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ORDERS, ALEXANDROFF SPACES AND DIGRAPHS

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Omeed AZEEZ tarafından YÜKSEK LİSANS tezi olarak sunulan "Order, Alexandroff Spaces and Digraphs" başlıklı bu çalışma Y.Ü. Lisansüstü Eğitim ve Öğretim Yönetmeliği ile Y.Ü. Fen Bilimleri Enstitüsü Eğitim ve Öğretim Yönergesi"nin ilgili hükümleri uyarınca tarafımızdan değerlendirilerek savunmaya değer bulunmuş ve 15 / 05 / 2015 tarihinde yapılan tez savunma sınavında aday oybirliği/oyçokluğu ile başarılı bulunmuştur.

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ÖZET

Sıralamalar, Alexandroff Uzayları ve Digraflar

Hazırlayan: Omeed Asaad AZEEZ, Matematik Yüksek Lisans

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Bu tez esas olarak önsıralamalar, graflar ve dıgraflar gibi bazı başka kavramlarla bağlantılı olan Alexandroff Uzaylarını ele alıyor.

Birinci bölüm, bir sonraki bölümde kullanılan, topoloji, sıralama ve minimal açık kümeler hakkında bazı temel bilgileri veriyor.

İkinci bölümde, Alexandroff Uzayları minimal açık kümeler kullanılarak tanımlanıyor ve önsıralama, kısmı sıralama tanımları ve de esas konuyla (Alexandroff Uzayları) ve T_0- Alexandroff Uzayları ile bağlantıları veriliyor. Sonra iki Alexandroff uzaynın çarpımı, Hausdorff Alexandroff Uzayları gibi bazı yeni uzaylar oluşturuluyor ve bölüm dönüşümü, bölüm uzayı ve indirgenemezliğin tanımları veriliyor.

Son bölüm graflarla ilgili kavramları tartışıyor ve onları topoloji ve Alexandroff Uzayları ile ilişkilendiriyor; digraf ve geçişken digraf tanımları tanıtılıyor ve Alexandroff Uzayları ile aralarındaki bağıntı değerlendiriliyor.

ATTESTATION

I, Omeed Asaad Azeez do hereby attest that I have completed my thesis to get Master’s (MSc) degree in Mathematics and I do hereby declare and attest that this is solely my work.

Sincerely,

Signature

Date: 15 / 05 /2015

Abstract

ORDERS, Alexandroff Spaces and Digraphs

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This thesis mainly deals with Alexandroff spaces which are related to some concepts like preorders, graphs and digraphs.

The first chapter gives some basic notions about topology, order and minimal open sets, which are used in the next chapter.

In the second chapter, Alexandroff spaces are defined by using minimal open sets, and definitions of preorder, partial order as well as their connection with the main subject(Alexandroff spaces) and T0-Alexandroff spaces are given.

Then some new spaces such as the product of two Alexandroff spaces and Hausdorff Alexandroff spaces are constructed, and the definitions of a quotient map and quotient space and irreduicibility are given.

The last chapter discusses notions about graphs and relates them to topology and Alexandroff spaces; the definitions of digraph and transitive digraph are introduced, and relation between these and Alexandroff spaces is considered.

ACKNOWLEDGMENTS

Firstly, I am grateful to the almighty God for helping me to complete this thesis.

I wish to thank my thesis's advisor Prof. Dr. Mehmet Terziler for the time he spent with me and his valuable guidance, encouragement and inspiration. My sincere thanks go to the teachers at the Department of Mathematics in the Institute of Natural and Applied Sciences. I like to send my special gratitude to KRG for their support.

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Introduction

Alexandroff Spaces were first introduced in 1937 by P.S. Alexandroff under the name "discrete spaces". The name is now forgotten because the name discrete spaces are used for topological spaces in which every subset is open.

Alexandroff showed in [2] that preorders or quasiorders on a set are equivalent to the topologies on in which arbitrary intersections of open sets are open. In other words, Alexandroff topologies on a set are in one-to-one correspondence with preorders on .

Alexandroff topologies have several characterizations. Let ( ) be a topological space. Then the following are equivalent:

(1) Open and closed characterizations:

Open set. An arbitrary intersection of open sets in is open.

Closed set. An arbitrary union of closed sets in is closed.

(2) Neighborhood characterization:

Every point of has a smallest neighborhood.

(3) Preorder characterizations:

Specialization preorder. Given the preorder on , is the finest topology satisfying if and only if (* +).

Open up-set. There is a preorder on such that the open sets of are precisely those that are upward closed.

Closed down-set. There is a preorder on such that the closed sets of are precisely those that are downward closed.

In 1980s, Alexandroff spaces were rediscovered when the concept of "finite generation" was applied to general topology and the name "finitely generated spaces" was adopted for them; actually, these spaces have all the properties of finite spaces. A systematic investigation of Alexandroff spaces from the point of general topology which had been neglected since the original paper [2] of Alexandroff was taken up by Arenas [3].

Inspired by the use of Alexandroff topologies in computer science, digital topology, graph theory, applied mathematicians and physicists in the late 1990s began investigating the Alexandroff topology corresponding to “causal sets” (locally finite sets with a partial order) which arise from a preorder defined on “spacetime” (fundamentally discrete) modeling causality.

In this thesis, we review almost all works done on Alexandroff spaces. In chapter 2, we relate preorders on a set to Alexandroff spaces, and partial orders to T_0 – Alexandroff spaces. Then we construct new Alexandroff spaces from given ones, making use of the works from [18], [7], [22] and [3].

In chapter 3, we study Alexandroff spaces in connection with digraphs. Essentially, we state and prove in Theorem 3.3.1 that there is a one-to-one correspondence between the set of all Alexandroff topologies and the set of all transitive digraphs on a set (not necessarily finite).

Finally, defining the “converse” of digraph we state and prove Theorem 3.3.5 concerning the “closed topology” induced by an Alexandroff topology.

Chapter 1

Basic Definitions, Concepts and Results

In this chapter, we introduce basic definitions and concepts to make easier the reading of the thesis.

1.1 Topological Spaces

There are several equivalent definitions of a topology. Among them we can mention *neighborhoods definition* whose axiomatization is due to F.Hausdorff, *Kuratowski closure definition, open sets definition,* and using de Morgan's laws *closed sets definition.* The most commonly used is that in term of open sets though the most intuitive is that in the terms of neighborhoods. These concepts can be found in any text book on topology, for example, see [11] and [19].

Definition 1.1.1 A topological space is a set together with a collection of subsets of , called open sets, satisfying the following axioms:

T_1. The empty set and itself are open.

T_2. Any union of open sets is open.

T_3. The intersection of any finite number of open sets is open.

The collection is also called a topology on .

Examples 1.1.2

(a) The trivial (indiscrete) topology consists of .

(b) Given a set is called the discrete topology on .

(c) The Sierpinski space (or the connected two-point set) is a finite topological space with two points, only one of which is closed. It is the smallest example of a topological space which is neither trivial nor discrete. More precisely, a Sierpinski space is a topological space S whose underlying set is and open sets are

(d) Given an infinite set , a cofinite topology on is a topology in which the open sets are the empty set and the sets whose complements are finite.

Note that the trivial topology is the coarsest topology and the discrete topology is the finest topology on a set.

Definition 1.1.3 Let be a topological space.

(i) It is a T_0 – space or Kolmogorov space if for every pair of distinct points of , at least one of them has an open neighborhood not containing the other.

The intuitive meaning of this condition is that the points of are topologically distinguishable, that is, for any in , there is an open set which contains one of these points and not the other.

(ii) It is T_1 –space or a Fréchet space, if for every pair of distinct points of each has a neighborhood not containing the other.

(iii) It is a T_2 –space or a Hausdorff space, or a separated space, if for every pair of distinct points and , there exist a neighborhood of and a neighborhood of y such that and are disjoint.

Proposition 1.1.4 Let be a topological space. Then the following conditions are equivalent:

(a) is a T_1 –space.

(b) is a T_0 –space.

(c) The singleton is closed set.

(d) Every subset of is the intersection of all open sets containing it.

(e) Every finite set is closed.

(f) Every finite set is open.

Proposition 1.1.5

Let be a topological space. Then the following conditions are equivalent:

(a) is a T_2 –space.

(b) Any singleton is equal to the intersection of all closed neighborhoods of . (A closed neighborhood of is a closed set that contains an open set containing .)

(c) The diagonal is closed as a subset of the product space

Examples and Counterexamples 1.1.6

1. Nearly all topological spaces considered in mathematics are T_0; in particular, T_1 and T_2 are T_0 – space.
2. The trivial topology is not a T_0 –space; no points are distinguishable.
3. Sierpinski space is a simple example of a topology that is T_0 but not T_1.
4. The cofinite topology on an infinite set is a simple example of a topology that is T_1 but not T_2. (That's the case, because no two open sets of the cofinite topology are distinct!).
5. All metric spaces are T_2 –space.

Definition 1.1.7 Let be a topological space. A collection of subsets of is said to form a basis for the space if the following two conditions are met:

B_1. The union of all elements of is the whole space .

B_2. Any finite intersection of elements from is itself a union of the members of .

Examples 1.1.8

1. For any topological space, the collection of all open sets is a basis. (That's because any open subset of a topological space can be expressed as a union of size one).
2. For a discrete space, the collection of singletons forms a basis (since every open subset of a discrete topological space is a union of singletons).
3. A basis for the usual topology on Euclidean space is the open balls.
4. The f*amily*is a basis for the indiscrete topology on

Definition 1.1.9 Let be a topological space. A subcollection of is said to be a subbase or subbasis of if satisfies one of the two following equivalent conditions:

S_1. generates , that is, is the smallest topology containing

S_2. The collection of open sets consisting of all finite intersections of elements of together with and the empty set, forms a basis for . (This means that every non-empty proper open set in can be written as a union of finite intersections of elements of .)

Examples 1.1.10

1. The usual topolog on the real nu bers has a subbase consisting of all semi-infinite open intervals either of the form , where and are real numbers. Together, these generate the usual topology, since, for , the intersections generate the usual topology.
2. A second subbase is formed by taking the subfamily where and are rational.

1.2 **Orders and topology**

In the 1937, P.Alexandroff showed in [2] that quasiorders or preorders on a set X are equivalent to the topologies on X which are closed under arbitrary intersections. Then, being an Alexandroff space restricts the type of topological spaces we are studying in general. So in order for an Alexandroff space to be useful, we must find mathematical interesting examples of its applications. Clearly, any discrete space is an Alexandroff space. Also a discrete space is Hausdorff Alexandroff space and vice versa. Furthermore, if is Hausdorff, then the associated quasiorder is equality. What is new is that non-Hausdorff topological spaces are finding numerous applications today in the areas of computer science, (see [16], [1]).

Definition 1.2.1 Let be any non-empty set and let be a binary relation on . Then

(i) is a quasiorder (or a preorder) on if it is reflexive and transitive.
(ii) is a partial order on if it is antisymmetric and quasiorder on
(iii) is a total (or linear) order on if is a partial order and for any two elements in either or that is, any two elements are comparable with respect to .
(iv) is the discrete order on if
(v) is an equivalence relation on , if it is symmetric and quasiorder.

From now on we shall denote by , and call to be a partially ordered or quasiordered set, or even a poset or a qoset for simplicity.

Definition 1.2.2 Let be a poset or qoset and let be a subset of . Then

(i) is a down-set (or a decreasing set, or order ideal) if, whenever and , we have .

(ii) is an up-set (or an increasing set, order filter) if, whenever and , we have .

Note that some authors use the terms *decreasing hole* for down-set and *increasing hole* for up-set.

Thus we may define the following:

and ,

and

Down-sets (up-sets) of the form are called principal and denoted also by

Proposition 1.2.3 Let be a poset or a qoset. Then

1. is the smallest down-set containing .
2. is down-set if and only if ⊠

Lemma 1.2.4

Let be a poset and be a down-set of . Then the following assertions are equivalent:

1. . ⊠
2. .
3. .

More on order ideals can be found, for example, in [9].

Here we recall the following well-known definition and fact.

Definition 1.2.5 Given a non-empty set X, let be a subcollection of the subsets of . If

(i) for any ,

(ii) ,

(iii) ,

then is called a partition of ; the members of are called equivalence classes or blocks.

If is a partition of and is a partial order on , then is called a partially ordered partition, or a popartition of .

Example 1.2.6 Let be the partition of real line . Order by

if and only if

where is the usual order on . Then is a popartition of . In fact, is a totally since is a total order on , ordered partition of .

Theorem 1.2.7

There is a natural correspondence between an equivalence relation on a set and a partition of

As it is pointed out in [21], there is a correspondence between a quasiorder on a set and a popartition of .

Proposition 1.2.8 There is a correspondence between a quasiorder on a set and a popartition of .

Proof First, suppose that is a quasiorder on . Define a relation on X by

if and only if and .

Clearly, is an equivalence relation on . Thus there is a corresponding partition of , by Theorem 1.2.7. Let be that partition. The elements of are equivalence classes of the form , put

if and only if there exist with .

Now is a partial order on the equivalence classes of . Hence any quasiorder on generates a popartition of .

Conversely, if is a popartion of , then define the relation on by

if and only if .

Clearly, is a quasiorder. Thus, from any popartition of we can obtain a quasiorder on and vice – versa. This completes the proof. ⊠

In 1937, Alexandroff [2] noted another occurrence of quasiorder in terms of topological concepts. Quasiorders on a finite set are identical to the topologies on that set!

A topology on a set is called *principal topology* if arbitrary intersections of open sets are open sets. Quasioreders on an arbitrary set are identical to the principal topologies on that set! (See [5].)

Let be a topological space. The *specialization order* on is defined by

if and only if

where denotes the closure of in . This definition means that every open neighborhood of contains .

Proposition 1.2.9 The specialization order is quasiorder on .

Proof Since for each we have , it follows that , i.e. is reflexive.

Now let and for . Then and imply

, hence , so . Thus is transitive. ⊠

Now if is a quasiorder on , then the set of all down-sets of is a principal topology on , called specialization topology associated with . Since the complement of down-sets in a qoset are up-sets, the closure of a set relative to the specialization topology is the smallest up-set of that contains . More on principal topologies can be found, for example, in [10]. Principal topologies are called *Alexandroff discrete topologies* or A – topologies in [11]. A – topologies are precisely the topology in which every point has a minimal neighborhood, as we show in chapter 2.

We shall mainly be concerned with T_0 – Alexandroff spaces. Thus, the specialization topology associated with quasiorder is T_0 if and only if is anti-symmetric, i.e. if and only if is a partial order; indeed, whenever is such a topology, then is T_0 if and only if for every pair , either or , that is, either or .

The following is obvious.

Proposition 1.2.10 Let be a topological space. Then

(a) If it is an Alexandroff space, then the only T_1 – topology on is the discrete topology.

(b) If it is a finite T_0 – space, then there is such that is closed.

As it is pointed out in [8], *at least for finite case* we can relate quasiorders and partial orders on a set to the topologies on .

Given a set we shall denote by

: the set of quasiorders on ,

the set of partial orders on ,

T the set of topologies on ,

T_0 the set of T_0 - topologies on

Theorem 1.2.11 Let be a finite set with elements. Then

(a) There is a one - to - one correspondence between and T .
(b) There is a one - to - one correspondence between and T_0 .

Proof

(a) Suppose that is a quasiorder on . Define a topology on which has a basis of open sets of the form for . This is indeed a basis, since . Thus, there corresponds a topology to such given quasiorder .
Conversely, assume that is a given topology on . Then define a relation on by:
if and only If , where .
Clearly, the relation is quasiorder on . Thus, from a topology on we obtain a quasiorder on .

(b) Using the correspondence in (a), it suffices to show that a partial order on corresponds to a T_0 –space and vice - versa.

Let be a partial order on and with . Then either and are incomparable or , or .

Case 1: and incomparable. Then and .

Case 2: . Then and .

Case 3: . Then and .

Thus in all cases we have T_0 –space.

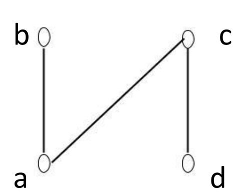
Conversely, suppose that a T_0 – topology is given on , and let with and . Then belongs to every open set containing and belongs to every open set containing . Therefore , hence is antisymmetric, that is, is a partial order. ⊠

Remark 1.2.12

1. The proof of Theorem 1.2.11 is not valid for *finite* sets, because two correspondences in (a) might no longer be inverses of each other: arbitrary intersections of open sets are not open, in general.
2. For an infinite set , the above correspondence holds only for Alexandroff spaces on .
3. In connection to this theorem, Proposition 1.2.10 says that the only poset (finite or infinite) corresponding to T_1 – space is the antichain and that every finite poset has a maximal element!

Example 1.2.13 (Illustrating Theorem 1.2.11(b))

Let and , , . This poset has the following diagram:



From this poset, we have the corresponding T_0 – Alexandroff space

{ }

with the *minimal* basis { }.

Now, let's revers the process: Consider T_0 – Alexandroff space , where

{ }.

Then we will find the specialization order on :

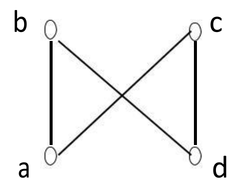
if and only if

It is easily found that , , and .

Thus we have

, , , .

The corresponding poset on has the diagram



Definition 1.2.14 A topological space is called a *locally Sierpinski space*, if every point has a neighborhood homeomorphic to a Sierpinski space.

This means that for , there exists , , called Sierpinski set, such that the relative topology on is a copy of Sierpinski topology.

Note that Sierpinski sets are open and together with isolated points, they form a basis for open subsets of .

The following can be found in [6].

Example 1.2.15 A locally Sierpinski space is T_0 – Alexandroff space.

Example 1.2.16 If is a locally Sierpinski space and is a discrete space, then is a locally Sierpinski space.

1.3 **Minimal open sets**

When we shall study Alexandroff spaces in Chapter 2, we shall need minimal open sets. This section is dealing with those sets. For details, see [20].

Let be a topological space.

Definition 1.3.1

A nonempty subset of is said to be a minimal open set if any open set contained in is the empty set or itself.

Proposition 1.3.2

(a) Let be a minimal open set and be an open set. Then or .

(b) Two minimal open sets are either disjoint or coincide.

Proof

(a) Assume, to the contrary, that
Since is minimal and by Definition 1.3.1, we must have , hence .

(b) Given two minimal open sets and , if , then by (a) we have and , so . ⊠

Proposition 1.3.3 Let be a minimal open set and . Then for any neighborhood of .

Proof Assume, to the contrary, containing . . By hypothesis is an open set This Thus, and a minimal open set. ⊠ contradicts the fact that is

Proposition 1.3.4 Let be a minimal open set.
Then for .

Proof First, we have , by Proposition 1.3.3. But, as is an open neighborhood of , we have . Hence we get
. ⊠

Proposition 1.3.5 Let be a minimal open set and let . Then or for any neighborhood of .

Proof This follows from Proposition 1.3.2. ⊠

Corollary 1.3.6 Let be a minimal open set and .
Let . Then or .

Proof Put . If , for any open neighborhood of , then . Therefore , otherwise there exists an open neighborhood of such that then we have ⊠

Proposition 1.3.7 Let be any nonempty *finite* set. Then there exists at least one *finite* minimal open set such that .

Proof If is a minimal set, we may have . So assume that is not a minimal open set. Then there exists a finite open set such that .

If is a minimal open set, we may set . If is not a minimal open set, there exists a finite open set such that . Containing this process, we obtain a sequence of open sets

But was finite, so this process repeats only finitely. Hence, eventually, we get a minimal open set for some positive integer , and proof is complete. ⊠

Definition 1.3.8

A topological space is said to be a *locally finite space,* if each of its elements is contained in a finite open set.

Proposition 1.3.9 Let be a locally finite space and a nonempty open set. Then there exists at least a *finite* minimal open set such that .

Proof Since is nonempty, it contains an element . is locally finite by hypothesis, hence there is a finite open set such that . Because and is finite, it follows that we have a minimal open set such that by Proposition 1.3.7. ⊠

Corollary 1.3.10 Let be a collection of nonempty sets and be a nonempty finite open set. Then is a finite open set. ⊠

Corollary 1.3.11 Any locally finite space is an Alexandroff space. ⊠

Chapter 2

ALEXANDROFF SPACES

As we mentioned in Chapter 1, Alexandroff spaces were introduced by P.S.Alexandroff (see [2]) under the name *discrete* spaces as topological spaces in which arbitrary intersections of open sets are open. Discrete spaces today are those special Alexandroff spaces in which every point is isolated; hence every set is open and closed. Accordingly, finite spaces and locally finite spaces (those in which each point has a finite open neighborhood) are clearly Alexandroff spaces.

In this chapter, we discuss the basic properties of Alexandroff spaces. Then we relate preorders (quasiorder) on a set to Alexandroff spaces, and partial orders to T_0 – Alexandroff spaces. Finally, we show how to construct new Alexandroff spaces from given ones. To do this, we mainly make use of [14] as well as [18], [7], [22], and [3].

2.1 **Topological *properties* of Alexandroff spaces**

In this section, we study spaces that have topologies with a stronger condition, namely, arbitrary intersections of open sets are open. This condition is a big restriction, since important spaces such as Euclidean spaces do not satisfy this property. Thus, for an Alexandroff space to be a mathematically interesting, it is crucial that it possesses properties that are not necessarily shared by a standard topological space.

Definition 2.1.1 Let be a topological space. Then is called an Alexandroff space, or sometimes A - space, if arbitrary intersections of open sets are open.

An immediate consequence of this definition is the following.

Proposition 2.1.2 Any discrete topological space is an Alexandroff space.

Proof This is obvious, since in a discrete space any subset is open.

The following result is also clear.

Proposition 2.1.3 A metric space is an Alexandroff space if and only if has the discrete topology.

Proof Suppose that is an Alexandroff space and let . Then the open balls

(−) are open in , where is a natural number. Since is Alexandroff, any

intersection of (−) is open, so (−) is open. But by the property

of the metric, (−) . Thus, we have shown that one – point sets, singletons, are open. Hence has the discrete topology.

Conversely, if has the discrete topology, then it is Alexandroff by Proposition 2.1.2. ⊠

Definition 2.1.1 is not too useful for proving theorems about Alexandroff spaces. We will use a different, yet equivalent definition in terms of minimal open neighborhoods. Recall that in a topological space , a neighborhood of a point is a set containing an open subset which contains .

Theorem 2.1.4 is an Alexandroff space if and only if each point in has a minimal open neighborhood.

Proof Suppose that is an Alexandroff space and let be any element in . Let . Take for . Then is an open neighborhood of , because is Alexandroff. On the other hand, by its definition is a minimal open neighborhood of .

Conversely, assume that each has a minimal open neighborhood . We prove that is Alexandroff space. Let , where is open in for each . If , then is open and we are done. If , then has at least one element; so for , we have for all . Hence, for all because is the minimal open neighborhood of .

Thus, we have . This means that open set is an open set, since it contains an is Alexandroff. ⊠
around each of its points. Therefore,

We denote by the set of all open neighborhoods of . Thus , that is, is the minimum element of ordered by inclusion.

Remark 2.1.5 Let be a topological space and let be a subset of . Then the closure of is defined by the set

When , we will denote simply by .

Note that for every , we have

.

The following fact gathers some easy properties of Alexandroff spaces (see [19]).

Fact If is an Alexandroff space, then for all and we have:

(1) .
(2) .
(3) .

Also we recall that a basis for a topology is *minimal* if it does not contain any basis as a subfamily.

Theorem 2.1.6 (see [5]). Let be an Alexandroff space. Then the basis is the unique minimal basis for .

Proof Fact (1) states that is a basis for . We now prove that is the unique minimal basis of . So let be any basis of . We show that . For , we have for some , by hypothesis that is a basis. Thus, there exists such that ; consequently, by

Fact (2). Since , we have . So we have proved that for every and hence . That shows the minimality of , since every subbase is in fact equal to . That proves also that is the unique minimal basis, because , and so is not minimal, for every basis ⊠

We now construct some examples of Alexandroff spaces (see [14]).

Example 2.1.7 Take and , where is the set of real numbers and is the set of integers. Then is an Alexandroff space with where . Note that for any two minimal open neighborhoods , we have that and are disjoint.

Example 2.1.8 (An Alexandroff space on)

Take and let }. Here is the closed ball with center 0 and radius , and . For is a minimal set in containing . Then is a basis for an Alexandroff topology .

2.2 **Alexandroff spaces, Quasiorder and partial orders**

In this section, we relate Alexandroff spaces to quasiordered sets, and T_0 – Alexandroff spaces to partially ordered sets. We first give two definitions.

Definition 2.2.1 Let and be quasiordered (or partially ordered) sets. A map is order – preserving if implies .

Definition 2.2.2 Let and be topological spaces. A function is continuous if is open in for each open in .

Proposition 2.2.3 An Alexandroff space admits a quasiordered structure .

Proof For every , let . Then is open, since is Alexandroff. Now define the relation on by

if and only if .

Since , for each , we have , so is reflexive. Let and , then and imply ; so , showing that is transitive. Thus, to each Alexandroff space there corresponds a quasiordered structure .

Now we show that there corresponds an Alexandroff space to a given quasiordered set.

Proposition 2.2.4 Let be a quasiordered set and define . Then

(a) The family forms a basis on .
(b) The topology generated by that basis is Alexandroff.

Proof (a) Clearly since . Now if , i.e. and , then and . Thus, we have and , hence is basis.

(b) For every collection of open sets, let Then for any , we have for all . This implies that for all , that is, we have . Thus . Hence , showing that is Alexandroff. ⊠

In the context of quasiordered sets, we can give another proof of Theorem 2.1.6.

Proposition 2.2.5 Let be an Alexandroff space. Then for every , we have

In other words, is the smallest open set containing . In this case, is called the minimal basis for .

Proof For every , containing , write , where are basic sets. Then there exists such that , i.e. . The transitivity tells us that . Also clearly we have . Hence ⊠

Now we connect T_0 – Alexandroff spaces and posets.

Proposition 2.2.6 Let be a nonempty set. Then is a poset if and only if the corresponding topology on is T_0.

Proof We have and if and only if and , so . By Proposition 2.2.4, and are the smallest open sets containing and , respectively. Thus, is T_0 if and only if is a poset. ⊠

We put things together to obtain the following conclusion.

Theorem 2.2.7 For a set , the Alexandroff space topologies on are in 1 -1 correspondence with the quasiorders on . The topology on corresponding to is T_0 if and only if the relation is a partial order on . ⊠

Given an Alexandroff space , will denote the unique minimal basis. The following concern T_0 – Alexandroff spaces.

Lemma 2.2.8 If is T_0 – Alexandroff space, then , which associates to every , is a bijection.

Proof Note that the definition gives surjectivity of . Now, suppose . Then we have, for every since and are minimum in and . Hence, and and so by the fact that is T_0.

Theorem 2.2.9 Let and be two T_0 – Alexandroff spaces. The following assertions are equivalent:

(a) and are homeomorphic.

(b) There exists an order – isomorphism (with respect to inclusion) between and .

The proof of this theorem can be found in [19].

Apart from these correspondences, we have the following relation between Alexandroff spaces.

Proposition 2.2.10 Let and be Alexandroff spaces, i.e. quasiordered sets. Then a function is continuous if and only if it is order – preserving.

Proof Let be continuous and suppose that . Then and we have ( ) since is continuous. Thus, , which means that , so is order – preserving.

Conversely, let be an open set in . If , then . If , then and thus by hypothesis, which implies that , so that . Therefore, is the union of these and hence is open. ⊠

2.3 **New Alexandroff Spaces From Old Spaces**

In this section, we will construct new Alexandroff spaces from given ones. For more details, see [23], for example.

Theorem 2.3.1 If and are Alexandroff spaces, then product space is also an Alexandroff space, with , where and , respectively.

Proof

The space has a basis . Let . Then is in . We claim that is the minimal open set in containing . To see this, if , then and , so and . Therefore, . Thus, is minimal. To show is Alexandroff with this basis , take any and let be any open set in containing . Then where . But must be in for at least one , which means that . Hence, is a minimal open set containing . Thus, as is arbitrary, we have shown that each point of has a minimal open set, in other words, is Alexandroff and . ⊠

The following can be proved by using induction on and applying Theorem 2.3.1.

Corollary 2.3.2 If are Alexandroff spaces, then so is . Furthermore, , where .

Definition 2.3.3 Let be a topological space and let be a subset of . Then the collection defines a topology on , called the *induced topology*; is called a subspace of .

Theorem 2.3.4 If is a subspace of the Alexandroff space , then is an Alexandroff space, and , where .

Proof Let and suppose that is an open neighborhood of in . Then by the definition we have where is open in . Since is Alexandroff, this means that , so that . Hence is an Alexandroff space with . ⊠

Remark 2.3.5 Given a nonempty set , we can introduce three mutually equivalent but slightly different viewpoints of quotient sets.

1. The quotient set / , associated to a surjective function onto a nonempty set , is defined to be / .
2. The quotient set / , associated to an equivalence relation on , is the set of equivalence classes / where .
3. The quotient space / , associated to a partition of , is defined as / .

We use these remarks to define quotient spaces.

Definition 2.3.6 Let be a topological space, any nonempty set, and let be *surjective* function. Then

(a) The *quotient topology* / on , induced by and , is defined by / and is called the quotient space.

(b) A function is called a *quotient map* if it has the property that if and only if

Theorem 2.3.7 If / is a quotient space of the Alexandroff space , then / is an Alexandroff space.

Proof Let / be the quotient map. Consider an arbitrary intersection of open sets in / . Then we have

Since is a quotient map, it follows that is an open set in for each . Hence is open in because is Alexandroff, and therefore is open in / by the definition of quotient space. ⊠

Example 2.3.8 If is an Alexandroff space, then we can define an equivalence relation on by if and only if . We can then form the quotient space / , which is an Alexandroff space by Theorem 2.3.7.

Definition 2.3.9 Let be an Alexandroff space and let . Then is called *irreducible* if implies .

Theorem 2.3.10 / is a discrete space if and only if is irreducible for

Proof Suppose that / is discrete. If is the quotient map, then is open in for each . Then , because . Now if , then , which means . Therefore, , which gives . Now suppose , then . So we must have because , hence is irreducible.

Conversely, assume that is irreducible for all . Let , then , so . This gives . Now if , then , and since is irreducible we must have . Therefore, so that . Hence, we have , open in . This means that is open in / and so / is discrete. ⊠

For general topological spaces, the spaces that satisfy the Hausdorff property are the nicest spaces to study. However, the Hausdorff Alexandroff spaces are not very interesting to study as the following theorem shows.

Theorem 2.3.11 is a Hausdorff Alexandroff space if and only if for in , we have .

Proof If is Hausdorff, we can find disjoint open sets and of such that and . Then and , so and must be disjoint.

Conversely, to show that is Hausdorff, we can take and to be the disjoint open sets containing and , respectively. ⊠

Corollary 2.3.12 is a Hausdorff Alexandroff space if and only if is discrete.

Proof Suppose that is Hausdorff. Then we claim that . To see this, let . Then , which means that . And since , we must have by Theorem 2.3.11. Hence, is open in , so must be discrete.

Conversely, if is discrete, then it is clearly Hausdorff. ⊠

Chapter 3

ALEXANDROFF SPACES AND DIGRAPHS

In the previous chapters; we introduced some basic notions about topological spaces and Alexandroff spaces, and we related them with preorders (quasiorders) and specialization order. However, for our thesis these are not enough, so we will give some more details in this chapter.

In the first part of this chapter we will mention some well-known definitions and concepts about graphs; for example see [14], [10] and [4]. In the second part and third part, we relate some concepts of general topology and Alexandroff spaces with graphs and digraphs as it can be found in [16] and [15].

3.1 Some basic notions about graphs

Definition 3.1.1 A graph is a mathematical structure consisting of two finite sets and . The elements of are called *vertices* (or *nodes*), and the elements of are called *edges*. The set of all vertices is denoted by and the set of all edges is denoted by . Each edge has a set of one or two vertices associated to it, which are called its endpoints. An edge is said to join its endpoints.

Definition 3.1.2 A vertex joined by an edge to a vertex is said to be a *neighbor* of . The *(open) neighborhood* of a vertex in a graph denoted , is the set of all neighbors of and the *(closed) neighborhood* of is given by .

Definition 3.1.3 A *directed edge* (or *arc*) is an edge, one of whose endpoints is designated as the tail and whose other endpoint is designated as the head. An *arc* is said to be *directed* from its tail to its head.

Definition 3.1.4 An edge between two vertices creates a connection in two opposite senses at once. Assigning a direction makes one of these senses forward and the other backward. In a line drawing, the choice of forward direction is indicated by placing an arrow on an edge.

Definition 3.1.5 A *directed edge* (or *arc*) is an edge, one of the whose endpoints is designated as the tail, and whose other endpoint is designated as the head.

Definition 3.1.6 A *directed graph* (or *digraph*) is a graph each of whose edges is directed. A graph is *simple* if it has neither self-loops nor multi-edges. A *digraph is simple* if it has neither self-loops nor multi-edges.

Definition 3.1.7 *Adjacent vertices* are two vertices that are joined by an edge and *adjacent edges* are two edges that have an endpoint in common. If vertex is an endpoint of an edge , then is said to be *incident* on , and is *incident* on .

Definition 3.1.8 A *path* is a non-empty graph of the form and , where the are all distinct. The vertices and are linked by and are called its end vertices or ends; the vertices are the inner vertices of . The number of edges of a path is its length, and the path of length is denoted by . A non-empty graph is called *connected* if any two of its points are linked by a path in .

Definition 3.1.9 A graph is a subgraph of (with) if and .

3.2 **Topological spaces and Alexandroff spaces with graphs**

In this part; we will relate some Topological notions and Alexandroff spaces with graphs. One can find it in [15].

Definition 3.2.1 Let be a graph without isolated vertex and let be the set of all vertices adjacent to , where if and only if , for all and for all . We have a subbasis for a topology on , when and it is called *graphic* topology on G.

Proposition 3.2.2 Suppose that is a graph. Then is an Alexandroff space.

Proof We have to prove that arbitrary intersection of members of is open. Let . If , then for each . Hence, for each So, . Since is locally finite, and so are finite sets. If is infinite, then is empty. But is finite, then is the intersection of finitely many open sets. Hence, it is open. ⊠

Definition 3.2.3 Let be a graph. The minimal basis for the topological graph is denoted by where is the smallest open set containing , i.e. it is the intersection of all open sets containing .

Proposition 3.2.4 Let be a graph, then and is finite for every .

Proof Since is the smallest open set containing and is a subbasis for , we have for some subset of ; this implies that for each . Therefore . So . By the definition of , we have is finite for every . ⊠

Corollary 3.2.5 Let be a graph; for every , if and only if . ⊠

Definition 3.2.6 Let be a topological space. A subset of is clopen, if it is both open and closed in .

Proposition 3.2.7 Let be an Alexandroff space; if there is a point such that is both maximal and minimal in { }, then is clopen and is disconnected.

Proof We have to show that is closed or equivalently . Suppose , then there exists such that . So we get and

. By minimality of , we get . So we have and by maximality of , we get . Hence, , hence, . ⊠

Definition 3.2.8 A topological space is called *graphic*, if there is some (locally finite) graph with vertex set and without isolated vertex such that .

Proposition 3.2.9 Let be a finite topological space and let be the smallest open set containing for every . If for every , or then is graphic.

Proof We construct a graph as follows , for any . For every , let and be the smallest set containing in and the set of all adjacent vertices to in , respectively. We have to prove that [by Definition 3.2.8]. Let , we have { } for every . Therefore, if and only if if and only if { }. Suppose that . Hence, , otherwise; and so which is a contradiction. Therefore, but implies that which is a contradiction. So .

Conversely, if then . So if for some , then and so . This implies that and so . Therefore, . ⊠

Definition 3.2.10 Let be an Alexandroff topological space and be the smallest open set containing for every . Then is dense in if and only if for every .

In particular, . Since is a non-empty open set in , we have . Let , then .

Theorem 3.2.11 Let be an Alexandroff topological space, and . Then

1) If is minimal dense subset in then there exists a surjective function such that for every . In particular,

.

2) Conversely; if is a function such that for every , then is a minimal dense subset in . Specially, if and are minimal dense subsets in , then we have | | | |, where | | denotes the cardinality of .

Proof

1) By minimality of elements of , the intersection of every pair of distinct elements of is empty. Our claim is that has a single element for each . Since , there exists some , so and by minimality of , we have . Assume, to the contrary that . Then . Therefore, Hence, which contradicts the minimality of . Let be a single element of , for every . Suppose that ; we will show that , which implies and this will prove that is surjective. Assume, to the contrary, ; so there exists such that . If , then by above claim and so . Therefore, implies that for every . Thus, which is a contradiction.
2) For every , there exists an element such that and . Therefore, and so is dense in . Now, suppose that and . Let and . Then there exists such that ( ) . On one hand, implies that and on the other hand, we have ( ) which implies and so ( ) , which is a contradiction. Now, if is a minimal dense subset in , then there

exists a surjective function such that , for every by (1). Hence, must be injective because if ( ) for some , then and so by their minimality.

Therefore, | | | | is constant. ⊠

3.3 Alexandroff spaces with directed graph (digraph)

Given a set (not necessarily finite), let and denote the set of all topologies on , transitive digraphs and the set of all Alexandroff respectively.

We will prove that each transitive digraph determines a unique topology on , and conversely. It can be found in [16].

Theorem 3.3.1 There is a one - to - one correspondence between and .

Proof Given we associate with a digraph as follows. The vertex set of is and its edge set is defined by: for any two and in , will be adjacent to if and only if belongs to every *open* set containing . Clearly, is transitive and uniquely determined by the topology .

Conversely, let be any element of . Now the family of down-sets forms a base for a topology on , where , here represents edge in . We claim that is a member of .

Let be a family of open sets and ∩ . We want to show that for each . Let . If , then for each . Suppose . Since and being basic set, , we get for some . Further, as is transitive, and implies that , hence, we have . Thus, for each .

Hence, for each , the open set , which contains , is a subset of . This proves that is open. Therefore, we have that .

Now, by the construction of , it follows that is unique. So we have the following configurations:

or

We complete the proof by showing that the digraph coincides or is the same as the digraph induced by the topology ; this means that we have indeed a one – to – one correspondence between and .

Let be given and take . Since is transitive, if , then . In other words, belongs to whenever belongs . Since the family is a base for , it follows that belongs to every open set containing . This implies that is an edge in , i.e. .

For the converse, suppose that . Then belongs to because belongs to every open set containing . Hence, and this proves that , in other word, . ⊠

Remark 3.3.2 By definition, a digraph has no loops! A relation is reflexive if every vertex has a loop. Obviously, it makes no difference in the number of transitive relations whether every point (vertex) has a loop or no point does. Hence we have

Corollary 3.3.3 On a set (not necessarily finite), there are the same number of transitive relations and Alexandroff topologies.

In view of Theorem 3.3.1, every transitive digraph on a set induces a unique A - topology on , we denote it by ; conversely, every A – topology on induced a unique transitive digraph on , we denote it by .

Definition 3.3.4 The "converse" digraph of a digraph is the digraph having the same vertex set as that of such that for any two , is adjacent to in if and only if is adjacent to in . Clearly, if is transitive, then is transitive.

It is evident that for an A – topology on , the family consisting of all *closed* sets of also forms a topology on which is an A – topology. Call this topology on the *closed topology* induced by .

Theorem 3.3.5 Let be a transitive digraph and be its converse. Then

1) The transitive digraph on induced by the topology is ; that is, ( ) , and
2) The topology on induced by is the closed topology ; that is .

Proof We prove only (1); (2) can be proved on similar lines. Consider . Then is in and hence is in every open set in the topology containing . This implies that belongs to every closed set in (i.e., an open set in) containing . Indeed, suppose where is a closed set in and . Then , the complement of , is open in , and , which is not true. This proves that every edge in is an edge in . Similarly, one can show that every edge in is an edge in . Thus, we have . ⊠

Definition 3.3.6 Let and be two digraphs. A function is said to be a homomorphism if and only if for with , implies .

Theorem 3.3.7 Let and be two transitive digraphs and be a function. Then is homomorphism if and only if it is continuous from the topological space ( ) into ( ).

Proof CN . Suppose is a homomorphism of into . As we have seen in Theorem 3.3.1, the family of sets forms a base for , where , . To prove that is continuous, it is sufficient to show that, for every and for every , .

Let . Let . If , then . Suppose now that . Then as and is a homomorphism, . Since , it follows that either or . Now, implies as and is transitive. In both cases, . Thus, . This proves that , for all .

CS . Assume that is continuous from ( ) into ( ). Choose with , and . Let be an open set in containing . By the continuity of , is open, further it contains . Hence it should contain the point as , i.e. . Thus, whenever an open set in contains , it also contains . This implies that . This proves that is homomorphism. ⊠

Corollary 3.3.8 Let and be two transitive digraphs with the same vertex set . Then is a subgraph of if and only if is stronger than .

Proof CN . Suppose that is a subgraph of . Then the identity map is homomorphism from into . By Theorem 3.3.7, it follows that is a continuous function of ( ) onto ( ). Hence is stronger than .

CS . Assume that is stronger than . Then as the identity map from ( ) onto ( ) is continuous, by Theorem 3.3.7, it is a homomorphism of into . Further, as it is one – to – one, it follows that is a subgraph of . ⊠

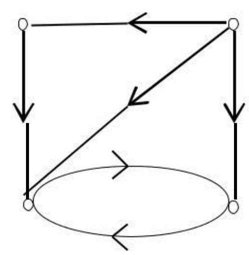
Finally, we can state the following obvious result:

Corollary 3.3.9 Let and be two transitive digraphs. is isomorphic to if and only if the topological space ( ) is homomorphic to ( ). ⊠

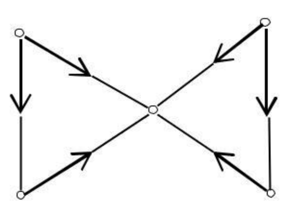
Example 3.3.9 Let and { }. Then we can determine associated with as follows: Vertex set is . Find its edge set .

. since belongs to every open sets and containing .

. By the same argument and . The other edges can be found similarly. Thus, we get the following transitive digraph , uniquely determined by the given :



Conversely, let and consider the graph (transitive) :



Determine the topology , uniquely determined by . For this we must first find . , , , , . Then is the topology associated with .

{ }.

Conclusion

T_0 – Alexandroff spaces can be investigated in terms of general topological concepts such as connected, locally path-connected, regular, separable, first countable, second countable, compact, para-compact spaces. Such an approach is given in [3].

Alexandroff spaces can also be considered as Functional Alexandroff Spaces and dynamical systems (see, for example, F.A.Z. Shirazi and N. Golistani, Functional Alexandroff Spaces, Hacettepe Journal of Mathematics and Statistics, Vol. 40 (4), (2011), 515-522).

Alexandroff spaces are also studied as a class of T_0 – A – spaces called upper bounded. This class contains the class of Artinian A – spaces (see, for example, H. Mahdi, On Upper Bounded T_0 – Alexandroff Spaces, Int.J. Contemp. Math. Sciences, Vol.9, 2014, no.81, 361-374).

Recently, as extensions of Alexandroff Spaces, Bi – Alexandroff Spaces are studied in connection with regular bitopological spaces (see, for example, Matutu, P., Bi – Alexandroff Spaces, Quaestiones Mathematicae, Vol. 30, Number 1, March 2007, pp. 57-65(g)).

Lastly, intensive development and research have been conducted in the field of Alexandroff Spaces in graph theory, and in particular in digital topology.

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