edges are colored the same color.

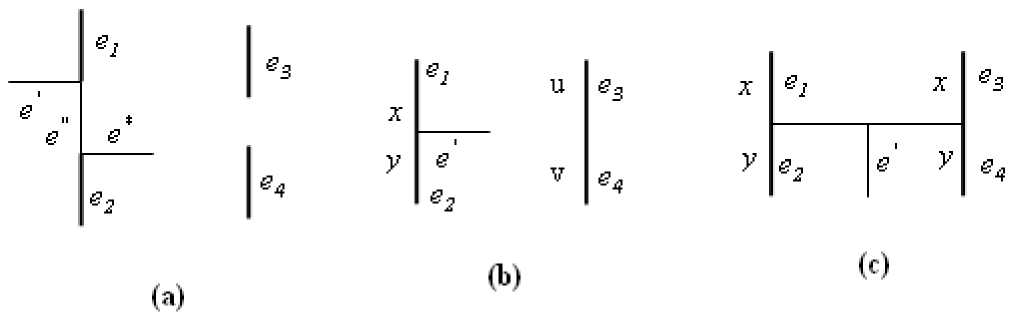

Figure 2.2

Therefore e' would have been a safe edge before Bob's move. Now Alice colors e' with color *u* or *v*, whichever is different from *x* and *y*. This completes the proof of the Theorem. ■

Let $G = (V, E)$ be a finite graph and let *L* be a linear order on the vertex set *V*. For a vertex $x \in V$, the back degree of *x* relative *to L* is defined as $|\{y \in V : xy \in E \text{ and } x > y \text{ in } L\}|$. The back degree *of L* is then the maximum back degree of vertices relative to *L*. The graph $G = (V, E)$ is said to be ***k* – degenerate** if there is a linear order *L* on *V* that has back degree *k* (Andres, 2007).

Theorem 2.1.8 (Bartnicki and Grytczuk, 2008): Let G be a graph whose edges can be partitioned into at most k – forests. Then

$$\chi_g^{'}(\mathrm{G}) \leq \Delta(G) + 3k - 1.$$

Proof: Suppose $D = (V,E)$ is a directed graph. For a vertex $x \in V$, let $E^{+}(x)$ denote the sets of edges out–going from x, and let $E^{-}(x)$ be the set of edges in-coming to x.

Let D be a directed graph obtained by orienting the edges of G so that $\Delta^{+}(D) \leq k$. Such orientation is clearly possible by the assumption on arboricity of G. We shall describe a strategy for Alice guaranteeing that at any moment of the game, for each uncolored (oriented) edge $e = xy$ there are at most $3k - 1$ colored edges incident x with. Since there are at most $\Delta - 1$ colored edges incident with y, there will always be a free color for e.

Suppose the edges of D are made of fluorescent lamps which are glowing when activated. Only Alice can activate the edges, and activated edges keep glowing till the end of the play. Bob can see which edges are glowing, but this will not help him too much. Alice applies the following "jumping rules" leading her to an edge she finally colors:

1. From an edge xy Alice can jump only to the set $E^{+}(y)$
2. Alice never jumps to a colored edge.
3. Each time Alice jumps into a non-glowing edge, she activates it.
4. If Alice jumps into a glowing edge, she colors it and stops.
5. If Alice has jumped to an edge xy and cannot make a further jump, she colors *xy* and stops.

Notice that if Alice jumps into some edges and continues jumping accordingly to the rules (1)-(5), she must eventually stop and color an edge. In her first turn Alice jumps into any edge and continues jumping till she colors an edge. Now, suppose Bob has just colored an edge xy. If xy is glowing then Alice jumps into any uncolored edge and continues jumping until she colors an edge. If xy is not glowing then Alice activates it and jumps to any edge in the set $E^{+}(y)$, provided there is at least one uncolored edge there, and continues jumping till she colors an edge. If

$E^+(y)$ has no uncolored edges, Alice again starts jumping from any other uncolored edge in D until she colors an edge.

Notice that after Alice's move only glowing edges can be colored, while after Bob's move there can be at most one colored non-glowing edge. Thus to prove the assertion it suffices to bound the number of *glowing* edges at the tail of any uncolored edge xy. Accordingly to the jumping rules it is clear that Alice can jump to the same edge xv at most twice (first time she activates it, second time she colors it). Hence, for each glowing edge xv in the set $E^+(x)$ there can be at most two glowing edges in the set $E^-(x)$ from which Alice could have jumped to xv. In the worst case, when each edge in $E^+(x)$ is colored (except xy) and xy glows, this gives a total of $k + 2(k - 1) + 1 = 3k - 1$ glowing edges around x. The proof is complete. ■

The *arboricity* Arb(G) of a graph G is the minimum number of forests that can cover the edges of G. $\text{Arb}(G) = \max\{e(H)/(v(H)-1): H \subset G\}$. Therefore if $Arb(G) = k$, then for any subgraph H of G, the average degree of the vertices of H is less than $2k$, and hence contains a vertex of degree $\leq 2k - 1$. It follows that G is $(2k - 1)$ – degenerate (Cai and Zhu, 2001).

Theorem 2.1.9 (Cai and Zhu, 2001):

If G has arboricity k, then $\chi'_g(G) = \Delta + 6k - 4$.

The smallest size of a color set C with which Alice has a winning strategy in the game played on G is called game chromatic index of G and denoted by χ'_{g_A} for the *first game* and χ'_{g_B} for the *second game* (Hochstättler, Andres and Schallück, 2010).In the first game, Alice has the first move, in the second game, Bob begins.

Theorem 2.1.10 (Hochstättler, Andres and Schallück, 2010): Let W_n be the n – wheel. Then

a) $\chi'_{g_A}(W_n) = n$ if $n \geq 6$.

b) $\chi'_{g_B}(W_n) = n$ if $n \geq 3$.

By easy calculations, one observes χ'_{g_A} (W_3) = 5, χ'_{g_A} (W_4), and χ'_{g_A} (W_5) = 6, Therefore by Theorem 2.1.4 the problem of determining the game chromatic index of wheels is completely solved. In particular, for large wheels, the game chromatic index equals to the trivial lower bound *n* for the game chromatic index.

Proof (a): We describe a winning strategy for Alice for the first game played on W_n, $n \geq 6$, with *n* colors. Here the situation is more complex since Alice has the disadvantage of the first move. However, Alice tries to act much in the way as in the strategy of the previous section.

Again, we number the spokes s_i and the rim edges r_i cyclically in such a way that s_i is adjacent to r_{i+1} and r_{i+2} where we take the incides modulo n, so that the spoke s_i and the rim edge r_i are independent for any $i = 0, \ldots, n-1$, since $n \geq 3$. During the game, Alice will keep in mind one special index i_0 and possibly change the special index several times. We denote s_{i_o} by s and r_{i_o} by *r*.

Alice's strategy is two-fold. The first part of Alice's strategy will consists of the first $n - 3$ moves of Alice and the first $n - 4$ moves of Bob. The second part concern the end-game of coloring the last seven edges.

In her first move, Alice chooses an index as special index and colors the spoke *s*. In the next $n - 4$ moves, she reacts on Bob's play in the following way: If Bob colors a spoke $s_i \neq s$ or a rim edge $r_i \neq r$, Alice answers by a matching move. If Bob colors *r* with a color *c*, Alice chooses a new special index i_0, so that s_{i_o} is uncolored and not adjacent to the old *r*, and colors s_{i_o} with *c* if *c* was a new color before Bob's move, otherwise with a new color. Note that there is such an index i_0, since the color *c* at rim *r* can block at most two spokes, but before. Alice plays her move there are still at least four uncolored spokes. By playing in this way, after Alice's k – th move, exactly *k* colors are used for spokes and at most $k - 1$ colors are used for rim edges, and the set of colors of the rim edges is a subset of the colors of the spokes. At that moment, there are three uncolored spokes and four uncolored rim edges, and, for any *i*, if the rim edge r_i is colored, then the spoke s_i is colored, too.

In the end-game, Alice has to avoid the situation that the last two uncolored spokes are blocked by a new color on the rim edge adjacent to both spokes or that the last uncolored spoke is blocked by a new color.

Proof (b): We describe a winning strategy for Alice for the second game played on W_n, $n \geq 3$, with n colors. We number the spokes s_i and the rim edges r_i cyclically in such a way that s_i is adjacent to r_{i+1} and r_{i+2} where we take the indices modulo n. Therefore the spoke s_i and the rim edge r_i are independent for any $i = 0, \ldots, n - 1$, since $n \geq 3$.

Alice's strategy is that after each of her moves, for any i, either s_i and r_i are colored both, or none of them is colored. She achieves this goal by matching moves. In a matching move, if Bob colors r_i (respectively s_i) with a new color, then Alice colors its partner s_i (respectively r_i) with the same colors, and if Bob colors r_i with a color which has already been used before, then Alice colors s_i with a new color.

Note that Bob cannot a spoke with an old color, since by this strategy the set of colors of the rim edges is a subset of the set of colors the spokes. After Alice's k-th move, exactly k colors are used for spokes. Thus Alice wins.

CHAPTER 3

THE GAME CHROMATIC NUMBER

The game chromatic number was introduced by Bodlaender in 1991 (Bodlaender, 1991). Let $G = (V, E)$ be a graph, and let C be a set of k colors. Consider the following game in which two players Alice and Bob take turns coloring the vertices of G with k colors. Each move consists of choosing an uncolored vertex of the graph and assigning to it a color from $\{1, . . . , k\}$ so that resulting coloring is proper, i.e, adjacent vertices get different colors. Alice wins if all the vertices of G are eventually colored Bob wins if at some point in the game the current partial coloring cannot be extended to a complete coloring of G, i.e, there is an uncolored vertex such that each of the k colors appears at least once in its neighborhood. We assume that Alice goes first (Chou, Wang and Zhu, 2000).

The game chromatic number of a graph $G = (V,E)$, denoted by $\chi_g(G)$, is the least number of a color set C for which Alice has a winning strategy in coloring G. This parameter is well-defined, since it is easy to see that Alice always wins if the number of colors is larger than the maximum degree of G (Bohman, Frieze and Sudakov, 2008).

Theorem 3.1 (Bodlaender, 1991): $\chi_g(T) \leq 5$ if T is a tree.

Faigle and Kern improved Bodlaender's bound for trees.

Theorem 3.2 (Faigle and Kern, 1993): If T is a tree, then $\chi_g(T) \leq 4$.

Proof: We will give a winning strategy for the coloring game described in the introduction using only 4 colors.

Initially, Player I chooses an arbitrary vertex r of T, which will, henceforth, be called the root, and assigns some color to it. During the whole game, Player I maintains a subtree T_0 of T that contains all the vertices colored so far. Player I initializes $T_0 = \{r\}$.

Suppose now that Player II has just moved by coloring vertex v. Let P be the (unique) directed path from r to v in T and let u be the last vertex P has in common with T_0.

Then Player I does the following:

(1) Update $T_0 := T_0 \cup P$.

(2) If u is uncolored, assign a feasible color to u.

(3) If u is colored and T_0 contains an uncolored vertex $v \in T_0$, assign a feasible color to v.

(4) If all vertices in T_0 are colored, color any vertex v adjacent to T_0 and update $T_0 := T_0 \cup \{v\}$.

It is clear that this strategy of Player I guarantees each player the existence of an uncolored vertex with at most 3 colored neighbors until the whole tree is colored. ■

Theorem 3.3 (Dinski and Zhu, 1999): The game chromatic number of a partial k – tree is most $(k+1)$ $(k+2)$. In particular, a series – paralel graph has game chromatic number at most 12.

Example:

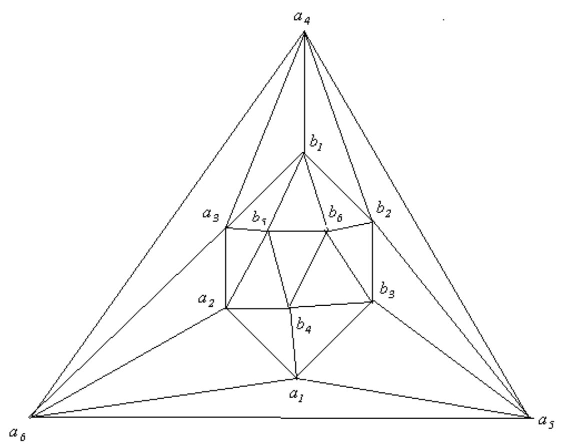

Figure 3.1

Consider the planar graph shown in Figure 3.1 This graph has game chromatic number 6. To see that the game chromatic number is at least 6, here is a winning strategy for Bob if the set X of colors is $\{1, 2, 3, 4, 5\}$. Note that for each $j = 1, 2, ..., 6$, the two–element set $\{a_j, b_j\}$ is a dominating set, i.e., every other node in the graph is adjacent to at least one of these two nodes. Each time Alice colors a node from $\{a_j, b_j\}$, say with color c, Bob responds by assigning color c to the other node in this set. It follows that c cannot be used by either player to color any other node in the graph. We leave it as an exercise to show that the game chromatic number is at most 6 (Kierstead and Trotter, 1992).

Theorem 3.4 (Chen, Schelp and Shreve, 1997):

Let P denote the Petersen graph then $\chi_g(P) = 4$.

3.1 Game Chromatic Number Of Cartesian Product Graphs

In 2007, Bartnicki, Bresăr, Grytczuk, Kovsě, Miechowicz, and Peterin studied the game chromatic number for the Cartesian product G $\times$ H of two graphs G and H (Bartnicki, Bresăr, Grytczuk, Kovsě, Miechowicz, and Peterin, 2007). They showed that the game chromatic number is not bounded over the family of Cartesian products of two complete bipartite graphs. Their result implies that the game chromatic number χ_g(G $\times$ H) is in general not bounded from above by a function of χ_g(G) and χ_g(H). Bartnicki et al. Also determined the exact values of $\chi_g(P_2 \times P_n)$, $\chi_g(P_2 \times C_n)$, and $\chi_g(P_2 \times K_n)$, where P_n, C_n, K_n denote the path graph, the cycle graph, and the complete graph on n vertices respectively.

The Cartesian product of graphs, given two graphs G and H, their Cartesian product $G \times H$ is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 = v_2$ and $u_1u_2 \in E(G)$, $v_1 v_2 \in E(H)$. The graphs G and H are called *factor graphs* of $G \times H$. Note that the Cartesian product operation is both commutative and associative up to isomorphism. Given a vertex $v \in V(H)$, the subgraph G_v of $G \times H$ induced by $\{(u,v): u \in V(G)\}$ is called a G – *fibers*; *H* – *fibers* are defined similarly (Sia, 2009).

More recently, in 2008, Zhu found a bound for the game chromatic number of the Cartesian product graph G $\times$ H in terms of the game coloring number and acyclic chromatic number of G and of H. Defining $\chi_g(\mathcal{G} \times \mathcal{H}) = sup\{\chi_g(G \times H): G \in \mathcal{G}, H \in \mathcal{H}\}$, Zhu's result implies that $\chi_g(\mathcal{F} \times \mathcal{F}) \leq 10$ and $\chi_g(\mathcal{P} \times \mathcal{P}) \leq 105$.

Let $K_{1,n}$ and W_n denote the star graph and wheel graph on *n* + 1 vertices respectively, and let $K_{m,n}$ denote the complete bipartite graph with parts of size *m* and *n*. In this section, we obtain exact values for the game chromatic number of additional Cartesian product graphs, namely the graphs $K_{1,n} \times P_n$, $K_{1,n} \times C_n$, $P_2 \times W_n$, and $P_2 \times K_{m,n}$ (Sia, 2009).

The values of the game chromatic numbers χ_g ($P_2 \times P_n$), χ_g ($P_2 \times C_n$), and χ_g ($P_2 \times K_n$), determined by Bartnicki et al. are equal to the trivial upper bounds obtained by considering the maximum vertex degree of the Cartesian product graph for the graphs $K_{1,n} \times P_n$, $K_{1,n} \times C_n$, $P_2 \times W_n$, however, we require a stronger upper bound, which is provided by the game coloring number of those graphs. This graph invariant is defined as follows. Suppose that Alice is completely color – blind: she cannot distinguish between any two colors. To accommodate Alice's disability, Alice and Bob modify the rules of the coloring game as follows. The players fix a positive integer *k* and, instead of coloring vertices, simply mark an unmarked vertex each turn. Bob wins if at some time some unmarked vertex has *k* marked neighbours, while Alice wins if this never occurs (Sia, 2009).

Theorem 3.1.1 (Grytczuk, Bartnicki, Bresar and Kovse, 2008): Let *n* be a positive integer. Then

$$\chi_g(K_2 \times P_n) = \begin{cases} 2 : for \;\; n = 1, \\ 3 : for \;\; 2 \le n \le 3, \\ 4 : for \;\; n \ge 4. \end{cases}$$

Proof. The result is clear for *n* = 1 where we have $K_2 \times P_1 = K_2$. For *n* = 2 we have $K_2 \times P_2 = C_4$ and again everything is clear. For *n* = 3, $K_2 \times P_3$ has two vertices of degree 3 and after the second move of Alice both vertices are colored. Since all remaining vertices have degree 2, the result is clear also for *n* = 3.

Assume $n \ge 4$. Denote the vertices of the two fibers of P_n with $v_1, v_2, \ldots, v_n$, and $v'_1, v'_2, \ldots, v'_n$. Suppose that only three colors {1, 2, 3} are available. By symmetry there are only two different cases for Alice's first move. If Alice starts in a vertex of degree 3 say v_2 with color 1, Bob responds with color 2 on the vertex v'_3. After Alice's next

move, at least one of v_3 and v_2' remains uncolored. Thus if v_3 remains uncolored, Bob colors v_4 with 3 and v_3 cannot be colored anymore (with these three colors).
Otherwise v_2' remains uncolored and Bob colors v_1' with color 3 and v_2' cannot be colored anymore.

Suppose Alice starts in a vertex of degree 2, say v_1, with color 1. Then Bob colors v_3' with 1. Note that Bob can force Alice to be the first to color a vertex from the set $P = \{v_1', v_2', v_2, v_3\}$ because there is even number of vertices v_i and v_i' for $i \geq 4$, and it is Alice's turn. Also the color 1 cannot be used on the vertices of P anymore. Vertices of P induce a path on four vertices. Since $\chi_g(P_4) = 3$, and only colors 2 and 3 may be used in P, we infer that Bob wins the game. By the trivial upper bound, the proof is complete. ■

Theorem 3.1.2(Grytczuk, Bartnicki, Bresar and Kovse, 2008):

For an integer $n \geq 3$, $\chi_g(K_2 \times C_4) = 4$.

Proof. Denote the vertices of the one fibers of P_n with $v_1, v_2, \ldots, v_n$, and $v_1', v_2', \ldots, v_n'$. Suppose that only three colors {1, 2, 3} are available. By symmetry there is only one possibility for Alice's first move, say v_2 with color 1. For $n \geq 5$ Bob responds on v_3' with 2 and Alice cannot deal with threats on both v_3 and v_2', so Bob forces fourth color in his next move. For $n = 4$ this strategy does not work, since Alice can color v_1 with 3 in her second move. But $K_2 \times C_4 = Q_3 = K_{4,4} \times M$ (M-perfect matching), hence $\chi_g(K_2 \times C_4) = 4$ as well. For $n = 3$ Bob responds on v_3' with 1, what forces Alice to use a new color and Bob easily wins the game. By the trivial upper bound, the proof is complete. ■

Theorem 3.1.3(Grytczuk, Bartnicki, Bresar and Kovse, 2008):

For a positive integer n, $\chi_g(K_2 \times K_n) = n + 1$.

Proof: For $n = 1, 2$ the result is clear. We describe a winning strategy for Bob with n colors for $n \geq 3$. In this first $n - 2$ moves Bob applies the following rules: (1) he always colors a vertex in the opposite fiber of K_n with the same color to the color used previously by Alice; (2) if possible he colors a vertex whose unique neighbour in the opposite K_n – fiber has been already colored. According to condition (1) of this strategy Alice is forced to use a new color in each her move, so after $n - 2$ moves exactly $n - 2$ colors will be used. Condition (2) guarantees that after $n - 2$ moves of each player all four uncolored vertices induce connected subgraph (either path P_4 or cycle C_4). Moreover Alice has to start the game on one of the four remaining vertices with only two last remaining colors. In both cases (P_4 or C_4) Bob wins the game by coloring the eligible vertex with last color. By the trivial upper bound, the proof is complete. ■

Proposition 3.1.1 (Guan and Zhu, 1999): Suppose that $G = (V, E)$ is a graph with $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $\chi_g(G) \le col_g \le col_g(G_1) + \Delta(G_2)$.

Proof: Alice plays according to the optimal strategy for G_1, so that at any point in the game any unmarked vertex has at most $col_g(G_1) - 1 + \Delta(G_2)$ marked neighbors. ■

Theorem 3.1.4 (Sia, 2009): For any Cartesian product graph $G \times H$, we have

$$\chi_g(G \times H) \le col_g(G \times H) \le col_g\left(\coprod_{V(H)} G\right) + \Delta(H).$$

Proof: In Proposition 3.1.1, set G_1 to be the union of all G – fibers and G_2 to be the union of all the F – *fibers*. ■

For ease of future reference, we note that for arbitrary positive integer and n, we have $col_g\left(\coprod_{i=1}^{l} S_n\right) = 2$ and $col_g\left(\coprod_{i=1}^{l} W_n\right) \le 4$. These bounds are attained by having Alice always mark the center vertex of the S_n – or W_n – *fiber* that Bob last played in, if it is unmarked.

Before stating our results, we introduce some final definitions and notational conventions. Suppose that Alice and Bob play the coloring game with k colors. We say that there is a *threat* to an uncolored vertex v if there are $k - 1$ colors in the neighborhood of v, and it is possible to color a vertex adjacent to v with the last color, so that all k colors would then appear in the neighborhood of v. The threat to the vertex v is said to be blocked it v is subsequently assigned a color, or it is no longer possible for v to have all k colors in its neighborhood. We shall also use the convention that color numbers are only used to differentiate distinct colors, and should not be regarded as ascribed to particular colors. For example, if only colors 1 and 2 have been used so far and we introduce a new color, color 3, then color 3 can refer to any color that is not the same as color 1 or color 2. Finally, we label figures in the following manner: vertices are labeled in the form "color (player, turn)," with the information in parentheses being omitted if the same configuration can be attained in multiple ways. A pair of asterisks indicates that Alice cannot block the threats to both vertices marked with asterisks in her next turn, so that Bob wins.

Proposition 3.1.2 (Sia, 2009): Let m and n be positive integers with $m \geq 2$. Then

$$\chi_g(K_{1,m} \times P_1) = 2;$$

$$\chi_g(K_{1,m} \times P_2) = 3;$$

$$\chi_g(K_{1,m} \times C_n) = \chi_g(K_{1,m} \times P_n) = 4 \text{ for } n \geq 3.$$

Proof: The result is clear for the graphs $K_{1,m} \times P_1$ and $K_{1,m} \times P_2$. For $n \geq 3$, we obtain the upper bounds $\chi_g(K_{1,m} \times P_n) \leq 4$ and $\chi_g(K_{1,m} \times C_n) \leq 4$ by taking G = $K_{1,m}$ and H = P_n or C_n in Theorem 3.2.1. It remains to show that Bob can win with three or fewer colors whenever $n \geq 3$. Observe that if Bob can force a subgraph of the form shown in Fig. 3.2 after his first turn, then he wins after his second turn, since Alice cannot block the threats to both the vertices marked with asterisks. It is easy to see that this can always be done for graphs of the form $K_{1,m} \times C_n$, $n \geq 3$. For graphs of the form $K_{1,m} \times P_n$, Bob cannot create this configuration after his first turn only if Alice makes her first move (say color 1) in the center vertex of one of the two "side" $K_{1,m}$–fibers (see Fig.3.3), or if n = 3 and Alice plays in a noncentral vertex of the middle $K_{1,m}$–fiber. Suppose that $n \geq 4$, so that we are in the former case. Bob should respond by playing color 2 in the center vertex of the $K_{1,m}$–fiber at a distance 2 away from Alice. This forces Alice to play color 3 in the unique vertex adjacent to both colored vertices, since this is the only way to block the threat to that vertex. Bob then plays color 3 as shown in Fig. 3.3.Alice cannot block the threats to both the vertices marked with asterisks, so Bob wins.

Finally, we are left with the graph $K_{1,m} \times P_3$. The case analysis in Fig.3.4 shows that Bob wins 3 colors.

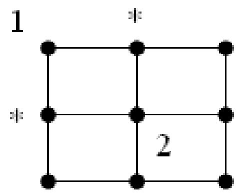

Figure 3.2: The configuration that Bob attempts to achieve after his first turn.

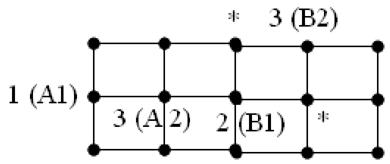

Figure 3.3: Bob's winning strategy for $K_{1,m} \times P_n$, $n \geq 4$. (The graph above is $K_{1,2} \times P_5$)

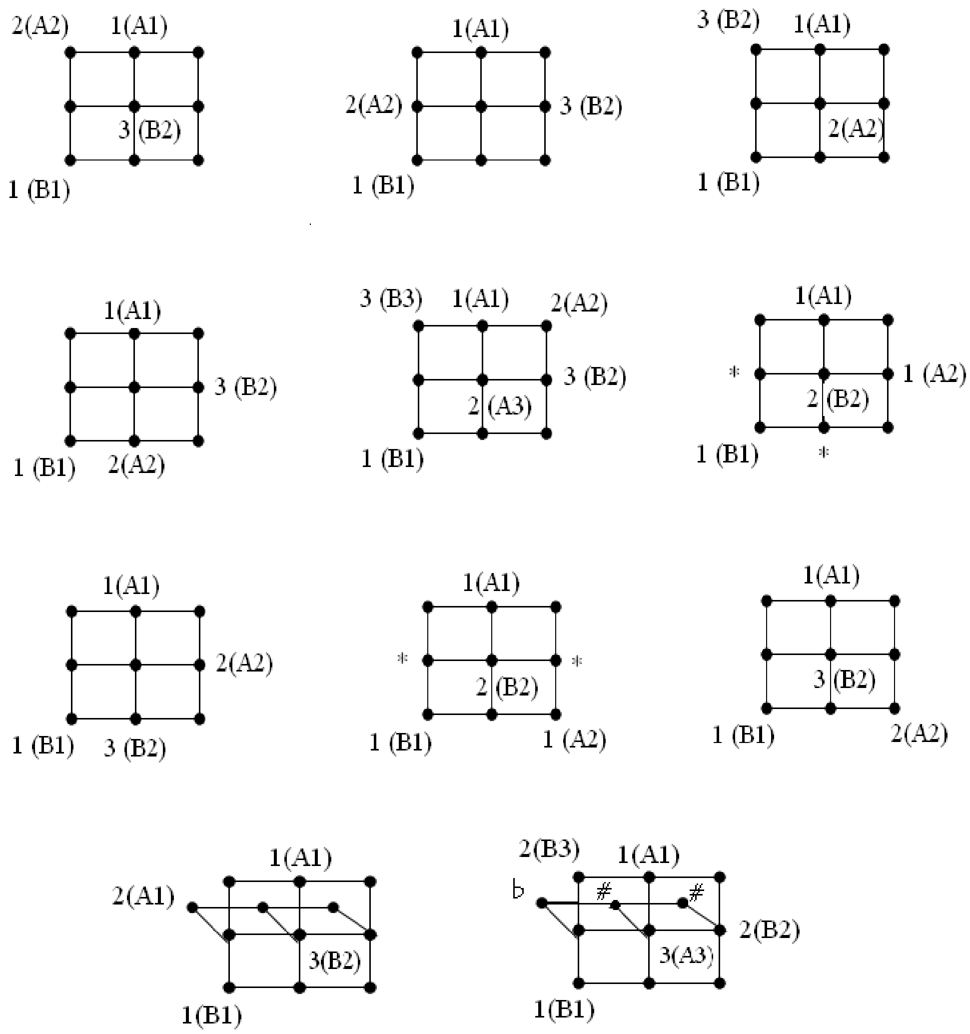

Figure 3.4: Bob's winning strategy for $K_{1,m} \times P_3$. The top nine cases show Bob's strategy when play is confined to a $K_{1,2} \times P_3$ subgraph, while the bottom two cases illustrate Bob's strategy when play is not confined to a $K_{1,2} \times P_3$ subgraph. A dagger (†) indicates that if Alice does not color that vertex on her specified turn, then Bob can play such that that vertex is adjacent to 3 distinct colors after his turn. In the bottom-right diagram, Alice either plays color 1 in the vertex marked with a flat (♭), or any color in a vertex marked with a sharp (#) on her second turn.

Proposition 3.1.3 [21]: For any integer $n \geq 9$, $\chi_g(P_2 \times W_n) = 5$.

Proof. First, we show that Bob has a winning strategy with four or fewer colors. We give Bob's winning strategy when there are exactly four colors; it will be easy to see that Bob can win using the same starting moves (and replacing color 4 by color 3, if necessary) when there are fewer than four colors. Denote the vertices of one W_n–fiber by $v_0, v_1, v_2, \ldots, v_n$, with v_0 being the center vertex and $v_1, v_2, \ldots, v_n$ the n-cycle, and denote the corresponding vertices of the other fiber by $v_0^{'}, v_1^{'}, v_2^{'}, \ldots, v_n^{'}$. Bob should respond to Alice's first move in such a way that we may assume without loss of generality that v_0 has color 1 and $v_1^{'}$ has color 2. If Alice responds with color 3 in $v_0^{'}$, then Bob plays color 4 in v_2. Alice cannot block the threats to both v_1 and $v_2^{'}$, so Bob wins.

Now suppose that Alice responds to Bob's first move by playing in some vertex that is not $v_0^{'}$. We may suppose without loss of generality that Alice plays in one of $v_1, v_n, v_{n-1}, \ldots, v_{[n/2]+1}, v_{[n/2]+2}^{'}, \ldots v_n^{'}$. If Alice plays in v_1 or v_n, then Bob responds with color 3 in $v_4^{'}$, as shown in Fig.3.5. This forces Alice to play color 4 in $v_0^{'}$; otherwise, Bob would win on his next turn by playing color 4 in one of the $v_i^{'}$. Bob then replies with color 2 in v_5, creating threats to v_4 and $v_5^{'}$. Alice cannot block both these threats, so Bob wins. On the other hand, if Alice does not play in v_1 or v_n, then Bob responds with color 3 in $v_2^{'}$, as shown in Fig.3.6 As before, this forces Alice to play color 4 in $v_0^{'}$. Bob then replies with color 2 in v_3, creating threats to v_2 and $v_3^{'}$. Once again, Alice cannot block both these threats, so Bob wins. Note that we have used the fact that $n \geq 9$ here, so that Alice's move on her second turn does not affect Bob's ability to threaten Alice after her third turn.

Finally, the upper bound $\chi_g(P_n \times W_n) \leq 5$ follows from taking $G = W_n$ and $H = P_2$. ■

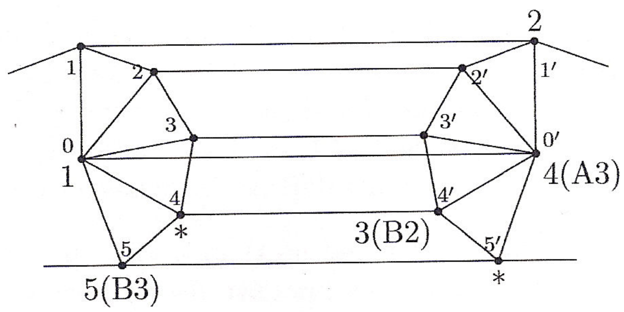

Figure 3.5: Bob's strategy if Alice plays in v_1 or v_n. Vertices of the form v_i are labeled by a small i, while vertices of the form $v_i^{'}$ are labeled by a small $i^{'}$.

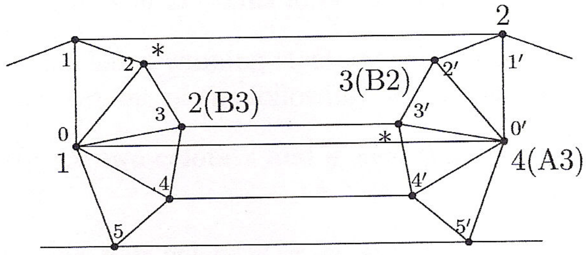

Figure 3.6: Bob's strategy if Alice does not play in v_1 or v_n.

Theorem 3.1.5(Grytczuk, Bartnicki, Bresar and Kovse, 2008): Let *n* be an arbitrary natural number. Then there exist natural numbers *k* and *m* such that

$\chi_g (K_{k,k} \times K_{m,n}) > n$.

3.2 Variation of the Game Chromatic Number

In this section, we introduced variation of the game chromatic number. We started the game coloring number then d – relaxed game chromatic number. The d – relaxed game chromatic number is a mixture of the concept of the coloring game and the concept of relaxed coloring of graphs. We next examined the relationship between acyclic game chromatic number and hereditary game chromatic number. Then, we introduced the relationship between acyclic chromatic number and chromatic number of oriented. Finally, we introduced incidence game chromatic number.

3.2.1. The Game Coloring Number

The concept of the game coloring number of a graph was first formally defined and investigated by Zhu (Zhu, 1999). The game coloring number, which gives an upper bound for the game chromatic number. It is defined through a two-person game as follows: two person, say Alice and Bob alternately select vertices of *G*, to form a linear ordering *L* of the vertices of G, so that $x \leq y$ in *L* if *x* is selected before *y*. The *back degree* of a vertex *x* with respect to *L* is the number of neighbours of *x* which precedes x in *L*, and the back degree of *L* is the maximum of the back degrees of the vertices of G with respect to *L*. Alice's goal is to minimize the back degree of *L* and Bob's goal is to maximize it. *The game coloring number* $col_g(G)$ of *G* is equal to $k + 1$, where k is the back degree of L when both Alice and Bob use their optimal strategies in playing the game (Chou, Wang and Zhu, 2000).

It is easy to see that $\chi_g(G) \leq col_g(G)$ for any graph G. And the argument that shows that $\chi_g(G) \leq col_g(G)$ also shows that $\chi_g^{(d)}(G) \leq col_g(G)$. For any forest T, $\chi_g(T) \leq col_g(T) \leq 4$; for every partial k-tree graph G, $\chi_g(G) \leq col_g(G) \leq 3k+2$; etc (Chou, Wang and Zhu, 2000).

3.2.2 d – Relaxed Game Chromatic Number

The d–relaxed game chromatic number of a graph is another variation of the chromatic number of a graph. Suppose $d \geq 0$ is an integer. In a d–relaxed game coloring (played on a graph G with color set C), a color $i \in C$ is legal for an uncolored vertex $x \in V(G)$ if by coloring x with color i, each vertex of color i is adjacent to at most d vertices of color $i_{(d)}$. The d–relaxed game chromatic number $\chi_g^{(d)}(G)$ of G is the least cardinality of a color set C for which Alice has a winning strategy for the d–relaxed coloring game played on G with color set C. If G is a forest and $d \geq 1$ then $\chi_g^{(d)}(G) \leq 3$ (Chou, Wang and Zhu, 2000).

The concept of the relaxed coloring game is a mixture of the concept of the coloring game and the concept of relaxed coloring (also called improper coloring or defective coloring) of graphs. Given two integers $k \geq 1$ and $d \geq 0$, a graph G is called (k, d)–colorable if its vertices can be colored with k colors in such a way that each vertex is adjacent to at most d vertices with the same color as itself (Chou, Wang and Zhu, 2000).

The relaxed game chromatic number, besides the fact that $\chi_g^{(d)}(G) \leq col_g(G)$, it is not clear whether the game coloring number can be used to derive better upper bounds for $\chi_g^{(d)}(G)$ when $d \geq 1$. For example, given an integer $k \geq 1$, it is unknown if there exists an integer d such that for any graph G with $col_g(G) \leq k$, $\chi_g^{(d)}(G) \leq k-1$ (Chou, Wang and Zhu, 2000).

3.2.3 Acyclic Game Chromatic Number

The acyclic chromatic number of a graph was introduced by Grünbaum. It was conjectured by Grünbaum, and proved by Borodin, that the maximum acyclic chromatic number of a planar graph is equal to 5 (Dinski and Zhu, 1999).

The acyclic chromatic number of a graph is another variation of the chromatic number of a graph. Suppose $G = (V, E)$ is a graph. The *acyclic* chromatic number of G, denoted by $\chi_a(G)$, is the least number t so that the vertices of G, can be colored by t colors in such a way that each color class is an independent set, and the subgraph of G induced by any two color classes is acyclic, i.e., the union of every two color classes induced a forest (Dinski and Zhu, 1999).

Theorem 3.2.1 (Dinski and Zhu, 1999): Let G be a graph. If $\chi_a(G) \le k,$ then $\chi_g(G) \le k\ (k+1)$.

3.2.4 Hereditary Game Chromatic Number

The *hereditary* game chromatic number of G, denoted by $\chi_{hg}(G)$, as the maximum of the game chromatic numbers of its subgraphs, i.e. (Dinski and Zhu, 1999),

$$\chi_{hg}(G) = \max\{\chi_{\mathrm{g}}(H) : \text{H is a subgraphs of G}\}.$$

It follows from the definition that $\chi_{hg}(G) \ge \chi_{\mathrm{g}}(G)$ for any graph G. Since the acyclic chromatic number of a graph is a monotonic parameter, Theorem 3.2.1 can be strengthened to the following:

Theorem 3.2.2(Dinski and Zhu, 1999): If $\chi_a(G) \le k,$ then $\chi_{\mathrm{hg}}(G) \le k\,(k+1)$.

Thus if a class of graphs have bounded acyclic number, then they also have bounded hereditary game chromatic number. It seems that the converse is also true, i.e., a class of graphs of bounded hereditary chromatic number, may also have bounded acyclic chromatic number.

3.2.5 Oriented Game Chromatic Number

The oriented chromatic number of a graph is another variation of chromatic number of graphs. Suppose $\vec{G}$ is an oriented graph. Then the *oriented* chromatic number $\chi_o(\vec{G})$ of $\vec{G}$ is the least number *t* such that the vertices of $\vec{G}$ can be colored by *t* colors in such a way that each color class is an independent set, and for any two color classes, say *U* and U', all the edges are in the same direction, i.e., either all the edges are from *U* to U', or all the edges from U' to *U* (Chou, Wang and Zhu, 2000). The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of $\chi_o(\vec{G})$ among all the orientations $\vec{G}$ of *G* (Dinski and Zhu, 1999).

Theorem 3.2.3 (Dinski and Zhu, 1999): If a graph *G* has oriented chromatic number *k*, then

$$\chi_g(G) \leq \mathrm{k}\, 2^{\mathrm{k}-1}(\mathrm{k}\, 2^{\mathrm{k}-1}+1).$$

The relationship between acyclic chromatic number and chromatic number of oriented as follows:

Theorem 3.2.4 (Dinski and Zhu, 1999): If a graph *G* has acyclic chromatic number *k*, then

$$\chi_o(G) \leq \mathrm{k}\ 2^{\mathrm{k}-1}.$$

Theorem 3.2.5 (Dinski and Zhu, 1999): If a graph *G* has oriented chromatic number *k*, then

$$\chi_a(G) \leq k^2 + k^{3+\lceil log_2 k \rceil+1}.$$

3.2.6 Incidence Game Chromatic Number

This is a competitive version of the incidence coloring number introduced by Brualdi and Massey (Andres, 2007).

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The set of incidences of G is defined as

$$I = \{(v, e) \in V \times E \mid \text{v is incident with } e\}.$$

Two distinct incidences (v, e), $(w, f) \in I$ are adjacent if $(v, f) \in I$ or $(w, e) \in I$. This means in particular, if either $v = w$ or $e = f$, then the incidences (v, e) and (w, f) are adjacent.

Consider the following game which is played on I with a color set C. Two players, Alice and Bob, alternately color an incidence with a color from C in such a way that incidences that are adjacent receive distinct colors. The game ends when this is not possible any more. Alice wins if every incidence is colored at the end of the game, otherwise Bob wins. The smallest number of colors, so that Alice has a winning strategy for the game played on I, is called *incidence game chromatic number* $\iota_g(G)$ of G (Andres, 2007).

Theorem 3.2.6 (Andres, 2007): Let C_k be the cycle with k vertices, then $\iota_g(C_k) = 5$ for $k \geq 7$.

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