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DERIVATIONS ON LATTICE IMPLICATION ALGEBRA

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Kabul ve Onay Sayfası

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ÖZET

KAFES IMPLICATION (ÇIKARIM) CEBİRLERİNDE TÜREVLER

ALTINDAĞ, Öncül

Yüksek Lisans Tezi , Matematik Bölümü

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Bu tez esas olarak üç bölümden oluşmaktadır. İlk bölümde kafes implication (çıkarım) cebirleri ile ilgili önbilgiler ve ilgili özellikler verilmiştir.

İkinci bölümde, kafes implication (çıkarım) cebirlerinde türev ve f-türev tanımı verilmiş ve ilgili özellikleri listelenmiştir.

Üçüncü bölümde, kafes implication (çıkarım) cebirlerinde simetrik ikili türev tanımı verilmiştir. Kafes implication (çıkarım) cebirlerinde verilen üç dönüşümün de simetrik ikili türev özelliğini taşıdığı görülmüştür.Daha sonra simetrik ikili türevin bazı önemli özellikleri listelenmiş ve ıspatlanmıştır. Ayrıca, kafes implication (çıkarım) cebirlerinde simetrik ikili türev aracılığı ile Fix ve Kernel kümeleri tanımlanmış ve bu kümelere ait ana özellikler çalışılmış ve ıspatlanmıştır.

ABSTRACT

DERIVATIONS ON LATTICE IMPLICATION ALGEBRA

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MSc. in Mathematics

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This thesis consists of three parts. In the first part preliminaries about the lattice implication algebras and their properties are given.

In the second part, the notions of derivation and f-derivation of lattice implication algebras are defined and all properties related are listed.

In the third part, the notion of symmetric bi-derivation in lattice implication algebras is defined. Three examples of maps in lattice implication algebras have been checked to see that they really have the properties of symmetric bi-derivation in lattice implication algebra. Then some important properties of these symmetric bi-derivations are listed and proved. Moreover, the Fix set and the Kernel are defined on lattice implication algebras for the symmetric bi-derivations and main properties of these sets are studied and proved.

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INTRODUCTION

The concept of lattice implication algebra was proposed by Y. Xu [Y. Xu,1993], in order to establish an alternative logic knowledge representation. In his paper, Xu combined lattice and implication algebras and created a new algebraic structure. The lattice implication algebra so constructed has two main elements: the lattice, defined to describe uncertainities, and the implication operator designed to describe the way of human's reasoning. From another point, we can say that Xu aimed to provide a logical foundation for uncertain information processing theory, the incomparability in uncertain information in the reasoning and establised a logical system with truth value in a relatively general lattice.

Meanwhile, "non-classical logic" is the name given to formal systems that differ in a significiant way from standart logical systems. Several ways exists to do this; like extensions, deriviations and variations. The aim of these departures is to make it possible to construct different models of logical consequence and logical truth. The lattice implication algebra; as mentioned above, is a combination of algebraic lattice and implication algebra. Here, let us define briefly these two concepts; an algebraic lattice is a poset which is locally and finetely presentable as a category. Or in other words, it is a complete lattice in which every element is the supremum of the compact elements below it. An implication algebra is an abstract algebra containing a nonempty set and a binary operation on this set, called the implication product, satisfying some defined properties.

Lattice implication algebra, being an important non-classical algebra, has been studied by many researchers. Y. Xu and K. Y. Qin [Y. Xu and Y. Qin,1992] discussed the properties of lattice implication H-algebras and gave some equivalent conditions about H lattice implication algebras. Y. Xu and K. Y. Qin [Y.Xu and Y. Qin, 1993] defined the notion of filters in a lattice implication algebra and obtained their properties.

Lee and Kim introduced in [Sang Deok Lee and Kyung Ho Kim, 2013] the notion of derivation in lattice implication algebra and considered its properties. Then Yon and Kim introduced in [Yong Ho Yon and Kyung Ho Kim, 2013] the notion of f-derivation in lattice implication algebra similarly.

In this paper we introduced the notion of symmetric bi-derivation in lattice implication algebra. We gave the properties of a symmetric bi-derivation D in lattice implication algebra, and also the properties of its trace. We also defined the fixed set and the Kernel of the map, and showed that every filter in the lattice implication L is D-invariant for D being a symmetric bi-derivation.

1. Preliminaries

In this part, in order to facilitate the readability of the thesis some basic definitions and properties of lattice implication algebras that are used in proofs are given to ensure ease of application. This is done in conjunction with references they received.

Definition 1.1. A *lattice implication algebra* is an algebra $(L; \wedge, \vee, ' \rightarrow, 0, 1)$ of type $(2, 2, 1, 2, 0, 0)$ where $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms for all $x, y, z \in L$ (Xu, 1993):

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(I2) $x \rightarrow x = 1$.

(I3) $x \rightarrow y = y' \rightarrow x'$.

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$.

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

Definition 1.2. If L satisfies conditions (I1)-(I5), we say that L is a *quasi lattice implication algebra*. A lattice implication algebra L is called *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

Remark 1.3. We can define a partial ordering "$\leq$" on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

Properties 1.4. *In a lattice implication algebra L, the following hold [Y. Xu, 1993]:*

(U1) $0 \rightarrow x = 1, 1 \rightarrow x = x$ and $x \rightarrow 1 = 1$.

(U2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

(U3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.

(U4) $x' = x \to 0$.

(U5) $x \vee y = (x \to y) \to y$.

(U6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$.

(U7) $x \leq (x \to y) \to y$.

In a lattice H implication algebra L , the following hold:

(U8) $x \to (x \to y) = x \to y$.

(U9) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Definition 1.5. A non-empty subset F of a lattice implication algebra L is called a *filter of* L if it satisfies:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$.

2. Derivations and f-Derivations of Lattice Implication Algebras

2.1. Derivations of Lattice Implication Algebras.

Lee and Kim defined first the derivations of lattice implication algebras in their paper *"On Derivations of Lattice Implication Algebras"* in 2013.

Definition 2.1. Let L be a lattice implication algebra. A map $d : L \to L$ is a *derivation* of L if

$$d(x \to y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$

Here is an example given by Lee and Kim;

Example 2.1. *Let* $L := \{0, a, b, c, 1\}$*. Define the partial order on* L *as* $0 < a < b < c < 1$*, and define*

$$x \wedge y := min\{x, y\}, x \vee y := max\{x, y\}$$

for all $x, y \in L$ *and* ” $'$ ” *and* ” $\to$ ” *as follows:*

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	b	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L; \wedge, \vee, ' \to, 0, 1)$ *is a lattice implication algebra. Define a map* $d : L \to L$ *will be defined as :*

$$d(x) = \begin{cases} 1, & \text{if } x = c, 1 \\ b, & \text{if } x = a, \\ a, & \text{if } x = 0, \\ c, & \text{if } x = b \end{cases}$$

It is easy to check that d *is a derivation of lattice implication algebra* L*.*

Lee And Kim listed and proved in *[”On Derivations of Lattice Implication Algebras”,2013]* the following properties about derivations on lattice implication algebras. In what follows L is a lattice implication algebra and d is a derivation of L unless otherwise specified.

Proposition 2.2. *Let* d *be a derivation of* L*, then* $d(1) = 1$*.*

Proposition 2.3. $d(x) = d(x) \vee x$ *and as a result* $x \leq d(x)$ *for all* $x \in L$

Proposition 2.4. *Let* f *be an expansive map on* L *i.e.,* $x \leq f(x)$ *for all* $x \in L$*. Then* $f(x) \to y \leq x \to f(y)$ *for all* $x, y \in L$

Theorem 2.5. *Let* d *be a map on* L*. Then the following identities are equivalent:*
i) d *is a derivation of* L*.*
ii) $d(x \to y) = x \to d(y)$ *for all* $x, y \in L$

Proposition 2.6. *If* $d_1, d_2, d_3, ..., d_n$ *are derivations of* L*, then* $d_1 o d_2 o d_3 o....o d_n$ *is a derivation of* L

Definition 2.7. Let d be a derivation of the lattice implication algebra L. We can define a set Fix_d(L) by

$$Fix_d(\text{L}) := \{x \in L | d(x) = x\} \text{ for all } x \in L.$$

Moreover if $x \in Fix_d($ L $)$ then $(dodod....d)(x) = x$.

Proposition 2.8. *Let L is a lattice implication algebra and d is a derivation of L, then we have the following properties about the fixed set:*

(i) If $x \in L$ *and* $y \in Fix_d(L)$ *then* $x \to y \in Fix_d(L)$.
(ii) If $y \in Fix_d(L)$ *then* $x \vee y \in Fix_d(L)$ *for all* $x \in L$.

Proposition 2.9. *If* $x \leq y$ *and* $x \in Fix_d(L)$ *then* $y \in Fix_d(L)$.

Proposition 2.10. *If* $x \leq y$ *implies* $d(x) \leq d(y)$ *for all* $x, y \in L$*, then d is called an isotone derivation. In addition, if d is an endomorphism, d is an isotone derivation.*

Proposition 2.11. *The derivation* $d : L \to L$ *defines on the lattice implication algebra L is an identity map if it satisfies* $x \to d(y) = d(x) \to y$ *for all* $x, y \in L$.

Theorem 2.12. *The derivation d of the lattice implication algebra L is one to one if and only if d is an identity derivation.*

Theorem 2.13. *The map* $d_a : L \to L$ *, defined for all* $x \in L$ *as* $d_a(x) = a \to x$ *for* $a \in L$ *where L is a lattice implication algebra is a derivation of L for each* $x \in L$.

Proposition 2.14. *For every* $a \in L$*, the simple derivation* d_a *defined above is an endomorphism if L is a lattice H implication algebra.*

Definition 2.15. For L a lattice implication algebra and a derivation d of it , one can define a Kerd by $Kerd = \{x \in L | d(x) = 1\}$.

Proposition 2.16. *If d is an endomorphism on L, then Kerd is a filter of L.*

Proposition 2.17. *If* $y \in Kerd$*, then* $x \vee y \in Kerd$ *for all* $x \in L$.

Proposition 2.18. *If* $x \leq y$ *and* $x \in Kerd$*, then* $y \in Kerd$.

Proposition 2.19. *If* $y \in Kerd$*, then we have* $x \to y \in Kerd$ *for all* $x \in L$.

Definition 2.20. Let L be a lattice implication algebra. A non-empty subset F of L is said to be d-invariant if $d(F) \subseteq F$ where $d(F) = \{d(x) | x \in F\}$

Theorem 2.21. *Every filter F of the lattice implication algebra L is d-invariant.*

2.2. f-Derivations of Lattice Implication Algebras

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After Lee and Kim; Yon and Kim then defined the f- derivations of lattice implication algebras in their paper in 2013 *"On f-Derivations of Lattice Implication Algebras"*.

Definition 2.22. Let L be a lattice implication algebra. A map $d : L \to L$ is a *derivation* of L if

$$d(x \to y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$.

In addition let f be a map between the implication algebras L_1 and L_2 defined as an *implication homomorphism* , that is, $f(x \to y) = f(x) \to f(y)\ \forall x, y \in L_1$.

If f is an implication endomorphism of L, the map $d : L \to L$ is a *f-derivation* of L if it satisfies the identity

$$d(x \to y) = (f(x) \to d(y)) \vee (d(x) \to f(y)) \text{ for all } x, y \in L$$

Here is an example given by Yon and Kim;

Example 2.2. *Let $L := \{0, a, b, 1\}$ be a bounded lattice, let us define for all $x, y \in L$ and " ′ " and " $\to$ " as follows:*

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

If we define a map $f : L \to L$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, a \\ 1, & \text{if } x = b, 1 \end{cases}$$

then this map f is an implication endomorphism. And define a map $d : L \to L$ by

$$d(x) = \begin{cases} b, & \text{if } x = 0, a \\ 1, & \text{if } x = b, 1 \end{cases}$$

It is easy to check that d is an f-derivation of lattice implication algebra L. Notice that d is not a derivation of L since $d(b \to 0)$ do not satisfy the equality defined above for the map d to be a derivation.

Yon and Kim listed and proved in their paper published in 2013 *"On f-Derivations of Lattice Implication Algebras"* the following properties about f-derivations of lattice implication algebras. In what follows L is a lattice implication algebra and d is a f-derivation of L unless otherwise specified.

Proposition 2.23. *Let d be a f-derivation of L then $d(1) = 1$.*

Proposition 2.24. *$d(x) = d(x) \vee f(x)$ and as a result we have;*

i) $f(x) \leq d(x)$ for all $x \in L$
ii) $f(x) \vee f(y) \leq d(x) \vee d(y)$ for all $x, y \in L$

Theorem 2.25. *Let d be a f-derivation on L. Then,*
$d(x \to y) = f(x) \to d(y)$ for all $x, y \in L$

Proposition 2.26. *Let d be an f-derivation of L, if it satisfies $d(x \to y) = d(x) \to f(y)$ for all $x, y \in L$ then $d(x) = f(x)$ and moreover $d = f$.*

Definition 2.27. If $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$, then d is called *isotone f-derivation.*

Proposition 2.28. *If d is an isotone f-derivation, then $d(x) \vee d(y) \leq d(x \vee y)$ for all $x, y \in L$*

Definition 2.29. Let d be a f-derivation of the lattice implication algebra L. We can define a set Fix_d(L) by

$$Fix_d(\mathrm{L}) := \{x \in L | d(x) = f(x)\} \text{ for all } x \in L.$$

Clearly $1 \in Fix_d$(L).

Proposition 2.30. *Let L be a lattice implication algebra and d be an f-derivation of L, then we have the following properties about the fixed set:*

(i) If $x \in L$ and $y \in Fix_d(\ L\ )$ then $x \to y \in Fix_d(\ L\ )$.
(ii) If $y \in Fix_d(\ L\ )$ then $x \vee y \in Fix_d(\ L\ )$ for all $x \in L$.

Proposition 2.31. *If $x \leq y$ and $x \in Fix_d(\ L\ )$ then $y \in Fix_d(\ L\ )$.*

Definition 2.32. For L a lattice implication algebra and an f-derivation d of it, one can define a Kerd by $Kerd = \{x \in L | d(x) = 1\}$.

Proposition 2.33. *If d is an endomorphism of L then Kerd is a filter of L.*

Proposition 2.34. *I $y \in Kerd$ then $x \vee y \in Kerd$ for all $x \in L$.*

Proposition 2.35. *If $x \leq y$ and $x \in Kerd$ then $y \in Kerd$.*

Proposition 2.36. *If $y \in Kerd$ then we have $x \to y \in Kerd$ for all $x \in L$.*

Definition 2.37. Let L be a lattice implication algebra. A nonempty subset F of L is called a *normal filter* if it $1 \in F$ and $x \in L$ and $y \in F$ imply $x \to y \in F$

Proposition 2.38. *As a result of the above definitions we can state that the sets $Fix_d(L)$ and Kerd are normal filters of L.*

3. The Symmetric Bi-Derivation of Lattice Implication Algebras

Definition 3.1. Let L be a lattice implication algebra. A mapping $D(.,.) : L \times L \to L$ is called symmetric if the equality $D(x, y) = D(y, x)$ holds for all $x, y \in L$

The following definition introduces the notion of symmetric bi-derivation for lattice implication algebras.

Definition 3.2. Let L be a lattice implication algebra and let $D(.,.) : L \times L \to L$ be a symmetric mapping. We call D a *symmetric bi-derivation* of L if it satisfies

$D(x \to y, z) = (x \to D(y, z)) \vee (D(x, z) \to y)$ for all $x, y, z \in L$.

It is clear that a symmetric bi-derivation of L also satisfies

$D(x, y \to z) = (D(x, y) \to z) \vee (y \to D(x, z))$ for all $x, y, z \in L$.

Example 3.1. *Let $L := \{0, a, b, 1\}$ be a set with the Cayley table;*

x	x'
0	1
a	b
b	a
1	0

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$ we have $x' = x \rightarrow 0$ and the operations $\vee$ and $\wedge$ on L are defined as : $x \vee y = (x \rightarrow y) \rightarrow y$ and $x \wedge y = ((x' \rightarrow y') \rightarrow y')'$. Then $(L; \wedge, \vee, ', \rightarrow, 0, 1)$ is a lattice implication algebra. The mapping $D(.,.) : L \times L \rightarrow L$ will be defined by $D(x, y) = x' \rightarrow (y' \rightarrow a)$ for every $x, y \in L$, i.e.,

$$D(x,y) = \begin{cases} a, & \text{if } x = 0 \text{ and } y = 0, \\ b, & \text{if } (x = a \text{ and } y = 0) \text{ or } (x = 0 \text{ and } y = a), \\ 1, & \text{otherwise} \end{cases}$$

Then we can see that D is a symmetric bi-derivation of L.

Example 3.2. *Let $L := \{0, a, b, 1\}$ be a set with the Cayley table;*

x	x'
0	1
a	b
b	a
1	0

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$ we have $x' = x \rightarrow 0$ and the operations $\vee$ and $\wedge$ on L are defined as : $x \vee y = (x \rightarrow y) \rightarrow y$ and $x \wedge y = ((x' \rightarrow y') \rightarrow y')'$. Then $(L; \wedge, \vee, ', \rightarrow, 0, 1)$ is a lattice implication algebra. The mapping $D(.,.) : L \times L \rightarrow L$ will be defined as :

$$D(x,y) = \begin{cases} x, & \text{if } y = 0, \\ y, & \text{if } x = 0, \\ b, & \text{if } x = a \text{ and } y = a, \\ 1, & \text{otherwise} \end{cases}$$

It is easy to check that D is a symmetric bi-derivation of L.

Example 3.3. *Let $(B; \wedge, \vee, ', 0, 1)$ be a Boolean algebra. If we define*

$$x \rightarrow y = x' \vee y$$

for every $x, y \in B$, then $(B; \wedge, \vee, ', \rightarrow, 0, 1)$ is a lattice implication algebra. For a fixed element a in B, if we define a map $D : B \times B \rightarrow B$ by

$$D(x, y) = x \vee y \vee a$$

for every $x, y \in B$, then D is a symmetric bi-derivation of B.

Definition 3.3. Let L be a lattice implication algebra and let $D(.,.) : L \times L \to L$ be a symmetric mapping. A mapping $d : L \to L$ defined for all $x \in L$ by $d(x) = D(x, x)$ is called the trace of D.

Proposition 3.4. *Let L be a lattice implication algebra and d be the trace of symmetric bi-derivation D of L. Then the followings hold:*

i) $D(1, x) = 1$ *for all* $x \in L$.

ii) $d(1) = 1$

Proof. i) Let $x, y \in L$. Then we have

$$\begin{aligned} D(1, x) &= D(x \to 1, x) \\ &= (x \to D(1, x)) \vee (D(x, x) \to 1) \\ &= (x \to D(1, x)) \vee 1 = 1 \end{aligned}$$

Therefore we get $D(1, x) = 1$.

ii) It is clear from i).

□

Proposition 3.5. *Let L be a lattice implication algebra and d be the trace of symmetric bi-derivation D of L. Then the followings hold:*

i) $D(x, y) = D(x, y) \vee x$ *for all* $x, y \in L$

ii) $d(x) = d(x) \vee x$ *for all* $x \in L$.

Proof. i) Let $x, y \in L$. Then we have

$$\begin{aligned} D(x, y) &= D(1 \to x, y) \\ &= (1 \to D(x, y)) \vee (D(1, y) \to x) \\ &= D(x, y) \vee (1 \to x) \\ &= D(x, y) \vee x \end{aligned}$$

So we have $D(x, y) = D(x, y) \vee x$.
Also, we can get $x \leq D(x, y)$ for all $x, y \in L$.

ii)Let $x \in L$. Then we have

$$\begin{aligned} d(x) &= D(x, x) \\ &= D(1 \to x, x) \\ &= D(x, x) \vee (1 \to x) \\ &= d(x) \vee x \end{aligned}$$

We proved $d(x) = d(x) \vee x$ and as above we can obtain $x \leq d(x)$ for all $x \in L$. □

Proposition 3.6. *Let L be a lattice implication algebra and d be the trace of symmetric bi-derivation D of L. Then*

$$d(x) \to y \leq x \to y \leq x \to d(y) \text{ for all } x, y \in L.$$

Proof. Let L be a lattice implication algebra and d be the trace of symmetric bi-derivation D of L.

From Proposition 3.3(ii) we have $x \leq d(x)$ and $y \leq d(y)$.

Using (U3); $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.

we obtain $d(x) \to y \leq x \to y$ and $x \to y \leq x \to d(y)$.

We can conclude $d(x) \to y \leq x \to y \leq x \to d(y)$ for all $x, y \in L$.

□

Proposition 3.7. *Let L be a lattice implication algebra and d be the trace of symmetric bi-derivation D of L. Then*

i)$D(d(x) \to x, x) = 1$

ii) $d(d(x) \to x) = 1 = d(x \to d(x))$

Proof. i) By using the definition of symmetric bi-derivation D of a lattice implication algebra and properties of a lattice implication algebras (I2), (U5) and (U1)we have,

$$\begin{aligned} D(d(x) \to x, x) &= (d(x) \to D(x,x)) \vee (D(d(x), x) \to x) \\ &= (d(x) \to d(x)) \vee (D(d(x), x) \to x) \\ &= (1 \to (D(d(x), x) \to x)) \to (D(d(x), x) \to x) \\ &= (D(d(x), x) \to x) \to (D(d(x), x) \to x) \\ &= 1 \end{aligned}$$

ii) By using the definition of trace of a symmetric bi-derivation D of a lattice implication algebra and properties of a lattice implication algebras, we have

$$\begin{aligned} d(d(x) \to x) &= D(d(x) \to x, d(x) \to x) \\ &= (d(x) \to D(x, d(x) \to x)) \vee (D(d(x), d(x) \to x) \to x) \\ &= (d(x) \to ((d(x) \to d(x)) \vee (D(x, d(x)) \to x))) \vee (D(d(x), d(x) \to x) \to x) \\ &= (d(x) \to (1 \vee (D(x, d(x)) \to x))) \vee (D(d(x), d(x) \to x) \to x) \\ &= (d(x) \to 1)) \vee (D(d(x), d(x) \to x) \to x) \\ &= 1 \vee (D(d(x), d(x) \to x) \to x) \\ &= 1 \end{aligned}$$

□

This proves the first part $d(d(x) \to x) = 1$.
The second equality is obvious since $d(x \to d(x)) = d(1) = 1$ which was the result that we obtained at the end of the proof of Proposition 3.3 ii).

Proposition 3.8. *Let a symmetric map $D : L \times L \to L$ be defined by*

$$D(x \to y, z) = x \to D(y, z) \text{ on } L \times L.$$

Then D is a symmetric bi-derivation of $L \times L$.

Proof. For all $y \in L$, $D(1, y) = D(D(1, y) \to 1, y) = D(1, y) \to D(1, y) = 1$. It follows

$$x \to D(x, y) = D(x \to x, y) = D(1, y) = 1.$$

for every $x, y \in L$. Since $x \leq D(x, z)$ and $y \leq D(y, z)$, we have

$$D(x, z) \to y \leq x \to y \leq x \to D(y, z).$$

Hence $D(x \to y, z) = x \to D(y, z) = (x \to D(y, z)) \vee (D(x, z) \to y)$, and D is a bi-derivation. □

Proposition 3.9. *If $D : L \times L \to L$ is a symmetric bi-derivation, then D satisfies $D(x \to y, z) = x \to D(y, z)$*

Proof. Let D be a symmetric bi-derivation and $x, y, z \in L$. Since $x \leq D(x, z)$ and $y \leq D(y, z)$, we have

$$D(x, z) \to y \leq x \to y \leq x \to D(y, z).$$

Hence $D(x \to y, z) = (x \to D(y, z)) \vee (D(x, z) \to y) = x \to D(y, z)$. □

As a consequence of Propositions 3.6 and 3.7 we can state the following theorem.

Theorem 3.10. *A map $D : L \times L \to L$ is a symmetric bi-derivation if and only if D is a symmetric map and it satisfies $D(x \to y, z) = x \to D(y, z)$ for every $x, y, z \in L$.*

Proposition 3.11. *A map D being a symmetric bi derivation defined on the lattice implication algebra L satisfies the following:*

$$D(x, y \to z) = y \to D(x, z)$$

for all $x, y, z \in L$.

Proof. We will make use of the previous theorem 3.8 and the fact that D is symmetric.

$$\begin{aligned} D(x, y \to z) &= D(y \to z, x) \\ &= y \to D(z, x) \\ &= y \to D(x, z) \end{aligned}$$

□

Proposition 3.12. *A map D being a symmetric bi derivation defined on the lattice implication algebra L satisfies the following: $D(x,y) = x' \to (y' \to D(0,0))$ for every $x, y \in L$. That is, the value of D is determined by $D(0,0)$.*

Proof. For any $x, y \in L$,

$$D(x,y) = D(x'', y'') = D(x' \to 0, y' \to 0) = x' \to (y' \to D(0,0))$$

. □

Proposition 3.13. *Let L be a lattice implication algebra where d is the trace of symmetric bi-derivation D of L. Then*

$$d(x \to y) = x \to [x \to d(y)]$$

Proof.

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) \\ &= x \to D(y, x \to y) \\ &= x \to D(x \to y, y) \\ &= x \to (x \to D(y,y)) \\ &= x \to (x \to d(y)) \end{aligned}$$

□

Furthermore, in a lattice H implication algebra with the additional equality $x \to (x \to y) = x \to y$ we get

$$d(x \to y) = x \to d(y)$$

Definition 3.14. Let D be a symmetric bi-derivation of the lattice implication algebra L and d be the trace of D. We can define a set $Fix_D(\mathrm{L})$ by

$$Fix_D(\mathrm{L}) := \{x \in L | d(x) = x\}$$

Proposition 3.15. *Let d be the trace of the symmetric bi-derivation D of the lattice implication algebra L, then we have;*

$$(dodod....od)(x) = x$$

Proof. The proof is obvious by the definition of the trace of D. □

Proposition 3.16. *Let D be a symmetric bi-derivation of the lattice H implication algebra L.*
i) if $x \in L$, $y \in Fix_D(L)$ then $x \to y \in Fix_D(L)$.
ii) if $y \in Fix_D(L)$ then $x \vee y \in Fix_D(L)$ $\forall x \in L$.

Proof. i) Let D be a symmetric bi-derivation of the lattice H implication algebra. By using Proposition 3.9 and the fact that $y \in Fix_D(L)$ we have
$d(x \to y) = x \to d(y) = x \to y$. Therefore, we get $x \to y \in Fix_D(L)$.

ii) Let D be a symmetric bi-derivation of the lattice H implication algebra. By using Proposition 3.9 and the fact that we have $y \in Fix_D(L)$ we have

$$\begin{aligned} d(x \vee y) &= d((x \to y) \to y) \\ &= (x \to y) \to d(y) \\ &= (x \to y) \to y \end{aligned}$$

Hence $d(x \vee y) = x \vee y$. Therefore, we get $x \vee y \in Fix_D(L)$.

□

Proposition 3.17. *Let D be a symmetric bi-derivation of the lattice H implication algebra L; for $x, y \in L$*

If $x \leq y$ and $x \in Fix_D(L)$ then $y \in Fix_D(L)$.

Proof. We have $x, y \in L$ and $x \leq y$ so that $x \to y = 1$ and $x \in Fix_D$(L) .

$$\begin{aligned} d(y) &= d(1 \to y) \\ &= d((x \to y) \to y) \\ &= d((y \to x) \to x) \\ &= d(y \vee x) \\ &= y \vee x \end{aligned}$$

by prop 3.11 $x \in Fix_D(L)$ implies $y \vee x \in Fix_D(L)$

$$\begin{aligned} d(y) &= (y \to x) \to x \\ &= (x \to y) \to y \\ &= 1 \to y \\ &= y \end{aligned}$$

So we get $d(y) = y$ and we have proved $y \in Fix_D$(L).

□

Definition 3.18. Let L be a lattice implication algebra. A nonempty subset A of L is said to be D-invariant if $D(A, A) \subseteq A$ where $D(A, A) = \{D(x, y) | x, y \in A\}$.

Proposition 3.19. *Let D be a symmetric bi-derivation of the lattice implication algebra L. Then every filter A is D-invariant.*

Proof. Let $y \in D(A, A)$ then $y = D(x, z)$ for some $x, z \in A$. We have $x \leq D(x, z)$ and $z \leq D(x, z)$ from Proposition 3.3. So $x \to D(x, z) = 1$ and $z \to D(x, z) = 1$. Since $x, z \in A$ and A is a filter we have $D(A, A) \subseteq A$.

□

Definition 3.20. Let D be a symmetric bi-derivation of the lattice implication algebra L, and let d be the trace of D. We can define $KerD$;

$$KerD := \{x \in L | D(x, x) = d(x) = 1\}$$

Proposition 3.21. *Let D be a symmetric bi-derivation of the lattice implication algebra L, and let d be the trace of D.*

$$\textit{If } y \in KerD \textit{ then } x \vee y \in KerD \ \forall x \in L.$$

Proof. $y \in KerD$ and using the definition of symmetric bi-derivation D of lattice implication algebra we have

Since

$$\begin{aligned} D(y, x \vee y) &= D(x \vee y, y) \\ &= D((x \to y) \to y, y) \\ &= ((x \to y) \to D(y, y)) \vee (D(x \to y, y) \to y) \\ &= ((x \to y) \to 1)) \vee (D(x \to y, y) \to y) \\ &= 1 \vee (D(x \to y, y) \to y) \\ &= 1 \end{aligned}$$

we have $D(y, x \vee y) = 1$.
Therefore,

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y) \\ &= D((x \to y) \to y, x \vee y) \\ &= ((x \to y) \to D(y, x \vee y)) \vee (D(x \to y, x \vee y) \to y) \\ &= ((x \to y) \to 1) \vee (D(x \to y, x \vee y) \to y) \\ &= 1 \vee (D(x \to y, x \vee y) \to y) \\ &= 1 \end{aligned}$$

Hence, we get the result that is $x \vee y \in KerD$, $\forall x \in L$

□

Definition 3.22. Let L be a lattice implication algebra. Then for a fixed element $a \in L$ let us define a map $d_a : L \to L$ such that $d_a(x) = D(x, a)$ for every $x \in L$.

Theorem 3.23. *For each $a \in L$ the map d_a defined above is a derivation of L.*

Proof. For a fixed element $a \in L$ let us define a map $d_a : L \to L$ such that $d_a(x) = D(x, a)$ for every $x \in L$.

$$d_a(x) = D(x, a)$$
$$d_a(x \to y) = D(x \to y, a)$$
$$= (x \to D(y, a)) \wedge (D(x, a) \to y)$$
$$= (x \to d_a(y)) \wedge (d_a(x) \to y)$$

So d_a is a derivation of L. So we can say that for each element $a \in L$ the map d_a defined above is a derivation of L.

□

Proposition 3.24. *Let D be a symmetric bi-derivation of the lattice implication algebra L. Then $D(x \vee y, z) = D(x, z) \vee D(y, z)$ and $D(x \wedge y, z) = D(x, z) \wedge D(y, z)$ for every $x, y, z \in L$.*

Proof. Let $x, y, z \in L$. Then we have

$$\begin{aligned} D(x \vee y, z) &= D(x'' \vee y'', z) \\ &= D((x' \wedge y')', z) \\ &= D((x' \wedge y') \to 0, z) \\ &= (x' \wedge y') \to D(0, z) \\ &= (x' \to D(0, z)) \vee (y' \to D(0, z)) \\ &= D(x' \to 0, z) \vee D(y' \to 0, z) \\ &= D(x'', z) \vee D(y'', z) \\ &= D(x, z) \vee D(y, z) \end{aligned}$$

We can prove the case of meet in the similar way. □

Proposition 3.25. *Let D be a symmetric bi-derivation of the lattice implication algebra L. D is monotone, That is if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $D(x_1, y_1) \leq D(x_2, y_2)$, for every $x_1, x_2, y_1, y_2 \in L$.*

Proof. $D(x_1 \vee x_2, y_1) = D(x_1, y_1) \vee D(x_2, y_1)$ from Proposition 3.24
since $x_1 \leq x_2$ we have $x_1 \vee x_2 = x_2$
$D(x_2, y_1) = D(x_1, y_1) \vee D(x_2, y_1)$
$D(x_1, y_1) \leq D(x_2, y_1)$ (*)

$D(y_1 \vee y_2, x_2) = D(y_1, x_2) \vee D(y_2, x_2)$ from Proposition 3.24
since $y_1 \leq y_2$ we have $y_1 \vee y_2 = y_2$
$D(x_2, y_1) = D(x_2, y_1) \vee D(x_2, y_2)$
$D(x_2, y_1) \leq D(x_2, y_2)$ (* *)

We get $D(x_1, y_1) \leq D(x_2, y_2)$ by taking (*) and (* *) together.

□

Proposition 3.26. *Let D be a symmetric bi-derivation of the lattice implication algebra L. Then*

$$D(x', x) = D(x, x') = 1 \textit{ for every } x \in L$$

Proof.

$$D(x', x) = D(x \to 0, x) = x \to D(0, x) = x \to D(x, 0) = D(x \to x, 0) = D(1, 0) = 1$$

□

Proposition 3.27. *Let D be a symmetric bi-derivation of the lattice implication algebra L.*

$$D(y, x) = 1 \textit{ for every } x, y \in L \textit{ with } x' \leq y$$

Proof. We know that $x' \leq y$ implies $x' \vee y = y$.

$$D(y, x) = D(x' \vee y, x) = D(x', x) \vee D(y, x) = 1 \vee D(y, x) = 1$$

□

CONCLUSION

The aim of this work was to study maps on lattice implication algebras and more specifically the derivations and f-derivations defined on implication algebras. Then this work aims to define a new type of derivation in lattice implication algebras, the notion of symmetric bi-derivations in this algebraic structure. First of all in the first part, some basic definitions needed for the readability of the work are given about the lattice implication algebras. Then in the second part, the notions of derivation and f-derivation in lattice implication algebras; introduced respectively by Lee and Kim [Sang Deok Lee and Kyung Ho Kim, 2013] and Yon and Kim [Yong Ho Yon and Kyung Ho Kim, 2013]are observed. Main properties of these maps are listed in this part. In the third part, the notion of symmetric bi-derivation of lattice implication algebras is defined; examples satisfying its properties are listed. Then some theorems and propositions that these symmetric bi-derivations satisfied in other algebraic structures like B-algebras [Ayar and Firat] and in lattices [Çeven, 2009] are proved for lattice implication algebras. Moreover, the properties of the symmetric bi-derivation D in lattice implication algebra, and also the properties of its trace are given;also are defined the fixed set and the Kernel of the map. The next step of this work can be some more detailed studies about other types of derivations in lattice implication algebras, generalized derivations can be for example studied in this algebraic structure.

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