YASAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE

MASTER THESIS

THE LAPLACE TRANSFORMATION APPLIED TO DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Presentation Date: 29.06.2015

Bornova-İZMİR

2015

This study titled "The Laplace Transformation Applied to Differential Equations and their Applications" and presented as Master's Thesis by Seçil Dereli has been evaluated in compliance with the relevant provisions of Y.U Graduate Education and Training Regulation and Y.U Institute of Science Education and Training Direction and jury members written below have decided for the defence of this thesis and it has been declared by consensus / majority of votes that the candidate has succeeded in thesis defence examination dated 29. 06. 2015

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ABSTRACT

THE LAPLACE TRANSFORMATION APPLIED TO DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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MSc in Mathematics
Supervisor: Assist. Prof. Dr. Refet POLAT
June 2015, 68 pages

In the first section of this thesis, an introduction about Mathematics, Laplace transformation, and Differential Equation is given. In the second chapter, there is basic information about the Laplace transform and inverse Laplace transform. In the third chapter, inspected articles about Laplace transform, their comparison and results are given. In the fourth chapter, Applications of Laplace transformation are given.

Keywords: Laplace transform, Differential Equation, inverse Laplace transform

ÖZET

LAPLACE DÖNÜŞÜMÜNÜN DİFERANSİYEL DENKLEMLERE UYGULANMASI VE UYGULAMALARI

DERELİ, Seçil

Yüksek Lisans Tezi, Matematik Bölümü
Tez Danışmanı: Yard. Doç. Dr. Refet POLAT
Haziran 2015, 68 sayfa

Bu tezin ilk bölümünde Matematik, Laplace dönüşümü ve diferansiyel denklem hakkında bir giriş verilmiştir. İkinci bölümde Laplace dönüşümü ve ters Laplace dönüşümü hakkında temel bilgiler vardır. Üçüncü bölümde Laplace dönüşümü ile alakalı incelenen makaleler, karşılaştırmaları ve sonuçları verilmiştir. Dördüncü bölümde ise Laplace dönüşümünün uygulamaları verilmiştir.

Anahtar Kelimeler: Laplace dönüşümü, Diferansiyel Denklem, ters Laplace dönüşümü

TEXT OF OATH

I declare and honestly confirm that my study titled "THE LAPLACE TRANSFORMATION APPLIED TO DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS", and presented as Master's Thesis has been written without applying to any assistance inconsistent with scientific ethics and traditions and all sources I have benefited from are listed in bibliography and I have benefited from these sources by means of making references.

ACKNOWLEDGEMENTS

I would like to thank to my supervisor Assist. Prof. Dr. Refet POLAT for his help on my thesis.

I also thank my parents for the unceasing encouragement, support and attention. I am also grateful to my friend Naz Ersever, who supported me throught this venture.

Seçil Dereli
İzmir, 2015

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1. INTRODUCTION

Mathematics is all around the world, also effective object. People tried to understand environment since civilizations have occurred. It was necessary to apply mathematics to improve daily life. It means not only does it every process and patterns has that occurred in the world, but also a good understanding of it will help in people's life.

During the history, most civilizations had great effects on advance mathematics. The ancient Egyptians, Sumerians and Chinese had found writing numbers and could perform calculations. In ancient Egypt, they had solved different kind of problems to build pyramids. Egyptian arithmetic also found on continuing groups of ten, was relatively simple.

Moreover, around 10 to 11 thousand years ago, the people of Mesopotamia used clay tokens to indicate amounts of grain, oil, etc. for trade. In order to calculate, scribes would use clay tokens that indicated different amounts, and abacus-like operations of combining and trading (Rudman, 2007). Also with Flood of Nile River forced people to develop mathematics.

The most ancient mathematical texts available are Plimpton 322 (Babylonian mathematics 1900 BC) (Friberg, 1981), the Rhind Mathematical Papyrus (Egyptian mathematics 2000-1800 BC) (Neugebauer, 1969) and the Moscow Mathematical Papyrus (Egyptian mathematics 1890 BC). All of these ancient texts refer the assumed Pythagorean Theorem, which seems to be the most ancient and widespread mathematical development after basic arithmetic and geometry.

So the theorem of Pythagorean is considered to be the origin of the mathematics and geometry. And it is obvious that Pythagorean Theorem had been used for engineering. It is used to construct better buildings. In consideration of ancient people we do the similar thing at the same time. We apply theorems to mathematics for the best technology.

Theories are the essential in mathematics. Most of the theorems help people to simplify and utilize many sciences that use mathematics like engineering, physics, chemistry, and economics etc. As Pythagorean, Laplace is an important theorem in modern mathematics. So many physical processes in nature are described by differential and integral equation with initial conditions or boundary conditions. Integral transforms not only helped in developing the theory of such equations but also provided methods to solve these equations.

Actually commonly called “modern” mathematics is not really so modern as one might suppose. Big part of it was in fact developed more than fifty years ago. What makes it different is that only in the last generation has it had a serious effect on the undergraduate teaching of mathematics in universities, and only in the last several years has its impact on school mathematics begun to be felt. Some such time-lag is natural and inevitable; new mathematics is usually created as a research tool, and as long as it is nothing more there is no compelling reason to seek the simplest way of expounding it or to re-examine more elementary mathematics in its light (Smithies, 1963).

From the early days of the calculus subject of differential equations has been an area of great theoretical research and practical applications, and it continues to be so in our day. This much stated several questions naturally arise. Just what is a differential equation and what does it signify? Where and how do differential equations originate and of what use are they? Confronted with a differential equation, what does one do with it, how does one do it, and what are the results of such activity? These questions indicate three major aspects of the subject: theory, method, and application. In following paragraphs it is going to be introduced to the reader to the basic aspects of the subject and at the same time give a brief survey of the three aspects just mentioned. In the fourth chapter, we shall find answers to the general questions raised above, answers that will become more and more meaningful as we proceed with the study of differential equations in the following chapters (Ross, 2004).

There are ordinary and partial differential equations in this study for Laplace transform. A partial differential equation is an equation involving an unknown function of two or more variables and certain of its partial derivatives (Evans, 1997). Ordinary and partial differential equations describe the way certain quantities vary with time, such as the current in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through an insulated conductor. These equations are generally coupled with initial conditions that describe the state of the system at time t = 0. A very powerful technique for solving these problems is that of the Laplace transform, which literally transforms the original differential equation into an elementary algebraic expression. This latter can then simply be transformed once again, into the solution of the original problem. This technique is known as the “Laplace transform method” (Schiff, 1999).

One of the most important integral transforms is the Laplace transform. This well-known integral transform was first used by Laplace in 1812 when he was working on probability theory. Since that time many works have been devoted to the study of the properties of the Laplace transform and its various applications in many fields of science. Bateman in 1910 used the modern Laplace transform followed by Bemotien in 1920. This transform gained a more modern approach. In 1920 when doeth applied this transform on differential integro-differential equation. Since most of the physical evolve in time in semi finite or infinite domains, Laplace transform conspired with Fourier transform has shown a strong analytical method to solve partial differential equations obtained when dealing with their processes. But the one variable Laplace transform is not capable of solving this equations alone (Basheer, 2015)

In mathematics Laplace Equation is a second order or partial differential equation named after Pierre-Simon Laplace who was among the most influential scientists in history (Gillispie, 2000) and who first studied its properties (Evans, 1997). Pierre-Simon, marquis de Laplace (23 March 1749 – 5 March 1827) was an influential French scholar whose work was also pivotal to the development of mathematics, statistics, physics, and astronomy.(Stigler, 1986)

In this study, application of Laplace Theorem to differential equations is going to be discussed. First thing about that are differential equations. The subject of differential equations constitutes a large and very important branch of modern mathematics. So a short information about modern mathematics is involved in. And then basic information about differential equations are given.

In the next chapter we lay down the foundations of the theory and the basic properties of the Laplace transform.

2. BASIC DEFINITION OF LAPLACE AND INVERSE LAPLACE TRANSFORMATION

Firstly, the definition of Laplace transformation is defined as follows;

"Suppose >0 that is a real or complex-valued function of the (time) variable and is a real or complex parameter". We define the

Laplace transform of as

$$\infty$$

$$F(s) = \mathcal{L}\ (\ ) = \ - \ (\ )$$

$$0$$

$$= \lim \qquad (\ ) \qquad - \qquad (2.1)$$

$$\to\infty$$

$$0$$

whenever the limit exist (as a finite number). When it does, the integral (2.1) is said to *converge.* If the limit does not exist, integral is $\mathcal{L}$ said $()$ to *diverge* and there is no Laplace transform defined for . The notation will also be used to denote the Laplace transform of, and the integral is the ordinary Riemann (improper) integral.

The parameter belongs to some domain on the real line or in the complex plane. We will choose *s* appropriately so as to ensure the convergence of the Laplace integral (2.1). In a mathematical and technical sense, the domain of is quite important. However, in a practical sense, when differential equations are solved, domain of *s* is routinely ignored. When *s* is complex, we

The symbol $\mathcal{L}$ is the *Laplace transformation,* which acts on functions $= (\ )$ and generates a new function, $(\ ) = \mathcal{L}\ (\ )$ (Schiff, 1999).

Some examples of Laplace transform are given below.

Example 2.1

In mathematics, this example is useful to transform differential equations. All the examples below are subject of mathematics. In this project, applications of Laplace transform in engineering problems are going to be examined by using examples.

If ( ) ≡ 1 for ≥ 0, then

$$\mathcal{L} \quad � = � \; - \; 1 \quad 0$$

$$= \lim � \; \_\_ � \; � \; \to\infty - \; 0$$

$$\to\infty \quad -$$

$$= \frac{1}{\ }$$

provided of course that s > 0 (if s is real). Thus we have

$$\mathcal{L}(1) = \frac{1}{\ } \quad (\; > 0).$$

(Schiff, 1999)

Some basic specifications of Laplace transformation are given below.

2.1 Basic Specifications of Laplace Transformation

Basic specifications and differences of Laplace transformation are given in this chapter. Because, they will be used in following examples or applications.

□ If ( ) =for ≥ 0(constant)

ℒ ( )� = �∞ − = �∞ −

0 0

= ∞ = ∞

= � = �0 = − �1 �0

= − (0 −

1) =

ℒ{ } =

□ If ( ) = for ≥ 0 (constant)

ℒ{ } = �∞ − = �∞ −(−)

0 0

−(−) ∞ 1 −(−) ∞

= �−(−)�0 = − − � 1 �0

1 1

= − − (0 − 1) = −

1

Two Laplace transforms are briefly summarized above. More detailed information is given in Table 2.1 Other examples about the subject are given in the following.

2.2 Laplace Transforms of Some Elementary Functions

Laplace transformation has so many specific functions. Only the functions which are going to be used in examples or applications are given in table below.

Laplace transformation is helpful to get fast and easily approachable results in problems with differential equations. It shortens the solution time. Therefore there are numerous applications of the numerical Laplace transform in wave propagation, structural dynamics, viscoelasticy, heat conduction, fluid dynamics and other areas of applied mechanics. (Narayanan and Beskos, 1982)

	( )	ℒ{ ( )} = ( )
1	= 0,1,2,3 ...	! / > 0 ! = 1.2.3.4 ...
2		–
3	sin	2 + 2
4	cos	2 + 2
5	sinh	2 − 2
6	cosh	2 − 2

Table 2.1 Laplace transforms of some elementary functions (Spiegel, 1965)

2.3 Some Important Properties of Laplace Transformation

Theorem below is for sufficient conditions for existence of Laplace transforms.

$0 \leq \leq$ (Spiegel, for $>$, then its Laplace transform

Theorem 2.1 If F(t) is sectionally continuous in every fini e interval and of exponential order 1965).

Properties of Laplace transform in theorems and their examples are given accordingly in the list below. In the following list of theorems we assume, unless otherwise stated, that all functions satisfy the conditions of *Theorem 2.1* so that their Laplace transform exist.

2.3.1 Linearity property

In this regard the first property which is going to be discussed is linearity property. This property is as below.

Theorem 2.2 If $_1$ and $_2$ are any constant while $_1$ $_2$ are functions with Laplace transform $_1()$ and $_2()$ respectively, then $()$ and $()$

$$\mathcal{L}\{\,_1\,_1(\,)+\,_2\,_2(\,)\}=\,_1\mathcal{L}\{\,_1(\,)\}+\,_2\mathcal{L}\{\,_2(\,)\}=\,_1\,_1(\,)+\,_2\,_2(\,)$$

The result is easily extended to more than two functions (Spiegel, 1965).

Example 2.2

Three examples of linearity property are given below and also it is discussed how to apply this property in the last example.

$$\mathcal{L}\{4\,^2-3\,2+5\,^-\}=4\mathcal{L}\{\,^2\}-3\mathcal{L}\{\,2\,\}+5\mathcal{L}\{\,^-\}=4\left(\frac{2!}{^3}\right)-3\left(\,^2\right)+$$

$$4\left(\right)+5\left(+\frac{1}{1}\right)$$

8 3 5

= 3 −

Laplace ℒ () ()

The symbol , which transforms into , is often called the *Laplace transformation operator*. Because of the property of ℒ expressed in this theorem, we say that ℒ is a *linear operator* or that it has the *linearity property* (Spiegel, 1965).

Example 2.3

$$\mathcal{L}\{ 3 - 8 + 1\} = \mathcal{L}\{ 3\} - \mathcal{L}\{8 \} + \mathcal{L}\{1\}$$

$$= \frac{3!}{4} - 8\frac{1}{2} + \frac{1}{}$$

2.3.2 First translation or shifting property

Second property which is going to be discussed is first translation or shifting property. This property is as below.

Theorem 2.3 If

$$\mathcal{L}\{ ()\} = (), \text{then}$$

$$\mathcal{L}\{ ()\} = (-).$$

(Spiegel, 1965)

Example 2.4

Since

$$\mathcal{L}\{ 3\} = \frac{3!}{4},$$

we have

$$\mathcal{L}\{ 5 \quad 3 \} = \frac{3!}{(- 5)4} = \frac{6}{(- 5)4}.$$

Example 2.5

Since

we have $\mathcal{L}\{ \quad 2 \} = \quad 2+4$,

$-$ $\quad + 1 \quad + 1$

$\mathcal{L}\{ \quad 2 \} = (\ +1)2 \quad +4 = 2+2+5$

2.3.3 Second translation or shifting property

Third property which is going to be discussed is second translation or shifting property. This property is as below.

Theorem 2.4 If $\mathcal{L}\{ \ ( )\} = \ ( )$ and $( ) = \quad 0 \quad <$ $(-) > $, then

$$\mathcal{L}\{ \ ( )\} = \quad - \quad ( ).$$

(Spiegel, 1965)

Example 2.6

Since $\{ \ 3\} = 3!\ 4 = 6\ 4$,

the Laplace transform of the function

$$( ) = \begin{cases} (-2)3 & > 2 \\ 0 & < 2 \end{cases}$$

is equal to

$$\overline{\quad 4 \quad}.$$

2.3.4 Change of scale property

Fourth property which is going to be discussed is change of scale property. This property is as below.

Theorem 2.5 If $\mathcal{L}\{\ (\ )\} =$ $(\ )$, then

$$\mathcal{L}\{\ (\quad)\} = \frac{1}{\ }\ \ .$$

(Spiegel, 1965)

Example 2.7

Since

$$\mathcal{L}\{\ \} = \frac{1}{\ ^2 + 1}$$

we have

$$\mathcal{L}\{\ 3\ \} = \frac{1}{3}\ \frac{1}{\left(\frac{\ }{3}\right)^2 + 1} = \frac{3}{\ ^2 + 9}.$$

(Spiegel, 1965)

Example 2.8

$$\mathcal{L}\{4\ ^2 + 3\cosh 4\ \} = 4\mathcal{L}\{\ ^2\ \} + 3\mathcal{L}\{\cosh 4\ \}$$

$$= 4\ \frac{1}{\ -2} + 3\ \frac{4\qquad 3}{\ }$$

$$7\ ^2 \qquad -6\ -64$$

$$= \frac{\ }{(\ -2)(\ ^2 - 16)}$$

(Stroud, 2003)

2.4 Laplace Transformation of Derivatives

This property is the most commonly used property in engineering. It is particularly important for this study.

Using the property of derivatives, we would like to propose the Laplace transform of Euler-Cauchy equation, and find the solution of Euler-Cauchy equation represented by differential operator using Laplace transform (Kim, 2013).

In the application chapter of this study this property will be the first step to start. This property is as below.

Theorem 2.6 If $\mathcal{L}\{\ (\ )\} =\ (\ )$, then

$$\mathcal{L}\ (\ )(\ ) = \quad (\ ) - \ {}^{-1}\ (0) - \ {}^{-2}\ '(0) - \cdots - \ {}^{(\ -2)}(0) - \ {}^{(\ -1)}(0)$$

If $\ '\ \ {}^{-1}$ are continuous for and of exponential order for while is sectionally continuous for (Spiegel, 1965). This is the general theorem for Laplace transformation of derivatives. Some theorem is going to be derived below by this theorem.

Theorem 2.7 If $\mathcal{L}\{\ (\ )\} =\ (\ )$, then

$$\mathcal{L}\{\ '(\ )\} =\ (\ ) -\ (0)$$

while $(\ )$ $0 \leqq \ \leqq$ $>$

if is continuous for and of exponential order for is sectionally continuous for (Spiegel, 1965).

Example 2.9

If $(\ ) = \ 3\ ,$ then

and we have

$$\mathcal{L}\{\ (\ )\} = \ \frac{}{2 + 9}$$

$$\mathcal{L}\{\ '(\ )\} = \mathcal{L}\{-3\ \ 3\ \} = \ �\ \frac{}{2 + 9}\ � - 1 = -\frac{9}{2 + 9}$$

This method is useful in finding Laplace transforms without integration (Spiegel, 1965).

Theorem 2.8 If in *Theorem 2.7*, $(\ )$ fails to be continuous at $= 0$ but

$\to 0$

which may or may not exist, then

$$\mathcal{L}\{\ '(\ )\} = \ (\ ) - \ (0 +).$$

(Spiegel, 1965)

Theorem 2.9 If in *Theorem 2.7*, $(\ )$ fails to be continuous at $=\ ,$ then

$$\mathcal{L}\{\ '(\ )\} = \ (\ ) - \ (0) - - \ \{\ (\ +) - \ (\ -)\}.$$

Where $(\ +) - \ (\ -)$ is sometimes called the *jump* at the discontinuity, $=$. For more than one discontinuity, appropriate modifications can be made (Spiegel, 1965).

Theorem 2.10 If $\mathcal{L}\{\ (\ )\} = \ \ (\ ),$ then

$\mathcal{L}\{\ ''(\ )\} = \ 2\ (\ ) - \ (0) - \ '(0)$

$(\ )$ $'(\ )$ $0 \leqq \ \leqq$

If and are continuous for and of exponential order for is sectionally continuous for

equation $(\ )$ $'(\ )$

If and have discontinuities, appropriate modification of this can be made as in *Theorem 2-8 and 2-9* (Spiegel, 1965).

Example 2.10

Determine the Laplace transform of the Laguerre polynomials, defined by

$-$

$\mathcal{L}(\ ) = \ !\ ($ $),$ $= 0,1,2, \ldots$

Let $(\ ) = \ \ -$. Then

$!$ �.

1

$(\ )$

First, we find by Theorem 2.6, and subsequently the first translation theorem coupled with multiplication by theorem,

$(\ )$ $!$

$\mathcal{L}$ � $=$ $\mathcal{L}(\ ) = (\ +1)$ $+1.$

It follows that

$) > 1),$

1 $(\ )$ $(\ -1)$

$\mathcal{L}\{\ (\ )\} = \mathcal{L}$ � $!$ � $=$ $+1$ $(\ \ ($

again by the first translation theorem.

2.5 Laplace Transformation of Integrals

Not only can the Laplace transform be differentiated, but it can be integrated as well (Schiff, 1999).

$\mathcal{L}\{\ (\ )\} = (\ )$ the $(\ )$

. ***Theorem 2.11*** If , then the division of by corresponding to integration of with respect to between limits 0 and

To prove this theorem, we have from the definition of the Laplace transform the following equation:

$$\mathcal{L} �� (\ ) � = �^{\infty}_{0} - ��_{0} (\ ) �_{0}$$

By making the following substitutions

$$= �_{\circ} (\ ) , \qquad = -$$

$$= (\ ) , \qquad = - - /$$

and integrating by parts, we obtain

$$\mathcal{L} �_{0} (\ ) = - \frac{}{} �_{0} (\ ) �^{\infty} + \frac{1}{} \quad �^{\infty}_{0\ 0} - (\ ) .$$

By substitution of the limits, the first term on the right side of this equation is zero, and we arrive at the result

$$\mathcal{L} �_{0} (\ ) = \frac{1}{} \quad �^{\infty}_{0} - (\ ) = \frac{1}{} (\ )$$

which agrees with the statement of the theorem (Thomson, 1960).

As a result;

$\mathcal{L}$ �� ( ) � $=$ ( ).

0

(Spiegel, 1965)

Example 2.11

Since

2

we have $\mathcal{L}\{\ 2\ \} =$ 2 + 4 ,

$\mathcal{L}$ �� 2 � = (2² + 4)

0

as can be verified directly (Spiegel, 1965).

2.5.1 Multiplication by

Theorem 2.12 If $\mathcal{L}\{\ (\ )\} =$ ( ), then

$\mathcal{L}\{\quad (\ )\} = (-1)$ ( ) $= (-1)$ ( )( )

(Spiegel, 1965)

Example 2.12

Since

1

2

we have $\mathcal{L}\{\quad\} =$ − 2 ,

$\mathcal{L}\{\ 2\ \} = -$ 1 � = 1 / 2

− 2 (− 2)

$\mathcal{L}\{ \ ^2 \ ^2 \ \} = \ ^2 \quad 1 \quad = \underline{\quad 2 \quad}$

$-2 \qquad (\ -3)$

(Spiegel, 1965)

2.5.2 Division by

Theorem 2.13 If $\mathcal{L}\{ \ ( )\} = \ ( )$, then

$\mathcal{L} \ ( ) = \ ^{\infty} \ ( )$

provided

lim

$\to 0 \ ( )$ exists

(Spiegel, 1965)

Example 2.13

Since $\quad 2 \ 1 \quad$ and $\quad$ we have

$\mathcal{L}\{ \ \} = \quad +1 \qquad \lim \qquad = 1,$

$\to 0$

$\mathcal{L} \qquad = \ ^{\infty} \qquad 2 \qquad = \tan^{-1}(1/ \ )$

$+ 1$

(Spiegel, 1965)

2.6 Inverse Laplace Transformation

In mathematics we use inverse Laplace transformation generally to get differential equations from the exponential functions that we examine. In this chapter inverse Laplace transformation is going to discussed.

2.6.1 Definition of Inverse Laplace Transform

In order to apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform (Schiff, 1999). This procedure, together with the shifting theorem, is often sufficient for the determination of the inverse transformation (Thomson, 1960).

then is called an *inverse Laplace*

If the Laplace transform of a function is , if ,
transformation $\mathcal{L}\{()\}$ $\mathcal{L}$ *transform* of (s) and we write symbolically F(t)= where is called the *inverse Laplace* operator (Spiegel, 1965).

Also the function can be expressed as the Bromwich contour integral over given by

$$() = \mathcal{L}^{-1}\{\ \}() = \frac{1}{2} \int_{-\infty}^{+\infty} ()$$

where is equal to any number such that the location of all singularities of has a real part less than (Zecchin et al., 2012)

Example 2.14

Since
we can write $\mathcal{L}\{\ \} = \frac{}{+3}$

$$\mathcal{L}\ -3\ \ 1$$

$$1\ \ -1\ \ -3$$

2.6.2 Uniqueness of Inverse Laplace transforms

follows $\mathcal{L}\{()\} = ()$ $\mathcal{L}\{() + ()\} =$ is(zero,). it is clear From this it the same Laplace

Since the Laplace transform of a null function

that, if then also e ()

that we can have two different functions with

transform.

Example 2.15

The two different functions;

1( ) = 3 and 2 0 = 1

( ) = ◆ −3 ≠ 1

have the same Laplace transform, which is [1/(+ 3)].

If we allow null function, we see that the inverse Laplace transform is unique. It is unique, however, if we disallow null function (which do not in general arise in cases of physical interest).This result is indicated in

( )

Theorem 2.14 (Lerch's theorem) If we restrict ourselves to functions £ which are sectionally continuous in every finite interval and of exponential order for , then inverse Laplace transform of , i.e. is unique. We shall always assume such uniqueness unless otherwise stated (Spiegel, 1965).

2.7 Partial Fractions

The method of partial fractions is a technique for decomposing functions like Y(s) above so that the inverse transform can be determined in a straightforward manner.

In many applications of the Laplace (transform) it becomes necessary to find the inverse of a particular transform, . Typically it is a function that is not immediately recognizable as the Laplace transform of some elementary function, such as

1

( ) =

real), where the goal is to Ω(e.g., ( ) >).

for s confined to some region Just as in calculus (for integrate such a function, the procedure required here is to decompose the function into partial fractions.

In the preceding example, we can decompose ( )into the sum of two fractional expressions:

1

$(-2)(- 3) = -2 + -3$

that is,

$1 = (-3) + (-2).$ (2.1)

$[1 \text{ and } (-3) + (-2)]$

are equal for all in Ω , except polynomials are

= 2= 3, the

Since (2.1) equates two polynomials that possibly for two identically equal for all values of . This follows from the fact

that two polynomials of degree that are equal at more than points are identically equal.

Thus, if $= 2$, $= -1$, and if $= 3$, $= 1$, so that

1 1 1

$( ) = (-2)(-3) = - -2 + -3 .$

Finally,

$( ) = \mathcal{L}^{-1}\{ ( )\} = \mathcal{L}^{-1} - \frac{1}{-2} + \mathcal{L}^{-1} \frac{1}{-3}$

$= - 2 + 3 .$

(Schiff, 1999)

2.7.1 Partial Fraction Decompositions

We will be concerned with quotient of two polynomials, namely a rational function

( ) = ( )

have no common () () () () ()

where the degree of is greater than degree of , and and

factors. Then can be expressed as a finite sum of partial fractions.

- For each linear factor of the form + of ( ), there corresponds a partial fraction of the form

constant

- For each repeated linear factor of the form (+) , there corresponds a partial fraction of the form.

1 2

(+)2 + ⋯ + (+) ,1 , 2 , …, constants.

- For every quadratic factor of the form 2 + + , therecorresponds a partial fraction of the form

+

2 + + , constants.

- For every repeated quadratic factor of the form 2 , there corresponds a partial fraction of the form (+ +)

1 + 1 2 + 2 +

2 + + (2 + +)2 + ⋯ + (2 + +) ,

1, 2 , 2, … , constants.
, … , ,1

The object is to determine the constants once the polynomial () / () has been represented by partial fraction decomposition. This can be achieved by several different methods (Schiff, 1999).

Example 2.16

$$\frac{1}{(\ -2)(\ -3)} = \frac{}{-2} + \frac{}{-3}$$

Or

$$1 = (\ -3) + (\ -2)$$

as we have already seen. Since this is a polynomial identity valid for all , we may equate the coefficients of like powers of on each side of the equals sign.

Thus, for , ; and for 0 , Solving these two equations simultaneously, as before (Schiff, 1999).

2.8 Some Inverse Laplace Transform

The following results follow at once from corresponding entries

Example 2.17

Find

$$^{-1}\left\{\frac{1}{2+6+13}\right\}.$$

Write

$$\frac{1}{2} = \frac{1}{(+\ )2+4} = \frac{1}{2} \frac{2}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{+6+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(+3)2+2}\right\} = \frac{1}{2} \ {}^{-3}\ \sin 2$$

(Ross, 2004)

Example 2.18

Find

-1 $+ 1$

Write $\mathcal{L}$ $2(- 1)$.

$+ 1$

or $2(- 1) = + 2 + - 1$,

$= 1$. $= 2$ $+ 1 = (- 1) + (- 1) + 2$,

which is an identity for all values of . Setting 1 gives 1 ; setting , and so

gives . Equating the $2 = 0$ $= -1$

Whence

$$\mathcal{L} \; 2(- 1) = -2\mathcal{L} \; - \mathcal{L} \; 2 + 2\mathcal{L} \; - 1$$

$$= -2 - + 2 .$$

Example 2.19

Find

-1 $2 \; 2$

$\mathcal{L}$ $(2 + 1)(- 1)2$.

We have

$2 \; 2 \qquad \pm$

or $$\frac{(2}{+1)(-1)2} = \frac{+1}{2} + \frac{+}{-1} + \frac{}{(-1)2},$$

$$2\,2 = (+)(- 1)2 + (2 + 1)(- 1) + (2 + 1)$$

Settings $= 1$ gives $= 1$. Also setting $= 0$ gives $0 = - +$, or

$$-1 = - .$$

Equating coefficients of 3 and , respectively,

$$0 = +$$

$= 1$

$$0 = -2 + .$$

These last two equation simply shows Then from the first equation, Therefore, ; finally, the second equation $= 0.$

$$\mathcal{L}^{-1}\left\{\frac{2\,2}{(2+1)(-1)2}\right\} = -\mathcal{L}^{-1}\left\{\frac{}{2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(-1)2}\right\}$$

$$= -\cos + + .$$

(Schiff, 1999)

	( )	$\mathcal{L}^{-1}\{\ (\ )\} =\ (\ )$
1	! / +1 ! = 1.2.3.4 ...	= 0,1,2,3 ...
2	–	
3	2 + 2	sin
4	2 + 2	cos
5	2 − 2	sinh
6	2 − 2	cosh

Table 2.2 Table of inverse Laplace transforms (Spiegel, 1965)

2.9 Some Important Properties of Inverse Laplace Transforms

In the following list we have indicated various important properties inverse Laplace transforms. Note the analogy of Properties 2.8 with the corresponding properties.

2.9.1 Linearity property

Theorem 2.15 If $_1$ and $_2$ are any constant while $_1()$ and $_2()$ are Laplace transform of $_1$ and $_2$ respectively, then

$$\mathcal{L}^{-1}\{\ _1\ _1(\ ) + \ _2\ _2(\ )\} = \ _1\mathcal{L}^{-1}\{\ _1(\ )\} + \ _2\mathcal{L}^{-1}\{\ _2(\ )\} = \ _1\ _1(\ ) + \ _2\ _2(\ )$$

The result is easily extended to more than two functions (Spiegel, 1965).

Example 2.20

$$\mathcal{L}^{-1}\ 4 + 3 + 2\ 5$$

$$= 4\mathcal{L}^{-1}\ 1 + 3\mathcal{L}^{-1}\ 2 + 5\mathcal{L}^{-1}\ 21$$

$$2 \qquad 5$$

Because of this property we can say that $\mathcal{L}^{-1}$ is a *linear operator* or that it has the *linearity property*.

2.9.2 First translation or shifting property

Theorem 2.16 If $\mathcal{L}^{-1}\{ ( )\} =$ $( )$, then

$$\mathcal{L}^{-1}\{ (-)\} = ( )$$

(Spiegel, 1965)

Example 2.21

Since

$$1 \quad \underline{1}$$

$$-1$$

we have $\mathcal{L}$ ◆ 2 +4◆ = 2 sin 2 ,

$$-1 \quad 1 \quad -1 \quad 1 \quad \underline{1}$$

$$\mathcal{L} ◆\ 2 - 2 + 5 ◆ = \quad \mathcal{L} ◆ (- 1)^2 + 4 ◆ = 2 \quad \sin 2$$

2.9.3 Second translation or shifting property

Theorem 2.17 If $\mathcal{L}^{-1}\{ ( )\} = ( )$, then

$$-1 \quad (-) \quad >$$

$$\mathcal{L} \quad \{ \quad ( )\} = ◆\ 0 \quad <$$

(Spiegel, 1965)

Example 2.22

Since $\mathcal{L}^{-1}$ ◆ 2 1 + 1 ◆ = sin , we have

$$\mathcal{L}^{-1}\left\{ \quad /3 \quad + 1 \right\} = \begin{cases} \sin(- /3) & \text{if } > /3 \\ 0 & \text{if } < 3 \end{cases}.$$

2.9.4 Change of scale property

Theorem 2.18 If $\mathcal{L}^{-1}\{ ( )\} =$ $( )$, then

$$\mathcal{L}^{-1}\{ (\quad)\} = \frac{1}{\ } \left(\ \right)$$

(Spiegel, 1965)

Example 2.23

Since $\mathcal{L}^{-1}\left\{ \frac{\ }{\ ^2 + } \right\} = \cos 4$, we have

$$\mathcal{L}^{-1}\left\{ \frac{2}{2^2 + 16} \right\} = \frac{1}{2}\cos \frac{4}{2} = \frac{1}{2} \cos 2$$

as is verified directly.

2.10 Inverse Laplace Transform of Derivatives

Laplace transform of derivatives is examined. Now inverse Laplace transform of derivatives is going to be examined in the same way.

$\mathcal{L}^{-1}\{ ( )\} =$ $( )$, then

$$\mathcal{L}^{-1}\{ ( )\} = \qquad \mathcal{L}^{-1}\left\{ \quad ( ) \right\} = (-1) \quad ( )$$

(Spiegel, 1965)

Example 2.24

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\left\{\frac{1}{s^2+1}\right\} = -\frac{2s}{(s^2+1)^2}$, we have

$$\mathcal{L}^{-1}\left\{-\frac{2s}{(s^2+1)^2}\right\} = -t\sin t$$

or

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t.$$

2.11 Inverse Laplace Transform of Integrals

Theorem 2.20 If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\} = (-1)^n t^n F(t)$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\left\{\frac{1}{s^2+1}\right\} = -\frac{2s}{(s^2+1)^2}$, we have

$$\mathcal{L}^{-1}\left\{-\frac{2s}{(s^2+1)^2}\right\} = -t\sin t$$

or

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t.$$

(Spiegel, 1965)

2.11.1 Multiplication by s

Theorem 2.21 If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and then $F(0) = 0$, then

$\mathcal{L}^{-1}\{\ \ (\ )\} = \ \ '(\ )$

Thus multiplication by has the effect of differentiating ( ). If (0) ≠ 0, then

$$\mathcal{L}^{-1}\{\ \ (\ ) - \ \ (0)\} = \ \ '(\ )$$

or

$$\mathcal{L}^{-1}\{\ \ (\ )\} = \ \ '(\ ) + \ \ (0)\ (\ )$$

where ( ) is the Dirac delta function or unit impulse function.

(Spiegel, 1965)

Theorem 2.22 If $\mathcal{L}^{-1}\{\ (\ )\} = \ (\ )$, then

$$\mathcal{L}^{-1}\ \{\ \underline{(\ )}\ \} = \int\ (\ )$$

$_0$

( ) from ^{0}to .

by (or multiplication by 1/) has effect of *integrating*

2.11.2 Division by

Thus division

(Spiegel, 1965)

Example 2.25 Since

$-1 \quad 1 \quad \underline{1}$

we have $\mathcal{L}$ $\{\ \overline{2} + 4\} = \ 2 \sin 2\ ,$

$-1 \quad 1 \quad \underline{1} \quad \underline{1}$

$$\mathcal{L}\ \{\ (\ 2 + 4)\} = \int_0 \quad 2 \sin 2 = 4 \quad (1 - \cos 2\ )$$

Generalizations to $\mathcal{L}^{-1}\{\ ()/\quad\}$, $= 2,3, \ldots$, are possible

2.12 Convolution Theorem

Clarkson and Pritchard (1992) had used the advantage of convolution theorem of Laplace transform. With MACSYMA program they applied inverse Laplace transform without carrying out any integration. This is a significant advantage for MACSYMA since recursive inverse Laplace transforms are much more readily computed by symbolic algebra than are repeated integrations.

In our study, another important procedure in connection with the use of tables of transforms is that furnished by the so-called convolution theorem which we shall state below. We first define the convolution of two functions and .

Definition

Let and be two functions that are piecewise continuous on every finite closed interval $0 \le \le$ and of exponential order. The function denoted by and defined by

$$() * () = \int_0 () (-) \qquad (2.2)$$

is called the *convolution* of the functions of and .

Let us change the variable of integration of integration in equation (2.2)

by means of the substitution $= -$, We have

$$() * () = \int_0 () (-) \qquad = -\int^0 (-) ()$$

$$= \int_0 () (-) = () * ().$$

Thus we have shown that

is $0 \le \le$

$* = *$

Suppose that both and are piecewise continuous on every finite closed interval and of exponential order $0 \le \le$. Then it can be shown that also piecewise continuous on every finite closed interval and of exponantial order , where is any positive number. Thus (Ross, 2004).

Example 2.26

$$\mathcal{L}^{-1}\left\{\frac{1}{(^2+1)}\right\}$$

Find using the convolution.

Solution:

We write $1/(^2+1)$ as product $()\,()$, where $() = 1/$ and $()$.

$$\mathcal{L}^{-1}\left\{\frac{1}{(^2+1)}\right\} = () * () = \int_0 1 \cdot \sin(-) ,$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(^2+1)}\right\} = () * () = \int_0 \sin \cdot 1 .$$

The second of these two integrals is slightly simpler. Evaluating it, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(\;^{2}+1)}\right\} = 1 - \cos$$

3. ARTICLES ABOUT LAPLACE TRANSFORM

Three articles of Laplace transformation in different disciplines are inspected and they have been formed a basis for the study.

Articles are so limited in the subject of Laplace transformation. There are a few articles about Applications of Laplace transform with differential equations. The reason of choosing these articles is to create variety of applications in Laplace transform.

Lately, articles about this subject are about computer programs. The greatest reason of this can be interpreted as the developments in engineering and computer applications in last 30 years.

3.1 Numerical Operational Methods for Time-Dependent Linear Problems

Narayanan and Beskos's article is published at International Journal for Numerical Methods in Engineering in 1982. A general and systematic discussion on the use of the operational method of Laplace transform for numerically solving complex time-dependent linear problems is presented in this article. The use of operational methods is, sometimes, very much useful to ”guess” specific theorems in various fields of analysis. Operational method is a technique by which problems in analysis, in particular differential equations, are transformed into algebraic problems, usually the problem of solving a polynomial equation. The most commonly used method is Laplace transformation. Since this article compares Fourier and Laplace operational methods, this article is helpful for those who looks for the suitable method for their research.

Another term in this article is time-dependent linear problem as we know from the definition of Laplace transform (See Page: 5) which means an equation whose dependent is time. To give an example Laplace transform

�( ) = �∞ −   ( )

where is the Laplace transform parameter and the parameter of the function which is being transformed t is time. Since transformation methods are generally using for time-dependent problems, complex time-dependent problems are given as examples and applications in this study to apply transformation method of Laplace.

This article took place in this study since it has a wide research and some applications in subject of Laplace transform. Therefore Narayanan and Beskos (1982) listed an almost complete bibliography on the numerical inversion of Laplace transform and its applications between the years 1934 and 1976 in their article, it is especially important for this study. And also Narayanan and Beskos have demonstrated us the generality and advantages of the Laplace transformation method against other known methods.

Narayanan and Beskos (1982) systematically discussed eight existing methods of numerical inversion of the Laplace transform with respect to their use, range of applicability, accuracy and computational efficiency on the basis of some framework vibration problems. These methods are as following:

- Interpolation-collocation methods
  - *Method of maximum degree of precision (3.1.1)*

    This method establishes a quadrature formula for the integral ( ) 1 + ∞ − ∞ , which has the maximum degree of precision relative to a certain class of rational functions.
  - *Schapery's collocation method (3.1.2)*

    Schapery's method proposed for linear viscoelastic stress analysis for a minimum approximation of error.
  - *Multidata method of Cost and Becker (3.1.3)*

    This is a modification of the Schapery's method. It aims the same with Schapery's method but the advantage of

Cost and Becker's method() is better computation for a better approximation of .

- Methods based on expansion in orthogonal functions
  - *Method based on expansion in trigonometric functions (3.2.1)*

    of on a

    In this method is determined in terms of the values finite sequence. It is used for trigonometric functions.
  - *Method based on expansion in Legendre polynomials (3.2.2)*

    This method is same with the other but it is for polynomials.
  - *Method based on expansion in Laguerre polynomials (3.2.3)*

    This method works for generally exponential harmonic functions which includes both trigonometric and polynomial terms.
- Methods based on Fourier transform
  - *Method based on Fast Fourier Transform (3.3.1)*

    Fast Fourier Transform algorithm of Cooley and Tukey is used in this method. According to article an extensive treatment of Fast Fourier Transform can be be found in the book of Brigham (The Fast Fourier Transform).
  - *Method based on the sine-cosine transform (3.3.2)*

    This method is an efficient improvement of the method of Dubner and Abate, which is based on the finite Fourier cosine transform.

Methods 3.1.2, 3.1.3, 3.2.1 and 3.2.2 work with real data, i.e. with real values of the Laplace transform parameter s, while methods 3.1.1, 3.2.3, 3.3.1 and 3.3.2 work with complex data. This has important implications on the acccuracy and efficiency of the methods (Narayanan and Beskos, 1982).

They also briefly described other applications of the Laplace transform method in conjunction with the finite element method or the boundary integral equation method in the areas of earthquake dynamic response of frameworks, thermally induced beam vibrations, forced vibrations of cylindrical shells, dynamic stress concentrations around holes in plates and viscoelastic stress analysis to demonstrate the generality and advantages of the method against other known methods.

The difference of this article compared to others is that Narayanan and Beskos confronted the operational methods in engineering problems.

According to Narayanan and Beskos, (1982) there are numerous applications of the numerical Laplace transform in wave propagation, structural dynamics, viscoelasticity, heat conduction, fluid dynamics and other areas of applied mechanics. So the applications of Laplace transform are used for the same reason with this study.

Finally, applications in applied mechanics which are connected with the solution of complex time-dependent, linear problems described by their governing equations (usually partial differential equations) as well as their initial and boundary conditions (Narayanan and Beskos, 1982). Narayanan and Beskos (1982) have accomplished the solution of these problems numerically in three steps as follows:

1. Apply Laplace transform with respect to time and reduce the problem to a time-dependent one.

2. Formulate and solve the problem in the Laplace domain by any numerical method, such as the finite element method , the finite difference method or the boundary integral equation method.

3. Numerically invert the transformed solution to obtain the time domain solution.

In conclusion of this article, use of the numerical Laplace transform for time-dependent, large-order linear problems offers a simple, straightforward and uniform method of solution by reducing the complexity of the problem through the reduction by one of the number of their independent variables, at the same time taking care of the initial conditions and easily handling time-dependent boundary conditions (Narayanan and Beskos, 1982).

This article includes a detailed applications and research about which method of Laplace transformation can be used for which kind of problem. So it can lead this project when solving problems at applications chapter.

3.2 A Laplace Transform Solution of Schrödinger's Equation Using Symbolic Algebra

Clarkson and Pritchard article is published at International Journal for Quantum Chemistry in 1992. Since Schrödinger equation is a time-dependent equation it can be transformed with Laplace transformation method. That is why this article took place in this project. A general solution to the one-dimensional time-independent Schrodinger equation is derived using the properties of the Laplace transform in this study.

The Schrödinger differential equation

$$\mathcal{L}\{\ (\ ,\ )\} = -\ ''(\ ,\ ) +\ (\ )\ (\ ,\ )$$

$$=\ (\ ,\ )$$

with T-periodic real-valued potential parameter is assumed here known with a continuous as of Hill type

bounded function and a real () involving the solution to (Khmelnytskaya and

Schrodinger's equation in a periodic domain including internal arise in the rotations in

study of many different areas of physics, () = (+ 2)

molecules, the kinetics of unimolecular isomerization reactions, and the motion of electrons in a metal or in semiconductors. Schrödinger equation is generally used for wave funcitons. In these applications, we need to solve for the eigenvalues of the equations (Clarkson and Pritchard, 1992)

Clarkson and Pritchard (1992) used inverse Laplace transform for the solution of eigenvalues of Mathieu's equation which is

$$2 \quad + [\quad - 2 \cos(2\ )] \quad = 0.$$

$$2$$

To describe this equation shortly, the Mathieu equation is a special case of a linear second order homogeneous differential equation, such as occurs in many applications in physics and engineering (Ruby, 1996). They compared eigenvalues obtained by Laplace transform method and eigenvalues obtained by Hill's Method. So they found that it can be only transformed accurately with Laplace transform. Fourier coefficients are, however, not distinct for this equation.

Clarkson and Pritchard (1992) had used the advantage of Laplace transform convolution theorem, which is discussed in former chapter of this study, with.Macsyma program they applied inverse Laplace transform without carrying out any integrations. This is a significant advantage for Macsyma since recursive inverse Laplace transforms are much more readily computed by symbolic algebra than are repeated integrations.

Macsyma is an interactive symbolic, numerical and graphical mathematical problem solving tool. Macsyma offers symbolic and numeric manipulation and solution capabilities in algebra, calculus and numerical analysis; 2D and 3D report-quality graphics interactive scientific notebooks; and a user programming environment.

Although they have derived a closed-form analytical solution for Schrödinger's equation, from a practical point of view, the most difficult part remains, and (that) is to evaluate the inverse Laplace transform of each term of the series . Under certain circumstances, these terms will be amenable to partial fraction decomposition and further analytical progress can be made. If this is not the case, they can still use considerations of parity to simplify the problem by separating the even and the odd solutions and, thus, reducing by half the amount of work that has to be done in taking the inverse Laplace transform.

So what is even and odd solutions?

3.2.1 Even and Odd Solutions

If(the→ potential—) function ( ) is an even function of with respect to parity , then the Hamiltonian will be even, and if the boundary conditions are either even or odd, then the eigenfunctions will also be of definite parity, either even or odd. Most bound-state problems fall into this category (Clarkson and Pritchard, 1999).

In certain boundary conditions function is solved in this article and an auxiliary function is defined. The purpose of this auxiliary function is to assist Clarkson and Pritchard in carrying out a partial fraction decomposition, () compact, which will greatly ease the process of taking the inverse Laplace transform to calculate . Although the explicit definitions of are not particularly they can be easily defined using recursion (Clarkson and Pritchard, 1999).

With partial fractions property, auxiliary function can be transformed by inverse Laplace transformation method.

Finally, if Clarkson and Pritchard can obtain the inverse() Laplace transform of the function to give them the approximation to , they can arrive at the general solution for Schrödinger equation (Clarkson and Pritchard, 1999).

In this article Clarkson and Pritchard have presented a Laplace transform method of solution of Schrödinger's equation that can be applied to any potential that can be expressed as a finite Fourier series. Unlike Neumann series solutions that recast the equation as an integral equation, this method requires no recursive integration of a kernel and relies instead on properties of the inverse Laplace transform convolution theorem. In a special case where the absolute value of the poles of the transformed potential was distinct, then a complete closed-form solution was obtained. In other cases, the symbolic and algebraic manipulation package Macsyma showed itself to be capable of carrying out the required inverse Laplace transforms.

The results of this method were compared with the results obtained by a comparable analytical method in this article which is Hill's method. The Laplace transform method was found to be simpler and more accurate than Hill's method in the article of Clarkson and Pritchard, with the six terms of the Hill's expansion necessary to reach the accuracy of three terms of the Laplace transform method. Also, unlike Hill's method, the Laplace transform method can be used to solve for the eigenfunctions, as well as the eigenvalues (Clarkson and Pritchard, 1999).

Likely the article of Clarkson and Pritchard, in this study Laplace transformation method has been used at applications for engineering. We all know an engineer’s duty is to improve people’s life quality by using natural sources. This study can be used for engineering students to learn solving complex time dependent problems.

3.3 Inverse Laplace Transform for Transient-State Fluid Line Network Simulation

Zecchin et al.’s article is published at Journal of Engineering Mechanics in 2012. This paper presents a study combining the new Laplace-domain input/output (I/O) model derived from the network admittance matrix with the Fourier series expansion numerical inverse Laplace transform to serve as a time-domain simulation model. A series of theorems are presented demonstrating the stability of the I/O model, which is important for the construction of the numerical inverse Laplace transform method.

Finally, an article which is also about engineering problems is going to be inspected in this chapter. 2 articles are chosen to define the advantages of

Laplace transformation method compared to others. It is discussed that in the first chapter Laplace transform is a powerful method which can be used in partial differential equations vary with time. And also with Laplace transform, complex differential equations can be solved by transforming them into simple equations. Zecchin et al. (2012) transformed some partial differential equations, which is about fluid line that has been developed by Bernoulli, with Laplace transform and used them to get better results with respect to stability, accuracy and computational efficiency.

Firstly I would like to cite theorems of this article;

∈ Λ

[(, Λ),]

Theorem 3.3.1. The network admittance matrix *matrices for each* *for the network* *is strictly passive if the link admittance are strictly passive.*

∈ Λ

Theorem 3.3.2. For a given network *for all links* *, the I/O transfer matrix* *are strictly passive.*

is stable if all the admittance matrices [(, Λ),]

Applications of these theorems are discussed below.

Theorem 3.3.1 is conditional on the strict passivity of the link admittance functions. This was demonstrated in former study of Zecchin to be conditional on the strict passivity of the resistive and compliance functions and . For all they do not create physically realizable models, and are strictly passive as

energy (Zecchin et al., 2012).

In brief, focus of this article has been on the use of the linear Laplace-domain network model from previous research of Zecchin as an alternative time-domain hydraulic simulator by way of the numerical inverse Laplace transform. But they also compared numerical inverse Laplace transform with method of characteristics (a technique for solving hyperbolic partial differential equations) for a range of networks and pipeline functions. Zecchin et al. used numerical inverse Laplace Transform for pipeline systems.

So what is numerical inverse Laplace transform?

= ℒ{ }

Given the Laplace transformable function f with Laplace transform , the function can be expressed as the Bromwich contour integral over given by

( ) = ℒ−1{ }( ) = � 1 � � +∞ ( )

= 2 −∞

where any number such that the location of all singularities of has a real less than . For the application of interest here, the Laplace-domain

function for the th output is given by Θ ( )

= ( ) = ( ) �Ψ ( )�

where th row of . From Theorem 3.3.2, it is known that all elemental functions of are stable.

After expressing the fundamentals of this article, let’s examine the part about Laplace transformation in this article. Inverse Laplace transformation is not so important for our study but as we see from this article, inverse Laplace transformation has an important place in solutions of engineering problems. So it was necessary to inspect this paper in our study.

Much of the research literature has focused on the use of discrete partial differential equation solvers; however, there has been significant interest on the development of time-domain models based on the inverse Laplace transform of the Laplace-domain solutions of the fluid line equations (Zecchin et al., 2012).

What Zecchin et al. did is like basic Laplace transformation. To give an example;

The Laplace-domain expression of the I/O transfer function is given by

$$\Psi(\ ) \qquad \Theta(\ ) \tag{3.3.1}$$

where = × I/O network transfer matrix.

And what is that I/O model?

By the convolution theorem (2.12) of the inverse Laplace transform, the time-domain Input/Output model representation of equation (3.3.1) is given by

$$\Psi(\ ) = h(\ -\ )\ \Theta(\ )$$

$$\Theta(\ ) \qquad 0 \qquad \Psi(\ )$$

where the lowercase symbols are the time-domain counterparts of their Laplace Transforms.

The primary interest within this article is the suitability of the linear numerical inverse Laplace transform approach for the time-domain simulation of pipeline networks were composed of both linear and nonlinear pipes. The important issues pertaining to the suitability of the numerical inverse Laplace transform method are (1) the accuracy of the method to approximate the true dynamics and (2) the relative computational efficiency of the method with respect to alternative simulation approaches. To undertake this analysis, many numerical experiments were undertaken comparing the proposed numerical inverse Laplace transform method combining many equations in the article with the commonly used method of characteristics approach (Zecchin et al., 2012). Within the experiments, 20 different case studies were considered that composed of four different networks with five different pipeline models. These and the adopted parameter settings for the numerical inverse Laplace transform are outlined in the article.

The approach presented here is entirely novel in that it couples the Laplace-domain I/O model from previous work of Zecchin et al. in a computationally efficient way with the Fourier series expansion numerical inverse Laplace transform from Abate and Whitt (1995). Heuristics from the previous work of Zecchin et al. were successfully used in the application of the numerical inverse Laplace transform to 20 different case studies in this article (four different networks in five different pipe types). The focus of the studies as on the accuracy and computational efficiency of the proposed numerical inverse Laplace transform.

For the cases considered, the numerical inverse Laplace transform was found to provide accurate approximations for all case studies, even networks with nonlinear pipe types. The accuracy was observed to be greater for the more highly dissipative networks. For large networks, numerical inverse Laplace transform was found to be computationally efficient compared with the method of characteristics. This relative efficiency was observed to be especially true for the case studies with more complex pipe types involving convolution operations, as these operations exert little additional computational time on the numerical inverse Laplace transform. In addition to the computational efficiency, the numerical inverse Laplace transform possesses the desirable property that it correctly captures wave propagation delays without the need for fine computational grids. As such, the numerical inverse Laplace transform represents a worthy alternative approach for modeling networks, particularly in cases where a limited number of measurement points are of interest, and the networks involve pipes with greatly varying propagation delays.

In the fourth chapter there are some examples about engineering problems that includes Laplace transform.

3.4 Comparison and Results

Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. The subject of differential equations possesses a large and important area of application. In this project engineering part of it had been discussed. There are so many applications of Laplace transformation in engineering problems in the articles above. By applying Laplace transform to linear ordinary differential equations in essential engineering problems, they transform into simple algebraic problems. As you see from the articles, Laplace is just a method for transformation, it is a tool. Nevertheless it is necessary for solving complex time-dependent differential equations in engineering.

It finds very wide applications in various areas of physics, structural engineering, mechanical engineering, electrical engineering, control engineering, optics, mathematics and signal processing. 3 articles about Laplace transform that is applied to differential equations are given above. Zecchin et al. used that method for modeling fluid line network simulation, Clarkson and Pritchard used Laplace transformation for mathematic applications, and Narayanan and Beskos discussed the applications of this method in various areas in their study.

By inspecting all these articles, these implications can be made for Laplace transform;

- Results with Laplace transformation is more accurate than Fourier series in engineering applications
- Laplace transform can be used for either designing engineering systems or examining designed systems for development
- Laplace transformation is compatible for computer programs. That makes it applicable for this century, so we will see Laplace transformation also in the future

To understand this method in detail, information and applications of Laplace transformation method are given in next chapter.

4. APPLICATIONS AT DIFFERENTIAL EQUATIONS WITH LAPLACE TRANSFORMS

Problems involving ordinary differential equations can be solved directly by an elementary knowledge of the Laplace transformation, whereas problems leading to partial differential equations require some knowledge of the complex variable theory for thorough understanding. The study of the complex variable basis of the Laplace transform method is strongly urged, since it offers a more general approach covering cases for which the elementary method is frequently inadequate (Thomson, 1960).

The method of Laplace transformation offers a powerful technique for the fields of applied mathematics. The class of functions that can be treated is extensive and includes those involved in many physical problems. In contrast to the classical method, which requires the general solution to be fitted to the initial or boundary conditions, these conditions are automatically incorporated in the Laplace transform solution for any arbitrary or prescribed excitation. Solutions for impulsive types of excitation and excitation of arbitrary nature can be concisely written in the Laplace transform notation. In fact, the method of Laplace transformation provides a great deal of insight into the nature of the solution, prior to its final completion (Thomson, 1960).

Schiff (1999) produced a general procedure for the Laplace transform method for solving ordinary differential equations can be summarized by the following steps.

(i) Take the Laplace transform of both sides of the equation. This results in what is called the transformed equation.

(ii) Obtain an equation variable . , where ( ) is an algebraic

(iii) expression in the $\mathcal{L}( ) = ( )$.

Apply the inverse

In this study, Laplace transformation method is applied at differential equations. We can use differential equations for mathematics and engineering problems. They both examined in this chapter.

4.1 Application at Differential Equations

In this chapter some problems which include differential equations are given and their solutions are given below them to show how *Laplace transform* can be applied at differential equations.

The derivative theorem in the form of Theorem 2.6 opens up the possibility of utilizing the Laplace transform as a tool for solving ordinary differential equations. Numerous applications of the Laplace transform to ordinary differential equations will be found in ensuing sections (Schiff, 1999).

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. For example, suppose that we wish to solve the second order linear differential equation

$$2$$

or

$$'' + ' + = (\ ) \qquad (4.1)$$

where and are constants, subject to the *initial* or *boundary conditions*

$$(0) = \ , \qquad '(0) = \qquad (4.2)$$

where and are given constants, On taking the Laplace transform of determination of $\mathcal{L}\{\ (\ )\} = \ (\ )$. The required solution is then obtained by finding the inverse Laplace transform of y(s). The method is easily extended to higher order differential equations in problems below (Spiegel, 1965).

In following questions applications of Laplace transform is going to be varied. Not only for mathematical questions but also basic engineering problems which can be solved by differential equations.

Example 4.1 $' - 3 \quad = \quad 2$ subject to $(0) = 1$

we have

$$\mathcal{L}\{\ \} = \quad , \mathcal{L}\{\ '\} = \quad - \ (0)$$

$$\mathcal{L}\{\ '\} - 3\mathcal{L}\{\ \} = \mathcal{L}\{\ 2\ \}$$

$$- \ (0) \ - 3 \ = \ \frac{1}{-2}$$

$$[\ - 3] \ = \ \frac{1}{-2} \ + 1$$

$$\mathcal{L}^{-1} \ = \ \mathcal{L}^{-1} \ \frac{1}{(-2)(-3)} + \ \mathcal{L}^{-1} \ \frac{1}{-3}$$

$$= \mathcal{L}^{-1} \ 1 \quad \mathcal{L}^{-1} \ 1 \quad \mathcal{L}^{-1} \ 1$$

$$= \ 2$$

Example 4.2

$$'' + \quad = \ , \quad (0) = 1, \ '(0) = -2$$

Taking the Laplace transform of both sides of the differential equation and using the given conditions,

we have,

$$\mathcal{L}\{ \ \} = \ , \mathcal{L}\{ \ ''\} = \ 2 \ - \quad (0) - \ '(0)$$

$$\mathcal{L}\{ ''\} + \mathcal{L}\{ \ \} = \mathcal{L}\{ \ \},$$

$$2 - (0) - \quad '(0) + \ = 1 \ 2$$

$$2 - \ + 2 + \ = 1 \ 2$$

$$= \mathcal{L}\{ \ \} = \quad 1 \quad 2(\ 2 + 1) \quad + \ 2 \quad - 2 \quad + 1$$

$$1 \quad 3 \quad = \ 2 + \quad + 1$$

$$= \mathcal{L}^{-1} \ 1 \quad 2 \quad + \ 2 + 1 \quad - \ 2 \ 3$$

3 sin . Then = + cos, − 3sin = 1 − sin − 3 cos , = − cos +

Check 4.2 $''$ $'$ $''$ and the function obtained is the required solution (Spiegel, 1965).

Example 4.3

$$'' - 3 \ ' + 2 \ = 4 \ 2 \ , \ (0) = -3, \quad '(0) = 5$$

Solution 4.3:

$\mathcal{L}\{ ''\} - 3\mathcal{L}\{ '\} + 2\mathcal{L}\{ \} = 4\mathcal{L}\{ 2 \}$

$^2 \quad ' (0)\} \quad - 3\{ \; - \; (0)\} + 2 \; = \; 4$

$^2 \; + 3 \; - 5\} - 3\{ \; + 3\} \; + 2 \; = \; 4$

$(^2 \qquad 4$

$= \frac{4}{(2 - 3 + 2)(2 -)} + \frac{14 - 3}{}$

$= \frac{-3 \, 2 + 20 \; - 24}{(- 1)(- 2)2}$

$= \frac{-7}{-1} + \frac{4}{-2} + \frac{4}{(- 2)2}$

Thus

$= \mathcal{L} + -2 + (-2)2 = -7 \; +4 \; +4$

$-1 \quad -7 \quad 4 \quad 4 \quad 2 \quad 2$

which can be verified as the solution (Spiegel, 1965).

Example 4.4 Find the Laplace transform of $( ) = [\cos(3)]2$

Using the trigonometric identity

$\cos^2 \quad \frac{1}{}(1 + \cos(6t))$

$2 \quad 1$

$\mathcal{L}(\cos \quad 2(2 \quad + 36)$

Example 4.5 Find the Laplace transform of $( ) = \sin(2) \cos(2)$

Using the trigonometric identity

$$\sin(2\ )\cos(2\ ) = \frac{1}{2\sin 4}$$

$$\mathcal{L}\{\sin(2\ )\cos(2\ )\} = \frac{4}{2(\ 2+16)}$$

$$= \frac{2}{\ 2+16}$$

Example 4.6 Solve the initial-value problem

$$'-2\ = 5\ , \qquad (0) = 3$$

Step 1. Taking the Laplace transform of both sides of the differential equation, we have

$$\mathcal{L}\{\ '\} - 2\mathcal{L}\{\ \} = \mathcal{L}\{\ 5\ \}$$

since

$$\mathcal{L}\{\ '\} = \quad - \ (0)$$

differential equation becomes this,

$$-3-2 = \frac{1}{\ -5}$$

4.2 Applications in Engineering Problems

During the past decade, the numerical operational methods of Fourier and Laplace transforms have been successfully applied to a variety of linear, time-dependent applied mechanics problems (Narayanan and Beskos, 1982). In this chapter it is going to be examined that how we apply Laplace transform method to engineering problems.

Now that we have introduced the Laplace transform, let us see what we can do with it. Please keep in mind that with the Laplace transform we actually have one of the most powerful mathematical tools for analysis, synthesis, and design. Being able to look at circuits and systems in the -domain can help us to understand how our circuits and systems really function. In this chapter firstly we will take an in-depth look at how easy it is to work with circuits in the -domain. And then we will examine physical systems. Actually that is a wonderful thing about the physical universe in which we live; the same differential equations can be used to describe any linear circuit, heat conduction, system, or process. The key is the term *linear* (Alexander and Sadiku, 2009).

So let this section describe the applications of Laplace transforms in the areas of science and engineering.

Many engineering systems described by differential equation via Laplace transformation (Hsiao, 2014). In this chapter we will examine the applications of Laplace transform at differential equations in engineering problems starting with the applications at electrical circuits. After that, applications of Laplace transform at other branches are given by examples to improve variety of applications.

Laplace transform's application at Electrical Circuits is going to be examined in following. According to Alexander and Sadiku, (2009) The Laplace transform is an integral transformation of a function from the So in this part time domain into the complex frequency domain, giving . () of this study Laplace transform is going to be () transformation. As examined below it is so easy to solve complex boundary-value problems with Laplace transformation.

: inductance (constant)

: resistance (constant) : capacitance (constant) : current

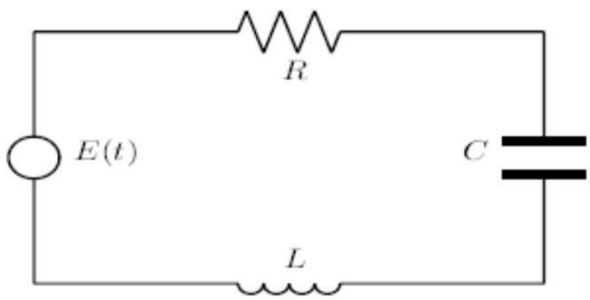


Figure 4.1 RCL circuit (From Schiff, 1999)

respectively, where Kirchoff's voltage (/) (1/) ∫ I(τ) dτ

In the (RCL) circuit in Figure 4.1, the voltage drops across the inductor,

resistor, and capacitor are given by , , and ( ) , that is,

law states that *the sum of the*0 *voltage drops across the individual components equals the impressed voltage*, ,

$$+ \; + \frac{1}{} \int ( ) \; = ( ). \qquad (4.1)$$

0

(4.1) as ( ) = ∫0 ( )

Setting (the charge of the condenser), we can write

2

$$2 + \qquad + \qquad = ( ) \qquad (4.2)$$

since = ∫ . This will be the basis of some of the electrical circuit problems throughout the sequel (Schiff, 1999).

As introduced above it is easy to apply Laplace transform at engineering problems such as electrical circuits.

At first, engineering problems with ordinary differential equations are going to be solved. Secondly, boundary-value problems involving partial differential equations will be examined in this chapter. Now let's see other applications of Laplace transform.

4.2.1 Ordinary Differential Equations

Example 4.7 Application to Beams

0 A beam (see Figure 4.2) which is hinged at its ends and carries a uniform load $= 0$ per unit length. Find the deflection at any point.

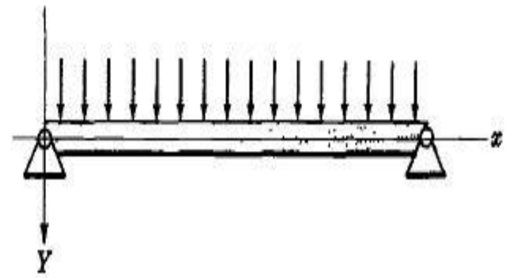


Figure 4.2 Beam (From Spiegel, 1965)

The differential equation and boundary conditions are

$$\frac{40}{4} \quad = \qquad 0 < \quad < 1 \tag{4.3}$$

$$(0) = 0, \quad ''(0) = 0, \quad ( ) = 0, \quad ''( ) = 0 \tag{4.4}$$

Taking Laplace transforms of both sides of (4.3), we have if

$$4 \quad - \quad 3 \ (0) \quad - \quad 2 \ ' (0) - \quad '' (0) - \quad ''' (0) = \frac{0}{}$$

Using the first two conditions in (4.4) and the unknown conditions

$$\frac{1}{} \quad \frac{2}{} \quad 0$$

Then inverting,

$$= \frac{}{2} + \frac{}{4} + \frac{}{5}$$

$$\frac{}{2\ 3} \quad \frac{0}{4} \qquad \frac{}{2\ 3} \quad \frac{0}{4}$$

$$( ) = {}_1 + \frac{}{3!} + \frac{}{4!} = {}_1 + \frac{}{6} + \frac{}{24}$$

From the last conditions in (4.4), we find

$$\underline{0\ 3} \qquad \underline{0}$$

$$_1 = 24 , \qquad _2 = -2$$

Thus the required deflection is

$$() = 24\ (\;\; ^0 \;\; ^3 \;\; -2 \;\; ^3 \;\; + \;\; ^4 \;\;) = \;\; ^0 \;\; 24(-)(\;\; ^2 + - \;\; ^2 \;\;)$$

(Spiegel, 1965)

And also according to Narayanan and Beskos, (1982) there are numerous applications of the numerical Laplace transform in wave propagation, structural dynamics, viscoelasticity, heat conduction, fluid dynamics and other areas of applied mechanics.

Laplace transform method can be used to solve for the eigenfunctions, as well as the eigenvalues (Clarkson et al., 1992) which are the the components of exponential-harmonic functions. Exponential-harmonic functions are essential for engineering problems below.

4.2.2 Boundary-Value Problems

Various problems in science and engineering, when formulated mathematically, lead to partial differential equations involving one or more unknown functions together with certain prescribed conditions on the functions which arise from the physical situation (Spiegel, 1965).

The conditions are called boundary conditions. The problem of finding solutions to the equations which satisfy the boundary conditions is called a boundary-value problem (Spiegel, 1965).

Applications of Laplace transform in solution of boundary-value problems are examined below.

Laplace transform can be applied in a solution of a boundary value problem with

-1

$( ) + \quad -1 \qquad ( ) \quad -1 \qquad + \cdots + \; 1 + \; 0 = ( )$

differential equation and

$(0) = \; 0 \quad ,'(0) = \; 1 \quad , \quad \ldots \;, \; -1(0) = \; -1$

boundary values are given.

and $\mathcal{L}\{( )\} = ( ) \qquad ( ) \qquad (- \; 1) \qquad \geq 0$

$\mathcal{L}\,�\;� = ( ) - \; -1 \; 0$

For , if and derivatives are continuos for is exponential

$\ldots \; -1$

$( )$

is found. So $\mathcal{L}\{$ (can)} be found by calculating the derivatives that is found and

writing with in the differential equation. With

$( ) = \mathcal{L}^{-1}\{\;( )\}$

transform, solution of differential equation is found (Yaşar, 2005).

For , if and derivatives are continuos for

Let’s see with some examples.

Example 4.8 Suppose that the current in an electrical circuit satisfies

$+ \qquad = \; 0$

,

where , , 0, and are constants. Find $= ( )$ for > 0 if $(0) = 0$. Taking the Laplace transform, 0

that is, $\mathcal{L}(\ ) + \ \mathcal{L}(\ ) =$ 2 + 2,

0

$\mathcal{L}(\ ) = (\ +\ )(\ 2\ +\ 2).$

Considering partial fractions

0 �

+

$\mathcal{L}(\ ) = (\ +\ �\ )(\ 2\ +\ 2) =\ +\ �\ +\ 2\ +\ 2,$

we find that = 2 2+ 2, = 2 2+ 2, = 2 2+ 2

0 − 0 0 ,

and so

$(\ ) =$ 2 0 2 − + 2 0 2 sin− 2 0 ω 2 cos ωt.

+ + ω + (Schiff, 1999)

Example 4.9 Suppose that the current in the electrical circuit depicted in Figure 4.2 satisfies

+ 1 � ( ) τ = E,

where , ; and are positive constants, (0) = 0. Then

$\mathcal{L}(\ )$

$\mathcal{L}(\ ) +\ = ,$

implying

$$\mathcal{L}(\ ) = \quad 2 \quad = \quad 2$$

Thus,

$$(\ ) = \quad \quad \sqrt{\ }\,1$$

(Schiff, 1999)

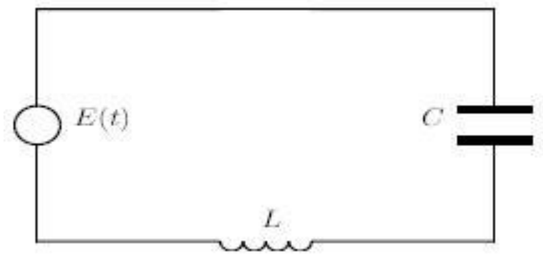
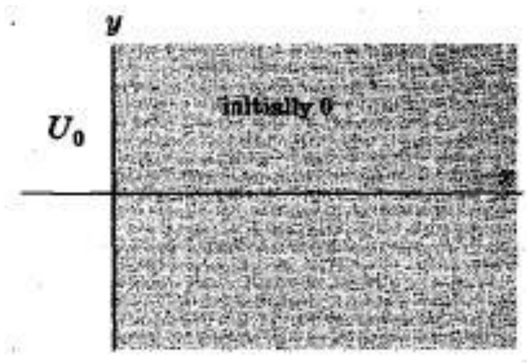


Figure 4.3 Electrical circuit (From Schiff, 1999)

Example 4.10 Heat Conduction



A semi-infinite solid > 0 at temperature zero. At time > 0, a constant temperature $_0 = 0$ is applied $= 0$ and maintained at the face. Find the temperature at any point of the solid at any later time > 0.

Figure 4.4 Semi-infinite solid (From Spiegel, 1965)

To solve this (example) one dimensional heat conduction equation must be known. Here is the temperature in a solid at position at time . The assumed constant , called the *diffusivity,* is equal to / where the *thermal conductivity* , the *specific heat* and the *density* (mass per unit volume) are constant. The amount of heat per unit area per unit time conducted

across a plane is given by(,) (Spiegel, 1965).

The boundary-value problem for the determination of the temperature is (,) at any point and any time > 0, > 0

$) = {}_0,$ $|\ (,)| <$

2

=2

$(, 0) = 0,$ (0,

where the last condition expresses the requirement that the temperature is bounded for all and . Taking Laplace transforms, we find

2 2

$- (, 0) = 2$ or $2 - = 0$ (4.5)

where

0

$(0,) = \mathcal{L}\{ (0,)\} =$ (4.6)

and $= (,)$ is required to be bounded.

Solving (4.5), we find

$(,) = {}_1 \quad - / + {}_2 - /$

Then we choose $_1 = 0$ so that is bounded as $\to \infty$, and we have

$(,) = {}_2 - /$

From (4.6) we have $_2 = {}_0/$, so that

$(,) = \frac{0}{} - /$

We find

$(,) = {}_0\text{erfc}\,( ) = {}_0 (1 - \frac{2}{\sqrt{}} \int_0^{/2\sqrt{}} {}^{-2})$. (Spiegel,1965)

Example 4.11

Work Example 4.10 if at $= 0$ the temperature applied is given by $( ),\ > 0$.

The boundary-value problem in this case is the same as in the preceding problem except that the boundary condition $_0$ is replaced by (4.3) of Example 4.7 that and so

$(0,) = ( )$ $_2 = ( )$ $( )$ $( )$

$(,) = ( ) - /$

Now since

$-1 \quad - / \quad -3/2 \quad - 2/4$

$\mathcal{L} \quad ( ) = \frac{}{2\sqrt{}}$

Hence by the convolution theorem,

$(,) = \int_0 \frac{}{2\sqrt{}} -2^3 -4^2 (-)$

$= \frac{2}{} \int^{\infty} - 2 (- \frac{2}{4 2})$

on letting $= 2/4$.

(Spiegel,1965)

Example 4.12 The Vibrating String

0 =. 0

An infinitely long string having one end at is initially at rest on the transverse displacement given by

axis. The end undergoes a periodic = 0

Find the displacement of any point on the string at any time.

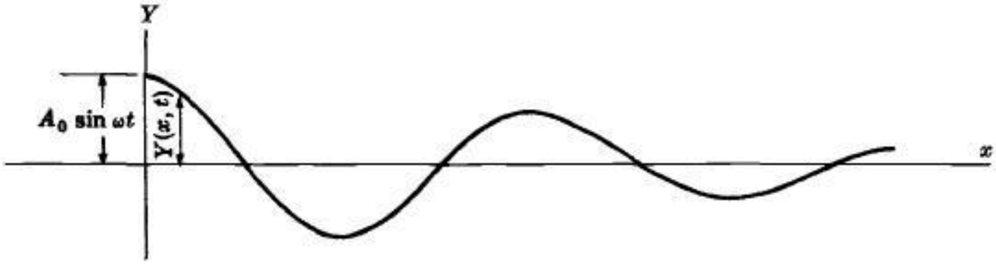


Figure 4.5 Infinitely long string (From Spiegel, 1965)

time , (,)

If is the transverse displacement of the string at any point at any then the boundary-value problem is

$$\frac{2}{2} = {}^{2}\ \frac{2}{2} \qquad > 0, \quad > 0$$

$$(\,, 0) = 0, \qquad (\,, 0) = 0, \qquad (0,) = {}_0\sin \ , \qquad | \ (\,,)| <$$

where the last condition specifies that the displacement is bounded.

Taking Laplace transforms, we find, if $(\,,) = \mathcal{L}\{ \ (\,,)\}$,

$$^{2} - (\,, 0) - (\,, 0) = {}^{2}\ \frac{2}{\ }\ \frac{22}{\ } \quad 2\text{ or} \quad = 0$$

0

$$(0,) = \frac{\ }{2} \ + \frac{\ }{2}, \quad (\,,) \text{ is bounded.} \qquad (4.7)$$

The general solution of the differential equation is

$$(,) = 1 \quad / + 2 \qquad - \ /$$

From the condition on boundedness, we must have $1 = 0$. Then

$$(,) = 2 - \ /$$

From the first condition in (4.7), $2 = 0 \ /(2 + 2)$. Then

$$0 \qquad - \ /$$

and so

$$0 \sin (-) \qquad > / .$$

$$= / \qquad (,) = �0 \qquad < /$$

This means physically that a point of the string stays at rest until the time . Thereafter it undergoes but lags behind it in time by the amount . The constant is the speed with which the wave travels (Spiegel, 1965).

5. CONCLUSION

Many physical processes in nature envolve with time in semi-infinite or infinite domains. Since these processes are described by both ordinary and partial differential equations, solving such equations is of great importance. Since the Laplace transforms transform a differential equation to an algebraic one, Laplace transform is considered to be a strong tool for solving partial differential equations that appear in various fields of science and engineering. Besides, the Laplace transforms method is considered to be the easiest methods used to solve such equations because unlike the other methods used less and uncomplicated calculations are needed.

In conclusion, our main aim is to explain Laplace methods and indicate its calculation and solution on most widely-used area on mathematics and engineering. In this thesis, it is explained and exemplified that Laplace is one of the useful theorems while solving problems related to differential equations.

This thesis can be considered as a survey on Laplace transform that probably will be a reference for scientists working on mathematics and other sciences directly related to the mathematics like engineering, physics, chemistry and economics etc.

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