YASAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MASTER THESIS

ARTINIAN WEAKLY SUPPLEMENTED MODULES

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APPROVAL PAGE

This study, title "Artinian Weakly Supplemrnted Modules" and presented as Master Thesis by Aram A. ABDULKAREEM has been evaluated in compliance with the provisions of Yaşar University Graduate Education and Training Regulation and Yaşar University Institute of Science Education and Training Direction. The jury members below have decided for the defence of this thesis and it has been declared by consensus/majority of the votes that the candidate has succeeded in his thesis defence examination dated 27th May, 2015.

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ABSTRACT

ARTINIAN WEAKLY SUPPLEMENTED MODULES

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In this thesis artinian weakly supplemented modules and totally artinian weakly supplemented modules are defined and some properties of these modules are studied. It is proved that homomorphic image, small cover and finite sum of artinin weakly supplemented modules are artinian weakly supplemented, but infinite direct sum of artinian weakly supplemented modules need not be artinian weakly supplemented. A factor module of totally artinian weakly supplemented modules is also totally artinian weakly supplemented. A module is artinian weakly supplemented (totally artinian weakly suplemented) if and only if a factor of it by a linear compact submodule is artinian weakly supplemented (totally artinian weakly suplemented).

Keywords:- Supplemented, Weakly supplemented, Artinian weakly supplemented, totally artinian weakly supplemented.

ÖZET

ARTİN ZAYIF TÜMLENEN MODÜLLER

ABDULKAREEM, Aram
Matematik Yüksek Lisans
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Mayıs 2015, 30 sayfa

Bu tezde Artin zayıf tümlenen modüller ve tümden Artin zayıf tümlenen modüller tanımlanmış ve bu modüllerin bazı özellikleri incelenmiştir. Artin zayıf tümlenen modüllerin homomorf görüntüsü, küçük örtüleri ve sonlu toplamlarının Artin zayıf tümlenen modüller olduğu, fakat Artin zayıf tümlenen modüllerin sonsuz toplamının Artin zayıf tümlenen modül olmayabileceği kanıtlanmıştır. Tümden Artin zayıf tümlenen modüllerin faktör modülleri de tümden Artin zayıf tümlenendir. Bir modülün Artin zayıf tümlenen (tümden Artin zayıf tümlenen modül) olması için bu modülün bir lineer kompakt altmodüle göre faktör modülünün Artin zayıf tümlenen modül (tümden Artin zayıf tümlenen modül) olması gerek ve yeterlidir.

Anahtar sözcükler: Tümlenen, Zayıf Tümlenen, Artin Zayıf Tümlenen, Tümden Artin Zayıf Tümlenen.

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TEXT OF OATH

I, Aram A. ABDULKAREEM do declare and honestly confirm that my study, titled "Artinian Weakly Supplemented Module" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions,that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

Student Name and Signature

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INTRODUCTION

Supplement submodules and supplemented modules play an important role in the investigation of modules. Different types of supplemented modules (weakly supplemented, cofinitely supplemented etc.) are used to study deep properties of modules and their submodules. In this thesis we define and study artinian weakly supplemented modules, that is modules M whose submodules N with M/N artinian have weak supplements.

The thesis consists of three chapters. In the first chapter we give some basic definitions, examples and theorems related to abelian groups, modules, small submodules and some specific modules as artinian, hollow, uniserial and linear compact. Also it provides the definition of supplement submodule, which services us a means to define supplemented, weakly supplemented, totally weak supplemented and cofinitely weak supplemented modules and gives some properties of these modules.

In the second chapter we give the definition of artinian weakly supplemented module, and prove that the homomorphic image, small cover, finite direct sum and finitely M-generated of artinian weakly supplemented are also artinian weakly supplemented. On the other hand we show by example that an infinite direct sum of artinian weakly supplemented modules may not be artinian weakly supplemented.

In the third chapter, we define totally artinian weakly supplemented modules and prove that every homomorphic image of totally artinian weakly supplemented module is totally artinian weakly supplemented. We give an example showing that artinian weakly supplemented modules need not be totally artinian weakly supplemented. We also prove that an R-module M is totally artinian weakly supplemented (artinian weakly supplemented) if and only if M/K is totally artinian weakly supplemented (artinian weakly supplemented) for a linear compact submodule K.

CHAPTER 1

PRELIMINARIES

In this chapter, we give some basic definitions, examples and theorems related to abelian groups, modules and some specific modules as artinian, hollow, uniserial and linear compact. Also it gives the definition of supplemented, weakly supplemented, totally weak supplemented and cofinitely weak supplemented modules with some properties of these modules.

1.1. Abelian Groups

Definition 1.1 *A group is a nonempty set G on which a binary operation is defined* $(a, b) \to a * b$ *satisfying the following properties:*
Closure:If a and b belong to G, then $a * b$ *is also in G;*
Associativity: $a * (b * c) = (a * b) * c$ *for all* $a, b, c \in G$*;*
Identity: There is an element e in G (which is called a neutral element) such that $a * e = e * a = a$ *for all a in G;*
Inverse:If a is in G there is an element a^t *in G such that* $a * a^t = a^t * a = e$*.*

A group G is *abelian* if the binary operation is commutative, i.e., $a * b = b * a$ for all a, b in G. We will consider only abelian groups and use additive notation, that is the operation will be denoted by + and the inverse element of a by $-a$.

Definition 1.2 *Let G be a group, and let H be a subset of G. Then H is called a subgroup of G if H is itself a group, under the operation induced by G.*

Definition 1.3 *Let G be a group, and let a be any element of G. The set* $(a) = \{x \in G \mid x = na \text{ for some } n \in \mathbb{Z}\}$ *is called the cyclic subgroup generated by a. The group G is called a cyclic group if there exists an element* $a \in G$ *such that* $G = (a)$*. In this case a is called a generator of G.*

Definition 1.4 *Let a be an element of the group G. If there exists a positive integer n such that* $na = e$*, then a is said to have finite order, and the smallest such positive integer* *is called the order of a, denoted by $o(a)$. If there does not exist a positive integer n such that $na = e$, then a is said to have infinite order.*

Definition 1.5 *A is called torsion or periodic group, if every element of A is of finite order.*

For a group A and an integer $n > 0$, let $A[n] = \{a \in A \mid na = 0\}$. Thus $g \in A[n]$ if and only if $o(g)|n$. Clearly, $A[n]$ is subgroup of A.

Definition 1.6 *A group A is called n-bounded if it satisfies $nG = 0$. A group A is bounded if it is n-bounded for some n* Ç 0.

Definition 1.7 *A group D is called divisible if $n|a$ for all $a \in D$ and all positive integers n.*

Thus a group D is divisible if and only if $nD = D$ for every positive n. The groups Q, $Z_{(p^\infty)}$ and Q/Z are examples for divisible groups, but a direct sum of cyclic groups is not divisible.

1.2. Module and submodule

Definition 1.8 *Let R be a ring with identity 1, M be an abelian group and $f : R \times M \longrightarrow M$, ($f(r, m) = rm$) be a function where $r \in R$, $m \in M$. Then M is called a left R-module (or a module in brief) if the following are satisfied:*

(i) $r(m + n) = rm + rn$ *for every* $r \in R$ *and* $m, n \in M$.

(ii) $(r + s)m = rm + sm$ *for every* $r, s \in R$ *and* $m \in M$.

(iii) $(rs)m = r(sm)$ *for every* $r, s \in R$ *and* $m \in M$.

(iv) $1.m = m$ *for every* $m \in M$.

Definition 1.9 *A subset N of an R-module M is called a submodule if N itself is a module with respect to the same operations. Notation: $N \leq M$.*

Definition 1.10 *Let M be an R-module and N be a submodule of M. The set of cosets $M/N = \{x + N \mid x \in M\}$ is a module relative to the addition and scalar multiplication defined by $(x + N) + (y + N) = (x + y) + N$, $r(x + N) = rx + N$. The resulting module M/N is called a factor module of M by N.*

Lemma 1.1 *(Modular Law) Let K, N, L be submodules of M and* $N \subseteq K$ *then*

$$K \cap (N + L) = N + K \cap L.$$

Proof ($\subseteq$) Let $k \in K \cap (N + L)$. Then k can be represented as $k = n + l$ for some $n \in N$, $l \in L$. Since $N \subseteq K$, $n \in K$ we have $l = k - n \in K + N \subseteq K + K = K$. Hence $l \in K \cap L$ and $k = n + l \in N + K \cap L$.
($\supseteq$) Obvious. □

1.3. Isomorphism

Definition 1.11 *If M and N are two modules, then a function* $f : M \longrightarrow N$ *is a homomorphism in case for all* $r, s \in R$ *and all* $x, y \in M$

$$f(rx + sy) = r f(x) + s f(y)$$

If N = M, then the homomorphism f is called endomorphism.

Definition 1.12 *A homomorphism* $f : M \longrightarrow N$ *is called an epimorphism if it is onto. It is called a monomorphism if it is one-to-one .*

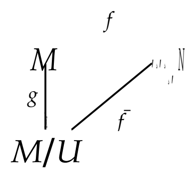
Definition 1.13 *Kernel of f:* $\text{Ker} f = \{m \in M \mid f(m) = 0\} \subseteq M$. *Image of f:* $\text{Im} f = \{ f(m) \mid m \in M\} \subseteq N$.

So it can be easily verified that f is an epimorphism if and only if $\text{Im} f = N$, and f is an monomorphism if and only if $\text{Ker} f = 0$.

Definition 1.14 *A homomorphism f is called an isomorphism if it is both an epimorphism and a monomorphism (i.e. it is a bijection).*

Theorem 1.1 ***Factorization Theorem***

Let $f : M \longrightarrow N$ *be a homomorphism of R-module. If U is a submodule of M with* $U \leq \text{Ker} f$, *then there is a unique homomorphism* $\bar{f} : M/U \longrightarrow N$ *with* $f = g\bar{f}$, *i.e. the following diagram is commutative:*



Moreover, $\text{Im} f = \text{Im} \bar{f}$ *and* $\text{Ker} \bar{f} = \text{Ker} f/U$.

Theorem 1.2 ***Fundamental Homomorphism Theorem***

Let M and N be left R-modules and $f : M \longrightarrow N$ be a homomorphism, then

$$M/\operatorname{Ker} f \cong \operatorname{Im} f.$$

If f is an epimorphism, then $M/\operatorname{Ker} f \cong N$.

Theorem 1.3 ***Second Isomorphism Theorem***

If N and K are submodules of M, then

$$(N + K)/K \cong N/(N \cap K).$$

Theorem 1.4 ***Third Isomorphism Theorem***

If $K \leq N \leq M$, then

$$(M/K)/(N/K) \cong M/N.$$

Definition 1.15 *An R-module M is called finitely generated if there exist elements $m_1, m_2, \ldots, m_r \in M$ so that each $m \in M$ can be written as*

$$m = a_1m_1 + a_2m_2 + \ldots + a_rm_r$$

for some $a_1, a_2, \ldots, a_r \in R$. The elements $m_1, m_2, \ldots, m_r$ are called generators of M.

One important property should be pointed out immediately: Any factor of a finitely generated module is also finitely generated. Indeed, if $m_1, \ldots, m_r$ generate M then the cosets $m_1 + N, \ldots, m_r + N$ generate the factor module M/N for every $N \leq M$.

Definition 1.16 *Let M be an R-module and $\{N_i \mid i \in I\}$ be a set of submodules of M. $M = \bigoplus_{i\in I} N_i$ is called internal direct sum (or direct sum) if the following conditions hold:*

1. $M = \sum_{i\in I} N_i$

2. For every $j \in I$, $N_j \cap \sum_{i\neq j} N_i = 0$

Then $M = \bigoplus_{i\in I} N_i$ is also said to be a decomposition of M.

Definition 1.17 *Let M be an R-module. A submodule A is called direct summand of M if $M = A \oplus B$ for some submodule $B \subseteq M$.*

Definition 1.18 *Let M be an R-module. Then an R-module N is called (finitely) M-generated if it is a homomorphic image of a (finite) direct sum of copies of M.*

Definition 1.19 *A sequence*

$$\cdots \xrightarrow{} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{} \cdots$$

of modules $\{M_n\}_{n\in \mathbb{Z}}$ *and homomorphisms* $\{f_n\}_{n\in \mathbb{Z}}$ *is exact if* $\operatorname{Im} f_{n+1} = \operatorname{Ker} f_n$ *for each* $n \in \mathbb{Z}$.

A sequence $0 \longrightarrow A \xrightarrow{f} B$ of R-modules is exact if and only if f is one-to-one, and a sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is onto. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is said a short exact sequence. In this case, f is a monomorphism and g is an epimorphism, so $\operatorname{Im} f \div A$ and $C \div B/\operatorname{Im} f$. Thus we can assume that $A \leq B$ and say $C \div B/A$.

1.4. Small Submodule and Radical

Definition 1.20 *Let M be an R-module. A submodule K of M is small (superfluous) in M if for all proper submodules L of M,* $L + K$ Ç M *holds. Small submodule is denoted by* $K \ll M$.

Definition 1.21 *An epimorphism* $f : M \longrightarrow N$ *is called small if Ker* $f \ll M$. *In this case a module M is called a small cover of a module N with small epimorphism* f.

Lemma 1.2 *(Wisbauer, 1991) Let M, N and L be R-module. Then:*

1. An epimorphism $f : M \longrightarrow N$ *is small if and only if every homomorphism* $g : L \longrightarrow M$ *with epimorphism* fg *is epimorphism.*

2. If $f : M \longrightarrow N$; $g : N \longrightarrow L$ *are two epimorphism, then* $g \circ f$ *is small if and only if* f *and* g *are small, i.e.* $\operatorname{Ker}(g \circ f) \ll M$ *if and only if* $\operatorname{Ker} f \ll M$ *and* $\operatorname{Ker} g \ll N$

Definition 1.22 *Let M be an R-module. A submodule N of M is called a minimal (simple) if N* Ç 0 *and there is no proper non-zero submodule of N, that if* $0 \leq N \leq M$ *then either* $N = 0$ *or* $N = M$. *N is called a maximal submodule of M if N is proper and there exists no other proper submodule K of M such that N* Ç *K, that if* $N \leq K \leq M$ *then either* $K = N$ *or* $K = M$.

Definition 1.23 *Let* $(T_\alpha)_{\alpha \in A}$ *be an indexed set of simple (minimal) submodules of M. If M is the direct sum of this set, then*

$$M = \bigoplus_{A} T_\alpha$$

is a semisimple decomposition of M. A module M is called semisimple in case it has a semisimple decomposition.

Theorem 1.5 *For an R-module M, the following statements are equivalent:*

(a) M is semisimple;

(b) M is generated by simple modules;

(c) M is the sum of some set of simple modules;

(d) M is the sum of its minimal (simple) submodules;

(e) Every submodule of M is a direct summand.

Definition 1.24 *Let M be an R-module. The radical of M is the sum of all small submodules of M, equivalently intersection of all maximal submodules of M. The radical of M is denoted by* $\mathrm{Rad}(M)$.

Lemma 1.3 *Let M be an R-module and* $K \subseteq L$ *and* L_i $(1 \leq i \leq n)$ *be submodules of M for some positive integer n. Then the following hold:*

1. $L \ll M$ *if and only if* $L + N \subsetneq M$ *for any proper submodule N of M.*
2. *Let* $L \ll M$*, then any submodule of L is also small in M.*
3. $L \ll M$ *if and only if* $K \ll M$ *and* $L/K \ll M/K$.
4. $L_1 + L_2 + \cdots + L_n \ll M$ *if and only if* $L_i \ll M$ $(1 \leq i \leq n)$.
5. *If* M' *is an R-module and* $\phi : M \longrightarrow M'$ *is a homomorphism, then* $\phi(L) \ll M'$ *whenever* $L \ll M$.
6. *If L is a direct summand of M, then* $K \ll L$ *if and only if* $K \ll M$.

Proof 1. $(\Rightarrow)$ Suppose that $L + N = M$, then by definition of small submodule $N = M$.

$(\Leftarrow)$ Assume that $L + N = M$ but by assumption this is true just for the case $N = M$.

2. Let $K \leq L$ and $K + X = M$. Then we get $L + X = M$. Since $L \ll M$, it follows that $X = M$ and this implies $K \ll M$.

3. $(\Rightarrow)$ Let $K + N = M$ for some $N \leq M$. Since $K \leq L$ we have $L + N = M$. Thus $N = M$ since $L \ll M$. Hence $K \ll M$.

Let $L/K + T/K = M/K$ for some $T \leq M$ containing K. Then $L + T = M$. Since $L \ll M$ we have $T = M$ and this implies that $T/K = M/K$. Thus $L/K \ll M/K$.
($\Leftarrow$) Let $L + N = M$ for some submodule N of M. Thus $L/K + (N + K)/K = M/K$. Since $L/K \ll M/K$, $(N + K)/K = M/K \Rightarrow N + K = M$. Since $K \ll M$, $N = M$. Hence $L \ll M$.

4. ($\Rightarrow$) Let $L_i + N = M$ for some submodule N of M. For $i \neq j$ ($j = 1, 2, \ldots, n$), $L_1 + L_2 + \cdots + L_i + \cdots + L_n + N = M$. By hypothesis, $L_1 + L_2 + \cdots + L_n \ll M$, so $N = M$ then $\sum_{j\neq i} L_j + N = M$, therefore $L_i \ll M$.
($\Leftarrow$) Let each $L_i \ll M$ and $L_1 + L_2 + \cdots + L_n + N = M$. Since $L_1 \ll M$, $L_2 + \cdots + L_n + N = M$. Then since $L_2 \ll M$, $L_3 + \cdots + L_n + N = M$. Continuing in this way we get $N = M$, therefore $L_1 + L_2 + \cdots + L_n \ll M$.

5. Let $\phi(L) + N = M'$ for some submodules $N \subseteq M'$ and $L \subseteq M$. $M = \phi^{-1}(M') = \phi^{-1}(\phi(L) + N) = \phi^{-1}(\phi(L)) + \phi^{-1}(N) = (L + \operatorname{Ker}\phi) + \phi^{-1}(N) = L + \phi^{-1}(N)$. Since $L \ll M$, $\phi^{-1}(N) = M$. $M' = \phi(L) + N \subseteq \phi(M) + N = \phi(\phi^{-1}(N)) + N \subseteq N$, so $M' = N$. Hence $\phi(L) \ll M'$.

6. ($\Rightarrow$) Let $K + T = M$ for some submodule T of M. Then $(K + T) \cap L = L$. By Modular Law, $K + (T \cap L) = L$. Since $K \ll L$, $T \cap L = L \Rightarrow L \subseteq T$. Since $K \subseteq L$, $K \subseteq T$, i.e. $M = K + T = T \Rightarrow M = T \Rightarrow K \ll M$.
($\Leftarrow$) Let $K \ll M$. Suppose L is a direct summand of M. There exists a submodule N of M such that $L + N = M$ and $L \cap N = 0$. Let $K + T = L$ for some submodule T of L. $M = L + N = K + T + N$. Since $K \ll M$, $T + N = M$. Then by Modular Law $L = (T + N) \cap L = T + N \cap L$. Since $N \cap L = 0$, $L = T$, therefore $K \ll L$. □

1.5. Artinian Modules

Definition 1.25 *An R-module M is called artinian if every non-empty set of submodules has a minimal element.*

Theorem 1.6 *(Kasch, 1982)Let M be an R-module and A be a submodule of M. The following properties are equivalent:*

1. *M is artinian.*
2. *A and M/A are artinian.*
3. *Every descending chain $A_1 \supset A_2 \supset A_3 \supset \cdots$ of submodules of M satisfy descending*

chain condition, i.e. every descending chain of submodules of M is stationary.

4. Every factor module of M is finitely cogenerated.

5. In every set $\{A_i \mid i \in I\} \neq \emptyset$ *of submodules* $A_i \subset M$ *there is a finite subset* $\{A_i \mid i \in I_0\}$ *(i.e. finite* $I_0 \subset I$*) with*

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I_0} A_i.$$

Example 1.1 *1. A module with only finitely many submodules is artinian. In particular, finite abelian groups are artinian over* $\mathbb{Z}$*.*

2. Finite dimensional vector spaces V are artinian, as every submodule is a subspace with dimension less that or equal to dim(V). However, infinite dimensional vector spaces are not artinian. For, suppose V is infinite dimensional vector space over the field K with basis $\mathcal{B}$*. Let* $u_i \in \mathcal{B}$ *for all* $i \in \mathbb{N}$ *be distinct elements in* $\mathcal{B}$*. Define* $U_n = (u_n, u_{n+1}, \ldots)$ *as a subspace of V . Then* $U_1 \supsetneq U_2 \supsetneq \cdots$ *is an infinite descending chain of subspaces.*

3. $\mathbb{Z}_{\mathbb{Z}}$ *is not artinian.*

$$\mathbb{Z} \supsetneq (2) \supsetneq (4) \supsetneq (8) \supsetneq \cdots$$

is a non-stationary descending chain of $\mathbb{Z}$*-submodules of* $\mathbb{Z}$*.*

Corollary 1.1 *Every non-zero artinian module contains a (simple) minimal submodule.*

Proof Let M be a non-zero artinian R-module. Let $\mathcal{F}$ be a family of all proper submodules of M. Then $(0) \in \mathcal{F} \Rightarrow \mathcal{F} \neq \emptyset$. Then $\mathcal{F}$ has a minimal element, say, N. Clearly N is a minimal submodule of M. □

Corollary 1.2 *Sum of finitely many artinian modules is artinian.*

Proof Let $M_1, M_2, \ldots, M_n$ be artinian modules over the same ring R. Let $M = \sum_{i=1}^{n} M_i$. Proof is by induction on n. For $n = 1$, there is nothing to prove. Suppose the result holds for $m < n$. Now $M = \sum_{i=1}^{n-1} M_i + M_n$. Let $N = \sum_{i=1}^{n-1} M_i$. Then by induction hypothesis N is artinian. Now

$$\frac{M}{N} = \frac{N + M_n}{N} \cong \frac{M_n}{N \cap M_n}$$

is artinian. As M/N and N are artinian, by Theorem 1.6, we get M is artinian. □

Note that this implies that the direct sum of finitely many artinian modules is artinian. However, this does not hold for an infinite direct sum.

1.6. Hollow, Uniserial and Linearly compact Modules

Definition 1.26 *Let M be an R-module. If every proper submodule of M is small in M, then M is called a hollow module.*

Proposition 1.1 *(Clark et al., 2006) For M the following are equivalent:*

1. M is hollow.

2. Every non-zero factor module of M is indecomposable.

3. For any non-zero modules K and N and any morphisms $K \xrightarrow{f} M \xrightarrow{g} N$*, if* $f \circ g$ *is surjective, then both f and g are surjective.*

Note that factor module of hollow modules are again hollow.

Definition 1.27 *A module is called local if it has a largest proper submodule. Equivalently, a module is local if and only if it is cyclic, non-zero, and has a unique maximal proper submodule.*

Definition 1.28 *Let M be an R-module. M is called uniserial if its submodules are linearly ordered by inclusion.*

Definition 1.29 *A submodule K of a nonzero module M is said to large or essential (K Ð M) if K ∩ L Ç 0 for every nonzero submodule L ≤ M. If all nonzero submodules of M are large in M, then M is called uniform.*

The following proposition gives some characterizations of uniserial module.

Proposition 1.2 *(Clark et al., 2006) Let M be an R-module. The following are equivalent:*

(a) M is uniserial;

(b) Every factor module of M is uniform;

(c) Every factor module of M has zero or simple socle;

(d) Every submodule of M is hollow;

(e) Every finitely generated submodule of M is local;

(f) Every submodule of M has at most one maximal submodule.

Definition 1.30 *Let M be an R-module. M is called linearly compact if for every family of cosets* $\{x_i + M_i\}_6$, $x_i \in M$, *and submodules* $M_i \subset M$ *(with* M/M_i *finitely cogenerated) such that the intersection of any finitely many of these cosets is not empty, then also* $\bigcap_6 (x_i + M_i)$ Ç ∅.

The following lemma gives some properties of linearly compact modules.

Lemma 1.4 *(Wisbauer, 1991) Let N be a submodule of the R-module M.*

1. Assume N to be linearly compact and $M_{i \in 6}$ *to be an inverse family of submodules of M. Then*

$$N + \cap_6 M_i = \cap_6 (N + M_i)$$

2. M is linearly compact if and only if N and M/N are linearly compact.

1.7. Supplement and Supplemented Modules

Definition 1.31 *Let U be a submodule of the R-module M. A submodule V of M is called a supplement or addition complement of U in M if V is a minimal element in the set of submodules* $L \subset M$ *with* $U + L = M$.

Remark 1.1 *Zero submodule is a trivial supplement of every module.*

Lemma 1.5 *V is a supplement of U in M if and only if* $U + V = M$ *and* $U \cap V \ll V$.

Proof (⇒) Let V be a supplement of U in M such that $M = U + V$. Suppose $(U \cap V) + X = V$ for some $X \subseteq V$, then $M = U + V = U + (U \cap V) + X = U + X$. By minimality of V, $X = V$. Thus $U \cap V \ll V$.

(⇐) Let $M = U + V$ and $U \cap V \ll V$. Suppose $M = U + Y$ for some $Y \subseteq V$. $V = M \cap V = (U + Y) \cap V = (U \cap V) + Y$ by Modular Law. Then $Y = V$ since $U \cap V \ll V$. Hence V is a supplement of U in M. □

The following propositions gives some properties of supplement.

Proposition 1.3 *(Wisbauer, 1991) Let* U, V *submodules of* R*-module* M*. Assume* V *to be a supplement of* U *in* M*. Then:*

1. *If* $W + V = M$ *for some* $W \subset U$*, then* V *is a supplement of* W *also.*
2. *If* M *is finitely generated, then* V *is also finitely generated.*
3. *If* U *is a maximal submodule of* M*, then* V *is cyclic and* $U \cap V = \mathrm{Rad}(V)$ *is a (the unique) maximal submodule of* V*.*
4. *If* $K \ll M$ *then,* V *is a supplement of* $U + K$*.*
5. *If* $K \ll M$*, then* $V \cap K \ll V$ *and* $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$*.*
6. *If* $\mathrm{Rad}(M) \ll M$*, then* U *is contained in a maximal submodule of* M*.*
7. *If* $L \subseteq U$, $V + L/L$ *is a supplement of* U/L *in* M/L*.*
8. *If* $\mathrm{Rad}(M) \ll M$ *or* $\mathrm{Rad}(M) \subseteq U$ *and* $p : M \longrightarrow M/\mathrm{Rad}(M)$ *is canonical epimorphism, then* $M/\mathrm{Rad}(M) = p(U) \oplus p(V)$*.*

Definition 1.32 *If for every* $V \subset M$ *with* $U + V = M$ *there is a supplement* V^t *of* U *such that* $V^t \subseteq V$*, then it is said that* U *has ample supplements in* M*.*

Lemma 1.6 *(Wisbauer, 1991) Let* U *be a linearly compact submodule of an* R*-module* M*. Then* U *has ample supplements in* M*.*

Proof Let $U, V \subseteq M$ such that U is linearly compact and $M = U + V$. Define $\Gamma = \{V^t \subseteq V \mid U + V^t = M\}$. Γ ç 0 since $V \in \Gamma$. Take a chain $\{V_\lambda\}$ in Γ. It is an inverse family of submodules V_λ since $\{V_\lambda\}$ is a chain. $\bigcap V_\lambda$ is a lower bound for $\{V_\lambda\}$. $U + (\bigcap V_\lambda) = \bigcap (U + V_\lambda) = M$ by the property of linearly compact module. Thus $\bigcap V_\lambda \in \Gamma$. By Zorn's Lemma there is a minimal element K in Γ such that $M = U + K$ so K is a supplement of U and $K \subseteq V$. Hence U has ample supplements in M. □

Definition 1.33 *Let* M *be an* R*-module. If every submodule of* M *has a supplement,* M *is called a supplemented module.*

Clearly semisimple modules are supplemented. Every artinian module is supplemented. Really, if U is a submodule of M then there is a submodule V of M such that $U + V = M$. Suppose there is a submodule V_0 of V with $U + V_0 = M$,

$U + V_1 = M$ and $V_1 \subset V_0 \cdots$ continuing in this way we have a descending chain of submodules $V \geq V_0 \geq V_1 \geq \cdots \geq V_n \cdots$ But since M is artinian, this sequence is finite. Denote the module at the end by V_n. Therefore V_n is a supplement of U in M. Note that hollow module is supplemented since every proper submodule is small. Since local module is hollow, a local module is supplemented.

The following proposition gives some properties of supplemented modules.

Proposition 1.4 *(Wisbauer, 1991) For an R-module M, the following properties hold:*
(i) Let U and V be submodules of M such that U is supplemented and U + V has a supplement in M. Then V has a supplement in M.
(ii) If $M = M_1 + M_2$ with M_1 and M_2 are supplemented modules, then M is also supplemented.
(iii) If M is supplemented, then
(a) Every finitely M-generated module is supplemented.
(b) $M/\operatorname{Rad}(M)$ is semisimple.

Proposition 1.5 *(Top, 2007)Let R be a ring and M be an R module with $N \subseteq M$. If in the exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

N, M/N are supplemented and N has a supplement in every H with $N \subseteq H \subseteq M$, then M is supplemented.

Definition 1.34 *A submodule N of an R-module M is called cofinite if the factor module M/N is finitely generated.*

Definition 1.35 *An R-module M is called cofinitely supplemented if every cofinite submodule of M has a supplement in M.*

Clearly supplemented modules are cofinitely supplemented. Moreover, the class of cofinitely supplemented is closed under homomorphic image and any direct sum by (Alizade et al., 2001). Let M be any module, then $Loc(M)$ will denote the sum of all local submodules of M and $Cof(M)$ the sum of all cofinitely supplemented submodules of M. The following Theorem give a characterization of cofinitely supplemented module.

Theorem 1.7 *(Alizade et al., 2001) Let R be any ring. The following statements are equivalent for an R-module M.*

1. *M is cofinitely supplemented.*
2. *Every maximal submodule of M has a supplement in M.*
3. *The module $\frac{M}{Loc(M)}$ does not contain a maximal submodule.*
4. *The module $\frac{M}{Cof(M)}$ does not contain a maximal submodule.*

1.8. Weakly Supplemented Modules

Definition 1.36 *Let M be an R-module and U, V submodule of M. Then V is called a weak supplement of U in M, if $U + V = M$ and $U \cap V \ll M$.*

Definition 1.37 *Let M be an R-module. M is called a weakly supplemented module if every submodule of M has a weak supplement in M.*

Example 1.2 *Supplemented, artinian, semisimple, linearly compact, uniserial and hollow modules are weakly supplemented modules.*

Lemma 1.7 *(Alizade and Büyükaşık, 2003) If $f : M \to N$ is a homomorphism and a submodule L containing Ker f is a weak supplement in M, then $f(L)$ is a weak supplement in $f(M)$.*

Proof If L is a weak supplement of K in M then $f(M) = f(L + K) = f(L) + f(K)$ and since $L \cap K \ll M$, we have $f(L \cap K) \ll f(M)$ by Lemma 1.3(5). As $K \supseteq Ker f$, $f(L) \cap f(K) = f(L \cap K)$. So $f(L)$ is a weak supplement of $f(K)$ in $f(M)$. □

Proposition 1.6 *(Alizade and Bü yü kaşık, 2003) If K is a weak supplement of N in a module M and $T \ll M$, then K is weak supplement of $N + T$ in M as well.*

Proof Let $f : M \to (M/N) \oplus (M/K)$ be defined by $f(m) = (m + N, m + K)$ and $g : (M/N) \oplus (M/K) \to (M/(N + T)) \oplus (M/K)$ be defined by $g(m + N, m^t + K) = (m+N+T, m^t+K)$. Then f is an epimorphism as $M = N+K$ and $\mathrm{Ker} f = N \cap K \ll M$ as K is a weak supplement of N in M. So f is a small epimorphism. Now $\mathrm{Ker}\, g = (N + T)/N \oplus 0$ and $(N + T)/N = \sigma(T) \ll (M/N)$ since $T \ll M$, where $\sigma : M \to M/N$ is the canonical epimorphism. Therefore g is a small epimorphism. By Lemma 1.2(1) , fg is a small epimorphism, i.e. $(N + T) \cap K = \mathrm{Ker}(fg) \ll M$. Clearly $(N + T) + K = M$, so K is a weak supplement of $N + T$ in M. □

Lemma 1.8 *(Alizade and Büyükaşık, 2003) If $f : M \to N$ is a small epimorphism, then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.*

Proof If L is a weak supplement of K in M then by Proposition 1.6, $L + \operatorname{Ker} f$ is also a weak supplement of K and by Lemma 1.7, $f(L) = f(L + \operatorname{Ker} f)$ is a weak supplement in N. Now let $f(L)$ be a weak supplement of a submodule T of N, i. e. $N = f(L) + T$ and $f(L) \cap T \ll N$. Then $M = L + f^{-1}(T)$. It follows from the proof of Corollary 9.1.5 in (Kasch) that the inverse image of a small submodule of N is small in M. So $L \cap f^{-1}(T) \leq f^{-1}(f(L) \cap T) \ll N$. Thus $f^{-1}(T)$ is a weak supplement of L. □

Proposition 1.7 *(Clark et al., 2006) The class of weakly supplemented modules is closed under homomorphic images, finite direct sums and small covers.*

Example 1.3 $\mathbb{Q}/\mathbb{Z}$ *is a weakly supplemented* $\mathbb{Z}$*-module.*
Firstly write $M := \mathbb{Q}/\mathbb{Z} = \bigoplus_p M_p$ as the direct sum of its (prime) p-components $M_p := \mathbb{Z}_{p^\infty}$. Every submodule N of M is of the form $N = \bigoplus N_p$ where $N_p = N \cap M_p \subseteq M_p$ are the p-components of N. Since M_p is hollow, either $N_p = M_p$ or $N_p \ll M_p$. Thus $N \ll M$ if and only if $N_p \neq M_p$ for all p. If N is not small in M, set $\Lambda = \{p \mid N_p \neq M_p\}$ and $L := \bigoplus_{p \in \Lambda} M_p$. Then $N + L = M$ and $N \cap L = \bigoplus_{p \in \Lambda} N_p \ll M$. Hence L is a weak supplement of N in M.

Example 1.4 $\mathbb{Q}$ *is a weakly supplemented* $\mathbb{Z}$*-module.*
Since $\mathbb{Q}$ *is a small cover of the weakly supplemented module* $\mathbb{Q}/\mathbb{Z}$*, by Proposition 1.7* $\mathbb{Q}$ *is also weakly supplemented module.*

Definition 1.38 *An R-module M is called cofinitely weak supplemented module if every cofinite submodule has (is) a weak supplement.*

Clearly cofinitely supplemented modules and weakly supplemented modules are cofinitely weak supplemented. Obviously any finitely generated modules is weakly supplemented if and only if it is cofinitely weak supplemented. In addition, by (Alizada and Büyükaşık, 2003) the class of cofinitely weak supplemented modules is closed under homomorphic image, direct sums, and small covers.The following theorem give a characterization of cofinitely weak supplemented module.

Theorem 1.8 *(Alizade and Bü yü kaşık, 2003) For a module M the following statements are equivalent.*

1. M is a cofinitely weak supplemented module,

2. Every maximal submodule of M has a weak supplement,

3. $M/cws(M)$ has no maximal submodules, where $cws(M)$ is the sum of all weak supplements of maximal submodules of M.

1.9. Totally Weakly Supplemented Modules

Definition 1.39 *An R-module M is said to be totally weakly supplemented if every submodule of M is weakly supplemented.*

Example 1.5 *Artinian, linearly compact, uniserial and semisimple modules are totally weakly supplemented modules.*

Lemma 1.9 *Every factor module of totally weak supplemented module is totally weak supplemented.*

Proof Let M be a totally weak supplemented module and N/K be a submodule of M/K for some submodule N which contains K. Since M is totally weak supplemented N is weakly supplemented. Hence N/K is weakly supplemented as a factor module of weakly supplemented module. Therefore M/K is totally weak supplemented module. □

Totally weak supplemented modules are weakly supplemented but converse does not hold in general. The $\mathbb{Z}$-module $\mathbb{Q}$ is weakly supplemented. Suppose that $\mathbb{Z}$ is weakly supplemented. Take any integer $n > 1$. Then $n\mathbb{Z}$ has a weak supplement $m\mathbb{Z}$ in $\mathbb{Z}$, that is $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$ and $[n, m]\mathbb{Z} = n\mathbb{Z} \cap m\mathbb{Z} \ll \mathbb{Z}$. Take any prime integer p that does not divide $[n, m]$. Then $[n, m]\mathbb{Z} + p\mathbb{Z} = \mathbb{Z}$, so $[n, m]\mathbb{Z}$ is not small in $\mathbb{Z}$. Contradiction, so the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not weakly supplemented, therefore $\mathbb{Q}$ is not totally weak supplemented.

CHAPTER 2

ARTINIAN WEAKLY SUPPLEMENTED MODULES

In this chapter, we define artinian weakly supplemented module, and prove that the class of artinian weakly supplemented modules is closed under homomorphic image, small cover, finite direct sum and finitely M-generated. Also we prove by example that an infinite direct sum of artinian weakly supplemented modules may not be artinian weakly supplemented. In addition we give theorem such as example to show that artinian weakly supplemented module will not be weakly supplemented in general.

Definition 2.1 *Let M be an R-module. A module M is said to be artinian weakly supplemented module if for every submodule N of M where M/N is artinian N has a weak supplement in M.*

Example 2.1 *Supplemented, artinian, semisimple, linearly compact, uniserial and hollow modules are artinian weakly supplemented modules.*

Proposition 2.1 *Every factor module of an artinian weakly supplemented module is artinian weakly supplemented.*

Proof Let M be an artinian weakly supplemented module and N be a submodule of M. Suppose that $L/N \leq M/N$ where $N \leq L \leq M$ and $(M/N)/(L/N)$ is artinian. Note that $M/L \div (M/N)/(L/N)$, by third isomorphism theorem, so M/L is artinian. Since M is artinian weakly supplemented, L has a weak supplement K in M, i.e. $M = L + K$ and $L \cap K \ll M$. It follows that

$$M/N = (L + K)/N = L/N + (K + N)/N$$

Since $L \cap K \ll M$ and by Lemma 1.3(5)

$$L/N \cap (K + N)/N = (L \cap (K + N))/N = (N + (K \cap L))/N \ll M/N$$

Therefore M/N is artinian weakly supplemented. □

Corollary 2.1 *A homomorphic image of an artinian weakly supplemented is artinian weakly supplemented.*

Proposition 2.2 *A small cover of an artinian weakly supplemented module is artinian weakly supplemented module.*

Proof Let N be an artinian weakly supplemented module and $f : M \longrightarrow N$ be a small epimorphism. Let L be submodule of M such that M/L is artinian. By factor theorem we have the epimorphism $g : M/L \longrightarrow N/f(L)$, therefore $N/f(L)$ is artinian and since N is artinian weakly supplemented, $f(L)$ has weak supplement in N. By Lemma 1.8, L also has a weak supplement in M. Thus M is artinian weakly supplemented. □

Corollary 2.2 *Let $N \ll M$ and M/N be artinian weakly supplemented. Then M is artinian weakly supplemented.*

Corollary 2.3 *Suppose that M is an R-module with $RadM \ll M$ and $M/RadM$ is artinian weakly supplemented. Then M is artinian weakly supplemented.*

Lemma 2.1 *Let N and L be submodules of R-module M with artinian weakly supplemented L and artinian M/N. If $N + L$ has a weak supplement in M, then N also has a weak supplement in M.*

Proof Let K be a weak supplement of $N + L$ in M, i.e.

$$M = K + N + L \text{ and } K \cap (N + L) \ll M$$

By 2ed and 3rd isomorphism theorem

$$\frac{L}{L \cap (N + K)} \div \frac{L + N + K}{N + K} = \frac{M}{N + K} \div \frac{M/N}{(N + K)/N}$$

The last module is a factor of artinian module, hence $L/(L \cap (N + K)$ is artinian. Since L is artinian weakly supplemented, then $L \cap (N + K)$ has a weak supplement H in L, i.e.

$$L = H + [L \cap (N + K)] \quad \text{and}$$

$$H \cap L \cap (N + K) = H \cap (N + K) \ll L$$

Now

$$M = K + N + L = K + N + H + L \cap (N + K) = N + (H + K)$$

and

$$N \cap (H + K) \le [H \cap (N + K)] + [K \cap (N + H)]$$
$$\le [H \cap (N + K)] + [K \cap (N + L)] \ll M$$

Therefore $H + K$ is a weak supplement for N in M. □

Proposition 2.3 *Let $M = M_1+M_2$, where M_1 and M_2 are artinian weakly supplemented, then M is artinian weakly supplemented.*

Proof Let N be a submodule of M such that M/N is artinian. Then $M = N + M_1 + M_2$. Note that

$$\frac{M}{M_2 + N} \div \frac{M/N}{(M_2 + N)/N}$$

Thus $M/(M_2+N)$ is artinian. Since 0 (zero) submodule is a trivial weak supplement of $N + M_1 + M_2$ and M_1 is artinian weakly supplemented, thus $N + M_2$ has a weak supplement in M by Lemma 2.1. Since $M_2 + N$ has a weak supplement and M_2 is artinian weakly supplemented again by Lemma 2.1 N has a weak supplement in M. □

Corollary 2.4 *Every finite direct sum of artinian weakly supplemented modules is artinian weakly supplemented.*

Proposition 2.4 *Let M be an R-module. If M is an artinian weakly supplemented, then every finitely M-generated module is artinian weakly supplemented.*

Proof Let N be a finitely M-generated module. Then there exists an epimorphism $f : M^n \longrightarrow N$ for some positive integer n.

$$M^n = \bigoplus_{i=1}^{n} M_i \qquad , M_i = M$$

Since M is artinian weak supplemented, by Corollary 2.4, M^n is artinian weakly supplemented and by Corollary 2.1 , N is artinian weakly supplemented. □

Lemma 2.2 *Let p be a prime integer, $A = \oplus_{i=1}^{\infty} (a_i)$ where $o(a_i) = p^i$ and $B \ll A$. Then B is bounded.*

Proof Suppose that B is unbounded. As a subgroup of the direct sum of cyclic groups B is also a direct sum of cyclic groups: $B = \oplus(b_i)$ Each (b_i) can be embeded into Z_{p^∞} by $b_i \rightarrowtail c_i$. Then there is a homomorphism $f : B \to Z_{p^\infty}$ with $f(b_i) = c_i$. Since B is unbounded, f is an epimorphism. Z_{p^∞} is injective, hence f can be extended to $g : A \to Z_{p^\infty}$. Since $g(B) = f(B) = Im f = Z_{p^\infty}$,

$$A = g^{-1}(Z_{p^\infty}) = g^{-1}(g(B)) = B + ker g$$

So we have got a contradiction with $B \ll A$. □

The following example shows that an infinite direct sum of artinian weakly supplemented modules need not be artinian weakly supplemented.

Example 2.2 *Let p be a prime integer and* $A = \oplus_{i=1}^{\infty}(a_i)$ *where* $o(a_i) = p^i$. *Each* (a_i) *is a hollow, so is supplemented, therefore artinian weakly supplemented. We will prove that A is not artinian weakly supplemented.*

Proof Suppose that A is artinian weakly supplemented. Each (a_i) can be embeded into Z_{p^∞} by $a_i \rightarrowtail c_i$. Therefore there is a homomorphism

$$f : A \to Z_{p^\infty} \quad \text{with} \quad f(a_i) = c_i$$

Clearly f is an epimorphism. Let $K = ker f$. Since $A/K \div Z_{p^\infty}$ is artinian, K has a weak supplement L in A, that is $K + L = A$ and $K \cap L \ll A$.

$$f(L) = f(K + L) = f(A) = Z_{p^\infty}$$

therefore L is unbounded. By Lemma 2.2 $K \cap L$ is bounded, that is $p^n(K \cap L) = 0$ for some $n \in Z^+$. Then $K \cap L \leq L[p^n]$, therefore there is an epimorphism $L/(K \cap L) \longrightarrow L/L[p^n]$. But

$$L/(K \cap L) \div (L + K)/K = A/K \div Z_{p^\infty}$$

so $L/L[p^n]$ is divisible. $L \leq A = \oplus_{i=1}^{\infty}(a_i)$, therefore L is also a direct sum of cyclic p-groups; $L = \oplus(c_i)$ with $o(c_i) = p^{m_i}$. Then

$$L[p^n] = \oplus(c_i)[p^n] = \oplus_{m_i > n}(p^{m_i - n}c_i)$$

therefore

$$L/L[p^n] \div \oplus_{m_i > n}(c_i)/(p^{m_i - n}c_i) \div \oplus_{m_i > n}(d_i)$$

where $o(d_i) = p^{m_i - n}$. So $L/L[p^n]$ is reduced. Then $L/L[p^n] = 0$, that is $L = L[p^n]$ is bounded. Contradiction. So A is not artinian weakly supplemented. □

The following theorem gives an example of an artinian weakly supplemented Z-module that is not weakly supplemented.

Theorem 2.1 $\mathrm{Q}^{(\mathrm{N})}$ *is an artinian weakly supplemented* Z*-module, that is not weakly supplemented.*

Proof Let $M = \mathrm{Q}^{(\mathrm{N})}$ and $K \leq M$ such that M/K is artinian. Then there is a finite set J such that $M = K + \mathrm{Q}^{(J)}$. Since J is finite and Q is weakly supplemented, then $\mathrm{Q}^{(J)}$ is weakly supplemented. Let L be a weak supplement of $K \cap \mathrm{Q}^{(J)}$ in $\mathrm{Q}^{(J)}$. Then

$$L + K \cap \mathrm{Q}^{(J)} = \mathrm{Q}^{(J)} \text{ and } L \cap K \cap \mathrm{Q}^{(J)} = K \cap L \ll \mathrm{Q}^{(J)}$$

We get

$$M = K + \mathrm{Q}^{(J)} = K + L + K \cap \mathrm{Q}^{(J)} = K + L$$

and

$$K \cap L \ll \mathrm{Q}^{(J)} \leq M$$

Hence L is a weak supplement of K in M, therefore M is artinian weakly supplemented.

Now, suppose that $\mathrm{Q}^{(\mathrm{N})}$ is weakly supplemented. Let $\mathrm{Z}^{(\mathrm{N})} \leq \mathrm{Q}^{(\mathrm{N})}$. Then

$$\mathrm{Q}^{(\mathrm{N})}/\mathrm{Z}^{(\mathrm{N})} \div (\mathrm{Q}/\mathrm{Z})^{(\mathrm{N})}$$

is weakly supplemented, because factor modules of weakly supplemented are weakly supplemented. $\mathrm{Q}^{(\mathrm{N})}/\mathrm{Z}^{(\mathrm{N})}$ is divisible and torsion. It has a direct summand isomorphic to $(\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$. We claim that $(\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$ is not weakly supplemented; Suppose the contrary. Let $X = \oplus_{i=1}^{\infty} (a_i)$, $|a_i| = p$. Then $X \leq (\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$. So it has a weak supplement, say A. Then $X + A = (\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$ and $X \cap A \ll (\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$. Since $X \cap A$ is small and torsion, it is bounded, that is $p^n(X \cap A) = 0$ for some $n \in \mathrm{Z}^+$. Since

$$\frac{X}{X \cap A} \oplus \frac{A}{X \cap A} \div \frac{(\mathrm{Z}_{p^\infty})^{(\mathrm{N})}}{X \cap A}$$

is divisible, $X/(X \cap A)$ is divisible. Then $p(X/(X \cap A)) = X/(X \cap A)$. Therefore $(pX+(X\cap A))/(X\cap A) = X/(X\cap A)$, so $pX+(X\cap A) = X$. Then $p^n(pX+(X\cap A)) = p^nX$, that is $p(p^nX) = p^nX$. It means that p^nX is divisible by p. But it is a p-group, so p^nX is a divisible subgroup of X. Since X is reduced, $p^nX = 0$, that is X is bounded. Contradiction. So $(\mathrm{Z}_{p^\infty})^{(\mathrm{N})}$ is not weakly supplemented, therefore $\mathrm{Q}^{(\mathrm{N})}$ also is not weakly supplemented. □

Lemma 2.3 *Let M be an artinian weakly supplemented module. For every submodule $N/RadM$ of $M/RadM$ with $(M/RadM)/(N/RadM)$ artinian, $N/RadM$ is a direct summand.*

Proof Note that

$$\frac{M}{N} \div \frac{M/RadM}{N/RadM} \text{ is artinian}$$

Since M is artinian weakly supplemented, then N has a weak supplement K in M, i.e. $M = N + K$ and $N \cap K \ll M$. Since $M = N + K + RadM$ we have

$$\frac{M}{RadM} = \frac{N}{RadM} + \frac{K + Radm}{RadM}$$

Since $N \cap K \leq RadM$, $N \cap (K + RadM) = (N \cap K) + RadM \leq RadM$. Then from

$$\frac{N}{RadM} \cap \frac{K + RadM}{RadM} = \frac{N \cap (K + RadM)}{RadM} = \frac{RadM}{RadM}$$

It follows that

$$\frac{M}{RadM} = \frac{N}{RadM} \oplus \frac{K + RadM}{RadM}$$

as required. □

Theorem 2.2 *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of R-modules L, M, N. If L and N are artinian weakly supplemented and L has a weak supplement in M, then M is artinian weakly supplemented.*

Proof Without loss of generality we will assume $L \leq M$. Let S be a weak supplement of L in M, i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$$M/(L \cap S) \div L/(L \cap S) \oplus S/(L \cap S)$$

$L/(L \cap S)$ is artinian weakly supplemented as a factor module of L which is artinian weakly supplemented. On the other hand

$$S/(L \cap S) \div (S + L)/L = M/L \div N$$

is also artinian weakly supplemented. Then $M/(L \cap S)$ is artinian weakly supplemented module as a sum of artinian weakly supplemented modules. Therefore M is an artinian weakly supplemented by Proposition 2.2. □

Proposition 2.5 *Let M be an R-module. M is artinian weakly supplemented if and only if M/K is artinian weakly supplemented for a linearly compact submodule K of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following exact sequence:

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0.$$

with K linearly compact and M/K artinian weakly supplemented. Since K is linearly compact it is artinian weakly supplemented. By Lemma 1.6, K has an ample supplement in M, therefore K has a weak supplement in M. Hence M is artinian weakly supplemented by Theorem 2.2. □

Proposition 2.6 *Let M be an R-module. M is artinian weakly supplemented if and only if M/U is artinian weakly supplemented for a uniserial submodule U of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following short exact sequence:

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

Since U is uniserial, it is hollow by Proposition 1.2 so U artinian weakly supplemented.

Case 1: If $U \ll M$, then M is artinian weakly supplemented by Proposition 2.2.

Case 2: If $U \not\ll M$, then there is a proper submodule N of M such that $U + N = M$. Since $U \cap N \leq U$ and U is hollow, every proper submodule is small in U, i.e. $U \cap N \ll U$ so $U \cap N \ll M$. Thus U has a weak supplement in M. Hence M is artinian weakly supplemented by Theorem 2.2. □

CHAPTER 3

TOTALLY ARTINIAN WEAKLY SUPPLEMENTED MODULES

In this final chapter, we introduce the definition of totally artinian weakly supplemented modules and prove that every homomorphic image of totally artinian weakly supplemented module is totally artinian weakly supplemented. We give an example showing that artinian weakly supplemented modules need not be totally artinian weakly supplemented. We also prove that an R-module M is totally artinian weakly supplemented if and only if M/K is totally artinian weakly supplemented for a linear compact submodule K as well as for uniserial submodule.

Definition 3.1 *An R-module M is said to be totally artinian weakly supplemented, if every submodule of M is artinian weakly supplemented.*

Example 3.1 *Artinian, semisimple, linearly compact and uniserial modules are totally artinian weakly supplemented modules.*

Lemma 3.1 *Every factor module of a totally artinian weakly supplemented module is totally artinian weakly supplemented.*

Proof Let M be a totally artinian weakly supplemented module and N/K be a submodule of M/K for some submodule N which contains K. Since M is totally artinian weakly supplemented, N is artinian weakly supplemented. Hence N/K is artinian weakly supplemented as a factor module of artinian weakly supplemented module. Therefore M/K is totally artinian weakly supplemented module.

□

Corollary 3.1 *Every homomorphic image of a totally artinian weakly supplemented module is totally artinian weakly supplemented module.*

Totally artinian weakly supplemented modules are artinian weakly supplemented but converse does not hold in general. The Z-module Q is weakly supplemented, so it is artinian weakly supplemented. But $\mathbb{Z}$-module $\mathbb{Q}$ is not totally artinian weakly supplemented. For this take the submodule $\mathbb{Z}$ of $\mathbb{Q}$. We must show that $\mathbb{Z}$ is not artinian weakly supplemented. Take any integer $n > 1$. $\mathbb{Z}/n\mathbb{Z}$ is artinian. Suppose that $n\mathbb{Z}$ has a weak supplement $m\mathbb{Z}$ in $\mathbb{Z}$, that is $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$ and $[n, m]\mathbb{Z} = n\mathbb{Z} \cap m\mathbb{Z} \ll \mathbb{Z}$. Take any prime integer p that does not divide $[n, m]$. Then $[n, m]\mathbb{Z} + p\mathbb{Z} = \mathbb{Z}$, so $[n, m]\mathbb{Z}$ is not small in $\mathbb{Z}$. Contradiction, so the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not artinian weakly supplemented, therefore $\mathbb{Z}$-module $\mathbb{Q}$ is not totally artinian weakly supplemented.

Proposition 3.1 *Let K be a linearly compact submodule of an R-module M. Then M is totally artinian weakly supplemented if and only if M/K is totally atinian weakly supplemented.*

Proof (⇒) Clear.

(⇐) Let M/K be totally artinian weakly supplemented,where K is linearly compact submodule of M. Consider the following exact sequence:

$$0 \dashrightarrow K \dashrightarrow M \dashrightarrow M/K \dashrightarrow 0.$$

Take a submodule N of M.

Case 1: If $N \leq K$, then N is artinian weakly supplemented since it is a linearly compact submodule.

Case 2: If $N \not\subset K$, then

$$(N + K)/K \div N/(N \cap K).$$

Hence we have the following exact sequence:

$$0 \dashrightarrow N \cap K \dashrightarrow N \dashrightarrow N/(N \cap K) \dashrightarrow 0.$$

Since $N \cap K$ is a submodule of a linearly compact module, $N \cap K$ is linearly compact so it is artinian weakly supplemented. Since M/K is totally artinian weakly supplemented and $N/(N \cap K) \div (N + K)/K$, then $N/(N \cap K)$ is artinian weakly supplemented as isomorphic to a submodule of M/K. Hence N is artinian weakly supplemented by Proposition 2.5. □

Proposition 3.2 *Let M be an R-module. M is totally artinian weakly supplemented if and only if M/U is totally artinian weakly supplemented for a uniserial submodule U of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following exact sequence:

$$0 \dashrightarrow U \dashrightarrow M \dashrightarrow M/U \dashrightarrow 0.$$

Take a submodule N of M. If $N \leq U$, then N is artinin weakly supplemented because submodules of uniserial modules are uniserial and uniserial modules are artinian weakly supplemented. If $N \not\subset U$, then

$$(N + U)/U \div N/(N \cap U)$$

Hence we have the following exact sequence:

$$0 \dashrightarrow N \cap U \dashrightarrow N \dashrightarrow N/(N \cap U) \dashrightarrow 0.$$

Since $N \cap U$ is uniserial, it is artinian weakly supplemented and $N/(N \cap U)$ is isomorphic to a submodule of M/U so $N/(N\cap U)$ is artinian weakly supplemented. Therefore N is artinian weakly supplemented by Proposition 2.6. □

In chapter 2, we proved that if M and N are artinian weakly supplemented modules, then the module $M + N$ is also artinian weakly supplemented. Clearly this implies that any finite direct sum of artinian weakly supplemented is also artinian weakly supplemented module. This raises an obvious question: suppose that M and N are totally artinian weakly supplemented modules, is then $M \oplus N$ totally artinian weakly supplemented? We begin to deal with this question by considering the case when one of M, N is semisimple.

Proposition 3.3 *Let $M = M_1 \oplus M_2$ be the direct sum of submodules M_1,M_2 such that M_2 is semisimple. Then M is totally artinian weakly supplemented if and only if M_1 is totally artinian weakly supplemented.*

Proof The necessity follows from by Corollary 3.1. Conversely, suppose that M_1 is totally artinian weakly supplemented. Let N be a submodule of M. Since M_2 is semisimple, then $M_2 = (N \cap M_2) \oplus L$ for some submodule L of M_2. It follows that

$$M = M_1 + M_2 = M_1 \oplus [(N \cap M_2) \oplus L]$$

and hance

$$N = N \cap M = N \cap [M_1 \oplus (N \cap M_2) \oplus L]$$

by modular law

$$N = (N \cap M_2) \oplus [N \cap (M_1 \oplus L)]$$

Now consider the submodule $H = N \cap (M_1 \oplus L)$ of $M_1 \oplus L$. Note that $H \cap L = N \cap (M_1 \oplus L) \cap L = N \cap L = 0$. So H embeds in M_1. By hypothesis, H is artinian weakly supplemented. Since M_2 is semisimple, $N \cap M_2$ is artinian weakly supplemented. Therefore N is artinian weakly supplemented by Proposition 2.3. Thus M is totally artinian weakly supplemented. □

Corollary 3.2 *Let $M = M_1 \oplus M_2 \oplus M_3$ be a direct sum of submodules M_1,M_2 and M_3 such that M_2 linearly compact and M_3 is semisimple, then M is totally artinian weakly supplemented if and only if M_1 is totally artinian weakly supplemented.*

Proof (⇒) Clear by Lemma 3.1.

(⇐) Suppose that M_1 is totally artinian weakly supplemented. Note that

$$M/M_2 = (M_1 \oplus M_2 \oplus M_3)/M_2 = M_1 \oplus M_3$$

Since M_1 is totally artinian weakly supplemented and M_3 is semisimple, then $M_1 \oplus M_3$ is totally artinian weakly supplemented by Proposition 2.3. Hence M/M_2 is totally artinian weakly supplemented. Since M_2 linearly compact, then M is totally artinian weakly supplemented by Proposition 3.1 □

Definition 3.2 *Let M be an R-module. The annihilator of M is* $\text{ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$.

Lemma 3.2 *(Smith, 2000) Let a module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of submodules M_i $(1 \leq i \leq n)$, for some $n \geq 2$, such that $R = \text{ann}(M_i) + \text{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then*

$$N = (N \cap M_1) \oplus \cdots \oplus (N \cap M_n)$$

for every submodule N of M.

Lemma 3.3 *Let R be a noetherian ring and $M = M_1 \oplus M_2 \oplus \ldots\ldots \oplus M_n$ be a direct sum of totally artinian weakly supplemented submodules $M_i(1 \leq i \leq n)$ for some $n \geq 2$. Let $R = ann(M_i) + ann(M_j)$ for all $1 \leq i < j \leq n$, then M is totally artinian weakly supplemented.*

Proof Let U and V be two submodules of M such that U/V is artinian. By Lemma 3.2

$$U = (U \cap M_1) \oplus (U \cap M_2) \oplus \ldots\ldots. \oplus (U \cap M_n)$$

and

$$V = (V \cap M_1) \oplus (V \cap M_2) \oplus \ldots\ldots. \oplus (V \cap M_n)$$

Since $U/V \div \oplus(U \cap M_i)/(V \cap M_i)$, then for every $1 \le i \le n$,$(U \cap M_i)/(V \cap M_i)$ is artinian. By assumption, $U \cap M_i$ is artinian weakly supplemented, then there exists a weak supplement K_i of $V \cap M_i$ in $U \cap M_i$. Let $K = K_1 \oplus K_2 \oplus \ldots.. \oplus K_n$. Then for every $1 \le i \le n$, $U \cap M_i = (V \cap M_i) + K_i$. Now

$$U = [(V \cap M_1) + K_1] \oplus [(V \cap M_2) + K_2]\ldots.. \oplus [(V \cap M_n) + K_n]$$

$$= [(V \cap M_1) + \ldots. + (V \cap M_n)] + [K_1 + \ldots. + K_n] = V + K$$

Also,

$$V \cap K = (V \cap K_1) \oplus \ldots. \oplus (V \cap K_n)$$

Hence by Lemma 1.3 (4), K is a weak supplement of V in U. Therefore U is artinian weakly supplemented. □

CONCLUSION

In this thesis we define artinian weakly supplemented and totally artinian weakly supplemented modules, and we reached some properties of these modules. As a result of this study, we have artinian weakly supplemented is closed under homomorphic image, small cover and finite sum. Also we obtained that artinian weakly supplemented (totally artinian weakly supplemented) can be characterized in terms of factor module of them by linear compact submodule.

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