YAŞAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MASTER THESIS

ON GENERALIZED METRIC SPACES

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Presentation Date: 24.05.2016

Bornova-İZMİR
2016

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ON GENERALIZED METRIC SPACES

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MSc in Department of Mathematics
Supervisor: Assist. Prof. Dr. Esra DALAN YILDIRIM
May 2016, 40 pages

This thesis consists, mainly, of four chapters.

In the first chapter, the topic of the thesis is introduced and in the second chapter, in order to clarify the reading of thesis, some types of metric space definitions, the concepts of convergence and completeness are given.

In the third chapter, after given necessary knowledge on G-metric spaces, which are a generalization of metric space, we investigate fundamental properties of G-metrics. Also, we introduce G- metric topology formed by means of G-metric. And then, we study some properties of concepts such as G-convergence, G-continuity and G-completeness. All the concepts occuring in this chapter are illustrated by original examples.

In the last chapter, we handle some fixed point theorems on G-metric spaces.

Keywords: G-metric spaces, metric spaces, quasi-metric spaces, fixed point.

ÖZET

GENELLEŞTİRİLMİŞ METRİK UZAYLAR ÜZERİNE

Cansu SAKARYA
Yüksek Lisans Tezi, Matematik Bölümü
Tez Danışmanı: Yrd. Doç. Dr. Esra DALAN YILDIRIM
Mayıs 2016, 40 sayfa

Bu tez esas olarak dört bölümden oluşmaktadır.

Birinci bölümde tez konusu tanıtılmış, ikinci bölümde ise tezin anlaşılabilir olması için çeşitli metrik uzay tanımları, yakınsaklık ve tamlık kavramları verilmiştir.

Üçüncü bölümde, bir metrik uzayın genelleştirilmesi olan G-metrik uzaylar üzerine bilgi verilerek G-metriğin temel özellikleri incelenmiştir. Ayrıca G-metrik yardımıyla oluşturulan G-metrik topoloji tanıtılmıştır. Daha sonra, G-yakınsaklık, G-süreklilik ve G-tamlık kavramlarına yer verilerek bu kavramlara ait bazı özellikler çalışılmıştır. Bölümde geçen tüm kavramlara ait özgün örneklerle çalışma desteklenmiştir.

Son bölümde G-metrik uzaylardaki bazı sabit nokta teoremleri ele alınmıştır.

Anahtar Sözcükler: G-metrik uzaylar, metrik uzaylar, quasi-metrik uzaylar, sabit nokta.

ACKNOWLEDGEMENTS

I would like to thank to my supervisor Assist. Prof. Dr.Esra Dalan Yıldırım for her support and help on my thesis. And also thank to my family.

Cansu SAKARYA
İzmir, 2016

TEXT OF OATH

I declare and honestly confirm that my study, titled "On Generalized Metric Spaces" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

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INDEX OF SYMBOLS AND ABBREVIATIONS

<u>Symbols</u>	<u>Explanations</u>
$\mathbb{R}$	The Set of Real Numbers
$\mathbb{N}$	The Set of Natural Numbers
τ	Topology
d	Metric
D	D-Metric
G	Generalized Metric
$\tau(G)$	G-Metric Topology

1. INTRODUCTION

The concept of metric spaces is basic in the study of topology and functional analysis. Metric spaces are crucial because they play an important role in the development of the fixed point theory.

Metric spaces may have different generalizations such as given in (Gahler, 1963; Gahler, 1966) and (Dhage, 1992; Dhage, 2000). In 1960's, Gahler (Gahler, 1963; Gahler, 1966) introduced the concept of 2-metric space as a generalization of usual notion of metric space (X, d). And then, Ha et al (Ha et al, 1988) realized that 2-metric may not be a continuous function while metric is a continuous function. This amounts to saying that there is no relation between these two functions.

In 1992, Dhage (Dhage, 1992; Dhage, 2000) proposed a new class of generalized metric space called D-metric space. In his ensuring papers, he tried to develop topological structures in such papers, claimed that D-metrics provide a generalization of metric functions and continued to present several fixed point results.

However, Mustafa and Sims (Mustafa and Sims, 2003) showed that most of Dhage's claims concerning the topological properties of D-metric spaces are not correct.

In 2006, they (Mustafa and Sims, 2006) introduced a new class of generalized metric spaces which are called G-metric spaces, as a generalization of a metric space (X, d). Consequently, many fixed point results on such spaces appeared, for example, in (Mustafa et al, 2008; Mustafa et al, 2009; Mustafa and Sims, 2009; Jleli and Samet, 2012).

In this thesis, we carefully read all papers mentioned and provided original examples, and also proved where we encountered a gap found in the papers.

2. PRELIMINARIES

Troughout this chapter necessary topics are given.

Definition 2.1. A metric space is a pair (X, d), where X is a set and d is a metric on X, that is, a function defined on $X\ x\ X$ such that for all $x, y, z \in X$ we have:

(M1) d is real valued, finite and nonnegative,
(M2) $d(x, y) = 0$ if and only if $x = y$,
(M3) $d(x, y) = d(y, x)$ (symmetry),
(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

(Kreyszig, 1978)

Definition 2.2. Let X be a nonempty set and $d: X\ x\ X \rightarrow [0, \infty)$ be a given function which satisfies the following properties:

(1) $\mathrm{d(x, y)} = 0$ if and only if $x = y$,
(2) $\mathrm{d(x, y)} \leq \mathrm{d(x, z)} + \mathrm{d(z, y)}$ for any points $x, y, z \in X$.

Then d is called a quasi-metric and the pair (X, d) is called quasi-metric space.

(Jleli and Samet, 2012)

Definition 2.3. Let X be a nonempty set and $\mathbb{R}$ denote the set of real numbers. A function $d: X\ x\ X\ x\ X \longrightarrow \mathbb{R}^{+}$ is said to be a 2-metric on X if it satisfies the following properties:

(A1) For every distinct points $x, y \in X$, there is $z \in X$, such that
$d(x, y, z) \neq 0$,
(A2) $d(x, y, z) = 0$ if two of the triple $x, y, z \in X$ are equal,

(A3) $d(x, y, z) = d(x, z, y) = \cdots$ (symmetry in all three variables),

(A4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$, for all $x, y, z, a \in X$.

The set X equipped with such a 2-metric is called a 2-metric space.

(Gahler, 1963)

Definition 2.4. Let X be a nonempty set and $\mathbb{R}$ denote the set of real numbers. A function $D: X\ x\ X\ x\ X \longrightarrow \mathbb{R}$ is said to be a D-metric on X if it satisfies the following properties:

(D1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$,

(D2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(D3) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry in all three variables),

(D4) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The set X together with such a D-metric is called D-metric space, and denoted by (X, D).

(Dhage, 1992)

Definition 2.5. A sequence (x_n) in a metric space (X, d) is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0,$$

and x is called the limit of (x_n) and we write

$$\lim_{n\to\infty} x_n = x$$

or, simply,

$$x_n \to x$$

We say that (x_n) converges to x or has the limit x. If (x_n) is not convergent, it is said to be divergent.

(Kreyszig, 1978)

Definition 2.6. A sequence (x_n) in a metric space (X, d) is said to be Cauchy if for every $\varepsilon > 0$ there is a natural number $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$.

(Kreyszig, 1978)

Definition 2.7. The metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

(Kreyszig, 1978)

3. G-METRIC SPACES

3.1. G-Metric and G-Metric Topology

Definition 3.1.1. Let X be a nonempty set, and let $G: X\,x\,X\,x\,X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(G1) $G(x, y, z) = 0 \; if \; x = y = z,$

(G2) $0 < G(x, x, y)\;; for\; all\; x, y \in X,$ with $x \neq y,$

(G3) $G(x, x, y) \leq G(x, y, z) for\; all\; x, y, z \in X$ with $z \neq y,$

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z),\; for\; all\; x, y, z, a \in X$ (rectangle inequality),

then the function *G* is called a generalized metric, or, more specifically, a G-metric on *X*, and the pair (X, G) is a G-metric space.

(Mustafa and Sims, 2006)

Example 3.1.1. If $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y, z in $\mathbb{R}^2$, then it satisfies all G-metric axioms.

(Mustafa and Sims, 2006)

Example 3.1.2. Let *X* be a nonempty set. We define $G: X\,x\,X\,x\,X \to \mathbb{R}^+$ by

$$G(x, y, z) = \begin{cases} 0, & all\; of\; the\; variables\; are\; equal\;. \\ 1, & two\; of\; varibles\; are\; equal\; and \\ & \;the\; remaining\; is\; distinct\;. \\ 2, & all\; of\; the\; variables\; are\; distinct. \end{cases}$$

for all $x, y, z \in X$. Then, *G* satisfies all of the G-metric axioms.

Definition 3.1.2. A G-metric space (X, G) is symmetric if

(G6) $G(x, y, y) = G(x, x, y)\ for\ all\ x, y \in X$.

(Mustafa and Sims, 2006)

Remark 3.1.1. The G-metric space (X, G) given in Example 3.1.2 is also symmetric.

Proposition 3.1.1. Let (X, G) be a G-metric space, then the following hold for all $x, y, z, a \in X$.

(1) $If\ G(x, y, z) = 0, then\ x = y = z,$

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z),$

(3) $G(x, y, y) \leq 2G(y, x, x),$

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z),$

(5) $G(x, y, z) \leq \frac{2}{3}\big(G(x, y, a) + G(x, a, z) + G(a, y, z)\big),$

(6) $G(x, y, z) \leq \big(G(x, a, a) + G(y, a, a) + G(z, a, a)\big),$

(7) $|G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(z, a, a)\},$

(8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z),$

(9) $|G(x, y, z) - G(y, z, z)| \leq \max\{G(x, z, z), G(z, x, x)\},$

(10) $|G(x, y, y) - G(y, x, x)| \leq \max\{G(y, x, x), G(x, y, y\}.$

(Mustafa and Sims, 2006)

Proof:

(1) We proceed by proving the contrapositive of the statement.

Case1: Let all of the variables be distinct. From (G2) and (G3), we have $0 < G(x, x, y) \leq G(x, y, z)$.

Case2: Let two of the variables be equal and the remaining be distinct. From (G4) and (G2), we get $G(x, y, z) > 0$.

Thus, $G(x, y, z) \neq 0$ for two cases.

(2) From (G4) and (G5), we get

$$G(x,x,y)+G(x,x,z)=G(y,x,x)+G(x,x,z)\geq G(y,x,z)=G(x,y,z).$$

(3) We know that $G(x,y,y)\leq G(x,x,y)+G(x,x,y)=2G(x,x,y)$ by (2). Then, we obtain $G(x,y,y)\leq 2G(y,x,x)$ from (G4).

(4) <u>Case 1:</u> Let $x\neq z$. Thus, we have

$$\begin{aligned} G(x,y,z) &\leq G(x,a,a)+G(a,y,z) \\ &= G(a,a,x)+G(a,y,z) \\ &\leq G(a,x,z)+G(a,y,z) \\ &= G(x,a,z)+G(a,y,z) \end{aligned}$$

from (G5), (G4), (G3) and (G4), respectively.

<u>Case 2:</u> Let $x=z$ and $y\neq a$. Then, we have

$$G(x,y,z)=G(x,y,x)=G(x,x,y)\leq G(x,y,a)\leq G(x,y,a)+G(x,a,x)$$

from (G4) and (G3). So, we obtain $G(x,y,z)\leq G(a,y,x)+G(x,a,x)=G(a,y,z)+G(x,a,z)$ by (G4).

<u>Case3:</u> Let $x=z$ and $y=a$. The proof is clear.

(5) By using (4), we have

$$G(x,y,z)\leq G(x,a,z)+G(a,y,z)$$

$$G(x,y,z)\leq G(x,a,y)+G(z,a,y)$$

$$G(x,y,z)\leq G(y,a,x)+G(z,a,x)$$

Thus, we get $3G(x,y,z)\leq 2(G(x,y,a)+G(x,a,z)+G(a,y,z))$ from (G4). So, we obtain

$$G(x,y,z)\leq \frac{2}{3}(G(x,y,a)+G(x,a,z)+G(a,y,z)).$$

(6) By using (G5), (2) and (G4), respectively, we have

$$\begin{aligned} G(x,y,z) &\leq G(x,a,a)+G(a,y,z) \\ &\leq G(x,a,a)+G(y,a,a)+G(z,a,a) \end{aligned}$$

$$G(x,y,z) \leq G(y,a,a) + G(a,x,z)$$
$$\leq G(y,a,a) + G(x,a,a) + G(z,a,a)$$
$$G(x,y,z) \leq G(z,a,a) + G(a,x,y)$$
$$\leq G(z,a,a) + G(y,a,a) + G(x,a,a)$$

Then, $3G(x,y,z) \leq 3G(x,a,a) + 3G(y,a,a) + 3G(z,a,a)$. Hence, we get
$$G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a).$$

(7) From (G5), we have $G(x,y,z) \leq G(z,a,a) + G(a,x,y)$. Thus,
$$G(x,y,z) - G(a,x,y) \leq G(z,a,a) \leq \max\{G(z,a,a), G(a,z,z)\}.$$
By the similar way we have $G(a,x,y) \leq G(a,z,z) + G(z,x,y)$ from (G5). Hence,
$$-\max\{G(z,a,a), G(a,z,z)\} \leq -G(a,z,z) \leq G(x,y,z) - G(a,x,y).$$
Therefore, we obtain
$$|G(x,y,z) - G(x,y,a)| \leq \max\{G(a,z,z), G(z,a,a)\}.$$

(8) Since $G(x,y,z) \leq G(a,x,z) + G(a,x,y)$ and $G(x,y,a) \leq G(z,x,a) + G(z,x,y)$ from (4), we obtain
$$-G(x,a,z) \leq G(x,y,z) - G(x,y,a) \leq G(x,a,z)$$
by using (G4). Hence, $|G(x,y,z) - G(x,y,a)| \leq G(x,a,z)$.

(9) Since $G(x,y,z) \leq G(y,z,z) + G(z,x,z)$ and $G(y,z,z) \leq G(z,x,x) + G(y,z,x)$ from (G5), we have
$$G(x,y,z) - G(y,z,z) \leq G(x,z,z) \leq \max\{G(x,z,z), G(z,x,x)\}$$
and
$$-\max\{G(x,z,z), G(z,x,x)\} \leq -G(z,x,x) \leq G(x,y,z) - G(y,z,z)$$
by using (G4).

Thus, we obtain
$$|G(x,y,z) - G(y,z,z)| \leq \max\{G(x,z,z), G(z,x,x)\}.$$

(10) The proof is done by a similar proof of (7) by using (3) and (G4). □

Proposition 3.1.2. Let (X, G) be a G-metric space and let $k > 0$, then G_1 and G_2 are G-metrics on X, where

(1) $G_1(x, y, z) = min\{k, G(x, y, z)\}$,

(2) $G_2(x, y, z) = \frac{G(x,y,z)}{k+G(x,y,z)}$.

(Mustafa and Sims, 2006)

Proof:

(1) (G1) Since $G(x, y, z) = 0$ when $x = y = z$, we have $G_1(x, y, z) = 0$.

(G2) Let $x \neq y$. $G_1(x, x, y) > 0$ since $G(x, x, y) > 0$ and $k > 0$.

(G3) Let $z \neq y$.

Case 1: Let $k \leq G(x, x, y)$. Since $G(x, x, y) \leq G(x, y, z)$, we have $G_1(x, x, y) = \min\{k, G(x, x, y)\} = k = \min\{k, G(x, y, z)\} = G_1(x, y, z)$.

Case 2: Let $G(x, x, y) < k$. Since $G(x, x, y) \leq G(x, y, z)$, we have $G_1(x, x, y) = \min\{k, G(x, x, y)\} = G(x, x, y) \leq \min\{k, G(x, y, z)\} = G_1(x, y, z)$.

Hence, we get $G_1(x, x, y) \leq G_1(x, y, z)$ by two cases.

(G4) Since G satisfies (G4), then G_1 also satisfies (G4).

(G5) Since $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, then we get

$$\begin{aligned} min\{k, G(x, y, z)\} &\leq min\{k, G(x, a, a) + G(a, y, z)\} \\ &\leq min\{k, G(x, a, a)\} + min\{k, G(a, y, z)\} \\ &= G_1(x, a, a) + G_1(a, y, z). \end{aligned}$$

(2) (G1) Since $G(x, y, z) = 0$ when $x = y = z$, we have $G_2(x, y, z) = 0$.

(G2) Let be $x \neq y$. Since $G(x, x, y) > 0$ and $k > 0$, we have $G_2(x, x, y) > 0$.

(G3) Let be $z \neq y$. Since $G(x, x, y) \leq G(x, y, z)$ and G_2 is increasing function, we have $G_2(x, x, y) \leq G_2(x, y, z)$.

(G4) It is obvious.

(G5) Since G satisfies (G5) and G_2 is increasing, we get

$$G_2(x, y, z) \leq G_2(x, a, a) + G_2(a, y, z).$$ □

Proposition 3.1.3. Let (X, G) be a G-metric space, $k > 0$ and $X = \bigcup_{i=1}^{n} A_i$ be any partition of X, then G_3 is a G-metric on X, where

$$G_3(x,y,z) = \begin{cases} G(x,y,z), & if\ for\ some\ i\ we\ have\ x,y,z \in A_i \\ k + G(x,y,z), & otherwise. \end{cases}$$

(Mustafa and Sims, 2006)

Proof: It is clear. □

Proposition 3.1.4: Let (X, G) be a G-metric space, then the following are equivalent.

(1) (X, G) is symmetric.
(2) $G(x,y,y) \leq G(x,y,a), for\ all\ x,y,a \in X.$
(3) $G(x,y,z) \leq G(x,y,a) + G(z,y,b), for\ all\ x,y,z,a,b \in X.$

(Mustafa and Sims, 2006)

Proof:
(1) ⇒ (2) Case1: Let $x \neq a$. $G(x,y,y) \leq G(x,,y,a)$ from (G3).
Case2: Let $x = a$. It is obvious since (X, G) is symmetric.

(2) ⇒ (3) By Proposition 3.1.1(2) and hypothesis, we get

$$G(x,y,z) \leq G(x,y,y) + G(y,y,z) \leq G(x,y,a) + G(z,y,b).$$

(3) ⇒ (1) From hypothesis and (G4), we get

$$G(x,y,y) \leq G(x,y,x) + G(y,y,y) = G(x,y,x) = G(x,x,y).$$

By similar way, we have

$$G(y,x,x) \leq G(y,x,y) + G(x,x,x) = G(y,x,y) = G(x,y,y).$$

Thus, we obtain $G(x,x,y) = G(x,y,y)$. That is, (X, G) is symmetric. □

Proposition 3.1.5. Let (X, d) be a metric space. Then $G_s(d)$ and $G_m(d)$ expressed as follows define G-metrics on X.

(1) $G_s(d)(x,y,z) = \frac{1}{3}(d(x,y) + d(y,z) + d(x,z))$,

(2) $G_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$.

(Mustafa and Sims, 2006)

Proof:

(1) (G1) Let $x = y = z$. Since d is a metric, $G_s(d)(x,y,z) = 0$.

(G2) Let $x \neq y$. We have $G_s(d)(x,x,y) > 0$ since $d(x,y) > 0$.

(G3) Case 1: Let $y \neq z$ and $x = y$. Thus,

$$G_s(d)(x,x,y) = 0 < G_s(d)(x,y,z).$$

Case 2: Let $x = z, z \neq y$ and $x \neq y$. Then,

$$G_s(d)(x,x,y) = G_s(d)(x,y,z).$$

Case 3: Let $x \neq z, z \neq y$ and $x \neq y$. By using (M4), we have

$$2d(x,y) \leq d(x,y) + d(x,z) + d(z,y).$$

Then,

$$G_s(d)(x,x,y) = \frac{2}{3}d(x,y) \leq \frac{1}{3}[d(x,y) + d(x,z) + d(z,y)]$$
$$= G_s(d)(x,y,z).$$

(G4) It is clear.

(G5) From (M4), we have

$$d(x,y) \leq d(x,a) + d(a,y),$$
$$d(x,z) \leq d(x,a) + d(a,z).$$

Thus,

$$G_s(d)(x,y,z) = \frac{1}{3}[d(x,y) + d(x,z) + d(z,y)]$$
$$\leq \frac{1}{3}[d(x,a) + d(a,y) + d(x,a) + d(a,z) + d(y,z)]$$
$$= G_s(d)(x,a,a) + G_s(d)(a,y,z).$$

(2) (G1) Let $x = y = z$. Since d is a metric, we have $G_m(d)(x,y,z) = 0$.

(G2) Let $x \neq y$. We have $G_m(d)(x,x,y) = d(x,y) > 0$.

(G3) Let $y \neq z$. We get

$$G_m(d)(x,x,y) = d(x,y)$$
$$\leq \max\{d(x,y), d(y,z), d(x,z)\} = G_m(d)(x,y,z).$$

(G4) It is obvious.

(G5) Case1: Let $G_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\} = d(x,y)$. Since d is a metric, we have

$$d(x,y) \leq d(x,a) + d(a,y) = G_m(d)(x,a,a) + d(a,y)$$
$$\leq G_m(d)(x,a,a) + \max\{d(a,y), d(y,z), d(a,z)\}$$
$$= G_m(d)(x,a,a) + G_m(d)(a,y,z).$$

The proofs of other cases are done by similar way. □

Example 3.1.3. Let d be a metric on ℝ defined by $d(x,y) = |x - y|$. Then, the following functions $G_s(d)$ and $G_m(d)$ are two G-metrics on ℝ.

(1) $G_s(d)(x,y,z) = \frac{1}{3}(|x-y| + |y-z| + |x-z|)$,

(2) $G_m(d)(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}$.

Proposition 3.1.6. Let (X, G) be a G-metric space. Then d_G define a metric on X.

$$d_G(x,y) = G(x,y,y) + G(x,x,y)$$

(Mustafa and Sims, 2006)

Proof:

(M1) Let $x \neq y$. $d_G(x,y) = G(x,y,y) + G(x,x,y) > 0$ from (G2).

(M2) Let $d_G(x,y) = 0$. Suppose $x \neq y$. By Proposition 3.1.1 (3), we have

$$G(x,y,y) + \frac{1}{2}G(x,y,y) = \frac{3}{2}G(x,y,y) \leq G(x,y,y) + G(x,x,y) = 0$$

Thus, $G(x,y,y) \leq 0$. This contradicts to (G2). Hence, our assumption is not true. That is, $x = y$. The converse is clear.

(M3) $d_G(x,y) = G(x,y,y) + G(x,x,y) = G(y,y,x) + G(y,x,x) = d_G(y,x)$ from (G4).

(M4) We have $G(x,y,y) \leq G(x,z,z) + G(z,y,y)$ and $G(x,x,y) \leq G(y,z,z) + G(z,x,x)$ by (G5). Then, we obtain

$G(x,y,y) + G(x,x,y) \leq G(x,z,z) + G(z,x,x) + G(z,y,y) + G(y,z,z)$.

Thus, $d_G(x,y) \leq d_G(x,z) + d_G(z,y)$. □

Example 3.1.4. Consider the G-metric space (X,G) given in Example 3.1.2. Then, the following function d_G is a metric on X.

$$d_G(x,y) = \begin{cases} 2, & x \neq y \\ 0, & x = y \end{cases}$$

Proposition 3.1.7. Let (X,G) be a G-metric space. Then, the following hold:

(1) $G(x,y,z) \leq G_s(d_G)(x,y,z) \leq 2G(x,y,z)$,

(2) $G(x,y,z) \leq G_m(d_G)(x,y,z) \leq 3G(x,y,z)$.

(Mustafa and Sims,2006)

Proof:

(1) From Proposition 3.1.1 (2), we have

$$G(x,y,z) \leq G(x,x,y) + G(x,x,z)$$
$$G(x,y,z) \leq G(y,y,x) + G(y,y,z)$$
$$G(x,y,z) \leq G(z,z,x) + G(z,z,y).$$

Thus, we get

$$\begin{aligned} G(x,y,z) &\leq \frac{1}{3}[G(x,x,y) + G(x,y,y) \\ &\quad + G(x,x,z) + G(z,z,x) + G(y,y,z) + G(z,z,y)] \\ &= \frac{1}{3}[d_G(x,y) + d_G(y,z) + d_G(x,z)] \\ &= G_s(d_G)(x,y,z). \end{aligned}$$

On the other hand,

<u>Case 1:</u> Let all variables be different. By (G3), we get

$$G(x,y,y) \leq G(x,y,z)$$

$$G(x,x,y) \leq G(x,y,z)$$
$$G(x,z,z) \leq G(x,z,y)$$
$$G(z,x,x) \leq G(z,x,y)$$
$$G(z,y,y) \leq G(z,x,y)$$
$$G(y,z,z) \leq G(y,z,x).$$

Thus, we have

$[G(x,x,y) + G(x,y,y) + G(x,x,z)+G(z,z,x) + G(y,y,z) + G(z,z,y)]$

$\leq 6G(x,y,z)$.

Hence,

$$\frac{1}{3}[d_G(x,y) + d_G(y,z) + d_G(x,z)] = G_s(d_G)(x,y,z) \leq 2G(x,y,z).$$

<u>Case 2:</u> Let $x = y$ and $y \neq z$. By Proposition 3.1.1 (3) and (G3), respectively, we have

$$G(x,z,z) \leq 2G(z,x,x) \leq 2G(x,y,z)$$
$$G(y,z,z) \leq 2G(z,y,y) \leq 2G(x,y,z)$$
$$G(z,x,x) \leq G(x,y,z)$$
$$G(z,y,y) \leq G(x,y,z)$$
$$G(x,y,y) = 0$$
$$G(x,x,y) = 0.$$

Then, we obtain

$\frac{1}{3}[G(x,x,y) + G(x,y,y) + G(x,x,z)+G(z,z,x) + G(y,y,z) + G(z,z,y)]$

$= \frac{1}{3}[d_G(x,y) + d_G(y,z) + d_G(x,z)] = G_s(d_G)(x,y,z) \leq 2G(x,y,z)$.

The other cases are proved similarly.

(2) By Proposition 3.1.1 (2), we obtain

$$G(x,y,z) \leq G(x,x,y) + G(x,x,z)$$
$$G(x,y,z) \leq G(y,y,x) + G(y,y,z)$$
$$G(x,y,z) \leq G(z,z,x) + G(z,z,y).$$

Then, we have

$$3G(x,y,z) \leq G(x,x,y) + G(x,x,z) + G(y,y,x) + G(y,y,z) + G(z,z,x) + G(z,z,y)$$

$$\leq 3\max\{G(x,x,y)+G(y,y,x), G(x,x,z)+G(z,z,x), G(y,y,z)+G(z,z,y)\}$$
$$= 3G_m(d_G)(x,y,z).$$

Thus, $G(x,y,z) \leq G_m(d_G)(x,y,z)$.

On the other hand, we get $G_m(d_G)(x,y,z) \leq 3G(x,y,z)$ from Proposition 3.1.1 (3) and (G3). □

Proposition 3.1.8. Let (X,d) be a metric space. Then, the following hold:

(1) $d_{G_s(d)}(x,y) = \frac{4}{3}d(x,y)$,
(2) $d_{G_m(d)}(x,y) = 2d(x,y)$.

(Mustafa and Sims, 2006)

Proof:

(1) $$d_{G_s(d)}(x,y) = G_s(d)(x,y,y) + G_s(d)(x,x,y) = \frac{1}{3}[d(x,y)+d(y,y)+d(x,y)] + \frac{1}{3}[d(x,x)+d(x,y)+d(x,y)] = \frac{4}{3}d(x,y).$$

(2) $$d_{G_m(d)}(x,y) = G_m(d)(x,y,y) + G_m(d)(x,x,y) = \max\{d(x,y), d(y,y), d(x,y)\} + \max\{d(x,x), d(x,y), d(x,y)\} = 2d(x,y).$$ □

Theorem 3.1.1. Let (X,G) be a G-metric space. The function $d: X\,x\,X \to [0,\infty)$ defined by $d(x,y) = G(x,y,y)$ satisfies the following properties:

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) \leq d(x,z) + d(z,y)$ for any points $x,y,z \in X$.

(Jleli and Samet,2012)

Proof:

(1) Let $d(x,y)=0$. By hypothesis $G(x,y,y)=0$. From Proposition 3.1.1 (1), we have $x=y$. Conversely, let $x=y$. Then, $d(x,y)=d(x,x)=G(x,x,x)=0$ by (G1).

(2) From (G5), we get

$$d(x,y)=G(x,y,y)\leq G(x,z,z)+G(z,y,y)=d(x,z)+d(z,y). \quad \square$$

Definition 3.1.3. Let (X,G) be a G-metric space, then for $x_0\in X,\ r>0$, the G-ball with centre x_0 and radius r is defined by

$$\mathrm{B_G(x_0,r)=\{y\in X\colon G(x_0,y,y)}<r\}$$

(Mustafa and Sims, 2006)

Example 3.1.5. Consider the G-metric space (X,G) given in Example 3.1.2. Then, we have

$$B_G(x_0,r)=\begin{cases} X & , \quad r>1 \\ \{x_0\} & , \quad r\leq 1 \end{cases}$$

for any $x_0\in X$.

Proposition 3.1.9. Let (X,G) be a G-metric space, then for any $x_0\in X$ and $r>0$, the following hold:

(1) If $G(x_0,x,y)<r$, then $x,y\in B_G(x_0,r)$,

(2) If $y\in B_G(x_0,r)$,then there exists a $\delta>0$ such that $B_G(y,\delta)\subseteq B_G(x_0,r)$.

(Mustafa and Sims, 2006)

Proof:

(1) It is obvious from (G3).

(2) Let $y\in B_G(x_0,r)$. Assume $a\in B_G(y,\delta)$. Then, we have $G(a,a,y)<\delta$. By (G5), we get $G(a,a,x_0)\leq G(x_0,y,y)+G(y,a,a)<G(x_0,y,y)+$

δ. Take $\delta = r - G(x_0, y, y)$. Thus, we obtain $G(a, a, x_0) < r$. Hence, $a \in B_G(x_0, r)$. □

Proposition 3.1.10. Let (X, G) be a G-metric space. Then $\mathfrak{B} = \{B_G(x, r) : x \in X, r > 0\}$ is the base of a G-metric topology $\tau(G)$ on X.

(Mustafa and Sims, 2006)

Proof:

(1) By the definition of a G-ball, $x \in B_G(x, r)$ for all $x \in X$. So, $\mathrm{X} = \bigcup_{x \epsilon X} B_G(x, r)$.

(2) Let $B_G(x, r_1), B_G(y, r_2) \in \mathfrak{B}$ and $a \in B_G(x, r_1) \cap B_G(y, r_2)$ for $a \in X$. From Proposition 3.1.9 (2), there exist $\delta_1, \delta_2 > 0$ such that $B_G(a, \delta_1) \subset B_G(x, r_1)$ and $B_G(a, \delta_2) \subset B_G(y, r_2)$. Let us choose a $\delta > 0$ such that $\delta = \min\{\delta_1, \delta_2\}$. Then, $a \in B_G(a, \delta) \subset B_G(x, r_1) \cap B_G(y, r_2)$. □

Proposition 3.1.11. Let (X, G) be a G-metric space, then for all $x_0 \in X$ and $r > 0$, we have

$$B_G(x_0, \tfrac{1}{3}r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r)$$

(Mustafa and Sims, 2006)

Proof:

Let $z \in B_G(x_0, \frac{1}{3}r)$. Then, $G(x_0, z, z) < \frac{1}{3}r$. By Proposition 3.1.1 (3), we have $G(z, x_0, x_0) \leq 2G(x_0, z, z) < \frac{2}{3}r$. So, $d_G(x_0, z) = G(x_0, z, z) + G(z, x_0, x_0) < r$. Hence, $z \in B_{d_G}(x_0, r)$. Therefore, $z \in B_G(x_0, r)$ since $G(z, z, x_0) < r$. □

Corollary 3.1.1. The G-metric topology $\tau(G)$ coincides with the metric topology induced by d_G.

(Mustafa and Sims, 2006)

3.2. G-Convergence and G-Continuity

Definition 3.2.1. Let (X, G) be a G-metric space. The sequence $(x_n) \subseteq X$ is G-convergent to x if it converges to x in the G-metric topology $\tau(G)$.

(Mustafa and Sims, 2006)

Proposition 3.2.1. Let (X, G) be a G-metric space, then for a sequence $(x_n) \subseteq X$ and a point $x \in X$, the following are equivalent.

(1) (x_n) is G-convergent to x.
(2) $d_G(x_n, x) \to 0$, as $n \to \infty$.
(3) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(4) $G(x_n, x, x) \to 0$, as $n \to \infty$.
(5) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

(Mustafa and Sims, 2006)

Proof:

(1) ⇔ (2): It is obvious from Proposition 3.1.11.

(2) ⇒ (3): Let $d_G(x_n, x) \to 0$ as $n \to \infty$. Then, for each $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $d_G(x_n, x) < \varepsilon$ whenever $n \geq N$. By Proposition 3.1.6, we have $G(x_n, x_n, x) \leq G(x_n, x_n, x) + G(x_n, x, x) < \varepsilon$. Thus, $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(2) ⇒ (4): The proof is similar to that of (2) ⇒ (3).

(3) ⇒ (4): It is clear since $G(x_n, x, x) \leq 2G(x_n, x_n, x)$ by Proposition 3.1.1 (3).

(4) ⇒ (5): It follows from Proposition 3.1.1 (2) since $G(x_m, x_n, x) \leq G(x_m, x_m, x_n) + G(x_m, x_m, x)$.

(5) ⇒ (2) Let $G(x_m, x, x) \to 0$ as $m, n \to \infty$. Since $d_G(x_n, x) = G(x_n, x, x) + G(x_n, x_n, x) \leq 2G(x_n, x_n, x) + G(x_n, x_n, x) = 3G(x_n, x_n, x)$ from Proposition 3.1.6 and Proposition 3.1.1 (3), we have $d_G(x_n, x) \to 0$ as $n \to \infty$. □

Definition 3.2.2. Let (X, G) and (X', G') be G-metric spaces and let $f: (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $x_0 \in X$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in X$; $G(x_0, x, x) < \delta$ implies $G'\big(f(x_0), f(x), f(x)\big) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $x_0 \in X$.

(Mustafa and Sims, 2006)

Example 3.2.1. Consider the G-metric G on $\mathbb{R}$ as in Example 3.1.3 (2). Let $f: (\mathbb{R}, G) \to (\mathbb{R}, G)$ be a function defined by $f(x) = 2x$. Then, f is G-continuous. Indeed, given $\varepsilon > 0$, we must find a number $\delta > 0$ such that $G\big(f(x), f(y), f(y)\big) < \varepsilon$ whenever $G(x, y, y) < \delta$ for all $x, y \in \mathbb{R}$. Since $G\big(f(x), f(y), f(y)\big) = \max\{|2x - 2y|, |2x - 2y|, |2y - 2y|\} < \varepsilon$, it follows that $G(x, y, y) = \max\{|x - y|, |x - y|, |y - y|\} < \frac{\varepsilon}{2}$. Hence, we have $\delta = \frac{\varepsilon}{2}$.

Proposition 3.2.2. Let (X, G) and (X', G') be G-metric spaces, then a function $f: X \to X'$ is G-continuous at a point $x_0 \in X$ if and only if it is G-sequentially continuous at x_0; that is, whenever (x_n) is G-convergent to x_0 we have that $(f(x_n))$ is G-convergent to $f(x_0)$.

(Mustafa and Sims, 2006)

Proof :

Let f be G-continuous at x_0 and $(x_n) \to x_0$. Since f is G-continuous at x_0, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $G(x_0, x, x) < \delta$ implies $G'\big(f(x_0), f(x), f(x)\big) < \varepsilon$. As $(x_n) \to x_0$, there exists a natural number N such that for all $n > N$, $G(x_n, x_n, x_0) < \delta$. By hypothesis, $G'\big(f(x_n), f(x_n), f(x_0)\big) < \varepsilon$ for all $n > N$. Hence, $\big(f(x_n)\big) \to f(x_0)$.

Conversely, assume that $(x_n) \to x_0$ implies $\big(f(x_n)\big) \to f(x_0)$ and f is not G-continuous at $x_0 \in X$. Then, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there is an $x \neq x_0$ satisfying $G(x_0, x, x) < \delta$ but $G'\big(f(x_0), f(x), f(x)\big) \geq \varepsilon$. In particular, for $\delta = 1/n$ there is an (x_n) satisfying $G(x_n, x_n, x_0) < \frac{1}{n}$ but $G'\big(f(x_n), f(x_n), f(x_0)\big) \geq \varepsilon$. Thus, $(x_n) \to x_0$ but $\big(f(x_n)\big)$ does not converge to $f(x_0)$. This is a contradiction. Hence, f is G-continuous. □

Proposition 3.2.3. Let (X, G) be a G-metric space, then the function G is jointly continuous at all three of its variables.

(Mustafa and Sims, 2006)

Proof:

Assume (x_k),(y_m) and (z_n) are G-convergent to $x, y\ and\ z$, respectively. From (G5), we get

$$G(x, y, z) \leq G(y, y_m, y_m) + G(y_m, x, z)$$
$$G(z, x, y_m) \leq G(x, x_k, x_k) + G(x_k, y_m, z)$$
$$G(z, x_k, y_m) \leq G(z, z_n, z_n) + G(z_n, y_m, x_k).$$

Thus,

$$G(x, y, z) - G(x_k, y_m, z_n) \leq G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n).$$

Similarly,

$$G(x_k, y_m, z_n) - G(x, y, z) \leq G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z).$$

By Proposition 3.1.1 (3), we have,

$$|G(x_k, y_m, z_n) - G(x, y, z)| \leq 2(G(x, x_k, x_k) + G(y, y_m, y_m) + G(z, z_n, z_n)).$$

Hence, $G(x_k, y_m, z_n) \to G(x, y, z)$, as $k, m, n \to \infty$ and the result follows from Proposition 3.2.2. □

Example 3.2.2. Let (X, G) be a G-metric space and G' be G-metric defined on $\mathbb{R}$ by $G'(a,b,c) = |a-b| + |b-c| + |a-c|$. Then, for the G-continuous functions $f, g: X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, $f+g$ and αf are G-continuous, where

$$f+g: X \to \mathbb{R}, \quad (f+g)(x) = f(x) + g(x)$$
$$\alpha f: X \to \mathbb{R}, \quad (\alpha f)(x) = \alpha f(x).$$

Indeed, given $\varepsilon > 0$, let the sequence $(x_n) \subseteq$ X be G-convergent to $x \in X$. Since f and g are G-continuous, there exists a natural number $N = N(\varepsilon)$ such that for each $n > N$, $G'\big(f(x_n), f(x_n), f(x)\big) = 2|f(x_n) - f(x)| < \frac{\varepsilon}{2}$ and $G'\big(g(x_n), g(x_n), g(x)\big) = 2|g(x_n) - g(x)| < \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned} G'\big((f+g)(x_n), (f+g)(x_n), (f+g)(x)\big) \\ &= 2|f(x_n) + g(x_n) - f(x) - g(x)| \\ &\leq 2(|f(x_n) - f(x)| + |g(x_n) - g(x)|) \\ &< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon \end{aligned}$$

Thus, $f+g$ is G-continuous.

Since f is G-continuous, there exists a natural number $N = N(\varepsilon)$ such that for each $n > N$,

$$G'\big(f(x_n), f(x_n), f(x)\big) = 2|f(x_n) - f(x)| < \frac{\varepsilon}{|\alpha|}.$$

Thus, $G'\big((\alpha f)(x_n), (\alpha f)(x_n), (\alpha f)(x)\big) = 2|\alpha f(x_n) - \alpha f(x)|$

$$\begin{aligned} &= 2|\alpha|.|f(x_n) - f(x)| \\ &< \frac{\varepsilon}{|\alpha|}.|\alpha| = \varepsilon. \end{aligned}$$

Hence, αf is G-continuous.

3.3. G-Completeness

Definition 3.3.1. Let (X, G) be a G-metric space, then a sequence $(x_n) \subseteq X$ is said to be G-Cauchy if for every $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$.

(Mustafa and Sims, 2006)

Proposition 3.3.1. Let (X, G) be a G-metric space. Then the following are equivalent.

(1) The sequence (x_n) is G-Cauchy.
(2) For every $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.
(3) (x_n) is a Cauchy sequence in the metric space (X, d_G).

(Mustafa and Sims, 2006)

Proof:

(1)⟹(2): It is obvious by (G3).

(2) ⇔ (3): It is clear from Proposition 3.1.6.

(2) ⟹(1): It is obvious by (G5) if we set $a = x_m$. □

Corollary 3.3.1. Every G-convergent sequence in any G-metric space is G-Cauchy.

(Mustafa and Sims, 2006)

Proof:

It is obvious by (G5) and Proposition 3.2.1. □

Corollary 3.3.2. If a G-Cauchy sequence in a G-metric space (X, G) contains a G-convergent subsequence, then the sequence itself is G-convergent.

(Mustafa and Sims, 2006)

Proof: It is clear. □

Definition 3.3.2. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

(Mustafa and Sims, 2006)

Proposition 3.3.2. A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.

(Mustafa and Sims, 2006)

Proof: It follows from Proposition 3.1.6 and Proposition 3.3.1. □

Theorem 3.3.1. Let (X, G) be a G-metric space. Let $d: X\ x\ X \rightarrow [0, \infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then, the following hold:

(1) (X, d) is a quasi-metric space,
(2) $(x_n) \subset X$ is G-convergent to $x \in X$ if and only if (x_n) is convergent to x in (X, d),
(3) $(x_n) \subset X$ is G-Cauchy if and only if (x_n) is Cauchy in (X, d),
(4) (X, G) is G-complete if and only if (X, d) is complete.

(Jleli and Samet,2012)

Proof:
(1) It is clear from Theorem 3.1.1.
(2) Let (x_n) be G-convergent to $x \in X$. By Proposition 3.2.1 (4), we have $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. So, (x_n) is convergent to x in (X, d). Similarly, the converse implication is performed again by using Proposition 3.2.1 (4).
(3) Let $(x_n) \subset X$ be G-Cauchy. Given $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$ from Proposition

3.3.1. Thus, $d(x_n, x_m) < \varepsilon$. That is, (x_n) is Cauchy in (X, d). The converse is clear from Proposition 3.3.1.

(4) The proof is obvious by (3) and the definitions of G-completeness and completeness. □

Remark 3.3.1. If (X, d) is a quasi-metric space then the function $\delta: X\ x\ X \rightarrow [0, \infty)$ defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$ is a metric on X.

(Jleli and Samet,2012)

Theorem 3.3.2. Let (X, G) be a G-metric space. Let $\delta: X\ x\ X \rightarrow [0, \infty)$ be the function defined by $\delta(x, y) = \max\{G(x, y, y), G(y, x, x)\}$. Then, the following hold:

(1) (X, δ) is a metric space,

(2) $(x_n) \subset X$ is G-convergent to $x \in X$ if and only if (x_n) is convergent to x in (X, δ),

(3) $(x_n) \subset X$ is G-Cauchy if and only if (x_n) is Cauchy in (X, δ),

(4) (X, G) is G-complete if and only if (X, δ) is complete.

(Jleli and Samet,2012)

Proof:

The proofs are clear since this theorem is a result of Theorem 3.3.1 and Remark 3.3.1. □

4. SOME FIXED POINT THEOREMS ON G-METRIC SPACES

Theorem 4.1. Let (X,G) be a complete G-metric space, and let $T: X \to X$ be a mapping satisfying one of the following conditions:

$$G\big(T(x),T(y),T(z)\big) \le aG(x,y,z) + bG\big(x,T(x),T(x)\big) + cG\big(y,T(y),T(y)\big) + dG\big(z,T(z),T(z)\big) \tag{4.1}$$

or

$$G\big(T(x),T(y),T(z)\big) \le aG(x,y,z) + bG\big(x,x,T(x)\big) + cG\big(y,y,T(y)\big) + dG\big(z,z,T(z)\big) \tag{4.2}$$

for all $x,y,z \in X$ where $0 \le a+b+c+d < 1$, then T has unique fixed point (say u, i.e., Tu=u), and T is G-continuous at u.

(Mustafa et al., 2008)

Proof:

Let T satisfy condition (4.1), then for all $x,y \in X$, we get

$$\begin{aligned} G(Tx,Ty,Ty) &\le aG(x,y,y) + bG(x,Tx,Tx) + cG(y,Ty,Ty) \\ &\quad + dG(y,Ty,Ty) \\ &= aG(x,y,y) + bG(x,Tx,Tx) \\ &\quad + (c+d)G(y,Ty,Ty) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} G(Ty,Tx,Tx) &\le aG(y,x,x) + bG(y,Ty,Ty) + cG(x,Tx,Tx) \\ &\quad + dG(x,Tx,Tx) \\ &= aG(y,x,x) + bG(y,Ty,Ty) + (c+d)G(x,Tx,Tx) \end{aligned} \tag{4.4}$$

Adding (4.3) and (4.4) side by side, we have

$$
\begin{aligned}
d_G(Tx,Ty) &= G(Tx,Ty,Ty) + G(Ty,Tx,Tx) \\
&\leq ad_G(x,y) + b\big(G(x,Tx,Tx) + G(y,Ty,Ty)\big) \\
&\quad +(c+d)(G(y,Ty,Ty) + G(x,Tx,Tx)) \\
&= ad_G(x,y) + (b+c+d)[G(x,Tx,Tx) + G(y,Ty,Ty)]
\end{aligned} \tag{4.5}
$$

If (X,G) is symmetric, then $d_G(x,y) = 2G(x,y,y)$ for each $x,y \in X$. So, from (4.5), we have

$$
d_G(Tx,Ty) \leq ad_G(x,y) + \frac{b+c+d}{2} d_G(x,Tx) + \frac{b+c+d}{2} d_G(y,Ty) \tag{4.6}
$$

for each $x,y \in X$.

Since $0 < a+b+c+d < 1$, the existence and uniqueness of the fixed point follows from well-known theorem in metric space (X,d_G).

If (X,G) is not symmetric, then $\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y)$ for each $x,y \in X$. From (4.5), we have

$$
d_G(Tx,Ty) \leq ad_G(x,y) + \frac{2}{3}(b+c+d)d_G(x,Tx) + \frac{2}{3}(b+c+d)d_G(y,Ty) \tag{4.7}
$$

for each $x,y \in X$.

Since $a + \frac{2}{3}(b+c+d) + \frac{2}{3}(b+c+d)$ may not be less than 1, we can not use metric to prove this case.

Let $x_0 \in X$ be an arbitrary point, and let define the sequence (x_n) by $x_n = T^n(x_0)$. From (4.1), we get

$$
\begin{aligned}
G(x_n,x_{n+1},x_{n+1}) \leq\; & aG(x_{n-1},x_n,x_n) + bG(x_{n-1},x_n,x_n) \\
& +(c+d)G(x_n,x_{n+1},x_{n+1})
\end{aligned} \tag{4.8}
$$

Then, we have

$$G(x_n, x_{n+1}, x_{n+1}) - (c+d)G(x_n, x_{n+1}, x_{n+1}) \leq (a+b)G(x_{n-1}, x_n, x_n).$$

Therefore, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1}, x_n, x_n) \tag{4.9}$$

Set $q = (a+b)/(1-(c+d))$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$. So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n).$$

If we continue, we will have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{4.10}$$

Then, for all $n, m \in N$ such that $n < m$, by (G5), we get

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0, x_1, x_1) \\ &\leq \tfrac{q^n}{1-q} G(x_0, x_1, x_1). \end{aligned} \tag{4.11}$$

That is $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$. Thus (x_n) is G-Cauchy sequence. Since (X, G) is G-complete, there exists $u \in X$ such that (x_n) G-converges to u.

Assume that $T(u) \neq u$, then from (4.3), we have

$$\begin{aligned} G\big(x_n, T(u), T(u)\big) &\leq aG(x_{n-1}, u, u) + bG(x_{n-1}, x_n, x_n) \\ &\quad +(c+d)G(u, T(u), T(u)). \end{aligned} \tag{4.12}$$

Taking the limit as $n \to \infty$, we have

$G\big(u, T(u), T(u)\big) \leq (c + d)G\big(u, T(u), T(u)\big)$ since $(x_n) \to u$. This is a contradiction. Thus, $u = T(u)$.

Suppose that $u \neq v$ such that $T(v) = v$, then

$$
\begin{aligned}
G(u, v, v) &\leq aG(u, v, v) + bG\big(u, T(u), T(u)\big) + (c + d)G\big(v, T(v), T(v)\big) \\
&= aG(u, v, v).
\end{aligned}
\tag{4.13}
$$

That is, $u = v$.

Let $(y_n) \subseteq X$ be a sequence such that $(y_n) \to u$ as $n \to \infty$.

Then, we get

$$
\begin{aligned}
G\big(u, T(y_n), T(y_n)\big) &\leq aG(u, y_n, y_n) + bG\big(u, T(u), T(u)\big) \\
&\quad + (c + d)G\big(y_n, T(y_n), T(y_n)\big) \\
&= aG(u, y_n, y_n) + (c + d)G\big(y_n, T(y_n), T(y_n)\big)
\end{aligned}
\tag{4.14}
$$

By (G5) and (4.14), we obtain

$$
\begin{aligned}
G(u, T(y_n), T(y_n)) &\leq aG(u, y_n, y_n) + (c + d)G(y_n, u, u) \\
&\quad + (\mathrm{c} + \mathrm{d})\mathrm{G}(\mathrm{u}, \mathrm{T}(y_n), T(y_n))
\end{aligned}
$$

Thus, we have

$$\big(1 - (c + d)\big)G\big(u, T(y_n), T(y_n)\big) \leq aG(u, y_n, y_n) + (c + d)G(y_n, u, u)$$

That is,

$$
\begin{aligned}
G\big(u, T(y_n), T(y_n)\big) &\leq (a/(1 - (c + d)))G(u, y_n, y_n) \\
&\quad + (c + d)/(1 - (c + d))G(y_n, u, u).
\end{aligned}
\tag{4.15}
$$

By taking the limit as $n \to \infty$, we have $G(u, T(y_n), T(y_n)) \to 0$ since $(y_n) \to u$. From Proposition 3.2.1, $T(y_n) \to u = Tu$. Thus, T is G-continuous at u by Proposition 3.2.2.

If T satisfies condition (4.2), the proof can be done by using similar argument. □

Corollary 4.1. Let (X,G) be a complete G-metric space and let $T: X \to X$ be a mapping satisfying one of the following conditions:

$$G\big(T^m(x), T^m(y), T^m(z)\big) \le \{aG(x,y,y) + bG\big(x, T^m(x), T^m(x)\big) + cG\big(y, T^m(y), T^m(y)\big) + dG\big(z, T^m(z), T^m(z)\big)\} \tag{4.16}$$

or

$$G\big(T^m(x), T^m(y), T^m(z)\big) \le \{aG(x,y,y) + bG\big(x,x,T^m(x)\big) + cG\big(y,y,T^m(y)\big) + dG\big(z,z,T^m(z)\big)\} \tag{4.17}$$

for all $x, y, z \in X$, where $0 \le a + b + c + d < 1$. Then T has a unique fixed point (say u), and T^m is G-continuous at u.

(Mustafa et al., 2008)

Proof:
From Theorem 4.1, we can say that T^m has unique fixed point (say u), that is, $T^m(u) = u$. Since $T(u) = T\big(T^m(u)\big) = T^{m+1}(u) = T^m(T(u))$, it follows that $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 4.2. Let (X,G) be a G-metric space and let $T: X \to X$ be a mapping such that T satisfies

(A1) $G(Tx,Ty,Tz) \le aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz)$ for all $x,y,z \in X$ where $0 < a + b + c < 1$,
(A2) T is G-continuous at a point $u \in X$,
(A3) there is $x \in X$; $\big(T^n(x)\big)$ has a subsequence $\big(T^{ni}(x)\big)$ G-converges to u.
Then u is a unique fixed point (i.e., $Tu = u$).

(Mustafa et al., 2009)

Proof:

By (A2) and Proposition 3.2.2, $\left(T^{ni+1}(x)\right)$ G-converges to $T(u)$. Assume that $T(u) \neq u$ and consider the two G-open balls $B_1 = B(u, \epsilon)$ and $B_2 = B(Tu, \epsilon)$ where $\epsilon < (1/6)\min\{G(u, Tu, Tu), G(Tu, u, u)\}$.

As $T^{ni}(x) \to u$ and $T^{ni+1}(x) \to Tu$, there exists a natural number $N_1 = N_1(\varepsilon)$ such that if $i > N_1$ implies $T^{ni}(x) \in B_1$ and $T^{ni+1}(x) \in B_2$. Hence, we must have

$$G\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right) > \epsilon \text{ for each } i > N_1. \qquad (4.18)$$

From (A1), we get

$$\begin{aligned} G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq\ & aG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right) \\ & + bG\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\right) \\ & + cG\left(T^{ni+2}(x), T^{ni+3}(x), T^{ni+3}(x)\right) \end{aligned} \qquad (4.19)$$

but, by (G3), we obtain

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\right) \leq G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right), \qquad (4.20)$$

$$G\left(T^{ni+2}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right). \qquad (4.21)$$

Then,

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq qG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right), \qquad (4.22)$$

where $q = a/(1-(b+c))$ and $q < 1$.

So, by inequalities (4.20) and (4.22), we get

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\right) \leq qG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right). \tag{4.23}$$

For $l > j > N_1$ and by (4.23), we have

$$\begin{aligned} G\left(T^{n_l}(x), T^{n_l+1}(x), T^{n_l+1}(x)\right) &\leq qG\left(T^{n_l-1}(x), T^{n_l}(x), T^{n_l}(x)\right) \\ &\leq q^2 G\left(T^{n_l-2}(x), T^{n_l-1}(x), T^{n_l-1}(x)\right) \\ &\vdots \\ &\leq q^{n_l-n_j} G\left(T^{n_j}(x), T^{n_j+1}(x), T^{n_j+1}(x)\right) \end{aligned} \tag{4.24}$$

So, as $l \to \infty$ we have $\lim G\left(T^{n_l}(x), T^{n_l+1}(x), T^{n_l+1}(x)\right) \leq 0$ which contradict (4.18). Hence $Tu = u$.

To prove uniqueness of u, suppose that u$\neq v$ such that $Tv = v$. Then, we have

$$G(u, v, v) = G(Tu, Tv, Tv) \leq aG(u, Tu, Tu) + (b + c)G(v, Tv, Tv) = 0. \tag{4.25}$$

which implies that $u = v$. □

Theorem 4.3. Let (X, G) be a complete G-metric space and let $T: X \to X$ be a mapping satisfying for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz) \tag{4.26}$$

where $0 < b + c + d < 1$, then T has a unique fixed point, say u, and T is G-continuous at u.

(Mustafa et al., 2009)

Proof: This theorem follows from Theorem 4.1 if $a = 0$ is taken. □

Example 4.1. Let $X = [0,1)$, $T(x) = x/4$ and $G(x,y,z) = \max\{|x - y|, |y - z|, |x - z|\}$. Then (X,G) is G-metric space but not complete since the sequence $x_n = 1 - 1/n$ is G-Cauchy which is not G-convergent in (X,G). However, conditions (2) and (3) in Theorem 4.2 are satisfied.

(Mustafa et al., 2009)

Theorem 4.4. Let (X,G) be a G-metric space and let $T: X \to X$ be a G-continuous mapping satisfying the following conditions:

(B1) $G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\}$ for all $x,y,z \in M$ where M is an everywhere dense subset of X (with respect to the topology of G-metric convergence) and $0 < k < 1/6$,

(B2) there is $x \in X$ such that $\left(T^n(x)\right) \to x_0$. Then x_0 is unique fixed point.

(Mustafa et al., 2009)

Proof:

Let $x,y,z \in X$.

<u>Case 1:</u> If $x,y,z \in X \backslash M$, let (x_n), (y_n), and (z_n) be sequences in M such that $(x_n) \to x$, $(y_n) \to y$ and $(z_n) \to z$. By (G5), we get

$$G(Tx,Ty,Tz) \leq G(Tx,Ty,Ty) + G(Ty,Ty,Tz), \tag{4.27}$$

Again, by using (G5) twice, we have

$$\begin{aligned} G(Tz,Ty,Ty) &\leq G(Tz,Tz_n,Tz_n) + G(Tz_n,Ty,Ty) \\ &\leq G(Tz,Tz_n,Tz_n) + G(Tz_n,Ty_n,Ty_n) + G(Ty_n,Ty,Ty) \end{aligned} \tag{4.28}$$

From (B1) and (G5), we obtain

$$\begin{aligned} G(Tz_n,Ty_n,Ty_n) &\leq k\{G(z_n,Tz_n,Tz_n) + 2G(y_n,Ty_n,Ty_n)\} \\ &\leq k\{G(z_n,z,z) + G(z,Tz,Tz) + G(Tz,Tz_n,Tz_n) \\ &\quad +2[G(y_n,y,y) + G(y,Ty,Ty) + G(Ty,Ty_n,Ty_n)]\} \end{aligned} \tag{4.29}$$

From (4.28) and (4.29),

$$G(Tz,Ty,Ty) \leq (1+k)G(Tz,Tz_n,Tz_n) + G(Ty_n,Ty,Ty) + kG(z_n,z,z) + 2kG(y_n,y,y) + 2kG(Ty,Ty_n,Ty_n) + kG(z,Tz,Tz) + 2kG(y,Ty,Ty) \tag{4.30}$$

In a similar way, we find

$$G(Tx,Ty,Ty) \leq (1+k)G(Tx,Tx_n,Tx_n) + G(Ty_n,Ty,Ty) + kG(x_n,x,x) + 2kG(y_n,y,y) + 2kG(Ty,Ty_n,Ty_n) + kG(x,Tx,Tx) + 2kG(y,Ty,Ty). \tag{4.31}$$

Hence by (4.30) and (4.31), we get

$$\begin{aligned} G(Tx,Ty,Tz) &\leq G(Tx,Ty,Ty) + G(Tz,Ty,Ty) \\ &\leq \{(1+k)G(Tx,Tx_n,Tx_n) + G(Ty_n,Ty,Ty) \\ &\quad + kG(x_n,x,x) + 2kG(y_n,y,y) + 2kG(Ty,Ty_n,Ty_n) \\ &\quad + kG(x,Tx,Tx) + 2kG(y,Ty,Ty)\} \\ &\quad + \{(1+k)G(Tz,Tz_n,Tz_n) + G(Ty_n,Ty,Ty) \\ &\quad + kG(z_n,z,z) + 2kG(y_n,y,y) \\ &\quad + 2kG(Ty,Ty_n,Ty_n) + kG(z,Tz,Tz) \\ &\quad + 2kG(y,Ty,Ty)\}. \end{aligned} \tag{4.32}$$

Taking the limit as $n \to \infty$, we have

$$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + 4G(y,Ty,Ty) + G(z,Tz,Tz)\}, \tag{4.33}$$

since T is G-continuous.

<u>Case 2:</u> If $x, y \in M$ and $z \in X \setminus M$, let (z_n) be a sequence in M such that $(z_n) \to z$. By using (G5) and (B1), we have

$$\begin{aligned} G(Tx,Ty,Tz) &\leq G(Tx,Ty,Ty) + G(Tz,Ty,Ty) \\ &\leq k\{G(x,Tx,Tx) + 2G(y,Ty,Ty)\} + G(Tz,Ty,Ty) \\ &\leq k\{G(x,Tx,Tx) + 2G(y,Ty,Ty)\} + G(Tz,Tz_n,Tz_n) + \\ &\quad G(Tz_n,Ty,Ty) \\ &\leq k\{G(x,Tx,Tx) + 2G(y,Ty,Ty)\} + G(Tz,Tz_n,Tz_n) + \end{aligned}$$

$$k\{G(z_nTz_n, Tz_n) + 2G(y, Ty, Ty)\}$$
$$\leq k\{G(x, Tx, Tx) + 2G(y, Ty, Ty)\} +$$
$$G(Tz, Tz_n, Tz_n) + k\{G(z_n, z, z) +$$
$$G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n) + 2G(y, Ty, Ty)\}$$
$$= k\{G(x, Tx, Tx) + 2G(y, Ty, Ty) + G(z_n, z, z) +$$
$$G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n) + 2G(y, Ty, Ty)\} +$$
$$G(Tz, Tz_n, Tz_n)$$
(4.34)

Taking the limit as $n \to \infty$, we get

$$G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz)\}$$
(4.35)

Case 3: If $y \in M$ and $x, z \in X \backslash M$, let (x_n) and (z_n) be a sequence in M such that $(x_n) \to x$ and $(z_n) \to z$. By (G5) and (B1), we have

$$G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty)$$
$$\leq G(Tx, Tx_n, Tx_n) + G(Tx_n, Ty, Ty) + G(Tz, Ty, Ty)$$
$$\leq G(Tx, Tx_n, Tx_n) + k\{G(x_n, Tx_n, Tx_n) + 2G(y, Ty, Ty)\}$$
$$+G(Tz, Ty, Ty)$$
$$\leq G(Tx, Tx_n, Tx_n) + k\{G(x_n, x, x) + G(x, Tx, Tx) +$$
$$G(Tx, Tx_n, Tx_n) + 2G(y, Ty, Ty)\} + G(Tz, Ty, Ty)$$
$$= (1 + k)G(Tx, Tx_n, Tx_n) + kG(x_n, x, x) +$$
$$kG(x, Tx, Tx) + 2kG(y, Ty, Ty) + G(Tz, Ty, Ty)$$
$$\leq (1 + k)G(Tx, Tx_n, Tx_n) + kG(x_n, x, x) + kG(x, Tx, Tx)$$
$$+ 2kG(y, Ty, Ty) + kG(z_n, z, z) + kG(z, Tz, Tz)$$
$$+ (1 + k)G(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty)$$
(4.36)

Since T is G-continuous, taking the limit as $n \to \infty$, we get

$$G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz)\}$$
(4.37)

Thus, for each $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz)$$

where $a = k,\ b = 4k,\ c = k$, and $a + b + c < 1$ since $0 < k < 1/6$. Then from Theorem 4.2, T has a unique fixed point. □

Theorem 4.5. Let (X, d) be a complete quasi-metric space and $T: X \to X$ be a mapping satisfiying

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

for all $x, y \in X$, where $\varphi: [0, \infty) \to [0, \infty)$ is continuous with $\varphi^{-1}(\{0\}) = \{0\}$. Then T has a unique fixed point.

(Jleli and Samet, 2012)

Theorem 4.6. Let (X, G) be a G-complete metric space and $T: X \to X$ be a mapping satisfiying

$$G(Tx, Ty, Ty) \leq G(x, y, y) - \varphi(G(x, y, y)),$$

for all $x, y \in X$, where $\varphi: [0, \infty) \to [0, \infty)$ is continuous with $\varphi^{-1}(\{0\}) = \{0\}$. Then T has a unique fixed point.

(Jleli and Samet, 2012)

Proof: The proof follows from Theorem 3.1.1 and Theorem 4.5. □

5. CONCLUSION

In this thesis, firstly, the concept of G-metric is introduced and its various properties are studied. Moreover, one studies how to deduce a G-metric from a metric and vice versa. Then, relationships between these metrics are investigated. Also, the definitions of G-metric topology and G-ball are given. After then, the concepts of G-convergence, G-continuity and G-completeness are handled, and several examples have supported the understanding of these concepts. At the end, some fixed point theorems on G-metric spaces are studied.

For a further research, one can obtain new fixed point theorems on such spaces.

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