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(MASTER THESIS)

RIESZ SPACES OF REAL CONTINUOUS FUNCTIONS

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KABUL VE ONAY SAYFASI

Emel AYDIN tarafından Yüksek Lisans tezi olarak sunulan "Riesz Spaces of Real Continuous Functions" başlıklı bu çalışma Y. Ü. Lisansüstü Eğitim ve Öğretim Yönetmeliği ile Y. Ü. Fen Bilimleri Enstitüsü Eğitim ve Öğretim Yönergesinin ilgili hükümleri uyarınca tarafımızdan değerlendirilerek savunmaya değer bulunmuş ve .... tarihinde yapılan tez savunma sınavında aday oybirliği-oyçokluğu ile başarılı bulunmuştur.

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Yüksek lisans tezi olarak sunduğum "Riesz Spaces on Real Continuous Functions" adlı çalışmanın tarafımdan bilimsel ahlak ve geleneklere aykırı düşecek bir yardıma başvurmaksızın yazıldığını ve yararlandığım eserlerin referanslarda gösterilenlerden oluştuğunu, bunlara atıf yapılarak yararlanılmış olduğunu belirtir ve bunu onurumla doğrularım.

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ÖZET

SÜREKLİ REEL FONKSİYONLARIN RİESZ UZAYI

AYDIN, Emel

Yüksek Lisans Tezi, Matematik Bölümü

Tez Danışmanı: Prof. Dr. Mehmet TERZİLER

İkinci Tez Danışmanı: Prof. Dr. Zafer ERCAN

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Bu tez dört bölümden oluşmaktadır. İlk üç bölüm Riesz Uzay Teorisi'nin bazı temel sonuçlarını ve sondaki ana bölüm için gerekli sonuçları içermektedir.

Son bölümün amacı: Riesz uzaylarının en ana örneği $C(X)$ uzaylarıdır. Sözle ifade edersek, topolojik bir X uzayı için, gerçel değerli sürekli fonksiyon uzaylarıdır. İyi bilinir ki, düzenli tam Riesz uzayı, eğer Sıralı Unit'e sahipse, kompakt bir Hausdorff uzayı olan X için $C(X)$ uzayı olarak da gösterilebilir. Bu Kakutani Gösterilim Teorisi olarak bilinir. Son bölümde, Kakutani Gösterilim Teoremi, Montalvo, F., Pulgarín, A., Requejo, B. (2006). sürekli gerçel fonksiyonların Riesz uzayı makalesinde genelleştirilmiştir. Temel olarak, bazı özellikler altında Riesz uzayı, bazı kompakt olması gerekmeyen topolojik X uzayları için, $C(X)$ uzayı olarak gösterilebilir.

Anahtar Kelimeler: Sıralı Vektor Uzayları, Kafes, Riesz Uzayı, Riesz Uzayında Idealler, Riesz Uzayında Hull-Kernel Topoloji, e-uniformly complete, e-semisimple, e-separating, 2-universally e-complete.

ABSTRACT

RIESZ SPACES OF REAL CONTINUOUS FUNCTIONS

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MSc. in Mathematics

Supervisor: Prof. Dr. Mehmet TERZİLER

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This thesis contains four chapters. The first three chapters contain some basic results of the Riesz Spaces Theory and some results which are necessary for the last main chapter.

The aim of the last chapter: One of the main example of Riesz spaces are $C(X)$-spaces, namely the space of real valued continuous functions on a topological space X. It is well known that a uniformly complete Riesz space can be represented as $C(X)$-space for some compact Hausdorff space if the Riesz space has an order unit. This is known as Kakutani Representation Theorem. In the last chapter, Kakutani Representation Theorem is generalized, via the paper Montalvo, F., Pulgarín, A., Requejo, B. (2006). Riesz spaces of real continuous functions.

Mainly, it will be shown that under certain conditions a Riesz space can be represented as $C(X)$-space for some topological space X, which is not necessarily compact.

Key words: Ordered Vector Space, Lattice, Riesz Space, Ideals in Riesz Space, Hull-Kernel Topology on Riesz Space, e-uniformly complete, e-semisimple, e-separating, 2-universally e-complete.

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Introduction

Throughout X denotes a completely regular Hausdorff space and $C(X)$ the set of real continuous functions on X. A characterization topology has been recently obtained in [12]. In this thesis we use some methods to characterize $C(X)$ at the level of the Archimedean Riesz spaces. The first and the most celebrated result in this direction was due to Yosida [16] in 1942: *An Archimedean Riesz Space E is isomorphic to $C(K)$ for some compact Hausdorff space K if and only if E contains a strong order unit $e > 0$ for which it is uniformly closed.* On spite of the numerous contributions realized, a very little effort has been made to extend this theory for more general topological spaces. Concretely, the unique approach that we have been able to find in the literature fort he completely regular case is that of Xiong [15] in 1989, although the solution is "external" in character requiring a very strong assumption (recall that by "internal" we mean either arithmetic conditions or assertions about certain subspaces). Supposing that an Archimedean Riesz space E contains a weak order unit $e > 0$, we define in an internal way to be e-uniformly complete and e-semisimple in order to represent E as a uniformly closed Riesz subspace of $C(X)$ for a convenient completely regular Hausdorff space X. Moreover, the space E^* of bounded elements of E is isomorphic to $C(K)$ for some compact Hausdorff space K. The main task in the paper consist in finding internal conditions for $C^*(X)$ and $C(K)$ to be isomorphic. We introduce the notion of e-separation to this aim. We conclude, by using an interesting property called 2-universal e-completeness which is closely related with to be E inverse-closed in $C(X)$.

Chapter 1

Riesz Spaces

1.1 Vector Spaces

In this section we will give the definition of real vector spaces.

Definition 1.1.1 *A real vector space is a triple $(X, +, \cdot)$ where X is a non-empty set, "$+ : X \times X \to X$" and "$\cdot : \mathbb{R} \times X \to X$" are functions satisfying the following axioms:*

We write $x + y$ and rx instead of $+(x, y)$ and $\cdot(r, x)$, respectively.

1) *For each $x, y, z \in X$ i.e, $(X, +)$ is an abelian group. We have:*
a) $x + y = y + x$ (*commutative under addition*);

b) $(x + y) + z = x + (y + z)$ (*associative under addition*);

c) *There exists an element* $0 \in L$, *called* zero element *such that* $x + 0 = x$;

d) *There exists an element* $-x$, *called the* negative *of* x, such that

$$x + (-x) = 0;$$

2) $\alpha, \beta \in \mathbb{R}$ **a)** $\alpha(\beta x) = (\alpha\beta)x$ (*associative under multiplication*);

b) $1x = x$ (*There is a scalar with multiplicative identity*).

3) **a)** $(\alpha + \beta)x = \alpha x + \beta x$ (*distributive over scalars*);

b) $\alpha(x + y) = \alpha x + \alpha y$ (*distributive over vectors*).

The fundemantal examples of the vector spaces are function spaces, for example; the set of real valued functions on a set X is a vector space under the following operations.

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (rf)(x) = rf(x).$$

In particular the sequence spaces l_∞, c, c_0, l_p spaces are examples of vector spaces.

1.2 Partially Ordered Sets

In this section we introduce the definition of a partial order on X.

Definition 1.2.1 *Let X be a non-empty set and $\leq \subseteq X \times X$. (We write $x \leq y$ whenever $(x, y) \in \leq$). $\leq$ is called a partial order on X if:*

$\forall x, y, z \in X$

i) $x \leq x$ *(reflexive)*;

ii) $x \leq y$ *and* yRz *implies* $x \leq z$ *($\leq$ is transitive)*;

iii) $x \leq y$ *and* $y \leq x$ *implies* $x = y$ *($\leq$ is anti-symmetric)*.

The set X, equipped with a partial order, is called a partially ordered set.

Example 1.2.1 *Let X be a non-empty set. Then X is a partially ordered set under the following order:*

$$x \leq y :\Leftrightarrow x = y.$$

Example 1.2.2 *The usual order on $\mathbb{R}$ is a partial order.*

Example 1.2.3 *Let X be a non-empty set and E be the set of real valued functions on X. X is a partially ordered set under the order*

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \quad in \quad \mathbb{R}, \forall x \in \mathbb{R}.$$

Theorem 1.2.1 *Let $(X, \leq)$ be a partially ordered set. Then, for each nonempty subset Y of X, $(Y, \leq_Y)$ is a partially ordered set, where*

$$\leq_Y = \leq \cap \, (Y \times Y),$$

that is, $\leq_Y$ is the restriction of $\leq$ into Y.

In the other sections some different examples of partially ordered sets will be given.

Definition 1.2.2 *Let X be a partially ordered set. A subset A of X is* bounded above *if there exists $x \in A$ such that $a \leq x$ for each $a \in A$. In this case, x is called an* upper bound *of A. Similarly, A is called* bounded below *if there exists $x \in X$ satisfying $x \leq a$ for each $a \in A$ and, in this case x is called a* lower bound *of A.*

Definition 1.2.3 *Let X be an ordered set and $A \subset X$.*
An element $x \in X$ is called the least upper bound *of A if x is an upper bound of A and $x \leq y$ for each upper bound y of A. In this case x called* supremum *of A and it is denoted by* $\sup A$.
An element $x \in X$ is called the greatest lower bound *of A if x is a lower bound of A and $y \leq x$ for each lower bound of A. In this case x called* infimum *of A and it is denoted by* $\inf A$.

In a partially ordered set X if $x = \sup A$ and $x \in A$ then x is called *maximum* of A. In this case we write $x = \max A$. For eaxmple in $\mathbb{R}$, $\max(0,1] = 1$, but there is no maximum of the open interval $(0,1)$.

If $x = \inf A$ and $x \in A$ then x is called *minimum* of A. In this case we write $x = \min A$. For eaxmple in $\mathbb{R}$, $\min[0,1] = 0$, but there is no minimum of the open interval $(0,1)$.

If v is the least upper bound of a subset $A \subset E$, then we may write

$$v = \sup(A) = \bigvee_{x \in A} x = \sup\{x : x \in A\}$$

If u is the greatest lower bound of A, then we will write

$$u = \inf(A) = \bigwedge_{x \in A} x = \inf\{x : x \in A\}$$

Definition 1.2.4 *An ordered set $(E, \leq)$ is called* lattice *if for any two elements $x, y \in E$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist.*

Example 1.2.4 *Let X be a non-empty set.*

(i) *If $\mathcal{L}$ is the set of all subsets of L ordered by inclusion, then $\mathcal{L}$ is a lattice which has X as the greatest element.*

(ii) *Suppose that X is infinite. Let $\mathcal{N}$ be the collection of all subsets A of X such that either A is finite or the complement A^c of A is finite. It is easy to see that $\mathcal{N}$ is a lattice. Now consider a subset $Y \subset X$ such that $Y \notin \mathcal{N}$. $B = \{\{x\} : x \in Y\}$ is a subset of $\mathcal{N}$ which does not have any least upper bound in $\mathcal{N}$.*

1.3 Ordered Vector Spaces

Definition 1.3.1 *Let E be a vector space and $\leq$ be a partial order on E. E is called an* ordered vector space *if the following axiom is satisfied:*

$$x \leq y \Rightarrow \alpha x + z \leq \alpha y + z, \ \forall \alpha \geq 0, \ z \in E.$$

Example 1.3.1 *The real numbers with the usual order is an ordered vector space.*

Example 1.3.2 *Let E be the set of polynomials on $[0, 1]$. Under the pointwise order is E an ordered vector space, but it is not a lattice.*

Example 1.3.3 *Let $C([0, 1])$ be the set of real valued continuous functions on $[0, 1]$. Under the pointwise order $C([0, 1])$ is an ordered vector space and it is a lattice.*

Example 1.3.4 *Let E be a vector space. And let $L(E, \mathbb{R})$ be the set of linear functionals on E. (A linear functional f on E is function from E into $\mathbb{R}$ such that*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \text{ for all } x, \ y \in E.)$$

Then $L(E, \mathbb{R})$ is an ordered vector space under the pointwise order.

1.4 Riesz Space

In 1928, at the International Mathematical Congress in Bologna, Italy, F. Riesz triggered the inverstigation of what is today called the theory of Riesz spaces. Soon after, in the mid-thirties, H. Freudenthal and L. V. Kantorovich indepently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz spaces. From then on the growth of the subject was rapid. In the forties and early fifties the Japanese school led by H. Nakano, T. Ogasawara, K. Yosida and the Russian school, let by L. V. Kantorovich, A. I. Judin, and B. Z. Vulikh, made fundamental contributions.

Definition 1.4.1 *Let E be an ordered vector space with the additional property. If $(E, \leq)$ is a lattice, then E is called a* Riesz space *(or* vector lattice*).*

Example 1.4.1 *Let $\mathbb{R}^n (n \geq 1)$ be the real linear space of all real n-tuples*

$$f = (f_1, \ \ldots, \ f_n)$$

with coordinatewise addition and multiplication by real numbers. If we define that $f \leq g$ means that $f_k \leq g_k$ holds for $1 \leq k \leq n$, then $\mathbb{R}^n$ is a Riesz space with respect to this introduced partial ordering.

Example 1.4.2 *If L is the real linear space of all real finite valued functions $f(x)$ on the arbitrary non-empty point set X with pointwise addition and multiplication by real constants, then L is a Riesz space with respect to the partial ordering introduced by defining that $f \leq g$ means that $f(x) \leq g(x)$ holds for every $x \in X$.*

Example 1.4.3 *Let $C(X)$ be the vector space of all real continuous functions on the topological space X. The space $C(X)$ is partially ordered by defining that $f \leq g$ holds whenever $f(x) \leq g(x)$ for all $x \in X$. Indeed,*

(i) $f \leq f$
($f(x) \leq f(x)$ for all $x \in X$). (reflexive)

(ii) $f \leq g$ *and* $g \leq f$
($f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all $x \in X$). Then $f(x) = g(x)$, $\forall x \in X$. So $f = g$ (anti-symmetric)

(iii) $f \leq g$ *and* $g \leq h$
($f(x) \leq g(x)$ and $g(x) \leq h(x)$ for all $x \in X$). Then $f(x) \leq h(x)$, $\forall x \in X$. So $f \leq h$ (transitive)

And $C(X)$ implies following axioms.

(i)

$$\begin{aligned} f \leq g &\Rightarrow f(x) \leq g(x),\ \forall x \in X. \\ &\Rightarrow f(x) + h(x) \leq g(x) + h(x),\ \forall h \geq 0,\ \forall x \in X. \\ &\Rightarrow f + h \leq g + h. \end{aligned}$$

(ii)

$$\begin{aligned} f \geq 0 &\Rightarrow f(x) \geq 0,\ \forall x \in X. \\ &\Rightarrow \alpha f(x) \geq 0,\ \forall x \in X,\ \alpha \geq 0, \\ &\Rightarrow \alpha f \geq 0. \end{aligned}$$

Then $C(X)$ is an ordered vector space.

Now, we should show that for $f,\, g \in C(X)$, $\sup\{f,\, g\}$ exists in $C(X)$. (Then follows from the following inequality:

$$|(f \vee g)(x) - (f \vee g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)|).$$

Hence $C(X)$ is a Riesz space.

Example 1.4.4 $A = \{f | f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b \textit{ continuous}\}$

A is an ordered vector space with pointwise ordering.

Let $f(x) = x,\ g(x) = -x \in A$

$sup\{f(x),\ g(x)\} = f(x) \vee g(x) = x \vee (-x) = |x|,\ \forall x \in \mathbb{R}$

$sup\{f(x),\ g(x)\} = |x| \notin A$

So A is not a Riesz space.

Example 1.4.5 *Let X be a non-empty set and let $B(X)$ be the collection of all bounded real valued functions defined on X. It is a simple and well-known fact that $B(X)$ is a vector space which is ordered by the positive cone*

$$B(X)_+ = \{f \in B(X) | f(t) \geq 0 \textit{ for all } t \in X\}$$

Thus $f \geq g$ holds if and only if $f - g \in B(X)_+$. Obviously,

$$(f \vee g)(t) = \max\{f(t),\ g(t)\} \textit{ and } (f \wedge g)(t) = \min\{f(t),\ g(t)\}$$

for every $t \in X$ and $f,\ g \in B(X)$. This shows that $B(X)$ is a Riesz space.

Example 1.4.6 *A function space is a vector space E of real valued functions on a set Ω such that for each pair $f,\ x \in E$ the functions*

$$[f \vee g](w) := \max\{f(w),\ g(w)\} \textit{ and } [f \wedge g](w) := \min\{f(w),\ g(w)\}$$

both belong to E. Clearly, every function space E with the pointwise ordering (i.e. *$f \leq g$ holds in E iff $f(w) \leq g(w)$ for all $w \in \Omega$) is a Riesz space.*

Theorem 1.4.1 *For arbitrary elements $x,\ y,\ z$ of a Riesz space, the following identities hold:(Aliprantis, 1985)*

1. $x \vee y = -[(-x) \wedge (-y)]$ *and* $x \wedge y = -[(-x) \vee (-y)]$;
2. $x + y = (x \wedge y) + (x \vee y)$;
3. $x + (y \vee z) = (x + y) \vee (x + z)$ *and* $x + (y \wedge z) = (x + y) \wedge (x + z)$; *and*
4. $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ *and* $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$ *for* $\alpha \geq 0$.

Proof: (1) From $x \leq x \vee y$ and $y \leq x \vee y$ we get

$$-(x \vee y) \leq -x \text{ and } -(x \vee y) \leq -y, \text{ and so } -(x \vee y) \leq (-x) \wedge (-y).$$

On the other hand, if

$$-x \geq z \text{ and } -y \geq z, \text{ then } -z \geq x \text{ and } -z \geq y,$$

and hence $-z \geq x \vee y$. That is, $-(x \vee y) \geq z$ holds, which shows that $-(x \vee y)$ is the infimum of the set $\{-x, -y\}$. Thus,

$$(-x) \wedge (-y) = -(x \vee y).$$

To get the identify for $x \wedge y$ replace x by $-x$ and y by $-y$ in the above proven identity.

(2) From $x \wedge y \leq y$ it follows that $x \leq x+y-x \wedge y$, and similarly $y \leq x+y-x \wedge y$. Hence,

$$x \vee y \leq x + y - x \wedge y, \text{ and so } x \wedge y + x \vee y \leq x + y.$$

On the other hand, from $y \leq x \vee y$ we see that $x + y - x \vee y \leq x$, and similarly,

$$x + y \leq x \wedge y + x \vee y,$$

and the identity follows.

(3) Clearly, $x + y \leq x + y \vee z$ and $x + z \leq x + y \vee z$, and thus

$$(x + y) \vee (x + z) \leq x + y \vee z.$$

On the other hand, we have $y = -x + (x + y) \leq -x + (x + y) \vee (x + z)$, and likewise

$$z \leq -x + (x + y) \vee (x + z), \text{ and so } y \vee z \leq -x + (x + y) \vee (x + z).$$

Therefore,

$$x + y \vee z \leq (x + y) \vee (x + z)$$

also holds, and thus

$$x + y \vee z = (x + y) \vee (x + z).$$

The other identity can be proven in a similar manner.

(4) Fix $\alpha > 0$.

Clearly, $(\alpha x) \vee (\alpha y) \leq \alpha(x \vee y)$. If $\alpha x \leq z$ and $\alpha y \leq z$ both hold, then

$$x \leq \alpha^{-1} z \text{ and } y \leq \alpha^{-1} z$$

also hold, and so $x \vee y \leq \alpha^{-1} z$.

This implies that $\alpha(x \vee y) \leq z$, which shows that $\alpha(x \vee y)$ is the supremum of the set $\{\alpha x, \alpha y\}$.

Therefore,

$$(\alpha x) \vee (\alpha y) = \alpha(x \vee y).$$

The other identity can be proven similarly.□

If A is a subset of a Riesz space for which $\sup A$ exists, then
(a) $\inf(-A)$ exists and

$$\inf(-A) = -\sup A;$$

(b) the supremum of the set $x + A := \{x + a : a \in A\}$ exists and

$$\sup(x + A) = x + \sup A;$$

(c) for each $\alpha \geq 0$ the supremum of the set $\alpha A := \{\alpha a : a \in A\}$ exists and

$$\sup(\alpha A) = \alpha \sup A.$$

Let E be a Riesz space. The set $\{x \in E : x \geq 0\}$ is called *positive cone* of E and it is denoted by E_+. In particular for each $x \in E$, we define

$$x^+ := x \vee 0,\ x^- := (-x) \vee 0, \text{ and } |x| := x \vee (-x).$$

The element x^+ is called the *positive part*, x^- the *negative part*, and $|x|$ the *absolute value* of x. The vectors x^+, x^-, and $|x|$ satisfy the following important properties.

Theorem 1.4.2 (*Aliprantis,1985*) *If x is an element of a Riesz space, then we have*

1. $x = x^+ - x^-$;

2. $|x| = x^+ + x^-$; *and*

3. $x^+ \wedge x^- = 0$.

Moreover, the decomposition in (1) *is unique in the sense that if* $x = y - z$ holds with $y \wedge z = 0$, then $y = x^+$ and $z = x^-$.

Proof: (1) From Theorem 1.4.1 we see that

$$x = x + 0 = x \vee 0 + x \wedge 0 = x \vee 0 - (-x) \vee 0 = x^+ - x^-.$$

(2) Using Theorem 1.4.1 and (1), we get

$$\begin{aligned} |x| &= x \vee (-x) = (2x) \vee 0 - x = 2(x \vee 0) - x \\ &= 2x^+ - x = 2x^+ - (x^+ - x^-) = x^+ + x^-. \end{aligned}$$

(3) Note that

$$\begin{aligned} x^+ \wedge x^- &= (x^+ - x^-) \wedge 0 + x^- = x \wedge 0 + x^- \\ &= -[(-x) \vee 0] + x^- = -x^- + x^- = 0. \end{aligned}$$

For the last part, let $x = y - z$ with $y \wedge z = 0$. Then by Theorem 1.4.1 we have

$$x^+ = (y - z) \vee 0 = y \vee z - z = (y + z - y \wedge z) - z = y.$$

Similarly, $x^- = z$. $\square$

Observe that if $T : E \to F$ is a positive operator between two Riesz spaces, then from $\pm x \leq |x|$ we see that $\pm Tx \leq T|x|$, and so

$$|Tx| \leq T|x|$$

holds for all $x \in E$.
In terms of the positive part, identity in Theorem 1.4.1(2) takes the following useful form:

$$x = (x - y)^+ + x \wedge y.$$

(To see this, note that

$$\begin{aligned} x &= x \vee y - y + x \wedge y = (x - y) \vee (y - y) + x \wedge y \\ &= (x - y) \vee 0 + x \wedge y = (x - y)^+ + x \wedge y. \end{aligned}$$

Regarding the absolute value, we have the following useful identities.)

Theorem 1.4.3 (*Aliprantis, 1985*) *If x and y are arbitrary elements in a Riesz space, then we have*

1. $x \vee y = \frac{1}{2}(x + y + |x - y|)$ *and* $x \wedge y = \frac{1}{2}(x + y - |x - y|)$;
2. $|x - y| = x \vee y - x \wedge y$;
3. $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|)$;
4. $|x| \wedge |y| = \frac{1}{2}||x + y| - |x - y||$.

Proof: (1) Note that

$$\begin{aligned} x + y + |x - y| &= x + y + (x - y) \vee (y - x) \\ &= [(x + y) + (x - y)] \vee [(x + y) + (y - x)] \\ &= (2x) \vee (2y) = 2(x \vee y). \end{aligned}$$

(2)Subtract the two identities in (1).

(3) Using (1), we have

$$\begin{aligned} |x + y| + |x - y| &= [x + y] \vee [-x - y] + |x - y| \\ &= [x + y + |x - y|] \vee [-x - y + |x - y|] \\ &= 2([x \vee y] \vee [(-x) \vee (-y)]) \\ &= 2([x \vee (-x)] \vee [y \vee (-y)]) \\ &= 2(|x| \vee |y|). \end{aligned}$$

(4) Using (1) and (3), we see that

$$\begin{aligned} ||x + y| - |x - y|| &= 2(|x + y| \vee |x - y|) - (|x + y| + |x - y|) \\ &= 2(|x| + |y|) - 2(|x| \vee |y|) \\ &= 2(|x| \wedge |y|). \square \end{aligned}$$

It should be noted that the formulas in (1) show that an ordered vector space is a Riesz space if and only if

$$|x| = x \vee (-x)$$

exists for each x.

In a Riesz space, two elements x and y are said to be *disjoint* (in symbols, $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. Note that by Theorem 4.2.1(4) we have $x \perp y$ if and only if $|x + y| = |x - y|$. Two subsets A and B of a Riesz space are called *disjoint* (denoted by $A \perp B$) whenever $a \perp b$ holds for all $a \in A$ and all $b \in B$. If A is a non-empty subset of a Riesz space E, then its *disjoint complement* A^d is defined by

$$A^d := \{x \in E : x \perp y \text{ for all } y \in A\}.$$

We write A^{dd} for $(A^d)^d$. Note that $A \cap A^d = \{0\}$.

If A and B are subsets of a Riesz space, then we shall write

$$\begin{aligned}
|A| &:= \{|a| : a \in A\}; \\
A^+ &:= \{a^+ : a \in A\}; \\
A^- &:= \{a^- : a \in A\}; \\
A \vee B &:= \{a \vee b : a \in A \text{ and } b \in B\}; \\
A \wedge B &:= \{a \wedge b : a \in A \text{ and } b \in B\}; \\
x \vee A &:= \{x \vee a : a \in A\}; \\
x \wedge A &:= \{x \wedge a : a \in A\}.
\end{aligned}$$

The next theorem tells us that every Riesz space satisfies the infinite distributive law.

Theorem 1.4.4 (*Aliprantis, 1985*) *Let A be non-empty subset of a Riesz space. If* $\sup A$ *exists, then* $\sup(x \wedge A)$ *exists for each x and*

$$\sup(x \wedge A) = x \wedge \sup A.$$

Similarly, if $\inf A$ *exists, then* $\inf(x \vee A)$ *exists for each x and*

$$\inf(x \vee A) = x \vee \inf A.$$

Proof: Assume that $\sup A$ exists, and let $y = \sup A$. Clearly, $x \wedge a \leq x \wedge y$ holds for all $a \in A$. Now let $x \wedge a \leq z$ for all $a \in A$. Since for each $a \in A$ we have

$$a = x \wedge a + x \vee a - x \leq z + x \vee y - x,$$

it follows that $y \leq z + x \vee y - x$.

That is, $x \wedge y = x + y - x \vee y \leq z$ holds, and this shows that $\sup(x \wedge A)$ exists and $\sup(x \wedge A) = x \wedge \sup A$ holds. The other statement can be proven in a similar manner.□

The next result includes most of the major inequalities that are used extensively.

Theorem 1.4.5 (*Aliprantis, 1985*) *For arbitrary elements x, y and z in a Riesz space the following inequalities hold:*

1. $||x| - |y|| \leq |x + y| \leq |x| + |y|$ (*the triangle inequality*);

2. $|x \vee z - y \vee z| \leq |x - y|$ *and* $x \wedge z - y \wedge z| \leq |x - y|$;

3. *If in addition x, y and z are all positive, then*

$$x \wedge (y + z) \leq x \wedge y + x \wedge z.$$

Proof: (1) Clearly, $x + y \leq |x| + |y|$ and $-x - y \leq |x| + |y|$ both hold. Thus, $|x + y| = (x + y) \vee (-x - y) \leq |x| + |y|$.
From this we see that $|x| = |(x + y) - y| \leq |x + y| + y$ and so $|x| - |y| \leq |x + y|$. Similarly, $|y| - |x| \leq |x + y|$ and hence $||x| - |y|| \leq |x + y|$ also holds.

(2) Note that

$$\begin{aligned} x \vee z - y \vee z &= [(x - z) \vee 0 + z] - [(y - z) \vee 0 + z] \\ &= (x - z)^+ - (y - z)^+ \\ &= [(x - y) + (y - z)]^+ - (y - z)^+ \\ &\leq [(x - y)^+ + (y - z)^+] - (y - z)^+ \\ &= (x - y)^+ \leq |x - y|. \end{aligned}$$

Similarly, $y \vee z - x \vee z \leq |x - y|$ and so $|x \vee z - y \vee z| \leq |x - y|$. The other inequality can be proven in a similar manner.

(3) Let $a = x \wedge (y + z)$. Then

$$a \leq x \text{ and } a \leq y + z \text{ and so } a - z \leq a \leq x$$

and $a - z \leq y$ from which it follows that $a - z \leq x \wedge y$. Thus, $a - x \wedge y \leq z$ and in view of $a - x \wedge y \leq a \leq x$, we see that $a - x \wedge y \leq x \wedge z$. That is, $a \leq x \wedge y + x \wedge z$ holds, as required. □

In particular, note that in a Riesz space we have

$$|x^+ - y^+| \leq |x - y|.$$

1.5 Archimedean Space

Definition 1.5.1 *The Riesz space E is said to be* Archimedean *if*

$$\inf \frac{1}{u} x = 0$$

holds for every $x \in E^+$.

Corollary 1.5.1 (*Zaanen, 1997*) *Let E be a Riesz space.*

(i) *The space E is Archimedean if and only if it is true for every $u \in E^+$ that $\inf\{\varepsilon_n u : n = 1, 2, \ldots\} = 0$ holds for any sequence (ε_n) of non-negative real numbers satisfying $\varepsilon_n \to 0$.*

(ii) *The space E is an Archimedean if and only if given u and v in E^+ such that $0 \leq nv \leq u$ for $n = 1, 2, \ldots$ it follows that $v = 0$. In other words, for any $u \geq 0$, $v = 0$, the sequence $(nv : n = 1, 2, \ldots)$ is not bounded above.*

Example 1.5.1 *$\mathbb{R}^2$ is an Archimedean space with coordinatewise ordering.*

Let (a, b), $(x_0, y_0) \in \mathbb{R}^2$

$$0 \leq (a, b) \leq \frac{1}{n} (x_0, y_0), \forall n \in \mathbb{N}$$
$$0 \leq (a, b) \leq inf(\frac{1}{n} (x_0, y_0)) = 0, \forall n \in \mathbb{N}$$
$$So\ a, b = 0$$

But $\mathbb{R}^2$ is not an Archimedean space with lexicographical ordering. i.e.

$$(x_1, y_1) \leq (x_2, y_2) :\Leftrightarrow x_1 < x_2 \ or\ (x_1 = x_2 \ and\ y_1 \leq y_2)$$

$(0, 1)$, $(1, 0) \in \mathbb{R}^2$,

$$(0, 0) \leq n(0, 1) \leq (1, 0), \forall n \in \mathbb{N} \ or\ (0, 0) \leq (0, 1) \leq \frac{1}{n} (1, 0), \forall n \in \mathbb{N}$$

Then $\inf(\frac{1}{n} (1, 0)) = (0, 0)$

But $(0, 1) \neq (0, 0)$

Example 1.5.2 *$C[0,\,1]$ is an Archimedean space. The ordering is given by*

$$f \leq g :\Leftrightarrow f(x) \leq g(x),\ \forall x \in [0,\,1]$$

Let $f,\ g \in C[0,\,1] = \{f | f : X \to \mathbb{R}$ continuous$\}$

$$\begin{aligned} nf \leq g &\Rightarrow f(x) \leq \frac{1}{n}\, g(x),\ \forall n \in \mathbb{N},\ \forall x \in [0,\,1] \\ &\Rightarrow f(x) = 0,\ \forall x \in [0,\,1] \\ &\Rightarrow f = 0 \end{aligned}$$

Example 1.5.3 *Assume that $n \geq 2$. We define the lexicographical order on $\mathbb{R}^n$ in the following way.*

$$x = (x_1, \ldots,\, x_n) \leq (y_1, \ldots,\, y_n) = y$$

if there exists $k \in \{0,\ \ldots,\, n\}$ such that

$$x_1 = y_1,\ \ldots,\ x_k = y_k \text{ and } x_{k+1} < y_{k+1}$$

It can easily be checked that $\mathbb{R}^n$ equipped with this order is a Riesz space. Further, it is totally ordered such that the order is non-Archimedean: If

$$x = (0,\,1,\,0,\ \ldots) \text{ and } y = (1,\,0,\ \ldots,\,0),$$

then $nx \leq y$ for every $n \in \mathbb{N}$.

1.6 Uniformly Convergent

Definition 1.6.1 *A sequence (x_n) in a Riesz space is said to be* decreasing *(in symbols, $x_\alpha \downarrow$) whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$. The notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf(x_\alpha) = x$ both hold.*

Definition 1.6.2 *It is said that the sequence $\{f_n\}$ in the Riesz space L is order convergent to the element $f \in L$ whenever there exists a sequence $p_n \downarrow 0$ in L such that $|f - f_n| < p_n$ holds for all n. This will be denoted by $f_n \to f$.*

Definition 1.6.3 *A sequence (f_n) in an Archimedean Riesz space E with an element $e > 0$ of E is said to be* e-uniformly Cauchy *if for each $\varepsilon > 0$ there exists*

$N \in \mathbb{N}$ such that $|f_n - f_m| \leq \varepsilon e$ for all n, $m > N$. The sequence $\{f_n\}_n$ is e-uniformly convergent *to some $f \in E$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n - f| \leq \varepsilon e$ for all $n > N$.*

E said to be e-uniformly complete *when*
(I) *Each e-uniformly Cauchy sequence of E is e-uniformly convergent to some element $f \in E$.*

A subset D of E is said to be e-uniformly dense *whenever for each $f \in E$, there exists a sequence in D which is e-uniformly convergent to f.*

In $C(X)$ to be 1-uniformly Cauchy, 1-uniformly convergent, 1-uniformly complete and 1-uniformly dense, coincides with the classical definitions of to be uniformly Cauchy, uniformly convergent, uniformly complete and uniformly dense, respectively. Furthermore, the set $C(X)^*$ of 1-bounded elements of $C(X)$ coincides with the space $C^*(X)$ of bounded functions of $C(X)$, and both $C(X)$ and $C^*(X)$ are uniformly complete.

1.7 Order Unit and Weak Order Unit

Definition 1.7.1 *An element $e > 0$ in a Riesz space E is said to be an* order unit *whenever for each $x \in E$ there exists some $\lambda > 0$ with $|x| \leq \lambda e$.*

Definition 1.7.2 *An element $e > 0$ of an Archimedean Riesz space E is said to be a* weak order unit *if*

$$|f| \wedge e = 0 \text{ implies } f = 0$$

.

Theorem 1.7.1 *Let E be a vector lattice and $e \in E$ be an order unit of E. Then e is a weak order unit.*

Proof: Let $x \geq 0$ be an element of E.

There exist $\lambda > 0$ such that $x \leq \lambda e$. (e is an order unit)

$$x \wedge x \leq \lambda e \wedge x$$
$$x \leq \lambda e \wedge x = 0 (x \wedge e = 0 \Rightarrow \lambda e \wedge x = 0)$$
$$0 \leq x \leq 0$$
$$x = 0$$

Hence e is a weak order unit.

Example 1.7.1 $C(0,\ 1) = \{f|f : (0,\ 1) \to \mathbb{R}$ *continuous functions*$\}$ *is a Riesz space under the pointwise order.*

There is no order unit of $C(0,\ 1)$. *Indeed, suppose that* $e \in C(0,\ 1)$. *We can suppose that* $0 < e$ *for each* $x \in (0,\ 1)$. (*Otherwise* e *can be repliced by* $e+1$)

Let $f \in C(0,\ 1)$ *be defined by* $f(x) = \frac{e(x)}{x}$. *Then there exist* $\lambda > 0$ *such that*

$$f \leq \lambda e.$$

Thus for each $x \in (0,\ 1)$, $f(x) = \frac{e(x)}{x} \leq \lambda e(x)$. *This implies* $\frac{1}{x} \leq \lambda$ *for each* $x \in (0,\ 1)$.

So there is no order unit of $C(0,\ 1)$.

On the other hand, the function $e(x) = 1$ *is a weak order unit of* $C(0,\ 1)$.

$C(X)$, endowed with the pointwise order, is an Archimedean Riesz space with the weak order unit 1.

Given an Archimedean Riesz space E with a weak order unit $e > 0$, the set

$$E^* = \{f \in E : |f| \leq ne \text{ for some } n \in \mathbb{N}\}$$

of *e-bounded elements* of E is a Riesz subspace of E also with the weak order unit e.

1.8 $C(X)$ Space

Definition 1.8.1 $C(X)$ *is a set of real continuous functions. We denote that* $C(X) = \{f|f : X \to \mathbb{R}$ *continuous*$\}$.

Theorem 1.8.1 *The Riesz spaces* $C(X)$ *is uniformly complete.* (*Zaanen, 1971*)

Proof: Let $0 \leq f_0 \in C(X)$ and let $(f_n : n = 1,\ 2,\ \ldots)$ be an f_0-uniform Cauchy sequence, i.e., for every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that

$$|f_m(x) - f_n(x)| \leq \varepsilon f_0(x), \text{ for } x \in X;\ m,\ n \geq n_0.$$

It follows that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. In order to prove that f is continuous, fix $x_0 \in X$ and $\varepsilon > 0$, and take a neighborhood

V of x_0 such that $|f_{n_0}(x_0) - f_{n_0}| \le \varepsilon$ holds for all $y \in V$ and $|f_{n_0}(x_0) - f_{n_0}(y)| \le \varepsilon$ holds for all $y \in V$ and $n_0 = n_0(\varepsilon)$. It follows now from

$$|f(x_0) - f(y)| \le |f_{n_0}(x_0) - f_{n_0}(y)| + \varepsilon\{f_0(x_0) + f_0(y)\} \le \varepsilon + \varepsilon\{2f_0(x_0) + \varepsilon\}$$

holding for all $y \in V$, that f is continuous. This shows that $C(X)$ is uniformly complete. □

Chapter 2

Ideals in Riesz Spaces

2.1 Ideals

Definition 2.1.1 (i) *A linear subspace V of L is called A* Riesz subspace *if f, $g \in V$ implies $f \vee g \in V$.*

(ii) *A linear subspace A of L is called an* order ideal (*or* ideal) *if $f \in A$, $g \in L$ and $|g| \leq |f|$ implies $g \in A$.*

(iii) *An ideal B of L is called a* band *if it follows from*

$$D \subset B,\ D \neq \emptyset \text{ and } f_0 = supD$$

existing in L that $f_0 \in B$.

It is obvious that a band is an ideal and that an ideal is a Riesz subspace.

Example 2.1.1 $\mathbb{R}^2$ *with ordering by*

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1 \leq x_2) \text{ or } (x_1 = x_2 \text{ and } y_1 \leq y_2)$$

Let $B = \{(0,\ x) \in \mathbb{R}^2\ ,\ \forall x \in \mathbb{R}\}$

Let take $(0,\ x)$, $(0,\ y) \in B$.

Firstly we should show that B is a subspace of $\mathbb{R}^2$. To show that

$$\begin{aligned}(0,\ x) + (0,\ y) &= (0,\ x+y)\\ \alpha(0,\ x) &= (0,\ \alpha x)\\ (0,\ x) \in B \Rightarrow |(0,\ x)| &= (0,\ x) \vee (0, -x)\\ &= (0,\ |x|) \in B\end{aligned}$$

Then $|(0,\ x)| \in B$.

So B is a subspace of $\mathbb{R}^2$.

Now, we should show that B is an ideal in $\mathbb{R}^2$.

$$(0,\ 0) \leq g \leq f\ ,\ (f \in B)$$
$$(0,\ 0) \leq g \leq (0,\ x)$$

According to ordering, g should be like $g = (0,\ z),\ z \in \mathbb{R}$.

Hence $g \in B$.

So B is an ideal in $\mathbb{R}^2$.

Let take $D \subset B$ such that $D = \{(0, y) : -1 \leq y \leq -3\}$.

$\sup D = (0,\ -1)$

$$x \leq (0,\ -1)\ ,\ \forall x \in D$$
$$(x,\ y) \leq (0,\ -1)$$
$$a \leq (x_0,\ y_0),\ \forall a \in D$$
$$(0,\ -1 - \frac{1}{n}) \leq (x_0,\ y_0)$$
$$(0,\ -1) \leq (x_0,\ y_0),\ \forall n \in \mathbb{N}$$

Hence D is a band in $\mathbb{R}^2$.

Definition 2.1.2 *The smallest ideal including a given non-empty subset D of E is called the ideal generated by D. If D consists of one-element f, the ideal generated by f is called a principal ideal.*

The ideal A_D generated by the non-empty subset D can easily be described explicity. Indeed, A_D consists of all $g \in E$ such that

$$|g| \leq |a_1 f_1| + \cdots + |a_n f_n|$$

holds for appropriate $f_1,\ \ldots,\ f_n \in D$ and appropriate real $a_1,\ \ldots,\ a_n$ where n is also variable, of course.
In particular, the principal ideal generated by the element $f \in E$ consists of all g satisfying $|g| \leq |af|$ for an appropriate real a.

2.2 Maximal Ideals

Definition 2.2.1 *An ideal I in L is said to be* proper *if $I \neq L$. A* maximal ideal *is a proper ideal that is maximal among the proper ideals.*

Lemma 2.2.1 *Let E be an Archimedean Riesz space with a weak order unit $e > 0$. If M is a maximal ideal of E which does not contain e, then every element of the quotient space E/M is a real multiple of $[e]$ (= class of the element e in the quotient space). As a consequence, the real vector space E/M is isomorphic to $\mathbb{R}$ by identifying $[e]$ with $1 \in \mathbb{R}$. (Montalvo, 2009)*

Remark: Given a homomorphism $x \in Hom_e(E, \mathbb{R})$ the set

$$M_x = \{f \in E : x(f) = 0\}$$

is a vector subspace of E because x is linear. Since equality $x(|f|) = |x(f)|$ holds, it follows that $f \in M_x$ if and only if $|f| \in M_x$. Hence, if $f \in M_x$ and $|g| \leq |f|$, then $0 \leq x(|g|) \leq x(|f|) = 0$ and therefore $|g| \in M_x$, which implies $g \in M_x$. Thus, M_x becomes an ideal of E. Furthermore, $e \notin M_x$ and E/M_x is isomorphic to $\mathbb{R}$, hence M_x is maximal. On the other hand, if we consider a maximal ideal M of E which does not contain e, then the quotient morphism $\pi_M : E \to E/M = \mathbb{R}$ belongs to $\text{Hom}_e(E, \mathbb{R})$. Thus, there is a one-to-one correspondence between $\text{Hom}_e(E, \mathbb{R})$ and the set consisting in maximal ideals of E which does not contain e. Accordingly, the kernel of the Riesz representation is the set consisting in the intersection of all the maximal ideals of E which do not contain e.

2.3 Maximal Ideals of $C(X)$

Example 2.3.1 *Let X be a topological space.*

$C(X) = \{f | f : X \to \mathbb{R} \text{ continuous}\}$

Let $A(x) = \{f \in C(X) : f(x_0) = 0\}$

We should check that, A is an ideal in $C(X)$. For this, we should show that:

$0 \leq |g| \leq |f|$; $f \in A \Rightarrow g \in A$?

$$\begin{aligned} f \in A \Rightarrow |f| \in A &\Rightarrow |f|(x_0) = 0 \\ &\Rightarrow |g|(x_0) = 0 \\ &\Rightarrow g \in A \end{aligned}$$

Hence A is an ideal.

So, is it maximal ideal?

Let $A \subseteq B$ *is an ideal.*

Let $f \in B/A \Rightarrow f(x_0) = 1$ *and let* $a \in A \Rightarrow g(x_0) = 0$

Then $(f - e)(x_0) = f(x_0) - e(x_0) = 0 \Rightarrow (f - e) \in A \Rightarrow (f - e) \in B.$

$$Let f - e = g \in B \Rightarrow e = f - g$$
$$\Rightarrow e \in B$$

Let $h \in C(X)$.

$|h| \leq ||h||e$, $B = C(X)$. *Then A is maximal ideal.*

$f : X \to \mathbb{R}$ *if* X *is compact and* f *is continuous then* $C(X)$ *has maximum value.*

Let X be a compact Hausdorff space. Then

$$\{M|M \text{ is a maximal ideal }\} = \{\{f \in C(X)|f(x) = 0\} : x \in X\}$$

If $M \subset C(X)$ *is maximal ideal, then* $\exists x_0 \in X$ *such that*

$$M = \{f \in C(X)|f(x_0) = 0\}$$

Let $M \subset C(X)$ *is a maximal ideal. Suppose that for each*

$$x \in X \ , \ \exists f_x \in M \ , \ f_x(x) > 0$$

$\forall x$, $\exists O_x \subset X$ *is open,* $x \in O_x$ *and* $\forall a \in O_x$, $0 < f_x(a)$

$$X = \cup_{x \in X} O_x, \ \exists x_1, \ x_2, \ \ldots, \ x_n \in X \ ; X = \cup_{i=1} O_{x_i}$$

$h = f_{x_1} + f_{x_2} + \cdots + f_{x_n}$, $\forall t \in X$, $h(t) > 0$?

$$h(t) = f_{x_1}(t) + f_{x_2}(t) + \cdots + f_{x_n}(t) > 0 \text{ because } \exists O_i \to t \in O_i$$

$f \in C(X)$, $\forall x$, $f(x) > 0$, $\exists \varepsilon > 0$ *such that* $f(x) \geq \varepsilon$, $\forall x \in X$

Since X is compact, there exists $min_{x \in X} f(x) = f(x_0) > 0$.

We choose $\varepsilon = f(x_0)$ *or* $\forall n \in \mathbb{N}$, $O_n = \{x \in X : f(x) > \frac{1}{n}\}$

$X = \cup_{k=1}^{n} O_k \Rightarrow O_n \subseteq O_{n+1}$ $X = O_n$, $n \in \mathbb{N}$, $f(x) > \frac{1}{n}$

2.4 Prime Ideal

Definition 2.4.1 *A proper ideal I of a vector lattice E is called prime if $x \in E$, $y \in E$ and $x \wedge y \in I$ implies $x \in I$ or $y \in I$*

Theorem 2.4.1 (*Zaanen, 1971*) (i) *The ideal P is prime if and only if, for any ideals A, B satisfying $A \cap B \subset P$, one at least of $A \subset P$ or $B \subset P$ holds.*

(ii) *If $x_0 \in X$ and P is an ideal in X, maximal with respect to the property of not containing x_0* (i.e. *any ideal $P \subset Q$ such that x_0 is no member of Q satisfies $Q = P$*), *then P is prime.*

(iii) *Every maximal ideal in X is prime.*

Proof: (i) Let P be prime, and let A, B be ideals such that $A \cap B \subset P$. If neither $A \subset P$ nor $B \subset P$ holds, there exists elements $x \in A$ and $y \in B$ such that x and y are no members of P. It follows that

$$x \wedge y \in A \cap B \subset P, \text{ so } x \in P \text{ or } y \in P$$

since P is prime. Contradiction. Hence, one at least of $A \subset P$ or $B \subset P$ holds.

Conversely, let the ideal P have the property that if A, B are ideals such that $A \cap B \subset P$, then $A \subset P$ or $B \subset P$.

We have to prove that $x \wedge y \in P$ implies $x \in P$ or $y \in P$. To this end, let A and B be the ideals generated by x and y respectively, i.e., $A = \{z : z \leq x\}$ and $B = \{z : z \leq y\}$.

Then $A \cap B = \{z : z \leq x \wedge y\}$, and so $A \cap B \subset P$ on account of $x \wedge y \in P$. It follows that $A \subset P$ or $B \subset P$, so $x \in P$ or $y \in P$.

(ii) Let $x_0 \in X$, and P is an ideal in X maximal with respect to the property of not containing x_0. If P is not prime, there exist elements y, z not in P, such that

$$y \wedge z \in P.$$

The element x_0 is in the ideal generated by P and y, so $x_0 = p_1 \vee y_1$ for some $p_1 \in P$ and some $y_1 \leq y$. Similarly, x_0 is in the ideal generated by P and z, so

$x_0 = p_2 \vee z_1$ for some $p_2 \in P$ and some $z_1 \leq z$. Setting $p_3 = p_1 \vee p_2$, we have $x_0 \leq p_3 \vee y_1$ and $x_0 \leq p_3 \vee z_1$, so

$$x_0 \leq (p_3 \vee y_1) \wedge (p_3 \vee z_1) = p_3 \vee (y_1 \wedge z_1) \in P,$$

which implies $x_0 \in P$, thus contradicting out hypotheses. It follows that P is prime.

(iii) If P is a maximal ideal, and x_0 is any element of X not in P, then P is maximal with respect to the property of nor containing x_0, so P is prime by part (ii). □

Chapter 3

Hull-Kernel Topology on Riesz Spaces

3.1 Riesz Homomorphism

Definition 3.1.1 *An operator $T : E \to F$ between two Riesz spaces is said to be a* lattice (*or* Riesz) homomorphism *whenever*

$$T(x \vee y) = T(x) \vee T(y)$$

holds for all $x,\ y \in E$.

Observe that every lattice homomorphism T is necessarily a positive operator. (If $x \in E^+$, then $T(x) = T(x \vee 0) = T(x) \vee T(0) = [T(x)]^+ \geq 0$ holds in F.) Also, it is important to note that the range of a lattice homomorphism is a Riesz subspace.

Given an Archimedean Riesz space E with a weak order unit $e > 0$ we may consider the set

$$Hom_e(E, \mathbb{R}) = \{x : E \to \mathbb{R} \text{ Riesz homomorphism} : x(e) = 1\}.$$

3.2 Kernel of Riesz Homomorphism

Theorem 3.2.1 *If A is an ideal in the Riesz space E, the quotient space E/A is a Riesz space with respect to the ordering defined by given $[f]$ and $[g]$ in E/A, we shall write $[f] \leq [g]$ whenever there exist elements $f_1 \in [f]$ and $g_1 \in [g]$ satisfying $f_1 \leq g_1$.(Zaanen, 1997)*

Proof: We show first that E/A is an ordered vector space. To do this we have to prove the following statements:

(i) It is clear that $[f] \leq [f]$ for every $[f]$.

(ii) If $[f] \le [g]$ and $[g] \le [h]$, let f_1 in $[f]$, g_1 in $[g]$ and h_1 in $[h]$ satisfy $f_1 \le g_1$ and $g_2 \le h_1$. Then

$$f_1 \le g_1 = g_2 + (g_1 - g_2) \le h_1 + (g_1 - g_2).$$

Since $g_1 - g_2 \in A$, we have $h_1 + (g_1 - g_2) \in [h]$, which shows that $[f] \le [h]$.

(iii) Let $[f] \le [g]$ as well as $[g] \le [f]$. Then there exist f_1, f_2 in $[f]$ and g_1, g_2 in $[g]$ such that $f_1 \le g_1$ and $f_2 \le g_2$. It follows that

$$0 \le g_1 - f_1 \le (g_1 - f_1) + (f_2 - g_2) = (f_2 - f_1) + (g_1 - g_2) \in A,$$

so $g_1 - f_1 \in A$, i.e., $[f] = [g]$.

We have a partial ordering in E/A, we prove now that the ordering is compatible with the vector space structure. It is obvious that $[f] \le [g]$ implies $\alpha[f] \le \alpha[g]$ for every $\alpha \ge 0$. Finally, to show that $[f] \le [g]$ implies

$$[f] + [h] \le [g] + [h],$$

choose $f \in [f]$ and $g \in [g]$ satisfying $f \le g$ and choose $h \in [h]$ arbitrarily. Then

$$f + h \le g + h, \text{ so } [f + h] \le [g + h],$$

i.e., $[f] + [h] \le [g] + [h]$.

It remains to show that E/A is a Riesz space with respect to the ordering. Precisely, we shall prove that $[f] \vee [g]$ exists for all $[f]$, $[g]$ and equal to $[f \vee g]$. It is evident that $[f \vee g] \ge [f]$ as well as $\ge [g]$, so it will be sufficient to show that any upper bound $[h]$ of $[f]$ and $[g]$ satisfies $[h] \ge [f \vee g]$. Given that $[h]$ is an upper bound, choose elements f, g, h in $[f]$, $[g]$, $[h]$ respectively. Then there exist q_1, q_2 in A such that

$$h \ge f + q_1 \text{ and } h \ge g + q_2,$$

so for $q = q_1 \wedge q_2 \in A$ we have $h \ge f + q$ and $h \ge g + q$. Therefore,

$$h \ge (f + q) \vee (g + q) = (f \vee g) + q.$$

Since $q \in A$, this shows that $[h] \ge [f \vee g]$, as desired. Note that it follows from $f + g = (f \vee g) + (f \wedge g)$ that $[f] \wedge [g]$ exists and satisfies

$$[f] \wedge [g] = [f \wedge g]$$

The Riesz space E/A is called the *quotient* Riesz space of E modulo the ideal A. The mapping $f \to [f]$ is a Riesz homomorphism of E onto E/A and A is the kernel of the homomorphism. This shows that every quotient space of E modulo some ideals is a Riesz homomorphic image of E.

Conversely, let T be a Riesz homomorphism of the Riesz space E into the Riesz space F. The image $T(E)$ of E is a Riesz subspace of F, so T is a Riesz homomorphism onto the Riesz space $T(E)$. Let A be the kernel of T. Then A is an ideal in E, and the mapping $[f] \to Tf$ is now a Riesz isomorphism of E/A onto $T(E)$. Hence E/A and $T(E)$ are Riesz isomorphic.

This shows that every Riesz homomorphic image of a Riesz space E is Riesz isomorphic to the quotient space of E modulo the kernel of the homomorphism.

3.3 Hull-Kernel Topology

Definition 3.3.1 *Let $\mathcal{M}$ denote the set of all maximal ideals of E. For any subset A of E^+ put*

$$A^{\Delta} = \{M \in \mathcal{M} : A \subset M\}$$

There exists a topology on $\mathcal{M}$ such that the sets $A^{\Delta}(A \subset E^+)$ are just the closed subsets of $\mathcal{M}$.

This topology is called the hull-kernel topology.

We shall denote by $Spec_e(E)$ the set $\mathrm{Hom}_e(E, \mathbb{R})$ equipped with the initial topology defined by E, that is, the weakest topology making continuous the functions

$$\widetilde{f} : Spec_e(E) \to \mathbb{R},\ x \to \widetilde{f}(x) = x(f)\ \ (f \in E).$$

The mapping $E \to C(Spec_e(E))$, $f \to \widetilde{f}$ is called the *Riesz representation* of E, and it is a Riesz homomorphism which maps e into 1.

It is well-known that the topological space $Spec_e(E)$ is a real compact space (in particular, completely regular and Hausdorff).

On $\mathrm{Hom}_e(E, \mathbb{R})$ it is also possible to consider the *hull-kernel topology* by choosing all sets of the form $\{coz(f) : f \in E\}$ as a subbase for such a topology (here

$$coz(f) = \{x \in \mathrm{Hom}_e(E, \mathbb{R}) : x(f) \neq 0\}$$

denotes the cozero-set of the element $f \in E$). Since

$$coz(f_1) \cap \ldots \cap coz(f_n) = coz(|f_1| \wedge \ldots \wedge |f_n|)$$

for any $f_1, \ldots, f_n \in E$, then $\{coz(f) : f \in E\}$ becomes a base for the hull-kernel topology. Next Lemma shows that both topologies coincide.

Lemma 3.3.1 *If E is an Archimedean Riesz space with a weak order unit $e > 0$, then both the initial topology defined by E and the hull-kernel topology coincide on* $\mathrm{Hom}_e(E, \mathbb{R})$. *((Montalvo, 2009)*

Proof: Fiven $f \in E$ and $\alpha, \beta \in \mathbb{R}$, a subbasic open set in $Spec_e(E)$ is

$$\begin{aligned} U &= \{x \in Spec_e(E) : \alpha < x(f) < \beta\} \\ &= \{x \in Spec_e(E) : x(f) \vee \alpha \neq \alpha\} \cap \{x \in Spec_e(E) : x(f) \vee \beta \neq \beta\} \\ &= coz(f \vee \alpha e - \alpha e) \cap coz(f \wedge \beta e - \beta e) \\ &= coz(|f \vee \alpha e - \alpha e| \wedge |f \wedge \beta e - \beta e|) \end{aligned}$$

and clearly $|f \vee \alpha e - \alpha e| \wedge |f \wedge \beta e - \beta e| \in E$

The following result is due to [16].

Lemma 3.3.2 *Let E be an Archimedean Riesz space with a weak order unit $e > 0$. The following statements hold:*

(i) *$Spec_e(E^*)$ is a non-empty compact Hausdorff space;*

(ii) *The Riesz representation $E^* \to C(Spec_e(E^*))$ is injective;*

(iii) *The image of E^* under its Riesz representation is uniformly dense in $C(Spec_e(E^*))$.*

Chapter 4

Representation of Riesz spaces as C(K) space

In this chapter, we will show that:

An Archimedean Rieesz space E is isomorphic to $C(X)$ for some completely regular Hausdorff space X if and only if there exists a weak order unit $e > 0$ for which E is e-uniformly complete, e-semisimple, e-separating and 2-universally e-complete.

4.1 e-semisimplicity

Definition 4.1.1 *Let E be an Archimedean Riesz space with a weak order unit $e > 0$. E is called* e-semisimple *if*

(II) The intersection of all the maximal ideals of E which do not contain e is 0.

To be E e-semisimple means that the image of E under the Riesz representation $E \to C(Spec_e(E))$ is a Riesz subspace of $C(Spec_e(E))$ isomorphic to E. It is clear that $C(X)$ is 1-semisimple since, as we have already mentioned, the Riesz representation $C(X) \to C(Spec_1(C(X))) = C(vX)$ is an isomorphism.

Proposition 4.1.1 *If E is an Archimedean Riesz space with a weak order unit $e > 0$, then the restriction morphism $Spec_e(E) \to Spec_e(E^*)$ becomes a homeomorphic embedding. Moreover, $Spec_e(E^*)$ is a compactification of $Spec_e(E)$ if and only if E is e-semisimple. (Montalvo, 2009)*

Proof: The injectivity of the restriction morphismis immediate: For any $x \in Spec_e(E)$ and $f \in E^+$, there exists $n \in \mathbb{N}$ such that $x(f) = x(f \wedge ne)$. Taking into account that the family $\{coz(f) : f \in E^*\}$ is a base for the topology of $Spec_e(E)$, we derive that such a topology must be the initial topology defined by the restriction morphism $Spec_e(E) \to Spec_e(E^*)$.

For the second statement, and since the equality $coz(f) = coz(|f| \wedge e))$ holds, it is clear that E is e-semisimple if and only if for $g \in E^*$, the condition $x(g) = 0$ for all $x \in Spec_e(E)$ implies $g = 0$. On the other hand, to be $Spec_e(E)$ dense in $Spec_e(E^*)$ is equivalent to assert that if $g \in E^*$ such that $x(g) = 0$ for all $x \in Spec_e(E)$, then

$$x(g) = 0 \text{ for all } x \in Spec_e(E^*).$$

By virtue of Lemma 3.3.2(ii), E^* is e-semisimple and therefore must be $g = 0$. We conclude that to be E e-semisimple is equivalent to be $Spec_e(E)$ dense in $Spec_e(E^*)$. □

Definition 4.1.2 *Let E be an Archimedean Riesz space with a weak order unit $e > 0$. A subset J of E is said to be a* vanished ideal *if it is an intersection of maximal ideals of E which do not contain e. We shall denote*

$$\mathcal{V}(E) = \{\textit{vanished ideals of } E\}.$$

By virtue of Lemma 3.3.1, when E is e-semisimple there is one-to-one correspondence between $\mathcal{V}(E)$ and the set consisting of nonempty closed subsets of $Spec_e(E)$ with the vanished ideal $I_F := \cap_{x\in F} M_x$ and any vanished ideal $J \in \mathcal{V}(E)$ with the non-empty closed subset $F_J := \cap_{f\in J} Z(f)$. We agree that E is a vanished ideal for which $F_E = \emptyset$ and $I_\emptyset = E$. On the one hand, the set consisting in closed subsets of $Spec_e(E)$ ordered by inclusion becomes a lattice: The infima and suprema of any finite collection of closed subsets is the intersection and the union of such closed subsets, respectively. Moreover, such a lattice has the smaller element $\emptyset$ and the bigger element $Spec_e(E)$. On the other hand, it is clear that the one-to-one correspondence between $\mathcal{V}(E)$ and the closes subsets of $Spec_e(E)$ reverses the inclusion. As a consequence we obtain:

Lemma 4.1.1 (*Montalvo, 2009*) *Let E be an Archimedean Riesz space with a weak order unit $e > 0$. If E is e-semisimple, then the set $\mathcal{V}(E)$ ordered by inclusion becomes a lattice with the smaller element 0 and the bigger element E.*

Definition 4.1.3 *We may define the following transitive relation on $\mathcal{V}(E)$:*

(a) *Given I, $J \in \mathcal{V}(E)$ we set $I < J$ in case there exists $H \in \mathcal{V}(E)$ such that $I \wedge H = 0$ and $H \vee J = E$.*

If E is e-semisimple, then $I \wedge H = 0$ if and only if $F_H \cup F_I = Spec_e(E)$ and $H \vee J = E$ if and only if $F_J \cap F_H = \emptyset$. Hence, the motivation for this notation is the following fact: $I < J$ if and only if $F_J \subseteq \overset{\circ}{F_I}$.

(b) *Given* $J \in \mathcal{V}(E)$ *we set* $J^{\perp} = \{f \in E : |f| \wedge |g| = 0 \text{ for all } g \in J\}$.

Since $|f| \wedge |g| = 0$ *if and only if* $f(coz(g)) = 0$, *it is clear that* $f \in J^{\perp}$ *if and only if*

$$0 = f(\cup_{g \in J} coz(g)) = f(Spec_e(E)/F_J)$$

and therefore $F_{J^{\perp}} = \overline{Spec_e(E)/F_J}$.

4.2 *e*-separation

Recall that non-empty subset S of a partially ordered set V is a *chain* whenever S endowed with the order induced by V becomes totally ordered.

Definition 4.2.1 *Let* E *be an Archimedean Riesz space with a weak order unit* $e > 0$. *A* separating chain *in* E *is a countable chain* $\mathcal{S}$ *of* $\mathcal{V}(E)$ *which satisfies*

(i) $\bigwedge \mathcal{S} = 0$ *and* $\bigvee \mathcal{S} = E$;
(ii) *If* I, $J \in \mathcal{S}$ *and* $I \subset J$, *then there exists* $H \in \mathcal{S}$ *such that* $I < H < J$.

E is said to be e-separating *in case*

(III) *For each two members* I, J *belonging to a separation chain in E, the inclusion* $I \subset J$ *implies that there exist* $f \in J$, $g \in I^{\perp}$ *such that* $f + g = e$.

The following lemma is due to Johnson and Mandelker, 1971. Recall that two subsets A and B of a topological space X are said to be *completely separated* if there exists $f \in C(X)$ such that $f(A) = 0$ and $f(B) = 1$.

Lemma 4.2.1 *Two subsets* A *and* B *of a topological space* X *are completely separated if and only if there exists a countable chain* $\mathcal{F}$ *of closed subsets of* X *satisfying the following conditions:(Montalvo, 2009)*

(i) $\cap \mathcal{F} = \emptyset$ *and* $\cup \mathcal{F} = X$;

(ii) *If* F, $G \in \mathcal{F}$ *and* $F \subset G$, *then there exists* $W \in \mathcal{F}$ *such that* $F \subseteq \overset{\circ}{W} \subseteq W \subseteq \overset{\circ}{G}$,

(iii) *There exist* C, $D \in \mathcal{F}$ *such that* $A \subseteq C \subset D \subseteq X/B$.

If $\mathcal{S}$ denotes a separating chain in $C(X)$ and $I, J \in \mathcal{S}$ are such that $I \subset J$, then $F_J \subset F_1 = vX(vX/F_1)$.

Accordingly with the previous lemma F_J and vX/F_1 are completely separated in vX. Taking into account the isomorphism $C(X) = C(vX)$, there exists $f \in C(X)$ such that $\widetilde{f}(F_1) = 0$ and $\widetilde{f}(vX/F_J) = 1$.

Then $\widetilde{f}\ \overline{(vX/F_J)} = 1$ because $\widetilde{f}$ is continuous, and we derive that $f \in I$ and $g = 1 - f \in J^{\perp}$ satisfy $f + g = 1$. Thus, $C(X)$ is 1-separating.

Theorem 4.2.1 *Let A be an Archimedean Riesz space with weak order unit $e > 0$ for which it is both e-semisimple and e-separating. If A, B is a pair of completely separated subsets of $Spec_e(E)$, then there exists*

$$h \in E^*,\, 0 \leq h \leq e$$

such that $x(h) = 0$ for every $x \in A$ and $x(h) = 1$ for every $x \in B$. (Montalvo, 2009)

Proof: Lemma 4.2.1 ensures that there exists a countable chain $\mathcal{F}$ of close subsets of $Spec_e(E)$ satisfying:

$$\cap\mathcal{F} = \emptyset \text{ and } \cup\mathcal{F} = Spec_e(E); \text{ if } F,\, G \in \mathcal{F}$$

and $f \subset G$, then there exists $W \in \mathcal{F}$ such that $F \subseteq \overset{\circ}{W} \subseteq W \subseteq \overset{\circ}{G}$; there exist C, $D \in \mathcal{F}$ such that

$$A \subseteq C \subset D \subseteq Spec_e(E)/B.$$

By virtue of the one-to-one correspondence between $\mathcal{V}(E)$ and the cloes subsets of $Spec_e(E)$, the family

$$\mathcal{S} = \{I_F : F \in \mathcal{F}\}$$

becomes a separating chain in E. Since I_D, $I_C \in \mathcal{S}$ and $I_D \subset I_C$, by hypothesis there exist $f \in I_C$ and $g \in I_D{}^{\perp}$ such that

$$f + g = e.$$

One the one hand, if $x \in A \subset C = F_{I_C}$, then $x(f) = 0$; on the other hand, if

$$x \in B \subset Spec_e(E)/D \subset \overline{Spec_e(E)/D} = F_{I_D{}^{\perp}},$$

then $x(g) = 0$ and therefore $x(f) = x(e - g) = 1 - x(g) = 1$. Considering $h = |f| \wedge e \in E^*$ we conclude the proof. □

4.3 2-universal e-completeness

Definition 4.3.1 *A sequence $(f_n)_n$ in an Archimedean Riesz space E with a weak order unit $e > 0$ is said to be 2-*disjoint *in case for each n, $|f_n| \wedge |f_k| \neq 0$ for at most two indices k distinct from n. E is said to be 2-*universally e-complete *provided that*

(IV) *For any 2-disjoint sequence $(f_n)_n$ of E such that for every maximal ideal M of E which does not contain e there is some m such that $f_m \notin M$, then $(f_n)_n$ has a least upper bound $\bigvee_n f_n$ in E.*

Theorem 4.3.1 (*Montalvo, 2009*) *Let E be an Archimedean Riesz space with a weak order unit $e > 0$ for which it is both e-semisimple and e-uniformly complete. The following conditions are equivalent:*

(i)*E is 2-universally e-complete;*

(ii) *If $f \in E$ is such that $x(f) \neq 0$ for every $x \in Spec_e(E)$, then the function $1/f \in C(Spec_e(E))$ belongs to E.*

Now, to verify that $C(X) = C(vX)$ is 2-universally 1-complete is immediate.

Main Result

Finally our main theorem yields:

Theorem 4.3.2 *An archimedean Riesz space E is isomorphic to $C(X)$ for some completely regular space X if and only if there exists a weak order unit $e > 0$ for which*
(I) *E is e-uniformly closed;*

(II) *E is e-semisimple;*

(III) *E is e-separating;*

(IV) *E is 2-universally e-complete.*

Proof: For the sufficiency, we have already seen along the paper that $C(X)$ satisfies (I)-(IV) for the weak order unit 1. For the necessity, condition (II) asserts that $Spec_e(E^*)$ is a compactification of $Spec_e(E)$ (see Proposition 4.1.1) If Z_1, Z_2 is a pair of disjoint zero-sets in $Spec_e(E)$, then conditions (II)-(III) ensure that there exists $h \in E^*$, $0 \leq h \leq e$ such that $h = 0$ on Z_1 and $h = 1$ on Z_2 (Theorem 4.2.1)

By (I), we have the isomorphism $E^* = c(Spec_e(E^*))$, and therefore

$$\{x \in Spec_e(E^*) : h(x) = 0\} \text{ and } \{x \in Spec_e(E^*) : h(x) = 1\}$$

are disjoint closed subsets in $Spec_e(E^*)$ containing Z_1 and Z_2 respectively. We have proven that disjoint zero-sets in $Spec_e(E)$ have disjoint closures in $Spec_e(E^*)$, which is equivalent to affirm that both $Spec_e(E^*)$ and $\beta Spec_e(E)$ are homeomorphic, and consequently we obtain the isomorphism $E^* = C^*(Spec_e(E))$.

Lastly, given $f \in C(Spec_e(E))$ both the function

$$f_1 = 1/(f^+ + 1) \text{ and } f_2 = 1/(f^- + 1)$$

belong to $C^*(Spec_e(E)) = E^*$ and satisfy $x(f_1) \neq 0$ and $x(f_2) \neq 0$ for every $x \in Spec_e(E)$.

By (II), (IV) and by applying Theorem 4.3.1, we have that $1/f_1$, $1/f_2 \in E$ and therefore

$$f = 1/f_1 - 1/f_2 \in E.$$

We conclude that E is isomorphic to $C(Spec_e(E))$.

References

[1] Aliprantis, C.D. and Burkinshaw O. (1985) Positive Operators. London: Academic Press, Inc.

[2] Birkhoff, G. (1948) Lattice Theory. New York City: American Mathematical Society, Volume XXV.

[3] De Jonge, E. and Van Rooij, A.C.M.(1977) Introduction To Riesz Spaces. Amsterdam: Mathematisch Centrum.

[4] Feldman, W.A. and Porter, J.F. (1981) The order topology for function lattices and realcompactness. Internat. L. Math. Math. Sci.**4**(2), 289-304.

[5] Fuchs, L. (1965-1966) Riesz vector spaces and Riesz algebras. Paris: Secrtariat Mathmatique.

[6] Jameson, G. (1970) Ordered Linear Spaces. Germany: Springer-Verlag.

[7] Johnson, D.G. and Mandelker, M. (1971) Separating chains in topological spaces. J. Lond. Math. Soc **4**(2), 289-304.

[8] Kelley, J.L. (1955) General Topology. Van Nostrand Reinhold, Princeton

[9] Luxemburg, W.A.J. and Zaanen, A.C. (1971) Riesz Spaces, Volume I. Amsterdam · London: North-Holland Publishing Company.

[10] Montalvo, F. and Pulgarin, A.A. and Requejo, B. (2006) 2-universally complete Riesz spaces and inverse-closed Riesz spaces. Indag.Math.New. Ser.**17**(2), 285-295.

[11] Montalvo, F. and Pulgarin, A.A. and Requejo, B. (2009) Riesz spaces of real continuous functions. Positivity. DOI10.1007 s11117-0031-6

[12] Montalvo, F. and Pulgarin, A.A. and Requejo, B. (2007) Zero-separating algebras of continuous functions. Topol.Appl. **154**(10), 2135-2141.

[13] Rudin, Walter. (1976) Principles of Mathematical Analysis. Singapore: McGraw-Hill, Inc.

[14] Schaefer, H.H. (1974) Banach Lattices and Positive Operators. New York: Springer-Verlag

[15] Xiong, H.Y. (1989) A characterization of Riesz spaces which are Riesz isomorphic to C(X) for some completely regular space X. Indag.Math. **51**(1),87-95.

[16] Yosida, K. (1942) On the representation of the vector lattices. Proc. Imp. Akad. Tokyo **18**, 339-342.

[17] Zaanen, A.C. (1997) Introduction to Operator Theory in Riesz Spaces. Germany: Springer-Verlag Berlin Heidelberg

CURRICULUM VITAE

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