YAŞAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MASTER THESIS

SOME TYPES OF CONTINUITY ON SPACES WITH MINIMAL STRUCTURES

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

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ABSTRACT

SOME TYPES OF CONTINUITY ON SPACES WITH MINIMAL STRUCTURES

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This thesis consists, essentially, of five chapters.

In the first chapter, the topic of the thesis is introduced and in the second chapter, in order to clarify the reading of the thesis, some types of open sets and some types of continuity in topological spaces are introduced.

In the third chapter, after giving necessary knowledge on spaces with minimal structures the interior and closure operators' properties are investigated in those spaces.

In the fourth chapter, we point out different kinds of open sets' definitions and their properties in these spaces. Also, while studying relationships between these sets a number of original illustrating examples are given.

In final chapter, we deal with some types of continuity between spaces with minimal structures and we examine the their fundamental properties and relations between each other.

Keywords: m-structure, m-open sets, m-α-open sets, m-semiopen sets, m- preopen sets, m-β-open sets, M-continuity, M-α-continuity, M-semicontinuity, M-precontinuity, M-β-continuity.

ÖZET

MİNİMAL YAPILI UZAYLARDA BAZI SÜREKLİLİK TÜRLERİ

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Bu tez esas olarak beş bölümden oluşmaktadır.

Birinci bölümde tez konusu tanıtılmış, ikinci bölümde ise tezin anlaşılabilir olması için topolojik uzaylardaki bazı açık küme türleri ve bazı süreklilik türleri tanıtılmıştır.

Üçüncü bölümde, minimal yapılı uzaylar üzerine bilgi verilerek bu uzaylardaki iç ve kapanış operatörlerinin özellikleri incelenmiştir.

Dördüncü bölümde, minimal yapılı uzaylardaki çeşitli açık küme türlerinin tanımlarına ve temel özelliklerine yer verilmiştir. Ayrıca, bu kümeler arasındaki ilişkiler incelenerek çalışma özgün örneklerle desteklenmiştir.

Son bölümde minimal yapılı uzaylar arasındaki bazı süreklilik türleri ele alınarak bunların temel özellikleri ve birbirleriyle ilişkileri çalışılmıştır.

Anahtar sözcükler: m-yapı, m-açık kümeler, m-α-açık kümeler, m-yarıaçık kümeler, m-ön açık kümeler, m-β-açık kümeler, *M*-süreklilik, *M*-α-süreklilik, *M*-yarısüreklilik, *M*-önsüreklilik, *M*-β-süreklilik.

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İlay BALKAN
İzmir, 2016

TEXT OF OATH

I declare and honestly confirm that my study, titled “SOME TYPES OF CONTINUITY ON SPACES WITH MINIMAL STRUCTURES” and presented as a Master’s Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefitedare listed in the bibliography, and that I have benefited from these sources by means of making references.

İlay BALKAN

26.01.2016

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INDEX OF SYMBOLS AND ABBREVIATIONS

Symbols	Explanations
$\wp(X)$	the power set of X
τ	topology
m_X	minimalstructure on X
$mcl(A)$	m-closure of A
$mint(A)$	m-interior of A
$\alpha(X)$	the family of all α-open set in X
$SO(X)$	the family of all semi-open set in X
$PO(X)$	the family of all pre-open set in X
$\beta(X)$	the family of all β-open set in X
$M\alpha(X)$	the family of m-α-open sets in X
$MSO(X)$	the family of m-semi-open sets in X
$MPO(X)$	the family of m-pre-open sets in X
$M\beta O(X)$	the family of m-β-open sets in X
$m\alpha cl(A)$	m-α-closure of A

INDEX OF SYMBOLS AND ABBREVIATIONS(continue)

Symbols	Explanations
$m\alpha int(A)$	m-α-interior of A
$mscl(A)$	m-semi-closure of A
$msint(A)$	m-semi-interior of A
$mpcl(A)$	m-pre-closure of A
$mpint(A)$	m-pre-interior of A
$m\beta cl(A)$	m-β-closure of A
$m\beta int(A)$	m-β-interior of A

1.INTRODUCTION

Maki (1996) introduced the concept of minimal structure which is more general than a topology, and using this concept he defined spaces with minimal structure. Moreover, he studied properties of closure and interior operators defined in those spaces. Then, Popa and Noiri (2000) defined M-continuous function's concept between spaces with minimal structures and obtained some characterizations and aspects of these functions. On the other hand, they gave the definitions of m-compactness and m-connectedness together with their properties.

Many mathematicians have defined some types of open sets and continuities which are generalizations of m-open sets and M-continuity, in spaces with minimal structures.

Min (2009) defined the concepts of m-semiopen sets, m-semi-interior and m-semi-closure operators; Min and Kim (2009) defined the concepts of m-preopen sets, m-pre-interior and m-pre-closure operators; Min (2010) defined the notions of m-α-open sets, m-α-interior and m-α-closure operators, and Nasef and Roy (2013) defined the concepts of m-β-open sets, m-β-interior and m-β-closure operators; and also they investigated some their fundamental properties.

Furthermore, people above have introduced M-semi-continuous, M-pre-continuous, M-α-continuous and M-β-continuous functions, and they obtained some characterizations of them. Then, they investigated the relationships between M-continuity and these new concepts.

In this thesis, we closely read all papers mentioned and provided original examples, and also proved where we encountered a gap given in the paper.

2.PRELIMINARIES

Throughout this chapter necessary topics are given.

Definition2.1: Let (X, τ) be a topological space and $A \subseteq X$.

a) A is called an α-open setif $A \subseteq Int(Cl(Int(A)))$ (Njastad, 1965)

b) A is called semi-open set if $A \subseteq CI(Int(A))$ (Levine, 1963)

c) A is called pre-open set if$A \subseteq Int(CI(A))$ (Mashhour et al.,1982)

d) A is called β–open if $A \subseteq CI(Int(CI(A)))$ (Abd El-Monsef et al. 1983**)**

The family of all α-open(semi-open, pre-open, β-open) sets in X is denoted by α(X)(SO(X), PO(X),β(X)).

Definition 2.2: Let (X, τ) be a topological space and $A \subseteq X$.

(1) The complement of an α-open set is said to be α-closed.

(Mashouret al, 1983)

(2) The complement of a semi-open set is said to be semi-closed.

(Crossley and Hildebrand, 1971)

(3) The complement of a pre-open set is said to be a pre-closed.

(El-Deeb et al,1983)

(4) The complement of a β-open set is said to be a β-closed.

(Abd El- Monsef et al,1983)

Definition 2.3: Let (X, τ) and (Y, σ) be two topological spaces, and let $f: (X, \tau) \longrightarrow (Y, \sigma)$ be a function.

a) f is called α-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists anα-open set U of X containing x such that $f(U) \subseteq V$.

(Mashhour, et al. 1983)

b) f is called $semi$-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a $semi$-open set U of X containing x such that $f(U) \subseteq V$.

(Levine, 1963)

c) f is called pre-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-open set U of X containing x such that $f(U) \subseteq V$.

(Mashhour, et al. 1982)

d) f is called β-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a β-open set U of X containing x such that $f(U) \subseteq V$.

(Abd El-Monsef et al., 1983)

3. MINIMAL STRUCTURES

Definition 3.1: A subfamily m_X of the power set $\wp(X)$ of a nonempty set X is called a minimal structure (briefly m-structure) on X if,

$$\emptyset \in m_X \text{ and } X \in m_X.$$

We denote a nonempty set X with a minimal structure m_X on it by (X, m_X). Each member of m_X is said to be m_X-open (briefly m-open) set and the complement of an m_X-open set is said to be m_X-closed (briefly m-closed) set.

(Maki,1996 and Popa and Noiri, 2000)

Remark 3.1: Let (X,τ) be a topological space. Then, the families $\tau, \alpha(X), SO(X), PO(X)$ and $\beta(X)$ are all minimal structures on X.

(Popa and Noiri, 2000)

Definition 3.2: Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-closure of A and the m-interior of A, denoted by $mcl(A)$ and $mint(A)$, respectively, are defined as follows:

$$mcl(A) = \bigcap\{F: A \subseteq F, X/F \in m_X\}$$

$$mint(A) = \bigcup\{G: G \subseteq A, G \in m_X\}$$

(Maki,1996)

Example 3.1: Let $X = \{a, b, c\}$ and let $m_X = \{\emptyset, \mathrm{X}, \{\mathrm{a}, \mathrm{b}\}, \{\mathrm{b}, \mathrm{c}\}\}$ be a minimal structure on X. Consider the set$A = \{a, b\}$ in X. Then, we have $mcl(A) = X$ and $mint(A) = \{a, b\}$.

Lemma 3.1: Let (X, m_X) be a space with a minimal structure m_X on X. For $A, B \subseteq X$, the following hold:

(1) $mcl(X\backslash A) = X\backslash mint(A)$ and $mint(X\backslash A) = X\backslash mcl(A)$,

(2) If $(X\backslash A) \in m_X$, then $mcl(A) = A$ and if $A \in m_X$, then $mint(A) = A$,

(3) $mcl(\emptyset) = \emptyset$, $mcl(X) = X$, $mint(\emptyset) = \emptyset$ and $mint(X) = X$,

(4) If $A \subseteq B$, then $mcl(A) \subseteq mcl(B)$ and $mint(A) \subseteq mint(B)$,

(5) $A \subseteq mcl(A)$ and $mint(A) \subseteq A$,

(6) $mcl\big(mcl(A)\big) = mcl(A)$ and $mint\big(mint(A)\big) = mint(A)$,

(Maki,1996)

Proof:

(1) Since $mint(A) = \cup \{G: G \subseteq A \; and \; G \in m_X \}$, we have $X \backslash mint(A) = \cap \{X/G : G \subseteq A and \; G \in m_X\} = \cap \{X/G : X/A \subseteq X/G \; and \; G \in m_X\} = mcl(X \backslash A)$. The proof of $X \backslash mcl(A) = mint(X \backslash A)$ is done by the similar way.

(2) The proofs are clear from the definitions of m-closure and m-interior.

(3) Since $\emptyset, X$ are both m-open and m-closed, it is obvious by(2).

(4) By hypothesis, we have $mcl(B) = \cap \{F: A \subseteq B \subseteq F$ and $X \backslash F \in m_X\} \supseteq mcl(A) = \cap \{H: A \subseteq H$ and $X \backslash H \in m_X\}$. The other proof is done by the similar way.

(5) It is obvious by the definitions of m-closure and m-interior.

(6) From (5) and the definition of m-closure, we have $mcl(mcl(A)) = \cap \{F: A \subseteq mcl(A) \subseteq F$ and $X \backslash F \in m_X\} = mcl(A)$. By the similar way, $mint\big(mint(A)\big) = mint(A)$ is proved.

The following examples show that the converse implications of (2) and (4) in Lemma 3.1 are not true in general.

Example 3.2:

(1) Let $X = \{a, b, c\}$ and let $m_X = \big\{\emptyset, X, \{a, b\}, \{b, c\}, \{c\}, \{a\}\big\}$ be a minimal structure on X. Consider two sets $A = \{b\}$and $B = \{a, c\}$. Then, we have $mcl(A) = A$ but $X \backslash A \notin m_X$. Also, we get $mint(B) = B$ but$B \notin m_X$.

(2) Let X = $\{1, 2, 3\}$ and let $m_X = \big\{\emptyset, X, \{1,2\}, \{2,3\}\big\}$ be a minimal structure on X. Consider three sets A = $\{1\}$, B = $\{2\}$ and C = $\{2,3\}$. Thus, $mcl(A) = \{1\} \subset mcl(B)$ =X but A $\nsubseteq$ B. Also, $mint\ (A) = \emptyset \subset mint(C) = \{2,3\}$ but A $\nsubseteq$ C.

Lemma 3.2: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then $x \in mcl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_x$ containing x.

(Popa and Noiri,2000)

Proof:

Necessity. Suppose that there is an m-open set U containing x such that $U \cap A = \emptyset$. Then, we have $A \subseteq X \backslash U$ and $X \backslash$U is m-closed. Thus, $mcl(A) \subseteq mcl(X \backslash U) = X \backslash U$ by Lemma3.1(2) and (4). Since $x \notin X \backslash U$, then $x \notin mcl(A)$.

Sufficiency. Assume $x \notin mcl(A)$. By the definition of m-closure, there is an m-closed set F not containing x such that $A \subseteq F$. Thus, $(X \backslash F) \cap A = \emptyset$ for an m-open set $X \backslash F$ containing x.

Definition 3.3: A function $f: (X, m_X) \rightarrow (Y, m_Y)$, where (X, m_X) and (Y, m_Y) are two spaces with minimal structures m_X and m_Y on X and Y, respectively, is said to be M-continuous if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_x$ containing x such that $f(U) \subseteq V$.

(Popa and Noiri,2000)

Theorem 3.1: For a function $f: (X, m_X) \rightarrow (Y, m_Y)$, the following properties are equivalent:

(1) f is M-continuous,

(2) $f^{-1}(V) = mint(f^{-1}(V))$ for every $V \in m_Y$,

(3) $f(mcl(A)) \subseteq mcl(f(A))$ for every $A \subseteq X$,

(4) $mcl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$ for every $B \subseteq Y$,

(5) $f^{-1}(mint(B)) \subseteq mint(f^{-1}(B))$ for every $B \subseteq Y$,

(6) $mcl(f^{-1}(K)) = f^{-1}(K)$ for every $K \subseteq Y$such that $(Y \backslash K) \in m_Y$.

(Popa and Noiri, 2000)

Proof:

(1)⟹(2) By Lemma 3.1(5), we have $mint(f^{-1}(V))\subseteq f^{-1}(V)$. So, we must show that $f^{-1}(V) \subseteq mint(f^{-1}(V))$. Let $V \in m_Y$ and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is M-continuous, there exists an m-open set Ucontaining x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$. Therefore, we have $x \in mint(f^{-1}(V))$.

(2)⟹(3) Let $A \subseteq X$. Assume that $x \in mcl(A)$ and $V \in m_Y$ containing $f(x)$. Thus, $x \in f^{-1}(V) = mint\big(f^{-1}(V)\big)$ from (2). By the definition of m-interior, there exists an m-open set U containing x such that $U \subseteq f^{-1}(V)$. Since $x \in mcl(A)$, then we get $U \cap A \neq \emptyset$. Hence, $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. That is $V \cap f(A) \neq \emptyset$. This shows that $f(x) \in mcl\big(f(A)\big)$.

(3)⟹(4) Let $B \subseteq Y$. By (3),we have $f(mcl(f^{-1}(B))) \subseteq mcl(f\big(f^{-1}(B)\big)) \subseteq mcl(B)$. Thus, we have $mcl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$.

(4)⟹(5) The proof is obvious from (4) and Lemma 3.1(1).

(5) ⟹ (6) Let K be an m-closed subset of Y. By (5), we have $f^{-1}(mint(Y\backslash K)) \subseteq mint(f^{-1}(Y\backslash K))$. Since K is m-closed, we get $f^{-1}(mint(Y\backslash K)) = X\backslash f^{-1}(K)$ and $mint\big(f^{-1}(Y\backslash K)\big) = X\backslash mcl(f^{-1}(K))$ from Lemma 3.1(1). This implies that $mcl(f^{-1}(K)) \subseteq f^{-1}(K)$. Also, we have $f^{-1}(K) \subseteq mcl(f^{-1}(K))$ by Lemma 3.1(5). Thus $(f^{-1}(K)) = mcl\big(f^{-1}(K)\big)$.

(6) ⟹ (1) Let $x \in X$ and $V \in m_Y$ containing $f(x)$. Then, we have $X\backslash f^{-1}(V) = f^{-1}(Y\backslash V) = mcl(f^{-1}(Y\backslash V) = mcl\big(X\backslash f^{-1}(V)\big) = X\backslash mint(f^{-1}(V))$ by (6) and Lemma 3.1(1). Since$f(x) \in V$, $x \in mint(f^{-1}(V))$. Hence, there exists an m-open set U containing x such that $U \subseteq f^{-1}(V)$. Thus, f is M-continuous.

Definition 3.4: A minimal structure m_X on a nonempty set X is said to have the property $(\mathcal{B})$ if the arbitrary union of m-open sets is$m$-open.

(Maki,1996)

Lemma 3.3: Let (X, m_X) be a space with a minimal structure m_X on X. Then, the following are eguivalent:

(1) m_X has the property $(\mathcal{B})$,

(2) If $mint(V) = V$, then $V \in m_X$,

(3) If $mcl(F) = F$, then $X \backslash F \in m_X$.

(Popa and Noiri, 2000)

Proof :

$(1) \Longrightarrow (2)$ Let $mint(V) = V$.By the definition of m-$interior$ and the property $(\mathcal{B})$, $mint(V)$ is m-open. Then, $V \in m_X$.

$(2) \Longrightarrow (1)$ Suppose that $U_i \in m_X$ for all $i \in I$. Let $V = \bigcup_{i \in I} U_i$. By Lemma3.1(5), we have $mint(V) \subseteq V$. So, we must show that $V \subseteq mint(V)$. Let $x \in V$. Then, there exists $i_0 \in I$ such that $x \in U_{i_0}$. Since U_{i_o} is m-open and $U_{i_0} \subseteq V$,then $U_{i_0} = mint(U_{i_0}) \subseteq mint(V)$ by Lemma 3.1(2) and (4). Thus, $x \in mint(V)$. Hence, we obtain $V \subseteq mint(V)$. By (2), we get $V \in m_X$.

$(2) \Longrightarrow (3)$ Let $mcl(F) = F$. Then, $X \backslash F = mint(X \backslash F)$ by Lemma 3.1(1). Thus, $X \backslash F \in m_X$ from (2).

$(3) \Longrightarrow (2)$ Let $mint(V) = V$. Then, we have $X \backslash V = mcl(X \backslash V)$ from Lemma 3.1(1). By (3), we have $V \in m_X$.

Corollary 3.1: Let (X, m_X) be a space with a minimal structure m_X satisfying the property $(\mathcal{B})$. For a function $f: (X, m_X) \rightarrow (Y, m_Y)$ the following are equivalent:

(1) f is M-continuous.

(2) $f^{-1}(V) \in m_X$ for every $V \in m_Y$.

(3) $X \backslash f^{-1}(F) \in m_X$for every subset F of Y such that $Y \backslash F \in m_Y$.

(Popa and Noiri, 2000)

The following example shows when the property$(\mathcal{B})$ does not hold, Corollary 3.1 may not be true.

Example 3.3: Let $X = \{a, b, c\}$ and $Y = \{1,2\}$. Consider two minimal structures defined as $m_X = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$ and $m_Y = \{\emptyset, Y, \{2\}\}$ on X and Y, respectively. Let $f: (X, m_X) \to (Y, m_Y)$ be a function defined by $f(a) = 1$ and $f(b) = f(c) = 2$. Then, f is M-continuous but $f^{-1}(\{2\}) = \{b, c\} \notin m_X$ when $\{2\} \in m_Y$.

4.DIFFERENT KINDS OF OPEN SETS ON SPACES WITH MINIMAL STRUCTURES

4.1 m-α-Open Sets

Definition 4.1.1: Let (X, m_X) be a space with a minimal structure m_X on X. A subset A of X is called an m-α-open set if $A \subseteq mint(mcl(mint(A)))$.The complement of an m-α-open set is called an$m$-α-closed set. The family of all m-α-open sets in X is denoted by $M\alpha(X)$.

(Min, 2010)

Remark 4.1.1: If the minimal structure m_X on a given non-empty set X is topology, then an m-α- open set is α-open.

(Min, 2010)

Proposition 4.1.1: Let(X, m_X) be a space with a minimal structure m_X on X. Then, every m-open set is m-α-open.

(Min, 2010)

Proof:

Let A be m-open. Then, we have $A = mint(A)$ from Lemma 3.1(2). Since $A \subseteq mcl(A)$, we get $A = mint(A) \subseteq mint(mcl(A)) = mint(mcl(mint(A)))$. Thus, A is m-α-open.

The following example shows that the converse implication of Proposition 4.1.1 is not true, in general.

Example 4.1.1: Let $X = \{1, 2, 3\}$ and let $m_X = \{\emptyset, X, \{1,2\}, \{2,3\}, \{1\}, \{3\}\}$be a minimal structure on X. Consider $A = \{1, 3\}$, then $mint\Big(mcl\big(mint(A)\big)\Big) = X$. Thus, A is m-α-open but it is not m-open.

Lemma 4.1.1: Let (X, m_X) be a space with minimal structure m_X on X and $A \subseteq X$. Then, A is an m-α-closed set if and only if $mcl(mint(mcl(A))) \subseteq A$.

(Min, 2010)

Proof:

Let A be m-α-closed. Then, we have $X \backslash A \subseteq mint(mcl(mint(X \backslash A)))$. By Lemma 3.1(1), we get $mint\Big(mcl\big(mint(X \backslash A)\big)\Big) = X \backslash mcl(mint(mcl(A)))$. Thus, $mcl(mint(mcl(A))) \subseteq A$. Converse implication is proved by the similar way.

Theorem 4.1.1: Let (X, m_X) be a space with minimal structure m_X on X. Any union of m-α-open set is m-α-open.

(Min, 2010)

Proof:

Let A_i be an m-α-open set for each $i \in I$. Then, we have $A_i \subseteq mint(mcl(mint(A_i))) \subseteq mint(mcl(mint(\bigcup_{i \in I} A_i)))$ for each $i \in I$ from Lemma 3.1(4). Thus, $\bigcup_{i \in I} A_i \subseteq mint(mcl(mint(\bigcup_{i \in I} A_i)))$. Hence, $\bigcup_{i \in I} A_i$ is an m-α-open set.

The following example shows that the intersection of any two m-α-open sets may not be m-α-open.

Example 4.1.2: Let $X = \{1,2,3,4\}$ and $m_X = \{\emptyset, X, \{1,2,3\}, \{3,4\}, \{4\}\}$ be a minimal structure on X. Then $\{1,2,3\}$ and $\{3,4\}$ are m-α-open sets but $\{1,2,3\} \cap \{3,4\} = \{3\}$ is not m-α-open.

Definition 4.1.2: Let (X, m_X) be a space with minimal structure m_X on X. For a subset A of X, the m-α-closure of A and the m-α-interior of A, denoted by $m\alpha cl(A)$ and $m\alpha int(A)$, respectively, are defined as the following:

$$macl(A) = \cap \{F: A \subseteq F, F \text{ is } m\text{-}\alpha\text{-closed in } X\}$$

$$maint(A) = \cup \{G: G \subseteq A, G \text{ is } m\text{-}\alpha\text{-open in } X\}$$

(Min, 2010)

Theorem 4.1.2: Let (X, m_X) be a space with minimal structure m_X on X and $A, B, F \subseteq X$. Then, the following hold.

(1) $maint(A) \subseteq A$ and $A \subseteq macl(A)$.

(2) If $A \subseteq B$, then $maint\ (A) \subseteq maint(B)$ and $macl(A) \subseteq macl(B)$.

(3) A is m-α-open iff $maint\ (A) = A$ and F is m-α-closed iff $macl(F) = F$.

(4) $maint(maint(A)) = maint(A)$ and $macl(macl(A)) = macl(A)$.

(5) $macl(X\backslash A) = X\backslash maint(A)$ and $maint(X\backslash A) = X\backslash macl(A)$.

(Min, 2010)

Proof:

The proofs of (1) and (2) are obvious from the definitions of m-α-interior and m-α-closure.

(3) If A is m-α-open, the proof is obvious from the definition of m-α-interior. Let $maint(A) = A$. Since any union of m-α-open sets is m-α-open from Theorem 4.1.1, A is m-α-open. By the similar way, if A is m-α-closed, it is clear. Let $macl(A) = A$. Since any intersection of m-α-closed sets is m-α-closed, A is m-α-closed.

(4) Since $maint(A)$ is m-α-open and $macl(A)$ is m-α-closed, we have $maint(maint(A)) = maint(A)$ and $macl(macl(A)) = macl(A)$ from (3).

(5) $X\backslash maint(A) = X\backslash \cup \{G: G \subseteq A, G \text{ is } m$-$\alpha$-open$\}$=$\cap \{X\backslash G: G \subseteq A, G \text{ is } m$-$\alpha$-open$\}$= $\cap \{X\backslash G: X\backslash A \subseteq X\backslash G, X\backslash G$ is m-α-closed$\}$ = $macl(X\backslash A)$. By the similar way, we have $maint(X\backslash A) = X\backslash macl(A)$.

The converse implication of (2) in Theorem 4.1.2 may not be true as shown in the following example.

Example 4.1.3: Let (X, m_X) be a space with a minimal structure m_X on X as in Example 4.1.1. Then, we get $M\alpha(X) = \{\emptyset, X, \{1,2\}, \{2,3\}, \{1\}, \{3\}, \{1,3\}\}$. Consider $A = \{2\}, B = \{1\}$ and $C = \{1,3\}$, then $m\alpha int(A) = \emptyset \subset m\alpha int(B) = \{1\}$ and $m\alpha cl(A) = \{2\} \subset m\alpha cl(C) = X$ but $A \nsubseteq B$ and $A \nsubseteq C$.

Theorem 4.1.3: Let (X, m_x) be a space with a minimal structure m_X on X and $A \subseteq X$. Then;

(1) $x \in m\alpha cl(A)$ if and only if $A \cap G \neq \emptyset$ for every m-α-open set G containing x.

(2) $x \in m\alpha int(A)$ if and only if there exists an$m$-α-open set Ucontaining x such that $U \subseteq A$.

(Min, 2010)

<u>Proof:</u>

(1) Suppose there exists an m-α-open set G containing x such that $A \cap G = \emptyset$. Then, $X \backslash G$ is m-α-closed and $A \subseteq X \backslash G$. Since $A \subseteq m\alpha cl(A) \subseteq X \backslash G$ and $x \notin X \backslash G$ implies $x \notin m\alpha cl(A)$. Converse implication is clear from the definition of m-α-closure.

(2) Suppose there exists an m-α-open set U containing x such that $U \subseteq A$. Since $U \subseteq m\alpha int(A) \subseteq A$ and $x \in U$ implies $x \in m\alpha int(A)$. The converse is obvious, from the definition of m-α-interior.

4.2 m-Semiopen Sets

Definition 4.2.1: Let (X, m_X) be a space with a minimal structure m_X on X. A subset A of X is called an m-semiopen set if $A \subseteq mcl(mint(A))$. The complement of an m-semiopen set is called m-semiclosed set. The family of all m-semiopen sets in X is denoted by $MSO(X)$.

(Min, 2009)

Remark 4.2.1: If the minimal structure m_x on a given nonempty set X is topology, then an m-semiopen set is semi-open.

(Min, 2009)

Proposition 4.2.1: Let (X, m_X) be a space with a minimal structure m_X on X. Then, every m-α-open set is m-semiopen.

Proof: Let A be an m-α-open set. Then, we have $A \subseteq mint(mcl(mint(A))) \subseteq mcl(mint(A))$ by Lemma 3.1(5). Thus, A is m-semiopen.

The following example shows that the converse of Proposition 4.2.1 is not true, in general.

Example 4.2.1:

Let $X = \{a, b, c\}$ and let $m_X = \{\emptyset, X, \{a\}, \{b\}\}$ be a minimal structure on X. Consider $A = \{a, c\}$, then A is m-semiopen but not m-α-open.

The following remark follows from Proposition 4.1.1 and Proposition 4.2.1.

Remark 4.2.2: Every m-open set is m-semiopen.

(Min, 2009)

Lemma 4.2.1: Let (X, m_X) be a space with minimal structure m_X on X and $A \subseteq X$. Then, A is an m-semiclosed set if and only if $mint(mcl(A)) \subseteq A$.

(Min, 2009)

Proof:

Let A be m-semiclosed. Then, we have $X \backslash A \subseteq mcl(mint(X \backslash A)) = mcl\big(X \backslash mcl(A)\big) = X \backslash (mint(mcl(A)))$ by Lemma 3.1(1). Thus, we obtain $mint(mcl(A)) \subseteq A$. The converse is done by similar way.

Theorem 4.2.1: Let (X, m_X) be a space with a minimal structure m_X on X. Any union of m-semiopen sets is$m$-semiopen.

(Min,2009)

Proof: Let A_i be an m-semiopen set for each $i \in I$. Thus, we get $A_i \subseteq mcl(mint(A_i)) \subseteq mcl(mint(\bigcup_{i\in I} A_i)))$ for each $i \in I$ by Lemma 3.1(4). Hence, we obtain $\bigcup_{i\in I} A_i \subseteq mcl(mint(\bigcup_{i\in I} A_i))$. So, $\bigcup_{i\in I} A_i$ is m-semiopen.

The intersection of any two m-semiopen sets may not be m-semiopen as seen from the following example.

Example 4.2.2: Let $X = \{1, 2, 3, 4\}$ and let$m_X = \{\emptyset, X, \{1, 4\}, \{1\}, \{4\}\}$ be a minimal structure on X. Consider $A = \{1, 3\}$ and $B = \{3, 4\}$, then $mcl\big(mint(A)\big) = \{1, 2, 3\}$ and $mcl\big(mint(B)\big) = \{2, 3, 4\}$. Thus, A and B are m-semiopen sets but $\{1,3\} \cap \{3,4\} = \{3\}$ is not m-semiopen since $mcl\big(mint(\{3\})\big) = \emptyset$.

Definition 4.2.2: Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-semi-closure of A and the m-semi-interior of A, denoted by $mscl(A)$ and $msint(A)$, respectively, are defined as the following:

$$mscl(A) = \cap \{F : A \subseteq F, F \ is \ m\text{-semiclosed in X}\}$$

$$msint(A) = \cup \{G : G \subseteq A, G \ is \ m\text{-semiopen in X}\}$$

(Min, 2009)

Theorem 4.2.2: Let (X, m_X) be a space with a minimal structure $m_X$on X and $A, B, F \subseteq X$. Then, the following hold:

(1) $msint(A) \subseteq A$ and $A \subseteq mscl(A)$.

(2) If $A \subseteq B$, then $msint(A) \subseteq msint(B)$ and $mscl(A) \subseteq mscl(B)$.

(3) A is m-semiopen iff $msint(A)$=A and F is m-semiclosed iff$mscl(F) = F$.

(4) $msint(msint(A)) = msint(A)$ and $mscl(mscl(A)) = mscl(A)$.

(5) $mscl(X\backslash A) = X\backslash msint(A)$ and $msint(X\backslash A) = X\backslash mscl(A)$.

(Min, 2009)

Proof:

(1) and (2) are obvious from the definitions of m-semi-interior and m-semi-closure.

(3) The proof is clear since any union of m-semiopen sets is m-semiopen from Theorem 4.2.1.

(4) By (3), it is clear since $msint(A)$ is m-semiopen and $mscl(A)$ is m-semiclosed.

(5) $X\backslash msint(A) = X\backslash \cup \{G: G \subseteq A, G\ is\ m\text{-semiopen}\}$
$= \cap \{X\backslash G: G \subseteq A, G\ is\ m\text{-semiopen}\} = \cap \{X\backslash G: X\backslash A \subseteq X\backslash G, X\backslash G \text{ is } m\text{-semiclosed}\} = mscl(X\backslash A)$. Also, we have $msint(X\backslash A) = X\backslash mscl(A)$ by the similar way.

The following example shows that the converse of (2) in Theorem 4.2.2 is not true, in general.

Example 4.2.3: Let $X = \{1,2,3\}$ and let $m_X = \{\emptyset, X, \{1,2\}, \{1,3\}\}$ be a minimal structure on X. Then, $MSO(X) = \{\emptyset, X, \{1,2\}, \{1,3\}\}$. Consider $A = \{1\}$, $B = \{2\}$ and $C = \{1,3\}$, then $msint(B) = \emptyset \subset msint(C) = \{1,3\}$ but $B \nsubseteq C$ and $mscl(B) = \{2\} \subset mscl(A) = X$ but $B \nsubseteq A$.

Theorem 4.2.3: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then,

(1) $x \in mscl(A)$ if and only if $A \cap G \neq \emptyset$ for every m-semiopen set G containing x.

(2) $x \in msint(A)$ if and only if there exists an m-semiopen set U containing x such that $U \subseteq A$.

(Min, 2009)

Proof:

(1) Assume there is an m-semiopen set G containing x such that $A \cap G = \emptyset$. Then, we have $A \subseteq X\backslash G$ such that $X\backslash G$ is m-semiclosed. Since $A \subseteq mscl(A) \subseteq X\backslash G$ and $x \notin X\backslash G$, we obtain $x \notin mscl(A)$. The converse is clear.

(2) Suppose there is an m-semiopen set U containing x such that $U \subseteq A$. Then, $U \subseteq msint(A) \subseteq A$ and $x \in U$ implies $x \in msint(A)$. The converse implication is obvious.

4.3 m-Preopen Sets

Definition 4.3.1: Let (X, m_X) be a space with a minimal structure m_X on X. A subset A of X is called an m-preopen set if $A \subseteq mint(mcl(A))$. The complement of an m-preopen set is called an m-preclosed set. The family of all m-preopen sets in X is denoted by MPO(X).

(Min and Kim, 2009)

Remark 4.3.1: If the minimal structure m_X on a given non-empty set X is a topology, then an m-preopen set is preopen.

Proposition 4.3.1: Let (X, m_X) be a space with a minimal structure m_X on X. Then, every m-α-open set is m-preopen.

Proof: Let A be an m-α-open set. Since $mint(A) \subseteq A$, we get $A \subseteq mint(mcl(mint(A))) \subseteq mint(mcl(A))$. Hence, A is m-preopen set.

The converse implications of Proposition 4.3.1 may not be true as shown in the following example.

Example 4.3.1: Let $X = \{1, 2, 3, 4\}$ and $m_X = \{\emptyset, X, \{1,2\}, \{1, 3, 4\}\}$ be a minimal structure on X. Consider $A = \{1, 3\}$, then A is m-preopen but it is not m-α-open.

The following remark follows from Proposition 4.1.1 and Proposition 4.3.1.

Remark 4.3.2: Every m-open set is m-preopen.

(Min and Kim, 2009)

Remark 4.3.3: m-preopenness and m-semiopenness are independent of each other.

(Min and Kim, 2009)

Example 4.3.2:

(1) Let (X, m_x) be a space with a minimal structure m_X on X and $A = \{1,3\}$ as in Example 4.3.1. Then, A is m-preopen but it is not m-semiopen.

(2) Let $X = \{1,2,3,4\}$ and $m_X = \{\emptyset, X, \{1\}, \{4\}, \{1,3\}\}$ be a minimal structure on X. Consider $B = \{1,2,3\}$, then B is m-semiopen but it is not m-preopen.

Lemma 4.3.1: Let (X, m_X) be a space with minimal structure m_X on X. Then, A is an m-preclosed set if and only if $mcl(mint(A)) \subseteq A$.

(Min and Kim, 2009)

Proof: It is similar to that of Lemma 4.1.1.

Theorem 4.3.1: Let (X, m_X) be a space with a minimal structure m_X on X. Any union of m-preopen sets is m-preopen.

(Min and Kim, 2009)

Proof: It is similar to that of Theorem 4.1.1.

The following example shows that the intersection of any two m-preopen sets may not be m-preopen set.

Example 4.3.3: Let $X = \{1, 2, 3, 4\}$ and let$m_X = \{\emptyset, X, \{1, 2, 3\}, \{3, 4\}\}$ be a minimal structure on X. $\{1, 2,3\}$ and $\{2, 4\}$ are m-preopen sets but $\{1,2,3\} \cap \{2,4\} = \{2\}$ is not m-preopen.

Definition 4.3.2: Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-pre-closure of A and m-pre-interior of A, denoted by $mpcl(A)$ and $mpint(A)$, respectively, are defined as the following:

$mpcl(A) = \cap \{F \subseteq X: A \subseteq F, F\ is\ m$-preclosed in X$\}$

$mpint(A) = \cup \{G \subseteq X: G \subseteq A, G\ is\ m$-preopen in X$\}$

(Min and Kim, 2009)

Theorem 4.3.2: Let (X, m_X) be a space with a minimal structure m_X on X and $A, B, F \subseteq X$. Then, the following hold;

(1) $mpint(A) \subseteq A$ and $A \subseteq mpcl(A)$.

(2) If $A \subseteq B$, then $mpint(A) \subseteq mpint(B)$ and $mpcl(A) \subseteq mpcl(B)$.

(3) A is m-preopen iff $mpint(A) = A$ and F is m-preclosed iff $mpcl(F) = F$.

(4) $mpint(mpint(A)) = mpint(A)$ and $mpcl(mpcl(A)) = mpcl(A)$.

(5) $mpcl(X \backslash A) = X \backslash mpint(A)$ and $mpint(X \backslash A) = X \backslash mpcl(A)$.

(Min and Kim, 2009)

Proof:

The proofs of (1) and (2) are clear from the definitions of m-pre-interior and m-pre-closure.

(3) If A is m-preopen, it is clear. Let $mpint(A) = A$. By Theorem 4.3.1, A is m-preopen. The other part of (3) is proved by the similar way.

(4) Since $mpint(A)$ is m-preopen and $mpcl(A)$ is m-preclosed, the proofs are obvious.

(5) $X \backslash mpint(A) = X \backslash \cup \{G: G \subseteq A, G$ is m-preopen$\} = \cap \{X \backslash G: G \subseteq A, G$ is m-preopen$\} = \cap \{X \backslash G: X \backslash A \subseteq X \backslash G, X \backslash G$ is m-preclosed$\} = mpcl(X \backslash A)$. By the similar way, we have $mpint(X \backslash A) = X \backslash mpcl(A)$.

The converse implication of (2) in Theorem 4.3.2 may not be true as shown in the following example.

Example 4.3.4: Let $X = \{1, 2, 3\}$ and let $m_X = \{\emptyset, X, \{1, 2\}, \{2,3\}\}$ be a minimal structure on X. Then, $MPO(X) = \{\emptyset, X, \{2\}, \{1,2\}, \{2,3\}, \{1,3\}\}$. Consider $A = \{1\}$ and $B = \{2,3\}$, then $mpint(A) = \emptyset \subset mpint(B) = \{2,3\}$ and $mpcl(A) = \{1\} \subset mpcl(B) = X$ but $A \nsubseteq B$.

Theorem 4.3.3: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then

(1) $x \in mpcl(A)$ if and only if $A \cap G \neq \emptyset$ for every m-preopen set G containing x.

(Min and Kim, 2009)

(2) $x \in mpint(A)$ if and only if there exists an m-preopen set U containing x such that $U \subseteq A$.

Proof:

(1) The proof is similar to that of Theorem 4.1.3(1).

(2) Assume that there exists an m-preopen set U containing x such that $U \subseteq A$. Since $U \subseteq mpint(A) \subseteq A$ and $x \in U$, we have $x \in mpint(A)$. The other part of (2) is clear from the definition of m-pre-interior.

4.4 m-β-Open Sets

Definition 4.4.1: Let (X, m_X) be a space with a minimal structure m_X on X. A subset A of X is called an m-β-open set if $A \subseteq mcl(mint(mcl(A)))$. The complement of an m-β-open set is called an m-β-closed set. The family of all m-β-open sets in X is denoted by $M\beta O(X)$.

(Vasques et. al., 2011)

Remark 4.4.1: If the minimal structure m_X on a given non-empty set X is a topology, then an m-β-open set is β-open set.

(Nasef and Roy, 2013)

Proposition 4.4.1: Let (X, m_X) be a space with a minimal structure m_X on X. Then,

(1) Every m-preopen set is m-β-open.

(2) Every m-semiopen set is m-β-open.

Proof:

(1) Let A be m-preopen set. Then, $A \subseteq mint\big(mcl(A)\big) \subseteq mcl(mint\big(mcl(A)\big))$. Hence, A is m-β-open.

(2) Let A be m-semiopen set. Since $A \subseteq mcl(A)$, we have $A \subseteq mcl(mint(A)) \subseteq mcl(mint\big(mcl(A)\big))$. Thus, A is m-β-open.

The following example shows that the converse implication of (1) and (2) of Proposition 4.4.1 may not be true.

Example 4.4.1: Let $X = \{1, 2, 3, 4\}$and let$m_X = \{\emptyset, X, \{1, 2, 3\}, \{3, 4\}, \{2\}\}$ be a minimal structure on X. Consider $A = \{1, 2\}$ and $B = \{1, 4\}$, then A and B are m-β-open but A is not m-preopen and B is not m-semiopen.

The following remark follows from Proposition 4.4.1, Remark 4.2.2 and Remark 4.3.2.

Remark 4.4.2: Every m-open set is m-β-open.

(Nasef and Roy, 2013)

Lemma 4.4.1: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then, A is an m-β-closed if and only if $mint(mcl(mint(A)) \subseteq A$.

(Nasef and Roy, 2013)

Proof: The proof is done by the similar way of Lemma 4.1.1 by using Lemma 3.1(1).

Theorem 4.4.1: Let (X, m_X) be a space with a minimal structure m_X on X. Any union of m-β-open sets is m-β-open.

(Nasef and Roy, 2013)

Proof: It is similar to that of Theorem 4.1.1.

The following example shows that the intersection of any two m-β-open sets may not be m-β-open set.

Example 4.4.2: Let (X, m_X) be a space with a minimal structure m_X on X as in Example 4.2.2. Consider $A = \{1,3\}$ and $B = \{3,4\}$, then A and B are m-β-open but $\{1,3\} \cap \{3,4\} = \{3\}$ is not m-β-open.

Definition 4.4.2: Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-β-closure of A and m-β-interior of A are denoted by $m\beta cl(A)$ and $m\beta int(A)$, respectively, are defined as the following;

$$m\beta cl(A) = \cap \{F: A \subseteq F, F is\ m\text{-}\beta\text{-closed in X}\}$$

$$m\beta int(A) = \cup \{G: G \subseteq A, G\ is\ m\text{-}\beta\text{-open in X}\}$$

(Nasef and Roy, 2013)

Theorem 4.4.2: Let (X, m_X) be a space with a minimal structure m_X on X and $A, B, F \subseteq X$. Then;

(1) $m\beta int(A) \subseteq A$ and $A \subseteq m\beta cl(A)$.

(2) If $A \subseteq B$, then $m\beta int(A) \subseteq m\beta int(B)$ and $m\beta cl(A) \subseteq m\beta cl(B)$.

(3) A is m-β-open iff $m\beta int(A) = A$ and F is$m$-β-closed iff $m\beta cl(F) = F$.

(4) $m\beta int(m\beta int(A)) = m\beta int(A)$ and $m\beta cl(m\beta cl(A)) = m\beta cl(A)$.

(5) $m\beta cl(X \backslash A) = X \backslash m\beta int(A)$ and $m\beta int(X \backslash A) = X \backslash m\beta cl(A)$.

(Nasef and Roy, 2013)

Proof:

The proofs of (1) and (2) are obvious from the definitions of m-β-interior and m-β-closure.

(3) If A is m-β-open, the proof is obvious. Let $m\beta int(A) = A$. By Theorem 4.4.1, A is m-β-open. Using similar way, the second part is proved.

(4) Since $m\beta int(A)$ is m-β-open and $m\beta cl(A)$ is m-β-closed, the proofs are obvious by (3).

(5) $X\setminus m\beta int(A) = X \setminus \cup \{G: G \subseteq A, G\ is\ m\text{-}\beta\text{-open}\}=$

$=\cap \{X\setminus G: G \subseteq A, G\ is\ m\text{-}\beta\text{-open}\}=\cap \{X\setminus G: X\setminus A \subseteq X\setminus G,\ X\setminus G$ is $m\text{-}\beta$-closed$\}$

$= m\beta cl(X\setminus A)$. By the similar way, we can proved $m\beta int(X\setminus A) = X\setminus m\beta cl(A)$.

The following example shows that the converse of (2) in Theorem 4.4.2 is not true, in general.

Example 4.4.3: Let $X = \{a, b, c\}$ and let $m_X = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ be a minimal structure on X. Then, $M\beta O(X) = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}, \{b\}\}$. Consider $A = \{a\}$ and $B = \{b, c\}$, then $m\beta int(A) = \emptyset \subset m\beta int(B) = \{b, c\}$ and $m\beta cl(A) = \{a\} \subset m\beta cl(B) = X$ but $A \nsubseteq B$.

Theorem 4.4.3: Let (X, m_X) be a space with a minimal structure m_x on X and $A \subseteq X$. Then,

(1) $x \in m\beta cl(A)$ if and only if $A \cap G \neq \emptyset$ for every $m\text{-}\beta$-open set G containing x.

(2) $x \in m\beta int(A)$ if and only if there exists an $m\text{-}\beta$-open set U containing x such that $U \subseteq A$.

(Nasef and Roy, 2013)

<u>Proof:</u> The proofs are similar to that of Theorem 4.1.3.

5.DIFFERENT KINDS OF CONTINUITIES BETWEEN SPACES WITH MINIMAL STRUCTURES

5.1. $\boldsymbol{M}$-$\boldsymbol{\alpha}$-Continuity

Definition 5.1.1: Let $f: (X, m_X) \to (Y, m_Y)$ be a function betweeen two spaces X and Y with minimal structures m_X and m_Y, respectively. Then f is said to be M-α-continuous if for each x and each m-open set V containing $f(x)$, there exists an m-α-open set U containing x such that $f(U) \subseteq V$.

(Min, 2010)

Remark 5.1.1: Let $f: (X, m_X) \to (Y, m_Y)$ be an M-α-continuous function betweeen two spaces X and Y with minimal structures $m_X$and m_Y, respectively. If the minimal structures m_X and m_Y are topologies on X and Y, respectively, then f is α-continuous.

(Min, 2010)

Proposition 5.1.1: Every M-continuous function is M-α-continuous.

(Min, 2010)

<u>Proof:</u> Since every m-open set is m-α-open, the proof is obvious.

The following example shows that the converse of Proposition 5.1.1 is not true, in general.

Example 5.1.1: Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Consider two minimal structures defined as follows $m_X = \{\emptyset, X, \{a\}, \{a, c, d\}\}$, $m_Y = \{\emptyset, Y, \{1, 2\}\}$ on X and Y, respectively. Let f: $(X, m_X) \to (Y, m_Y)$ be a function defined by $f(a) = 1, f(b) = f(c) = 2\ and\ f(d) = 3$. Then f is M-α-continuous but not M-continuous.

Theorem 5.1.1: Let $f:(X,m_X) \to (Y,m_Y)$ be a function betweeen two spaces X and Y with minimal structures m_X and m_Y,respectively. Then following are equivalent:

(1) f is M-α-continuous,

(2) $f^{-1}(V)$ is an m-α-open set for each m-open set V in Y,

(3) $f^{-1}(B)$ is an m-α-closed set for each m-closed set B in Y,

(4) $f(m\alpha cl(A)) \subseteq mcl(f(A))$ for $A \subseteq X$,

(5) $m\alpha cl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq m\alpha int(f^{-1}(B))$ for $B \subseteq Y$.

(Min, 2010)

Proof :

(1) $\Longrightarrow$ (2) Let V be an m-open set in Y and $x \in f^{-1}(V)$. Since f is$M$-α-continuous, there exist an m-α-open set U containing x such that$f(U) \subseteq V$. So, $x \in U \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Thus, $f^{-1}(V)$ is m-α-open since any union of m-α-open sets is m-α-open.

(2) $\Longrightarrow$ (3) Let B be an m-closed set in Y. Then $Y\backslash B$ is m-open in Y. By (2), $f^{-1}(Y\backslash B) = X\backslash f^{-1}(B)$ is m-α-open. Hence, $f^{-1}(B)$ is m-α-closed.

(3) $\Longrightarrow$ (4) Let $A \subseteq X$. Then, $mcl(f(A))) = \cap \{F \subseteq Y, f(A) \subseteq F$ and F is m-closed$\}$. So, $f^{-1}(mcl(f(A))) = \cap \{f^{-1}(F) \subseteq X: A \subseteq f^{-1}(F)$ and $f^{-1}(F)$ is m-α-closed$\} \supseteq m\alpha cl(A)$. Thus,$f(m\alpha cl(A)) \subseteq mcl(f(A))$.

(4)$\Longrightarrow$(5) Let $B \subseteq Y$. By (4), we have $f(m\alpha cl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$. Thus, $m\alpha cl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$.

(5) $\Longrightarrow$ (6) Let $B \subseteq Y$. Then,$f^{-1}(mint(B)) = f^{-1}(Y\backslash mcl(Y\backslash B)) = X\backslash f^{-1}(mcl(Y\backslash B))$. Since, $m\alpha cl(f^{-1}(Y\backslash B)) \subseteq f^{-1}(mcl(Y\backslash B))$ by (5), we have $f^{-1}(mint(B)) \subseteq X\backslash m\alpha cl(f^{-1}(Y\backslash B)) = m\alpha int(f^{-1}(B))$.

(6)⟹(1) Let V be an m-open set containing$f(x)$. By (6),we have $f^{-1}(mint(V)) \subseteq m\alpha int(f^{-1}(V))$. From Lemma3.1(2), $x \in f^{-1}(mint(V))$. Thus, $x \in m\alpha int(f^{-1}(V))$. By Theorem 4.1.3(2), there exists an m-α-open set U containing x such $U \subseteq f^{-1}(V)$. Hence f is M-α-continuous.

Lemma 5.1.1: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then;

(1) $mcl(mint(mcl(A))) \subseteq mcl(mint(mcl(m\alpha cl(A))) \subseteq m\alpha cl(A)$.

(2) $m\alpha int(A) \subseteq mint(mcl(mint(m\alpha int(A)))) \subseteq mint(mcl(mint(A)))$.

(Min, 2010)

Proof:

(1) Let $A \subseteq X$. Since $m\alpha cl(A)$ is an m-α-closed, $mcl(mint(mcl(m\alpha cl(A))) \subseteq m\alpha cl(A)$ by Lemma 4.1.1. Furthermore, we have $mcl(mint(mcl(A)) \subseteq mcl(mint(mcl(m\alpha cl(A)))$ since $A \subseteq m\alpha cl(A)$.

(2) Let $A \subseteq X$. Since $m\alpha int(A) = X \backslash m\alpha cl(X \backslash A)$, the proof is obvious.

Theorem 5.1.2: Let $f: (X, m_X) \to (Y, m_Y)$ be a function betweeen two spaces X and Y with minimal structures m_X and m_Y,respectively. Then, the following are equivalent:

(1) f is M-α-continuous,

(2) $f^{-1}(V) \subseteq mint(mcl(mint(f^{-1}(V))))$ for each m-open set V in Y,

(3) $mcl(mint(mcl(f^{-1}(F)))) \subseteq f^{-1}(F)$ for each m-closed set F in Y,

(4) $f(mcl(mint(mcl(A))) \subseteq mcl(f(A)$ for $A \subseteq X$,

(5) $mcl(mint(mcl(f^{-1}(B))) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq mint(mcl(mint(f^{-1}(B))))$ for $B \subseteq Y$.

(Min, 2010)

Proof:

(1) ⟺ (2)From Theorem 5.1.1 and definition of m-α-open set, the proof is clear.

(2) $\Leftrightarrow$ (3) From Theorem 5.1.1 and Lemma 4.1.1, the proof is obvious.

(3) $\Longrightarrow$ (4) Let $A \subseteq X$. By Theorem 5.1.1(4), we have $f(m\alpha cl(A)) \subseteq m\alpha cl(A)$. Then, by Lemma 5.1.1(1), we get $mcl(mint(mcl(A))) \subseteq m\alpha cl(A)$. Thus, $mcl\left(mint(mcl(A))\right) \subseteq m\alpha cl(A) \subseteq f^{-1}(mcl(f(A))$. Hence, $f(mcl(mint(mcl(A))) \subseteq mcl(f(A))$.

(4)$\Longrightarrow$(5)Let $B \subseteq Y$.Then,$f^{-1}(B) \subseteq X$. By(4), $f(mcl(mint(mcl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$.Hence $mcl(mint\left(mcl(f^{-1}(B))\right) \subseteq f^{-1}(mcl(B))$.

(5)$\Longrightarrow$(6) Let $B \subseteq Y$. Then, $f^{-1}(mint(B)) = f^{-1}(Y \backslash mcl(Y \backslash B)) = X \backslash f^{-1}(mcl(Y \backslash B))$. Since by (5), we have $mcl(mint\left(mcl(f^{-1}(Y \backslash B))\right)) = f^{-1}(mcl(Y \backslash B))$, we get $f^{-1}(mint(B)) \subseteq X \backslash mcl(mint(mcl(f^{-1}(Y \backslash B)))) = mint(mcl\left(mint(f^{-1}(B))\right))$.

(6)$\Longrightarrow$(1) Let V be an m-open set in Y. Since $mint(V) = V$,$f^{-1}(V) = f^{-1}(mint(V)) \subseteq mint(mcl(mint(f^{-1}(V)))$ from (6). Thus, f is M-α-continuous by (2).

5.2 *M*-Semicontinuity

Definition 5.2.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be a function between two spaces X and Y with minimal structures m_X and m_Y, respectively. Then f is said to be M-semicontinuous if for each x and each m-open set V contining $f(x)$, there exists an m-semiopen set U containing x such that $f(U) \subseteq V$.

(Min, 2009)

Remark 5.2.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be an M-semicontinuous function between two spaces X and Y with minimal structures m_X and m_Y, respectively. If the structures m_X and m_Y are topologies on X and Y, respectively, then f is semicontinuous.

(Min,2009)

Proposition 5.2.1: Every M-α-continuous function is M-semicontinuous.

Proof: The proof is obvious since every m-α-open set is m-semiopen.

The following example shows that the converse of Proposition 5.2.1 may not be true.

Example 5.2.1: Let $X = \{a, b, c\}$ and $Y = \{1,2\}$. Consider two minimal structures defined as follows $m_X = \{\emptyset, X, \{a\}, \{c\}\}$ and $m_Y = \{\emptyset, Y, \{1\}\}$ on X and Y, respectively. Let $f: (X, m_x) \longrightarrow (Y, m_Y)$ be a function defined by $f(a) = f(b) = 1$ and $f(c) = 2$. Then, f is M-semicontinuous but not M-α-continuous.

The following remark follows from Proposition 5.1.1 and Proposition 5.2.1.

Remark 5.2.2: Every M-continuous function is M-semicontinuous

(Min,2009)

Theorem 5.2.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be a function between two spaces X and Y with minimal structures m_X and m_Y, respectively. Then the following are equivalent:

(1) f is M-semicontinuous,

(2) $f^{-1}(V)$ is m-semiopen for each m-open set V in$Y$,

(3) $f^{-1}(B)$ is m-semiclosed for each m-closed set B in Y,

(4) $f(mscl(A)) \subseteq mcl(f(A))$ for $A \subseteq X$,

(5) $mscl(f^{-1}(B) \subseteq f^{-1}(mscl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq msint(f^{-1}(B))$ for $B \subseteq Y$.

(Min, 2009)

Proof:

(1)⟹(2) Since any union of m-semiopen sets is m-semiopen, the proof is obvious.

(2)⟹(3) Let B be an m-closed set in Y. Then, $Y\backslash B$ is m-open. From (2),$f^{-1}(Y\backslash B) = X\backslash f^{-1}(B)$ is m-semiopen. Thus, $f^{-1}(B)$ is m-semiclosed.

(3)⟹(4) Let $A \subseteq X$. Since, $mcl\big(f(A)\big) = \cap \{F \subseteq Y: f(A) \subseteq F$ and F is m-closed$\}$, then $f^{-1}\Big(mcl\big(f(A)\big)\Big) = \cap \{f^{-1}(F) \subseteq X: A \subseteq f^{-1}(F)$ and $f^{-1}(F)$ is m-semiclosed$\} \supseteq mscl(A)$. Hence, $mcl(f(A)) \supseteq f(mscl(A))$.

(4)⟹(5) Let$B \subseteq Y$. From (4), we get $f(mscl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$. Thus, $mscl\big(f^{-1}(B)\big) \subseteq f^{-1}\big(mcl(B)\big)$.

(5)⟹(6) Let $B \subseteq Y$. Then, $f^{-1}\big(mint(B)\big) = f^{-1}\big(Y\backslash mcl(Y\backslash B)\big) = X\backslash f^{-1}(mcl(Y\backslash B))$. By hypothesis, $f^{-1}\big(mint(B)\big) \subseteq X\backslash mscl\big(f^{-1}(Y\backslash B)\big) = msint(f^{-1}(B))$.

(6)⟹(1) Let V be an m-open set containing $f(x)$. From (6), we get $f^{-1}\big(mint(V)\big) \subseteq msint\big(f^{-1}(V)\big)$. Since $V = mint(V)$ by Lemma 3.1(2), then $x \in f^{-1}\big(mint(V)\big)$. Therefore, $x \in msint\big(f^{-1}(V)\big)$. So, there exists an m-semiopen set Ucontaining x such that $U \subseteq f^{-1}(V)$ from Theorem 4.2.3(2). Thus, f is M-semicontinuous.

Lemma 5.2.1: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then

(1) $mint(mcl(A)) \subseteq mint(mcl\big(mscl(A)\big)) \subseteq mscl(A)$.

(2) $msint(A) \subseteq mcl(mint\big(msint(A)\big)) \subseteq mint(mcl(A))$.

(Min,2009)

Proof:

(1) Let $A \subseteq X$. Since $mscl(A)$ is m-semiclosed, by Lemma 4.2.1, $mint(mcl(mscl(A))) \subseteq mscl(A)$. Also, since $A \subseteq mscl(A)$, we have $mint(mcl(A)) \subseteq mint(mcl(mscl(A)))$.

(2) It is obvious by Theorem 4.2.2(5).

Theorem 5.2.2: Let $f: (X, m_X) \to (Y, m_Y)$ be a function between two spacesX and Y with minimal structures m_X and m_Y, respectively. Then, the following are equivalent:

(1) f is M-semicontinuous,

(2) $f^{-1}(V) \subseteq mcl(mint(f^{-1}(V)))$ for each m-open set V in Y,

(3) $mint\left(mcl(f^{-1}(F))\right) \subseteq f^{-1}(F)$ for each m-closed set F in Y,

(4) $f\left(mint(mcl(A))\right) \subseteq mcl(f(A))$ for $A \subseteq X$.

(5) $mint(mcl(f^{-1}(B))) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$.

(6) $f^{-1}(mint(B)) \subseteq mcl(mint(f^{-1}(B)))$ for $B \subseteq Y$.

(Min,2009)

Proof:

(1)⟺(2) By Theorem 5.2.1 and definition of m-semiopen set, the proof is obvious.

(1)⟺(3) It is clear from Theorem 5.2.1 and Lemma 4.2.1.

(3)⟹(4) Let $A \subseteq X$. By Theorem 5.2.1(4) and Lemma 5.2.1(1), we have $mint(mcl(A)) \subseteq mscl(A) \subseteq f^{-1}\left(mcl(f(A))\right)$. Thus, $f(\text{mint}(mcl(A)) \subseteq mcl(f(A))$.

(4)⟹(5) Let $B \subseteq Y$. From (4), we get $f(mint(mcl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B)) \subseteq mcl(B)$. So, $mint(mcl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$.

(5)⟹(6) Let $B \subseteq Y$. Since $mint(B) = Y \setminus mcl(Y \setminus B)$, the proof is obvious.

(6)⟹(1) Let V be an m-open set in Y. Since V is m-open, we have $f^{-1}(V) = f^{-1}(mint(V)) \subseteq mcl(mint(f^{-1}(V)))$ by (6). Thus, f is M-semicontinuous via (2).

5.3 $\boldsymbol{M}$-Precontinuity

Definition 5.3.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be a function between two spaces X and Y with minimal structures m_X and m_Y, respectively. Then f is said to be M-precontinuous if for each x and each m-open set V contining $f(x)$, there exists an m-preopen set U containing x such that $f(U) \subseteq V$.

(Min and Kim, 2009)

Remark 5.3.1: Let $f: (X, m_X) \longrightarrow (Y, m_Y)$ be an M-precontinuous function between two spaces X and Y with minimal structures m_X and m_Y, respectively. If the minimal structures m_X and m_Y are topologies on X and Y, respectively, then f is precontinuous.

Proposition 5.3.1: Every M-α-continuous function is M-precontinuous.

Proof: Since every m-α-open set is m-preopen, it is clear.

The following example shows that the converse of Proposition 5.3.1 is not true, in general.

Example 5.3.1: Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider two minimal structures defined as $m_X = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ and$m_Y = \{\emptyset, Y, \{2\}\}$ on X and Y, respectively. Let $f: (X, m_X) \longrightarrow (Y, m_Y)$ be a function defined by $f(a) = f(c) = 2, f(b) = 1$. Then,$f$ is M-precontinuous but not M-α-continuous.

The following remark follows from Proposition 5.1.1 and Proposition 5.3.1

Remark 5.3.2: Every M-continuous function is M-precontinuous.

(Min and Kim, 2009)

Remark 5.3.3: M-precontinuity and M-semicontinuity are independent of each other.

(Min and Kim, 2009)

Example 5.3.2:

(1) Consider Example 5.3.1., then f is M-precontinuous but not M-semicontinuous.

(2) Let $X = \{1,2,3,4\}$ and $Y = \{a, b\}$. Consider two minimal structures $m_X = \{\emptyset, X, \{1\}, \{4\}\}$ and $m_Y = \{\emptyset, Y, \{a\}, \{b\}\}$ on X and Y, respectively. Let $f: (X, m_X) \longrightarrow (Y, m_Y)$ be a function defined by $f(1) = f(2) = f(3) = a$ and $f(4) = b$. Then, f is M-semicontinuous but not M-precontinuous.

Theorem 5.3.1: Let $f: (X, m_X) \longrightarrow (Y, m_Y)$ be a function between two spaces X and Y with minimal structures m_X and m_Y, respectively. Then following are equivalent:

(1) f is M-precontinuous,

(2) $f^{-1}(V)$is an m-preopen set for each m-open set V in Y,

(3) $f^{-1}(B)$is an m-preclosed set for each m-closed set B in Y,

(4) $f(mpcl(A)) \subseteq mcl(f(A))$ for $A \subseteq X$,

(5) $mpcl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq mpint(f^{-1}(B))$ for $B \subseteq Y$.

(Min and Kim,2009)

<u>Proof :</u>

(1)⟹(2) By Theorem 4.3.1, it is obvious.

(2)⟹(3) It is similar to that of Theorem 5.2.1.

(3)⟹(4) Let $A \subseteq X$. Then, we have$f^{-1}(mcl(f(A))) = f^{-1}(\cap\{F \subseteq Y, f(A) \subseteq F$ and F is m-closed$\}) \supseteq \cap \{$H ⊆ X: A ⊆ H and H is m-preclosed$\} = mpcl(A)$. Thus,$f(mpcl(A)) \subseteq mcl(f(A))$.

(4)⟹(5) Let $B \subseteq Y$. By(4), we have $f(mpcl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$. So, we obtain $mpcl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$.

(5)⟹(6) Let $B \subseteq Y$. Since, $mint(B) = Y \backslash mcl(Y \backslash B)$, we have $f^{-1}\big(mint(B)\big) = X \backslash f^{-1}(mcl(Y \backslash B))$.Then,
$f^{-1}(mint(B)) \subseteq mspint(f^{-1}(B))$ by (5) and Theorem 4.3.2(5).

(6)⟹(1) It follows from Theorem 4.3.3(2) and Lemma 3.1(2).

Lemma 5.3.1: Let (X, m_X) be a space with minimal structure m_X on X and $A \subseteq X$. Then;

(1) $mcl(mint(A)) \subseteq mcl(mint(mpcl(A)) \subseteq mpcl(A))$.

(2) $mpint(A) \subseteq mint(mcl(mpint(A))) \subseteq mint(mcl(A)))$.

(Min and Kim,2009)

<u>Proof:</u>

(1) Let $A \subseteq X$. Since $mpcl(A)$ is m-preclosed, by Lemma 4.3.1, $mcl(mint(mpcl(A)) \subseteq mpcl(A)$. Also, we have$mcl(mint(A)) \subseteq mcl(mint(mpcl(A)))$since $A \subseteq mpcl(A)$.

(2) It is similar to the proof of (1) since $mpint(X \backslash A) = X \backslash mpcl(A)$ for $A \subseteq X$.

Theorem 5.3.2: Let $f: (X, m_X) \longrightarrow (Y, m_Y)$be a function between two spaces X and Y with minimal structure m_X and m_Y, respectively. The following are equivalent:

(1) f is M-precontinuous,

(2) $f^{-1}(V) \subseteq mint(mcl(f^{-1}(V))))$ for each m-open set V in Y,

(3) $mcl(mint(f^{-1}(F))) \subseteq f^{-1}(F)$ for each m-closed set F in Y,

(4) $f\ (mcl(mint(A)) \subseteq mcl(f(A))$ for $A \subseteq\ X$,

(5) $mcl(mint(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq\ mint(mcl(f^{-1}(B)))$ for $B \subseteq Y$.

(Min and Kim, 2009)

Proof: The proofs are similar to the proof of Theorem 5.2.2.

5.4 M-β-Continuity

Definition 5.4.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be a function between two spaces X and Y with minimal structures m_{X} and m_Y, respectively. Then f is said to be M-β-continuous if for each x and each m-open set V containing $f(x)$,there exists an m-β-open set Ucontaining x such that $f(U) \subseteq V$.

(Nasef and Roy, 2013)

Remark 5.4.1: Let $f: (X, m_X) \rightarrow (Y, m_Y)$ be an M-β-continuous function between two spaces X and Y with minimal structures m_X and $m_{Y,}$ respectively. If the minimal structures m_X and $m_Y$are topologies on X and Y, respectively, then f is β-continuous.

(Nasef and Roy,2013)

Proposition 5.4.1:

(1) Every$M$-semicontinuous function is M-β-continuous.

(2) Every M-precontinuous function is M-β-continuous.

Proof: From Proposition 4.4.1, the proofs are obvious.

The following examples illustrate that the converse implications of (1) and (2) of Propositon 5.4.1 may not be true.

Examples 5.4.1:

(1) Consider Example 5.3.1., then f is M-β-continuous but not M-semicontinuous.

(2) Consider Example 5.3.2(2).,then f is M-β-continuous but not M-precontinuous.

The following remark follows from Proposition 5.4.1., Remark 5.2.2 and Remark 5.3.2.

Remark 5.4.2: Every M-continuous function is M-β-continuous.

(Nasef and Roy, 2013)

Theorem 5.4.1: Let $f:(X,m_X) \to (Y,m_Y)$ be a function between two spaces X and Y with minimal structures m_X and m_Y, respectively. Then the following are equivalent:

(1) f is M-β-continuous,

(2) $f^{-1}(V)$ is m-β-open for each m-open set V in Y,

(3) $f^{-1}(F)$ is m-β-closed for each m-closed set F in Y,

(4) $f(m\beta cl(A)) \subseteq mcl(f(A))$ for $A \subseteq X$,

(5) $m\beta cl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq m\beta int(f^{-1}(B))$ for $B \subseteq Y$.

(Nasef and Roy, 2013)

Proof:

(1)⟹(2) It follows from Theorem 4.4.1.

(2)⟹(3) Let F be an m-closed set in Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is m-β-open. Hence, $f^{-1}(F)$ is m-β-closed.

(3)⟹(4) Let $A \subseteq X$. Then, we have $f^{-1}(mcl(f(A)))$ =$f^{-1}(\cap \{F \subseteq Y: f(A) \subseteq F$ and F is m-closed in $Y\})$ =$\cap \{f^{-1}(F) \subseteq X: A \subseteq f^{-1}(F)$ and $f^{-1}(F)$ is m-β-closed in $X\} \supseteq m\beta cl(A)$. Thus $mcl(f(A)) \supseteq f(m\beta cl(A))$.

(4)⟹(5) Let $B \subseteq Y$. By (4), we have $f(m\beta cl(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$. Thus $m\beta cl(f^{-1}(B)) \subseteq f^{-1}(mcl(B))$.

(5)⟹(6) Let $B \subseteq Y$. By (5), we get $m\beta int(f^{-1}(B)) = X \backslash m\beta cl(f^{-1}(Y|B)) \supseteq X \backslash f^{-1}(mcl(Y \backslash B)) = f^{-1}(Y \backslash mcl(Y \backslash B)) = f^{-1}(mint(B))$.

(6)⟹(1) Let V be an m-open set containing $f(x)$. By Lemma 3.1(2) and (6), $x \in m\beta int(f^{-1}(V))$. Therefore, there exists an m-β-open set U containing x such that $U \subseteq f^{-1}(V)$ from Theorem 4.4.3(2). Hence, f is M-β-continuous.

Lemma 5.4.1: Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$.Then;

(i) $mint(mcl(mint(A))) \subseteq mint(mcl(mint(m\beta cl(A)))) \subseteq m\beta cl(A)$.

(ii) $m\beta int(A) \subseteq mcl(mint(mcl(m\beta int(A)))) \subseteq mint(mcl(A))$.

(Nasef and Roy, 2013)

Proof:

(i) It follows from Lemma 4.4.1.

(ii) It is obvious since $m\beta int(X \backslash A) = X \backslash m\beta cl(A)$ for $A \subseteq X$ from Theorem 4.4.2(5).

Theorem 5.4.2: Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be a function between two spaces X andY with minimal structures m_X and m_Y, respectively. Then the following areequivalent:

(1) f is M-β-continuous,

(2) $f^{-1}(V) \subseteq mcl(mint(mcl(f^{-1}(V))))$ for each m-open set V in Y,

(3) $mint(mcl(mint(f^{-1}(F)))) \subseteq f^{-1}(F)$ for each m-closed set F in Y,

(4) $f(mint(mcl(mint(A)))) \subseteq mcl(f(A))$ for $A \subseteq X$,

(5) $mint(mcl(mint(f^{-1}(B)))) \subseteq f^{-1}(mcl(B))$ for $B \subseteq Y$,

(6) $f^{-1}(mint(B)) \subseteq mcl(mint(mcl(f^{-1}(B))))$ for $B \subseteq Y$.

(Nasef and Roy, 2013)

Proof:

(1)⟺(2) The proof is obvious from Theorem 5.4.1 and the definiton of m-β-open set.

(1)⟺(3) By Theorem 5.4.1 and Lemma 4.4.1, it is clear.

(3)⟹(4) Let $A \subseteq X$. By Theorem 5.4.1(4) and Lemma 5.4.1(1), we have $mint(mcl(mint(A))) \subseteq m\beta cl(A) \subseteq f^{-1}(mcl(f(A)))$.Thus, $f(mint(mcl(mint(A)))) \subseteq mcl(f(A))$.

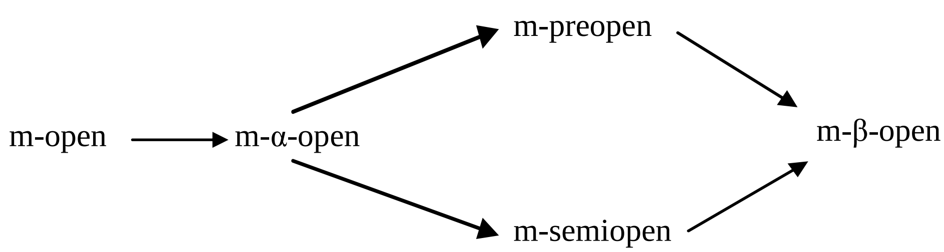
(4)⟹(5) Let $B \subseteq Y$. Since (4), we obtain $f(mint(mcl(mint(f^{-1}(B))) \subseteq mcl(f(f^{-1}(B))) \subseteq mcl(B)$. Hence,$mint(mcl(mint(f^{-1}(B))) \subseteq f^{-1}(mcl(B))$.

(5)⟹(6) It is clear.

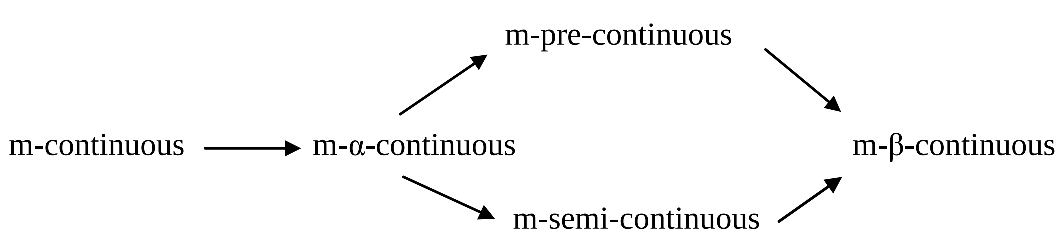
(6)⟹(1) Let V be an m-open set in Y. Then by (6), we have$f^{-1}(V) = f^{-1}(mint(V)) \subseteq mcl(mint(mcl(f^{-1}(V))))$. Hence, f is M-β-continuous.

6. CONCLUSION

In this thesis, first spaces with minimal structures are introduced and properties of m-interior and m-closure operators are investigated. Then, the definitions of m-α-open set, m-semiopen set, m-preopen set and m-β-open set are given. Related properties are investigated and relationships between eachother are shown in the following diagram.



Also, examples are given to show that opposite implications in the diagram do not hold. Besides, m-α-continuity, m-semi-continuity, m-pre-continuity and m-β-continuity are dealt. Investigating the transitions between those functions and relationships between M-continuous functions and continuity types above the following is obtained.



The converse implications in the diagram are shown to be not true by given examples.

For a further research, one can defined distinct types of sets and different types of continuity in spaces with minimal structures.

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