ÖZET

GRAF İŞLEMLERİ ÜZERİNDE ZEDELENEBİLİRLİK PARAMETRELERİ

KANDİLCİ, Saadet

Yüksek Lisans Tezi, Matematik Bölümü

Tez Danışmanı: Yrd. Doç. Dr. Refet POLAT
İkinci Danışmanı: Yrd. Doç. Dr. Gökşen BACAK TURAN

Bir iletişim ağında, belli merkezlerin ya da bağlantıların zarar görmesinden sonra, iletişim kesilene kadar geçen süredeki ağın dayanma gücünün ölçümüne, zedelenebilirlik değeri denir. Bir iletişim ağı, zedelenebilirlik değerinin belirlenebilmesi için, merkezleri bir grafın tepelerine, bağlantıları grafın ayrıtlarına karşılık gelecek şekilde bir graf ile modellenir. Bilinen zedelenebilirlik parametrelerinden bazıları Bağlantılılık, Bütünlük, Komşu Bütünlük, Rupture Derecesi, Komşu Rupture Derecesi, Toughness, Tenacity, Scattering Sayısı'dır.

Bu tezde komşu rupture dereceleri üzerine çalışılmıştır, bazı özel graflara işlemler uygulanmış ve komşu rupture dereceleri hesaplanmıştır. Son olarak total graflar ve tümleyenleri incelenmiş neighbor rupture dereceleri hesaplanmıştır.

Anahtar Sözcükler: Zedelenebilirlik, Rupture Derecesi, Komşu Rupture Derecesi, Graf İşlemleri, Total Graflar.

ABSTRACT

VULNERABILITY PARAMETERS ON GRAPH OPERATIONS

KANDİLCİ, Saadet

Thesis Adviser: Assist. Prof. Dr. Refet POLAT

Second Adviser: Assist. Prof. Dr. Gökşen BACAK TURAN

The vulnerability shows the resistance of the network until communication breakdown after the disruption of certain stations or communication links. A communication network is modelled by a graph to measure the vulnerability as stations corresponding to the vertices and communication links corresponding to the edges. The well-known vulnerability parameters are Connectivity, Integrity, Neighbor Integrity, Rupture Degree, Neighbor Rupture Degree, Toughness, Tenacity, Scattering Number etc.

In this thesis the information about neighbor rupture degree is given. Then neighbor rupture degree of some graph operations are obtained. Finally total graphs and complement of total graphs are drawn and their neighbor rupture degree is studied.

Keywords: Vulnerability, Rupture Degree, Neighbor Rupture Degree, Graph Operations, Total Graphs.

TEŞEKKÜR / ACKNOWLEDGEMENTS

Yüksek lisans tezimi hazırlarken bana rehberlik eden ve desteğini eksik etmeyen danışman hocam Sayın Yrd. Doç. Dr. Refet POLAT'a, bu tez çalışması boyunca bilgi ve deneyimlerini esirgemeyen yardımcı danışman hocam Sayın Yrd. Doç. Dr. Gökşen BACAK TURAN'a, tezin biçimlenmesinde değerli katkılarını aldığım Sayın Dr. Mehmet KURT'a, hazırlık aşamasında yardımcı olan başta annem olmak üzere tüm sevdiklerime teşekkürü bir borç bilirim.

Saadet KANDİLCİ

İÇİNDEKİLER

TABLE OF CONTENTS

Page

INDEX OF FIGURE

Figure

Page

1. INTRODUCTION

Since the ancient times, the limited facilities with unlimited needs coverage need, has been the most important of life. That any need has needed another one day by day, so, that provided people have been always in a new question with this mentality, perhaps no unforeseen technological developments have been seen and people have been offered the use of them for the last 50 years. These technological developments have caused the creation of new research fields.

For example, the computers which were used to the calculation of financial jobs for big companies are now used by people individually to meet the ordinary needs. The research which were done in the past to make the computer faster and smaller in size now are replaced by new research about such machines which are not called computer today and how to use them efficiently as a part of a world-wide network. In fact, that world-wide network consists of local networks. Therefore the establishment of the network structure in the case of limited possibilities has become more important in the view of type of the transmitted data and the search of transmission of the data.

Recently lots of various kinds of problems related with the data transmission can be given, more reliable transmission, better high quality transmission and more continuous transmission and etc.

In this work unpredictable problems which can occur on the network and their effects on it have been studied. A network can be break down completely or partially with unexpected reasons. If the data does not transmit to the desired location that means there is a problem on the system. This problem can block a treaty of billions of liras or could occur big problem for human's life. In these days the reliability and the vulnerability of networks are so important. For that reason ***graphs*** are taken as a model in the research area of reliability and vulnerability of the networks. Each network center is taken as a vertex and the connections of these vertices are edges of a graph.

A few questions can be asked at this point;

How can the reliability and the vulnerability of network be determined? What are the factors of the reliability and the vulnerability? Answers can be given with this example, let's think about the way that you are using every day to work. What can be done if there is a problem on that way? We have two choices;

- We may give up going to work although we have the risk of dismissal.
- We can look for another way to work.

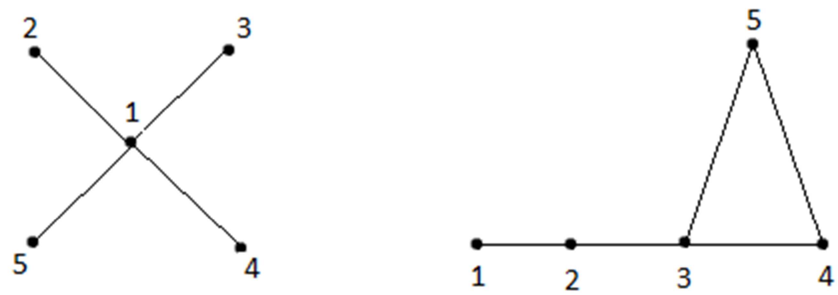
The question ‘if there is another way to reach work’ may come to our minds. In other words ‘Has the link connection between home and work completely broken down? ’. To answer these questions, we must know the dimensions of the problem between home and work, the vulnerability of the graph which represents the way between home and work should be searched.

1.1 Vulnerability Parameters of Graphs

In a communication network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. The well-known vulnerability parameters are Connectivity, Integrity, Neighbor Integrity, Rupture Degree, Neighbor Rupture Degree, Toughness, Tenacity, Scattering Number etc.

The well-known vulnerability parameter is connectivity which is defined by Harary in 1972.

Definition 1.1.1 (Harary, 1972) Connectivity k(G) is the minimum number of vertices that need to be removed in order to disconnect a graph.



$K_{1,4}$ $P_3 \cup K_3$

Figure1.1 Star graph $K_{1,4}$ and union of P_3 and K_3

When connectivity of $K_{1,4}$ and $P_3 \cup K_3$ are compared, it is obvious that both of the graphs have connectivity 1. If the vertex 1 from $K_{1,4}$ and vertex 2 from $P_3 \cup K_3$ are removed this makes these graphs disconnected. Although connectivity of these two graphs are equal, the effects of the removing vertices for the graph structures are different.

If vertex 1 in $K_{1,4}$ is removed all the vertices in the remaining graph are isolated but when vertex 2 is removed in $P_3 \cup K_3$, the graph is not detached completely. In this case it is clearly seen that we need more information about the components in the remaining graph to find the damage.

A question can be asked:

" Is connectivity an adequate measurement for vulnerability of any graph? "

The answer of the question can be found if the questions are answered that are listed in below.

1- The number of centers that are not functioning
2- The number of the connected networks
3- The size of the largest remaining group within which mutual communication can still occur
4- The number of centers which are connected to the disrupted vertices
5- The number of connected sub-networks which are still in communication after the disrupted centers and their neighbors are removed
6- The size of the largest remaining group in which the communication still occurs after removing the disrupted centers and its neighbors

The well-known connectivity parameter deals with the first question, but doesn't give any information about the rest questions. In 1987 Barefoot and Entringer introduced a new parameter, integrity, which deals with the questions 1 and 3.

Definition 1.1.2 (Barefoot and Entringer, 1987) The integrity of a graph G= (V, E) is defined by

$$I(G) = \min\{|S| + m(G-S)\};\ S \subset V(G)$$

where $m(G-S)$ denotes the order of largest component in $G-S$.

Although integrity deals with more questions than connectivity, it doesn't give any information about the number of components. A new vulnerability parameter, rupture degree, which related with component was introduced in 2005 by Li and Li.

Definition 1.1.3 (Li and Li, 2005) The rupture degree of an non-complete connected graph G is defined by

$$r(G)= \max\{w(G-S)-|S|-m(G-S): S \subset V(G),\ w(G-S) > 1\}$$

where w(G−S) denotes the number of components in the graph G−S and m(G−S) is the order of the largest component of G−S.

Connectivity, integrity, rupture degree and other parameters measure the vulnerability of graphs corresponding to a network. But most of the measurements are not interested in the effects of the neighbors of the disrupted vertices. A station or operative is captured; the adjacent stations will be betrayed and are therefore useless in the whole network. These networks are called *spy networks*.

Spy network is a system whose installation is hidden. For example, the spy whose name is A can communicate with spies B, C, D. If spy A dies, the other spies B, C or D can be inaccessible. Because of their inaccessibility their connections also become meaningless or if it is discovered that spy A actually services for another institution, B, C or D are thrown away from the system as a matter of safety. That means if a vertex is disrupted, all the other related vertices and connections of them can't be used anymore.

In this context, the concept of neighbor integrity was defined by Cozzens and Wu in 1996. Before the definition of neighbor integrity some information of neighborhood are needed.

Let G be a simple graph and let u be any vertex of G. The set $N(u) =\{u \in V(G) \mid v \neq u;$ v and u are adjacent$\}$ is the open neighborhood of u, and $N[u] =\{u\}\cup N(u)$ is the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood of u is removed from G. A set of vertices $S =\{u_1, u_2, \dots, u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. If S has been subverted from the graph G, then the remaining graph is called survival graph, denoted by G/S.

Definition 1.1.4 (Cozzens and Wu, 1996) The neighbor integrity of a graph G is defined by

$$NI(G) = \min\{|S| + c(G/S): S \subset V(G)\}$$

where S is any vertex subversion strategy of G and c(G/S) is the order of the largest component of G/S.

1.2 Neighbor Rupture Degree

As in the spy network example, there is a system which deals not only with the vertices but also deals with their neighbors in the below example.

Let's consider a company's distribution system as a graph structure. A firm has distributors in eighty one city and four sub-major distributors in every city and each of these distributors who sell products has 20 or 30 traders. All these traders reach to the final destination in other words reach to the markets. Each trader has approximately hundred markets in their portfolio. It means; if we begin from top;

F → city1, city2, …, city81

city1→ city1(d_1), city1(d_2), city1(d_3), city1(d_4)

city2→ city2(d_1), city2(d_2), city2(d_3), city2(d_4)

⋮

city81→ city81(d_1), city81(d_2), city81(d_3), city81(d_4)

city1(d_1) → city1($d_1 t_1$), city1($d_1 t_2$), city1($d_1 t_3$), ⋯ city1($d_1 t_{30}$)

⋮

city1(d_4) → city1($d_4 t_1$), city1($d_4 t_2$), city1($d_4 t_3$), ⋯ city1($d_4 t_{30}$)

If any distributor is disrupted, the traders who work with these distributors can't get their products. So these traders can't serve any products to their markets. In this system, if the vertex of distributor removes, the traders' vertices which are connected with this distributor vertex will be removed.

In this context, the concept of neighbor rupture degree was defined by Bacak-Turan and Kırlangıç in 2010.

Definition 1.2.1 (Bacak-Turan and Kırlangıç, 2010) The neighbor rupture degree of a non-complete connected graph G is defined to be

$$Nr(G) = \max\{w(G/S) - |S| - c(G/S) : S \subset V(G),\ w(G/S) \geq 1\}$$

where S is any vertex subversion strategy of G, w(G/S) is the number of connected components in G/S and c(G/S) is the maximum order of the components of G/S.

In particular, the neighbor rupture degree of a complete graph K_n is defined to be $Nr(K_n) = 1 - n$. A set $S \subset V(G)$ is said to be *Nr-set* of G if

$$Nr(G) = w(G/S) - |S| - c(G/S)$$

Example 1.2.1 Neighbor rupture degree of graph G in Figure 1.2.1 as shown below.

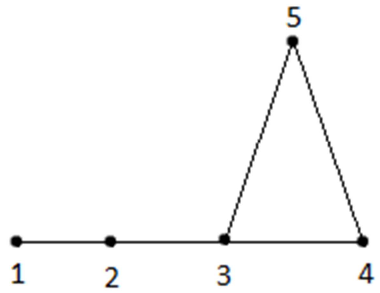


Figure 1.2 A graph of G

Solution: Let S be a subversion strategy of G.

If S = {1} then w(G/S) = 1 and c(G/S) = 3 thus we have $Nr_1(G) = -3$

If S = {2} then w(G/S) = 1 and c(G/S) = 2 thus we have $Nr_2(G) = -2$

If S = {3} then w(G/S) = 1 and c(G/S) = 1 thus we have $Nr_3(G) = -1$

If S = {4} or S = {5} then w(G/S) = 1 and c(G/S) = 2 thus we have $Nr_4(G) = -2$

If S = {1, 2} then w(G/S) = 1 and c(G/S) = 2 thus we have $Nr_5(G) = -3$

If S contains two or more vertices except (1, 2) then w(G/S) = 0 it contradicts to the definition

From the definition of neighbor rupture degree we have

$Nr(G) = \max\{Nr_1(G), Nr_2(G), Nr_3(G), Nr_4(G), Nr_5(G)\} = -1$

The results of neighbor rupture degree for some special graphs are listed in follows.

Theorem 1.2.1 (Bacak-Turan and Kırlangıç, 2010)

a) Let P_n be a path graph with n vertices and n ≥ 12,

$$Nr(P_n) = \begin{cases} 0, & n \equiv 1 \ (mod\ 4) \\ -1, & n \equiv 0, 2, 3 \ (mod 4) \end{cases}$$

b) Let C_n be a cycle graph with n vertices and n ≥ 15,

$$Nr(C_n) = \begin{cases} -1, & n \equiv 0 \ (mod\ 4) \\ -2, & n \equiv 1, 2, 3 \ (mod 4) \end{cases}$$

c) Let $K_{n1, n2, n3, \ldots, nk}$ be a k-partite graph

$$Nr(K_{n1,n2, \ldots ,nk}) = \max \{ n_1, n_2, n_3, \ldots, n_k\} - 3$$

d) Let $W_{1, n}$ be a wheel graph

$$Nr(W_{1, n}) = \begin{cases} -1, & n \equiv 1 \ (mod\ 4) \\ -2, & n \equiv 0, 2, 3 \ (mod 4) \end{cases}$$

The results of neighbor rupture for the upper and lower bound are listed in below.

Theorem 1.2.2 (Bacak-Turan and Kırlangıç, 2010) Let G be a graph of order n. Then

$$Nr(G) \geq n$$

Theorem 1.2.3 (Bacak-Turan and Kırlangıç, 2010) Let G be a graph of order n and *K(G)* be the neighbor connectivity of G. Then,

$$Nr(G) \leq n - 2K(G) - 1.$$

Theorem 1.2.4 (Bacak-Turan and Kırlangıç, 2010) Let G be a graph of order n. Then,

$$Nr(G) \leq \alpha(G) - NI(G).$$

Theorem 1.2.5 (Bacak-Turan and Kırlangıç, 2010) For any graph G, we have

$$Nr(G) \leq 2\alpha(G) - 2NI(G) - r(G).$$

Theorem 1.2.6 (Bacak-Turan and Kırlangıç, 2010) For any graph G,

$$Nr(G) \geq 3 - I(G) - NI(G) - r(G).$$

Corollary 1.2.1 (Bacak-Turan and Kırlangıç, 2010)

$$3 - I(G) - NI(G) - r(G) \leq Nr(G) \leq 2\alpha(G) - 2NI(G) - r(G)$$

2. NEIGHBOR RUPTURE DEGREE OF SOME GRAPH OPERATIONS

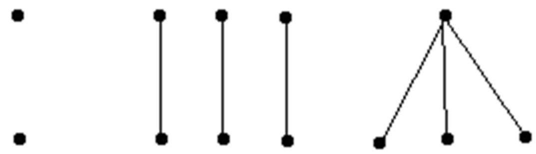
Operations on graphs produce new graphs from current ones. They may be separated into two major categories; unary operations and binary operations. Unary operations create a new graph from the current one. Binary operations create a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$.

2.1 Union of Graphs

In this part, neighbor rupture degree of union of some graphs are given.

The union $G = G_1 \cup G_2 \cup \ldots \cup G_i$ has $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_i)$ and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_i)$. If a graph G consists of k ($k \geq 2$) disjoint copies of a graph H, then we write G=kH (Chartrand and Lesniak, 1996).

Union of some graphs is given in Figure 2.1.1 below.



$2K_1 \cup 3K_2 \cup K_{1,3}$

Figure 2.1 Union of six graphs

Theorem 2.1.1 Let $G_1, G_2, G_3, \ldots, G_n$ be connected graphs, then

$$Nr(G_1 \cup G_2 \cup \ldots \cup G_n) \geq Nr(G_1) + Nr(G_2) + \ldots + Nr(G_n)$$

Proof Let $G = G_1 \cup G_2 \cup \ldots \cup G_n$ be the union of $G_1, G_2, \ldots, G_n$. Let $S_1, S_2, \ldots, S_n$ be Nr-sets of $G_1, G_2, \ldots, G_n$ respectively and let $S = S_1 \cup S_2 \cup \ldots \cup S_n$ be a subversion strategy of G. Then we obtain

$Nr(G) \geq w(G/(S_1 \cup S_2 \cup \ldots \cup S_n)) - |S_1 \cup S_2 \cup \ldots \cup S_n| - c(G/(S_1 \cup S_2 \cup \ldots \cup S_n))$

$= w(G_1/S_1) + w(G_2/S_2) + \ldots + w(G_n/S_n) - |S_1| - |S_2| - \ldots |S_n| - \max\{c(G_1/S_1), c(G_2/S_2), \ldots, c(G_n/S_n)\}$

$\geq w(G_1/S_1) + w(G_2/S_2) + \ldots + w(G_n/S_n) - |S_1| - |S_2| - \ldots |S_n| - c(G_1/S_1) - c(G_2/S_2) - \ldots - c(G_n/S_n)$

$= Nr(G_1) + Nr(G_2) + Nr(G_3) + \ldots + Nr(G_n)$.

Thus we have $Nr(G_1 \cup G_2 \cup \ldots \cup G_n) \geq Nr(G_1) + Nr(G_2) + \ldots + Nr(G_n)$. ■

Theorem 2.1.2 Let K_{n_1}, K_{n_2}, . . ., K_{n_m} be connected graphs with $n_1 \leq n_2 \leq \ldots \leq n_m$ and $n_{i+1} - n_i \geq 2; \forall i \in Z^+$. Then neighbor rupture degree is

$$Nr(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m}) = 2 - m - n_1.$$

Proof Let S be a subversion strategy of $K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m}$. Since these are complete graphs, it is obvious that S contains at most one vertex from each K_{n_i}.

If $|S| = k$, then $w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = m - k$ and $c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) \geq n_{m-k}$. Thus we have
$w((K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \ldots \cup K_{n_m})/S) - |S| - c((K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \ldots \cup K_{n_m})/S) \leq m - 2k - n_{m-k}$. Since $n_{i+1} - n_i \geq 2$; $n_{m-k} \geq n_1 + 2(m-k-1)$

$\leq m - 2k - n_1 - 2(m-k-1)$

$= 2 - m - n_1$

$\Rightarrow Nr \leq 2 - m - n_1$ . . . (1)

There exist S^* such that $|S^*| = m-1$, $w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = 1$ and $c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = n_1$. Then we have

$w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) - |S| - c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S)$

$= 2 - m - n_1$

$\Rightarrow Nr \geq 2 - m - n_1$ . . . (2)

From (1) and (2) we obtain $Nr(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m}) = 2 - m - n_1$. ■

Theorem 2.1.3 Let K_{n_1}, K_{n_2}, K_{n_3}, . . . , K_{n_m} be connected graphs with $n_1 \le n_2 \le n_3 \ldots \le n_m$ and for all i, $n_{i+1} - n_{i} \le 2$. Then neighbor rupture degree is

$$Nr(K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \ldots \cup K_{n_m}) = m - n_m.$$

Proof Let S be a subversion strategy of $K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m}$ and let |S|=k. Then we have $w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = m - k$ and $c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) \ge n_{m-k}$. Thus we get
$w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) - |S| - c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S)$
$\le m - 2k - n_{m-k}$

Let $f(k) = m - 2k - n_{m-k}$. Then $f(k+1) = m - 2(k+1) - n_{m-(k+1)}$ and
$f(k+1) - f(k) = -2 - n_{m-(k+1)} - n_{m-k}$. Since $n_{m-k} - n_{m-(k+1)} \le 2$ by the assumption, f(k+1) < f(k). Thus f is a decreasing function and takes its maximum value at k=0.
Hence we have $Nr \le m - n_m$ . . . (1)

There exist S^* such that $|S^*|=0$, $w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = m$ and
$c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) = n_m$. Then we get
$w((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S) - |S| - c((K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_m})/S)$
$= m - n_m$
Thus we have $Nr \ge m - n_m$ . . . (2)

From (1) and (2) we obtain $Nr = m - n_m$ ■

Corollary 2.1.1 $Nr(K_n \cup K_m) = \begin{cases} 2-n, & 2 > n-m \\ -m, & \text{otherwise} \end{cases}$

2.2. Join of Graph

Join operation is a binary operation. In this part neighbor rupture degree of join of some graphs are given.

The join G= $G_1 + G_2$ has V(G) = V(G_1) ∪ V(G_2) and

E(G) = E(G_1) ∪ E(G_2) ∪ { uv | u∈ V(G_1) and v∈ V(G_2)}

(Chartrand and Lesniak, 1996).

Join of graphs is given in Figure 2.2 below.

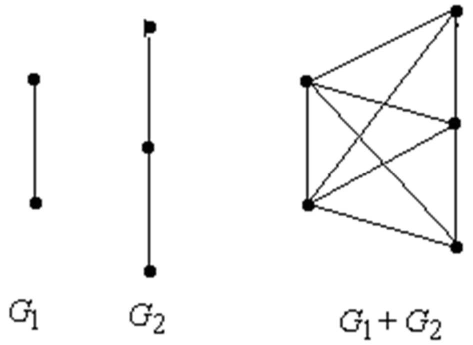


Figure 2.2 Simple graphs G_1, G_2 and their union

Theorem 2.2.1 Let G_1 and G_2 be two connected graphs then the neighbor rupture degree is

$$Nr(G_1 + G_2) = \max\{Nr(G_1), Nr(G_2)\}$$

Proof Let S be a subversion strategy of $G_1 + G_2$**.** There are three cases according to the elements of S.

Case1: Let $S = S_1 \subset V(G_1)$ be the Nr-set of G_1 such that $w(G_1/S_1) - |S_1| - c(G_1/S_1) = Nr(G_1)$. Since any elements from G_1 are adjacent to every element of G_2 in ($G_1 + G_2$), we have

$$w((G_1 + G_2)/S_1) - |S_1| - c((G_1 + G_2)/S_1) = w(G_1/S_1) - |S_1| - c(G_1/S_1) = Nr(G_1)$$

Case2: Let S= $S_2 \subset V(G_2)$ be Nr-set of G_2 such that $w(G_2/S_2) - |S_2| - c(G_2/S_2) = Nr(G_2)$. Since any elements from G_2 are adjacent to every element of G_1 in $(G_1 + G_2)$, we have

$w((G_1+G_2)/S_2) - |S_2| - c((G_1+G_2)/S_2) = w(G_2/S_2) - |S_2| - c(G_2/S_2) = Nr(G_2)$

Case3: Let $S \subset V(G_1) \cup V(G_2)$. Since S contains at least one vertex of $V(G_1)$ which is adjacent to all the vertices of $V(G_2)$ and S contains at least one vertex of $V(G_2)$ which is adjacent to all the vertices of $V(G_1)$ in $G_1 + G_2$, then $(G_1 + G_2)/S$ is empty. It contradicts to the definition of neighbor rupture degree.

Thus $Nr(G_1 + G_2) = \max\{Nr(G_1), Nr(G_2)\}$ ■

Theorem 2.2.2 Let G_1, G_2 and G_3 be connected graphs, then neighbor rupture degree is

$$Nr(G_1 + G_2 + G_3) = \max\{Nr(G_1 \cup G_3), Nr(G_2)\}$$

Proof: Let S be a subversion strategy of $G_1 + G_2 + G_3$.There are three cases according to elements of S.

Case1: Let $S \subset V(G_1 \cup G_3)$ be Nr-set of $G_1 \cup G_3$. Since every element of G_2 adjacent to every element from G_1 and G_3 we have

$$Nr(G_1 + G_2 + G_3) = Nr(G_1 \cup G_3)$$

Case2: Let $S \subset V(G_2)$ be Nr-set of G_2 . Since every element of G_2 adjacent to every element from G_1 and G_3 we have

$$Nr(G_1 + G_2 + G_3) = Nr(G_2)$$

Case3: Let $S \subset V(G_1 + G_2)$ or $S \subset V(G_1 + G_3)$ then $((G_1 + G_2 + G_3)/S)$ is empty which contradicts to definition of neighbor rupture degree.

Thus $Nr(G_1 + G_2 + G_3) = \max\{Nr(G_1 \cup G_3), Nr(G_2)\}$ ■

Corollary 2.2.1 If $G_2 \cong K_n$ then $Nr(G_1+G_2+G_3) \geq Nr(G_1)+Nr(G_3)$

Proof From the Theorem 2.2.2 we have $Nr(G_1+G_2+G_3)=Nr(G_1 \cup G_3)$ and from theorem 2.1.1 we obtain $Nr(G_1 \cup G_3) \geq Nr(G_1)+Nr(G_3)$ ■

■

Corollary 2.2.2 If $G = (K_{n_1}+K_{n_2}+K_{n_3})$ with $n_1 \geq n_2 \geq n_3$ then $Nr(G) = -n_3$

Proof From the Theorem 2.2.2 we have $Nr(G) = Nr(K_{n_3}) - 1 = 1 - n_3 - 1 = -n_3$ ■

2.3 Complement of Graphs

In this part, neighbor rupture degree of complement of some graphs are given. Complement of a graph is a unary operation.

The complement of a simple graph G is obtained by taking the vertices of G and joining two of them whenever they are not joined in G (Balakrishnan, 1995).

Complement of graph G is given in Figure 2.3 below.

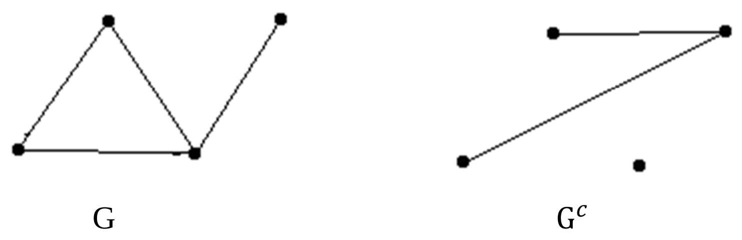


Figure 2.3 A simple graph G and its complement.

The complement K_n^c of the complete graph K_n has n vertices and no edges and is referred to as the empty graph of order n. A graph G is self-complementary if G^c=G.

Theorem 2.3.1 Let P_n be a path graph of order n. Then the neighbor rupture degree of complement of P_n is

$$\mathrm{Nr}\,(P_n^c) = -1$$

Proof: Let S be a subversion strategy of P_n^c and let S = {u} where u∈V(P_n).

Case1: If deg(u) = 1 in P_n then u is adjacent to all vertices in P_n^c except its neighbor in P_n. It means | N[u] |= n−1 in P_n^c then we have

$$w\,(P_n^c/S) - |S| - c(P_n^c/S) = 1-1-1 = -1$$

Case2: If deg(u) = 2 in P_n then u is adjacent to all vertices in P_n^c except its neighbors in P_n. It means |N[u]| = n−2 in P_n^c where the remaining two vertices are adjacent. Therefore

$$w\,(P_n^c/S) - |S| - c(P_n^c/S) = 1-1-2 = -2$$

Case3: If |S| ≥ 2 the remaining graph is empty. Therefore it contradicts to the definition of neighbor rupture degree.

From these three cases we have Nr (P_n^c) = −1. ■

Theorem 2.3.2 Let $W_{1,n}$ be a wheel graph of order n+1. Then the neighbor rupture degree of complement of $W_{1,n}$ is

$$\mathrm{Nr}\,(W_{1,n}^c) = -1$$

Proof Let u ⊂ V ($W_{1,n}$) be a centered vertex in $W_{1,n}$ with deg(u) = n and let v_i ⊂ V($W_{1,n}$) where i = 1,2,…,n with deg(v_i) = 3(i = 1,2,…,n). So u is isolated vertex in $W_{1,n}^c$ and |N[v_i]|= n−2 (i = 1,2,…n) in $W_{1,n}^c$.

Let S be subversion strategy of $W_{1,n}^c$. There are two cases according to the number of elements of S.

Case1: Let |S|=1

If S = {u} then w ($W_{1,n}^c$ /S) = 1, c ($W_{1,n}^c$ /S) = n. Thus we get

$$w(W_{1,n}^c/S) - |S| - c(W_{1,n}^c/S) = 1-1-n = -n$$

If S={v_i} where i=1,2,…,n-1 or n then w($W_{1,n}^c$ /S) = 2, c ($W_{1,n}^c$ /S) = 2. Thus we have

$$w(W_{1,n}^c/S) - |S| - c(W_{1,n}^c/S) = 2-2-1 = -1$$

Case2: Let $|S| \geq 2$

If S contains two adjacent vertices in $W_{1,n}$ the only remaining vertex in $W_{1,n}$/S is the vertex in the center. Therefore

$Nr \geq 1-|S|-1 = -|S| \geq -2$

If S contains two non-adjacent vertices in $W_{1,n}$ there will be at most two components with one vertex in $W_{1,n}$/S. Thus

$Nr \geq 2-|S|-1 = 1-|S| \geq -1$

From these cases we obtain Nr ($W_{1,n}^{c}$) = −1. ■

Theorem 2.3.3 Let $K_{m,n}$ (m<n) be a complete bipartite graph with |n-m|≥2. Then the neighbor rupture degree of $K_{m,n}^{c}$ is

$$Nr(K_{m,n}^{c}) = -\min\{m, n\}.$$

Proof It is obvious that $K_{m,n}^{c} = K_m \cup K_n$

Let the vertex $u \subset V(K_m)$ and the vertex $v \subset V(K_n)$ and let S be a subversion strategy of $K_m \cup K_n$. We have four cases according to the elements of S.

Case1: If |S|=0, then $c((K_m \cup K_n)/S) = \max\{m, n\}$ and $w((K_m \cup K_n)/S) = 2$. Therefore we have

$w((K_m \cup K_n)/S) - |S| - c((K_m \cup K_n)/S) = 2-0-\max\{m, n\} = 2-\max\{m, n\}$

Case2: If S = {u}, then $c((K_m \cup K_n)/S) = n$, $w((K_m \cup K_n)/S) = 1$. Thus we have

$w((K_m \cup K_n)/S) - |S| - c((K_m \cup K_n)/S) = = 1-1-n = -n$

Case3: If S = {v}, then $c((K_m \cup K_n)/S) = m$, $w((K_m \cup K_n)/S) = 1$. Therefore we have

$w((K_m \cup K_n)/S) - |S| - c((K_m \cup K_n)/S) = = 1-1-m = -m$

Case4: If S = {u, v} then $w((K_m \cup K_n)/S) = 0$ it contradicts to the definition of neighbor rupture degree.

Since |n-m|≥2 we have Nr ($K_{m,n}^{c}$) = −max {m, n}= − min{m, n}. ■

Corollary 2.3.1 Let $K_{1,n}$ be a star graph of order n+1. Then the neighbor rupture degree of $K_{1,n}^c$ is

$$\text{Nr}\,(K_{1,n}^c) = -1$$

2.4 Cartesian Product of Graph

In this part cartesian product operation are studied. Cartesian product is a binary operation.

The cartesian product G $= G_1 \times G_2$ has V(G) = V(G_1)×V(G_2), and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

$u_1 = v_1$ and $u_2 v_2 \in$ E(G_2)

or

$u_2 = v_2$ and $u_1 v_1 \in$ E(G_1). (Chartrand and Lesniak, 1996)

Cartesian product of graphs is given in Figure 2.4 below.

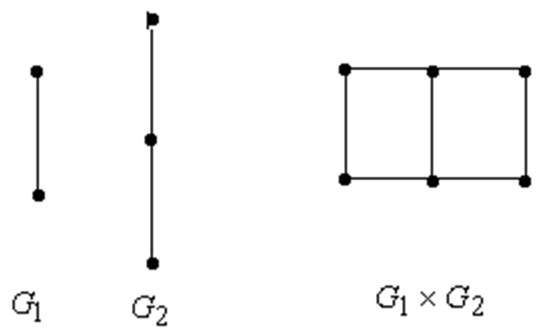


Figure 2.4 Two simple graphs G_1, G_2 and its Cartesian product

Theorem 2.4.1 Let $P_2 \times P_{3a}$ be a cartesian product with $a \in$ Z$^+$. Then neighbor rupture degree of $P_2 \times P_{3a}$ is

$$\text{Nr}(P_2 \times P_{3a}) = a - 1\ .$$

Proof Let S be a subversion strategy of $P_2 \times P_{3a}$ and |S|=r .There are two cases according to the number of elements in S.

Case1: Let $1 \le r \le a$. Then $N[S] \le 4r$ and $w \le 2r$.

Since c(($P_2 \times P_{3a}$)/S) $\ge \frac{|V(S)| - |N(S)|}{w} \ge \frac{2.(3a) - 4r}{2r} = \frac{3a}{r} - 2$ and

w(($P_2 \times P_{3a}$)/S) −|S|−c(($P_2 \times P_{3a}$)/S) $\le 2r - r - \left(\frac{3a}{r} - 2\right) = r - \frac{3a}{r+2}$

Let $f(r) = r - \frac{3a}{r+2}$. f is an increasing function since $f'(r) = 1 + \frac{3a}{r^2}$ >0. So it takes its maximum value at r =a. Then

$f(a) = a - \frac{3a}{a} + 2 = a - 1$. Hence

$$\mathrm{Nr}(P_2 \times P_{3a}) \le a - 1 \qquad \ldots (1)$$

Case2: If $a \le r \le |V(P_2 \times P_{3a})|$ then c (($P_2 \times P_{3a}$) /S) ≥ 1 and

w(($P_2 \times P_{3a}$)/S) $\le 2(3a) - 4a - (r - a) = 3a - r$ thus we obtain

w(($P_2 \times P_{3a}$)/S) −|S|−c(($P_2 \times P_{3a}$)/S) $\le 3a - r - r - 1 = 3a - 2r - 1$

Let f(r)= $3a - 2r - 1$. Since $f' < 0$ f is a decreasing function, so it takes its maximum value at r =a. Then $f(a) = 3a - 2a - 1 = a - 1$

$$\mathrm{Nr}(P_2 \times P_{3a}) \le a - 1 \qquad \ldots (2)$$

From (1) and (2) we have $\mathrm{Nr}(P_2 \times P_{3a}) \le a - 1$ … (3)

It is obvious that there exist S^* such that $|S^*| = a$, w(($P_2 \times P_{3a}$)/S^*)=$2a$ and c(($P_2 \times P_{3a}$)/S^*)=1 so $\mathrm{Nr}(P_2 \times P_{3a}) \ge a - 1$. . . . (4)

From (3) and (4) we have $\mathrm{Nr}(P_2 \times P_{3a}) = a - 1$. ■

Theorem 2.4.2 Let $P_2 \times P_{3a+1}$ be a cartesian product with $a \in Z^+$. Then neighbor rupture degree of $P_2 \times P_{3a+1}$ is

$$\mathrm{Nr}(P_2 \times P_{3a+1}) = a - 2 .$$

Proof Let S be a subversion strategy of $P_2 \times P_{3a+1}$ and let |S|=r .There are two cases according to the number of elements in S.

Case1: If $1 \le r \le a$ then $N[S] \le 4r$ and w(($P_2 \times P_{3a+1}$)/S) $\le$ 2r . Then

c(($P_2 \times P_{3a+1}$)/S) $\ge \frac{|V(S)|-|N(S)|}{w} \Rightarrow$ c(($P_2 \times P_{3a+1}$)/S) $\ge \frac{2(3a+1)-4r}{2r}$

$$= \frac{3a+1}{r} - 2.$$

Then we have w(($P_2 \times P_{3a+1}$)/S)-|S|-c(($P_2 \times P_{3a+1}$)/S) $\le r - \frac{3a+1}{r} + 2$

Let f(r)= $r - \frac{3a+1}{r} + 2$, since $f'(r) = 1 + \frac{3a+1}{r^2}$ >0 f(r) is an increasing function so it takes maximum value at r =a.

f(a)= $a - \frac{3a+1}{a} + 2 = a - 1 - \frac{1}{a} < a-1$, hence

$$Nr(P_2 \times P_{3a+1}) \le a-2 \quad ...(1)$$

Case2: If r=a+1 then w(($P_2 \times P_{3a+1}$)/S) $\le$ 2r and c(($P_2 \times P_{3a+1}$)/S) $\ge$1 thus we get

w(($P_2 \times P_{3a+1}$)/S) $-$|S|$-$c(($P_2 \times P_{3a+1}$)/S) $\le a-2$, hence

$$Nr(P_2 \times P_{3a+1}) \le a-2 \quad ...(2)$$

Case3: If a+2 $\le$ r $\le$|V($P_2 \times P_{3a+1}$)| then c(($P_2 \times P_{3a+1}$)/S)$\ge$1 and w(($P_2 \times P_{3a+1}$)/S) $\le$2(3a+1)$-$4$a$$-2-$(r$-$$a$$-$1)=3$a$$-$r+1 thus we get w(($P_2 \times P_{3a+1}$)/S) $-$|S|$-$c(($P_2 \times P_{3a+1}$)/S) $\le$3$a$$-$2r .

Let f(r)= 3$a$$-$2r, since $f' < 0$ f is a decreasing function so it takes maximum value at r=a+2.

f(a+2)=3$a$$-$2($a$+2)$\le$ $a$$-$4 , hence

$$Nr(P_2 \times P_{3a+1}) \le a-2 \quad ... (3)$$

From (1), (2) and (3) we have

$$Nr(P_2 \times P_{3a+1}) \le a-2 \quad ... (4)$$

It is obvious that there exist S such that |S|=a+1, w(($P_2 \times P_{3a+1}$)/S)=2a and c(($P_2 \times P_{3a+1}$)/S)=1 so Nr($P_2 \times P_{3a+1}$)$\ge$$a$$-$2 ... (5)

From (4) and (5) we have Nr($P_2 \times P_{3a+1}$)=$a$$-$2. ■

Theorem 2.4.3 Let $P_2 \times P_{3a+2}$ be a cartesian product with a$\in$ Z$^+$. Then neighbor rupture degree of $P_2 \times P_{3a+2}$ is

$$\text{Nr}\,(P_2 \times P_{3a+2}) = a-1$$

Proof Let S be a subversion strategy of $P_2 \times P_{3a+2}$ and let |S|=r .There are three cases according to the number of elements in S.

Case1: If $1 \le r \le a$ then $N[S] \le 4r$, $w \le 2r$ and $c \ge \frac{2(3a+2)-4r}{2r} = \frac{3a+2}{r} - 2$.

Thus we have

w(($P_2 \times P_{3a+2}$) /S)-|S|-c(($P_2 \times P_{3a+2}$) /S) $\le 2r-r-(\frac{3a+2}{r}-2)$

Let $f(r) = r-(\frac{3a+2}{r}-2)$ since $f' > 0$ f is an increasing function. So it takes maximum value at r =*a*.

$$f(a) = a - \frac{3a+2}{a} + 2 = a - 1 - \frac{2}{a} \le a-1$$

$$\text{Nr}(P_2 \times P_{3a+2}) \le a-1 \qquad \ldots (1)$$

Case2: If r =*a*+1, then w(($P_2 \times P_{3a+2}$) /S)$\le 2a+1$ and c(($P_2 \times P_{3a+2}$) /S)≥ 1 so we have

w(($P_2 \times P_{3a+2}$) /S) −|S|−c(($P_2 \times P_{3a+2}$) /S) $\le 2a+1-a-1-1=a-1$

$$\text{Nr}(P_2 \times P_{3a+2}) \le a-1 \qquad \ldots (2)$$

Case 3: If $a+2 \le |S| = r < |V(P_2 \times P_{3a+2})|$ then w(($P_2 \times P_{3a+2}$) /S) $\le$ $2(3a+2) - 4a - 3 - (r-a-1) = 3a+2-r$ and c(($P_2 \times P_{3a+2}$) /S)≥ 1 so we have

w(($P_2 \times P_{3a+2}$) /S) −|S|−c(($P_2 \times P_{3a+2}$) /S) $\le 3a+2-r-r-1$

Let $f(r) = 3a + 2 - 2r - 1$ since $f' < 0$, f is a decreasing function so it takes maximum value at r = *a*+2.

$f(a+2) = 3a+1-2(a+2) \le a-1$

$$\text{Nr}(P_2 \times P_{3a+2}) \le a-1 \qquad \ldots (3)$$

From (1), (2) and (3) we have $\text{Nr}(P_2 \times P_{3a+2}) \le a-1$ ... (4)

It is obvious that there exist S* such that |S*|=*a*+1, w(($P_2 \times P_{3a+2}$) /S*)=2*a*+1 and c(($P_2 \times P_{3a+2}$) /S*)=1 so $\text{Nr} \ge a-1$ ... (5)

From (4) and (5) we get Nr=$a-1$ ■

Theorem 2.4.4 Let K_m and K_n be two complete graphs with ($m \leq n$) then neighbor rupture degree of Cartesian product of K_m and K_n is

$$Nr(K_m \times K_n) = 1-n.$$

Proof Let S be a subversion strategy of $K_m \times K_n$ and let $|S| = r$. We have two cases according to the cardinality of S.

Case1: If $0 \leq r < m-1$ then $w((K_m \times K_n)/S) = 1$ and $c((K_m \times K_n)/S) \geq (m-r)(n-r)$ so we have

$w((K_m \times K_n)/S) - |S| - c((K_m \times K_n)/S) \leq 1 - r - (m-r)(n-r)$

Let $f(r) = 1 - r - mn - mr - nr + r^2$. Since f(r) is an decreasing function in $(0, m-1)$ it takes its maximum value at 0 and $f(0) = 1-mn$ thus we get

$$Nr(K_m \times K_n) \leq 1-mn \dots (1)$$

Case2: If $r = m-1$ then $w((K_m \times K_n)/S) = 1$ and $c((K_m \times K_n)/S) \geq n - m + 1$ so we have

$w((K_m \times K_n)/S) - |S| - c((K_m \times K_n)/S \leq 1 - (m-1) - (n-m+1) = 1-n$

$$Nr(K_m \times K_n) \leq 1-n \dots (2)$$

From (1) and (2) we have $Nr \leq 1-n \quad \dots \quad (3)$

There exist S such that $r = m-1$, $w((K_m \times K_n)/S) = 1$ and $c((K_m \times K_n)/S) = n-m+1$ thus we have

$$Nr \geq 1-n \quad \dots (4)$$

From (3) and (4) we get $Nr = 1-n$. ■

2.5 Tensor Product

Tensor product is a binary operation. In this part tensor product is applied in various graphs.

The tensor product $G_1 \otimes G_2$ of two simple graphs G_1 and G_2 is the graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ and where in (u_1,u_2) and (v_1,v_2) are adjacent in $G_1 \otimes G_2$ if, and only if, u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2
(Balakrishan and Ranganathan, 1999).

Tensor product of graphs is given in Figure 2.5 below.

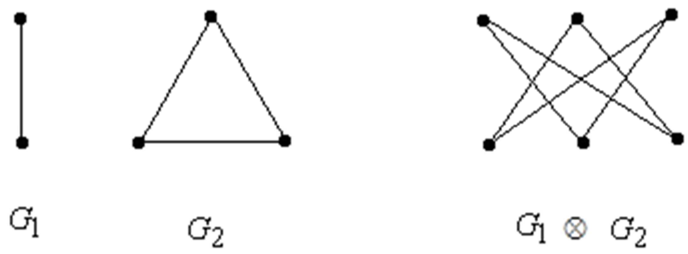


Figure 2.5 Simple graphs G_1, G_2 and their tensor product

Theorem 2.5.1 Let $P_3 \otimes P_n$ be a tensor product of P_3 and Pn and $n \equiv 0$ (mod4). Then neighbor rupture degree of $P_3 \otimes P_n$ is

$$Nr(P_3 \otimes P_n) = n-1$$

Proof Let S be a subversion strategy of $P_3 \otimes P_n$ and |S| = r be the number of removing vertices from $P_3 \otimes P_n$. There are two cases according to the number of elements in S.

Case 1: If $0 \le r \le (n/2)$ then $w((P_3 \otimes P_n)/S) \le n + r$ and $c((P_3 \otimes P_n)/S) \ge 1$ thus we have
$w((P_3 \otimes P_n)/S) - |S| - c((P_3 \otimes P_n)/S) \le n-1$ … (1)

Case 2: If $(n/2) \le r \le 3n$ then $w((P_3 \otimes P_n)/S) \le n + (n/2) - ((r - (n/2)) = 2n - r$ and $c((P_3 \otimes P_n)/S) \ge 1$ thus we have
$w((P_3 \otimes P_n)/S) - |S| - c((P_3 \otimes P_n)/S) \le 2n - 2r - 1$

Let $f(r) = 2n - 2r - 1$since $f'(r) < 0$ the function f(r) is a decreasing function so it takes its maximum value at $r = (n/2)$ and $f(n/2) = n-1$ … (2)

From (1) and (2) we get $Nr(P_3 \otimes P_n) \le n-1$ … (3)

There exist S^* such that $|S^*| = (n/2)$, $w((P_3 \otimes P_n)/S) = n + (n/2)$ and $c((P_3 \otimes P_n)/S) \ge 1$ thus we have $Nr(P_3 \otimes P_n) \ge n-1$ … (4)

From (3) and (4) we get $Nr(P_3 \otimes P_n) = n-1$. ■

Theorem 2.5.2 Let $P_3 \otimes P_n$ be a tensor product of P_3 and Pn and $n \neq 0$ (mod4). Then neighbor rupture degree of G is

$$Nr\,(P_3 \otimes P_n) = 2n - 2\lceil (n+1)/2 \rceil - 1$$

Proof Let S be a subversion strategy of $P_3 \otimes P_n$ and |S| = r be the number of removing vertices from $P_3 \otimes P_n$. There are two cases according to the number of elements in S.

Case1: If $0 \leq r \leq \lceil (n+1)/2 \rceil - 1$ then $w((P_3 \otimes P_n)/S) \leq 3r - 1$and $c((P_3 \otimes P_n)/S) \geq 3$. Thus we have
$w((P_3 \otimes P_n)/S) - |S| - c((P_3 \otimes P_n)/S) \leq 2r - 4$

Let f(r) = 2r -4 since $f'(r) > 0$ the function f(r) is an increasing function so it takes its maximum value at $r = \lceil (n+1)/2 \rceil - 1$ and
$f(\lceil (n+1)/2 \rceil - 1) = 2\lceil (n+1)/2 \rceil - 6$ so we have

$$Nr(P_3 \otimes P_n) \leq 2\lceil (n+1)/2 \rceil - 6$$

Case2: If $\lceil (n+1)/2 \rceil \leq r < 3n$ then $w((P_3 \otimes P_n)/S) \leq n + (n - r)$ and $c((P_3 \otimes P_n)/S) \geq 1$.Thus we have

$$w((P_3 \otimes P_n)/S) - |S| - c((P_3 \otimes P_n)/S) \leq 2n - 2r - 1.$$

Let f(r) = 2n – 2r -1since $f'(r) < 0$ the function f(r) is an decreasing function so it takes maximum value at $r = \lceil (n+1)/2 \rceil$ and $f(\lceil (n+1)/2 \rceil) = 2n - 2\lceil (n+1)/2 \rceil - 1$
From two cases we get $Nr \leq 2n - 2\lceil (n+1)/2 \rceil - 1$.
There exist S^* such that $|S^*| = \lceil (n+1)/2 \rceil$, $w((P_3 \otimes P_n)/S^*) = 2n - \lceil (n+1)/2 \rceil$ and $c((P_3 \otimes P_n)/S^*) = 1$ thus we have

$$Nr((P_3 \otimes P_n) \geq 2n - 2\lceil (n+1)/2 \rceil - 1$$

Since $Nr((P_3 \otimes P_n) \leq 2n - 2\lceil (n+1)/2 \rceil - 1$and $Nr((P_3 \otimes P_n) \geq 2n - 2\lceil (n+1)/2 \rceil - 1$ we obtain

$Nr\,(P_3 \otimes P_n) = 2n - 2\lceil (n+1)/2 \rceil - 1.$ ■

Theorem 2.5.3 Let the tensor product of K_m and K_n is $K_m \otimes K_n$ then neighbor rupture degree of $K_m \otimes K_n$ is

$$Nr\,(K_m \otimes K_n) = Nr\,(K_{m-1,n-1}) - 1 = \max\{\, m-4,\ n-4 \,\}$$

Proof Let (a, b) be any vertex of $K_m \otimes K_n$. The only vertices that are not adjacent to (a, b) in ($K_m \otimes K_n$) are; (a, c_j) with (j = 1,2…,n) and (d_i, b) with (i=1,2,…,m), where $c_j \in V(K_n)$ and $d_i \in V(K_m)$.

The vertices (a, c_j) are not adjacent to each other, neither the vertices (d_i, b). But these are adjacent to each other so

$$(K_m \otimes K_n) - (a,b) \cong K_{m-1,\, n-1}$$

$$Nr\,(K_m \otimes K_n) = Nr\,(K_{m-1\,,\,n-1}) - 1$$

$$= \max \{ m-3, n-3\} - 1$$ ■

2.6 Composition of Graphs

Composition operation is a binary operation. In this part neighbor rupture degree of composition of some graphs are studied.

The composition of simple graphs G and H is the simple graph G[H] with vertex set V(G)×V(H), in which (u, v) is adjacent (u′, v′) if and only if either u, u′ ∈ E(G) or u = u′ and v, v′ ∈ E(H). (Bondy and Murty, 1976)

Composition of graphs is given in Figure 2.6.

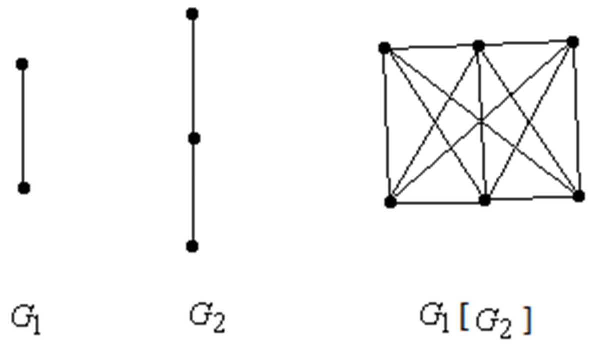


Figure 2.6 Composition of G_1 and G_2

Theorem 2.6.1 Let G be a composition of P_3 and in ($G = P_3[P_n]$) then neighbor rupture degree of $P_3[P_n]$ is

$$Nr\,(P_3[P_n]) = Nr\,(P_n)$$

Proof Let the vertex set of $P_3[P_n]$ be labeled as, P_n^{I}, P_n^{II} and P_n^{III} and let S be a subversion strategy of $P_3[P_n]$.

……… P_n^{I}

……… P_n^{II}

……… P_n^{III}

Case 1: Let S be a Nr-set of P_n^{I} The S- set removes all elements of P_n^{II} then we have

$$Nr(P_3[P_n]) = Nr(P_n^{I} \cup P_n^{III}) \geq Nr(P_n^{I}) + Nr(P_n^{III})$$
$$\geq Nr(P_n) + Nr(P_n) = 2\,Nr(P_n)$$

Case 2: Let S be a subversion strategy of $P_n^{''}$ The S- set removes all elements of P_n^{I} and P_n^{III} so it depends only P_n^{II}

$$Nr(P_3[P_n]) = Nr(P_n^{II}) = Nr(P_n)$$

Case 3: Let S be a subversion strategy of P_n^{I} and P_n^{II} Then S set removes all P_n^{I}, P_n^{II} and P_n^{III} then w = 0. It contradicts to the definition of neighbor rupture degree.

$$Nr(P_3[P_n]) = \max\{2\,Nr(P_n),\ Nr(P_n)\}$$

since $Nr(P_n) \leq 0$ then $\max\{2\,Nr(P_n),\ Nr(P_n)\} = Nr(P_n)$ ■

Theorem 2.6.2 Neighbor rupture degree of composition of K_m and any graph G is

$$Nr(K_m[G]) = Nr(G).$$

Proof Let the vertex set of $K_m[G]$ be labeled as, G^{I}, G^{II}, . . . ,G^{m}.

……… G^{I}

……… G^{II}

⋮

……… G^{m}

Let S be a subversion strategy of $K_m[G]$. We have two cases according to elements of S.

Case1: Let we choose one element from any vertex set G^i (i=1,2, .. ,m) , if $u \in V(G^i)$ and S={u} then it removes all of other vertex set. So it depends only G^i which we choose one element. Then we have

$Nr(K_m[G])=Nr(G^i)=Nr(G)$

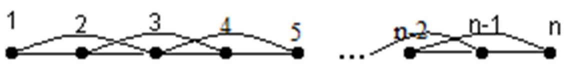
Case2: Let we choose two element from any vertex set G^i (i=1,2, .. ,m) and G^j (j=1, 2, …, m) with $i \neq j$. Then $(K_m[G])/S$ is empty set. It contradicts to the definition of neighbor rupture degree.

From two cases we obtain $Nr(K_m[G])=Nr(G)$. ■

2.7 Power of Graphs

A second power of a graph G is formed by adding an edge between all pairs of vertices of *G* with distance at most two. A *second power* of a graph is also called a square.

Square of path graph P_n is given in Figure 2.7.



P_n^2

Figure 2.7 Path graph and its square

Theorem 2.7.1 Neighbor rupture degree of P_n^2 (n > 6) is

$$Nr(P_n^2) = \begin{cases} 0, & n \equiv 1 \ (mod 6) \\ -1, & otherwise \end{cases}$$

Proof Let S be a subversion strategy of P_n^2 and let |S| = x. There are two cases according to the number of elements of S.

Case1: If $0 \le x \le \lceil \frac{n}{6} \rceil - 1$ then $w(P_n^2/S) \le x+1$, $c(P_n^2/S) \ge \frac{n-5x}{x+1}$ then we get

$$w(P_n^2/S) - |S| - c(P_n^2\backslash S) \le x+1-x-\frac{n-5x}{x+1} = 6 - \frac{n+5}{x+1}$$

Let $f(x) = 6 - \frac{n+5}{x+1}$

if $n \equiv 1 \pmod 6$, then $f(x) = 0$

otherwise $f(x) < 0$

Case2: If $x = \lceil \frac{n}{6} \rceil$ then $w(P_n^2/S) \leq x$, $c(P_n^2/S) \geq 1$ then we get

$w(P_n^2/S) - |S| - c(P_n^2/S) \leq x - x - 1 = -1$ therefore $Nr \leq -1$

Case3: If $\lceil \frac{n}{6} \rceil + 1 \leq x \leq n$ then $w(P_n^2/S) \leq x-1$, $c(P_n^2/S) \geq 1$ then we get

$w(P_n^2/S) - |S| - c(P_n^2 \backslash S) \leq x - 1 - x - 1 = -2$ therefore $Nr \leq -2$

According to three cases we have

$Nr(P_n^2) \leq 0$ where $n \equiv 1\ (mod6)$

$Nr(P_n^2) \leq -1$ otherwise

There exist S^* such that $|S^*| = \lceil \frac{n-1}{6} \rceil$, $w(P_n^2/S^*) = \lceil \frac{n}{6} \rceil$, $c(P_n^2/S^*) = 1$ then

$$w(P_n^2/S^*) - |S^*| - c(P_n^2/S^*)\} = \begin{cases} 0, & n \equiv 1\ (mod6) \\ 1, & otherwise \end{cases}$$

Thus we get the result,

$$Nr(P_n^2/S) = \max\{w(P_n^2/S) - |S| - c(P_n^2/S))\} = \begin{cases} 0, & n \equiv 1\ (mod6) \\ 1, & otherwise \end{cases}$$

■

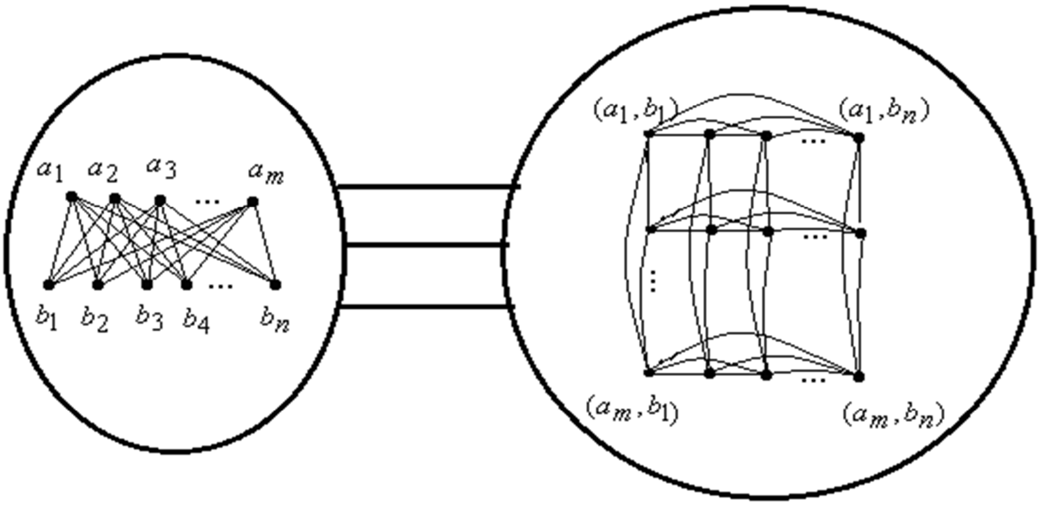
3. TOTAL GRAPHS AND COMPLEMENT OF TOTAL GRAPHS

3.1 Total Graphs

In this section we deal with the neighbor rupture degree of total of some special graphs.

Definition 3.1.1 (Gross and Yellen, 2004) The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The *total graph* T(G) has vertex set V(G) U E(G) and two vertices of T(G) are adjacent whenever they are neighbors in G.

Let $K_{m,n}$ be a complete bipartite graph then the total graph of complete bipartite graph $K_{m,n}^{+++}$ is given in Figure 3.1 below.



$K_{m,n}$ $L(K_{m,n})$

Figure 3.1 Total graph of complete bipartite graph

Let a_i and b_j be the vertices of $K_{m,n}$ and let (a_i, b_j) be vertices of $L(K_{m,n})$. In total graph edges between the vertices of $K_{m,n}$ and $L(K_{m,n})$ are as follows,

a_i (i=1, 2, ... or m) is joined to all (a_i, b_j) (j=1,2,…,and n) by an edge
b_j (j=1, 2, … or n) is joined to all (a_i, b_j) (i=1,2,…,and m) by an edge

Theorem 3.1.1 Neighbor rupture degree of total graph $K_{m,n}$ with (m ≤ n) is

$$\text{Nr}(K_{m,n}^{+++}) = n-2m-1$$

Proof Let S be a subversion strategy of $K_{m,n}^{+++}$ and let |S|=x. We have two cases according to cardinality of S.

Case1: If 1 ≤ x ≤ m−1 then w($K_{m,n}^{+++}\backslash S$) = 1and c($K_{m,n}^{+++}\backslash S$) ≥ (m−x)(n−x) +m + n−2x. Thus we have

$$w(K_{m,n}^{+++}\backslash S) - |S| - c(K_{m,n}^{+++}\backslash S) \le 1-x-(m-x)(n-x)-m-n+2x$$
$$\le 1-(mn-xm-xn+x^2)-m-n+x$$
$$\le 1-nm+nx+xm-m-n+x-x^2$$

Let f(x) = 1−nm +nx +xm−m−n +x +x^2 and f is an increasing function in 1 ≤ x < m−1 so it takes maximum value at x =m−1.

$$f(m-1) = 1-nm-n(m-1)+m(m-1)-m-n+m-1+(m-1)^2$$
$$= -2mn+m-1 \text{ therefore } \text{Nr}(K_{m,n}^{+++}) \le -2mn+m-1.$$

Case2: If x =m then w($K_{m,n}^{+++}\backslash S$) ≤ n−m and c($K_{m,n}^{+++}\backslash S$) ≥ 1. Thus we have

$$w(K_{m,n}^{+++}\backslash S) - |S-|c(K_{m,n}^{+++}\backslash S) \le n-m-m-1 = n-2m-1$$

According to two cases we have

$$\text{Nr}(K_{m,n}^{+++}) \le n-2m-1 \quad \dots \quad (1)$$

There exist S such that |S|=m, w($K_{m,n}^{+++}\backslash S$)=m−n and c($K_{m,n}^{+++}\backslash S$)=1 then we get

$$\text{Nr}(K_{m,n}^{+++}) \ge n-2m-1 \quad \dots \quad (2)$$

From 1 and 2 we have Nr($K_{m,n}^{+++}$) = n−2m−1. ■

Corollary 3.1.1 Let $K_{1,n}$ be star graph then the neighbor rupture degree of total graph of $K_{1,n}$ is

$$\text{Nr}\left(K_{1,n}^{+++}\right) = n-3.$$

Let P_n be path graph then the total graph of P_n, P_n^{+++} is given in Figure 3.2 below.

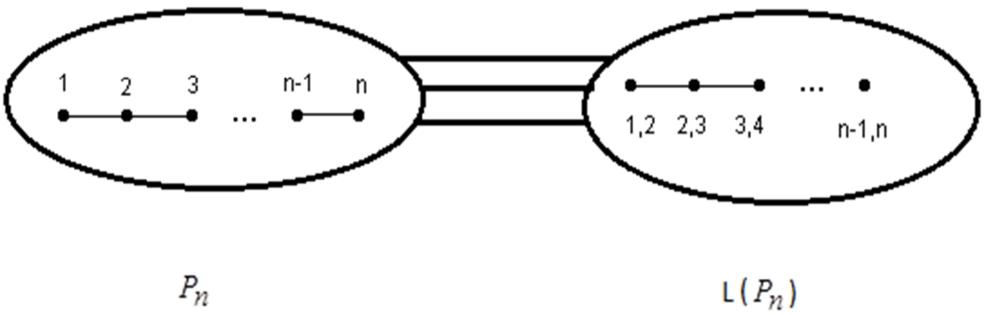


Figure 3.2 Total graph of path graph

Let i (i=1, 2, …, n-1, n) be the vertices of P_n and (i,i+1) be the vertices of L(P_n). In total graph the edges between the vertices of P_n and L(P_n) are as follows,

i (i=1, 2, …, n-1, n) is joint to all (i,i+1) by an edge

Theorem 3.1.2 Neighbor rupture degree of total graph of P_n is

$$\text{Nr}(P_n^{+++}) = \begin{cases} 0, & n \equiv 1(mod3) \\ -1, & otherwise \end{cases}$$

Proof Since $P_n^{+++} \approx P_{2n-1}^2$, by theorem … dan we have

$$\text{Nr}(P_t^2) = \begin{cases} 0, & n \equiv 1(mod6) \\ -1, & otherwise \end{cases}$$

Therefore we obtain the neighbor rupture degree of P_n^{+++}

$$\text{Nr}(P_n^{+++}) = \begin{cases} 0, & n \equiv 1(mod3) \\ -1, & otherwise \end{cases}$$

■

Let $C_{t,r}$ be comet graph then the total graph of comet $C_{t,r}$ ($C_{t,r}^{+++}$) is given in Figure 3.3 below.

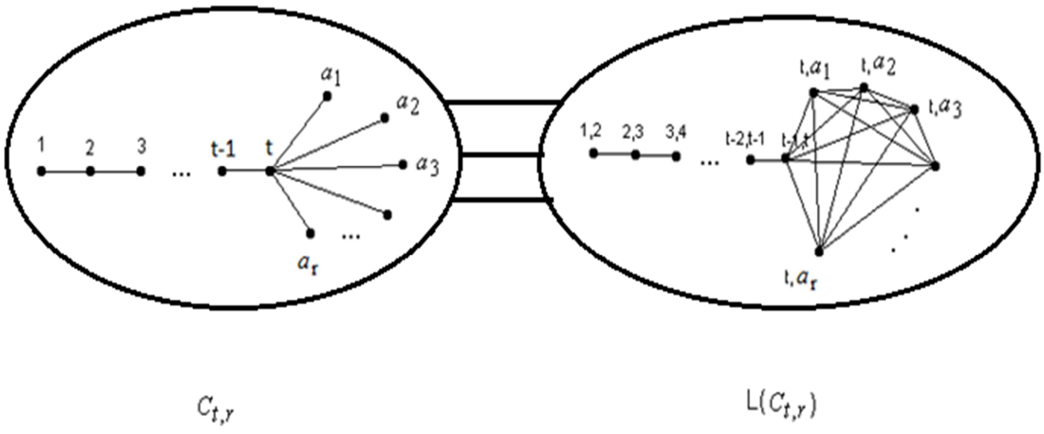


Figure 3.3 Total graph of comet graph

Let i (i=1, 2, …, t-1, t, $a_1, a_2, \ldots, a_r$) be vertices of $C_{t,r}$ and let (i,i+1), (t, a_i) be vertices of L($C_{t,r}$). In total graph the edges between the vertices of $C_{t,r}$ and L($C_{t,r}$) are as follows,

i (i=1, 2, …, t-1, t) is joined to all (i,i+1) and a_i is joined to all (t, a_i) by an edge

Theorem 3.1.3 If $C_{t,r}$ is a comet graph then the neighbor rupture degree of total graph of $C_{t,r}$ is

$$Nr(C_{t,r}^{+++}) = \begin{cases} r-1, & t \equiv 2\ (mod3) \\ r-2, & otherwise \end{cases}$$

Proof Let S be a subversion strategy of $C_{t,r}^{+++}$ and let |S|= x. There are three cases according to the elements of S.

Case1: Assume S={i} or {(i,i+1)} then the number of components w($C_{t,r}^{+++}$\S) ≤ x+1 and c(G\S) ≥ r. Therefore we get

$$w(C_{t,r}^{+++}\backslash S) - |S| - c(C_{t,r}^{+++}\backslash S) \leq x+1-x-r = 1-r.$$

Case2: Let S= {t} then $C_{t,r}^{+++}\backslash S \cong P_{2(t-2)}^{2}$

Thus we have Nr($C_{t,r}^{+++}$) = Nr ($P_{2(t-2)}^{2}$)-1 and since the neighbor rupture degree of $P_{2(t-2)}^{2}$

$$\text{Nr}\,(P_{2(t-2)}^{2}) = \begin{cases} 0, & t \equiv 3\ (mod6) \\ -1, & otherwise \end{cases}$$

we get

$$Nr(C_{t,r}^{+++}) = \begin{cases} -1, & t \equiv 2\ (mod3) \\ -2, & otherwise \end{cases}$$

Case3: Let S={(t, a_i)} then $C_{t,r}^{+++}\backslash S \cong P_{t-1}^{+++} \cup rK_1$

Thus we have Nr($C_{t,r}^{+++}$) =Nr(P_{t-1}^{+++})+ r – 1 and since the neighbor rupture degree of P_{t-1}^{+++}

$$Nr(P_{t-1}^{+++}) = \begin{cases} 0\,, & t \equiv 2\ (mod3) \\ -1, & otherwise \end{cases}$$

we get

$$Nr(C_{t,r}^{+++}) = \begin{cases} r-1, & t \equiv 2\ (mod3) \\ r-2, & otherwise \end{cases}$$

According to three cases we obtain

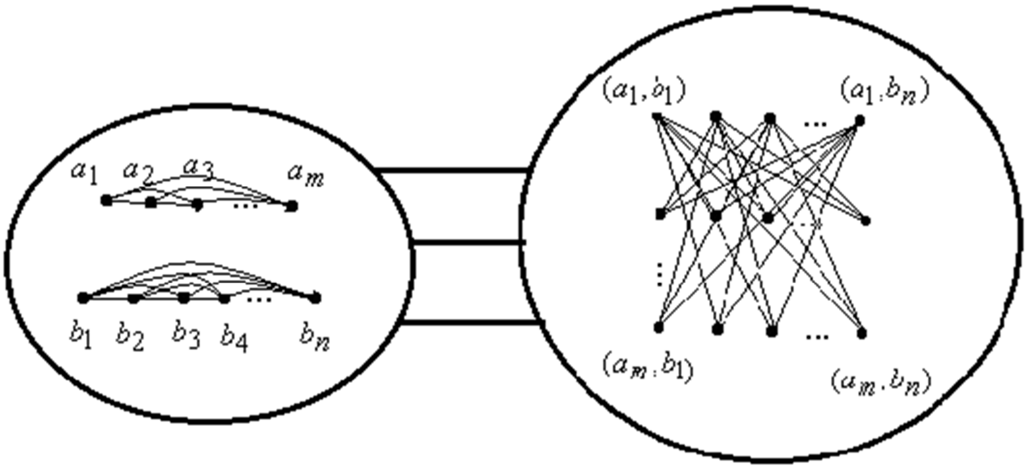
$$Nr(C_{t,r}^{+++}) = \begin{cases} r-1, & t \equiv 2\ (mod3) \\ r-2, & otherwise \end{cases}$$

■

3.2 Complement of Total Graphs

In this section we deal with the neighbor rupture degree of complement of total graphs.

Let $K_{m,n}$ be a complete bipartite graph. The complement of the total graph of complete bipartite graph $K_{m,n}^{---}$ is given in Figure 3.4



$K^c_{m,n} = K_m \cup K_n$ $(L(K_{m,n}))^c$

Figure 3.4 Complement of total graph of complete bipartite graph

Let a_i and b_j be the vertices of $K^c_{m,n}$ and let (a_i, b_j) be vertices of L $(K^c_{m,n})$. In total graph edges between the vertices of $K^c_{m,n}$ and L $(K^c_{m,n})$ are as follows

a_i (i=1,2,.. or m) is joined to all (a_k, b_l) since $k \neq i$ by an edge
b_j (j=1,2,…,or n) is joined to all (a_k, b_l) since $j \neq l$ by an edge

Theorem 3.2.1 Neighbor rupture degree of complement of total graph of $K_{m,n}$ is

$$\text{Nr}(\overline{K_{m,n}}) = n - 4.$$

Proof Let S be a subversion strategy of $\overline{K_{m,n}}$. We have four cases according to elements of S.

Case1: Let any vertex $a_i \subset V(K_m)$ and $S=\{a_i\}$ then $|S|=1$, w($\overline{K_{m,n}}$/S)=1, c($\overline{K_{m,n}}$/S)=2n thus we have w($\overline{K_{m,n}}$/S) $-|S|-$c($\overline{K_{m,n}}$/S)$= -2n$.

Case2: Let any vertex $b_j \subset V(K_n)$ and $S=\{b_j\}$ then $|S|=1$, w($\overline{K_{m,n}}$/S)=1,

c($\overline{K_{m,n}}$/S)=2m thus we have w($\overline{K_{m,n}}$/S) $-|S|-$c($\overline{K_{m,n}}$/S)$= -2m$.

Case3: Let any vertex $\{(a_i, b_j)\} \subset$ V(L $(K^c_{m,n})$) and $S=\{(a_i, b_j)\}$ then $|S|=1$, w($\overline{K_{m,n}}$/S) =1, c($\overline{K_{m,n}}$/S)=m+n+2 thus we have w($\overline{K_{m,n}}$/S) $-|S|-$c($\overline{K_{m,n}}$/S)$=-m-n-2$.

Case4: Let any two vertex $\{(a_i, b_j)\}$ and $\{(a_i, b_h)\}$ with h$\neq$j and S={ $(a_i, b_j), (a_i, b_h)$ } then $|S|$=2, w($\overline{K_{m,n}}$ /S)=n-1, c($\overline{K_{m,n}}$ /S)=1 then we have w($\overline{K_{m,n}}$ /S) $-|S|-$c($\overline{K_{m,n}}$ /S)= n$-$4.

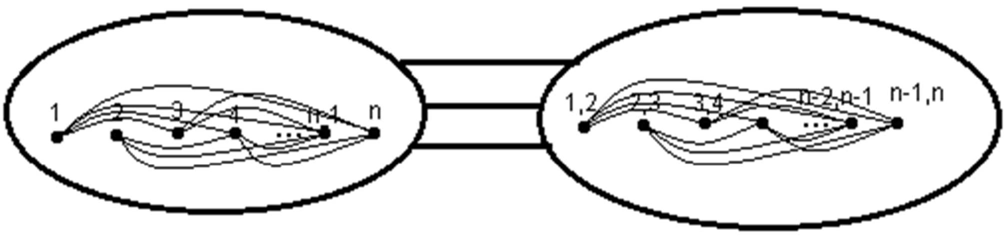
From four cases we have neighbor rupture degree of $\overline{K_{m,n}}$ **,** Nr($\overline{K_{m,n}}$) = n$-$4.

■

Corollary 3.2.1 Let $K_{1,n}$ be star graph then neighbor rupture degree is

$$\text{Nr}\left(\overline{K_{1,n}}\right)=\text{n-4}.$$

Let P_n be a path graph then complement of the total graph of path graph $\overline{P_n}$ is given in Figure 3.5 below.



P_n^c (L(P_n))^c

Figure 3.5 Complement of total graph of path graph

Let i (i=1, 2, …, n-1, n) be the vertices of P_n^c and (i,i+1) be the vertices of L(P_n^c) . In total graph the edges between the vertices of P_n^c and L(P_n^c) are as follows

i (i=1, 2, …, n-1, n) is joint to all (j,j+1) by an edge except i$\neq$j and i$\neq$j+1.

Theorem 3.2.2 Neighbor rupture degree of complement of total graph of path $\overline{P_n}$ is

$$\text{Nr}(\overline{P_n})=0.$$

Proof Let S be a subversion strategy of $\overline{P_n}$. There is four cases according to elements of S.

Case1: Let i$\subset$V(P_n^c) and S={i},i=1,2,… or n .

If deg(i)=1 then $|S|$=1, w($\overline{P_n}$ /S)=2 and c($\overline{P_n}$ /S)=1. Thus we get w($\overline{P_n}$ /S)$-|S|-$c($\overline{P_n}$ /S) =0.

If deg(i)=2 then $|S|$=1, w($\overline{P_n}$ /S)=2 and c($\overline{P_n}$ /S)=2. Thus we have

w($\overline{P_n}$ /S)$-|S|-$c($\overline{P_n}$ /S) = -1.

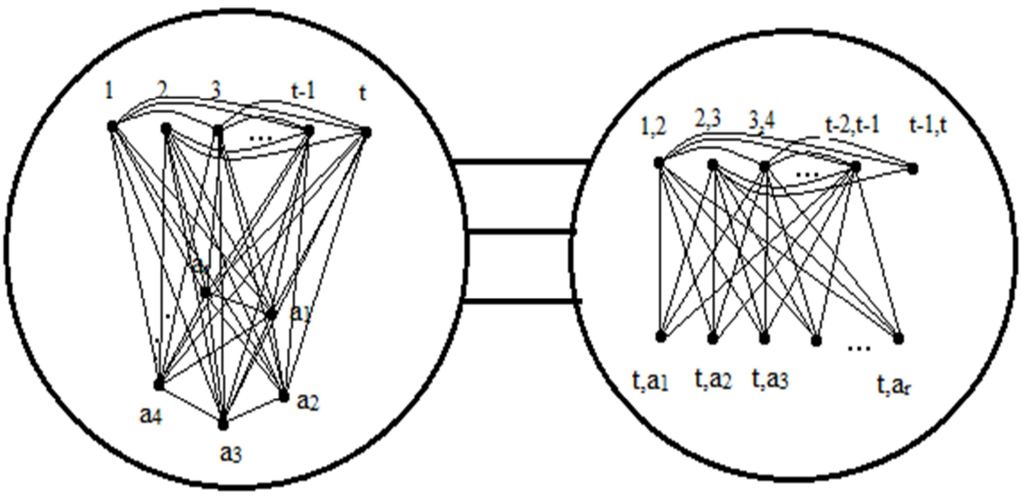
Case2: Let (i,i+1) $\subset$V($L(P_n)^c$) in $\overline{P_n}$ and let S ={(i,i+1)}

If deg (i,i+1)=1 then $|S|$=1, w($\overline{P_n}$ /S)=2 and c($\overline{P_n}$ /S)=2 thus we get w($\overline{P_n}$ /S)$-|S|-$ c($\overline{P_n}$ /S)= -1.

If deg (i,i+1)=2 then $|S|$=1,w($\overline{P_n}$ /S)=2 and c($\overline{P_n}$ /S)=2 thus we have w($\overline{P_n}$ /S)$-|S|-$ c($\overline{P_n}$ /S)= -1.

From these cases we have neighbor rupture degree of $\overline{P_n}$ Nr($\overline{P_n}$) = 0. ■

Let $C_{t,r}$ be a complete bipartite graph then the complement of total graph of comet graph $\overline{C_{t,r}}$ is given in Figure 3.6 below.



$C_{t,r}^c$ (L($C_{t,r}$))^c

Figure 3.6 Complement of total graph of comet graph

Let i (i=1, 2, …, t-1, t, $a_1, a_2, \dots, a_r$) be vertices of $C_{t,r}^c$ and let (i,i+1), (t, a_i) be the vertices of L($C_{t,r}^c$) . In total graph the edges between the vertices of $C_{t,r}^c$ and L($C_{t,r}^c$) are as follows

i (i=1, 2, …, t-1, t) is joined to all vertices of L($C_{t,r}^c$) except (i-1,i) and (i,i+1).
a_i is joined to all vertices of L($C_{t,r}^c$) except (t, a_i) by an edge.

Theorem 3.2.3 Let $C_{t,\,r}$ is a comet graph with r ≥ 4. The neighbor rupture degree of complement of total graph of $C_{t,\,r}$ is

$$\mathrm{Nr}(\overline{\overline{\overline{C_{t,r}}}}) = r-4.$$

Proof: Let S be a subversion strategy of $\overline{\overline{\overline{C_{t,r}}}}$. We have four cases according to elements of S.

Case1: Let i⊂V($C_{t,r}^c$) and S={i},i=1,2,… or n .

If deg(i)=1,then $|S|$=1,w($\overline{\overline{\overline{C_{t,r}}}}$ /S)=2 and c($\overline{\overline{\overline{C_{t,r}}}}$ /S)=1. Thus we get

$$w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = 0.$$

If deg(i) =2 then $|S|$=1,w($\overline{\overline{\overline{C_{t,r}}}}$ /S)=1 and c($\overline{\overline{\overline{C_{t,r}}}}$ /S) =4. Thus we have

$$w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = -4.$$

Case2: Let a_i ⊂V($C_{t,r}^c$) and S={a_i},i=1,2,… or r. Since S={a_i} we get w($\overline{\overline{\overline{C_{t,r}}}}$ /S)=1 and c($\overline{\overline{\overline{C_{t,r}}}}$ /S)=1. Therefore we have

$$w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = -1.$$

Case3: Let (i,i+1) ⊂V(L($C_{t,r}^c$)) in $\overline{\overline{\overline{C_{t,r}}}}$.

If deg (i,i+1)=1 then $|S|$=1,w($\overline{\overline{\overline{C_{t,r}}}}$/S)=2 and c($\overline{\overline{\overline{C_{t,r}}}}$/S) = 2. Thus we have

$$w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = -1.$$

If deg (i,i+1)=2 then $|S|$=1,w($\overline{\overline{\overline{C_{t,r}}}}$/S)=1 and c($\overline{\overline{\overline{C_{t,r}}}}$/S) = 4. Thus we obtain

$$w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = -4.$$

Case4: Let (a_t,b_i) ⊂V(L($C_{t,r}^c$)) in $\overline{\overline{\overline{C_{t,r}}}}$.

If S = {(a_t,b_i)} then we have $w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = -r$

If S = {(a_t,b_i), (a_t,b_j)} then we have $w(\overline{\overline{\overline{C_{t,r}}}}/S) - |S| - c(\overline{\overline{\overline{C_{t,r}}}}/S) = r-4$.

From four cases we have neighbor rupture degree of $\overline{\overline{\overline{C_{t,r}}}}$ is $\mathrm{Nr}(\overline{\overline{\overline{C_{t,r}}}}) = r-4$. ■

4.CONCLUSION

In this thesis, the vulnerability parameter which takes an important part of graph theory is discussed in full details. Firstly the related basic definitions are given. Subsequently neighbor rapture degree which is vulnerability parameter is examined. Unary and binary operations are examined in various graphs and their neighbor rapture degree is calculated. Finally, total graphs are examined, the totals of the special graphs and the complements of the totals of the special graphs are taken and the neighbor rapture degrees are found.

KAYNAKLAR DİZİNİ

Bacak Turan, G., 2010, Vulnerability Parameters in Graphs, PhD Thesis, Ege University.

Bacak Turan, G. ve Kırlangıç, A., 2010 Vulnerability Parameters in Graphs, 1st International Symposium on Computing in Science and Engineering, Turkey, 20s,.

Balakrishan, R. and Ranganathan, K., 1999, A Textbook of Graph Theory.

Barefoot, C.A. and Entringer, H.R., 1987, Vulnerability in graphs: a comparative
survey. J. Combin. Math. Combin. Comput., 25-33pp.

Bondy, J.A. and Murty, U.S.R., 1976, Graph Theory with Applications. London, Macmillan Press Ltd., 263p.

Chartrand, G. and Lesniak, L., 1996, Graphs and Digraphs, Chapman and Hall, California, 421p.

Cozzens, M.B. and Wu, S.S.Y., 1998, Vertex-neighbor-integrity of powers of cycles.
Ars Combin., 48, 257-270pp.

Gross, J.L and Yellen J., 2004, Handbook of Graph Theory.

Harary, F., 1972, Graph Theory, Addison-Wesley Publishing Company, 274p.

Kırlangıç, A, 2004, Graph operations and neighbor-integrity. Mathematica Bohemica,
3, 245-254pp.

Kırlangıç, A. and Ozan, A., 2000, Neighbor integrity of total graphs. International
Journal of Computer Mathematics, 23, 73-76pp.

Li, Y., and Li, X., 2004, Computing the rupture degrees of graphs. 7th Internat. Symp. Parallel Architectures, Algorithms and Networks, ISPAN 04.