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MASTER THESIS

APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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This thesis includes the analytical and numerical methods for solving first order ordinary differential equations. Starting with historical information about differential equations, we present the concepts for differential equations. In this thesis, we study the types of seperable, homogeneous, exact, linear and some special equations as anaytical methods. On the other hand, we aim the numerial methods that are called Euler, Improved Euler, Second-Order Runge-Kutta and Fourth-Order Runge-Kutta Methods. After meeting the methods, we present the mathematical models for the applications of first-order ordinary differential equations. Developing the mathematical models, we introduce the problems for first-order ordinary differential equations. These problems classify in different kind of areas such as engineering, chemistry, physics, economics and sociology in order. We start with mechanical problems. Secondly mixture problems follow up after the mechanical problems. We present cooling and warming problems and later on financial problems. Finally we introduce growth and decay problems. We discuss three problems for each areas and these problems are solved both analytically and numerically. In numerical solutions, we get the approximations for each problem. Therefore we focus on choosing the better numerical approximations between the given numerical methods. Since the approximation depends on the step size, we deal with different step size. Finally, we compare the approximations.

This thesis consists of 5 chapters which include all of these subjects

Keywords: Application, Runge-Kutta, Heun, Euler, first-order, analytical, numerical

ÖZET

ADİ DİFERANSİYEL DENKLEMLERİN UYGULAMALARI

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Bu tez birinci mertebeden adi diferansiyel denklemler için analitik ve nümerik çözüm yöntemlerini içerir. Diferansiyel denklemler hakkında tarihsel bilgilerle başlayarak, diferansiyel denklemlerin temel kavramlarını tanıtır. Bu çalışmada analitik yöntemler adına değişkenlerine ayrılabilen, homojen, tam, doğrusal ve özel denklem tiplerinden bahsediyoruz. Öte yandan nümerik yöntem olarak, Euler, Geliştirilmiş Euler, İkinci Mertebeden Runge-Kutta ve Dördüncü Mertebeden Runge-Kutta Yöntemlerini çalışıyoruz. Yöntemleri tanıttıktan sonra uygulamaların matematiksel modellerini tanıtıyoruz. Modellemeleri geliştirdikten sonra birinci mertebeden adi diferansiyel denklemlerin problemlerini sunuyoruz. Bu problemler mühendislik, kimya, fizik, ekonomi ve sosyoloji gibi çeşitli alanlardan seçilerek sınıflandırıldı. Mekanik problemlerden başlayıp ikinci olarak karışım problemlerini sunuyoruz. Isınma- soğuma problemlerini, finans problemleri takip ediyor. Son olarak büyüme ve çürüme problemlerini sunuyoruz. Her alanda üç problem çalışıyoruz ve problemleri analitik ve nümerik olarak çözüyoruz. Nümerik çözümler için nümerik yakınsamalar elde ediyoruz. Böylece verilen nümerik yöntemler arasından daha iyi yakınsayanını belirlemeye odaklanıyoruz. Yakınsama adım büyüklüğüyle ilişkili olduğundan, farklı adım büyüklükleriyle çalışıyoruz. Son olarak yakınsamaları kıyaslıyoruz.

Bu tez tüm bu konuları içeren 5 bölümden oluşmaktadır.

Anahtar sözcükler: Uygulama, Runge-Kutta, Heun, Euler, birinci mertebe, analitik, nümerik

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Melis Buse NİSA
İzmir, 2014

TEXT OF OATH

I declare and honestly confirm that my study, titled “Applications of Ordinary Differential Equations” and presented as a Master’s Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

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Abbreviations	Explanations
RK2	Second-Order Runge-Kutta Method
RK4	Fourth-Order Runge-Kutta Method
ODE	Ordinary Differential Equation

1 INTRODUCTION

Mathematics is an useful subject. In society, it has applications in lots of areas such as engineering, chemistry, physics, sociology, economics and many other disciplines.

A famous mathematician once said that the complete appreciation of mathematics requires an element of poetry, and it is true that the mathematics can offer the same sort of inspiration. The poet sees the essence behind the daily experience, the universe in agrain of sand, and the mathematicians sees the law working behind the parachute and the pendelum, the suspension bridge and the rolling motion of a wheel. The law is hidden in differential equations which is a branch of mathematics. The subject of differential equations constitutes a large and very important branch of modern mathematics. In word, an equation includes unknown functions and its derivatives is called differential equation. We will give another definition of differential equations in following section. Without knowing something about differential equations and methods of solving them it is difficult to appreciate the history of this important branch of mathematics. Further the development of differential equations is intimately interwoven with the general development of mathematics and can not be seperated from it. Nevertheless to provide some historical perspective, we indicate here some of the major trends in the history of the subject and identify the most prominent early contributors (Pickles, 2010).

Isaac Newton (1642-1727) did relatively work in differential equations as such his development of the calculus and elucidation of the basic principles of mechanics provided a basis for the application in the eighteenth century, most notably by Euler. Newton classified first order differential equations according to three forms $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x, y)$ (Boyce and Diprima, 2001).

Gottfried Leibniz (1646-1716) was mainly self-taught in mathematics since his interest in the subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, Leibniz was very concious of the power of good mathematical notation and was responsible for the notation dy/dx for the derivative and for the integral sign. He discovered the methods of seperation of variables in 1691, the reduction of homogeneous equations to seperable ones in 1691, and the procedure for solving first order linear equations (Boyce and Diprima, 2001).

The brother Jacob (1654-1705) and Johann (1667-1748) Bernoulli did much to develop methods of solving differential equations and to extend the range of their applications. With the aid of calculus, they solved number of problems in mechanics by formulating them as differential equations (Boyce and Diprima, 2001).

From the early days of the calculus the subject has been an area of great theoretical research and practical applications, and it continues to be so in our day. This much stated, several questions naturally arise. Just what is a differential equation and what does it signify? Where and how do differential equations originate and of what use are they? Confronted with a differential equation, what does one do with it, how does one do it, and what are the results of such activity? These questions indicate three major aspects of the subject: theory, method and application (Ross, 2004).

Let us now consider briefly where, and how, such equations actually originate. In this way we shall obtain some indication of the great variety of subjects to which theory and methods of differential equations may be applied (Ross, 2004).

Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering. We indicate a few such problems, which could easily extended to fill many pages. The problem of determining the motion of a projectile rocket, satellite, planet, charge of current in an electric circuit, the vibrations of a wire or a membrane, conduction of heat in a rod or a slab, and curves that have certain geometrical properties; the study of the rate of decomposition of a radioactive substance or the rate of growth of a population, reactions of chemicals and in economics (Ross, 2004).

The mathematical formulation of such problems give rise to differential equations. But it is not possible many time to get the exact solution for all of these problems analytically. Getting the exact solution for those types of all problems can be a challenge (Ross, 2004).

The numerical method is a solution technique that provides accurate approximation for the problem. The differential equations that resist solution by analytically led to the investigation of numerical methods for the problem. Therefore we need numerical methods for at least approximating the solution. Differential equations provide a rather more difficult problem. The basic method is to divide continuous time into discrete intervals, and to estimate the state of the system at the start of each interval. Thus the approximate solution changes through a series of steps. The crudest method for calculating the steps is to multiply the step length by the derivative at the start of the interval. This method is called Euler's method (Boyce and Diprima, 2001).

The greatest mathematician of the eighteenth century Leonhard Euler (1707-1783) was a student of Bernoulli. His interests ranged over all areas of mathematics and many fields of application. Of particular interest is his formulation of problems in mechanics. Lagrange said of Euler's work in mechanics "The first great work in which analysis is applied to the science of movement." Among other things Euler identified the condition for exactness of first order differential equations developed the theory of integrating factors, and gave the general solution of homogeneous linear equations with constant. (Boyce and Diprima, 2001).

In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on powerseries expansions (Boyce and Diprima, 2001).

In addition to Euler's method, Heun's method which is known also the modified Euler's method is a numerical approximation method for solving ordinary differential equations. This method is named by Karl Heun (1859-1929) (Boyce and Diprima, 2001).

By 1900 fairly effective numerical integration methods had been devised, but their implementation was severely restricted by the need to execute the computations by hand or with very primitive computing equipment. In the last 60 years the development of increasingly powerful and versatile computers has vastly enlarged the range of problems that can be investigated effectively by numerical methods. Within the past few years these two trends have come together. Computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Unexpected phenomena such as strange attractors, chaos, and fractals, have been discovered, are being intensively studied, and are leading to important new insights in a variety of applications (Boyce and Diprima, 2001).

More sophisticated techniques are presented by two German mathematicians Carl Runge (1856-1927) publishes the first Runge-Kutta method in performing the Runge-Kutta types of integration. Fourth order Runge-Kutta method that is described by Martin Kutta (1867-1944) is both commonly used and sufficiently accurate for most applications. It is always worth treating numerical solutions to differential equations with caution. Errors in calculating them may accumulate. Here, we will have the choose of numerical techniques such as Euler's method, Heun's Method, Second-Order Runge-Kutta and Fourth-Order Runge-Kutta (Roberts, 2010).

The study of differential equations in the twenty-first century remains a fertile source and fascinating and important unsolved problems. Finding and interpreting the solutions of these differential equations is therefore a central part of applied mathematics, and a thorough understanding of differential equations is essential for any applied mathematicians. (Roberts, 2010).

This study contains the basic concepts of differential equations such as definitions and classifications. In order to get familiar with the solution methods, both analytical and numerical methods are presented. Later on, the applications of ordinary differential equations and their numerical and analytical solutions are given. Here in this study, our first concern is approximating the solution and determining the best method for the given each problems in the light of given foregoing.

2 BASIC CONCEPTS IN DIFFERENTIAL EQUATIONS

Definition 2.1: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (Zill and Cullen, 2005).

Some examples of differential equations are given below

$$\frac{dy}{dx} = xy \qquad \frac{dy}{dx} + \frac{dy}{dz} = e^x$$

Differential equations are classified according to the specified properties that are given below

2.1 Classifications of Differential Equations

2.1.1 Classification by Type

Definition 2.1.1.1: If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an ordinary differential equation (Zill and Cullen, 2005).

Some examples of ordinary differential equations are given below

$$\frac{dy}{dx} + 5y = e^x \qquad \frac{dy}{dt} + \frac{dx}{dt} = 2x + y$$

Definition 2.1.1.2: An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a partial differential equation (Zill and Cullen, 2005).

Some examples of partial differential equations

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

2.1.2 Classification by Order and Degree

Definition 2.1.2.1: The order of a differential equation is the order of the highest derivative in the equation (Zill and Cullen, 2005).

Some examples of differential equations with different orders are given below

$$y''' - 2y'' + y' = \sin x \qquad \textit{order: 3}$$

$$(\frac{\partial^2 y}{\partial x^2})^3 - \frac{dy}{dx} = \ln x \qquad \textit{order: 2}$$

Definition 2.1.2.2: The degree of a differential equation is given by the exponent that is raised the highest derivative that occurs in the equation (Ross, 2004).

Some examples of differential equations with different orders and degrees are given below

$$x^2 y - y'x = 0 \qquad \textit{order: 1 degree: 1}$$

$$\frac{d^2 y}{dx^2} + 3x(\frac{dy}{dx})^4 - 3y = 0 \qquad \textit{order: 2 degree: 1}$$

$$(y'')^3 + (y')^5 - 4y = 4x + 1 \qquad \textit{order: 2 degree: 3}$$

2.1.3 Classification by Linearity

Definition 2.1.3.1: An n-th order ordinary differential equation $F(x, y, y', \dots y^{(n)}) = 0$ is said to be linear if F is linear in $y, y', \dots y^{(n)}$. This means an n-th order ordinary differential equation is linear when $F(x, y, y', \dots y^{(n)}) = 0$ is

$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2.1.3.1)$$

Two important special cases of (2.1.3.1) are linear first order and linear second order. But here we will only consider linear first order ordinary differential equations (Zill and Cullen, 2005).

In other words, in a differential equation if every dependent variables and the degree of every derivatives with any order is one and the dependent variables itself and the derivatives do not lie as multiplication then the equation is called linear differential equation (Ross, 2004).

Definition 2.1.3.2: A nonlinear ordinary differential equation is simply one that is not linear (Ross, 2004).

Some examples of linear and nonlinear differential equations are given below

$$(y-x)dx+4xdy=0 \qquad \textit{linear}$$

$$y''-2y'+y=0 \qquad \textit{linear}$$

$$(1-y)y'+2y=e^x \qquad \textit{nonlinear}$$

$$\frac{\partial^2 y}{\partial x^2}+\sin y=0 \qquad \textit{nonlinear}$$

$$\frac{d^2y}{dx^2}+3(\frac{dy}{dx})^2+y=0 \qquad \textit{linear}$$

2.1.4 Classification by Type of Coefficients

Definition 2.1.4.1: A linear differential equation has constant coefficients if the coefficients of $y, y', y'', \ldots$ are all constants (Ross, 2004).

Definition 2.1.4.2: A linear differential equation has variable coefficients if the $y, y', y'', \ldots$ are multiplied by any variable (Ross, 2004).

Some examples of differential equations with constant and variable coefficients are given below

$$y''+y'+y=0 \qquad \textit{with constant coefficients}$$

$$xy''-2x^2y'+y=x \qquad \textit{with variable coefficients}$$

2.1.5 Classification by Homogeneity

Definition 2.1.5.1: For the linear differential equation

$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$ if $g(x)$=0 then the equation is called homogeneous (Bronson and Costa, 2006).

Definition 2.1.5.2: If a differential equation is not homogeneous then it is nonhomogeneous (Bronson and Costa, 2006).

Some examples of homogeneous and nonhomogeneous differential equations are given below

$$y'' - 2y' + y = 0 \qquad \textit{homogeneous}$$

$$\frac{dy}{dx} + x^2 y = x - e^x \qquad \textit{nonhomogeneous}$$

2.2 Nature of Solutions

Definition 2.2.1: A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives (Ross, 2004).

Definition 2.2.2: A general solution to an n-th order differential equation is a solution in which the value of the constant, c in the solution, may vary (Ross, 2004).

Definition 2.2.3: Let f be a real function defined for all x in a real interval I and having n-th derivative for all $x \in I$. The function f is called an explicit solution of differential equation (Ross, 2004).

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}) = 0 \qquad (2.2.3.1)$$

if $F(x, f(x), f'(x), \dots, f^n(x)) = 0$ is defined for all $x \in I$ and

if $F(x, f(x), f'(x), \dots, f^n(x)) = 0$ for all $x \in I$.

That is, the substitution of $f(x)$ and its various derivatives, respectively (2.2.3.1) reduces to (2.2.3.1) an identity on I (Ross, 2004).

Definition 2.2.4: A relation $g(x, y) = 0$ is called an implicit solution of (2.2.3.1) if this relation defines at least one real function f of the variable x on an interval I such that this function is an explicit solution of (2.2.3.1) on this interval (Ross, 2004).

An example of an explicit solution and implicit solution is given below

Example 2.2.4.1: Consider the function f defined for all real x by

$$f(x) = 2\sin x + 3\cos x \qquad (2.2.4.1)$$

is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad (2.2.4.2)$$

for all real x. Observing that

$f'(x) = 2cosx - 3sinx$ and $f''(x) = -2sinx - 3cosx$.

Substituting into the differential equation,

$(-2sinx - 3cosx) + (2\sin x + 3\cos x) = 0$ which holds for all real x.

Thus 2.2.4.1 is an explicit solution of (2.2.4.2)

Example 2.2.4.2: The relation $x^2 + y^2 - 25 = 0$ (2.2.4.3) is an implicit solution of the differential equation

$$y\frac{dy}{dx} + x = 0 \qquad (2.2.4.4)$$

on the interval I defined by $-5 < x < 5$. For the relation (2.2.4.3) defines two real functions

$f_1(x) = \sqrt{25 - x^2}$ and $f_2(x) = -\sqrt{25 - x^2}$ respectively for all real $x \in I$ and both of these two functions are explicit solutions of the differential equation (2.2.4.4).

Choosing $f_1(x) = \sqrt{25 - x^2}$ and calculating $f'_1(x) = \frac{-x}{\sqrt{25-x^2}}$

for all real $x \in I$. Substituting $f_1(x)$ and $f'_1(x)$ into (2.2.4.4), we obtain the identity

$x + (\sqrt{25 - x^2})(\frac{-x}{\sqrt{25-x^2}}) = 0$ or $x - x = 0$

which holds for all real $x \in I$. Thus the function $f_1(x)$ is an explicit solution of (2.2.4.4) on interval I (Ross, 2004).

Definition 2.2.5: On some interval I containing x_0 the problem is given as $\frac{d^n y}{dx^n} = f(x, y, y', ..., y^{n-1})$ subject to $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{n-1}(x_0) = y_{n-1}$

where $y_0, y_1, ..., y_{n-1}$ are arbitrarily specified real constants, is called an initial value problem. The value $x_0, y(x_0) = y_0, y'(x_0) = y_1, ..., y^{n-1}(x_0) = y_{n-1}$ are called inital conditions (Zill and Cullen 2005).

In other words, if all of the associated supplementary conditions relate to one x value, the problem is called initial value problem. If the conditions relate to two different x values, the problem is called boundary value problem (Ross, 2004).

An example of inital value problem is given below

Example 2.2.5.1: Consider a solution for $y' = 2x$ at $y(1) = 4$.

Integrating both sides, we get

$$y = x^2 + c,$$

where c is an arbitrary constant.Applying the initial condition we obtain $c = 3$. Therefore

$$y = x^2 + 3,$$

satisfies the requirements for the initial value problem (Ross, 2004).

Example 2.2.5.2: Consider a solution for $\frac{d^2y}{dx^2} + y = 0$ at

$$y(0) = 1 \text{ and } y(\pi/2) = 5.$$

In this example we seek for a solution at two different x values, therefore it is a boundary value problem.

Since the given differential equation is a second order linear ordinary differential equation and the roots of the auxiliary equation are complex numbers.The general solution of the differential equation is

$$y(x) = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x)$$

Applying the boundary conditions,

$$y(x) = 5sinx + 3cosx$$

satisfies the requirements for the boundary value problem (Ross, 2004).

3 ANALYTICAL AND NUMERICAL METHODS FOR SOLVING FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

The methods for solving first order ordinary differential equations can be categorized into two main title called analytical and numerical methods. Previously analytical methods were a major method for solving differential equations. These are the methods where we can get the exact solution. In order to write down solutions, we can use our knowledge of calculus, algebra and other mathematical fields,. Analytical methods are generally useful to work for simple models , however they are not enough for complex mathematical expressions that can not be solved by hand. It takes too much time to get the exact solution and sometimes it is not possible to reach to a solution.

Therefore, numerical methods are discovered. Numerical methods are the name of the methods that we get the solutions using algorithms.These methods help us to get the approximate solutions for the mathematical models, even though it is complicated. Any complex mathematical model can be solved by using computer programmes and can be executed. By using numerical methods the approximations can be nearly exact.

3.1 Analytical Methods for Solving First Order Ordinary Differential Equations

Analytical solutions are sometimes called closed-form solution or explicit solution. An equation is said to be a closed-form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set. These solutions are obtained by analytical methods (Edwards and Penney, 1996).

Here, numerous analytical methods are presented. There are several methods for solving ordinary differential equations. The exact results can be obtained by definite procedures with the aid of the calculus.

In this section, the methods are given for solving first order differential equations. The most general first order differential equation can be written as

$$\frac{dy}{dx} = f(x, y) \tag{3.1.1}$$

There is no general formula for the solution to (3.1.1) but there are several classification in order to solve it. First, the equation is classified and then it can be solved.

Here are the numerous methods that are used for first order differential equations. (Edwards and Penney, 1996).

3.1.1 Seperable Differential Equations

The first-order differential equation $\frac{dy}{dx} = H(x, y)$ is called seperable provided that $H(x, y)$ can be written as the product of a function of x and a function of y.

$\frac{dy}{dx} = g(x)h(y) = \frac{g(x)}{f(y)}$ where $h(y) = \frac{1}{f(y)}$. In this case the variables x and y can be separated-isolated on opposite sides of an equation-by writing informally the equation.

$$f(y)dy = g(x)dx$$

which we understand to be concise notation for the differential equation

$$f(y)\frac{dy}{dx} = g(x)$$

It is easy to solve this special type of differential equation simply by integrating both sides with respect to x :

$$\int f(y(x))\frac{dy}{dx}dx = \int g(x)dx + C$$

equivalently,

$$\int f(y)dy = \int g(x)dx + C$$

All that is required is that the antiderivatives.

$F(y) = \int f(y)dy$ and $G(x) = \int g(x)dx$ can be found.

After doing the integrations the implicit solution that can be solved for an explicit solution is found (Edwards and Penney, 1996).

Remark: If the dependent and independent variables are in the case of addition or subtraction, then the equation is not seperable (Ross, 2004).

The example of seperable equations is given below

Example 3.1.1.1: Solve the initial value problem $\frac{dy}{dx} = -6xy$ where $y(0) = 7$

Solution 3.1.1.1: Informally, we divide both sides of the differential equation by y and multiply each side by dx to get

$$\frac{dy}{y} = -6xdx, \qquad y(0) = 7$$

Hence

$$\int \frac{dy}{y} = \int -6xdx$$

$$\ln|y| = -3x^2 + C$$

It is seen from the initial condition $y(0) = 7$ that $y(x)$ is positive near $x = 0$, so we may delete the absolute value symbols:

$$\ln y = -3x^2 + C$$

and hence

$$y(x) = e^{-3x^2 + C} = e^{-3x^2} e^C = Ae^{-3x^2}$$

where $A = e^C$. The condition $y(0) = 7$ yields $A = 7$, so desired solution is $y(x) = 7e^{-3x^2}$ (Edwards and Penney, 1996).

3.1.2 Homogeneous Differential Equations*:*

If the right-side of the equations $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of the ratio $\frac{y}{x}$ only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. It means $\frac{dy}{dx} = f(x, y)$ is homogeneous

if $f(x, y) = g(\frac{y}{x})$ or $f(x, y) = h(\frac{y}{x})$ (Ross, 2004).

Suppose that $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ then let $\frac{y}{x} = \theta$, then $y = x\theta$.

Taking the derivation with respect to x ,

$$\frac{dy}{dx} = \theta + x\frac{d\theta}{dx} = F(\theta)$$

Therefore

$$F(\theta) - \theta = x\frac{d\theta}{dx}$$

It is seen that the equation can be reduced to seperable.

$$\frac{d\theta}{F(\theta) - \theta} = \frac{dx}{x}$$

Remark: A dictionary definition of "homogeneous" is "of a similar kind or nature." Consider a differential equation of the form

$$Ax^n y^n + \frac{dy}{dx} = Bx^p y^q + Cx^r y^s$$

whose polynomial coefficient functions are "homogeneous" in the sense that each of their terms has the same total degree, $m + n = p + q = r + s = K$ (Boyce and Diprima 2001).

The example of homogeneous equations is given below

Example:3.1.2.1: Solve the differential equation that is $(x^2 + y^2)dx + 2xydy = 0$

Solution 3.1.2.1:

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy} = -(\frac{x^2}{2xy} + \frac{y^2}{2xy}) \quad \text{then} \quad \frac{dy}{dx} = -\frac{1}{2}(\frac{x}{y} + \frac{y}{x})$$

If it is written in terms of $\frac{y}{x}$, and getting $\frac{dy}{dx} = -\frac{1}{2}(\frac{1}{\frac{y}{x}} + \frac{y}{x})$

Now the equation can be written as $\frac{dy}{dx} = -\frac{1}{2}(\frac{1+(\frac{y}{x})^2}{\frac{y}{x}})$

If $\frac{y}{x} = \theta$, then the differentiation becomes $\frac{dy}{dx} = \theta + \frac{d\theta}{dx}x$

Substituting into the the equation $\theta + \frac{d\theta}{dx}x = -\frac{1}{2}(\frac{1+\theta^2}{\theta})$

Therefore, $\frac{dx}{x} + \frac{d\theta}{\theta + (\frac{1+\theta^2}{2\theta})} = 0$

After some manipulations,

$$\frac{dx}{x} + \frac{2\theta d\theta}{1+3\theta^2} = 0$$

Using substitution $\frac{y}{x} = \theta$

Seperating the variables, the solution is $\ln x + \frac{1}{3}\ln\left|1+3\left(\frac{y}{x}\right)^2\right| = \frac{1}{3}\ln c$. (Trench, 2001).

3.1.3 Equations Reducible to Homogeneous

The general form of an equation reducible to homogeneous is given as

$$(a_1X + b_1Y + c_1)dX + (a_2X + b_2Y + c_2)dY = 0.$$

Since $(a_1 * X + b_1 * Y)dX + (a_2 * X + b_2 * Y)dY = 0$ is homogeneous equation, solving

$a_1X + b_1Y = -c_1$ and $a_2X + b_2Y = -c_2$ gives the shifting condition (h, k),

$X = x + h$, $Y = y + k$ and $dY = dy$, $dX = dx$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Bronson and Costa 2006).

The example of an equation reducible to homogeneous is given below

Example 3.1.3.1: Solve the equation $(X - 2Y + 1)dX + (4X - 3Y - 6)dY = 0$.

Solution 3.1.3.1: The equation is not homogeneous. Therefore we need to reduce the equation into homogeneous equation.

For $X - 2Y = -1$ and $4X - 3Y = 6$, we get $X = 3$ and $Y = 2$.

Choosing the shifting as $(3,2)$

$$X = x + 3 \qquad Y = y + 2$$

Since $dY = dy$ and $dX = dx$ we can write

$$(x + 3 - 2(y + 2) + 1)dx + (4(x + 3) - 3(y + 2) - 6)dy = 0.$$

Simplifying the equation we get the homogeneous equation

$$(x - 2y)dx + (4x - 3y)dy = 0.$$

Setting $\frac{y}{x} = \theta$, then $y = x\theta$.

Taking the derivations $dy = \theta dx + xd\theta$.

Replacing the equation we get

$$(x-2\theta x)dx+[4x-3\theta x][\theta dx+xd\theta]=0 \text{ and}$$

$$[x-2\theta x+4x\theta-3\theta^2 x)dx+[4x-3\theta x]xd\theta=0$$

After some operations

$$x(1+2\theta-3\theta^2)dx+x^2(4-3\theta)d\theta=0$$

Dividing by x, we get

$$(1+2\theta-3\theta^2)dx=-x\ (4-3\theta)d\theta .$$

Now it is seen that the equation is seperable

$$\int\frac{dx}{x}-\int\frac{4-3\theta}{1-2\theta-3\theta^2}d\theta=0 .$$

Integrating both sides we get

$$\ln|x|-\frac{1}{2}\left|\ln 3\left(\frac{y}{x}\right)^2-2\frac{y}{x}-1\right|=\ln c$$

Substituting $\frac{y}{x}=\theta$, we get

$$\ln|x|-\frac{1}{2}\left|\ln 3\left(\frac{y}{x}\right)^2-2\frac{y}{x}-1\right|=\ln c \text{ (Bronson and Costa, 2006)}$$

3.1.4 Exact Differential Equations

The general form for first order ordinary differential equation is given below

$$M(x,y)dx+N(x,y)dy=0 \qquad (3.1.4.1)$$

if there exists $F(x,y)$, which $\frac{\delta F}{\delta x}=M$ and $\frac{\delta F}{\delta y}=N$ (3.1.4.2)

then (3.1.4.1) is called exact differential equation (Bronson. and Costa 2006).

Theorem 3.1.4.1: If $M, N, M_{y,} N_x$ are continuous $\alpha < t < \beta$, $\gamma \le y \le \delta$ interval $M_y = N_x$ and then (3.1.4.1) is exact differential equation. It means that there exist $F(x, y)$ which satisfy (3.1.4.2) (Bronson and Costa 2006).

Firstly, for any function $g(y)$, the function $F(x,y) = \int M(x,y)dx + g(y)$ satisfies the condition $\frac{\delta F}{\delta x} = M$. We plan to choose $g(y)$ so that

$$N = \frac{\delta F}{\delta y} = \left(\frac{\delta}{\delta y} \int M(x,y)dx \right) + g'(y) \quad (3.1.4.3)$$

Therefore

$$g'(y) = N - \frac{\delta}{\delta y} \int M(x,y)dx \quad (3.1.4.4)$$

So it is indeed found the desired function $g(y)$ by integrating the equation (3.1.4.4). Substituting this result in equation (3.1.4.3) to obtain

$$F(x,y) = \int M(x,y)dx + \left(\int N(x,y) - \frac{\delta}{\delta y} \int M(x,y)dx \right) dy$$

as the desired function with $F_x = M$ and $F_y = N$ (Bronson and Costa 2006).

Remark: The equation $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$ is a necessary condition that the differential equation be $M(x,y)dx + N(x,y)dy = 0$ be exact. (Edwards and Penney, 1996).

The example of exact equations is given below

Example 3.1.4.1: Solve the differential equation

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

Solution 3.1.4.1: Let $M(x,y) = 6xy - y^3$ and $N(x,y) = 4y + 3x^2 - 3xy^2$. The given equation is exact because

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}$$

Integrating $\frac{\partial F}{\partial x} = M(x, y)$ with respect to x ,we get

$$F(x, y) = \int (6xy - y^3)dx = 3x^2 y - xy^3 + g(y)$$

Then taking the derivation with respect to y and set $\frac{\partial F}{\partial y} = N(x, y)$. This yields

$$\frac{\partial F}{\partial y} = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

And it follows that $g'(y) = 4y$. Hence $g(y) = 2y^2 + C_1$.

Therefore, a general solution of the differential equation is defined implicitly by the equation

$$3x^2 y - xy^3 + 2y^2 = C \text{ (Bronson and Costa, 2006)}$$

3.1.5 Equations Reducible to Exact

If $M_y \neq N_x$ then the equation is not exact. If the equation is not exact, it is necessary to construct a new function $f(x)$ or $g(y)$ in order to get the integrating factor $\mu(x, y)$. Finding the integrating factor in two different ways are given below

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ and then } \mu(x, y) = e^{\int f(x)dx} \text{ or}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y) \text{ and then } \mu(x, y) = g(y) = e^{-\int g(y)dy}$$

By multiplying the differential equation by integrating factor the differential equation is reduced to exact differential equation (Bronson and Costa 2006).

The example of non-exact equations is given below

Example 3.1.5.1: Solve the differential equation

$$(2x^2 + y)dx + (x^2 y - x)dy = 0$$

Solution 3.1.5.1: Calculating that $M_y = 1$ $N_x = 2xy - 1$. Therefore $M_y \neq N_x$ which means the differential equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - 2xy + 1}{x^2 y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x} = f(x)$$

Now, the integrating factor $\mu(x, y)$ can be found by using $f(x)$

$$\mu(x, y) = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$$

Multiplying both sides by x^{-2},

$$x^{-2}(2x^2 + y)dx + x^{-2}(x^2 y - x)dy = 0,$$

and organizing the equation,

$$(2 + \frac{y}{x^2})dx + (y - \frac{1}{x})dy = 0$$

Realizing that the differential equation is exact.

$$M_y = \frac{1}{x^2} = N_x = \frac{1}{x^2}$$

Now, we observe that the differential equation is reduced to exact form.

According to exact differential equation procedure

$$f(x, y) = \int (2 + \frac{y}{x^2})dx$$

Integrating the equation

$$F'(x,y) = 2x - \frac{y}{x} + \varphi(y) = -\frac{1}{x} + \varphi'(y) = y - \frac{1}{x}\ .$$

Therefore
$$\varphi'(y) = y$$
$$\varphi(y) = \frac{y^2}{2} + c$$

The general form of the equation is found as

$$F(x,y) = 2x - \frac{y}{x} + \frac{y^2}{2} + c \text{ (Trench, 2001).}$$

3.1.6 First Order Linear Differential Equations

In Section (3.1.1) it was seen how to solve a separable differential equation by integrating after multiplying both sides by an appropriate factor. For instance, to solve the equation.

$$\frac{dy}{dx} = 2xy \tag{3.1.6.1}$$

Multiplying both sides by the factor $\frac{1}{y}$ to get

$$\frac{1}{y}\frac{dy}{dx} = 2x \tag{3.1.6.2}$$

Because each side of the equation in (3.1.6.2) is recognizable as a derivative (with respect to the independent variable x), all that remains are two simple integrations, which yield $\ln y = x^2 + C$. For this reason, the function $\rho(y) = 1/y$ is called an integrating factor for the original equation in (3.1.6.1)

An integrating factor for a differential equation is a function $\rho(x,y)$ such that the multiplication of each side of the differential equation by $\rho(x,y)$ yields an equation in which each side is recognizable as a derivative. With the aid of the appropriate integrating factor, there is a standard technique for solving the linear first-order equation.

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{3.1.6.3}$$

on an interval on which the coefficient functions. $P(x)$ and $Q(x)$ are continuous. Multiplying each side of in equation (3.1.6.3) by the integrating factor.

$$\rho(x)=e^{\int P(x)dx}$$

The result is

$$e^{\int P(x)dx}\frac{dy}{dx}+P(x)e^{\int P(x)dx}y=Q(x)e^{\int P(x)dx} \tag{3.1.6.4}$$

Because

$$D_x[\int P(x)dx]=P(x)$$

The left-hand side is the derivative of the product $y(x)e^{\int P(x)dx}$ so equation (3.1.6.4) is equivalent to

$$D_x[y(x)e^{\int P(x)dx}]=Q(x)e^{\int P(x)dx}$$

Integration of both sides of this equation gives

$$y(x)e^{\int P(x)dx}=\int(Q(x)e^{\int P(x)dx})dx+C$$

Finally, solving for *y*, we obtain the general solution of the linear first-order equation in

$$y(x)=e^{-\int P(x)dx}[\int(Q(x)e^{\int P(x)dx})dx+C]$$ (Edwards and Penney, 1996).

The example of first order linear equations is given below

Example 3.1.6.1: Solve the inital value problem

$$\frac{dy}{dx}-y=\frac{11}{8}e^{-x/3} \qquad y(0)=-1$$

Solution 3.1.6.1: Here, $P(x)=-1$ and $Q(x)=\frac{11}{8}e^{-x/3}$ so the integrating factor is

$$\rho(x)=e^{\int(-1)dx}=e^{-x}$$

Multiplication of both sides of the given equation by e^{-x} yields

$$e^{-x}\frac{dy}{dx}-e^{-x}y=\frac{11}{8}e^{-4x/3}$$

which is recognized as $\frac{d}{dx}e^{-x}y=\frac{11}{8}e^{-4x/3}$

Hence integration with respect to x gives

$$e^{-x}y=\int\frac{11}{8}e^{-4x/3}dx=-\frac{33}{32}e^{-4x/3}+C$$

and multiplication by e^{x} gives the general solution

$$y(x)=Ce^{x}-\frac{33}{32}e^{-x/3}$$

Substitution of $x=0$ and $y=-1$ now gives $C=\frac{1}{32}$ so the desired solution is

$y(x)=\frac{1}{32}e^{x}-\frac{33}{32}e^{-x/3}=\frac{1}{32}(e^{x}-33e^{-x/3})$ (Edwards and Penney, 1996).

3.1.7 Special Types of Differential Equations

We now consider a special type of equation that can be reduced to a linear equation after appropriate substitution.

3.1.7.1 Bernoulli Equations

A first order ordinary differential equation of the form

$$\frac{dy}{dx}+p(x)y=g(x)y^{n} \tag{3.1.7.1}$$

is called a Bernoulli equation.

If $n=0$, then Bernoulli equation is actually a linear equation or $n=1$, then Bernoulli equation is seperable. However if $n\neq 0$ or $n\neq 1$, we must proceed in a different manner.

Suppose that $n \neq 0$ or $n \neq 1$. Then we need to use the transformation $v = y^{1-n}$.

We first multiply (3.1.7.1) by y^{-n}, therefore we express the equation as

$$y^{-n}\frac{dy}{dx} + y^{1-n}p(x) = g(x) \tag{3.1.7.2}$$

If we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Substituting the variables, (3.1.6.2) transforms to

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)v = g(x)$$

equivalently,

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)g(x)$$

Letting $p_1(x) = (1-n)p(x)$ and $g_1(x) = (1-n)g(x)$

The equation may be written as

$$\frac{dv}{dx} + p_1(x)v = g_1(x)$$

which is linear in v (Ross, 2004).

The example of Bernoulli Equation is given below

Example 3.1.7.1.1: Solve $\frac{dy}{dx} + y = xy^3$.

Solution 3.1.7.1.1: This problem is a Bernoulli differential equation where $n = 3$
.

We first multiply the equation by y^{-3}. Therefore we obtain

$$y^{-3}\frac{dy}{dx}+y^{-2}=x\,.$$

Substituting $v=y^{1-n}=y^{1-3}=y^{-2}$, and differentiating $\frac{dv}{dx}=-2y^{-3}\frac{dy}{dx}$.

Preceeding the differential equation reduces to the linear equation as

$$-\frac{1}{2}\frac{dv}{dx}+v=x$$

Writing the equation in the standard form

$$\frac{dv}{dx}-2v=-2x \qquad (3.1.7.1.1)$$

The integrating factor is

$$e^{\int p(x)dx}=e^{-\int 2dx}=e^{-2x}$$

Multiplying (3.1.7.1.1) by e^{-2x},

$$e^{-2x}\frac{dv}{dx}-2e^{-2x}v=-2xe^{-2x}$$

Integrating, we find

$$e^{-2x}v=\frac{1}{2}e^{-2x}(2x+1)+c$$

Simplifying the equation we find

$$v=x+\frac{1}{2}+ce^{2x}$$

where c is an arbitrary constant.

Replacing $v = \frac{1}{y^2}$, we get

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x} \text{ (Ross, 2004)}$$

3.1.7.2 Riccati Equation

Suppose that $P(x)$, $Q(x)$ and $R(x)$ are continuous functions.

$$\frac{dy}{dx} + P(x)y = Q(x)y^2 + R(x)$$

is the general form of Riccati equation.

The general solution for this equation can not be found by directly, but if they give one or more special solution we can find the general solution.

If $y = y_1(x)$ is the special solution for Riccati diffential equation, then the general solution;

$$y = z(x) + y(x)$$

where z is the dependent and x is the independent variable then the equation can be reduced to Bernoulli equation, or,

$$y = y_1(x) + \frac{1}{u(x)}$$

where u is the dependent and x is the independent variable then the equation can be reduced to linear equation (Bronson and Costa 2006).

Example 3.1.7.2.1: Given Riccati equation

$\frac{dy}{dx} + P(x)y = y^2 + R(x)$ for two special solution $y_1 = x$ and $y_2 = x - 1$.

Find $P(x)$, $Q(x)$, $R(x)$ and solution of the equation.

Solution 3.1.7.2.1: Since

$$y_1 = x \text{, then } dy_1 = dx \text{ or } \frac{dy_1}{dx} = 1.$$

Therefore

$$\frac{dy}{dx} + P(x)y = Q(x)y^2 + R(x)$$

$$1 + P(x)x = x^2 + R(x) \quad (3.1.6.2.1)$$

$$y_2 = x - 1 \text{, then } dy_2 = dx \text{ or } \frac{dy_2}{dx} = 1.$$

Therefore,

$$1 + P(x)(x-1) = (x-1)^2 + R(x) \qquad (3.1.7.2.2)$$

Solving the equations (3.1.7.2.1) and (3.1.7.2.2) together , we have

$$P(x) = 2x - 1 \text{ and } R(x) = x^2 - x + 1$$

Therefore,

$$\frac{dy}{dx} + (2x-1)y = y^2 + x^2 - x + 1 \qquad (3.1.7.2.3)$$

Since $y_1 = x$, then $y = z + x$ and taking the derivation $y' = z' + 1$.

Substituting into the (3.1.7.2.3),

$$z' + 1 + (2x-1).(z+x) = (z+x)^2 + x^2 - x + 1$$

After some operations we have,

$$\text{If } z' - z = z^2 \text{ then } \frac{dz}{dx} = z + z^2,$$

Multiplying by z^{-2},

$$z^{-2}z' - z^{-1} = 1$$

Substituting now , $z^{-1} = v$ then $v' = -z^{-2}z'$, we obtain

$$-v' - v = 1 \text{ or } v' + v = -1.$$

Here, we realize that the equation is linear differential equation. Applying the linear differential equation procedures we have

$$v = e^{-\int dx}[\int e^{\int dx}.(-1)dx]$$

After some manipulations

$$v = e^{-x}(e^{-x} + c)$$

Replacing with z ,

$$\frac{1}{z} = e^{-x}(e^{-x} + c)$$

Since $z = y - x$,

$$\frac{1}{y - x} = e^{-x}(e^{-x} + c)$$ (Bronson and Costa 2006).

3.1.7.3 Clairout Equation

$$y = xy' + f(y')$$

is the general form of Clairout equation. The general solution for this equation is

$$y = cx + f(c).$$

If we substitute $y' = p$ into the equation,

$$y = xp + f(p)$$

First derivation of x can be obtained as

$$y' = p + x\frac{dp}{dx} + \frac{dp}{dx}f'(p)$$

Substituting again $y' = p$,

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx}f'(p)$$

equivalently,

$$0 = xp' + p'f'(p)$$

Therefore

$$p'(x + f'(p)) = 0$$

Here, we obtain $p' = 0$ or $x + f'(p) = 0$

If $p' = 0$, then $y' = c$ and $y = xc + f(c)$ is the general solution.

If $x + f'(p) = 0$, then p can be solved as a function of x and $y = xp + f(p)$ is singular solution (Bronson and Costa 2006).

Example 3.1.7.3.1: Find the singular and general solution of the problem

$$y = xy' + y' - (y')^2 .$$

Solution 3.1.7.3.1: Here we realize that the equation is Clairaut equation.

Substituting $y' = p$,

$$y = xp + p - p^2$$

and the derivation is

$$y' = p + xp' + p' - 2pp'$$

Substituting again $y' = p$

$$p = p + xp' + p' - 2pp'$$

Therefore

$$p'(x + 1 - 2p) = 0$$

Here we have $p'=0$ or $x+1-2p=0$

For $p'=0$, $p=c$ then $y=cx+x-c^2$ is the general solution.

For $x+1-2p=0$, then $y=xp+p-p^2$.

Using $x=2p-1$, $y=xp+p-p^2$ is reduced to $y=(2p-1)p+p-p^2$.

Simplifying the equation, we obtain $y=p^2$.

Therefore $y=(\frac{x+1}{2})^2$,

equivalently $4y=(x+1)^2$ is the singular solution (Bronson and Costa 2006).

3.2 Numerical Methods for Ordinary Differential Equations

In Section (3.1), we examined to solve first order ordinary differential equations analytically. As an alternative to analytical methods, we can consider the use of numerical methods. Because sometimes a solution of a differential equation may not be able to getit analytically. Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with hand. The role of numerical methods in mathematical models has increased efficiently with the development of fast and efficient computers. Using any computer programming language, a numerical method can be reduced to a numerical algorithm which is a set of rules for solving a problem in finite number of steps. The numerical algorithms can be easily implemented to any mathematical models. Therefore we conclude this chapter with a method by which we can solve the differential equations numerically. We are going to mention about the well-known and basic numerical methods such as Euler, Heun and second order Runge-Kutta(RK2) and fourth order Runge-Kutta(RK4).

3.2.1 Euler's Method (Tangent Line Method)

It is the exception rather than the rule when a differential equation of the general form

$$\frac{dy}{dx}=f(x,y)$$

can be solved exactly and explicitly by elementary methods. For example, consider the simple equation

$$\frac{dy}{dx} = e^{-x^2} \qquad (3.2.1.1)$$

A solution of equation (3.2.1.1) is simply an antiderivative of e^{-x^2}. But it is known that every antiderivative of $f(x) = e^{-x^2}$ is a nonelementary function. function-one that cannot be expressed as a finite combination of the familiar functions of elementary calculus. Hence no particular solution of equation (3.2.1.1) is finitely expressible in terms of elementary functions. To find a simple explicit formula for a solution of (3.2.1.1) is doomed to failure. As a possible alternative, an old-fashioned computer plotter can be programmed to draw a solution curve that starts to the inital point (x_0, y_0) and attempts to thread its way through the slope field of a given differential equation $y' = f(x, y)$.The procedure the plotter carries out can be described as follows (Edwards and Penney, 1996).

• The plotter pen starts at the initial point (x_0, y_0) and moves a tiny distance along the slope segment though (x_0, y_0). This takes it to the point (x_1, y_1).

•At (x_1, y_1) the pen changes direction, and now moves a tiny distance along the slope segment through this new starting point (x_1, y_1). This takes it to the next starting point (x_2, y_2).

•At (x_2, y_2) the pen changes direction again, and now moves a tiny distance along the slope segment through. (x_2, y_2) This takes it to the next starting point (x_3, y_3) (Edwards and Penney, 1996).

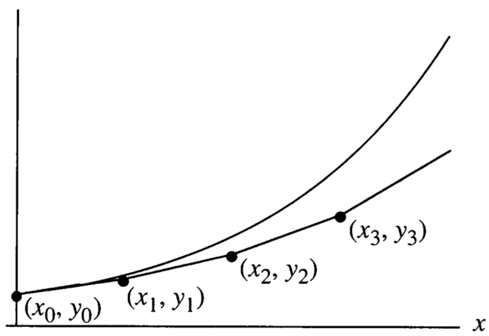

Figure 3.2.1.1: The first fewsteps in approximating a solution curve

Leonhard Euler--the great 18th-century mathematician for whom so many mathematical concepts, formulas, methods, and results are named did not have a computer plotter, and his idea was to do all this numerically rather than graphically.

In order to approximate the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0 \tag{3.2.1.2}$$

First, choosing a fixed (horizontal) step size h to use in making each step from one point to the next. Suppose that it is started at the initial point (x_0, y_0) and after n steps have reached the point (x_n, y_n). Then the step from (x_n, y_n) to the next point (x_{n+1}, y_{n+1}) is illustrated in Figure (3.2.1.2).

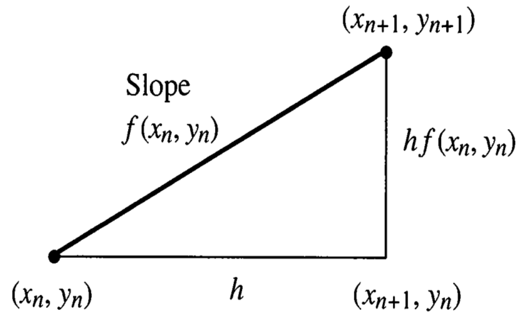

Figure 3.2.1.2: The step from (x_n, y_n) to (x_{n+1}, y_{n+1})

The slope of the direction segment through (x_n, y_n) is $m = f(x_n, y_n)$. Hence a horizontal change of h from x_n to x_{n+1} corresponds to a vertical change of $m.h = h.f(x_n, y_n)$ from y_n to y_{n+1}.

Therefore the coordinates of the new point (x_{n+1}, y_{n+1}) are given in terms of the old coordinates by

$$x_{n+1} = x_n + h \qquad y_{n+1} = y_n + hf(x_n, y_n) \tag{3.2.1.3}$$

Given the initial value problem in (3.2.1.2), Euler's method with step size h consists of starting with the initial point (x_0, y_0) and applying the formulas

$$x_1 = x_0 + h \quad y_1 = y_0 + h.f(x_0, y_0)$$

$$x_2 = x_1 + h \quad y_2 = y_1 + h.f(x_1, y_1)$$

$$x_3 = x_2 + h \quad y_3 = y_2 + h.f(x_2, y_2)$$

$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$

to calculate successive points (x_1, y_1), (x_2, y_2), (x_3, y_3),…on an approximate solution curve.(Edwards and Penney, 1996).

Example 3.2.1.1: Apply Euler's method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x + \frac{1}{5}y \qquad y(0) = -3$$

(a) first with step size $h = 1$ on the interval $[0,5]$,

(b) then with step size $h = 0.2$ on the interval $[0,1]$.

Solution 3.2.1.1: (a) With $x_0 = 0$, $y_0 = -3$, $f(x, y) = x + \frac{1}{5}y$ and $h = 1$ the iterative formula in (3.2.1.2) yields the approximate values

$$y_1 = y_0 + h[x_0 + \frac{1}{5}y_0] = (-3) + (1)(0 + \frac{1}{5}(-3) = -3.6$$

$$y_2 = y_1 + h[x_1 + \frac{1}{5}y_1] = (-3.6) + (1)(1 + \frac{1}{5}(-3.6) = -3.32$$

$$y_3 = y_2 + h[x_2 + \frac{1}{5}y_2] = (-3.32) + (1)(2 + \frac{1}{5}(-3.32) = -1.984$$

$$y_4 = y_3 + h[x_3 + \frac{1}{5}y_3] = (-1.984) + (1)(3 + \frac{1}{5}(-1.984) = 0.6192$$

$$y_5 = y_4 + h[x_4 + \frac{1}{5}y_4] = (0.6192) + (1)(4 + \frac{1}{5}(0.6192) \approx 4.7430$$

at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $x_5 = 5$. Note how the result of each calculation feeds into the next one. The resulting table of approximate values is

x	0	1	2	3	4	5
Approx.y	−3	−3.6	−3.32	−1.984	0.6192	4.7430

Figure:3.2.1.3: The table of approximations with Euler's method when $h = 1$

(b) Starting a fresh with $x_0 = 0$, $y_0 = -3$, $f(x, y) = x + \frac{1}{5}y$ the approximate values are

$$y_1 = y_0 + h[x_0 + \frac{1}{5}y_0] = (-3) + (0.2)[0 + \frac{1}{5}(-3)] = -3.12$$

$$y_2 = y_1 + h[x_1 + \frac{1}{5}y_1] = (-3.12) + (0.2)(0.2 + \frac{1}{5}(-3.12) \approx -3.205$$

$$y_3 = y_2 + h[x_2 + \frac{1}{5}y_2] = (-3.205) + (0.2)(0.4 + \frac{1}{5}(-3.205) \approx -3.253$$

$$y_4 = y_3 + h[x_3 + \frac{1}{5}y_3] = (-3.253) + (0.2)(0.6 + \frac{1}{5}(-3.253) \approx -3.263$$

$$y_5 = y_4 + h[x_4 + \frac{1}{5}y_4] = (-3.263) + (0.2)(0.8 + \frac{1}{5}(-3.263) \approx -3.234$$

At the points $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$ and $x_5 = 1$.

x	0	0.2	0.4	0.6	0.8	1
Approx.y	−3	−3.12	−3.205	−3.253	−3.263	−3.234

Figure 3.2.1.4: The table of approximation when $h = 0.2$

Figure 3.2.1.5 shows the graph of this approximation, together with the graphs of the Euler approximations obtained with step sizes $h = 0.2$ and 0.05, as well as the graph of the exact solution.

$$y(x) = e^{x/5} - 5x - 25$$

It is that decreasing the step size increases the accuracy, but with any single approximation, the accuracy decreases with distance from the initial point.

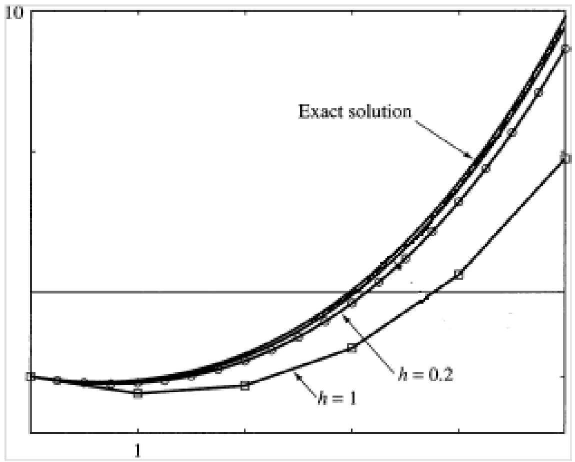

Figure 3.2.1.5: Graphs of Euler approximations with step sizes $h = 1$ and $h = 0.2$

High accuracy with Euler's method usually requires a very small step size and hence a larger number of steps than can reasonably be carried out by hand(Edwards and Penney, 1996).

3.2.2 The Improved Euler Method (Heun's Method)

Euler's method is rather unsymmetrical. It uses the predicted slope $k = f(x_n, y_n)$ of the graph of the solution at the left-hand endpoint of the interval $[x_n, x_n + h]$ as if it were the actual slope of the solution over that entire interval. Attention is now to a way in which increased accuracy can easily be obtained; it is known as the improved Euler method (Edwards and Penney, 2006).

Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0$$

suppose that after carrying out n steps with step size h we have computed the approximation

y_n to the actual value $y(x_n)$ of the solution at $x_n = x_0 + nh$. Euler method can be used to obtain a first estimate-which we now call u_{n+1} rather than y_{n+1} of the value of the solution at. $x_{n+1} = x_n + h$. Thus

$$u_{n+1} = y_n + h.f(x_n, y_n) = y_n + h.k_1$$

Now that $u_{n+1} \approx y(x_{n+1})$ has been computed, taking

$$k_2 = f(x_{n+1}, u_{n+1})$$

as a second estimate of the slope of the solution curve $y = y(x)$ at $x = x_{n+1}$.

Of course, the approximate slope $k_1 = f(x_n, y_n)$ at $x = x_{n+1}$ has already been calculated. Why not average these two slopes to obtain a more accurate estimate of the average slope of the solution curve over the entire subinterval $[x_n = x_{n+1}]$? This idea is the essence of the improved Euler method. Figure 3.2.2.1 shows the geometry behind this method (Edwards and Penney, 1996).

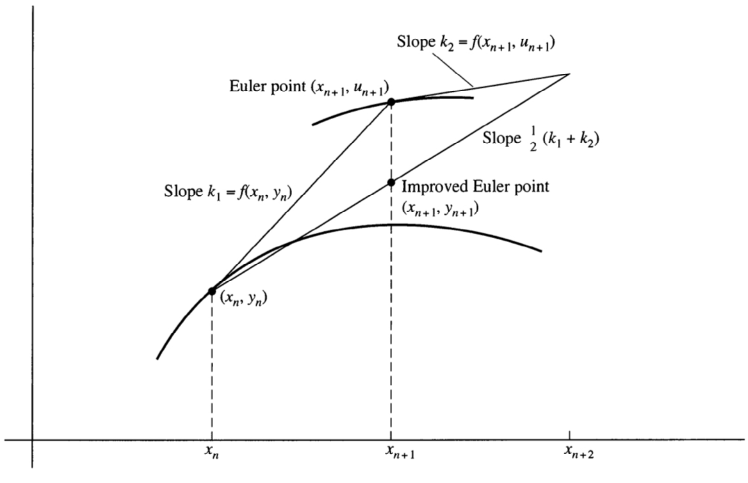

Figure 3.2.2.1: The Improved Euler Method geometrically

Remark: A predictor–corrector method is an algorithm that proceeds in two steps. The improved Euler method is one of a class of numerical techniques known as predictor-corrector methods. First a predictor u_{n+1} of the next y value is computed; then it is used to correct itself. Thus the improved Euler method with step size h consists of using the

predictor

$$u_{n+1} = y_n + h.f(x_n + y_n)$$

and the corrector

$$y_{n+1} = y_n + h.\frac{1}{2}[f(x_n, y_n) + f(x_{n+1}, u_{n+1})]$$

Iteratively to calculate succesive approximations y_1, y_2, y_3 …to the true values $y(x_1), y(x_2), y(x_3)$ … of the exact solution $y = y(x)$ at the points x_1, x_2, x_3 … respectively.

In other words, firstly, the prediction step calculates a rough approximation of the desired quantity. Second, the corrector step refines the initial approximation by using another means (Edwards and Penney, 1996).

Example 3.2.2.1: Apply Improved Euler's method to the initial value problem

$$\frac{dy}{dx} = x + y\,,\ y(0) = 1$$

with exact solution $y(x) = 2e^x - x - 1$.

Solution 3.2.2.1: The predictor-corrector formulas for the improved Euler method are

$$u_{n+1} = y_n + h.(x_n + y_n)$$

$$y_{n+1} = y_n + h.\frac{1}{2}[(x_n + y_n) + (x_{n+1} + u_{n+1})]$$

with step size $h = 0.1$ we calculate

$$u_1 = 1 + (0.1)(0 + 1) = 1.1$$

$$y_1 = 1 + (0.05)[(0 + 1) + (0.1 + 1.1)] = 1.11$$

$$u_2 = 1.11 + (0.1)(0.1 + 1.11) = 1.231$$

$$y_2 = 1.11 + (0.05)[(0.1 + 1.11) + (0.2 + 1.231)] = 1.24205$$

and so forth. The table in Figure 3.2.2.2 compares the results obtained using the improved Euler method with those obtained previously using the "unimproved" Euler method. When the same step size $h = 0.1$ is used, the error in the Euler approximation to $y(1)$ is 7.25% but the error in the improved Euler approximation is only 0.24%. (Edwards and Penney, 1996)

x	*Euler Method* $h = 0.1$ *Values of* y	*The Improved Euler Method* $h = 0.1$ *Values of* y	*Actual* y
0.1	1.1000	1.1100	1.1103
0.2	1.2200	1.2421	1.2428
0.3	1.3620	1.3985	1.3997
0.4	1.5282	1.5818	1.5836
0.5	1.7210	1.7949	1.7974
0.6	1.9431	2.0409	2.0442
0.7	2.1974	2.3231	2.3275
0.8	2.4872	2.6456	2.6511
0.9	2.8159	3.0124	3.0192
1.0	3.1875	3.4282	3.4366

Figure 3.2.2.2: Euler and improved Euler approximations

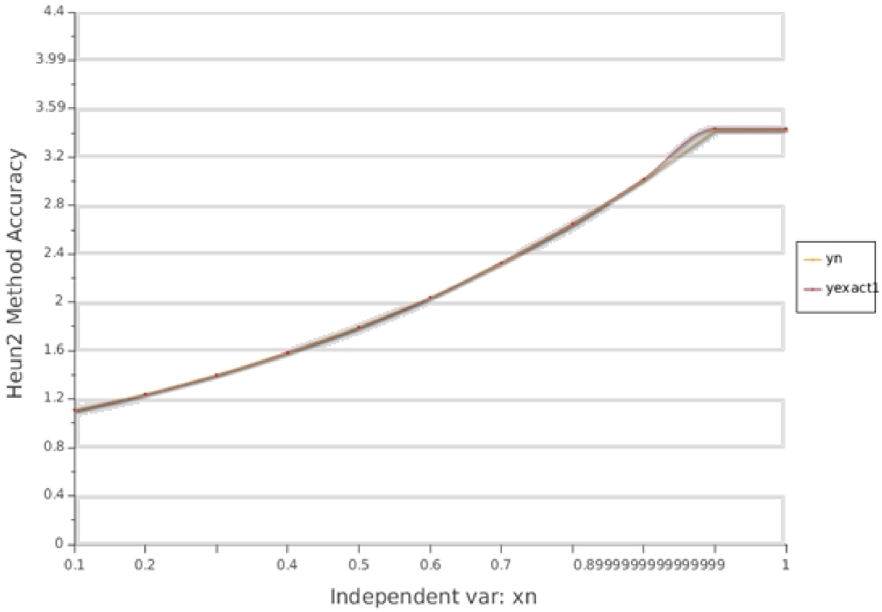

Figure 3.2.2.3: The graph of the example with Improved Euler Method when $h = 0.1$

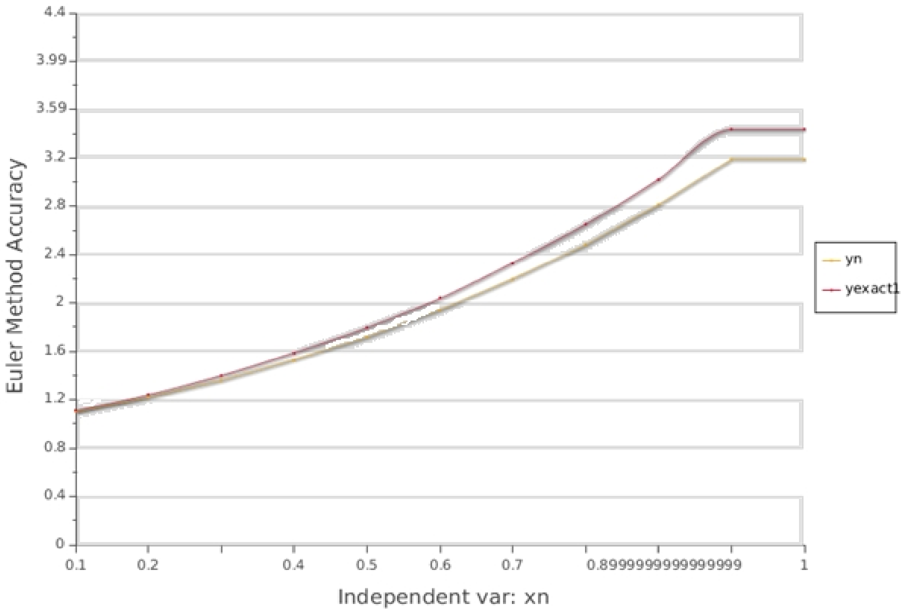

Figure 3.2.2.4: The graph of the example with Euler Method when $h = 0.1$

3.2.3 Runge-Kutta Methods

Fundamentally, all Runge-Kutta methods are generalizations of the basic Euler formula (3.2.1.3) in that the slope function f is replaced by a weighted average of slopes over the interval . $x_n \le x \le x_{n+1}$ That is,

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + ... w_m k_m) \qquad (3.2.3.1)$$

Here the weights w_i, $i = 1, 2, ..., m$ are constants that generally satisfy $(w_1 + w_2 + ... + w_m) = 1$ and each $k_i, i = 1, 2, ..., m$, is the function f evaluated at a selected point (x, y) for which $x_n \le x \le x_{n+1}$. It is seen that the k_i are defined recursively. The number m is called the order of the method. Observe that by taking $m = 1$, $w_1 = 1$ and $k_1 = f(x_n, y_n)$ and getting the familiar Euler Formula $y_{n+1} = y_n + h.f(x_n, y_n)$ .Hence Euler's method is said to be a first-order Runge-Kutta method (Zill and Cullen, 2005).

In general Runge-Kutta method gives more accurate result than Euler's method at the same step length. However, Runge-Kutta method is more difficult to use. Because Runge-Kutta methods requires more computation than Euler's method. There are several Runge-Kutta methods but here we will give two of them that is mostly used.

3.2.3.1 Second-Order Runge-Kutta Method

To further illustrate (3.2.3.1) we consider now a second-order Runge-Kutta procedure. This consists of finding constants or parameters w_1, w_2, α and β so that the Formula

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2) \qquad (3.2.3.1.1)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

So, how do we find the unknowns w_1, w_2, α, β .

For purpose, it suffices to say that this can be done whenever the constants satisfy

$$w_1 + w_2 = 1 \qquad w_2\alpha = \frac{1}{2} \qquad w_2\beta = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally the value of w_2, is chosen to evaluate the other three constants. The three values generally used for w_2 are $\frac{1}{2}, 1, \frac{2}{3}$ and known as Heun's Method, the midpoint method and Ralston's method (Zill and Cullen, 2005).

Remark: The choice $w_2 = \frac{1}{2}$ $w_1 = \frac{1}{2}$ $\alpha = 1$ and $\beta = 1$ and so (3.2.3.1.1) becomes $y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$ where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + h, y_n + hk_1)$. It gives us the explanation behind Heun's method (Zill and Cullen, 2005).

Example 3.2.3.1.1: Apply a Second-Order Runge-Kutta Method to the equation below and find $y(3)$ with $h = 1.5$, $x_0 = 0$, $y_0 = 5$.

$$\frac{dy}{dx} = 3e^{-x} - 0.4y$$

Solution 3.2.3.1.1: Here in this example,

$$k_1 = f(x_0, y_0) = f(0,5) = 3e^{-0} - 0.4(5) = 1$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1h) = f(0 + \frac{1}{2}(1.5), 5 + \frac{1}{2}(1)(1.5)) = f(0.25, 5.25)$$
$$= 3e^{-0.25} - 0.4(5.25) = -0.08829$$

$$y_1 = y_0 + k_2h = 5 + (-0.8829)(1.5) = 3.676 \approx y(1.5) \text{ since } x_1 = x_0 + h = 0 + 1.5 = 1.5$$

Therefore, $x_1 = 1.5$ and $y_1 = 3.676$.

$$k_1 = f(x_1, y_1) = f(1.5, 3.676) = 3e^{-1.5} - 0.4(3.676) = -0.8009$$

$$k_2 = f(x_1, \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h) = f(1.5 + \frac{1}{2}(1.5), 3.676 + \frac{1}{2}(-0.8009)(1.5))$$
$$= f(2.25, 3.075) = 3e^{-2.25} - 0.4(3.075) = -0.9131$$

$y_2 = y_1 + k_2 h = 3.676 + (-0.9138)(1.5) = 2.304) \approx y(3)$ since
$x_2 = x_1 + h = (1.5 + 1.5) = 3$

In conclusion, $y(3) \approx 2.304$.

x	*The Improved Euler Method* $h = 1.5$ *Values of* y	*Standard Runge-Kutta order2* *Values of* y	*Actual y*
0	5	5	5
1.5	4.3020	3.676	4.372465
3	2.8080	2.8089	2.763006

Figure 3.2.3.1.1: The table of approximation with Improved Euler and Standard Runge-Kutta

In this example the Improved Euler Method gives better approximation because as a second order Runge-Kutta method we have chosen the midpoint method (Zill and Cullen, 2005).

3.2.3.2 Fourth-Order Runge-Kutta Method

More often used is fourth-order Runge-Kutta procedure consists of finding parameters so that the formula

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \alpha_1 h, y_n + \beta_1 h k_1)$$

$$k_3 = f(x_n + \alpha_2 h, y_n + \beta_2 h k_1 + \beta_3 h k_2)$$

$$k_4 = f(x_n + \alpha_3 h, y_n + \beta_4 h k_1 + \beta_5 h k_2 + \beta_6 h k_3)$$

The most commonly used set of values for the parameters yields the following result:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \qquad (3.2.3.2.1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

While other fourth-order formulas are easily derived, the algorithm summarized in (3.2.3.2.1) is so widely used and recognized as a valuable computational tool it is often referred to as the fourth-order Runge-Kutta method or the classical Runge-Kutta method (Zill and Cullen, 2005).

Example 3.2.3.2.1: Use the Runge-Kutta method with $h = 0.1$ to obtain an approximation to $y(1.5)$ for the solution of $y' = 2xy$, $y(1) = 1$.

Solution 3.2.3.2.1: For the sake of illustration let us compute the case when $n = 0$. From (3.2.3.2.1) we find

$$k_1 = f(x_0, y_0) = 2x_0y_0 = 2$$

$$k_2 = f(x_0 + \frac{1}{2}(0.1), y_0 + \frac{1}{2}(0.1)2) = 2(x_0 + \frac{1}{2}(0.1))(y_0 + \frac{1}{2}(0.2)) = 2.31$$

$$k_3 = f(x_0 + \frac{1}{2}(0.1), y_0 + \frac{1}{2}(0.1)2.31) = 2((x_0 + \frac{1}{2}(0.1))(y_0 + \frac{1}{2}(0.231)) = 2.34255$$

$$k_4 = f(x_0 + 0.1, y_0 + (0.1)2.34255) = 2(x_0 + 0.1)(y_0 + 0.234255) = 2.715361$$

and therefore

$$y_1 = y_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.1}{6}(2 + 2(2.31) + 2(2.34255) + 2.715361) = 1.23367435$$

x_n	y_n	*Actual Value*
1.00	1.0000	1.0000
1.10	1.2337	1.2337
1.20	1.5527	1.5527
1.30.	1.9937	1.9937
1.40	2.6116	2.6117
1.50	3.4902	3.4904

Figure 3.2.3.2.1: The table of RK4 approximation with $h = 0.1$

The remaining calculations are summarized in Figure 3.2.3.2.1, whose entries are rounded to four decimal places.

x_n	*Euler*	*Improved Euler*	*Runge-Kutta (order 4)*	*Actual Value*
1.00	1.0000	1.0000	1.0000	1.0000
1.10	1.2000	1.2320	1.2337	1.2337
1.20	1.4640	1.5479	1.5527	1.5527
1.30.	1.8154	1.9832	1.9937	1.9937
1.40	2.2874	2.5908	2.6116	2.6117
1.50	2.9278	3.4509	3.4902	3.4904

Figure 3.2.3.2.2: The table of the comparison of the given numerical methods with $h = 0.1$

In Section 3.2, we presented the numerical methods. We started with one of the fundamental methods that is Euler. As an alternative method, we introduced Improved Euler Method. Runge-Kutta's methods are also important for soving ordinary differential equations. Therefore we mentioned not only two order Runge-Kutta but also fourth order of this method. Meeting each numerical solution, we concluded one example in order to get familiar with the method. Moreover for some examples, we compared the results on different methods (Zill and Cullen, 2005).

3.3 Errors in Numerical Approximations

In this section, we will explain the errors and the types of errors occcurs in numerical approximation.

The use of a numerical procedure, on an initial value problem raises a number of questions that must be answered before the approximate numerical solution can be accepted as satisfactory. One of these is the question of convergence. That is, as the step size h tends to zero, do the values of the numerical approximation $y_1, y_2, ... y_n, ...$ approach the corresponding values of the actual solution? If we assume that the answer is affirmative, there remains the important practical question of how rapidly the numerical approximation converges to the solution. In other words, how small a step size is needed in order to guarantee a given level of accuracy? We want to use a step size that is small enough to ensure the required accuracy, but not too small. An unnecessarily small step size slows down the calculations and in some cases may even cause a loss of accuracy (Boyce and Diprima, 2001).

There are three fundamental sources of error in approximating the solution of an initial value problem numerically.

- The formula, or algorithm, used in the calculations is an approximate one. For instance, the Euler formula uses straight-line approximations to the actual solution.
- Except for the first step, the input data used in the calculations are only approximations to the actual values of the solution at the specified points.
- The computer used for the calculations has finite precision; in other words, at each stage only a finite number of digits can be retained.

Let us temporarily assume that our computer can execute all computations exactly; that is, it can retain infinitely many digits (if necessary) at each step. Then the difference E_n between the solution $y = \phi(t)$ of the initial value problem and its numerical approximation y_n at the point $t = t_n$ is given by $E_n = \phi(t_n) - y_n$ The error *En* is known as the global truncation error. It arises entirely from the first two error sources listed above by applying an approximate formula to approximate data (Boyce and Diprima 2001).

However, in reality it is carried out the computations using finite precision arithmetic, which means that we can keep only a finite number of digits at each step. This leads to a round-off error R_n defined by $R_n = y_n - Y_n$ where Y_n is the value actually computed from the given numerical method. The absolute value of the total error in computing $\phi(t_n)$ is given by

$$|\phi(t_n) - Y_n| = |\phi(t_n) - y_n + y_n - Y_n|$$

Making use of the triangle inequality, we obtain

$$|\phi(t_n) - y_n| + |y_n - Y_n| \le |E_n| + |R_n|$$

Thus the total error is bounded by the sum of the absolute values of the global truncation and round-off errors. It is possible to obtain useful estimates of the global truncation error. The round-off error is more difficult to analyze, since it depends on the type of computer used, the sequence in which the computations are carried out, the method of rounding off, and so forth.

It is often useful to consider separately the part of the global truncation error that is due only to the use of an approximate formula.We can do this by assuming at the *n*th step that the input data are accurate, that $y_n = \phi(t_n)$ This error is known as the local truncation error (Boyce and Diprima, 2001).

Remark: An important consideration in using numerical methods to approximate the solution of an initial value problem is the stability of the method. Simply stated, a numerical method is stable if small changes in the initial condition result in only small changes in the computed solution. A numerical method is said to be unstable if it is not stable. The reason that stability considerations are important is that in each step after the first step of a numerical technique (Boyce and Diprima, 2001).

We are essentially starting over again with a new initial-value problem, where the initial condition is the approximate solution value computed in the preceding step. Because of the presence of round-off error, this value will almost certainly vary at least slightly from the true value of the solution. Besides round-off error, another common source of error occurs in the initial condition itself; in physical applications the data are often obtained by imprecise measurements (Zill and Cullen, 2005).

4 APPLICATIONS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this section we will consider the applications of first order ordinary differential equations. We will look for the solutions both anaytically and numerically. For the analytical solution, we will use the given analytical methods in Section 3.1 and for the numerical solution we will discuss the given methods in Section 3.2. In given examples we can get both analytical and numerical solution. However our concern is getting the best numerical method for each problem. Therefore, we will calculate not only the searching values but also spesific values for understanding the behaviour of the solution for each problem.

The solutions of the problems will give us to make comparisons between the methods. We will mention familiar mathematical models for first order ordinary differential equations. We will classify the applications in different kind of areas. Firstly, we will present the applications of mechanical problems which is a branch of engineering. These problems are based on Newton's Second Law. In these problems, we will deal with the acting force on a body or an object. Secondly, we will discuss a branch of chemistry called mixture problems. In mixture problems, there will be substance and we will add or subtract the substance at the specified rate. Then we will mention about cooling and warming problems related to physics. This mathematical model build according to Newton's Cooling Law. In these problems, we will observe the temperature of an object whether it is cooling or warming. After the dealing with cooling and warming problems, the financial problems that are branch of economics follow up. These problems are related to saving money and annual interest. This section ends up with the growth and decay problems. These problems are part of sociology. In these problems, we will mention logistic equation and bacterial populations.

Before starting the models, we should recognize the units. In applications there are three main sets of units in use for length, mass, force, and time: In unit systems, CGS including centimeter(cm), dyne(d) and gram(g); MKS including meter(m), newton (N) and kilogram(kg); British System including foot(ft), pound(lb), and slug(sl) (Trench, 2001).

4.1 Mechanical Problems

Before the applications in mechanics, let us remember some basic principles of the subject. The momentum of a body is defined to the product of the mass and the velocity. The velocity v and the momentum mv are vector quantities (Ross, 2004).

The time rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the direction of this resultant force according to Newton's Second Law (Ross,2004).

In mathematical language, the basic law of mechanics is given below

$$\frac{d}{dt}(mv) = KF$$

where m is the mass of a body , v is the velocity, F is the resultant force acting upon it and K is a constant of proportionality.

If the m is considered constant, this reduces to

$$m\frac{dv}{dt} = KF \quad \text{or} \quad a = K\frac{F}{m} \quad \text{or} \quad F = kma$$

where $k = \frac{1}{K}$ and $a = dv/dt$ is the acceleration of the body.

Obviously, the simplest system of the form for which $k = 1$. Then the system is reduced to $F = ma$. The instantaneous velocity of a body is the time rate of change of x, $v = \frac{dx}{dt}$ and the instantaneous acceleration is the time rate of change of v. Note that x, v and a are vector quantities.

Let us now apply Newton's second law to a freely falling body. If the mass of body is m and the weight of the body is w then the only force acting on the body is its weight and the acceleration is the gravity. Therefore $F = ma$ reduces to $w = mg$ .According to Newton's Second Law is $F = ma$, where acting forces on body $F = F_1 + F_2$, let $m = \frac{w}{g}$ and taking $g = 32$ (Ross, 2004).

Remark: g is the acceleration due to gravity at Earth's surface. This quantity has been determined experimentally. Approximate values of g are

g = 980 cm/s2 (CGS)

g = 9.8 m/s2 (MKS)

g = 32 ft/s2 (British)

Example 4.1.1: A skydiver equipped with parachute and other essential equipment falls from rest toward the earth. The total weight of the man plus the equipment is 160 lb. Before the parachute opens, the air resistance (in pounds) is numerically equal to $\frac{1}{2}v$, where v is the velocity (in feet per second). The parachute opens 5 secs after the fall begins. Find the velocity of the skydriver when parachute opens (Ross, 2004).

Analytical Solution 4.1.1*:* Since Newton's Second Law is $F = ma$, where acting forces on body $F = F_1 + F_2$, let $m = \frac{w}{g}$ and take $g = 32$. Since air resistance and weight are in different direction, we obtain

$$5\frac{dv}{dt} = 160 - \frac{1}{2}v \qquad (4.1.1)$$

Note that the skydriver was initally at rest $v = 0$ when $t = 0$ so $v(0) = 0$.

After the parachute opens, is formulated as

$$5\frac{dv}{dt} = 160 - \frac{5}{8}v^2 \text{ and } v(5) = v_1 \qquad (4.1.2)$$

Seperating variables on (4.1.1) we obtain

$$\frac{dv}{v - 320} = -\frac{1}{10}dt$$

after the integration it yields

$$\ln(v - 320) = -\frac{1}{10}t + c_0$$

which is simply equal to

$$v = 320 + ce^{-t/10}$$

which is valid for $0 \le t \le 5$

Applying the initial condition to (4.1.3) we get

$$v = 320(1 - e^{-t/10})$$

In particular when $t = 5$ we obtain

$$v_1 = 320(1 - e^{-1/2}) \approx 125.91018 \text{ ft/sec}$$

which is the velocity when parachute opens (Ross, 2004).

Numerical Solution 4.1.1: In this problem, we get the analytical solution above. Now, we need to approximate the solution numerically. Therefore we need to execute all the given methods and decide which one approximates better.

Here, we choose step size $h = 1$ an calculate to 5 decimal places. Since we try to find the fifth second, let us divide into 5 intervals.

t_n	*Euler (h=1)*	*Heun (h=1)*	*Runge-Kutta order 2 (h=1)*	*Runge-Kutta order 4 (h=1)*	v_{exact}
1	32	30.4	30.4	30.452	30.45203
2	60.8	57.912	57.912	58.00611	58.00616
3	86.72	82.81036	82.81036	82.93810	82.93817
4	110.048	105.34337	105.34337	105.49750	105.49759
5	131.0432	125.73575	125.73575	**125.91010**	**125.91019**

Figure 4.1.1.1: The table of approximation of Example 4.1.1

After calculating the approximation, it is seen that the closer approximation is obtained by RK4 but anyway we should make the error analysis.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
1	1.54797	0.05203	0.05203	0.00003
2	2.79384	0.09416	0.09416	0.00005
3	3.78183	0.12781	0.12781	0.00007
4	4.55041	0.15420	0.15420	0.00009
5	5.13301	0.17444	0.17444	0.00009

Figure 4.1.1.2: The table of error analysis of Example 4.1.1

The tables are shown that Euler,Heun and RK2 is far from the exact solution.The percent error is 4.07672% for Euler. On the other hand Heun and RK2 have 0.13854%, which means they give the same error for 5 decimal places. RK4 gives the best approximation for this example. Even though the error increased with the increasement of the point, the approximation is better comparing with the other methods. Because the percentage error is 0.00007%. for RK 4.

Example 4.1.2: An object weighing 48 lb is released from rest at the top of a plane metal slide that is inclined 30° to the horizontal. Air resistance in numerically equal to one-half the velocity and the coefficient of friction is one-quarter. What is the velocity of the object 2 sec after it is released? (Ross, 2004)

Analytical Solution 4.1.2: The line of motion is along the slide. We choose the origin at the top and the positive x direction down the slide. If we temporarily neglect the friction and air resistance, the forces acting upon the object A are its weight and and the normal force N exerted by the slide which acts in an upward direction perpendicular to the slide.

The componenents of the weight parallel and perpendicular to the slide have magnitude

$$48\sin 30° = 24 \text{ and } 48\cos 30 = 24\sqrt{3}$$

The components perpendicular to the slide are in equilibrium and hence the normal force N has magnitude $24\sqrt{3}$.

The component on the weight paralel to the plane since this force acts downward it is $F_1 = 24$

F_2, the frictional force having numerical value $\frac{1}{4}24\sqrt{3}$. Since this force acts upward direction along the side we have $F_2 = -6\sqrt{3}$.

The air resistance that is on negative direction has numerical value $F_3 = -\frac{1}{2}v$

If we apply Newton's second law, $F = F_1 + F_2 + F_3 = 24 - 6\sqrt{3} - \frac{1}{2}v$ and

$$m = \frac{w}{g} = \frac{48}{32}.$$

Thus we have the differential equation

$$\frac{3}{2}\frac{dv}{dt} = 24 - 6\sqrt{3} - \frac{1}{2}v$$

Since the object is released from rest, the initial condition is $v(0) = 0$. The differential equation is seperable.

Seperating the variables we have

$$\frac{dv}{48 - 12\sqrt{3} - v} = \frac{dt}{3}$$

Integrating and simplifying we find

$$v = 48 - 12\sqrt{3} - c_1 e^{-\frac{t}{3}}$$

The initial condition gives $c_1 = 48 - 12\sqrt{3}$.

Thus we obtain $v = (48 - 12\sqrt{3})(1 - e^{-\frac{t}{3}})$.

Letting $t = 2$, we have

$$v(2) = (48 - 12\sqrt{3})(1 - e^{-\frac{2}{3}}) \approx 10.2 \text{ ft/sec} \quad \text{(Ross, 2004)}$$

Numerical Solution 4.1.2: In this example the point that we want to reach is the time 2secs. We divide into four parts in order to show the numerical values until 2secs.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	v_{exact}
0.5	4.24342	4.17733	4.17733	4.17806	4.18146
1	7.82520	7.71347	7.71347	7.71471	7.72099
1.5	10.84852	10.70685	10.70685	10.70842	10.71714
2	13.40047	13.24077	13.24077	**13.24254**	**13.25332**

Figure 4.1.2.1: The table of approximations for Example 4.1.2

Figure 4.1.2.1 shows that even though *h=0.1* , Euler is still maximized the error with 1.11029%. Heun and RK2 give the same approximation with 0.09469% and finally we have 0.081338% for RK4.

t_n	*Euler (errror*	*Heun (error max)*	*Runge-Kutta*	*Runge-Kutta*

	max.)		order 2 (error max)	order 4 (error max)
0.5	0.06196	0.00413	0.00413	0.00340
1	0.10421	0.00751	0.00751	0.00628
1.5	0.13138	0.01029	0.01029	0.00871
2	0.14715	0.01325	0.01325	0.01078

Figure 4.1.2.2: The table of error analysis for Example 4.1.2

Therefore according to Figure 4.1.2.2, the error is minimized in RK4 and maximized in Euler method. Moreover we observe that when the points are increased, the error is increased for each method.

Example 4.1.3: A body weighing 8 lb falls from rest toward the earth from a great height. As it falls, air resistance act upon it, and we shall assume that this resistence (in pounds) is numerically equal to $2v$, where v is the velocity (in feet per second). Find the velocity and distance fallen at time t seconds. Determine the distance at $t=5$ (Ross, 2004).

Analytical Solution 4.1.3: Let F_1 be the weight, $8\,lb$, acting downward so that it is positive where F_2 is $2v$ and acting upward.

Using Newton's Second Law, $F=ma$ becomes,

$$m\frac{dv}{dt}=F_1+F_2.$$

Taking $g=32$, $m=\frac{w}{g}=\frac{8}{32}=\frac{1}{4}$

Therefore $\frac{1}{4}\frac{dv}{dt}=8-2v$ (4.1.2.1)

Since the body is initially at rest, we have the inital condition $v(0) = 0$.

(4.1.2.1) is seperable equation we can describe as

$$\frac{dv}{8-2v} = 4dt.$$

By integrating we find

$$-\frac{1}{2}\ln|8-2v| = 4t + c_0$$

Applying the initial condition we find $c_1 = 8$. Thus the veloctiy at time t is given as

$$v = 4(1-e^{-8t})$$

Now to determine the distance fallen at time t, we can write

$$\frac{dx}{dt} = 4(1-e^{-8t})$$

And noting that $x(0) = 0$. Integrating the above equation we get

$$x = 4(t + \frac{1}{8}e^{-8t}) + c_2$$

Since $x = 0$ and $t = 0$, we find $c_2 = -\frac{1}{2}$. Hence the distance fallen is given by

$$x = 4(t + \frac{1}{8}e^{-8t} - \frac{1}{8})$$

At time $t = 5$, the distance is $x \approx 19,5$ ft (Ross, 2004).

Numerical Solution 4.1.3: After the analytical solution above, we need to execute the methods. The step size is choosen as $h = 0.1$. We divide into 50 parts but in the table we will show only the integers.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	x_{exact}
1	3.27836	3.47379	3.49897	3.50009	3.50017
2	7.27361	7.47361	7.49897	7.49993	7.5
3	11.27361	11.47361	11.49880	11.49993	11.5
4	15.27361	15.47361	15.49880	15.49993	15.5
5	19.27361	19.47361	19.49880	**19.49993**	**19.5**

Figure 4.1.3.1: The table of approximation of Example 4.1.3

Figure 4.1.3.1 shows that Euler's method is far from the exact solution. Heun and RK2 give different approximations. Since RK2 involves different methods in it. Midpoint, Ralston and Heun are the examples of those methods. Here we understand that another RK2 method which is different than Heun gives better approximation.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
1	0.22631	0.02638	0.00119	0.00007
2	0.22639	0.02639	0.00119	0.00007
3	0.22639	0.02639	0.00119	0.00007
4	0.22639	0.02639	0.00119	0.00007
5	0.22639	0.02639	0.00119	0.00007

Figure 4.1.3.2: The table of error analysis of Example 4.1.3

From Figure 4.1.3.2, we understand that the errors are stable on the given points. Because we generally get the same approximation in every step for each method. Euler, Heun, RK2 and RK4 have the percentage error 1.16097%, 0.13533%, 0.00615% and 0.00036% in order. Therefore RK4 gave the best approximation according to the table of error analysis.

Until now, we observe that Euler, Heun and RK2 approximates slower. Therefore we decide to decrease the step size for these methods.

t_n	*Euler* *(h=0.01)*	*Heun* *(h=0.01)*	*Runge-Kutta order 2* *(h=0.01)*	*Runge-Kutta order 4* *(h=0.1)*	x_{exact}
1	3.47991	3.49990	3.49990	3.50009	3.50017
2	7.47973	7.49973	7.49973	7.49993	7.5
3	11.47973	11.49973	11.49973	11.49993	11.5
4	15.47973	15.49973	15.49973	15.49993	15.5
5	19.47973	**19.49973**	**19.49973**	**19.49993**	19.5

Figure 4.1.3.3: The table of comparison with different step size *h=0.01*

According to Figure 4.1.3.3, Euler still doesn't converge as much as RK4 despite the decreasement of step size. On the other hand , Heun and RK2 converges nearly RK4.

Fixing the step size of RK4, let us decrease the step size ten times more. We have the following numerical approximation

t_n	*Euler* *(h=0.001)*	*Heun* *(h=0.001)*	*Runge-Kutta order 2* *(h=0.001)*	*Runge-Kutta order 4* *(h=0.1)*	x_{exact}
1	3.49817	3.50017	3.50017	3.50009	3.50017
2	7.49800	7.5	7.5	7.49993	7.5
3	11.49800	11.5	11.5	11.49993	11.5
4	15.49800	15.5	15.5	15.49993	15.5
5	19.49800	**19.5**	**19.5**	**19.49993**	19.5

Figure 4.1.3.4: The table of comparison with different step size *h=0.001*

Observations from Figure 4.1.3.4 shows that we have the exact solution for Heun and RK2 when *h=0.001.* Heun and RK2 converges better than RK4 with the decreasement of the step size. However, we see that Euler doesn't converge as RK4. RK4 is efficient than Euler more than 100 times.

4.2 Mixture Problems

Letting x denote the amount of the substance S present at time t, the derivative $\frac{dx}{dt}$ denotes the rate of change of x with respect to t. If IN denotes the rate at which S enters the mixture OUT the rate at which it leaves, we have once at the basic equation

$$\frac{dx}{dt} = IN - OUT$$

Saltwater solutions with a given concentration is added at a specified rate to a tank that initially contains saltwater with a different concentration. To construct a tractable mathematical model for mixing problems we assume in our examples that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. (Ross, 2004)

Example 4.2.1: A tank initially contains $50gal$ at pure water. Starting at time $t=0$ a brine containing $2lb$ of dissolved salt per gallon flows into the tank at the rate of $3gal/\min$. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. How much salt is in the tank at any time $t>0$? How much salt is present at the end of $25\min$? (Ross, 2004)

Analytical Solution 4.2.1: If we apply the basic equation

$$\frac{dx}{dt}=IN-OUT$$

The brine flows in at the rate of $3gal/\min$ and each gallon contains $2lb$ of salt. Thus

$$IN=(2lb/gal)(3gal/\min)=6lb/\min$$

$$OUT=(\frac{x}{50}lb/gal)(3gal/\min)=\frac{3x}{50}lb/\min$$

Thus the differential equation for x as a function of t is

$$\frac{dx}{dt}=6-\frac{3x}{50}$$

Since initially there was no salt in the tank , $x(0)=0$.

The differential equation is both linear and seperable. Seperating the variables

$$\frac{dx}{100-x}=\frac{3}{50}dt$$

Integrating and simplifying we obtain

$$x=100+ce^{-3t/50}$$

Applying the initial condition $x=0$ at $t=0$, we have $c=-100$. Thus

$$x = 100(1 - e^{-3t/50})$$

which is the solution of the first part. At the end of $25\,\text{min}$

$$x(25) = 100(1 - e^{-1.5}) \approx 77.68698lb$$

After a long time means $t \to \infty$, and we observe that $x \to 100$ (Ross, 2004).

Numerical Solution 4.2.1: For this saltwater problem, we divided the interval into 50 parts but we will only show spesific points in the table in order to discover the behaviour of the numerical approximations.

t_n	*Euler (h=0.5)*	*Heun (h=0.5)*	*Runge-Kutta order 2 (h=0.5)*	*Runge-Kutta order 4 (h=0.5)*	x_{exact}
5	26.25759	25.91477	25.91477	25.91878	25.91818
10	45.62057	45.11378	45.11378	45.11884	45.11884
15	59.89929	59.33742	59.33742	59.34303	59.34303
20	70.42877	69.87503	69.87503	69.88058	69.88058
25	78.19346	77.68185	77.68185	**77.68698**	**77.68698**

Figure 4.2.1.1: The table of approximations for Example 4.2.1

In this example, Heun and RK2 give the same results. However this time, Euler approximates closer to these methods. Let us make the error analysis table in order to identify the efficiency of the methods.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
5	0.31484	0.00341	0.00341	0
10	0.50173	0.00505	0.00505	0
15	0.55626	0.00561	0.00561	0
20	0.54819	0.00555	0.00555	0
25	0.50648	0.00514	0.00514	0

Figure 4.2.1.2: The table of error analysis for Example 4.2.1

Euler has the percent error 0.65195%, Heun and RK2 have the percent error 0.00660%. On the other hand, RK4 still have the best approximation having the exact value for 5 decimal places.

Example 4.2.2: A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$ water that contains ½ pound of salt per gallon is poured into the tank at the rate of $4gal/\min$ and the mixture is drained from the tank at the same rate. Find a differential equation for the quantity $x(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $x(t)$ and find $\lim_{t\to\infty} x(t)$ (Trench, 2001).

Analytical Solution 4.2.2: To find a differential equation for x, we must use the given information to derive an expression for x' which is the rate of change of the quantity of salt in the tank changes with respect to time. Since the subtraction of the rate at which salt enters the tank and the rate by which it leaves is equal to the rate of change then we can write

$$(\frac{1}{2}lb/gal).(4gal/\min) = 21b/\min$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we're removing 1/150 of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing 1/150 of the salt per minute. Therefore, if there are $x(t)$ pounds of salt in the tank at time t, the rate out at any time t is $x(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

OUT=(concentration).(rate of flow out)

$$\frac{x(t)}{600}.4=\frac{x(t)}{150}$$

We can now write that

$$x'=2-\frac{x}{150}$$

which is a first order equation.

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of the equation are of the form $x=ue^{-t/150}$ where $u'e^{-t/150}=2$, so $u'=2e^{t/150}$. Hence $u=300e^{t/150}+c$ so

$$x=ue^{-t/150}=300+ce^{-t/150}$$

Since $x(0)=40$, $c=-260$;

$$x(t)=300-260e^{-t/150}$$

From the equation we see that $\lim_{t\to\infty}x(t)=300$ for any value of $x(0)$. This is intuitively reasonable, since the incoming solution contains ½ pound of salt per gallon and there are always 600 gallons of water in the tank (Trench, 2001).

Numerical Solution 4.2.2: In Example 4.2.2, we searched the limit when $t\to\infty$. Now, we search the results of numerical methods when $t\to\infty$.

t_n	*Euler* *(h=10)*	*Heun* *(h=10)*	*Runge-Kutta order 2* *(h=10)*	*Runge-Kutta order 4* *(h=10)*	x_{exact}
∞	299.99999	299.99999	299.99999	299.99999	300

Figure 4.2.2.1: The table of approximations for $t \to \infty$

We see that every method approximates same when t goes to infinity. Therefore the errors for each method will be same.

t_n	*Euler* *(errror max.)*	*Heun* *(error max)*	*Runge-Kutta order 2* *(error max)*	*Runge-Kutta order 4* *(error max)*
∞	0.00001	0.00001	0.00001	0.00001

Figure 4.2.2.2: The table of error analysis for $t \to \infty$

We see that all the given numerical methods approximate same in 5 decimal places. Therefore the error is the same and we have the percentage error that is 0.00000333%.

Example 4.2.3: A large tank initially contains $50gal$ of brine in which there is dissolved salt per galon flows into the tank at the rate of $5gal/\min$. The mixture is kept uniform by stirring and the stirred mixture simultaneously flows out at the slower rate of $3gal/\min$. How much salt is in the tank after $2.5\min$ (Ross, 2004).

Analytical Solution 4.2.3: In order to use the equation

$$\frac{dx}{dt} = IN - OUT$$

we should write IN and OUT.

$$IN = (2lb/gal)(5gal/\min) = 10lb/\min$$

Also once again

$$OUT = (Clb / gal)(3gal / \min)$$

where Clb / gal denotes the concentration. But here since the rate of outflow is different from that inflow, the concentration is noy quite so simple.

At time $t = 0$, the tank contains $50gal$ of brine. Since brine flows in at the slower rate of $3gal / \min$, there is a net gain $5 - 3 = 2gal / \min$ of brine in the tank. Thus at the end of t minutes the amount of brine in the tank is $50 + 2tgal$.

Hence the concentration at time t minutes is $\frac{x}{50+2t} 1b / gal$ and so

$$OUT = \frac{3x}{50+2t} lb / \min$$

Thus the differential equation becomes

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t}$$

Since there was initially $101b$ of salt in the tank, we have the initial condition $x(0) = 10$.

The differential equation is not seperable but it is linear. Putting it in standard form

$$\frac{dx}{dt} + \frac{3}{50+2t} x = 10$$

We find integrating factor as

$$\exp\left(\int \frac{3}{2t+50} dt \right) = (2t+50)^{3/2}$$

Multiplying through by this, we have

$$(2t+50)^{3/2}\frac{dx}{dt}+3(2t+50)^{1/2}x=10(2t+50)^{3/2}$$

or

$$\frac{d}{dt}[(2t+50)^{3/2}x]=10(2t+50)^{3/2}$$

Thus

$$(2t+50)^{3/2}x=2(2t+50)^{5/2}+c$$

or

$$x=4(t+25)+\frac{c}{(2t+50)^{3/2}}$$

Applying the initial condition $x=10$ at $t=0$ we find

$$10=100+\frac{c}{(50)^{3/2}}$$

or

$$c=-(90)(50)^{3/2}=-22500\sqrt{2}$$

Thus the amount of salt at any time $t>0$ is given by

$$x=4t+100-\frac{22500\sqrt{2}}{(2t+50)^{3/2}}$$

At time $t=2.5$, we have

$$x(2.5)=4(2.5)+100-\frac{22500\sqrt{2}}{(2(2.5)+50)^{3/2}}=31.989424516 \text{ gal (Ross, 2004).}$$

Numerical Solution 4.2.3: In this mixture problem, we use again 5 decimal places and we divided into 10 parts and the numerical values of the points are given in Figure 4.2.3.1.

t_n	*Euler (h=0.5)*	*Heun (h=0.5)*	*Runge-Kutta order 2 (h=0.5)*	*Runge-Kutta order 4 (h=0.5)*	x_{exact}
0.5	14.7	14.63382	14.63339	14.63404	14.63404
1	19.26765	19.14165	19.14083	19.14206	19.14206
1.5	23.71185	23.53275	23.53059	23.53233	23.53233
2	28.04076	27.81173	27.81026	27.81246	27.81246
2.5	32.26185	31.98856	31.98682	**31.98942**	**31.98942**

Figure 4.2.3.1: The table of approximations for Example 4.2.3

We observe that the error is 0.085163% for Euler. Heun and RK2 give different aprroximations having percentage error 0.002702% and 0.00813% in order when RK4 has the exact value.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
0.5	0.06596	0.00022	0.00065	0
1	0.12559	0.00041	0.00123	0
1.5	0.17952	0.00042	0.00174	0
2	0.22830	0.00073	0.00220	0
2.5	0.27243	0.00086	0.00260	0

Figure 4.2.3.2: The table of error analysis for Example 4.2.3

We see RK4 still approximates better than the given methods, however in this example we observe that Heun's and RK2's approximations are also very close to the exact value. Moreover on the searching point Heun approximates better than RK2.

Changing the step size, we investigate which method is efficient depending on the step size. Fixing the step size of RK4, we change the others' step size.

t_n	*Euler (h=0.0005)*	*Heun (h=0.001)*	*Runge-Kutta order 2 (h=0.001)*	*Runge-Kutta order 4 (h=0.1)*	x_{exact}
0.5	14.63404	14.63404	14.63404	14.63404	14.63404
1	19.14206	19.14206	19.14206	19.14206	19.14206
1.5	23.53233	23.53233	23.53233	23.53233	23.53233
2	27.81246	27.81246	27.81246	27.81246	27.81246
2.5	**31.98942**	**31.98942**	**31.98942**	**31.98942**	**31.98942**

Figure 4.2.3.3: The comparison table of the different step size *h=0.001*

From Figure 4.2.3.3 we can observe that RK4's approximation has the exact numerical values when *h=0.1.* On the other hand RK2 and Heun have the same approximations to the exact value when *h=0.001*. Moreover, we choose the step size of Euler as *h=0.0005* in this question. Then we get very close approximation to the exact value with Euler. Also if we decrease the step size of Euler two hundred times comparing with RK4, we get the same results with both methods.

4.3 Cooling and Warming Problems

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in medium with temperature $T_m(t)$ the rate of change of T at time t is proportional to $T(t) - T_m(t)$ thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m)$$

Here $k > 0$, since the temperature of the object must decrease if ,$T > T_m$ or increase if

$T < T_m$. We'll call k the temperature decay constant of the medium.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m

This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it's reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it's a huge cauldron of molten metal.

To solve the differential equation, we rewrite it as

$$T' + kT = kT_m$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$ so $u' = kT_m e^{kt}$. Hence

$$u' = T_m e^{kt} + c$$

Therefore,

$$T = ue^{-kt} = T_m + ce^{-kt}$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so the solution of the initial value problem is obtained as

$$T = T_m + (T_0 - T_m)e^{-kt}$$ (Trench, 2001).

Example 4.3.1: A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is the temperature after 8 minutes? (Trench, 2001).

Analytical Solution 4.3.1: Here $T_0 = 400$ and $T_m = 25$, so according to the solution of the initial value problem

$$T = 25 + 375e^{-kt}$$

We determine k from the stated condition that $T(4) = 200$; that is

$$T(4) = 200 = 25 + 375e^{-4k}$$

Hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4}\ln\frac{7}{15} = \frac{1}{4}\ln\frac{15}{7}$$

Substituting this into the equation yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\frac{15}{7}}$$

Therefore the temperature of the insulator after 8 minutes is

$$T(8) = 25 + 375e^{-2\ln\frac{15}{7}} = 25 + 375(\frac{7}{15})^2 \approx 107 \text{ °C (Trench, 2001).}$$

Numerical Solution 4.3.1: In this example, we deal with the numerical approximations of cooling and warming problems. For that reason choosing the step size *h=0.1* and dividing 80 parts, we can execute the methods.

In Figure 4.3.1.1 we only show the some spesific points and the searching points.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	T_{exact}
2	280.23352	281.17977	281.17977	281.17378	281.17378
4	198.71773	200.00820	200.00820	200	200
6	143.23623	144.55616	144.55616	144.54777	144.54777
8	105.47427	106.67432	106.67432	**106.66668**	**106.66668**

Figure 4.3.1.1: The table of approximations for Example 4.3.1

From Figure 4.3.1.1, it is seen that Heun and RK2 give the same approximations and RK4 has the same approximation as the exact.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
2	0.94026	0.00599	0.00599	0
4	1.28227	0.00820	0.00820	0
6	1.31154	0.00839	0.00839	0
8	1.19241	0.00764	0.00764	0

Figure 4.3.1.2: The table of error analysis for Example 4.3.1

Getting the table of error analysis and calculating the percent error, we have that Euler has the percentage error 1.11788% when Heun and RK2 have 0.00716%. On the other hand RK4 presents the exact value.

Example 4.3.2: When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F? (Zill and Cullen, 2005).

Analytical Solution 4.3.2: From the given informations, we can identify that $T_m = 70$ and $T(0) = 300$. The differential equation can be modelled as

$$\frac{dT}{dt} = k(T - 70)$$

and determine the value of *k* so that $T(3) = 200$.

The differential equation both linear and seperable. If we seperate variables

$$\frac{dT}{(T-70)} = kdt$$

yields

$$\ln|T - 70| = kt + c_1$$

and so $T = 70 + c_2 e^{kt}$. When $t = 0$ and $T = 300$,

$$300 = 70 + c_2 \text{ gives } c_2 = 230 .$$

Therefore $300 = 70 + 230e^{kt}$

Finally, the measurement $T(3) = 200$ leads to $e^{3k} = \frac{13}{23}$ or, $k = \frac{1}{3}\ln\frac{13}{23}$=-0.19018.

Thus $300 = 70 + 230e^{-0.19018t}$

We note that the solution actually furnishes no finite solution to *T*(*t*) =70, since $\lim_{t\to\infty} T(t) = 70$

Yet we intuitively expect the cake to reach room temperature after a reasonably long period of time. How long is “long”? Of course, we should not be disturbed by the fact that the model does not quite live up to our physical intuition. But Figure 4.3.2.1 and 4.3.2.2 clearly show that the cake will be approximately at room temperature in about one-half hour (Zill and Cullen, 2005).

$T(t)$	t (min)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

Figure 4.3.2.1: The table of the temperature of the cake changing with the time

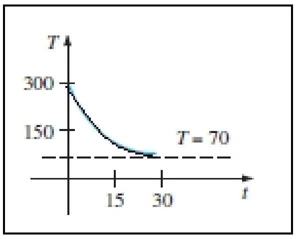

Figure 4.3.2.2: The graph of the changing the temperature of the cake with the time

Numerical Solution 4.3.2: For the example 4.3.2, we show the change of the temperature with time on Figure 4.3.2.1 and Figure 4.3.2.2. Now, we need to execute the numerical approximations for a cake problem. Therefore choosing the step size $h=0.1$, considering the points on Figure 4.3.2.1 and 4.3.2.2 and we will calculate when the temperature reaches to approximately 70° F.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	T_{exact}
20.1	74.84838	75.03138	75.03138	75.03021	75.03021
21.3	73.85061	74.00480	74.00480	74.00381	74.00381
22.8	72.88699	73.01090	73.01090	73.01011	73.01011
24.9	71.92897	72.01957	72.01957	72.01899	72.01899
28.6	70.94795	70.99926	70.99926	70.99893	70.99893
32.3	70.46585	70.49443	70.49443	**70.49424**	**70.49424**

Figure 4.3.2.3: The table of approximations for Example 4.3.2

In the numerical solution of this problem, we see that we can not reach to the specified temperature but we approximates it about half an hour.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
20.1	0.18183	0.00117	0.00117	0
21.3	0.15320	0.00099	0.00099	0
22.8	0.12312	0.00079	0.00079	0
24.9	0.09002	0.00058	0.00058	0
28.6	0.05098	0.00033	0.00033	0
32.3	0.02839	0.00019	0.00019	0

Figure 4.3.2.4: The table of error analysis for Example 4.3.2

At the spesified point 32.3, we have the percentage error 0.04028% for Euler; 0.00026% for RK2. Heun and RK2 give the same approximation. From Figure 4.3.2.3 and 4.3.2.4, it is seen that RK4 gives the better approximation comparing with the other methods.

Now our aim is to get closer to the given temperature more. Choosing the same step size we extend the time interval and we get the results as below.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	T_{exact}
199.90	70.00000000000037	70.00000000000037	70.00000000000037	70.00000000000037	70

Figure 4.3.2.5: The table of approximations for Example 4.3.2

We observe that we can not reach to the given temperature. We still approximates.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
199.90	3.694822225e-13	3.694822225e-13	3.694822225e-13	3.694822225e-13

Figure 4.3.2.6: The table of error analysis for Example 4.3.2

It is seen that after 199.90 minutes, we have approximately 70 for each method but not exactly.

Example 4.3.3: A metal bar at a temperature of $100°F$ is placed in a room at a constant temperature of $0°F$. If after 20 minutes the temperature of the bar is $50°F$. Find the time it will take the bar to reach a temperature of $25°F$ and the temperature of the bar after 10 minutes (Bronson and Costa 2006).

Analytical Solution 4.3.3: Here from the given information $T_m = 0$; the medium here is the room which is being held at a constant temperature of $0°F$.

Thus we have

$$\frac{dT}{dt} + kT = 0 \text{ whose solution is } T = ce^{-kt}$$

Since $T = 100$ at $t = 0$, it follows that $100 = ce^{-k(0)}$ or $c = 100$.

Substituting this value into the equation, we have

$$T = 100e^{-kt}$$

At $t = 20$, we are given that $T = 50$; hence substituting the values

$$50 = 100e^{-20k}$$

from which $k = -\frac{1}{20}\ln\frac{50}{100} = -\frac{1}{2}(-0.693) = 0.035$

Substituting k into the equation, we obtain the temperature of the bar at any time t

$$T = 100e^{-0.035t}$$

We require t when $T = 25$.

Therefore ,

$$25 = 100e^{-0.035t}$$

or

$$-0.035t = \ln\frac{1}{4}$$

Solving the equation we find that $t = 39.6\,\text{min}$.

We require T when $t = 10$. Substituting $t = 10$ into the equation and then solving for T, we find that

$T = 100e^{-0.035(10)} = 100(0.705) = 70.5°F$ (Bronson and Costa 2006).

Numerical Solution 4.3.3: In 4.3.3, we need to reach to the temperature $70°F$ and it is seen from the analytical solution , when $t = 10$ it can reach approximately to $70°F$**.** Choosing *h=0.1*, according to the numerical methods,the table can be shown below at some specified points.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	T_{exact}
2.5	91.60783	91.62190	91.62190	91.62189	91.62189
5	83.91994	83.94573	83.94573	83.94570	83.94570
7.5	76.87723	76.91268	76.91268	76.91264	76.91264
10	70.42556	70.46886	70.46886	**70.46881**	**70.46881**

Figure 4.3.3.1: The table of approximations for Example 4.3.3

It is seen that Heun and RK2 give the same results. Euler is still maximized the error. We have the exact value with RK4

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*

2.5	0.01406	0.00001	0.00001	0
5	0.02576	0.00003	0.00003	0
7.5	0.03541	0.00005	0.00005	0
10	0.04325	0.00005	0.00005	0

Figure 4.3.3.2: The table of error analysis for Example 4.3.3

According to the Figure 4.3.3.2 and Figure 4.3.3.1 we have the same approximations for Heun and RK2 methods at the searching point. Therefore we have the percentage error 0.00007165% for RK2 and Heun and 0.06184% for Euler's method.

t_n	*Euler (h=0.001)*	*Heun (h=0.001)*	*Runge-Kutta order 2 (h=0.001)*	*Runge-Kutta order 4 (h=0.1)*	T_{exact}
2.5	91.62175	91.62189	91.62189	91.62189	91.62189
5	83.94545	83.94570	83.94570	83.94570	83.94570
7.5	76.91229	76.91263	76.91263	76.91263	76.91263
10	70.48838	**70.46881**	**70.46881**	**70.46881**	**70.46881**

Figure 4.3.3.3: The table of approximation with different step size *h=0.001*

According to Figure 4.3.3.3, we have the exact value with Heun, RK2,RK4 and nearly exact with Euler. If the step size decreases 100 times for Heun and RK2, we have the exact value. However we still don't have the exact numerical value with Euler in spite of the step size.

4.4 Financial Problems

In financial problems, we will consider the rate problem for the money that is deposited in a bank.

Suppose that a sum of money is deposited in a bank or money fund that pays interest at an annual rate *r*. The value *S(t)* of the investment at any time *t* depends on the frequency with which interest is compounded as well as on the interest rate. Financial institutions have various policies concerning compounding: some compound monthly, some weekly, some even daily. If we assume that compounding takes place continuously, then we can set up an initial value problem that describes the growth of the investment.

The rate of change of the value of the investment is $\frac{dS}{dt}$ and this quantity is equal to the rate at which interest accrues, which is the interest rate *r* times the current value of the investment *S(t)*. Thus

$$\frac{dS}{dt} = rS$$

is the differential equation that governs the process. Suppose that we also know the value of the investment at some particular time,

Then the solution of the initial value problem gives the balance *S(t)* in the account at any time *t*. This initial value problem is readily solved, since the differential equation is both linear and separable. Consequently, by solving equations, we find that

$$S(t) = S_0 e^{rt}$$

let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate *k*,

$$\frac{dS}{dt} = rS + k$$

where k is positive for deposits and negative for withdrawals.

The differential equation is linear with the integrating factor e^{-rt}, so its general solution is

$$S(t) = ce^{rt} - (k/r)$$

where c is an arbitrary constant. To satisfy the initial condition, we must choos $c = S_0 + (k/r)$ Thus the solution of the initial value problem

$$S(t) = S_0 e^{rt} + (k/r)(e^{rt} - 1) \text{ (Boyce and Diprima, 2001)}$$

Example 4.4.1: If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously. The value of account after t years is

$$S(t) = 150e^{0.055t}$$

What is the value of the account after $t = 10$ years? (Boyce and Diprima, 2001).

Analytical Solution 4.4.1: Note that it's necessary to write the interest rate as decimal; thus $\tau = 0.055$. If we substitute $t = 10$ then we have

$$S(10) = 150e^{0.55} \approx \$259.98795 \text{ (Boyce and Diprima 2001).}$$

Numerical Solution 4.4.2: The interest rate is analytically found after 10 years. Using numerical methods, we divide into 100 parts and some spesific points are shown below.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	S_{exact}
2	167.39125	167.44162	167.44162	167.44171	167.44171

4	186.79888	186.91130	186.91130	186.91151	186.91151
6	208.45665	208.64487	208.64487	208.64522	208.64522
8	232.62546	232.90557	232.90557	232.90608	232.90608
10	259.59645	259.98723	259.98723	**259.98795**	**259.98795**

Figure 4.4.1.1: The table of approximations for Example 4.4.1

Here, we realize that RK4 again approximates to the exact value. Euler approximates further than the other methods. RK2 and Heun give the same results.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
2	0.05046	0.00009	0.00009	0
4	0.11263	0.00021	0.00021	0
6	0.18857	0.00035	0.00035	0
8	0.28062	0.00051	0.00051	0
10	0.39150	0.00072	0.00072	0

Figure 4.4.1.2 The table of error analysis for Example 4.4.1

Since we have the suitable step size, we get better approximations for each method. We have 0.15058% for Euler. On the other hand, we have 0.00028% for Heun and RK2 and we get the exact solution with RK4.

Example 4.4.2: A person places \$5000 in an account that accrues interest compounded continuously. Assuming no additional deposits or withdrawals how much will be in the account after seven years if the interest rate is a constant 8.5 percent for the first four years and a constant for the last three years? (Trench, 2001)

Analytical Solution: Let $S(t)$ denote the balance in the account at any time t. Initially, $S(0)=5000$ for the first four years $k=0.085$. Therefore

$$\frac{dS}{dt}-0.085S=0$$

Its solution is

$$S(t)=ce^{0.085t} \text{ for } 0\le t\le 4.$$

At $t=0$, $S(0)=5000$ which when substituted into the equation yields

$$5000=ce^{0.085(0)}=c$$

Therefore

$$S(t)=5000e^{0.085t}$$

Substituting $t=4$ into the equation, we find the balance after four years to be

$$S(4)=5000e^{0.085(4)}=5000(1.404948)=\$7024.74$$

This amount also represents the beginning balance for the last three-year period.

Over the last three years, the interest rate is 9.25 percent and the equation becomes

$$\frac{dS}{dt}-0.095S=0$$

Its solution is

$$S(t)=ce^{0.0925t}$$

At $t=4$, $S(4)=7024.74$, which when substituted into the equation yields

$$7024.74 = ce^{0.0925(4)} = c(1.447735)$$

or

$$c = 4852.23$$

and the equation becomes

$$S(t) = 4852.23e^{0.0925t}$$

Substituting $t = 7$ into the equation we find the balance after seven years to be

$$S(7) = 4852.23e^{0.0925(7)} = 4852.23(1.910758) = \$9271.44 \text{ (Trench, 2001).}$$

Numerical Solution 4.4.2: In this example, we show the balance after 7 years, dividing into 70 parts. In this question we only show the point that we want to search for.

t_n	*Euler (h=0.1)*	*Heun (h=0.1)*	*Runge-Kutta order 2 (h=0.1)*	*Runge-Kutta order 4 (h=0.1)*	S_{exact}
7	9259.61644	9271.39890	9271.39890	**9271.43533**	**9271.43709**

Figure 4.4.2.1: The table of approximations for Example 4.4.2

We see that we have closer approximation with RK4. Heun and RK2 give the same results while Euler is further.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*

7	11.82065	0.03819	0.03819	0.00176

Figure 4.4.2.2: The table of error analysis for Example 4.4.2

The percentage error that we have 1.11788% for Euler, 0.00716% for Heun and RK2 and 0.00000013% for RK4..

Example 4.4.3: Suppose that one opens an individual retirement account at age 25 and makes annual investments of $2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in individual retirement account at age 65? (Trench, 2001)

Analytical Solution 4.4.3: The information about the problem gives us $S_0 = 0$, r = 0.08, and k = $2000 and the question asks to determine *S(*40*)*. Therefore from the solution of the initial value problem, we have

$$\frac{dS}{dt} = 0.08S + 2000$$

Using the solution of initial value problem we have

$$S(t) = 0 + \frac{2000}{0.08}(e^{0.08t} - 1)$$

If we use $t = 40$,

$$S(40) = 25000(e^{3.2} - 1) = \$588313.25493 \text{ (Trench, 2001).}$$

Numerical Solution 4.4.3: After solving the problem analytically, let us now execute the given numerical methods. We deal with the money at age 65. Now the person is at age 25. Therefore 40 parts are enough for approximations which means *h=1.*

t_n	*Euler (h=1)*	*Heun (h=1)*	*Runge-Kutta order 2 (h=1)*	*Runge-Kutta order 4 (h=1)*	S_{exact}
35	28973.12493	30593.82070	30593.82070	30638.50900	30638.52321
45	91523.92860	98626.91601	98626.91601	98825.74734	98825.81061
55	226566.42223	249915.70410	249915.70410	250579.19831	250579.40952
65	518113.03742	586344.57448	586344.57448	**588312.62819**	**588313.25493**

Figure 4.4.3.1: The table of approximations for Example 4.4.3

According to the approximations of numerical methods we have the percentage error 11.93246% for Euler, 0.33463% for Heun and RK2 and 0.00011% for RK4.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
35	1665.39828	44.70251	44.70251	0.014214
45	7301.88201	198.89460	198.89460	0.06327
55	24012.98729	663.70541	663.70541	0.21120
65	70200.21751	1968.68044	1968.68044	0.62673

Figure 4.4.3.2: The table of error analysis for Example 4.4.3

From Figure 4.4.3.1 and 4..4.3.2 we have the closest approximation with RK4. Euler has great error comparing with other methods. We observe that the step size is not enough small to get accurate approximation. Therefore we decrease the step size.

t_n	*Euler* *(h=0.01)*	*Heun* *(h=0.01)*	*Runge-Kutta order 2* *(h=0.01)*	*Runge-Kutta order 4* *(h=0.01)*	S_{exact}
35	30620.73122	30638.51847	30638.51847	30638.52321	30638.52321
45	98746.62966	98825.78949	98825.78949	98825.81061	98825.81061
55	250315.12111	250579.33901	250579.33901	250579.40952	250579.40952
65	587529.13408	588313.04571	588313.04571	**588313.25493**	**588313.25493**

Figure 4.4.3.3: The table of approximation with different step size *h=0.01*

From Figure 4.4.3.3 we conclude that for this example, the step size should be chosen *h=0.01* for RK4. Because we get the exact value with RK4. However as we expect, this step size is still not enough for other methods. We should decrease the step size more.

4.5 Growth and Decay Problems

In certain problems, there can be lots of models. Population growth, nuclei decay, bacterial growth or decay are some of these situations. One of the most common mathematical models for a physical process is the exponential model.

The rate at which a quantity changes is a known function of the amount present and/or the time and it is desired to find quantity itself. If x denotes the amount of the quantity present at time t, then $\frac{dx}{dt}$ denotes the rate at which the quantity changes.

Let x be the amount of present after t years. Since that is proportional to the amount present we have

$$\frac{dx}{dt} = kx$$

where k is a constant of proportionality (Ross, 2004).

The solution of this differential equation can be obtained as $x(t) = Ce^{kt}$. If the initial value $x(t_0) = x_0$ then the solution for the initial value problem is $x = x_0 e^{k(t-t_0)}$. We say that a quantity satisfies this equation grows exponentially if $k > 0$ or decays exponentially if $k < 0$ (Ross, 2004).

Example 4.5.1: A radioactive substance has a half-life of 1620 years. If its mass is now 4grams, how much will be left 810 years from now? (Trench, 2001).

Remark: The half-life τ of a radioactive material is defined to be the time required for half of its mass to decay; that is if $x(t_0) = x_0$, then

$$x(\tau + t_0) = \frac{x_0}{2}$$

Since the radioactive materials decays at a rate prportional to the mass of the material present $k < 0$. Therefore the initial value problem can be modelled as

$$x = x_0 e^{-k(t-t_0)}$$

If $t = \tau + t_0$ is equivalent to $x_0 e^{-k\tau} = \frac{x_0}{2}$ then $e^{-k\tau} = \frac{1}{2}$.

Taking logarithms yields

$$-k\tau = \ln\frac{1}{2} = -\ln 2$$

So the half-life is $\tau = \frac{1}{k}\ln 2$.

The half-life is independent of t_0 and x_0, since it is determined by the properties of material, not by the amount of the material present at any particular time. (Trench, 2001)

Analytical Solution 4.5.1: From the given information, we can say that $t_0 = 0$ and $x_0 = 4$. Therefore the solution of the initial value problem can be reduced to

$$x = 4e^{-kt}$$

where we determine k from with $\tau = 1620$ years

$$k = \frac{\ln 2}{r} = \frac{\ln 2}{1620}$$

Substituting this into the equation we have

$$x = 4e^{-(t \ln 2)/1620}$$

Therefore the mass after 810 years will be

$$x(810) = 4e^{-(810 \ln 2)/1620} = 4e^{-(l n 2)/2} = 3.44106 \text{ g} \quad \text{(Trench, 2001).}$$

Numerical Solution 4.5.1: The numerical approximations of mass for a radioactive substance are given below. Here we choose *h=10*.

t_n	*Euler (h=10)*	*Heun (h=10)*	*Runge-Kutta order 2 (h=10)*	*Runge-Kutta order 4 (h=10)*	x_{exact}
250	3.81427	3.81444	3.81444	3.81444	3.81444
500	3.63716	3.63749	3.63749	3.63749	3.63749
750	3.46828	3.46875	3.46875	3.46875	3.46875
810	3.42893	**3.42943**	**3.42943**	**3.42943**	**3.42943**

Figure 4.5.1.1: The table of approximations for Example 4.5.1

According to Figure 4.5.1.1, although the step size is chosen as *h=10*, we get closer approximations for each method as expected. Heun, RK2 and RK4 give the same results. They have the exact value but Euler is the only one method that has different approximations with 0.01458% percentage error.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
250	0.00017	0	0	0
500	0.00033	0	0	0
750	0.00047	0	0	0
810	0.0005	0	0	0

Figure 4.5.1.2: The table of error analysis for Example 4.5.2

Even though Euler can not get the exact value in this problem. We note that percent error is very small for Euler.

Example 4.5.2: The population x of a certain city satisfies the logistic law

$$\frac{dx}{dt}=\frac{1}{100}x-\frac{1}{(10)^8}x^2$$

where time t is measuredin years. Given that the population of this city is 100000 in 1980. Determine what will the population be in 2000? Assuming the differential equation $t>1980$, how large will the population ultimately be? (Ross, 2004)

Analytical Solution 4.5.2: We must solve the seperable equation according to the initial condition $x(1980)=100000$.

Seperating the variables we have

$$\frac{dx}{(10)^{-2}x-(10)^{-8}x^2}=dt$$

Using partial fractions, this becomes

$$100\left[\frac{1}{x}+\frac{(10)^{-6}}{1-(10)^{-6}x}\right]dx = dt$$

Integrating and assuming $0 < x \ < (10)^{6}$, we obtain

$$100\left[\ln x - \ln\left[1-(10)^{-6}x\right]\right]t + c_1$$

and hence

$$\ln\left[\frac{x}{1-(10)^{-6}x}\right] = \frac{1}{100}t + c_2$$

Thus we find

$$\frac{x}{1-(10)^{-6}x} = ce^{t/100}$$

Solving this for x, we finally obtain

$$x = \frac{ce^{t/100}}{1+(10)^{-6}ce^{t/100}}$$

Now applying the initial condition, we have

$$(10)^{5} = \frac{ce^{19.8}}{1+(10)^{-6}ce^{19.8}}$$

From which we obtain

$$c = \frac{(10)^{5}}{e^{19.8}[1-(10)^{5}(10)^{-6}]} = \frac{(10)^{6}}{9e^{19.8}}$$

Substituting c into the equation we have

$$x = \frac{(10)^{6}}{1+9e^{19.8-t/100}}$$

This gives the population x as a function of time for $t > 1980$.

If we let $t = 2000$, we obtain

$$x = \frac{(10)^6}{1 + 9e^{-0.2}} \approx 119.495$$

The other question asks how large the population ultimately be, assuming the differential equation applies for all $t > 1980$, we find

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{(10)^6}{1 + 9e^{19.8 - t/100}} = (10)^6 = 1000000$$ (Ross, 2004).

Numerical Solution 4.5.2: Now, we will calculate the numerical values for the population problem that is given above assuming that *h=1*.

t_n	*Euler* (*h=1*)	*Heun* (*h=1*)	*Runge-Kutta order 2* (*h=1*)	*Runge-Kutta order 4* (*h=1*)	x_{exact}
5	104572.32943	104590.80466	104590.81174	104590.86083	104590.86083
10	109328.54923	109366.75445	109366.76947	109366.87039	109366.87039
15	114273.72962	114332.94346	114332.96736	114333.12284	114333.12284
20	119412.86479	119494.38520	119494.41898	**119494.63171**	**119494.63171**

Figure 4.5.2.1: The table of approximations for Example 4.5.2

We still see that Heun and RK2 give the same results for 5 decimal places. Also in Figure 4.2.1.1, we see that RK4 give the same results as in the exact value. The percentage error 0.06843% for Euler, 0.00021% for Heun and RK2.

t_n	*Euler* (*errror max.*)	*Heun* (*error max*)	*Runge-Kutta order 2*	*Runge-Kutta order 4*

			(error max)	*(error max)*
5	18.5314	0.05617	0.04909	0
10	38.32116	0.11594	0.10092	0
15	59.39322	0.17938	0.15548	0
20	81.76692	0.24651	0.21273	0

Figure 4.5.2.2: The table of error analysis for Example 4.5.2

Figure 4.5.2.2 shows that RK4 gives the exact solution for 5 decimal places. It means the best method for this example is RK4.

The other part of the question asks how the population ultimately be. Therefore we need to approximate to infinity. Executing all the given methods we have a table as below

t_n	*Euler (h=1)*	*Heun (h=1)*	*Runge-Kutta order 2 (h=1)*	*Runge-Kutta order 4 (h=1)*	x_{exact}
∞	999999.99999	999999.99999	999999.99999	999999.99999	1000000

Figure 4.5.2.3: The table of approximations when $t \to \infty$

We see that each method approximates the same for 5 decimal places when $t \to \infty$.Their percentage error is same and numerically equal to 9.99995972e-10%.

Example 4.5.3: Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t that the population at time $t = 0$ (hours,h) was 1000, and that the population doubled after 1 h. What is the predicted number of bacteria at time 1.5 (Edwards and Penney, 1996).

Analytical Solution 4.5.3: We know that the time rate of change of a population $P(t)$ can be modelled as

$$\frac{dP}{dt} = kP$$

Addititional information gives us

$$P(0) = 1000 = Ce^0 = C \text{ and } P(1) = 2000 = Ce^k$$

It follows that $C = 1000$ and $e^k = 2$ so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P$$

Substitution of $k = \ln 2$ and $C = 1000$ yields the solution

$$P(t) = 1000.e^{(\ln 2)t} = 1000.2^t$$

We can use this solution to predict future populations of bacteria colonfy. $P(1.5)$ is one of these prediction that can be found as

$P(1.5) = 1000.2^{3/2} \approx 2828.42712$ (Edwards and Penney, 1996).

Numerical Solution 4.5.3: In this example, choosing *h=0.5*, we calculate the numerical approximations and show these approximations for three spesific points.

t_n	*Euler (h=0.5)*	*Heun (h=0.5)*	*Runge-Kutta order 2 (h=0.5)*	*Runge-Kutta order 4 (h=0.5)*	P_{exact}
0.5	1346.5735	1406.63009	1406.63009	1414.16924	1414.21356
1	1813.26019	1978.60822	1978.60823	1999.87463	2000
1.5	2441.68812	2783.16988	2783.16988	**2828.16118**	**2828.42712**

Figure 4.5.3.1: The table of numerical approximations for Example 4.5.3

From Figure 4.5.3.1, we note that RK2 and Heun have the same approximations there are big differences on approximations of Euler, Heun and RK4.

t_n	*Euler (errror max.)*	*Heun (error max)*	*Runge-Kutta order 2 (error max)*	*Runge-Kutta order 4 (error max)*
0.5	67.64006	7.58346	7.58346	0.04432
1	186.73980	21.39177	21.39177	0.12537
1.5	386.73900	45.25724	45.25724	0.26594

Figure 4.5.3.2: The table of error analysis for Example 4.5.3

Figure 4.5.3.2 shows that we have the percentage error 13.67329% for Euler which means it is really far from the exact solution, however the percentage error is 1.60009% for Heun and RK2 and 0.00940% for RK4

Now, we change the step size except for RK4 and assume that *h=0.01*.

t_n	*Euler (h=0.01)*	*Heun (h=0.01)*	*Runge-Kutta order 2 (h=0.01)*	*Runge-Kutta order 4 (h=0.5)*	P_{exact}
0.5	1412.52360	1414.20953	1414.20953	1414.16924	1414.21356
1	1995.22291	1999.98860	1999.98860	1999.87463	2000
1.5	2818.29945	2828.40293	2828.40293	**2828.16118**	**2828.42712**

Figure 4.5.3.3: The table of comparison with different step size *h=0.01*

According to Figure 4.5.3.3 Euler with *h=0.01* still does not converge as RK4. However RK2 and Heun approximately converges as RK4.

t_n	*Euler (h=0.001)*	*Heun (h=0.001)*	*Runge-Kutta order 2 (h=0.001)*	*Runge-Kutta order 4 (h=0.1)*	P_{exact}
0.5	1414.04366	1414.21340	1414.21340	1414.21335	1414.21356
1	1999.51947	1999.99953	1999.99953	1999.99939	2000
1.5	2827.40782	**2828.42612**	**2828.42612**	**2828.42582**	**2828.42712**

Figure 4.5.3.4: The table of comparison with different step size *h=0.001*

From Figure 4.5.3.4, we observe that Heun and RK2 give better approximation than RK4 for the specified step size however Euler is still not closer as much as the others.

The examples end up here. Until now, we got some results and we also made comparisons between the methods. In the following section, we will mention these conclusions and comparisons.

5 CONCLUSION

In this section, we will evaluate the results of the fourth section. In the examples of Section 4, we solved the application of first order ordinary differential equations analytically and numerically by the given methods in Section 3.1 and Section 3.2. The solutions help us to make comparison between methods. These comparisons give us the evaluations below.

In numerical results, we observe that the error depends on the step size. Decreasing the step size and increasing the step numbers give more accurate approximation.

Euler's Method is deficient comparing with the other methods, when the step size is large. In some cases, we reached to the error 13% with Euler. Therefore if Euler method will be chosen, then the step size would decrease as much as possible. In the example that we have most step numbers, we get the error 0.04% with Euler but we should remember that increasement of the step number causes more and more execution time.

Heun and Second-Order Runge-Kutta Method usually give the same results. This is an expectation considering that Heun is an example of Second-Order Runge-Kutta Method. However in some cases their results are different. It is related to the chose of a Second-Order Runge-Kutta method. Heun is an example of Second-Order Runge-Kutta Method but it is not the best one. Therefore if another Second-Order Runge-Kutta method different than Heun give more accurate solution, we used that one.

In some problems, we approximated to infinity. In these situations, all methods with same step size give the same results and same error but the execution time can differ.

Fourth-Order Runge-Kutta is the most efficient one considering the step size and correspondingly execution time. We get very close approximation in 7 problems Moreover, we get the exact value in 8 problems. We get the error 0.08% at most.

According to approximations with different step size, we observed that Fourth-Order Runge-Kutta generally needs a hundred times less step size than Heun and Second-Order Runge-Kutta. On the other hand, Euler generally needs more than one hundred step size in order to catch the Fourth-Order Runge-Kutta Method approximation.

In conclusion, our main aim is to make comparison between numerical methods. While making this comparison we discover that step size is an important factor in order to get efficient the numerical approximation to the problem. Changing the step size, we evaluate the results. This results shows that decreasing the step size provides to get better approximation for each method. We also observe that the order is very important for numerical approximations. Therefore Fourth-Order Runge-Kutta method give the best approximation for the applications of first-order ordinary differential equations. Finally, we can conclude that the more we increase the order, the more we get accurate approximation.

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