T.C.

YAŞAR UNIVERSITY

INSTITUTE OF NATURAL AND APPLIED SCIENCES

MASTER THESIS

EXACT SOLUTION METHODS FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Cem OĞUZ

Supervisor

Assist. Prof. Dr. Shahlar Maharramov

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Assist. Prof. Dr. Ahmet Yıldırım

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YEMİN METNİ

Yüksek Lisans tezi olarak sunduğum " Exact Solution Methods For Solving Partial Differential Equations " adlı çalışmamın tarafımdan bilimsel ahlak ve geleneklere aykırı düşecek bir yardıma başvurmaksızın yazıldığını ve yararlandığım kaynakların kaynakçada gösterilenlerden oluştuğunu, bunlara atıf yapılarak yararlanılmış olduğunu belirtir ve bunu onurumla doğrularım.

Cem OĞUZ

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I am also grateful to my family for their confidence to me and for their endless supporting. This thesis is dedicated to my family...

ABSTRACT

Master Thesis

EXACT SOLUTION METHODS FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Cem OĞUZ

Yaşar University

Institute of Natural and Applied Sciences

It is well known that most of the phenomena that arise in mathematical, physics and engineering fields can be described by partial differential equations (PDEs). Therefore, partial differential equations are a useful tool for modelling. But as our analysis or differential methods are not enough to solve partial differential equations**,** these equations cause us to search new methods or develop old methods to solve ones.

In this thesis, we discussed the Series Methods like Adomian Method, Variational Iteration Method and Homotopy Perturbation Methods, and Solitary Methods such as G'/G Expansion Methods, e^x function Method, the Sine–Cosine Method and the Homogeneous Balance Method, which are the recently developed methods, illustating the so-called methods on some problems by implementing.

Keywords: Partial Differential Equations, Series Methods, Solitary Methods.

ÖZET

Yüksek Lisans Tezi

KISMI DİFERANSİYEL DENKLEMLERİ ÇÖZMEK İÇİN TAM ÇÖZÜM YÖNTEMLERİ

Cem OĞUZ

Yaşar Üniversitesi

Fen Bilimleri Enstitüsü

Matematik fizik ve mühendislik alanlarında ortaya çıkan doğal olayların kısmı diferansiyel denklemlerle (KDD) ifade edilebileceğini hepimiz biliyoruz. Bu nedenle KDDlerin bunların modellemek için yaralı bir araçtır. Ama bizim analiz ve diferansiyel bilgimiz, kısmı diferansiyel deklemleri çözmek için yetersiz olduğundan son yıllarda bu denklemler bizi başka metotlar bulmaya ya da eskil metotları geliştirmeye yöneltmislerdir.

Bu tezde, son yıllarda geliştirilmiş yöntemlerden olan Adomian, Varyasyonel İterasyon ve Homotopi Pertürbasyon Metodu gibi seri metotları ile G'/G genişletme, e^x fonksiyon, Sin-Cosünüs ve Homojen Denge Metodu gibi solitary metotlarını inceleyip, sözü geçen metotları örnekler üzerinde açıkladık.

Anahtar Sözcükler: Kısmı diferansiyel denklemler, Adomian Metodu, Varyasyonel İterasyon Metodu, Homotopi Pertürbasyon Metodu, G'/G Genişletme, e^x Fonksiyon Metodu, Sin-Cosünüs Metodu, Homojen Denge Metodu.

CONTENTS

ABBREVIATIONS

PDE	Partial Differential Equation
PDEs	Partial Differential Equations
NPDE	Nonlinear Partial Differential Equation
NLEEs	Nonlinear evolution equations
DE	Differential Equation
ODE	Ordinary Differential Equation
KG	The Klein–Gordon
BH	The Burgers–Huxley
KdV	The Korteweg de-Vries
mKdV	The *modified KdV*
KP	The Kadomtsev-Petviashvili
NLS	The Nonlinear Schrodinge*r*
Eq.	Equation
Eqs.	Equations
ADM	The Adomian decomposition method
VIM	Variational Iteration Method
HPM	Homotopy Perturbation Method
HB	Homogeneous Balance
HBM	Homogeneous Balance Method
WTC	Wiess, Tabor, Carnevale
CK	Clarkson, Kruskal

1 INTRODUCTION

That most of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs) is well known. In physics for example, the heat flow and the wave propagation phenomena, in ecology, most population models, and the dispersion of a chemically reactive material are all well described by partial differential equations. Besides, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are controlled within its domain of validity by partial differential equations.

Partial differential equations have become a useful tool for denoting these natural phenomena of science and engineering models. Therefore, it becomes increasingly important and popular to be familiar with all traditional and recently developed methods for solving partial differential equations, and the implementation of these methods.

Our analysis or differential methods are not enough to solve partial differential equations along with the given conditions that characterize the initial conditions and the boundary conditions of the dependent variable, so these equations cause us to search new methods or develop old methods to solve ones. In this thesis we will compare the series methods with the solitary methods and extend homogen balance method (HBM) that is one of the new methods, but first let us remark about PDEs.

1.1 GENERAL INFORMATION

1.1.1 Definitions of Partial Differential Equations

Partial differential equations are equations that involve partial derivatives of an unknown function u(x, y, z,…). In the case of ordinary differential equations, the unknown function u(x) (or u(y)) depends on a single independent variable x (or y). In contrast, for partial differential equations the unknown function depends on two or more independent variables. Then a partial differential equation for a function u (x, y, . . .) is a relationship between u and its partial derivatives $u_x, u_y, u_{xy}, u_{yy},..$ and can be written as

$$F(x, y, u, u_x u_y, u_{xx}, u_{xy}, u_{yy},..) = 0 \qquad (1.1.1)$$

Where F is some function x,y, and the variables are independent variables and u (x, y,…) is called a dependent variable. In (1.1.1) we have used the subscript notation for the partial differentiation, i.e.,

$$u_x = \frac{\partial u}{\partial x}, \text{ and } u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \text{ and so on.}$$

We will always assume that the unknown function u is sufficiently well behaved so that all necessary partial derivatives exist and corresponding mixed partial derivatives are equal, e.g.,

$$u_{xy} = u_{yx}, \quad u_{xxy} = u_{xyx}, \quad \text{and so on.}$$

As in the case of ordinary differential equations (ODEs), we define the order of the partial DE (1.1.1) to be the highest order partial derivative appearing in the equation. For example, for first-order PDEs, in these equations the highest order becomes one and they has the following form

$$F(x, y, z, ..., t, u, u_x, u_y, u_z, ..., u_t) = 0. \tag{1.1.2}$$

Similarly, the general second-order partial differential equation in several independent variables x, y, z, ..., t has the following form

$$\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{z}, ..., \mathrm{t}, \mathrm{u}, \mathrm{u}_\mathrm{x}, \mathrm{u}_\mathrm{y}, \mathrm{u}_\mathrm{z}, ..., \mathrm{u}_\mathrm{t}, \mathrm{u}_\mathrm{xx}, \mathrm{u}_\mathrm{xy}, ..., \mathrm{u}_\mathrm{xt}, \mathrm{u}_\mathrm{yy}, \mathrm{u}_\mathrm{yz}, ..., \mathrm{u}_\mathrm{yt}, ..., \mathrm{u}_\mathrm{tt}) = 0, \tag{1.1.3}$$

and so on for higher-order equations. In Eq.(1.1.3), $u_x, u_y, u_z, ..., u_t$ are first partial derivatives and $u_{xx}, u_{xy}, ..., u_{xt}, u_{yy}, u_{yz}, ..., u_{yt}, ..., u_{tt}$ are second partial derivatives. The following are examples of PDEs:

$$u_x + u_y = 4u_t - uu_x \qquad (first-order)$$
$$u_x^2 + u_y^2 = 0 \qquad (first-order)$$

$$u_{xz} + 2uu_y - 4z = 0 \qquad (second-order)$$
$$5u_x u_t - 3zu_y + 2u_{zt} = 0 \quad (second-order \quad for \quad u_{zt}, \quad not \quad u_x u_t)$$

$$u_t + uu_z + u_{zzz} = c \qquad (third-order \quad for \quad u_{zzz})$$

Some Concepts Of Linear, Quasi-linear and nonlinear

A partial differential equation is called linear if:

* The power of the dependent variable and each partial derivative contained in the equation is one

** The coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables.

However, if any of these conditions is not satisfied, the equation is called nonlinear.

If these coefficients are additionally functions of u which do not produce or otherwise involve derivatives, the equation is called quasi-linear. As coefficients of such equations don't hold condition (**) (but power of coefficients is 1) , they are nonlinear PDEs. An equation which is not linear is called a nonlinear equation.

Now let us give some examples:

$$u_{xx} - 7u_{xy} + 6u_{yy} = u \qquad linear \tag{1.1.3}$$

$$u_{xx} + 2uu_y - 4u^2 = 0 \qquad quasi-linear \tag{1.1.4}$$

$$u_{zz} + u^2 u_y + u_{yyzz} \qquad nonlinear \tag{1.1.5}$$

The equation (1.1.4) above is not quasi-linear due to u^2, since the concept of quasi linear is related to equations' coefficients.

Some Important Classical Linear Model Equations

Stated before, linear partial differential equations arise in a wide variety of scientific applications, such as the diffusion equation and the wave equation. Examples here are equations of most common interest.

1. The wave equations is

$$u_{tt} - c^2 \nabla^2 u = 0,$$

Where u=u(x,y,z,t) has three space variables x,y,z and time variable t, that is , it is in three dimensional space

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and c is constant. For example, The wave equation in one dimensional space is given by

$$u_{tt} = c^2 u_{xx}.$$

2. The heat equation or diffusion equation is

$$u_t - k\nabla^2 u = 0,$$

where k is the constant of diffusivity. The heat equation in one dimensional space is given by

$$u_t = k\ u_{xx}.$$

3. The Laplace equation is

$$\nabla^2 u = 0,$$

4. The Poisson equation is

$$\nabla^2 u = f(x, y, z),$$

where f (x, y, z) is a given function describing a source or sink.

5. The Telegraph equation is in general form

$$u_{tt} - c^2 u_{xx} + au_t + bu = 0,$$

where a, b, and c are constants.

6. The Klein–Gordon (or KG) equation is

$$\underbrace{\frac{\partial^2 \varphi}{\partial t^2} - c^2 \nabla^2 \varphi}_{\square\varphi} + (\frac{mc^2}{h})^2 \varphi \equiv 0,$$

Where $\square\varphi$ is called d'Alebertian operator, $h = 2\pi h$ is Planck constant, and m is a constant mass of particle.

7. The time-independent Schrödinger equation in quantum mechanics is

$$\left(\frac{h^2}{2m}\right)\nabla^2 \varphi + (E - V)\phi = 0,$$

where $h = 2\pi h$ is the Planck constant, m is the mass of the particle whose wave function is φ(x, y, z, t), E is a constant, and V is the potential energy. If V = 0, reduces to the Helmholtz equation.

Some Nonlinear Partial Differential Equations

In what follows we list some of the well-known nonlinear models that are of great interest.

1. The Advection Equation

It is given by

$$u_t + uu_x = f(x,t) \quad x \in R,\ t > 0$$

2. The Burger Equation

It has two main type

a) The *Burgers equation* is given by

$$u_t + uu_x = \alpha u_{xx},\ x \in R,\ t > 0$$

where α is the kinematic viscosity. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics.

b) The Burgers–Huxley (BH) equation

$$u_t + \alpha uu_x - vuu_{xx} = \beta(1-u)(u-\gamma)u,$$

where $\alpha, \beta \geq 0,\ \gamma(0 < \gamma < 1)$, and ν are parameters, describes the interaction between convection, diffusion, and reaction. When α = 0, the equation above reduces to the Hodgkin and Huxley equation.

3. The Korteweg de-Vries

Its well known types are as follows:

a) The Korteweg de-Vries (KdV) equation is given by

$$u_t + auu_x + bu_{xxx} = 0,\ x \in R,\ t > 0$$

Where a and b are constant, it is a simple and useful model for describing the long time evolution of dispersive wave phenomena.

b) The *modified KdV equation* (*mKdV*) is given by

$$\text{g} u_t + 6u^2 u_x - u_{xxx} = 0 \, , x \in R, \; t > 0$$

describes nonlinear acoustic waves in an anharmonic lattice and Alfven waves in a collisionless plasma.

4. The Liouville equation

It is given by

$$u_{tt} - u_{xx} = e^{\pm u} \, .$$

5. The Fisher equation

It is given by

$$u_t = v u_{xx} + u(k - u) \, , \quad x \in R, \; t > 0$$

where ν, k, and κ are constants, is used as a nonlinear model equation to study wave propagation in a large number of biological and chemical systems.

6. The Kadomtsev-Petviashvili (KP) equation

It is given by

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \, ,$$

which is a two-dimensional generalization of the KdV equation to describe slowly varying nonlinear waves in a dispersive medium. The equation with σ^2 = +1 arises in the study of weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water.

7. The K(n,n) equation

It is given by

$$u_t + a(u^n)_x + b(u^n)_{xx} = 0, \qquad n > 1 \, .$$

8. The *Nonlinear Schrodinger* (*NLS*) *equation*

It is isgiven by

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \; i = \sqrt{-1}$$

where γ is a constant, describes the evolution of water waves.

9. The Boussinesq equation

It is given by

$$u_{tt} - u_{xx} + (3u^2)_{xx} - u_{xxxx} = 0$$

describes one-dimensional weakly nonlinear dispersive water waves propagating in both positive and negative x-directions.

The above-mentioned nonlinear partial differential equations. are important and many give rise to solitary wave solutions.

Well-posed PDEs

A partial differential equation is said to be well-posed if a unique solution that satisfies the equation and the prescribed conditions exists, and provided that the unique solution obtained is stable. The solution to a PDE is said to be stable if a small change in the conditions or the coefficients of the PDE results in a small change in the solution.

The Canonical Forms

In this part, using coordinate transformations we shall reduce PDEs to second-order linear PDEs which are known as canonical forms. These transformed equations sometimes can be solved rather easily. Here the concept of characteristic of second-order partial DEs plays an important role.

A second order linear partial differential equation in two independent variables x and y in its general form is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y Fu = G,$$

where A,B,C,D,E,F, and G are constants or functions of the variables x and y. A second order partial differential equation is usually classified into three basic classes of equations:

Case 1: $B^2 - 4AC > 0$

Hyperbolic equation is an equation that satisfies the property $B^2 - 4AC > 0$. Examples of hyperbolic equations are wave propagation equations.

Case 2 : $B^2 - 4AC = 0$

Parabolic equation is an equation that satisfies the property $B^2 - 4AC = 0$.Examples of parabolic equations are heat flow and diffusion processes equations.

Case 3: $B^2 - 4AC < 0$

Elliptic equation is an equation that satisfies the property $B^2 - 4AC < 0$. Examples of elliptic equations are Laplace's equation and Schrodinger equation.

Examples: Let us classify the following second order partial differential equations as parabolic, hyperbolic or elliptic:

a) $t^2 u_{tt} - x^2 u_{xx} = 0$

b) $tu_{tt} + u_{xx} = t^2$

c) $u_{xx} - 6u_{xy} + 9u_{yy} + 2u_x + 3u_y - u = 0$

Solutions

a) $A = t^2, B = 0, C = -x^2$.

This means that

$$B^2 - 4AC = 4t^2 x^2 > 0.$$

Hence, the equation in (a) is hyperbolic everywhere except on the t and x–axes.

b) A=t, B=0, C=1. Then

$$B^2 - 4AC = -4t.$$

Since $B^2 - 4AC = -4t$, equation in (b) is elliptic in the half-plane t > 0.Otherwise equation become hyperbolic for t<0.

c) We observed that $B^2 - 4AC = 0$ for A=1, B=-6,C=9, so the equation in (c) is parabolic.

First-Order, Quasi-Linear Equations

Many problems in mathematical, physical, and engineering sciences deal with the formulation and the solution of PDEs. In general, first-order partial differential equations are useful to solve or make simpler second, third or higher order equations.

In this part, we shall give some concepts and definitions concerned with the first-order partial differential equations.

Basic Concepts and Definitions for First-Order, Quasi-Linear Equations

The most general first-order nonlinear partial differential equation in two independent variables x and y has the form

$$F(x,y,u,u_x,u_y)=0,\ (x,y)\in D\subset R^2, \qquad (1.1.6)$$

where F is a given functions of its arguments, and U=u(x,y) is an unknown functions with variables x,y in D. By using notations $u_x = p$ and $u_y = q$, $F(x,y,u,u_x,u_y)=0$ takes the following form

$$F(x,y,u,p,q)=0 \qquad (1.1.7)$$

More formally, writing this equation in the following operator is possible

$$L_\chi u(\chi)=f(\chi),$$

where L_χ is a partial differential equations operator and f(χ) is a given function in two or more independent variables χ =(x,y,). If L_χ is not linear operator, then we call equation (1.1.7) nonlinear partial differential equations. Also, if f(χ) =0, the equation (1.1.6) or (1.1.7) is called homogen equation, and in the opposite case, i.e, $f(\chi)\neq 0$, it is called nonlinear partial differential equation.

Quasi-Linear: Equation (1.1.6) is called quasi-linear partial differential equation if it is linear in first-order derivatives of unknown function u=u(x,y), its general form is as follows;

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \qquad (1.1.8)$$

where a,b and c are functions of x,y,u or their powers. An important special case of these equations, besides, is that of linear equations.

Examples:

$$uu_x + u_y + u/2 = 0, \qquad uu_x + u_t + nu^2 = 0,$$

$$(3x-u)u_x + (3y-u)u_y = x+y, \qquad (y^2-u^2)u_x - xyu_y = xu.$$

Semilinear: Equation (1.1.8) is called a semilinear partial differential equations if its coefficients a and b are independent of u, and hence, the semilinear equation can be expressed in the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u^n) \qquad n > 1 \tag{1.1.9}$$

Note that if n=1, sometimes equation (1.1.9) can turn into linear form. (See 1.1.10)

Examples:

$$xu_x + yu_y = u^3, \tag{1.1.10}$$

$$(x+1)^2 u_x + (y+1)^2 u_y = (x+y)u^2 \tag{1.1.11}$$

Semilinear and quasi-linear partial differential equatios can be considered as nonlinear PDEs.

2. NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Introduction

The fact that many physical, chemical and biological problems are described by the interaction of convection and diffusion and by the interaction of diffusionand reaction processes is well known. In determinig these phenomena, nonlinear partial differential equations, mainly such as Burgers', nonlinear Schrödinger, Fisher and KDV equations and so, are of use.

In this chapter, first, we shall explain some of nonlinear PDEs aforementioned and later give some information to solve these equation as numeric or exact.

2.1 THE CERTAIN TYPES OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Burgers' equations

It is thought that Burgers equation, describing both structure and features of shock waves, traffic flow and turbulent fluid in a channel, is one of the basic model equations in fluid mechanics. These equations show the coupling between diffusion and convection processes. Its standart form is given by

$$u_t + uu_x = vu_{xx}, \qquad t > 0, \tag{2.1.1}$$

where v is a constant defining the kinematics viscosity. If v=0, the equation is called inviscid Burgers equation, governing gas dynamics.

Now, let us show how the equation is obtained; We remark the differential form of the nonlinear equation

$$\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{2.1.2}$$

To search structure of discontinus solution, we suppose a functional relation q = Q(ρ), ignoring discontinuity for ρ and q, so it would be a better approximation to assume that q is a function of the density gradient p_x as well as ρ. A simple model is to take

$$q = Q(p) - vp_x, \tag{2.1.3}$$

where ν is a positive constant. Substituting (2.1.3) into (2.1.2), we obtain the nonlinear diffusion equation

$$p_t + c(p)p_x = vp_{xx}, \qquad (2.1.4)$$

where c(p)=Q'(p).

Multipling (2.1.4) by c(ρ) to obtain

$$\begin{aligned} c_t + cc_x &= vc'(p)p_{xx} \\ &= \left\{c_{xx} - c''(p)p_x^2\right\}. \end{aligned} \qquad (2.1.5)$$

If Q(ρ) is a quadratic function in ρ, then c(ρ) is linear in ρ, and $c''(p) = 0$. Consequently, (2.1.5) becomes

$$c_t + cc_x = vc_{xx}.$$

If c is replaced by the fluid velocity field u (x, t), we o obtain the well-known Burgers equation. Thus, it can be seen that the Burgers equation is a balance between time evolution, nonlinearity,

Fisher's Equations

In mathematics, Fisher's equation, also known as the Fisher-Kolmogorov equation and the Fisher-KPP equation, named after R. A. Fisher and A. N. Kolmogorov, is the partial differential equation. This equation encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in one dimensional habitual, neutron population in a nuclear reaction describes the logistic growth-diffusion process and the wave propagation of an advantageous gene in a population has the form

$$u_t - vu_{xx} = ku(1 - \frac{u}{\kappa}), \qquad (2.1.6)$$

Where v and k are diffusion constant and the linear growth rate, respectively. Besides, they are positive. κ(>0) is the carrying capacity of enviroment. The term $f(u) = ku(1 - \frac{u}{\kappa})$ shows a nonlinear growth rate and vanishes when u= κ,that is, the value κ is effect in determinig the population's destiny. For example, if $u \geq \kappa$,the population decreases and eventually disappers.

The Fisher equation is a particular case of a general model equation, called the nonlinear reaction-diffusion equation, which can be obtained by introducing the net growth rate f (x, t, u) so that it takes the form

$$u_t - vu_{xx} = f(x,t,u),\ x \in R, \qquad t > 0 \tag{2.1.7}$$

The term f refered to as a source or reaction term represents the birth-death process in an ecological context, many physical, biological and chemical problems and so on. The spread of animal or plant populations and the evolution of neutron populations in a nuclear reactorare described by tis equation (2.1.7), where f represents the net growth rate.

We study nondimensional form of Fisher's equation. If we use nondimensional quantities x*,t*,u* defined by $x^* = x(k/v)^{1/2}$. Then Fisher's equation (2.1.6) takes the following form

$$u_t - u_{xx} = u(1-u),$$

where $x^* = \sqrt{\frac{y}{k}}, \qquad t^* = kt, u^* = \kappa^{-1}u$. and $\sqrt{\frac{v}{k}}, k^{-1}$ and κ represent the lenghtscale and population scale, respectively. When the conditions u=0 and u=1, the equation turns into homogeneous problem, which represent unstable and stable solutions, respectively.

Nonlinear Schröndinger Equation

The nonlinear Schrödinger equation (NLS) is a nonlinear version of Schrödinger's equation. It is a classical field equation with applications to optics and water waves. Unlike the Schrödinger equation, it never describes the time evolution of a quantum state. It is an example of an integrable model. Its standart form is defined by

$$iu_t + u_{xx} + \gamma |u|^2 = 0,$$

where γ is constant and u(x,t) is complex. The equation generally exhibits solitary type solutions. A soliton, or solitary wave, is a wave where the speed of propagation is independent of the amplitude of the wave. Solitons usually occur in fluid mechanics. The nonlinear Schrodinger equations that are commonly used are given by

$$iu_t + u_{xx} \pm 2|u|^2 u = 0.$$

Depending on the constant γ, other forms of NSE, therefore, are used as well. The nonlinear Schrodinger equation will be handled differently in this section by using some methods like the variational iteration method.

The KDV Equation

The KdV equation derived by Korteweg and de Vries to describe shallow water waves of long wavelength and small amplitude is a nonlinear evolution equation that models a diversity of important finite amplitude dispersive wave phenomena. It has also been used to describe a number of important physical phenomena such as acoustic waves in a harmonic crystal and ion-acoustic waves in plasmas. The KdV equation is a nonlinear, dispersive partial differential equation for a function u of two real variables, space x and time t. Its standart form is

$$u_t + auu_x + u_{xxx} = 0,$$

where the derivative u_t characterizes the time evolution of the wave propagating in one direction, the nonlinear term uu_x describes the steepening of the wave, and the linear term u_{xxx} accounts for the spreading or dispersion of the wave, and u(x,t) represents the water's free surface in non-dimensionl variables.

There are many varieties of this equation with several connections to physical problems. Some of these are as follows:

$$u_t + u_{xxx} - 6uu_x + u/2t = 0, \qquad \text{(KDV Cylindirical)}$$

$$u_t + (u_{xx} - 2\eta u^3 - 3u(u_x)^2 / 2(\eta + u^2))_x = 0, \qquad \text{(KDV deformed)}$$

$$u_t + u_{xxx} + f_x(u) = 0, \qquad \text{(KDV Generalized)}$$

$$u_t + u_{xxx} - 6u^2 u_x + u/t = 0, \qquad \text{(KDV Sperical)}$$

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} + 3w\omega_{xx} \\ \omega_t &= 3u_x\omega + 6u\omega_x - 4\omega_{xxx} \end{aligned} \qquad \text{(KDV Super) and so on.}$$

The Klein-Gordon equation

The Klein–Gordon equation (Klein–Fock–Gordon equation or sometimes Klein–Gordon–Fock equation) is a relativistic version of the Schrödinger equation. The equation appearing in relativistic physics, nonlinear optics and plasma physic is considered one of the most important mathematical method and used to describe dispersive wave phenonome in general.

It arises in physic in linear or nonlinear. The standart form of the nonlinear Klein-Gordon Equation is given by

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = h(x,t),$$

with initial conditions

$$u(x,0) = f(x), \; u_t(x,0) = g(x),$$

where a is a constant, h(x,t) is a source term and F(u(x,t)) is a nonlinear function of u(x,t). This equation has some varieties like Sine-Gordon and Sinh-Gordon. For example, The sine-gordon equation appared first in differential geometry. As it appears in many physical phenomena such as the propagation of magnetic flux and the stability of fluid motions, this equation became the focus of a lot of research work. Its standart form is

$$u_{tt} - c^2 u_{xx} + a \sin u = 0, \quad \text{u(x,0)=f(x)}, \quad u_t(x,0) = g(x),$$

where c and a are constants. Here, several methods can be used to solve these equations.

3 SOLUTION METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

In this part, we shall examine new methods to solve linear or nonlinear PDEs, particularly nonlinear, homogen or nonhomogen. As PDEs are in most of the phenomena and used to model ones, solving these equations are of great importance.

This part is about introducing the recently developed methods to handle partial differential equations in an accessible manner. Some of the traditional techniques are stil being used as well. Now we divide these new methods into two main heads.

The first is called The series Solution Methods such as Adomian method, Variational Iteration Method and Homotopy Perturbation Methods.

The second is called The Solitons Solution Methods like G'/G expansion methods, e^x function method, the sine–cosine method and the homogeneous balance method.

3.1 THE SERIES SOLUTION METHODS

3.1.1 Adomian Decomposition Method

The Adomian decomposition method (ADM) is a non-numerical method for solving nonlinear differential equations, both ordinary and partial. The general direction of this work is towards a unified theory for partial differential equations (PDE). The method was developed by George Adomian, chair of the Center for Applied Mathematics at the University of Georgia. This method is a such kind of series solution. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral

equations. The decomposition method demonstrates fast convergence of the solution and also provides several significant advantages.

The method for linear DEs is as follows:

The Adomian decomposition method consists of decomposing the unknown function u(x,y) of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y), \qquad (3.1.1)$$

where the componets $u_n(x,y),\ n\geq 0$ are to be determined in a recursive manner. To give a clear view of ADM, we first consider the linear differential equation written in an operator form by

$$Lu+Ru=g, \qquad (3.1.2)$$

where L is, mostly, the higher order derivative than R, assumed to be invertible, and R is linear differential operator, and g is source. We next apply the inverse operator L^{-1} to both sides of equation (3.1.2) and using the given condition to obtain

$$u=f-L^{-1}(Ru), \qquad (3.1.3)$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. Substituting

$u=\sum_{n=0}^{\infty}u_n$ which ADM defines solution u by an infinites series into both sides of (3.1.3) leads to

$$\sum_{n=0}^{\infty}u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty}u_n\right)\right). \tag{3.1.4}$$

To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2\ \cdots$, it is important to note that Adomian method suggests that the zeroth component u_0 is usually defined by the function f described above. Accordingly, the formal recursive relation is defined by

$$\begin{aligned} &u_0 = f, \\ &u_{k+1} = -L^{-1}(R(u_k)), \qquad k \geq 0, \end{aligned}$$

Or equivalently

$$\begin{aligned} &u_0 = f, \\ &u_1 = -L^{-1}(R(u_0)), \\ &u_2 = -L^{-1}(R(u_1)), \\ &u_3 = -L^{-1}(R(u_2)), \\ &\quad\vdots \end{aligned}$$

Having determined these components, we then substitute it into $u=\sum_{n=0}^{\infty}u_n$ to obtain the solution in a series form.

Many researches show that ADM converges very rapidly to that solution, requires less computational work and has been emphasized in many works over some methods. Now, we shall show us how the method works on some examples:

Example 3.1.1.1:

Use the Adomian decomposition method to solve he initial-boundary value problem

$$\begin{aligned} &\text{PDE} \qquad u_t = u_{xx}, \qquad 0 < x < \pi, t > 0, \\ &\text{BC} \qquad \begin{matrix} u(0,t) = 0, & t \geq 0, \\ u(\pi,t) = 0, & t \geq 0, \end{matrix} \\ &\text{IC} \qquad u(x,0) = \sin x. \end{aligned} \tag{3.1.5}$$

First, we rewrite the equation above in an operator form by

$$L_t u(x,t) = L_x u(x,t), \tag{3.1.6}$$

where the differential operators are defined by

$$L_t = \frac{\partial}{\partial t}, \qquad L_x = \frac{\partial^2}{\partial x^2}.$$

It is obvious that the integral operators L_t^{-1} and L_x^{-1} exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L_t^{-1}(.) = \int_0^t (.)dt, \qquad L_x^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

This means that

$$L_t^{-1} L_t u(x,t) = \int_0^t u_t(x,t)dt = u(x,t) - u(x,0). \tag{3.1.7}$$

Where other variable, i.e, x is considred as constant as integrating.

Or similarly,

$$\begin{aligned} L_x^{-1} L_x u(x,t) &= \int_0^x \int_0^x u_{xx}(x,t)dxdx = \int_0^x (u_x(x,t) - u_x(0,t))dx \\ &= u(x,t) - u(0,t) + xu_x(0,t), \end{aligned} \tag{3.1.8}$$

where $u_x(0,t)$ is constant, for it does not include variable x.

Applying L_t^{-1} *or* L_x^{-1} to both sides of (3.1.6) and using the initial condition we find

$$\begin{aligned} & L_t^{-1} L_t u(x,t) = L_t^{-1}(L_x u(x,t)), \\ & u(x,t) - u(x,0) = L_t^{-1}(L_x u(x,t)), \qquad \text{for operator } L_t^{-1} \\ & u(x,t) - \sin x = L_t^{-1}(L_x u(x,t)). \end{aligned} \tag{3.1.9}$$

Note that while the some of PDEs can be solved with both operators, some are solved for only operator, since in some operator arised partial derivative(s) like $u_x(0,t)$ in (3.1.8) or similar expressions, and if the value or values of these partial derivatives don not exist in initial or boundary, this equation cannot solved with this operator.

The decomposition method defines the unknown function u(x,t) into a sum of components defined by the series

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \qquad (3.1.10)$$

Substituting the series (3.1.10) into both sides of last expression in (3.1.9) yields

$$\sum_{n=0}^{\infty}u_n(x,t)=\sin x+L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\right),$$

or equivalently

$$u_0+u_1+u_2+...=\sin x+L_t^{-1}(L_x(u_0+u_1+u_2+...)).$$

The decomposition method suggests that the zeroth component u_0(x,t) is identified by the terms arising from the initial/boundary conditions and from source terms. The remaining components $u_0,u_1,u_2...$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$u_0(x,t)=\sin x,$$
$$u_{k+1}=L_t^{-1}(L_x(u_k)).$$

From the recurrence scheme, we obtain

for k=0, $$u_0(x,t)=\sin x,$$
$$u_1=L_t^{-1}(L_x(u_0))=L_t^{-1}(L_x(\sin x))=L_t^{-1}(-\sin x)=-t\sin x$$

$$u_1(x,t)=-t\sin x,$$
fork=1, $$u_2=L_t^{-1}(L_x(u_1))=L_t^{-1}(L_x(-t\sin x))=L_t^{-1}(t\sin x)=\frac{1}{2!}t^2\sin x,$$
$$\vdots$$

Consequently, the solution u(x, t) in a series form is given by

$$\begin{aligned}u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+...\\&=\sin x\left(1-t+\frac{1}{2!}t^2-...\right),\end{aligned}$$

and its solution in a closed form by

$$u(x,t)=e^{-t}\sin x,$$

obtained upon using the Taylor expansion of e^{-t}.

The method for nonlinear DEs is as follows:

The Adomian decomposition method will be applied in this part to nonlinear partial differential equations. An important remark should be made here concerning the representation of the nonlinear terms that appear in the equation. Although the linear term u is expressed as an infinite series of components, the Adomian decomposition method requires a special representation for the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2$, etc. which appear in the equation. In the following, the Adomian scheme for calculating representation of nonlinear terms will be introduced in details.

Calculation of Adomian Polynomials

The unknown linear function u may be represented by the decomposition series $u = \sum_{n=0}^{\infty} u_n$, while the nonlinear term F(u), such as $u^2, u^3, u^4, \sin u, e^u$, uu_x, u_x^2...etc. can be expressed by an infinite series of the so-called Adomian polynomials in the form

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n),$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. In calculating Adomian polynomials exist several methods in literature. For the nonlinear term F(u) to be as term breaking homogeneous of DE or coefficient of DE or only the alone term, Adomian polynomials A_n can be evaluated by using the following expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2, ...$$

Assuming that the nonlinear function is F(u), the general formula above can be simplified as follows;

For n=0, $\quad A_0 = \frac{1}{0!} \frac{d^0}{d\lambda^0} \left[F\left(\sum_{i=0}^{0} \lambda^i u_i \right) \right]_{\lambda=0} = F(u_0),$

Forn=1, $$A_1=\frac{1}{1!}\frac{d^1}{d\lambda^1}\left[F\left(\sum_{i=0}^{1}\lambda^i u_i\right)\right]\bigg|_{\lambda=0}=\frac{d}{d\lambda}\left[F\left(\sum_{i=0}^{1}\lambda^i u_i\right)\right]\bigg|_{\lambda=0}$$

$$A_1=\frac{d}{d\lambda}\left[F\left(u_0+\lambda u_1\right)\right]\big|_{\lambda=0}=\left[u_1F'\left(u_0+\lambda u_1\right)\right]\big|_{\lambda=0}=u_1F'\left(u_0\right),$$

For n=2, $A_2=u_2F'(u_0)+\frac{1}{2!}u_1^2F''(u_0),$

For n=3, $A_3=u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3!}u_1^3F'''(u_0).$

Other polynomials can be generated in a similar manner. One important observation can be made here. It is clear that A_0 depends only on u_0, A_1 depends only u_1 and u_2, A_3 depends only on u_0, u_1 and u_2. The same conclusion holds for other polynomials. In the following, we will calculate Adomian polynomials for several forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

Calculation of Adomian Polynomials A_n

We can divide Adomian Polynomials A_n into five main parts

1. Nonlinear Polynomials

The polynomials can be obtained as follows by using formula:

Case 1: F(u)=u ^{2}

$A_0=F(u_0)=u_0^2,$
$A_1=u_1F'(u_0)=2u_0u_1,$

$A_2=2u_0u_2+u_1^2,$
$A_3=2u_0u_3+2u_1u_2.$

Case 2: F(u)=u^3

$A_0=F(u_0)=u_0^3,$
$A_1=u_1F'(u_0)=3u_0^2u_1,$

$A_2=3u_0^2u_2+3u_0u_1^2$
$A_3=3u_0^2u_3+6u_1u_2u_0+u_1^3.$

2. Nonlinear Derivatives

The polynomials can be obtained as follows:

Case 1: F(u)=$(u_x)^2$

$A_0=F(u_0)=u_{0_x}^2,$
$A_1=2u_{0_x}u_{1_x},$

Case 2: F(u)=uu_x

$A_0=F(u_0)=u_0u_{0_x},$
$A_1=u_{0_x}u_1+u_0u_{1_x},$

$$A_2 = 2u_{0_x} u_{2_x} + u_{1_x}^2 \qquad A_2 = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$

$$A_3 = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x} \qquad A_3 = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0.$$

3. Trigonometric and Hyperbolic Nonlinearity

The polynomials can be obtained as follows

Case1: F(u)=sinu

$$A_0 = \sin u_0,$$
$$A_1 = u_1 \cos u_0,$$
$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0,$$
$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0.$$

Case2: F(u)=sinhu

$$A_0 = \sinh u_0,$$
$$A_1 = u_1 \cosh u_0,$$
$$A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0,$$
$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 - \frac{1}{3!} u_1^3 \cosh u_0.$$

4. Exponentional Nonlinearity

Case 1: F(u)= e^u

$$A_0 = e^{u_0},$$
$$A_1 = u_1 e^{u_0},$$
$$A_2 = (u_2 + \frac{1}{2!} u_1^2) e^{u_0},$$
$$A_3 = (u_3 + u_2 u_1 + \frac{1}{3!} u_1^3) u_1 e^{u_0}.$$

Case 2: F(u)= e^{-u}

$$A_0 = e^{-u_0},$$
$$A_1 = -u_1 e^{u_0},$$
$$A_2 = (-u_2 + \frac{1}{2!} u_1^2) e^{-u_0},$$
$$A_3 = (-u_3 + u_2 u_1 - \frac{1}{3!} u_1^3) u_1 e^{-u_0}.$$

5. Logarithmic Nonlinearity

Case1: F(u)=lnu,

u>0

$$A_0 = \ln u_0,$$
$$A_1 = \frac{u_1}{u_0},$$
$$A_2 = \frac{u_2}{u_0} - \frac{1}{2} \frac{u_1^2}{u_0^2},$$
$$A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3} \frac{u_1^3}{u_0^3}.$$

Case2: F(u)=ln(u+1),

$-1 < u \leq 1$

$$A_0 = \ln(1 + u_0),$$
$$A_1 = \frac{u_1}{1 + u_0},$$
$$A_2 = \frac{u_2}{u_0 + 1} - \frac{1}{2} \frac{u_1^2}{(1 + u_0)^2},$$
$$A_3 = \frac{u_3}{u_0 + 1} - \frac{u_1 u_2}{(1 + u_0)^2} + \frac{1}{3} \frac{u_1^3}{(u_0 + 1)^3}.$$

Example 3.1.1.2: Solve the following nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = (xt)^2, \qquad u(x,0) = 0 \quad u_t(x,0) = x. \tag{3.1.11}$$

Solution 3.1.1.2

In an operator form, the equation (3.1.11) can be written as

$$L_t u - L_x u + u^2 = x^2 t^2, \tag{3.1.12}$$

where the differential operators L_t and L_x are

$$L_t = \frac{\partial^2}{\partial t^2}, L_x = \frac{\partial^2}{\partial x^2}, \text{ respectively.}$$

The unknown linear function u may be represented by the decomposition series $u = \sum_{n=0}^{\infty} u_n$ and function F(u)=u^2 can be represented by the series of Adomian Polynomials given by

$$u^2 = \sum_{n=0}^{\infty} A_n .$$

Operating with L_t^{-1} on the equation (3.1.12) and using the initial condition $u(x,0) = 0, \quad u_t(x,0) = x$ we obtain

$$u(x,t) = xt + \frac{1}{12} x^2 t^4 + L_t^{-1}\left(u_{xx}\right) - L_t^{-1}(u^2). \tag{3.1.13}$$

Using the decomposition series and the series of Adomian Polynomials in place of u and u^2 in Eq. (3.1.13), respectively give

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{1}{12} x^2 t^4 + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right) - L_t^{-1}(\sum_{n=0}^{\infty} A_n), \tag{3.1.14}$$

From equation (3.1.14) is obtained the recursive relation

$$u_0(x,t) = xt,$$
$$u_{k+1}(x,t) = \frac{x^2 t^4}{12} + L_t^{-1}(u_{k_{XX}}) - L_t^{-1} A_k, k \geq 0,$$

where A_k are Adomian polynomials that represent the nonlinear term u^2, and given by

$$A_0 = F(u_0) = u_0^2 = x^2t^2, \qquad A_2 = 2u_0u_2 + u_1^2$$
$$A_1 = u_1F'(u_0) = 2u_0u_1 \qquad A_3 = 2u_0u_3 + 2u_1u_2$$

which give

$$u_0(x,t) = xt,$$

for k=1, $$u_1(x,t) = \frac{x^2t^4}{12} + L_t^{-1}(u_{0_{XX}}) - L_t^{-1}A_0, A_0 = x^2t^2$$
$$u_1(x,t) = \frac{x^2t^4}{12} + L_t^{-1}(0) - \frac{x^2t^4}{12} + \frac{x^2(0)^4}{12} = 0,$$

for k≥ 1, $u_{k+1}(x,t) = 0.$

The equation's solution

$$u(x,t) = xt$$

is readily obtained.

Example 3.1.1.3: Solve the following nonlinear KDV equation

$$u_t - 6uu_x + u_{xxx} = 0,\ x \in R,$$
$$u(x,0) = f(x) = 6x. \qquad (3.1.15)$$

Solution 3.1.1.3

Operating with L_t^{-1} to (3.1.15) we obtain

$$u = 6x - L_t^{-1}(u_{k_{XXX}}) + 6L_t^{-1}(uu_x).$$

The unknown linear function u may be represented by the decomposition series $u = \sum_{n=0}^{\infty} u_n$ and function F(u)=uu_x can too given by

$$uu_x = \sum_{n=0}^{\infty} A_n.$$

Using the decomposition series and the series of Adomian Polynomials in place of u and uu_x in Eq., respectively

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}\left(6\left(\sum_{n=0}^{\infty} A_n\right)\right) - L_t^{-1}(L_x(\sum_{n=0}^{\infty} u_n), \tag{3.1.16}$$

Equation (3.1.16) gives the recursive relation

$$\begin{aligned} &u_0(x,t) = 6x, \\ &u_1(x,t) = -L_t^{-1}(u_{k_{xxx}}) + 6L_t^{-1}A_k,\ k \geq 0, \end{aligned} \tag{3.1.17}$$

where A_k are Adomian polynomials that represent the nonlinear term uu_x, and given by

$$\begin{aligned} &A_0 = F(u_0) = u_0 u_{0_x}, \\ &A_1 = u_{0_x} u_1 + u_0 u_{1_x}, \\ &A_2 = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ &A_3 = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0, \end{aligned} \tag{3.1.18}$$

which gives

for k=0,
$$\begin{cases} u_0(x,t) = 6x, \\ u_1(x,t) = -L_t^{-1}(u_{0_{\text{xxx}}}) + 6L_t^{-1}A_0, A_0 = 36x, \\ u_1(x,t) = -L_t^{-1}(0) + 6.36xt = 6^3 xt. \end{cases}$$

for k=1,
$$\begin{cases} u_1(x,t) = 6^3 xt, \\ u_2(x,t) = -L_t^{-1}(u_{1_{xxx}}) + 6L_t^{-1}A_1, \\ u_2(x,t) = -L_t^{-1}(0) + (6^4 xt + 6^4 xt)/2 = 6^5 xt^2. \end{cases}$$

for k=2, $u_3(x,t) = 6^7 xt^3.$

The solution in a series form is given by

$$u_3(x,t) = 6x(1 + (36t) \ + (36t)^2 + (36t)^3 + ...) \tag{3.1.19}$$

and in a closed form by

$$u(x,t) = \frac{6x}{1-36t}, \quad |36t| < 1. \tag{3.1.20}$$

3.1.2 VARIATIONAL ITERATION METHOD

Introduction

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.

Recently introduced variational iteration method (VIM) by Ji-Huan He, which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proven successful in deriving analytical solutions of linear and nonlinear differential equations.

The variational iteration method was successfully applied to Burger's and coupled Burger's equations, to Schruodinger-KdV, generalized KdV and shallow water equations, to linear Helmholtz partial differential equation. Linear and nonlinear wave equations, KdV, K(2,2), Burgers, and cubic Boussinesq equations have been solved using the variational iteration method.

He's variational iteration method

For the purpose of illustration of the methodology to the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form

$$Lu + Nu = g(x, t),$$

where L is a linear operator, N is a nonlinear operator and g(x,t) is the source inhomogeneous term. According to the variational iteration method, using generalized Lagrange multipliers, we can construct a correction functional as follows

$$u_{n+1} = u_n + \int_0^t \lambda(s)\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right]ds,$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, $\widetilde{u_n}$ is considered as a restricted variation i.e., $\delta\widetilde{u_n} = 0$. Moreover for each variable in the equation, we can write a correction functional. After writing a correction functional, to determine optimally the Lagrange multiplier λ, we shall use integration by parts. The successive approximations $u_{n+1}(x,t),\ n \geq 0$ of the solution u(x, t) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 which is one of the initial/boundary conditions of DE in general. Consequently, the solution

$$\mathrm{u(x,t)} = \lim_{n\to\infty} u_n .$$

Applications Of VIM

Now for applying VIM, we will give some examples

Example 3.1.2.1: We shall us that the same result can be obtained whatever correction functional to chose. Consider the following partial differential equation

$$\frac{\partial p}{\partial t}+\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=0, \quad (x,y)\in\Omega,\ t>0$$

$$\text{BC:} \quad \left.\begin{aligned} &p(t,0,y)=-4+y^2\\ &\frac{\partial p}{\partial x}(t,0,y)=0\end{aligned}\right\rangle \text{(3.1.a), with respect to x} \qquad (3.1.21)$$

$$\text{BC:} \quad \left.\begin{aligned} &p(t,x,0)=-4+x^2\\ &\frac{\partial p}{\partial y}(t,x,0)=0\end{aligned}\right\rangle \text{(3.1.b), with respect to y}$$

$$\text{IC:} \quad p(0,x,y)=x^2+y^2$$

Solution 3.1.2.1: To solve this equation (3.1.21), we will use VIM. According to variable t, the correction functional for the Eq. (3.1.21) reads in the form

$$p_{n+1}(t,x,y)=p_n(t,x,y)+\int_0^t \lambda(s)\left[\frac{\partial p_n(s,x,y)}{\partial s}+\frac{\partial^2 \tilde{p}_n(s,x,y)}{\partial x^2}+\frac{\partial^2 \tilde{p}_n(s,x,y)}{\partial y^2}\right]ds, \qquad (3.1.22)$$

Moreover, as the unknown function is a funtion in 3 variables x, y, and t, we can write each correction function for every one variable. That is,

$$\begin{aligned} p_{n+1}(t,x,y)&=p_n(t,x,y)\\ &+\int_0^x \lambda_2(M)\left[\frac{\partial \widetilde{p_n}(t,M,y)}{\partial t}+\frac{\partial^2 p_n(t,M,y)}{\partial M^2}+\frac{\partial^2 \widetilde{p_n}(t,M,y)}{\partial y^2}\right]dM, \end{aligned} \qquad (3.1.23)$$

and correction funtional

$p_{n+1}(t,x,y)=$

$$p_n(t,x,y)+\int_0^y \lambda_3(z)\left[\frac{\partial \widetilde{p_n}(t,x,z)}{\partial t}+\frac{\partial^2 \widetilde{p_n}(t,x,z)}{\partial x^2}+\frac{\partial^2 p_n(t,x,z)}{\partial z^2}\right]dz \qquad (3.1.24)$$

where corection functions are formed for x and y, respectively and from every one correction function, we obtain the same result. Here, we will use functional (3.1.22). Taking variation with respect to the independent variable p_n yields

$$\delta p_{n+1}=\delta p_n+\delta\int_0^t \lambda_1(s)\left[\frac{\partial p_n(s,x,y)}{\partial s}+\frac{\partial^2 \tilde{p}_n(s,x,y)}{\partial x^2}+\frac{\partial^2 \tilde{p}_n(s,x,y)}{\partial y^2}\right]ds,$$

noticing that $\delta \widetilde{p_n}=0$, then becomes

$$\delta p_{n+1}=\delta p_n+\int_0^t \lambda_1 \delta \frac{\partial p_n}{\partial s}ds.$$

To simply this statement above indeed, to determine optimally the Lagrange multiplier λ , using integration by parts gives

$$=\delta p_n+\lambda_1\delta p_n \mid_{s=t} -\int_0^t \lambda_1^{'}(s)\delta p_n ds ,$$

Making the coeffcients of δp_n to 0, its stationary conditions are obtained:

$$\left.\begin{array}{l}\delta p_n => 1+\lambda_1(s)\mid_{s=t}=0,\\ \delta p_n => \lambda^{'}_1(s)=0.\end{array}\right. \text{ out of integral} \qquad (3.1.25)$$

From these two equations (3.1.25), we find $\lambda(s)=-1$, substituting this value of the Lagrange multiplier into the correction function (3.1.22) gives the iteration formula

$$p_{n+1}=p_n+\int_0^t (-1)\left[\frac{\partial p_n(s,x,y)}{\partial s}+\frac{\partial^2 p_n(s,x,y)}{\partial x^2}+\frac{\partial^2 p_n(s,x,y)}{\partial y^2}\right]ds$$

or formula

$$p_{n+1}=p_n-\int_0^t \left[\frac{\partial p_n(s,x,y)}{\partial s}+\frac{\partial^2 p_n(s,x,y)}{\partial x^2}+\frac{\partial^2 p_n(s,x,y)}{\partial y^2}\right]ds. \qquad (3.1.26)$$

Choosing the zeroth approximation $p_0(x,t)$ as $p(0,x,y)=x^2+y^2$, applying the zeroth approximation $p_0(x,t)$ into (3.1.26) we obtain the successive approximations.

For n=0; $$p_1(t,x,y)=p_0(t,x,y)-\int_0^t \left[\frac{\partial p_0}{\partial s}+\frac{\partial^2 p_0}{\partial x^2}+\frac{\partial^2 p_0}{\partial y^2}\right]ds,$$

$$p_1=x^2+y^2-\int_0^t [0+2+2]ds$$

$$p_1(t,x,y)=-4t+y^2+x^2,$$

For n=1; $$p_2(t,x,y)=p_1(t,x,y)-\int_0^t \left[\frac{\partial p_1}{\partial s}+\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial y^2}\right]ds,$$

$$p_2=-4t+y^2+x^2-\int_0^t [-4+2+2]ds,$$

$$p_2=-4t+y^2+x^2.$$

Its solution is $-4t+y^2+x^2$. If we select correction funtional (3.1.23) and the zeroth approximation $p_0(t,y)$ as $p(t,0,y)=-4t+y^2$, using the iteration, we obtain again

$$p_1(t,x,y)=p_0-\int_0^x (m-x)\left(\frac{\partial p_0}{\partial t}+\frac{\partial^2 p_0}{\partial \mu^2}+\frac{\partial^2 p_0}{\partial y^2}\right)dm \tag{3.1.27}$$

$$=-4t+y^2-2\int_0^x (m-x)dm=-4t+y^2+x^2 \tag{3.1.28}$$

where $\lambda_2(m)=(m-x)$. Again we obtained the same solution. Note that for every correction functional, we can chose the zeroth approximation which contains all initial or boundary conditions instead of different zeroth approximation, that is, general expression satisfies all ones.

Example 3.1.2.2: Solve the following nonlinear KDV equation

$$\begin{aligned} &u_t-6uu_x+u_{xxx}=0, x\in R, \\ &u(x,0)=f(x)=6x. \end{aligned} \tag{3.1.29}$$

Solution 3.1.2.2:

Following the analysis presented above we obtain the correction functional

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda_1(s)\left(\frac{\delta\partial u_n(s,x)}{\partial s} - \delta 6 u_n \frac{\delta\partial \widetilde{u_n}(s,x)}{\partial x} + \frac{\delta\partial^3 \widetilde{u_n}(s,x)}{\partial x^3}\right) ds, \qquad (3.1.30)$$

noticing that $\delta\widetilde{u_n} = 0$,then

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda_1(s)\left[\frac{\delta\partial u_n(s,x)}{\delta\partial s}\right] ds. \qquad (3.1.31)$$

To simply this statement (3.1.31), to determine optimally the Lagrange multiplier λ, using integration by parts we obtain

$$= \delta u_n + \lambda_1 \delta u_n \mid_{s=t} - \int_0^t \lambda_1^{'}(s)\delta u_n ds.$$

Making the coeffcients of δu_n to 0, its stationary conditions are obtained as follows:

$$\begin{aligned} &\delta u_n => 1 + \lambda_1(s) \mid_{s=t} = 0, \\ &\delta u_n => \lambda^{'}_1(s) = 0. \end{aligned} \qquad (3.1.32)$$

From these two equations (3.1.32), we find $\lambda(s) = -1$, substituting this value of the Lagrange multiplier into the correction function (3.1.30) gives the iteration formula

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(s,x)}{\partial s} - 6 u_n \frac{\partial u_n(s,x)}{\partial x} + \frac{\partial^3 u_n(s,x)}{\partial x^3}\right) ds. \qquad (3.1.33)$$

Choosing the zeroth approximation $u_0(x,t)$ as $u_0(x,t) = 6x$, applying the zeroth approximation $u_0(x,t)$ into (3.1.33) we obtain the successive approximations

$$u_0(x,t) = 6x$$

$$u_1(x,t) = 6x + 6^3 xt,$$

$$u_2(x,t) = 6x + 6^3 xt + 6^5 xt^2 + 93312 xt^3,$$

$$u_3(x,t) = 6x + 6^3 xt + 6^5 xt^2 + 6^7 xt^3,$$

$$\vdots$$

$$u_n(x,t) = 6x(1 + (36t) + (36t)^2 + (36t)^3 + ...).$$

This gives exact solution by

$$u(x,t)=\frac{6x}{1-36t},\quad |36t|<1. \tag{3.1.34}$$

Example 3.1.2.3: Solve the following wave equation

$$\text{PDE: } u_t = c(u_x + u_y) = 0, \quad c>0 \tag{3.1.35}$$

$$\text{BC: } \begin{array}{l} u(0,y,t)=\sin\left[\pi\left(\dfrac{y-2ct}{\ell}\right)\right], \\ u(x,0,t)=\sin\left[\pi\left(\dfrac{x-2ct}{\ell}\right)\right], \end{array} \tag{3.1.35a}$$

$$\text{IC: } \quad u(x,y,0)=\sin\left[\pi\left(\frac{x+y}{\ell}\right)\right]. \tag{3.1.35b}$$

Solution 3.1.2.3:

Following the analysis presented above we obtain the correction functional (3.1.36)

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t \lambda(s)\left[\frac{\partial u_n(x,y,s)}{\partial s}+c\left(\frac{\partial \tilde{u}_n(x,y,s)}{\partial x}+\frac{\partial \tilde{u}_n(x,y,s)}{\partial y}\right)\right]ds$$

noticing that $\delta\widetilde{u_n}=0$, then yields

$$\delta u_{n+1}=\delta u_n+\int_0^t \lambda(s)\delta\frac{\partial un}{\partial s}ds. \tag{3.1.37}$$

To simply this statement (3.1.37), to determine optimally the Lagrange multiplier λ, using integration by parts yields we obtain

$$\delta u_{n+1}=\delta u_n+\lambda\delta u_n\,\Big|_{s=t}+\int_0^t \lambda^{'}\delta\partial u_n ds.$$

Making the coeffcients of δu_n to 0, its stationary conditions are obtained as follows:

$$\begin{array}{l} \delta u_n : 1+\lambda(s)\,\Big|_{s=t}=0, \\ \delta u_n : \lambda^{'}(s)=0. \end{array} \tag{3.1.38}$$

From these equations (3.1.38), we find $\lambda(s)=-1$, substituting this value of the Lagrange multiplier into the correction function gives the iteration formula (3.1.39)

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t\left[\frac{\partial u_n(x,y,s)}{\partial s}+c\left(\frac{\partial u_n(x,y,s)}{\partial x}+\frac{\partial u_n(x,y,s)}{\partial y}\right)\right]ds \qquad (3.1.39)$$

Choosing the zeroth approximation $u_0(x,y,0)$ as $u(x,y,o)=\sin\left[x\left(\frac{x+y}{\ell}\right)\right]$, applying the zeroth approximation $u_0(x,y,0)$ into (3.1.39) we obtain the successive approximations;

For n=0, $$u_1(x,y,t)=u_o(x,y,t)-\int_0^t\left[\frac{\partial u_o(x,y,s)}{\partial s}+c\left(\frac{\partial u_o(x,y,s)}{\partial x}+\frac{\partial u_o(x,y,s)}{\partial y}\right)\right]ds$$
$$=\sin\left[\pi\frac{(x+y)}{\ell}\right]-\int_0^t\left[0+\frac{2\pi c}{\ell}\cdot\cos\left[\frac{\pi(x+y)}{\ell}\right]\right]ds$$
$$=\sin\left[\pi\frac{(x+y)}{\ell}\right]-\frac{2\pi ct}{\ell}\cdot\cos\left[\frac{\pi(x+y)}{\ell}\right],$$

For n=1, $$u_2(x,y,t)=u_1(x,y,t)-\int_0^t\left[\frac{\partial u_1(x,y,s)}{\partial s}+c\left(\frac{\partial u_1(x,y,s)}{\partial x}+\frac{\partial u_1(x,y,s)}{\partial y}\right)\right]ds$$
$$u_2=u_1-\int_0^t\left[\frac{-2\pi c}{\ell}.\cos\frac{\pi(x+y)}{\ell}\right]ds$$
$$-\int_0^t c\left[\frac{\pi}{\ell}.\cos\frac{\pi(x+y)}{\ell}+\frac{2\pi^2 cs}{\ell}\cdot\sin\frac{\pi(x+y)}{\ell}\right]ds$$
$$-\int_0^t c\left[\frac{\pi}{\ell}\cos\frac{\pi(x+y)}{\ell}+\frac{2\pi^2 cs}{\ell}\cdot\sin\frac{\pi(x+y)}{\ell}\right]ds$$

$$u_2=\sin\frac{\pi(x+y)}{\ell}\left[1-\frac{4\pi^2c^2t^2}{2.\ell^2}\right]-\frac{2\pi ct}{\ell}\cos\frac{\pi(x+y)}{\ell}$$

For n=2, $$u_3(x,y,t)=u_2-\int_0^t\left[\frac{\partial u_2(x,y,s)}{\partial s}+c\left(\frac{\partial u_2(x,y,s)}{\partial x}+\frac{\partial u_2(x,y,s)}{\partial y}\right)\right]ds$$

$$u_3=\sin\frac{\pi(x+y)}{\ell}\left[1-\frac{1}{2!}\left(\frac{2c\pi t}{\ell}\right)^2\right]$$
$$-\cos\frac{\pi(x+y)}{\ell}\left[\frac{2c\pi t}{\ell}-\frac{1}{3!}\left(\frac{2c\pi t}{\ell}\right)^3\right],$$

$$\vdots$$

Generalizing this gives

$$u_n = \sin\frac{\pi(x+y)}{\ell}\left[1-\frac{1}{2!}\left(\frac{2c\pi t}{\ell}\right)^2+\frac{1}{4!}\left(\frac{2c\pi t}{\ell}\right)^4+\ldots\right]$$
$$-\cos\frac{\pi(x+y)}{\ell}\left[2c\pi t-\frac{1}{3!}\left(\frac{2c\pi t}{\ell}\right)^3-\frac{1}{5!}\left(\frac{2c\pi t}{\ell}\right)^4\ldots\right]$$

This gives exact solution (3.1.40) by

$$u(x,t)=\sin\left(\frac{\pi x}{y}\right)\cos\left(\frac{c\pi t}{\ell}\right)-\cos\left(\frac{\pi x}{y}\right)\sin\left(\frac{c\pi t}{\ell}\right). \qquad (3.1.40)$$

A comparison between the variational iteration method and Adomian decomposition method

Comparing these two methods, the two methods are powerful and efficient methods that both give approximations of higher accuracy and closed form solutions if existing.

He's variational iteration method gives several successive approximations through using the iteration of the correction functional. However, Adomian decomposition method provides the components of the exact solution. Moreover, the VIM requires the evaluation of the Lagrangian multiplier λ, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations.

More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials, hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For nonlinear equations that arise frequently to express nonlinear phenomenon, He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

3.1.3 HOMOTOPY PERTURBATION METHOD

Introduction

Perturbation method proposed and improved by He is one of the well-known methods for solving nonlinear problems analytically. It is based on the existence of small/large parameters, the so-called perturbation quantities. However, many nonlinear problems don not contain such kind of perturbation quantities. In general, the perturbation method is valid only for weakly nonlinear problems, so we apply He's homotopy perturbation method to find the approximation solution and numerical solutions of these equatios that don not contain such kind of perturbation quantities. No matter what kind differential equations are, the method yields a very rapid convergence of the solution series in the most cases. Comparing with other methods, The homotopy perturbation method (HPM) is useful to obtain exact and efficiently, accurately and easily approximate solutions of linear and nonlinear differential equations.

Basic Idea of HPM

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique. To illustrate the HPM, He considered the following nonlinear differential equation:

$$A(u) = f(r),\ r \in \Omega, \tag{3.1.41}$$

with boundary conditions $B(u, \partial u / \partial n) = 0, \quad r \in \Gamma.$

Here A is a general differential operator, B is a boundary operator, $\mathrm{f(r)}$ is a known analytical function, Γ is the boundary of the domain Ω and $\partial u / \partial n$ denoted differential along the normal drawn outwards from Ω. The operator A can be divided into two parts M and N. Therefore (3.1.41) can be rewritten as follows:

$$\mathrm{M(u)+N(u)} = f(r). \tag{3.1.42}$$

He constructed a homotopy $v(r, p): \Omega x[0,1] \to R$ v(r,p) which satisfies

$$\mathrm{H}(v, p) = (1-p)\left[M(v) - M(y_0)\right] + p\left[A(v) - f(r)\right] = 0, \tag{3.1.43}$$

or

$$\mathrm{H}(v, p) = M(v) - M(y_0) + pM(y_0) + p\left[N(v) - f(r)\right] = 0, \tag{3.1.44}$$

where $r \in \Omega$ and p $\in$ [0, 1] is an imbedding parameter, y_0 is an initial approximation of (3.1.41). Hence, it is obvious that

$$\mathrm{H}(v, 0) = M(v) - M(y_0) = 0, \tag{3.1.45}$$

and

$$\mathrm{H}(v,1)=M(v)-M(y_0)+1.M(y_0)+1.\left[N(v)-f(r)\right]=0,$$
$$\mathrm{H}(v,p)=\underbrace{M(v)+N(v)}_{f(v)}-f(r)=0,$$

From (3.1.41), $A(u)=f(r)$,

$$\mathrm{H}(v,1)=A(v)-M(y_0)=0, \tag{3.1.46}$$

and the changing process of p from 0 to 1, is just that o f H(v,p) from M(v)-M(y_0) to A(v)−f(r). In topology, that is called deformation, M(v)-M (y_0) and A(v)-f (r) are called homotopic. Applying the perturbation technique, due to the fact that $0\le p\le 1$ can be considered as small parameter, we can assume that the solution (3.1.43) or (3.1.44) can be expressed as a series in p as follows:

$$v(t)=v_0(t)+p\ v_1(t)+p^2v_2(t)+p^3v_3(t)+\ldots \tag{3.1.47}$$

When p→ 1, (3.1.43) or (3.1.44) corresponding to (3.1.42) and (3.1.47) becomes the approximation solution of (3.1.42), i.e.,

$$u=\lim_{p\to 1}v=v_0(t)+v_1(t)+v_2(t)+v_3(t)+\ldots \tag{3.1.48}$$

Note that the series (3.1.48) is convergence for most cases, and also the rate of convergence depends on A(v). Now, we give some examples for applications of this method.

Example 3.1.3.1

We first consider a special case of the homogeneous non-linear KdV equation:

$$u_t-6uu_x+u_{xxx}=0, \tag{3.1.49}$$

with initial condition

$$u(x,0)=g(x)=\frac{x}{6}. \tag{3.1.49a}$$

Solution 3.1.3.1

First, we divide the equation (3.1.49) into two parts L and N such that

$$\underbrace{u_t}_{L}-\underbrace{6uu_x+u_{xxx}}_{N}=0.$$

To solve equation (3.1.49-49a) by homotopy perturbation method, using property $\mathrm{H}(v,p)=L(v)-L(y_0)+p\left[N(v)+L(y_0)-f(r)\right]=0,$

we construct the following homotopy:

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}\right)=p\left(6u\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3}-\frac{\partial u_0}{\partial t}\right). \tag{3.1.50}$$

Assume the solution of Equation (3.1.49-49a) in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+..... \tag{3.1.51}$$

Substituting (3.1.51) into only function u in Eq. (3.1.50), u_0 is remained in same form.

$$\left(\frac{\partial}{\partial t}(u_0+pu_1+.....)-\frac{\partial u_0}{\partial t}\right)=p\left(6(u_0+pu_1+p^2u_2+...)\frac{\partial}{\partial x}(u_0+pu_1+...)\right)$$
$$-p\left(\frac{\partial^3}{\partial x^3}(u_0+pu_1+...)-\frac{\partial u_0}{\partial t}\right), \tag{3.1.52}$$

Rearraging the statement (3.1.52) yields

$$\left(\frac{\partial u_0}{\partial t}+p^1\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+p^3\frac{\partial u_3}{\partial t}+...-\frac{\partial u_0}{\partial t}\right)=$$
$$p\left(6(u_0+pu_1+p^2u_2+p^3u_3+...)(\frac{\partial u_0}{\partial t}+p^1\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+p^3\frac{\partial u_3}{\partial t}+...)\right)$$
$$-p\left(\frac{\partial^3 u_0}{\partial x^3}+p^1\frac{\partial^3 u_1}{\partial x^3}+p^2\frac{\partial^3 u_2}{\partial x^3}+p^3\frac{\partial^3 u_3}{\partial x^3}+...-\frac{\partial u_0}{\partial t}\right),$$

collecting terms of the same power of p we obtain

$$\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}+p\left(\frac{\partial u_1}{\partial t}-6u_0\frac{\partial u_0}{\partial x}+\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial u_0}{\partial t}\right)$$
$$+p^2\left(\frac{\partial u_2}{\partial t}-6\left(u_1\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_1}{\partial x}\right)+\frac{\partial^3 u_1}{\partial x^3}\right) \tag{3.1.53}$$
$$+p^3\left(\frac{\partial u_3}{\partial x}-6\left(u_2\frac{\partial u_0}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_0\frac{\partial u_2}{\partial x}\right)+\frac{\partial^3 u_2}{\partial x^3}\right)+....=0.$$

where $p\neq 0$. Equating coefficients of the powers of p to 0 gives respectively, (1), (2), (3)…:

(1) $\quad p^0: \ \dfrac{\partial u_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0,$

As from p^0 the value $u_0(x,t)$ can not be calculated, we choose initial value as value of $u_0(x,t)$, i.e $u_0(x,t) = \dfrac{x}{6}.$ Moreover if we obtained an equation u_0 from p^0, we would use it in p^1 for finding $u_1(x,t)$ and arbitary choosing u_0 would also be not necessary.

(2) $\quad p^1: \ \dfrac{\partial u_1}{\partial t} = 6u_0 \dfrac{\partial u_0}{\partial x} - \dfrac{\partial^3 u_0}{\partial x^3} - \dfrac{\partial u_0}{\partial t},$

using $u_0(x,t) = \dfrac{x}{6},$ we find:

$$u_1 = \frac{xt}{6} + f(t), \tag{3.1.54}$$

We will make up homogeneus initial/boundary condition like $u_k(0,t) = 0,$ $k = 1,2...$ for each term $u_1, u_2, u_3, ...$ to find term(s) arising after taking integral, since this will enable us to calculate terms $u_1, u_2, u_3, ...$ in easy way. Then using $u_k(0,t) = 0,$ with $u_1(0,t) = 0,$ we obtain

$$u_1(0,t) = \frac{0t}{6} + f(t) = 0, \ f(t) = 0 \quad then \quad u_1(x,t) = \frac{xt}{6}, \tag{3.1.55}$$

(3) $\quad p^2: \ \dfrac{\partial u_2}{\partial t} = 6\left(u_1 \dfrac{\partial u_0}{\partial x} + u_0 \dfrac{\partial u_1}{\partial x} \right) - \dfrac{\partial^3 u_1}{\partial x^3},$

using homogeneus initial/boundary condition $u_2(0,t) = 0$ and $u_1(x,t) = \dfrac{xt}{6}$ gives

$$u_2(x,t) = \frac{xt^2}{6}. \tag{3.1.56}$$

Similary, we find $u_3(x,t)$ as

$$u_3(x,t) = \frac{xt^3}{6}. \tag{3.1.57}$$

The solution of (3.1.49-49a) can be obtained by setting $p=1$ in Eq. (3.1.51)

$$u = u_0 + u_1 + u_2 + u_3 + ... \tag{3.1.58}$$

Substituting u_0, u_1, u_2, u_3 into Eq. (3.1.58) we have

$$u(x,t) = \frac{x}{6} + \frac{xt}{6} + \frac{xt^2}{6} + \frac{xt^3}{6} + ...$$

This series has the closed form

$$u(x,t) = \frac{x}{6(1-t)}, \tag{3.1.59}$$

which is the exact solution of the problem

Example 3.1.3.2:

We will examine the Fisher equation, i.e.

$$u_t = u_{xx} + u(1-u), \tag{3.1.60}$$

subject to a constant initial condition

$$u(x,0) = \lambda. \tag{3.1.61}$$

Solution 3.1.3.2:

In order to solve Eq. (3.1.60), using HPM, we construct the following homotopy for this equation:

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left(\frac{\partial^2 u}{\partial x^2} - u(1-u) - \frac{\partial u_0}{\partial t}\right) \tag{3.1.62}$$

Assume the solution of equation (3.1.60) in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \tag{3.1.63}$$

Substituting $u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ...$ into Eq. (3.1.62) and equating the coefficients of like powers of p to 0, we get following set of differential equations

$$p^0: \ \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad u_0(x,t) = \lambda,$$

$$p^1: \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} + u_0 - u_0^2, \ u_1(x,0) = 0,$$

$$p^2 : \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} - u_1 u_0 + u_1(1-u_0), \qquad u_2(x,0) = 0,$$

$$p^3 : \frac{\partial u_3}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} - u_0 u_2 - u_1^2 + u_2(1-u_0), \qquad u_3(x,0) = 0,$$

$$\vdots$$

Solving the systems accordingly, thus we obtain,

$$\begin{aligned} &u_0(x,t) = \lambda, \\ &u_1(x,t) = \lambda(1-\lambda)t, \end{aligned} \tag{3.1.64}$$

$$\begin{aligned} &u_2(x,t) = \lambda(1-\lambda)(1-2\lambda)\, t^2/2!, \\ &u_3(x,t) = \lambda(1-\lambda)(1-6\lambda+6\lambda^2)\, t^3/3!, \end{aligned} \tag{3.1.65}$$

$$\vdots$$

By setting p =1 in Eq. (3.1.63), the solution of (3.1.60) can be obtained as $u = u_0 + u_1 + u_2 + u_3 + \ldots$ Thus, in view of (3.1.63) and (3.1.65) the solution in a series form is given by

$$\begin{aligned} u(x,t) = \lambda + \lambda(1-\lambda)t + \lambda(1-\lambda)(1-2\lambda)\frac{t^2}{2!} \\ + \lambda(1-\lambda)(1-6\lambda+6\lambda^2)\frac{t^3}{3!} + \ldots \end{aligned} \tag{3.1.66}$$

And using some algebra with the aid of symbolic computation tool, the solution in a closed form is given by

$$u(x,t) = \frac{\lambda e^t}{1-\lambda+\lambda e^t}, \tag{3.1.67}$$

which is the exact solution of the problem.

3.2 SOLITARY SOLUTION METHODS

In recent years, the Nonlinear evolution equations (NLEEs) which are the subject of study in various branches of mathematical physical sciences such as physics, biology, chemistry, etc.,play an important role in solving non-linear PDEs have been investigated by many authors who are interested in non-linear physical phenomena. Many powerful methods have been presented by those authors such as the homogeneous balance method, the hyperbolic tangent expansion method, the trial function method, the tanh-method , the non-linear transform method, the inverse scattering transform, the Backlund transform, the Hirota's Bilinear method , the Generalized Riccati equation, the Weierstrass Elliptic Function Method, the Sine–Cosine method, the Jacobi Elliptic Function Expansion, the truncated Painleve expansion, the F-expansion method, the rank analysis method, the Ansatz method , the exp-function expansion method … and so on. Here, we will introduce some of the so-called methods such as G'/G expansion methods, Exp(x) function method, the Sine–Cosine method and the homogeneous balance method to us.

The greatest property of the four methods as opposed to those of in the previous chapter is not to have requirement for initial/boundary conditions or initial trial functions at the outset, and can obtain a general solution without approximation. Moreover in these methods to take advantage of a computer algebra system like maple makes solving problem simpler. Disadvantage of these methods is to use Homogeneous Balance.

3.2.1 G'/G EXPANSION METHOD

Introduction

The G'/G-expansion method, first introduced by Wang et al., has become widely used to search for various exact solutions of NLEEs . This method is firstly proposed by which the traveling wave solutions of non-linear equations are obtained. The main idea of this method is that the traveling wave solutions of non-linear equations can be expressed by a polynomial in G'/G, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \omega G(\xi) = 0,$ where $\xi = \sum_{i=1}^{N} k_i x_i$, where N is the number of the unknown function and k_i's are constants. The degree of this polynomial in (G'/G) that we assume the solution of the unknown function can be determined by considering the homogeneous balance between the highest order derivatives and the non-linear terms appearing in the given non-linear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

Very lately, the G'/G-expansion method playing an important role in expressing the traveling wave solutions are improved the method to deal with evolution equations with variable coefficients, and devised an algorithm for using the method to solve nonlinear differential-difference equations modified the method to derive traveling wave solutions for Whitham-Broer-Kaup-like equations.

Although many efforts have been devoted to find various methods to solve (integrable or non-integrable) NLEEs, there is no unified method. As stated before , the main merits of the G'/G-expansion method over the other methods are that it gives more general solutions with some free parameters which, by suitable choice of the parameters, turn out to be some known solutions gained by the existing methods. Besides, (I) in all finite difference and finite element methods, it is necessary to have boundary and initial conditions. However, the G'/G -expansion method handles NLEEs in a direct manner with no requirement for initial/boundary conditions or initial trial functions at the outset. It obtains a general solution with free parameters that can be determined via boundary and/or initial conditions; (II) most of the methods give solutions in a series form and it becomes essential to investigate the convergence of approximation series. For example, the Adomian decomposition method depends only on the initial conditions and obtains a solution in a series which converges to the exact solution of the problem. But, with the G'/G-expansion method, one may obtain a general solution without approximation, (III) it serves as a powerful technique to integrate the NLEEs, even if the Painleve test of integrability fails; (IV) and finally,the solution procedure, using a computer algebra system like maple, matlab, is of utter simplicity.

Description of the G'/G -expansion method

Before we explain this method, we will make a remark about second-order linear differential equation. A second-order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^{2x}}+Q(x)\frac{dy}{dx}+R(x)y=G(x),$$

or

$$P(x)y''+Q(x)y'+R(x)y=G(x),$$

where P,Q ,R , and G are continuous functions. In the case G(X)=0, for all x, such equations are called homogeneous linear equations. Now, we explain the G'/G - expansion method:

The objective of this section is to outline the use of the G'/G -expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose we have a nonlinear PDE for u(x,t), in the form

$$P(u,u_x,u_t,u_{xx},u_{xt},...)$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives.In follow, we will tell application of the method step by step:

Step 1:

Using the transformation $u(x,t)=u(\xi)$, $\xi=x+ct$ or $\xi=x-ct$, reduce the partial differential equation $P(u,u_x,u_t,u_{xx},u_{xt},...)=0$ to an ordinary differential equation for $u(\xi)$, where we suppose that the transformation permits us reducing a partial differential equation to an ordinary differential equation, i.e,

$$P(u, cu', u', u'', cu', c^2u'', ...) = 0.$$

Later, if possible, integrating the reduced equation term by term one or more times as long as eq can not be integrated.the integration constant(s) can be set to 0 for simply.

Step 2:

Suppose that the solution of the reduced equation can be expressed by a polynomial in (G'/G) as follows

$$u = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{3.2.1}$$

Where a_0 and a_i, for i=1,2,…, N are constant and integer. Also, $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \omega G(\xi) = 0,$$

Or equivalently

$$\frac{d}{d\xi}\left(\frac{G'}{G}\right) = -\left(\frac{G'}{G}\right)^2 - \lambda\left(\frac{G'}{G}\right) - \omega, \tag{3.2.2}$$

Where λ *and* ω are contants. General solution of the equation (3.2.2) is

$$\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{|\lambda^2 - 4\omega|}}{2}\left(\dfrac{C_1 \sinh B\xi + C_2 \cosh B\xi}{C_1 \cosh B\xi + C_2 \sinh B\xi}\right), & \lambda^2 - 4\omega > 0 \\ \dfrac{\sqrt{4\omega - \lambda^2}}{2}\left(\dfrac{-C_1 \sinh B\xi + C_2 \cosh B\xi}{C_1 \cosh B\xi + C_2 \sinh B\xi}\right), & \lambda^2 - 4\omega < 0, \\ \dfrac{kC_2}{C_1 + C_2 x}, & \lambda^2 - 4\omega = 0 \end{cases} \tag{3.2.3}$$

where B=$\dfrac{\sqrt{|\lambda^2 - 4\omega|}}{2}$, and the positive integer N can be determined by considering the homogeneous balance the highest order derivatives or the highest degree of the linear term and highest order nonlinear appearing in ODE. Formulating this statement:

$$nonlinear \quad term \begin{cases} \text{k : degree of the underivative term in front of nonlinear term,} \\ \text{l : order of nonlinear tem} \end{cases}$$

$$linear \quad term \begin{cases} \text{s: degree of the linear term,} \\ \text{p: order of linear term,} \end{cases}$$

Balance $\rightarrow$ kxN+N+l=sxN+N+p.

For example:

Balancing $\dfrac{d^3u}{d\xi^3}$ and $u\dfrac{du}{d\xi}$ gives N+3= N+N+1→ N=2.

Balancing $\dfrac{d^3u}{d\xi^3}$ and $\left(\dfrac{du}{d\xi}\right)^2$ yields N+3 =2(N+1)→N=1.

If N doesn't become integer, say 1/p, then apply the transformation $u = v^{1/p}$ to the reduced equation, later applying step 2 again for finding a new value N.

Step 3:

Substituting the solution of Eq. (3.2.1) together with Eq. (3.2.2) into the reduced equation yields an algebraic equation involving powers of (G'/G). (The reason of using Eq. (3.2.2) is to prevent that the the reduced equation include derivatives of (G'/G) after substituting). Equating the coefficients of each power of (G'/G) to 0 gives a system of algebraic equations for a_i, λ, ω, and c. Then, we solve the system with software like maple to determine these constants. Next, using these constants we get solutions of the reduced equation, depending on the sign of $\Delta = \lambda^2 - 4\omega.$

Example 3.2.1.1:

Considering the Boussinesq equation

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0. \tag{3.2.3}$$

Using the transformation $\xi = x - ct,$ we reduce the equation to ODE

$$(c^2 - 1)u'' - (u^2)'' + u^{(4)} = 0. \tag{3.2.4}$$

By twice integrating (3.2.4), we find

$$(c^2 - 1)u - u^2 + u'' = 0, \tag{3.2.5}$$

Balancing $u'' \quad with \quad u^2$ in Eq. (3.2.5) gives

N+2=2N → N=2.

Suppose that the solution of the Boussinesq can be expressed by a polynomial in (G'/G) as follows for N=2

$$u=\sum_{i=0}^{2}a_i\left(\frac{G'}{G}\right)^i=u(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, a_2=0, \quad (3.2.6)$$

which satisfies second-order linear differential equation (SOLDE)

$$\frac{d}{d\xi}\left(\frac{G'}{G}\right)=-\left(\frac{G'}{G}\right)^2-\lambda\left(\frac{G'}{G}\right)^2-\omega,$$

For simply, if we write A in place of $\left(\frac{G'}{G}\right)$, polynomial and second-order linear differential equation become respectively;

$$u(\xi)=a_0+a_1A+a_2A^2, \tag{3.2.7a}$$

$$\frac{d}{d\xi}A=A'=-A^2-\lambda A-\omega. \tag{3.2.7b}$$

If (3.2.7a) is a solution of the Eq. (3.2.5), then it satisfies Eq. (3.2.5), this means

$$u^2(\xi)=a_2^2A^4+2a_2a_1A^3+(a_1^2+2a_0a_1)A^2+2a_0a_1A+a_0^2, \tag{3.2.8}$$

$$u'(\xi)=a_1A'+2a_2AA',$$

$$u'(\xi)=A'(a_1+2a_2A),$$

$$\begin{aligned}u'(\xi)&=-(A^2+\lambda A+\omega)(a_1+2a_2A),\\ u'(\xi)&=-\left(2a_2A^3+A^2(2a_2\lambda+a_1)+A(2a_2\omega+a_1\lambda)+a_1\omega\right),\end{aligned} \tag{3.2.9}$$

$$u''(\xi)=a_1A''+2a_2AA''+2a_2(A')^2,$$

$$u''(\xi)=A''(a_1+2a_2A)+2a_2(A')^2, \tag{3.2.10}$$

Substituting Eqs. (3.2.8), (3.2.9) and (3.2.10) together with Eq. (3.2.7b) into (3.2.5), collecting the coefficients of A^i (i=0,1,..4) and set it to 0 we obtain the system

$$\begin{aligned}&6a_2-a_2^2=0,\\ &10a_2\lambda-2a_1a_2+2a_1=0,\end{aligned}$$

$$\begin{aligned}&-a_2+c^2a_2+3a_1\lambda+4a_2\lambda^2-a_1^2+8a_2\omega-2a_0a_2=0,\\ &-2a_0a_1-a_1+c^2a_1+2a_1\omega+a_1\lambda^2+6a_2\lambda\omega=0,\end{aligned}$$

$$c^2a_0 - a_0^2 - a_0 + 2a_2\omega^2 + a_1\lambda\omega = 0.$$

Solving this system by maple gives

$$a_0 = 6\omega, \quad a_1 = 6\lambda, \quad a_2 = 6, \quad c = \sqrt{1+4\omega-\lambda^2}, \qquad (3.2.11a)$$

or

$$a_0 = \lambda^2 + 2\omega, \quad a_1 = 6\lambda, \quad a_2 = 6, \quad c = \sqrt{1-4\omega+\lambda^2}, \qquad (3.2.11b)$$

where λ *and* ω are arbitary constants. By substituting equations (3.2.11a) and (3.2.11a) into (3.2.7a), we have general solutions of Eq. (3.2.5). Using (3.2.3), we have three solutions of the Boussinesq equation as follows according to $\Delta = \lambda^2 - 4\omega$.

A) When $\lambda^2 - 4\omega > 0$,

$$u_1 = \frac{3}{2}(\lambda^2-4\omega)\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi}{C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi + C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi}\right)^2 - \frac{3}{2}\lambda^2 + 6\omega, \qquad (3.2.12a)$$

where $\xi = x - \sqrt{1+4\omega-\lambda^2}t$.

Or

$$u_2 = \frac{3}{2}(\lambda^2-4\omega)\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi}{C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi + C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\omega}\xi}\right)^2 - \frac{3}{2}\lambda^2 + \lambda^2 + 2\omega, \qquad (3.2.12b)$$

where $\xi = x - \sqrt{1+4\omega+\lambda^2}t$.

B) When $\lambda^2 - 4\omega < 0$,

$$u_3 = \frac{3}{2}(\lambda^2-4\omega)\left(\frac{-C_1\sinh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi + C_2\cosh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi}{C_2\sinh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi + C_1\cosh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi}\right)^2 - \frac{3}{2}\lambda^2 + 6\omega, \qquad (3.2.13a)$$

where $\xi = x - \sqrt{1+4\omega-\lambda^2}t$.

Or

$$u_4 = \frac{3}{2}(\lambda^2 - 4\omega)\left(\frac{-C_1 \sinh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi + C_2 \cosh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi}{C_2 \sinh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi + C_1 \cosh\frac{1}{2}\sqrt{-\lambda^2+4\omega}\xi}\right)^2 - \frac{3}{2}\lambda^2 + \lambda^2 + 2\omega, \qquad (3.2.13b)$$

where $\xi = x - \sqrt{1+4\omega+\lambda^2}t.$

C) When $\lambda^2 - 4\omega = 0,$

$$u_5(\xi) = \frac{6C_2^2}{(C_1+C_2)^2} - \frac{3}{2}\lambda^2 + 6\omega, \qquad (3.2.14a)$$

where $\xi = x - \sqrt{1+4\omega-\lambda^2}t.$

Or

$$u_6(\xi) = \frac{6C_2^2}{(C_1+C_2)^2} - \frac{3}{2}\lambda^2 + \lambda^2 + 2\omega, \qquad (3.2.14b)$$

where $\xi = x - \sqrt{1+4\omega+\lambda^2}t.$

In particular, if $C_1 \neq 0,\ C_2 = 0,\ \lambda > 0,\ \omega = 0$ then after writing these values into Eq (3.2.12), u_1 becomes

$$u_1 = -\frac{3}{2}\lambda^2 \sec h^2 \frac{\lambda}{2}\xi, \qquad (3.2.15a)$$

and u_2 becomes

$$u_2 = \frac{1}{2}\lambda^2\left(3\tanh^2\frac{\lambda}{2}\xi - 1\right), \qquad (3.2.15b)$$

which are solitary solitions.

Example 3.2.1.2

Now, we consider the modified Zakharov-Kuznetsov equation

$$u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0, \qquad (3.2.16)$$

Introducing a complex variation ξ defined as $\xi = x + y - ct$ (transformation), we have

$$\frac{\partial u}{\partial t}=\frac{du}{d\xi}\frac{\partial \xi}{\partial t}=u'(-c),\ \frac{\partial u}{\partial x}=\frac{du}{d\xi}\frac{\partial \xi}{\partial x}=u'.1,\ u_{xyy}=u''', \tag{3.2.17}$$

$$\begin{aligned} u_{xxx} &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{du}{d\xi}\frac{\partial \xi}{\partial x}\right) \\ &= \frac{\partial^2}{\partial x^2}\left(u'.1\right)=\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{du'}{d\xi}\frac{\partial \xi}{\partial x}\right)...=u'''.1, \end{aligned} \tag{3.2.18}$$

Substituting these Eqs. (3.2.17) and (3.2.18) into Eq. (3.2.16) we obtain a reduced ODE

$$-cu'+u^2u'+2u^{(3)}=0,$$

Once integrating we find

$$-3cu+u^3+6u''=0, \tag{3.2.19}$$

We can determine value of N by balancing u^3 *and* u'' in integrated ODE (3.2.19). N+2=3N →N=1. We can suppose that solution of ODE is of the form

$$u(\xi)=a_0+a_1\left(G'/G\right), \tag{3.2.20}$$

or $u(\xi)=a_0+a_1A,$ (writing A in place of (G'/G) for simply). Moreover $u(\xi)=a_0+a_1\left(G'/G\right)$ is an Eq. that satisfies the following OLDE, i.e,

$$\frac{d}{d\xi}\left(\frac{G'}{G}\right)=-\left(\frac{G'}{G}\right)^2-\lambda\left(\frac{G'}{G}\right)^2-\omega, \tag{3.2.21}$$

or equivalent to $\frac{d}{d\xi}A=A'=-A^2-\lambda A-\omega,$ called equivalent Eq., shortly.

If $u(\xi)$ is a solution of ODE (3.2.21), then it satisfies (3.2.21)

$$u^3(\xi)=\left(a_0+a_1A\right)^3,$$

$$u^3(\xi)=a_1^3A^3+3a_0a_1^2A^2+3a_0^3a_1\,A+a_0^3, \tag{3.2.22a}$$

T o find u'', integrating u twice yields

$$\begin{aligned} u(\xi)&=a_0+a_1A, \\ u'(\xi)&=a_1A', \end{aligned}$$

$u'(\xi) = -a_1(A^2 + \lambda A + \omega)$ (from equivalent Eq.) integrating u once gives

$$u''(\xi) = 2a_1A^3 + 3a_1A^2 + (2a_1\omega + a_1\lambda^2)A + a_1\lambda\omega. \qquad (3.2.22b)$$

Substituting Eqs. (3.2.22a) and (3.2.22b) into Eq. (3.2.19), collecting the coefficients of A^i (i=0, 1, ... 4) and setting it to 0, we obtain the system

$$\begin{aligned} &12a_1 + a_1^3 = 0, \\ &3a_0a_1^2 + 18a_1\lambda = 0, \\ &3a_0^2a_1 - 3ca_1 + 6a_1\lambda^2 + 12a_1\omega = 0, \\ &-3ca_0 + a_0^3 + 6a_1\lambda = 0. \end{aligned} \qquad (3.2.23)$$

Solving this system by maple gives

$$a_1 = \pm\sqrt{3}i, a_1 = 2\pm\sqrt{3}i, c = 4\omega - \lambda^2, \qquad (3.2.24)$$

Where λ *and* ω are constants from equivalent Eq. Substituting values (3.2.24) into Eq. (3.2.20), we obtain polynomial

$$u(\xi) = \pm\sqrt{3}i \pm 2\sqrt{3}i\left(G'/G\right), \qquad (3.2.25)$$

where $\xi = x + y + \left(\lambda^2 - 4\omega\right)t$. Substituting the general solution of SOLD into Eq. (3.2.3), we have three solutions as follows according to $\Delta = \lambda^2 - 4\omega$:

A) When $\lambda^2 - 4\omega > 0$,

$$u_{1,2} = \pm i\sqrt{3(\lambda^2 - 4\omega)}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\omega}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\omega}\xi}{C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\omega}\xi + C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\omega}\xi}\right) \pm\sqrt{3}i, \qquad (3.2.26a)$$

B) When $\lambda^2 - 4\omega < 0$,

$$u_{3,4} = \pm i\sqrt{3(4\omega - \lambda^2)}\left(\frac{-C_1\sinh\frac{1}{2}\sqrt{4\omega - \lambda^2}\xi + C_2\cosh\frac{1}{2}\sqrt{4\omega - \lambda^2}\xi}{C_2\sinh\frac{1}{2}\sqrt{4\omega - \lambda^2}\xi + C_1\cosh\frac{1}{2}\sqrt{4\omega - \lambda^2}\xi}\right) \pm\sqrt{3}i, \qquad (3.2.26b)$$

C) When $\lambda^2 - 4\omega = 0$,

$$u_{5,6}(\xi) = \frac{\pm\sqrt{3}iC_2^2}{(C_1 + C_2)^2}. \qquad (3.2.26c)$$

In particular, if, $C_1 \neq 0,\ C_2 = 0,\ \lambda > 0,\ \omega = 0,$ then after writing these values into Eq. (3.2.26a), $u_{1,2}$ becomes

$$u_{1,2} = \pm\sqrt{3}i\lambda \tanh\frac{\lambda}{2}\xi \pm \sqrt{3}i\,. \tag{3.2.27}$$

And after writing $C_1 \neq 0, C_2 = 0, \lambda > 0, \omega = 0$ into Eq. (3.2.26b), $u_{3,4}$ becomes

$$u_{1,2} = \pm\sqrt{3}\lambda \tan\frac{\lambda}{2}\xi \pm \sqrt{3}i, \tag{3.2.28}$$

which are solitary solitions of the modified Zakharov-Kuznetsov equation and where $\xi = x + y + \left(\lambda^2 - 4\omega\right)t.$

3.2.2 THE SINE-COSINE METHOD

Introduction

The nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, chemistry, biology, fluid dynamics, plasma, optical fibers and other areas of engineering. One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods that look for exact solutions for nonlinear evolution equations.

Therefore, exact solution methods of nonlinear evolution equations have become more and more important resulting in methods like Variation Iteration Method, Homotopy Perturbation Method, Exp-Function method, the sine–cosine method, the homogeneous balance method, tanh–sech method and Extended tanh–coth method. Most of exact solutions have been obtained by these methods, including the solitary wave solutions, shock wave solutions, periodic wave solutions, and the like. We have summarized the main steps of using sine–cosine method, as listed below:

Step 1:

Using the transformation $u(x,t) = u(\xi)$, $\xi = x - ct$, reduce the partial differential equation $P(u, u_x, u_t, u_{xx}, u_{xt}, ...)$ =0 to an an ordinary differential equation for $u(\xi)$

$$P(U, -cU', U', U'', -cU', c^2uU'', ...) = 0\,. \tag{3.2.29}$$

Later, if possible, integrating the reduced equation term by term one or more times as long as eq cannot be integrated. The arisen integration constant(s) can be set to 0 for simply.

Step 2:

The solutions of many nonlinear equations can be expressed in the form

$$U(\xi) = u(x,t) = \lambda \sin^{\beta}(\omega\xi), \quad |\omega\xi| < \frac{\pi}{2}, \tag{3.2.30a}$$

or in the form

$$U(\xi) = u(x,t) = \lambda \cos^{\beta}(\omega\xi), \quad |\omega\xi| < \frac{\pi}{2\omega}, \tag{3.2.31a}$$

where λ, ω and β are parameters to be determined later. ω and c are constants. We use

$$\begin{aligned} U(\xi) &= \lambda \sin^{\beta}(\omega\xi), \\ U^{n}(\xi) &= \lambda^{n} \sin^{\beta n}(\omega\xi), \end{aligned} \tag{3.2.30b}$$

$$\begin{aligned} U(\xi) &= \lambda co\,s^{\beta}(\omega\xi), \\ U^{n}(\xi) &= \lambda^{n} co\,s^{\beta n}(\omega\xi), \end{aligned} \tag{3.2.31b}$$

and the derivatives of (3.2.30b) and (3.2.31b) become

$$\begin{aligned} \left(U^{n}\right)_{\xi} &= n\omega\beta\lambda^{n} \cos(\omega\xi)\sin^{n\beta-1}(\omega\xi), \\ \left(U^{n}\right)_{\xi\xi} &= -n^{2}\omega^{2}\beta^{2}\lambda^{n} \sin^{n\beta}(\omega\xi) + n\omega^{2}\lambda^{n}\beta(n\beta-1)\sin^{n\beta-2}(\omega\xi), \\ \left(U^{n}\right)_{\xi\xi\xi} &= n\omega^{3}\beta\lambda^{n}(n\beta^{2} - 3\beta + 2)\sin^{n\beta-3}(\omega\xi)\cos^{3}(\omega\xi) + \\ &\quad n\omega^{3}\lambda^{n}\beta(3n\beta-2)\sin^{n\beta-1}(\omega\xi)\cos^{3}(\omega\xi), \end{aligned} \tag{3.2.30c}$$

or

$$\begin{aligned} \left(U^{n}\right)_{\xi} &= -n\omega\beta\lambda^{n} \sin(\omega\xi)\cos^{n\beta-1}(\omega\xi), \\ \left(U^{n}\right)_{\xi\xi} &= -n^{2}\omega^{2}\beta^{2}\lambda^{n} \cos^{n\beta}(\omega\xi) + n\omega^{2}\lambda^{n}\beta(n\beta-1)\cos^{n\beta-2}(\omega\xi), \\ \left(U^{n}\right)_{\xi\xi\xi} &= n\omega^{3}\beta\lambda^{n}(-n^{2}\beta^{2} + 3\beta n - 2)\cos^{n\beta-3}(\omega\xi)\sin^{3}(\omega\xi) + \\ &\quad n\omega^{3}\lambda^{n}\beta(3n\beta-2)\sin^{n\beta-1}(\omega\xi)\sin(\omega\xi), \end{aligned} \tag{3.2.31c}$$

and so for the other derivatives.

Step 3:

After substituting the equations (3.2.30b) and (3.2.30c) or (3.2.31b) and (3.2.31c) above into the reduced equation (3.2.29), we balance the terms of the cosine functions or balance the terms of the sine functions and solving the resulting system of algebraic equations by using the computerized symbolic calculations. We next collect all terms with same power in $\sin^k(\omega\xi) \quad or \quad \cos^k(\omega\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns ω, β and λ. All possible values of the parameters l, b and k have been obtained.

Example 3.2.2.1:

Now, we consider the ubiquitous KdV equation in dimensionless

$$u_t + auu_x + u_{xxx} = 0. \qquad (3.2.32)$$

Using the transformation $\xi = x - ct$, we reduce the KdV equation to ODE

$$-cu + \frac{a}{2}u^2 + u'' = 0. \qquad (3.2.33)$$

Substituting the cosine assumption into the reduced equation (3.2.33) gives

$$-c\lambda\cos^{\beta}(\omega\xi) + \frac{a}{2}\lambda^2\cos^{2\beta}(\omega\xi) -$$
$$\lambda\omega^2\beta^2\cos^{\beta}(\omega\xi) + c\lambda\omega^2\beta(\beta-1)\cos^{\beta-2}(\omega\xi) = 0.$$

We should also use the balance between the exponents of the cosine functions. This means that Eq. (3.2.32) is satisfied only if the following system of algebraic equations holds

$$\beta = \beta \to \omega^2\beta^2\lambda \_ -c\lambda,$$
$$2\beta = \beta - 2 \to \frac{a}{2}\lambda^2 = -c\lambda\omega^2\beta(\beta-1),$$
$$\beta - 1 \neq 0.$$

This in turn gives

$$\beta = -2, \omega = \frac{1}{2}\sqrt{-c}, \lambda = \frac{3c}{a}. \qquad (3.2.34)$$

The results (3.2.34) can be easily obtained if we also use the sine assumption. Consequently, we obtain the following soliton solutions

$$u_1(x,t)=\frac{3c}{a}\sec h^2\left[\frac{\sqrt{c}}{2}(x-ct)\right],\ c>0, \qquad (3.2.35)$$

$$u_2(x,t)=-\frac{3c}{a}\csc h^2\left[\frac{\sqrt{c}}{2}(x-ct)\right],\ c>0, \qquad (3.2.36)$$

$$u_3(x,t)=\frac{3c}{a}\sec^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right],\ c>0, \qquad (3.2.37)$$

Example 3.2.2.2:

Now, we consider the modified KdV (mKdV) equation

$$u_t+u^2u_x+u_{xxx}=0. \qquad (3.2.38)$$

Substituting the wave variable ξ = x−ct into the mKdV equation and integrating once we find

$$-cu+\frac{a}{3}u^3+u''=0, \qquad (3.2.39)$$

We use the sine-cosine method, substituting the cosine assumption into the reduced equation gives

$$-c\lambda\cos^{\beta}(\omega\xi)+\frac{a}{3}\lambda^3\cos^{3\beta}(\omega\xi)-$$
$$\lambda\omega^2\beta^2\cos^{\beta}(\omega\xi)+c\lambda\omega^2\beta(\beta-1)\cos^{\beta-2}(\omega\xi)=0.$$

We balance the exponents of the cosine functions. Equation (3.2.39) is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned}&\beta=\beta\to\omega^2\beta^2\lambda=-c\lambda,\\&3\beta=\beta-2\to\frac{a}{3}\lambda^3=-c\lambda\omega^2\beta(\beta-1),\\&\beta-1\neq0.\end{aligned} \qquad (3.2.40)$$

which leads to

$$\beta=-1,\ \omega=\sqrt{-c},\ \lambda=\frac{6c}{a}. \qquad (3.2.41)$$

This in turn gives the periodic solutions for c < 0, a< 0:

$$u_1(x,t)=\sqrt{\frac{6c}{a}}\sec\left[\sqrt{-c}(x-ct)\right],\ c<0,\ a<0, \tag{3.2.42}$$

$$u_2(x,t)=\sqrt{\frac{6c}{a}}csc\left[\sqrt{-c}(x-ct)\right],\ c<0,\ a<0, \tag{3.2.43}$$

However, for c > 0, a > 0, we obtain the soliton solution

$$u_3(x,t)=\sqrt{\frac{6c}{a}}\sec h\left[\sqrt{c}(x-ct)\right]. \tag{3.2.44}$$

3.2.3 THE EXP-FUNCTION METHOD

Introduction

The Exp-function method was proposed in 2006 by He and Wu to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations. It has been demonstrated that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving high-dimensional nonlinear evolutions in mathematical physics. The Exp-function method has been exploited for the determination of exact solutions of many nonlinear differential equations.

Recently, several alternative modifications of the Exp-function method have been developed. The tanh, extended tanh, improved tanh and generalized tanh function methods sech and rational Exp-function method, the modified simplest equation method and many similar standard function methods are also successfully used for the construction of solutions of nonlinear differential equations.

Basic idea of Exp-function method

To explain beter, We consider the general nonlinear partial differential equation of the type

$P(u,u_x,u_t,u_{xx},u_{xt},...)$=0. For such a PDE, we now summarize the main steps as follows

Step 1: Using the transformation $u(x,t)=u(\xi)$, $\xi=x+ct$ or $\xi=x-ct$, reduce the partial differential equation $P(u,u_x,u_t,u_{xx},u_{xt},...)$=0 to an ordinary differential equation for $u(\xi)$, i.e,

$$P(u, cu', u', u'', cu', c^2u'', ...) = 0. \quad (3.2.45)$$

Later, if possible, integrating the reduced equation term by term one or more times as long as eq cannot be integrated. The integration constant(s) can be set to 0 for simply.

Step 2: Assume that the solution $u(\xi)$ of the reduced Eq. (3.2.45) is in the form

$$u(\xi) = \frac{a_c \exp(c\xi) + ... + a_{-d} \exp(-d\xi)}{a_p \exp(p\xi) + ... + a_{-q} \exp(-q\xi)}, \quad (3.2.46)$$

where c, d, p and q are unknown positive integers, we can determine these numbers by using homogeneous balance according to exp-function (3.2.46) such that to determine c and d, we balance the linear term of highest order of equation with the highest order nonlinear term, similarly to determine p and q, we balance the linear term of lowest order of equation with the lowest order nonlinear term

Step 3: After determining c, d, p and q, substitute (3.2.46) into the reduced eq. (3.2.45) and equating the coefficients of exp(nξ) to zero, obtain a system of algebraic equations for a_n, b_n k and w. Then, solve the system with the aid of a computer algebra system such as mathematica, maple or matlab to determine these constants.

Step 4: Substitute the values in the previous step into expression (3.2.46) and find the traveling wave solutions of the reduced Eq. Then, to check the correctness of the solutions, it is necessary to substitute them into the original equation.

Example 3.2.3.1:

Consider the following equation

$$u_t = u_{xx} + \lambda_1 u\, u_x + \lambda_2 u + u^2. \quad (3.2.47)$$

Introducing a transformation as $\eta = kx + wt$ we can covert equation (3.2.47) into ordinary differential equations

$$wu' - k^2u'' - \lambda_1 kuu' - \lambda_2 u + \lambda_3 u^2 = 0 \quad (3.2.48)$$

where the prime denotes the derivative with respect to η. The solution of the equation can be expressed in the form

$$u(\xi)=\frac{a_c\exp(c\eta)+...+a_{-d}\exp(-d\eta)}{a_p\exp(p\eta)+...+a_{-q}\exp(-q\eta)}, \tag{3.2.49}$$

To determine the values of c, p, d and q, we balance the linear term of highest order of equation (3.2.48) with the highest order nonlinear term

$$u''=\frac{c_1\exp\left[\left(3p+c\right)\eta\right]+...}{c_2\exp\left[\left(4p\right)\eta\right]+...} \tag{3.2.50}$$

and

$$u'u=\frac{c_3\exp\left[\left(2p+2c\right)\eta\right]+...}{c_4\exp\left[\left(4p\right)\eta\right]+...} \tag{3.2.51}$$

where c_i are determined coefficients only for simplicity. To determine the values of c and p, balancing the highest order of exp-function like (3.2.50) and (3.2.51), we have

3p+c=2p+2c

which in turn gives p=c.

To determine the values of d and q, we balance the terms of lowest order of exp-function (3.2.49)

$$u''=\frac{...+d_1\exp\left[\left(-d-3q\right)\eta\right]}{....+d_2\exp\left[\left(-4p\right)\eta\right]}, \tag{3.2.52}$$

and

$$u'u=\frac{...+d_3\exp\left[\left(2p+2c\right)\eta\right]}{...+d_4\exp\left[\left(4p\right)\eta\right]}, \tag{3.2.53}$$

where d_i are determined coefficients only for simplicity. Now, using (3.2.52) and (3.2.53), we have

\- d-3q=-2d-2q

which in turn gives q=d.

For simplicity, we set p=c=1 and q=d=1 then the trial solution, equation (3.2.49) reduces to

$$u(\xi)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{b_1\exp(\eta)+a_1+b_{-1}\exp(-\eta)}, \tag{3.2.54}$$

Substituting equation (3.2.54) into equation (3.2.48), we have

$$\frac{1}{A}[c_3\exp(3\eta)+c_2\exp(2\eta)+c_1\exp(\eta)+c_0+c_{-1}\exp(-\eta)$$
$$+c_{-2}\exp(-2\eta)+c_{-3}\exp(-3\eta)]=0.$$

where

$$A=(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta))^3,$$
$$c_{-3}=-\lambda_2a_{-1}b_{-1}^2-\lambda_3a_{-1}^2b_{-1},$$

$$c_{-2}=wa_0b_{-1}^2-\lambda_2a_0^2b_{-1}-\lambda_3a_{-1}^2b_0-k^2a_0b_{-1}^2-wa_{-1}b_{-1}b_0+k^2a_{-1}b_{-1}b_0$$
$$-\lambda_1ka_{-1}b_{-1}a_0+\lambda_1ka_{-1}^2b_0-wa_{-1}b_{-1}b_0-2\lambda_2a_{-1}b_{-1}b_0-2\lambda_3a_{-1}b_{-1}a_0,$$

$$c_{-1}=-wa_{-1}b_0^2-k^2a_{-1}b_0^2+wa_0b_{-1}b_0-\lambda_2a_{-1}b_0^2-\lambda_3a_0^2b_{-1}+k^2a_0b_{-1}b_0$$
$$-\lambda_1ka_{-1}b_0a_0-\lambda_1ka_0^2b_{-1}-2\lambda_2a_0b_{-1}b_0-2\lambda_3a_{-1}b_0a_0-2wa_{-1}b_1b_{-1}$$
$$+2wa_1b_{-1}^2-4k^2a_1b_{-1}^2+4k^2a_{-1}b_1b_{-1}-\lambda_2a_1b_{-1}^2-\lambda_3a_{-1}^2b_1$$
$$-2\lambda_1ka_1b_{-1}a_{-1}+2\lambda_1ka_{-1}^2b_1-2\lambda_2a_{-1}b_1b_{-1}-2\lambda_3a_1b_{-1}a_{-1},$$

$$c_0=-\lambda_3a_0^2b_0-\lambda_2a_0b_0^2-3wa_{-1}b_1b_0+3wa_1b_{-1}b_0-3k^2a_{-1}b_1b_0+6k^2a_0b_1b_{-1}$$
$$-2\lambda_2a_{-1}b_1b_0-2\lambda_2a_0b_1b_{-1}-2\lambda_2a_1b_{-1}b_0-2\lambda_3a_0b_1a_{-1}-2\lambda_3a_1b_0a_{-1}$$
$$-2\lambda_3a_0b_{-1}a_1+3\lambda_1kb_1a_{-1}a_0-3\lambda_1ka_0b_{-1}a_1,$$

$$c_1=wa_1b_0^2-k^2a_1b_0^2-wa_0b_1b_0-\lambda_2a_1b_0^2-\lambda_3a_0^2b_1+k^2a_0b_1b_0$$
$$-\lambda_1ka_1b_0a_0+\lambda_1ka_0^2b_1-2\lambda_2a_0b_1b_0-2\lambda_3a_1b_0a_0-2wa_{-1}b_1^2$$
$$+2wa_1b_{-1}+4k^2a_1b_1b_{-1}-\lambda_2a_{-1}b_1^2-\lambda_3a_1^2b_{-1}$$
$$+2\lambda_1ka_1b_1a_{-1}-2\lambda_1ka_1^2b_{-1}-2\lambda_2a_1b_1b_{-1}-2\lambda_3a_1b_1a_{-1},$$

$$c_2=-wa_0b_1^2-\lambda_2a_0^2b_1-\lambda_3a_1^2b_0-k^2a_0b_1^2+wa_1b_1b_0+k^2a_1b_1b_0$$
$$+\lambda_1ka_1b_1a_0-\lambda_1ka_1^2b_0-2\lambda_2a_1b_1b_0-2\lambda_3a_1b_1a_0,$$

$$c_3=-\lambda_2a_1b_1^2-\lambda_3a_1^2b_1.$$

Equating the coefficients of $\exp(\eta)$ to be zero, we obtain

$$\{c_{-3}=0\quad c_{-2}=0\quad c_{-1}=0\quad c_0=0\quad c_1=0\quad c_2=0\}. \tag{3.2.55}$$

Solution of (3.2.55) will yield

$$a_0 = \frac{a_1 b_{-1}}{b_0}, \quad a_{-1} = 0, \quad b_1 = 0, \quad w = \frac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2}, \quad k = \frac{\lambda_3}{\lambda_1}. \tag{3.2.56}$$

We, therefore, obtained the following generalized solitary solution u(x,t) of the original equation

$$u(x,t) = \frac{\dfrac{a_1 b_{-1}}{b_0} + a_1 \exp\left(\dfrac{\lambda_3}{\lambda_1} x + \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right)}{b_0 + b_1 \exp\left(-\dfrac{\lambda_3}{\lambda_1} x - \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right)}, \tag{3.2.57}$$

where a_1, b_0, b_{-1}, λ_1, λ_2 and λ_3 are real numbers.

.

$$u(x,t) = \frac{a_1 \cosh\left(\dfrac{\lambda_3}{\lambda_1} x - \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right) - a_1 \sinh\left(\dfrac{\lambda_3}{\lambda_1} x - \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right) + \dfrac{a_1 b_{-1}}{b_0}}{b_{-1} \cosh\left(\dfrac{\lambda_3}{\lambda_1} x - \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right) + b_{-1} \sinh\left(-\dfrac{\lambda_3}{\lambda_1} x - \dfrac{\lambda_2 \lambda_1^2 + \lambda_3^2}{\lambda_1^2} t\right) + b_0}. \tag{3.2.58}$$

3.2.4 THE HOMOGENEOUS BALANCE METHOD

Introduction

Many phenomena in physics and other fields are described by nonlinear evolution equations. When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact traveling wave solutions have to be explored

The investigation of the exact traveling wave solutions of nonlinear evolutions equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, elastic media, optical fibers, etc.

In recent years, the homogeneous balance (HB) method has been widely applied to derive the nonlinear transformation and exact solutions (especially the solitary wave solutions) and auto Bäcklund transformations as well as the similarity reductions of nonlinear partial differential equations (PDEs) in mathematical physics.

Wang et al. , Khalfallah applied the (HB) method to obtain the new exact traveling wave solutions of a given nonlinear partial differential equations. Fan showed that there is a close connection among the HB method, Wiess, Tabor, Carnevale (WTC) method and Clarkson, Kruskal (CK) method.

As the mathematical models of complex physical phenomena, nonlinear evolution equations are involved in many fields from physics to biology, chemistry, engineer, plasma physics, optical fibers and solid state physics etc. Many methods were developed for finding the exact traveling wave solutions of nonlinear evolutions equations, such as Hirota's method, Backlund and Darboux transformation, Painlevé expansions, Homogeneous balance method, Jacobi elliptic function, Extended tanh-function method, F-expansion method and extended F-expansion method.

Basic idea of HBM

We have used the homogeneous balance method to obtain solutions of nonlinear differential equations. We now summarize the key steps as follows:

First, we consider the partial differential equation $P(u,u_x,u_t,u_{xx},u_{xt},...)=0$

Step 1:

Using the transformation $u(x,t)=u(\xi)$, $\xi=x+ct$ or $\xi=x-ct$, reduce the partial differential equation $P(u,u_x,u_t,u_{xx},u_{xt},...)=0$ to an ordinary differential equation for $u(\xi)$,where we suppose that the transformation permits us reducing a partial differential equation to an ordinary differential equation ,i.e,

$$P(u,cu',u',u'',cu',c^2u'',...)=0. \qquad (3.2.59)$$

Later, if possible, integrating the reduced equation term by term one or more times as long as eq cannot be integrated. The integration constant(s) can be set to 0 for simply.

Step 2 :

Suppose that the solution of the reduced equation can be expressed by

$$u=\sum_{i=0}^{N}a_i\phi^{i}, \qquad (3.2.60)$$

where a_0 and a_i, for i=1,2,…, N are constant and integer. Also, $\phi(\xi)$ satisfies the Riccati equation:

$$\phi' = k(1-\phi^2) \text{ or } \phi' = a\phi^2 + b\phi + c, \tag{3.2.61}$$

which has general solution $\varphi(\xi)=\tanh(k\xi),\ \varphi(\xi)=\coth(k\xi),$ where k is contant. The positive integer N can be determined by considering the homogeneous balance the highest order derivatives or the highest degree of the linear term and highest order nonlinear appearing in ODE. Formulating this statement:

$$nonlinear\quad term\begin{cases}\text{k : degree of the underivative term in front of nonlinear term,}\\ \text{l: order of nonlinear tem}\end{cases},$$

$$linear\quad term\begin{cases}\text{s: degree of the linear term,}\\ \text{p: order of linear term,}\end{cases}$$

Balance $\rightarrow$ kxN+N+l=sxN+N+p.

If N doesn't become integer, say 1/p, then apply the transformation $u = v^{1/p}$ to the reduced equation, later applying step 2 again for finding N.

Step 3:

Substituting the solution of Eq. (3.2.60) together with Eq. (3.2.61) into the reduced equation yields an algebraic equation involving powers of ϕ. (The reason of using Eq. (3.2.61) is to prevent that the the reduced equation include derivatives of ϕ after substituting). Equating the coefficients of each power of ϕ^i to 0 gives a system of algebraic equations for a, b, and c. Then, we solve the system with software like maple to determine these constants. Next, we get solutions of the reduced equation.

Example 3.2.4.1:

We consider the burger-Kdv equation for HB

$$u_t + uu_x + \beta u_{xxx} = 0. \tag{3.2.62}$$

Let us consider the traveling wave solutions $u(\xi), \xi = x - \lambda t$,then Eq. above becomes

$$\beta u''' - \alpha u'' + uu' - \lambda u' = 0. \tag{3.2.63}$$

Once integration yields

$$\beta u''-\alpha u'+\frac{1}{2}u'-\lambda u=0. \tag{3.2.64}$$

We suppose that the solutions of (3.2.64) have the form given by

$$u=\sum_{i=0}^{N}a_i\phi^{i}, \tag{3.2.65}$$

To determine N, by balancing u'' with u^2 gives

N+2=2N→N=2.

For N=2, (3.2.65) becomes

$$u=a_0+a_1\phi+a_2\phi^2. \tag{3.2.66}$$

$u=a_0+a_1\phi+a_2\phi^2$ satisfies the Riccati Eq. $\phi'=k(1-\phi^2)$. If this is a solution of reduced eq. (3.2.64), then it satisfies (3.2.64).

$$u'=a_1k+2a_2k\,\phi-a_1k\,\phi^2-2a_2k\,\phi^3,$$

$$u''=2a_2k^2-2a_1k^2\phi-8a_2k^2\phi^2+2a_1k^2\phi^3+6a_2k^2\phi^4, \tag{3.2.67}$$

$$u^2=a_0^2+a_1^2\phi^2+a_2^2\phi^4+2a_0a_1\phi+2a_2a_1\phi^3+2a_0a_2\phi^2.$$

Substituting Eqs above into the reduced ode, collecting all terms with the same order of ϕ and setting the coefficients of each order of ϕ to 0 yields a set of algebra equationswith respect to a_0,a_1,a_2,k and λ:

$$6\beta a_2k^2+\frac{1}{2}a_2^2=0,$$
$$2\beta a_1k^2+2\alpha a_2k+a_1a_2=0,$$

$$-8\beta a_2k^2+\alpha a_1k+\frac{1}{2}a_1^2+a_0a_2-\lambda a_2=0,$$
$$-2\beta a_1k^2+2\alpha a_2k+a_1a_0-\lambda a_1=0,$$

$$2\beta a_2k^2-\alpha a_1k+\frac{1}{2}a_0^2-\lambda a_0=0.$$

Which have solutions

$$a_0=\frac{9\alpha^2}{25\beta},\quad a_1=\pm\frac{6\alpha^2}{25\beta},\quad a_2=-\frac{3\alpha^2}{25\beta},\quad k=\mp\frac{\alpha}{10\beta},\quad \lambda=\frac{6\alpha^2}{25\beta} \tag{3.2.68}$$

and

$$a_0=-\frac{3\alpha^2}{25\beta},\quad a_1=\pm\frac{6\alpha^2}{25\beta},\quad a_2=-\frac{3\alpha^2}{25\beta},\quad k=\mp\frac{\alpha}{10\beta},\quad \lambda=\frac{6\alpha^2}{25\beta}. \tag{3.2.69}$$

Substituting the solutions (3.2.68) and(3.2.69) into Eq. (42) gives two traveling front solutions and two singular traveling wave solutions,

$$u_1=\pm\frac{6\alpha^2}{25\beta}-\frac{6\alpha^2}{25\beta}\tanh\left[\frac{\alpha}{10\beta}\left(x\mp\frac{6\alpha^2}{25\beta}t\right)\right]+\frac{3\alpha^2}{25\beta}\operatorname{sech}^2\left[\frac{\alpha}{10\beta}\left(x\mp\frac{6\alpha^2}{25\beta}t\right)\right], \tag{3.2.69}$$

and

$$u_2=\pm\frac{6\alpha^2}{25\beta}-\frac{6\alpha^2}{25\beta}\coth\left[\frac{\alpha}{10\beta}\left(x\mp\frac{6\alpha^2}{25\beta}t\right)\right]-\frac{3\alpha^2}{25\beta}\operatorname{csch}^2\left[\frac{\alpha}{10\beta}\left(x\mp\frac{6\alpha^2}{25\beta}t\right)\right]. \tag{3.2.70}$$

Example 3.2.4.2:

In order to seek explicit and exact special solutions of the generalized long–short wave equations

$$\begin{aligned} &iS_t+S_{xx}=\alpha LS \\ &L_t+\beta(|S|^2)_x=0, \end{aligned} \tag{3.2.71}$$

we firstly introduce a gauge transformation

$$S(x,t)=e^{i(kx+wt+\xi_0)}\varphi(x,t), \tag{3.2.72}$$

where $\varphi(x,t)$ is a real-valued function, k, w are two real constants to be determined, ξ_0 is an arbitrary constant. Substituting (3.2.72) into (3.2.71), we have

$$\varphi_{xx}-(w+\alpha k^2)\varphi-L\varphi=0, \tag{3.2.73}$$

$$\varphi_t+2\alpha k\varphi_x=0, \tag{3.2.74}$$

$$L_x+2\beta\varphi\varphi_x=0. \tag{3.2.75}$$

In view of (3.2.74), we suppose

$$\varphi(x,t) = \varphi(\xi) = \varphi(x - 2\alpha kt + \xi_1), \tag{3.2.76}$$

where ξ_1 is an arbitrary constant. Substituting (3.2.76) into (3.2.73), we infer that

$$L(x,t) = \frac{\varphi_{xx} - (w + \alpha k^2)\varphi}{\varphi} = \frac{\varphi^{"}(\xi)}{\varphi(\xi)} - (w + \alpha k^2). \tag{3.2.77}$$

Therefore, we can also assume

$$L(x,t) = \psi(\xi) = \psi(x - 2\alpha kt + \xi_1). \tag{3.2.78}$$

Substituting (3.2.78) into (3.2.75) and integrating the resultant equation yields

$$\psi(\xi) = \frac{\beta\varphi^2(\xi)}{2\alpha k} - C, \tag{3.2.79}$$

where C is an integration constant. Substituting (3.2.79) into (3.2.73), we obtain

$$\varphi^{"}(\xi) - (w + \alpha k^2 - C)\varphi(\xi) - \left(\frac{\beta}{2\alpha k}\right)\varphi^3(\xi) = 0. \tag{3.2.80}$$

Set l $= -(w + k^2 - C)$, m $= -\frac{\beta}{2\alpha k}$, then (3.2.80) becomes

$$\varphi^{"}(\xi) + l\varphi(\xi) + m\varphi^3(\xi) = 0. \tag{3.2.81}$$

We seek the solution of (3.2.81) in the form

$$\varphi = \sum_{i=0}^{N} a_i \phi^i, \tag{3.2.82}$$

where q_i are constants to be determined later and ϕ satisfy the Riccati equation

$$\phi' = a + b\phi + c\phi^2. \tag{3.2.83}$$

Where a, b and c are constants. If balancing $\phi"$ with ϕ^3 in Eq. (3.2.81), it is easy to show that N=1. Therefore, we use the ansatz

$$\varphi = q_0 + q_1\phi, \tag{3.2.84}$$

substituting Eq. (3.2.84) into Eq. (3.2.81) along with (3.2.83) and collecting all terms with the same power in ϕ^i (i=0, 1, 2, 3, 4) yields a set of algebraic system for q_0, q_1 ,1 and m, namely

$$q_1 ba + mq_0^3 + lq_0 = 0,$$

$$b^2 q_1 + 2caq_1 + 3mq_0^2 q_1 + lq_1 = 0,$$

$$3q_1 bc + 3mq_0 q_1^2 = 0,$$

$$2q_1 c^2 + mq_1^3 = 0,$$

for which, with the aid of matlab, we find

$$q_0 = \frac{1}{2b^2} - 2ca, \quad q_1 = -\sqrt[3]{2c^2}. \tag{3.2.85}$$

It is to be noted that the Riccati equation (3.2.83) can be solved using the homogeneous balance method.

Case: I. Let $\phi = \sum_{i=0}^{N} b_i \tanh^i \xi$. Balancing ϕ' with ϕ^2 leads to

$$\phi = b_0 + b_1 \tanh \xi. \tag{3.2.86}$$

Substituting Eq. (3.2.86) into (3.2.83), we have the following solution of Eq. (3.2.83)

$$\phi = -\frac{1}{2c}(b + 2\tanh \xi), \quad ac = \frac{b^2}{4} - 1. \tag{3.2.87}$$

Substituting Eqs. (3.2.87) and (3.2.85) into (3.2.84), (3.2.78) and (3.2.72), we have the following new traveling wave solution of Eq. (3.2.71), respectively.

$$L(x,t) = 2\frac{\sqrt[3]{2c^2}}{c}\tanh(\xi)(1-\tanh^2(\xi))\left(\frac{1}{2b^2} - 2ca - \frac{\sqrt[3]{2c^2}}{2c}(b + 2\tanh(\xi)),\right)^{-1} - (w + \alpha k^2), \tag{3.2.88}$$

$$S(x,t) = \left(\frac{1}{2b^2} - 2ca - \frac{\sqrt[3]{2c^2}}{2c}(b + 2\tanh(x - 2\alpha kt + \xi_1)),\right) e^{i(kx + wt + \xi_0)}, \tag{3.2.89}$$

Similarly, let $\phi = \sum_{i=0}^{N} b_i \coth^i \xi$, then we obtain the following new traveling wave soliton solutions of Eq. (3.2.71).

$$S(x,t)=\left(\frac{1}{2b^2}-2ca-\frac{\sqrt[3]{2c^2}}{2c}(b+2\coth(x-2\alpha kt+\xi_1)),\right)e^{i(kx+wt+\xi_0)}, \tag{3.2.90}$$

$$L(x,t)=2\frac{\sqrt[3]{2c^2}}{c}\coth(\xi)(1-\coth^2(\xi))-\left(\frac{1}{2b^2}-2ca-\frac{\sqrt[3]{2c^2}}{2c}(b+2\coth(\xi)),\right)^{-1}-(w+\alpha k^2) \tag{3.2.91}$$

Case: II. When c=1, b = 0, the Riccati equation (3.2.83) has the following solutions

$$\phi=\begin{cases}-\sqrt{-a}\tanh(\sqrt{-a}\xi), & a<0\\ -1/\xi, & a=0\\ \sqrt{a}\tanh(\sqrt{a}\xi), & a>0\end{cases} \tag{3.2.92}$$

When a < 0, we have

$$S(x,t)=\left(\frac{1}{2b^2}-2ca+\sqrt[3]{2c^2}(\sqrt{-a}\tanh\sqrt{-a}(x-2\alpha kt+\xi_1))\right)e^{i(kx+wt+\xi_0)}, \tag{3.2.93}$$

$$L(x,t)=-2\sqrt[3]{-4c^4a^3}\tanh\sqrt{-a}(\xi)(1-\tanh^2\sqrt{-a}(\xi))\left(\frac{1}{2b^2}-2ca+\sqrt[3]{2c^2}(\sqrt{-a}\tanh\sqrt{-a}(\xi))\right)^{-1}-(w+\alpha k^2). \tag{3.2.94}$$

where $\xi=x-2\alpha kt+\xi_1$.

When a= 0, we have

$$S(x,t)=\left(\frac{1}{2b^2}-2ca+\frac{\sqrt[3]{2c^2}}{(x-2\alpha kt+\xi_1)}\right)e^{i(kx+wt+\xi_0)}, \tag{3.2.95}$$

$$L(x,t)=\frac{2\sqrt[3]{2c^2}}{(x-2\alpha kt+\xi_1)^3}\left(\frac{1}{2b^2}-2ca+\frac{\sqrt[3]{2c^2}}{(x-2\alpha kt+\xi_1)}\right)^{-1}-(w+\alpha k^2). \tag{3.2.96}$$

When a > 0, we have

$$S(x,t)=\left(\frac{1}{2b^2}-2ca-\sqrt[3]{2c^2}(\sqrt{a}\tan\sqrt{a}(x-2\alpha kt+\xi_1)),\right)e^{i(kx+wt+\xi_0)}, \tag{3.2.97}$$

$$L(x,t)=2\sqrt[3]{2c^2}\sqrt{a^3}\tanh\sqrt{a}(\xi)(1+\tanh^2\sqrt{a}(\xi))\left(\left(\frac{1}{2b^2}-2ca-\sqrt[3]{2c^2}(\sqrt{a}\tan\sqrt{a}(\xi)),\right)\right)^{-1} \qquad (3.2.98)$$
$$-(w+\alpha k^2),$$

which contain a periodic-like solutions. Where $\xi = x-2\alpha kt+\xi_1$.

RESULTS

The recent methods are far faster than the old methods like method of seperation of variables and fourier series method in terms of both simplying of callculating and the converging to solution. Moreover, we can solve problems that we can not solve them with old method. In last years, it was found the most of these methods to solve NPDEs, considering this in future it will be found newer, faster methods convering to solution of nonlinear partial differential equations

ADVICES

In this thesis, we were interested in the new method in a restricted way, that is, out of the methods we dissused are many method to solve nonlinear partial differential equation. It can be learn these method from database in internet. Also, if it asked to know about PDEs more, you can refered to the book called ‘Nonlinear Partial Differential Equations for Scientist and Engineers, by Lokenath Debnath’. For more examples about methods which we handled, you can refered to papers relating the so-called methods.

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