YAŞAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MASTER THESIS

CONTROL OF M/Coxian-2/s MAKE-TO-STOCK PRODUCTION SYSTEMS

Özgün ÖZTÜRK

Thesis Advisor: Asst. Prof. Dr. Önder BULUT

Department of Industrial Engineering

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2016

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

Asst. Prof. Dr. Önder BULUT (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

Prof. Dr. Murat FADILOĞLU

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

Asst. Prof. Dr. Zeynep SARGUT

Prof. Dr. Cüneyt GÜZELİŞ
Director of the Graduate School

ABSTRACT

CONTROL OF M/Coxian-2/s MAKE-TO-STOCK PRODUCTION SYSTEMS

ÖZTÜRK, Özgün
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In this thesis, we consider a make-to-stock production environment with multiple processing channels, several customer classes, fixed production start-up costs and lost sales. Demands of customer classes are generated from independent Poisson processes. Processing times are assumed to be independent two-phase Coxian random variables. Each phase of Coxian distribution is an exponential random variable corresponding to a specific stage in production and there is a certain visiting probability from phase-one to phase-two. Phase-type processing time assumption allows to model a system with a rework/inspection operation. The problem is to control the production and allocate the on hand inventory among different customer classes. We extend the production-inventory control literature by considering phase-type production times, several customer classes, parallel production channels and start-up cost in a single model. First, the dynamic programming formulation is developed and optimal production and rationing policies are characterized under average system cost criterion. Furthermore, a dynamic rationing policy and several production policies are proposed and their performance analyses are carried out. The final contribution of this thesis is to propose a new method, based on renewal theory, to calculate the long-run average system cost under the optimal production and static rationing policies when there is a single processing channel.

Keywords: make-to-stock, inventory-production control, phase-type processing times, multiple production channels, start-up cost.

ÖZET

M/Coxian-2/s STOĞA-ÜRETİM SİSTEMLERİNİN KONTROLÜ

Özgün ÖZTÜRK
Yüksek Lisans Tezi, Endüstri Mühendisliği Bölümü
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Bu tezde sabit hazırlık ve kayıp satış maliyetlerini içeren, paralel üretim kanalları ve birden çok müşteri sınıfının bulunduğu stoğa-üretim sistemlerinde üretim ve stok tayınlama kontrolü ele alınmaktadır. Müşteri taleplerinin bağımsız Poisson süreçleri uyarınca geldiği ve üretim zamanlarının iki-aşamalı Coxian dağılıma sahip olduğu varsayılmıştır. Coxian dağılımının her bir aşaması Üssel rassal değişkeni olmakla birlikte bu aşamalar üretimin belirli bir fazına tekabül etmektedir. Ayrıca, birinci üretim aşamasından ikinci aşamaya belirli bir olasılık ile geçilmektedir. Bu çalışmada dikkate alınan faz-tipi üretim zamanları, yeniden işleme/kontrol operasyonlarının modellenmesine imkan vermektedir. Problem, üretim kontrolü ve eldeki envanterin müşteri sınıfları arasında ayrıştırılmasını kapsamaktadır. Faz-tipi üretim zamanları, farklı müşteri sınıfları, paralel üretim kanalları ve sabit hazırlık maliyeti tek bir modelde ele alınarak üretim-envanter kontrolü literatürü genişletilmektedir. İlk olarak, dinamik programlama formülasyonu geliştirilmiş ve en iyi üretim ve tayınlama politikaları, ortalama sistem maliyeti kriteri baz alınarak karakterize edilmiştir. Sistem durum bilgisinin kullanıldığı dinamik bir stok tayınlama politikası ile alternatif üretim politikaları önerilmiş, performans analizleri yapılmıştır. Son olarak, yenileme teorisi baz alınarak geliştirilen yeni yöntem ile tek üretim kanallı sistem için en iyi üretim politikası ve statik tayınlama politikası altında ortalama sistem maliyeti hesaplanmıştır.

Anahtar sözcükler: stoğa üretim, üretim-envanter kontrolü, faz-tipi üretim zamanları, paralel üretim kanalları, hazırlık maliyeti.

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Özgün ÖZTÜRK
İzmir, 2016

TEXT OF OATH

I declare and honestly confirm that my study, titled “Control of M/Coxian-2/s Make-to-Stock Production Systems” and presented as a Master’s Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

Özgün ÖZTÜRK
İzmir, 2016

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INDEX OF SYMBOLS AND ABBREVIATIONS

Symbols	Explanations
μ_i	Production rate of phase i
β	Certain visiting probability
λ_j	Demand rate of customer class j
c_j	Lost sale cost of customer class j
h	Holding cost rate
K	Fixed start-up cost
α	Discount rate
ν	Uniform transition rate
s	Number of available production channels/servers
$X_i(t)$	Number of active channels at stage i at time t
$X_3(t)$	Inventory level at time t
$u_p^{(x_1,x_2,x_3)}$	Production decision for a given state
$u_{r_j}^{(x_1,x_2,x_3)}$	Rationing decision of customer class j for a given state
X^*	Lower production control level
X^{**}	Upper production control level

Abbreviations

CTMC	Continuous Time Markov Chain
ETTP	Expected Time to Produce
FCFS	First Come First Served
ILP	Inventory Level Policy
IPP	Inventory Position Policy
MDP	Markov Decision Process
MIPP	Modified Inventory Position Policy
MWSL	Modified Work Storage Level
OP	Optimal Production Policy
RA	Renewal Analysis
WSL	Work Storage Level

1 INTRODUCTION

In this thesis, we consider a make-to-stock production environment with multiple processing channels, several customer classes, fixed production start-up costs and lost sales. In a make-to-stock production system, there is always a tradeoff between excess inventory, shortages and production costs. Production control is the main tool handling this tradeoff and providing cost effective operation. However, nowadays, in addition to the production control, customers are also differentiated based on their service level requirements or lost sale costs. Almost all the parties in a supply chain develop stock reservation strategies in anticipation of future demand arrival of their privileged customers. Generally, the idea behind the differentiation is to manage the variation among customers in order to provide effective service.

In general, in a make-to-stock environment, a production control decision requires starting production at the right time and producing with the optimum number of channels to provide sufficient amount of products. A stock reservation strategy is also required for the inventory allocation among the several customer classes. In the literature and practice such strategies are referred as inventory rationing strategies. Inventory rationing reserves some portion of the inventory in anticipation of demand arrivals from the customer classes having higher priorities by rejecting demands from the other classes when the *inventory status* drops below certain threshold levels corresponding to different classes. Here, *inventory status* refers a function of the state variables that keep track of the required system information such as inventory level, number of outstanding production orders and their ages. The form of the optimal inventory status function would change from system to system but it is still unknown even for most of the basic inventory or production-inventory settings. Therefore, most of the studies in the literature either assumes that inventory status equals inventory level, which is referred as static rationing, or they propose approximate functional forms including other system variables, which is referred as dynamic rationing. Bulut and Fadiloglu (2011), Liu and Zhang (2015), and Özkan (2016) provide extensive discussions on the optimality of rationing policies for inventory and production-inventory systems, respectively.

In order to better understand our problem, we explain it using a supply chain illustrated in Figure 1.1. Suppose there is a specific type of product which is delivered to the end customers through the supply chain described in the future. The first echelon of the chain is for the raw material suppliers that provides necessary materials to manufacturers. At the second stage, manufacturers process raw materials and deliver finished products to the retailers where the end customers have access to the products. Let the manufacturers have multiple processing or assembly channels and at each alternative channel there is also a rework/inspection operation. All these manufacturers are actually the customers of raw material suppliers. Similarly, retailers are the customers of manufacturers. It is better for raw material suppliers, manufacturers and retailers to ration their inventories by classifying their customers. For instance, for a specific raw material supplier, some of its customers, that is some of the manufacturers, might be more valuable than the others. This value might come from their high market shares, high shortage costs/service level requirements or their long term contracts. It is more cost effective for the raw material supplier to reserve some inventory for this class of valuable customers. Thus, at all the levels of the chain (excluding the end customers) all the parties would develop their production and rationing strategies to operate their own systems by balancing holding, shortage and production costs.

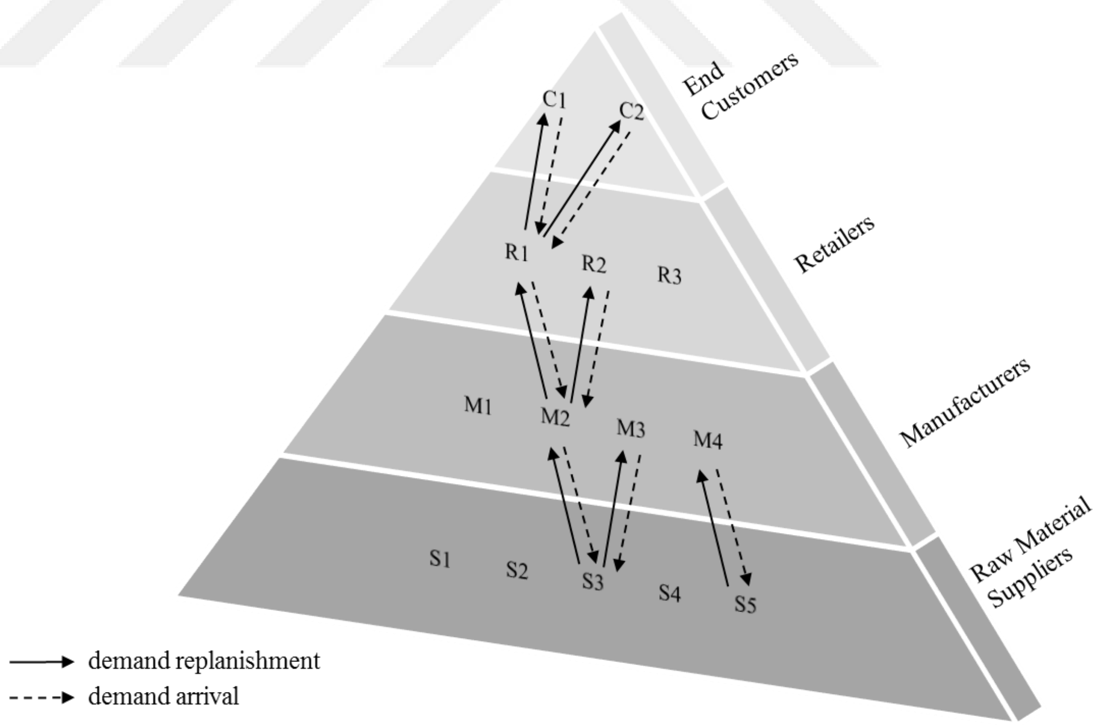

Figure 1.1 An Explanatory Example of a Supply Chain Network

Production times have different structures in different industries and companies. Due to the nature of the production environment and its technology, production times might have zero, moderate or high variance. In order to consider the systems with rework, we assume phase-type production times at each alternative channel. In specific, production times are assumed to be independent 2-phase Coxian random variables. In practice, a production channel might be considered as a machine such that rework operation is also done on the same machine whenever needed. Furthermore, a channel might be a worker/operator of a labor-intensive system who performs the main operation and inspection/rework once in a while. A busy worker would be either busy at the first phase (main operation) or at the second phase (inspection/rework) at any given time. We also assume demands from different customer classes arrive according to independent stationary Poisson processes. In the make-to-stock production literature, phase-type processing times, several customer classes and multiple production channels are not yet studied at the same time.

The rationale behind 2-phase Coxian processing times extension is the following: i. the second stage of the production process (the second phase of Coxian random variable) can be considered as a rework/inspection operation, ii. since 2-phase Coxian consists of Exponential stages, the study directly extends Bulut and Fadıloğlu (2011) that assumes a single exponential stage, iii. Our study is a multi-server extension of Lee and Hong (2003) that considers Coxian processing times for single channel system and focuses solely on static policies, iv. when the production rates are equal at each stage of Coxian and all the items certainly visit the second stage, the model turns out to be one that enables us to study the multi-server systems with Erlangian processing times, v. Using Coxian production times we preserve the Markovian structure and are able to use Markov Decision Process (MDP) techniques. The representation of a production channel feeding the inventory after a processing time that is a two-phase Coxian random variable is shown in Figure 1.2 below.

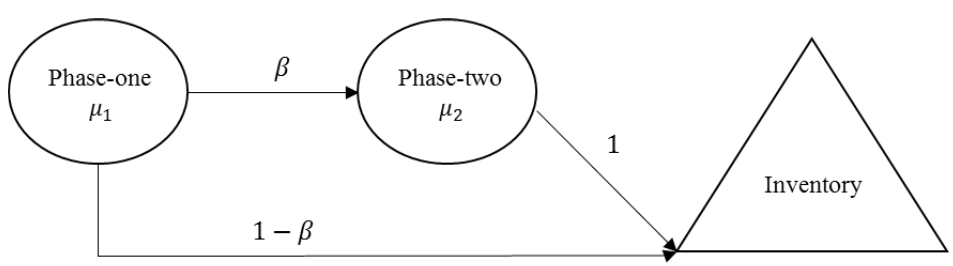

Figure 1.2 Representation of a production channel of our system

Visiting probability of second stage, β, enables us to work on more general systems than the ones having exponential processing times, which is a classical assumption in the literature. Different values of β, μ_1 and μ_2 corresponds to systems with different rework characteristics and processing time moments.

Due to the above assumptions, we model the system as an $M/Coxian-2/s$ make-to-stock queue with several demand classes, fixed start-up costs and lost sales. In the terminology of production-inventory control literature, the classical Kendall Lee queueing notation is used for the models of make-to-stock systems. However, the meaning of the queuing notation is slightly different in the make-to-stock environment. In our case, *M* denotes Markovian inter-demand arrival times but the arrived demands do not enter a queue and trigger a production order, instead they are either directly satisfied from the inventory or lost/rejected and immediately leave the system. The second entry in the notation, which is "Coxian-2" in our case, is for the production time distribution. The inventory is replenished using *s* many available production channels (servers) according to a production policy in anticipation of the future demand arrivals. That is, Coxian-2 is not the "service" time of each demand arrival, it is the replenishment lead time of any production order triggered according to the policy.

Our study extends the related literature in several aspects. Initially, optimal or approximate dynamic stock rationing policies for both single and multiple channel systems having the production channel structure shown in Figure 1.2. has not been touched yet. Even though the structure of the optimal production policy is known for the single server systems, which is a base stock policy, there is no study in the literature on the production control of multi-channel systems. Lee and Hong (2003) is the only study considering the similar production structure but they assume single channel and static rationing. In this thesis, we both characterize the optimal production and rationing policies of $M/Coxian-2/s$ make-to-stock queue and conduct performance analyses for several alternative production and rationing policies. In addition to these, we also study the effect of fixed production (start-up) cost on the structure of the optimal production and rationing policies.

We provide the literature review in Section 2. Section 3 is devoted to dynamic programming formulation of the problem, characterization of the optimal production and rationing policies, and an extensive numerical study depicting the effect of system parameters on the optimal policies. We propose a new dynamic rationing policy and test its performance in Section 4. Section 5 is for the performance analysis of several alternative production policies each using the information carried by the system state vector in a different way. In Section 6, using a renewal approach, we calculate the long-run expected average system cost under optimal production and static rationing policies. Section 7 concludes the thesis and provides directions for future research.

2 LITERATURE REVIEW

In this chapter, we review the production and inventory control literature in the make-to-stock environment. This problem is first attacked by considering the systems having single production channel and single customer/demand class. Analyses are mostly based on queueing theory techniques. Interestingly, the early studies consider the fixed startup or shut-down costs. More recent studies extend the literature either by considering multiple customer classes or multiple production channels without fixed costs. Another common feature of the recent studies is the Markovian structure that enables them to develop Markov Decision Process (MDP) formulation for the control of make-to-stock systems. In our study, preserving the Markovian structure, we consider multiple customer classes, multiple production channels allowing reworks and fixed start-up costs at the same time.

Gavish and Graves (1980) is the first to study the production-inventory problem assuming single channel, fixed and deterministic production times, independent Exponential inter-demand-arrival times, and backorders. He modeled the problem as a $M/D/1$ make-to-stock queue in the infinite horizon under the time-average cost criterion. This first study is actually the extension of Heyman (1968) and Sobel (1969) studies to the make-to-stock production environment. Heyman (1968) and Sobel (1969) study $M/G/1$ and $G/G/1$ queueing systems, respectively, operating with server start-up and shut-down costs, and unit service and queue-time costs. For both of the settings, it is shown that the optimal policy is a *two critical number policy* denoted by (S, s) policy and (M, m) policy in Heyman (1968) and Sobel (1969), respectively. If the queue length is less than or equal to m (or s), service is not provided until queue length increases to M (or S). Service is triggered when the queue length is M and continued until it drops to m again. Although the analyses of Heyman (1968) and Sobel (1969) are specific for the queueing environment, we believe that their setting covers the production control for make-to-order systems.

The optimal policy structure, which is a *two critical number policy*, is preserved in the make-to-stock production environment setting of Gavish and Graves (1980). However, the control parameters of the policy are defined on the inventory level: start production when the inventory level hits to the lower control level and continue until it hits to the upper control level.

Gavish and Graves (1981) extends the findings of Gavish and Graves (1980) to the general service time setting modelled as an $M/G/1$ make-to-stock queue. Graves and Keilson (1981) again studies a single machine, single customer class production-inventory system with backordering and start-up costs, and extends the literature by considering a compound Poisson demand structure where demand size is another exponential random variable. They show that the structure of the policy is still a two-critical-number policy denoted by (I^{*}, I^{**}). In addition to this study, there are also several other studies considering compound arrivals:

Moreover, there are several studies in production/inventory environment that consider compound arrivals: Doshi et al. (1978) analyzes a continuous review production/inventory system in backorders and lost sales environment. Demands are assumed to be compound Poisson arrival process and fixed cost is incurred for each switch over the production rate. They consider two-critical-levels for the control, i.e. if upper level is reached, production rate is switched from fast to slow, if inventory level is below the lower level, then production rate is switched from slow to fast. De Kok et al. (1984) considers similar problem to Doshi et al. (1978) by developing switch-over level approximations. In addition to these studies, Altiok (1989) studies phase-type unit production rate with compound Poisson demand process. He controls production using continuous review (R, r) policy and calculates steady-state probabilities in order to obtain minimum cost for given R and r values in both backordering and lost sales cases. On the other hand, Lee and Srinivasan (1989) considers a single production channel with fixed startup cost. Demand arrivals are according to a Poisson process and processing times are assumed to be arbitrary distribution. Backordering cost is occurred whenever inventory is not available. They propose a renewal analysis in order to calculate expected cost. In consideration of two-critical-level policy, they define a production and non-production period and calculate the expected cost during periods. Since the horizon is infinite, accumulated cost during periods converges to a value. Afterwards, they obtain expected system cost. They also extend their work by considering compound Poisson process (Lee and Srinivasan, 1991).

After a while, researchers apply MDP analysis since the structure of the problems is Markovian. Except for Lee and Hong (2003), the literature assumes that there is no fixed production/setup cost. In addition, production is triggered by a single server except that Bulut and Fadıloğlu (2011). To the best of our knowledge, there is no such a study that considers multiple production channels and processing times different from Exponential random variables simultaneously.

Ha (1997a) is the first study that uses MDP techniques in problem modeling. Demand arrives according to a Poisson process and production times are assumed to be independent Exponential random variables. The study considers make-to-stock production environment with single production channel, multiple demand classes and lost sales without fixed startup cost. Since multiple demand classes term is considered, he first defines stock rationing problem in production environment. In study of Ha (1997a), demand classes are differentiated based on their lost sales cost. Another way to ration customers is to classify them based on their service level requirements, it can be seen such studies in the literature. Ha (1997a) models the problem as an $M/M/1$ make-to-stock queue and he shows that base-stock policy is optimal production control policy. He also shows that stock-reservation policy is optimal for rationing inventory. This policy indicates that each demand class has a rationing level and it is optimal to satisfy a demand from a class if the inventory level is greater than rationing level of that class. He proposes a stationary analysis of the system based on two demand classes and performs comparisons with FCFS policy to test the power of the rationing. Ha (1997b) considers an $M/M/1$ make-to-stock queue with two demand classes and backorders. In case demand is not satisfied, customers join the backorder queues of their classes. Customers are differentiated by their waiting cost i.e. high priority customer classes have higher waiting cost. Ha (1997b) defines a two-variable system state such that inventory level and number of class-two backorders since negative inventory level implies the number of class-one backorders. He shows that base-stock policy is optimal policy for production control and static-threshold level policy is optimal for rationing. Vericourt et al. (2002) addresses the extension of Ha (1997b) by considering n different customer classes. Bulut and Fadıloğlu (2011) contributes the literature with multiple production channels. Bulut and Fadıloğlu (2011) extends the work of Ha (1997a) and model the problem as an $M/M/s$ make-to-stock queue with multiple demand classes and lost sales. System state includes inventory level and number of active channels at a given time. They show that optimal production policy is a state-dependent base-stock policy and optimal rationing policy is a threshold type policy which is a function of active servers. There is a threshold inventory level for each class and it is optimal to satisfy incoming demand from a customer class above the threshold level of that class, otherwise it is optimal to reject. Speaking of threshold level, it is optimal to satisfy a demand from class-one, whenever there is an on-hand inventory. They also embed full order and partial order cancellation flexibility to the model and perform stationary analysis under a base-stock policy. Özkan (2016) extends the study of Bulut and Fadıloğlu (2011) by adding fixed start-up cost to the $M/M/s$ make-to-stock environment. Thus far, production times are assumed to be exponential random variables. Since exponential distribution has a memoryless property, current production status does not provide an information except from multi-server cases such that Bulut and Fadıloğlu (2011). Ha (2000) analyzes a make-to-stock queue with Erlangian processing times that allow to keep track of current status of the production. Ha (2000) assumes multiple demand classes and lost sales and problem is modeled as an $M/E_k/1$ make-to-stock queue. System state keeps the number of completed stages and inventory level at a given time and state variables define the work storage level (WSL). He shows that a critical work level policy is optimal for production control and inventory rationing control. It is optimal to produce if the WSL is below the critical work level and not to produce otherwise. It is optimal to satisfy a demand of a class if the WSL is above critical work storage level of that class and reject otherwise. Gayon et al. (2009) differs from Ha (2000) with a backordering assumption. It is shown that it is optimal to produce if the WSL is lower than a given threshold level and not to produce otherwise. Also optimal rationing policy is characterized by n customer classes work storage rationing thresholds. Lee and Hong (2003) is the study that considers non-exponential processing times and fixed start-up cost. They analyze a production system controlled by two-critical levels i.e. (s, S) type policy, multiple demand classes and lost sales. Single channel production environment is considered and processing times are assumed to be two-phase Coxian random variables. Problem is modeled as continuous time Markov Chain and average operating cost is obtained via steady-state probabilities. System state covers the inventory level and the current phase of the production. They propose a heuristic algorithm to obtain rationing levels for customer classes under static rationing.

Inventory rationing problem is initiated by Veinott (1965) in a backordering environment. Ordering policy is the base-stock policy and there are different service levels between customer classes. Topkis (1968) shows that the optimal inventory rationing policy is a dynamic threshold policy for periodic review systems with zero lead time. Nahmias and Demmy (1981) considers a military depot in a backordering environment and describe service levels under static rationing for an (r, Q) continuous system. Deshpande et al. (2003) considers two demand classes and backorders and proposes an approach for static threshold levels for an (r, Q) continuous system. Later on, Fadıloğlu and Bulut (2010) analyzes a dynamic rationing policy for continuous review inventory systems called Rationing with Exponential Replenishment Flow (RERF). It is shown that policy depends on the ages and the numbers of outstanding orders. In recent times, Pang et al. (2014) considers a make-to-stock production environment with multiple demand classes, lost sales and no fixed cost. Batch demand arrival is permitted and arbitrary, phase-type and Erlangian processing times are considered. They show that optimal rationing levels are time-dependent. Liu and Zhang (2015) studies an inventory system with two demand classes and backordering. They propose an approximate closed-form expression for dynamic critical levels. Liu et al. (2015) performs a two-step approach based on certainty equivalence principle for multiple demand classes. They obtain closed-form expressions for rationing thresholds.

3 THE MODEL AND THE ANALYSES OF OPTIMAL POLICIES

In this chapter, it is characterized both optimal production and rationing policies in environment of single product, fixed start-up cost, multiple parallel production channels, multiple customer classes and lost sales. It is assumed that demands arrive according to a stationary Poisson process with rate $\lambda_j, j \in \{1, \dots, N\}$ for a customer class j. Since there are several customer classes, they are differentiated based on their lost sale costs. For each unsatisfied demand of customer class j, lost sale cost c_j is incurred. Without loss of generality, it is assumed that $c_1 \geq \cdots \geq c_N$. Processing times are assumed to be two-phase Coxian random variables. Each phase of Coxian distribution is exponentially distributed with rates μ_1 and μ_2 respectively and there is a certain visiting probability $\beta \in [0,1]$ from phase-one to phase-two (see Figure 1.2 for the illustration of a production channel having 2-phase Coxian processing times).

Triggered production is started at phase-one. After processing at phase-one with rate μ_1, it is either passed to phase-two with probability β, processed there with rate μ_2 and places in the inventory or bypassed with probability $1 - \beta$ and takes place in the inventory. Fixed cost of each activated server is K, holding cost per item in the inventory is h and production cost rate is p. Discount rate is denoted by α and there is no order cancellation. Based on the aforementioned assumptions, the system is modelled as $M/Coxian - 2/s$ make-to-stock queue.

Dynamic programming based modelling approach is provided in Section 3.1, optimal production and rationing policies via numerical studies are introduced in Section 3.2 and it is explained how optimal production/rationing decisions and average cost criterion are affected by system parameters in Section 3.3.

3.1 Dynamic Programming Formulation

System state is defined with three variables to keep track of the events: $X_i(t), \forall_{i \in \{1,2\}}$ denotes the number of active servers at i^{th} stage at time t and $X_3(t)$ denotes the inventory level at time t. Events are production completion at phase-one, production completion at phase-two and demand arrival from a customer class. Based on the definition, system state space is

$$SS = \left\{ \left(X_1(t), X_2(t), X_3(t) \right) \mid \sum_{i=1}^{2} X_i(t) \le s, X_i(t) \in Z^+ \cup \{0\}, \forall i = 1,2,3 \right\} \quad (3.1)$$

where s is the number of available servers. Since there are Markovian policies in the space of optimal policies, through the Markovian property, decision can be made in either stage completion or demand arrival. For this reason, system state definition $\left(X_1(t), X_2(t), X_3(t)\right)$ is used regardless of time dimension as (x_1, x_2, x_3). Since the original problem is continuous time Markov process, it is converted to the equivalent discrete time problem via uniformization technique by Lippman (1975). The uniform transition rate is defined as $\nu = \sum_{i=1}^{N} \lambda_j + s(\mu_1 + \mu_2)$.

Production decision is denoted by u_p where $u_p \in \{x_1, x_2, \dots, (x_1 + s - x_2)\}$ and rationing decision for customer class j is denoted by u_{r_j} where $u_{r_j} \in \{0,1\}, j = 1,2, \dots, N$. Production decision is upper bounded by number of available servers and lower bound of the decision is number of active servers at stage-one since there is no order cancellation. For the rationing decision, if $u_{r_j} = 0$, then incoming demand of class j is rejected, if $u_{r_j} = 1$, demand is satisfied. Based on the definitions, optimal cost-to-go function is written by

$$\begin{aligned} J(x_1, x_2, x_3) = \frac{1}{\nu + \alpha} \min_{x_1 \le u \le s - x_2} \{ & hx_3 + p(u + x_2) + K(u - x_1) \\ & + (s(\mu_1 + \mu_2) - u\mu_1 - x_2\mu_2) J(u, x_2, x_3) \\ & + u\mu_1(\beta J(u - 1, x_2 + 1, x_3) \\ & + (1 - \beta) min\{J(u - 1, x_2, x_3 + 1), J(u, x_2, x_3 + 1)\}) \\ & + x_2\mu_2 min\{J(u, x_2 - 1, x_3 + 1), J(u + 1, x_2 - 1, x_3 + 1)\} \\ & + T_R(u, x_2, x_3)\} \end{aligned} \quad (3.2)$$

where $T_R(x_1, x_2, x_3) = \sum_{j=1}^{N} T_{R_j}(x_1, x_2, x_3)\,, j \in \{1,2, \dots, N\}$,

$$T_{R_j}(x_1, x_2, x_3) = \begin{cases} \lambda_j \min\{J(x_1, x_2, x_3 - 1), c_j + J(x_1, x_2, x_3)\}, \ x_3 > 0 \\ \lambda_j \left(c_j + J(x_1, x_2, 0) \right), \ x_3 = 0 \end{cases} \quad (3.3)$$

In equation (3.2), expected discounted cost is calculated for a given system state based on production decision minimization. Holding cost is charged for each unit in inventory, production cost is charged for total number of active servers and fixed startup cost is required for each activated server. Due to the uniformization, the term

$(s(\mu_1 + \mu_2) - u\mu_1 - x_2\mu_2)J(u, x_2, x_3)$ is necessary for the fictious self-transitions. It is because any system state (x_1, x_2, x_3) directly turns to (u, x_2, x_3) when a production decision is occurred. Production is completed at stage-one with rate $u\mu_1$ and passed through stage-two with probability β and bypassed stage-two with probability $1 - \beta$. In case of visiting second stage with probability β, inventory level remains the same because an item leaves the stage-one, gets into stage-two and current production is not finished yet. In case of leaving the system with probability $1 - \beta$, second stage is not visited and inventory level is increased by one unit. Since there is a production completion, next production decision is either continuing with the remaining number of active channels, i.e. $u - 1$, or keeping the finished channel active, i.e. u, without paying start-up cost. The optimal decision is the one that provides minimum cost. Additionally, production is completed at stage-two with rate $x_2\mu_2$. In that case, a finished item is added to the inventory and optimal production decision is either producing with the remaining channels or continuing with the finished channel in addition to the remaining ones because of the fixed start-up cost.

In equation (3.3), $T_{R_j}(x_1, x_2, x_3)$ corresponds the rationing decision for demand class j. Demand is occurred with rate λ_j, rationing operator decides whether to satisfy the demand from class j or not if there is on-hand inventory, otherwise incoming demand from class j is rejected.

Although we develop the dynamic programming formulation based on expected discounted cost criterion, we use average cost criterion in our numerical studies as Ha (1997a, 2000) and Lee and Hong (2003) in the literature. Thus, we easily obtain the average system cost for a given policy via Continuous Time Markov Chain (CTMC) analysis. Consideration of average cost whilst eliminating the determination of discount rate, allows the cost of all visited states to converge to the same average cost value. In order to obtain an average system cost, we revise the equation (3.2) by using value iteration algorithm and setting discount rate to be zero additively. In this case, optimal cost-to-go function value is divided by the number of events. We consider the absolute value of difference between average cost of all feasible states with one another as a termination criterion for the value iteration. Value iteration terminates when the absolute value of difference is smaller than predetermined epsilon value and expected average cost is obtained. By means of this criterion, costs of whole states converge to the same value with the epsilon unit of deviation. Figure 3.1 shows the pseudo code of value iteration algorithm where i takes value 0 if the cost criterion is discounted and value 1 if the criterion is average and k represents the current step.

$Value\ iteration\ (i)$:
$k = 0$
$Assign\ an\ estimated\ value\ for\ J_0$
$While\ (difference > epsilon)$
$k = k + 1$
$Loop: For\ all\ states$
$Loop: u = \{set\ of\ all\ possible\ production\ decisions\}$
$RD = Rationing\ decision(u)$
$Calculate\ J_k^{cand}(J_{k-1}, state(u), RD)$
$End\ loop$
$J_k(state) = \min_u(J_k^{cand}(J_{k-1}, state(u), RD))$
$End\ loop$
$If\ i = 0$
$difference = max\ |J_k\ (state) - J_{k-1}(state)|$
$End\ loop$
$If\ i = 1$
$$difference = \max_{state \in SS} \max_{state' \in SS/\{state\}} \left| \frac{J_k\ (state)}{k} - \frac{J_k(state')}{k} \right|$$
$End\ loop$

Figure 3.1 Value Iteration Algorithm Pseudo Code

Since we obtain the average system cost, we give the numerical characterization of the optimal production and rationing policies in Section 3.2. After that, we analyze the effect of system parameters on the optimal policies in Section 3.3.

3.2 Characterization of the Optimal Production and Rationing Policies

In this chapter, we introduce the optimal production and rationing decisions under average system cost. System state space is bounded by the inventory level and numerical studies consider two customer classes $(c_1 \geq c_2)$. We define the setting as a vector such that $(K, s, \mu_1, \mu_2, \beta, h, \lambda_1, \lambda_2, c_1, c_2)$ where K is the fixed start-up cost, s is the number of available servers, μ_1, μ_2, β are Coxian parameters, λ_1, λ_2 are demand rates from class 1 and 2 respectively and c_1, c_2 are lost sale costs for related customer classes.

First analysis for the system considers a single production channel with 2-phase Coxian processing times and system setting is set to be $(0, 1, 5, 2.5, 0.3, 2, 3, 2, 10, 3)$.

Table 3.1 shows the optimal production decisions under average cost criterion when the number of available server is one. As it is well known for a single server system with fixed cost and general processing times, *two-critical number policy* (X^*, X^{**}) is optimal for production control. As it is seen from the table, there are two-critical levels and when fixed start-up cost is set to be zero, behavior of the production decisions become base-stock policy as well as critical numbers are equal to each other and stands for the base-stock level S. In the case of $K = 0$, it is seen in Table 3.1 that $X^* = X^{**} = 6$, i.e. $S = 6$ since there is no fixed start-up cost. On the other hand, critical values are obtained as $X^* = 4$, $X^{**} = 9$ in the case of $K = 2$.

Table 3.1 Optimal Production Decisions ($\boldsymbol{s = 1}$)

$K = 0$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	1	1	1	1	1	1	1	0	0	0	0	0
[0,1]	1	1	1	1	1	1	1	0	0	0	0	0
[1,0]	1	1	1	1	1	1	1	0	0	0	0	0
$K = 2$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	1	1	1	1	1	0	0	0	0	0	0	0
[0,1]	1	1	1	1	1	1	1	1	1	1	0	0
[1,0]	1	1	1	1	1	1	1	1	1	1	0	0

Table 3.2 Optimal Rationing Decisions of Class-one ($\boldsymbol{s = 1}$)

$K = 0$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	0	1	1	1	1	1	1	1	1	1	1	1
[0,1]	0	1	1	1	1	1	1	1	1	1	1	1
[1,0]	0	1	1	1	1	1	1	1	1	1	1	1
$K = 2$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	0	1	1	1	1	1	1	1	1	1	1	1
[0,1]	0	1	1	1	1	1	1	1	1	1	1	1
[1,0]	0	1	1	1	1	1	1	1	1	1	1	1

Optimal rationing decisions of customer class-one and class-two are expressed in Table 3.2 and 3.3 respectively. According to the Table 3.2, it is optimal to satisfy a demand from customer class-one when there is an on-hand inventory for a single channel system. The dynamic structure of rationing decisions for customer class-two can be seen in Table 3.3. As stated in the table, it is optimal to satisfy a demand from class-two if the state is [1,0,3]. On the other hand, it is optimal to reject the demand if the state is [0,1,3]. Production states [0,0] and [1,0] have the same rationing decisions since triggered production switches from state [0,0] to state [1,0] instantaneously. Briefly, different rationing decisions may occur in the same inventory levels due to the current production status.

Table 3.3 Optimal Rationing Decisions of Class-two ($s = 1$)

$K = 0$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	0	0	0	1	1	1	1	1	1	1	1	1
[0,1]	0	0	0	0	1	1	1	1	1	1	1	1
[1,0]	0	0	0	1	1	1	1	1	1	1	1	1
$K = 2$												
State	Inventory Level											
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
[0,0]	0	0	0	1	1	1	1	1	1	1	1	1
[0,1]	0	0	0	0	1	1	1	1	1	1	1	1
[1,0]	0	0	0	1	1	1	1	1	1	1	1	1

Although optimal production policy is well-defined for the single channel environment, it has not been characterized the multiple channel production with Coxian processing times in make-to-stock environment yet. Table 3.4 shows the optimal production decisions with a given setting $(0,2,5,2.5,0.3,2,3,2,15,3)$. In addition, optimal rationing decisions are presented in Table 3.5 for both customer classes.

As it is seen from the Table 3.4, optimal production decisions seem to be highly dynamic. Rows represent the production stages and columns show the inventory level. Any intersection of the rows and columns, production decisions are indicated for a given state $[x_1, x_2, x_3]$. At the very beginning, at state [0,0,0], production is triggered by activating two servers $(u_p = 2)$ but at state [0,1,0], production decision becomes one $(u_p = 1)$. Referring to Table 3.4, it is seen that production decisions remain the same for some states. When all the servers are active, production decision becomes the number of active server at first stage, i.e. states [0,2,0], [1,1,0] and [2,0,0] in Table 3.4, since there is no cancellation or no more channels to be activated.

Table 3.4 Optimal Production Decisions ($s = 2$)

$K = 0$							$K = 2$						
State	Inventory Level						State	Inventory Level					
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	2	2	2	2	0	0	*[0,0]*	2	2	1	1	0	0
[0,1]	1	1	1	1	0	0	*[0,1]*	1	1	1	0	0	0
[0,2]	0	0	0	0	0	0	*[0,2]*	0	0	0	0	0	0
[1,0]	2	2	2	2	1	1	*[1,0]*	2	2	1	1	1	1
[1,1]	1	1	1	1	1	1	*[1,1]*	1	1	1	1	1	1
[2,0]	2	2	2	2	2	2	*[2,0]*	2	2	2	2	2	2

Table 3.5 Optimal Rationing Decisions ($s = 2$)

$K = 0$													
Class-1							Class-2						
State	Inventory Level						State	Inventory Level					
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	0	1	1	1	1	1	*[0,0]*	0	0	1	1	1	1
[0,1]	0	1	1	1	1	1	*[0,1]*	0	0	0	1	1	1
[0,2]	0	1	1	1	1	1	*[0,2]*	0	0	0	1	1	1
[1,0]	0	1	1	1	1	1	*[1,0]*	0	0	1	1	1	1
[1,1]	0	1	1	1	1	1	*[1,1]*	0	0	0	1	1	1
[2,0]	0	1	1	1	1	1	*[2,0]*	0	0	1	1	1	1

Table 3.5 shows the optimal rationing decisions for both customer classes. Whenever there is an on-hand inventory, it is optimal to satisfy incoming demand form class-one at any production stage as it is seen from the left hand side of the Table 3.5. Right hand side of the table shows that optimal rationing decisions for class-two depend on the system state.

In addition, optimal decisions under discounted cost criterion may take different values from the optimal decisions under average cost for a given setting, however these two criteria have the same characteristics. For instance, base-stock policy is optimal production policy for $M/Coxian-2/1$ under both discounted and average cost consideration.

3.3 Effect of System Parameters on Optimal Policies

In this section, we explain the effect of system parameters on optimal production and rationing decisions. Since there is no optimal production or rationing characterization of $M/Coxian-2/s$ make-to-stock systems with fixed start-up cost, 2-phase Coxian parameters (μ_1, μ_2, β), number of server (s) and start-up cost (K) are major parameters of this study. A base setting is determined such that $(K, s, \mu_1, \mu_2, \beta, h, \lambda_1, \lambda_2, c_1, c_2) = (0,2,2,5,0.5,2,2,2,25,3)$ for numerical studies in this chapter.

Fixed startup cost (K) is incurred for each activated channel at phase-one since production is triggered at phase-one. Table 3.6 shows the optimal production and rationing decisions while K is increasing. At state [0,0,0], production starts with two active servers $(u_p = 2)$ regardless from the value of K, but the general perspective is to activate less channel while K is getting high. Consider the state [1,0,4]; two production channel is active when there is no start-up cost, however a single channel is active at the production when the start-up cost is positive. Suppose s-many channels are activated at stage-one currently. If the new production decision is the same with the number of active channels at that stage, then fixed start-up cost is not incurred because of the continuation.

Table 3.6 Effect of Fixed Startup Cost on Optimal Policies

Optimal Production Decisions																												
State	K = 0							K = 1							K = 2							K = 8						
	Inventory Level							Inventory Level							Inventory Level							Inventory Level						
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	2	2	2	2	2	1	0	2	2	2	2	1	1	0	2	2	2	2	1	1	0	2	2	2	2	1	1	0
[0,2]	1	1	1	1	1	0	0	1	1	1	1	0	0	0	1	1	1	0	0	0	0	1	1	1	0	0	0	0
[1,0]	2	2	2	2	2	1	1	2	2	2	2	1	1	1	2	2	2	2	1	1	1	2	2	2	2	1	1	1
Optimal Rationing Decisions																												
[0,0]	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1
[0,2]	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1
[1,0]	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1

When the start-up cost increases, the production decision is to continue with the same number of channels instead of activating a channel later in order to avoid to the fixed cost. Optimal rationing decisions in Table 3.6 shows that increasing K causes lower rationing level for second customer class. In that case, system behavior is to produce more when there is high start-up cost, because reactivating a channel becomes costly when start-up cost is high. Higher production reduces the rejection level class-two. Also system considers the tradeoff between start-up cost and lost sale cost of second class. Increasing fixed cost also causes increasing average system cost.

Effect of number of production channels (s) is shown in Table 3.7. As the number of server increases, system state space increases. On the basis of this information, we truncate the table and show the optimal decisions for common production states. The detailed production and rationing decisions in Table 3.7 are shown in Appendix 1. First of all, production is finished at lower inventory levels when the number of channel is increased. Increasing channel causes higher production rate and it is easy to reach a specific inventory level. Optimal production decision at the very beginning, state [0,0,0], is to activate as many as channel by considering availability except from the decision where $s = 9$. Optimal production decision is equal to the number of available server when $s = 8$, but optimal decision remains the same when $s = 9$. Although number of available channel is practically infinite, it is not optimal to activate more than 8 channel for this setting. After this point, model turns out to be a typical inventory system. As it is seen in Figure 3.2, average system cost decreases and converges to a value while the number of channels is increasing because providing more available channel does not increase the system cost.

Table 3.7 Effect of Production Channels on Optimal Policies

Optimal Production Decisions																												
State	$s = 3$							$s = 6$							$s = 8$							$s = 9$						
	Inventory Level							Inventory Level							Inventory Level							Inventory Level						
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	3	3	3	3	1	0	0	6	6	3	1	0	0	0	8	5	3	1	0	0	0	8	5	3	1	0	0	0
[0,2]	1	1	1	0	0	0	0	4	2	0	0	0	0	0	5	2	0	0	0	0	0	5	2	0	0	0	0	0
[1,0]	3	3	3	3	1	1	1	6	6	3	1	1	1	1	8	5	3	1	1	1	1	8	5	3	1	1	1	1
Optimal Rationing Decisions																												
[0,0]	0	0	0	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1
[0,2]	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1
[1,0]	0	0	0	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1

Optimal rationing decisions are for the benefit of second customer class. Since production rate increases, a finished product is achieved rapidly and incoming demand from clas-two is satisfied at a lower inventory level.

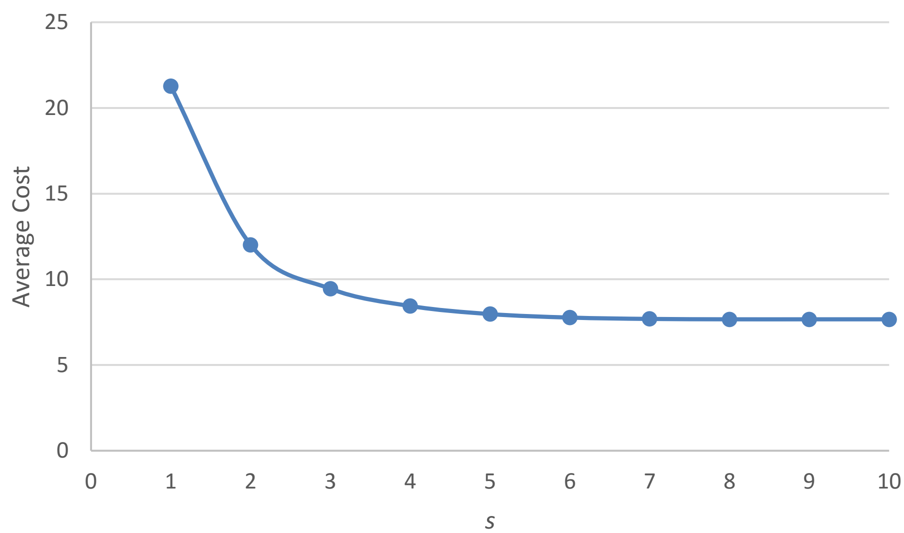

Figure 3.2 Average Costs with Increasing Number of Channels

Table 3.8 shows that system holds fewer items in the inventory while holding cost (h) is increasing. It is shown that there is a non-increasing trend in optimal production decisions for any system state while holding cost is increasing. It is also seen that the rationing level of class-two is decreased.

Table 3.8 Effect of Holding Cost on Optimal Policies

Optimal Production Decisions																					
State	$h = 2$							$h = 3$							$h = 4$						
	Inventory Level							Inventory Level							Inventory Level						
$[x_1, x_2]$	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6
[0,0]	2	2	2	2	2	1	0	2	2	2	2	1	0	0	2	2	2	2	0	0	0
[0,1]	1	1	1	1	1	0	0	1	1	1	1	0	0	0	1	1	1	0	0	0	0
[1,0]	2	2	2	2	2	1	1	2	2	2	2	1	1	1	2	2	2	2	1	1	1
Optimal Rationing Decisions																					
[0,0]	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1
[0,1]	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	1	1	1	1	1
[1,0]	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1

Production rates (μ_1, μ_2) have also significant effect on the optimal decisions. We examine the effect of production rates in twofold. Table 3.9 show the optimal decisions in the case of $\mu_1 \geq \mu_2$. Particularly, it is seen that optimal production decisions have non-increasing trend while μ_1/μ_2 is increasing. For a given inventory level, production decisions in any state $[x_1, x_2]$ are non-increasing. An increment in the production rate of stage-one causes lower rejection level for the demand of customer class-two. As production rate increases, expected time to finish for an item decreases and placing an item in inventory becomes rapid.

Table 3.9 Effect of Production Rates on Optimal Policies ($\boldsymbol{\mu_1} \geq \boldsymbol{\mu_2}$)

Optimal Production Decisions																								
State	$\mu_1 = 2, \mu_2 = 2$								$\mu_1 = 6, \mu_2 = 2$								$\mu_1 = 8, \mu_2 = 2$							
	Inventory Level								Inventory Level								Inventory Level							
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
[0,0]	2	2	2	2	2	2	2	0	2	2	2	2	0	0	0	0	2	2	2	2	0	0	0	0
[0,1]	1	1	1	1	1	1	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
[1,0]	2	2	2	2	2	2	2	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1
Optimal Rationing Decisions																								
[0,0]	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1	1
[0,1]	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1
[1,0]	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1	1

Table 3.10 Effect of Production Rates on Optimal Policies ($\boldsymbol{\mu_1} < \boldsymbol{\mu_2}$)

Optimal Production Decisions																								
State	$\mu_1 = 2, \mu_2 = 4$								$\mu_1 = 2, \mu_2 = 6$								$\mu_1 = 2, \mu_2 = 8$							
	Inventory Level								Inventory Level								Inventory Level							
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
[0,0]	2	2	2	2	2	2	0	0	2	2	2	2	2	1	0	0	2	2	2	2	2	1	0	0
[0,1]	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0
[1,0]	2	2	2	2	2	2	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1
Optimal Rationing Decisions																								
[0,0]	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1
[0,1]	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1
[1,0]	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1

Table 3.10 shows the optimal decisions for the case $\mu_1 < \mu_2$. Increasing the production rate of stage-two makes the optimal production decisions non-increasing but an increment in μ_2 does not affect the system as much as the increment in μ_1

because visiting the second-stage is probabilistic, i.e. $\beta \in [0,1]$. On the other hand, an item is processed at stage-one certainly. Since the expected time to finish for an item decreases, incoming demand of a customer class-two is satisfied in earlier levels.

Table 3.11 shows the effect of demand rates (λ_1, λ_2) on the optimal policies by keeping the total demand constant. The ratio of λ_1 and λ_2 is chosen as 0.6, 1 and 1.67 respectively. In the sense of production decision, production is finished at higher inventory levels while λ_1/λ_2 is increasing. In the setting shown in Table 3.11, it is optimal to increase production amount in order to satisfy incoming demand of customer class-one because demand rate of that class increases. Although the total demand rate remains unchanged, λ_2 is relatively getting smaller than λ_1. While λ_2 is relatively getting smaller, rationing level of demand class-two is getting higher in anticipation of future demand arrival from customer class-one.

Table 3.11 Effect of Demand Rates on Optimal Policies

Optimal Production Decisions																								
State	$\lambda_1 = 1.5, \lambda_2 = 2.5$								$\lambda_1 = 2, \lambda_2 = 2$								$\lambda_1 = 2.5, \lambda_2 = 1.5$							
	Inventory Level								Inventory Level								Inventory Level							
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
[0,0]	2	2	2	2	2	0	0	0	2	2	2	2	2	1	0	0	2	2	2	2	2	2	1	0
[0,1]	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0
[1,0]	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	1	2	2	2	2	2	2	1	1
Optimal Rationing Decisions																								
[0,0]	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
[0,1]	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1
[1,0]	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1

Hereinbefore, we ration the inventory based on the lost sale cost of demand classes and it is optimal to satisfy incoming demand of class-one if there is an on-hand inventory. In Table 3.12, we increase the gap between lost sale costs of demand classes (c_1, c_2). It is optimal to produce more when lost sale cost of class-one is increased in order not to stock out. In addition, inventory is reserved for the prioritized customer and it is optimal to reject the demand of lower prioritized customer while c_1 is increasing. As a remark, it is likely to prevent rationing by setting c_1 and c_2 to the same value. In that case, inventory is allocated with respect to the first come first served policy.

Table 3.12 Effect of Lost Sale Costs on Optimal Policies

Optimal Production Decisions																												
State	$c_1 = 10, c_2 = 3$							$c_1 = 15, c_2 = 3$							$c_1 = 20, c_2 = 3$							$c_1 = 25, c_2 = 3$						
	Inventory Level							Inventory Level							Inventory Level							Inventory Level						
$[x_1, x_2]$	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6
[0,0]	2	2	2	2	1	0	0	2	2	2	2	2	0	0	2	2	2	2	2	0	0	2	2	2	2	2	1	0
[0,1]	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0
[1,0]	2	2	2	2	1	1	1	2	2	2	2	2	1	1	2	2	2	2	2	1	1	2	2	2	2	2	1	1
Optimal Rationing Decisions																												
[0,0]	0	0	1	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1
[0,1]	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1
[1,0]	0	0	1	1	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1

Visiting probability (β) is one of the essential parameters of the 2-phase Coxian processing times because it allows us to analyze $M/M/s$ by eliminating second phase $(\beta = 0)$ and $M/E_2/s$ by visiting second phase with probability one $(\beta = 0)$. We first explain the effect of β in a single channel system, then we extend our scope to the s-many parallel production channels. We set the parameters such that $K = 0$, $s = 1$, $[\mu_1, \mu_2] = [5, 2.5]$, $h = 2$, $[\lambda_1, \lambda_2] = [2, 2]$ and $[c_1, c_2] = [25, 3]$. Recalling the base stock policy is optimal production policy for $M/Coxian - 2/1$, we show the base-stock levels for given β values in Figure 3.3. While β is increasing, it is more likely to visit second phase and higher β causes higher expected time to produce. Nevertheless, it takes more time to put an item to the inventory and base stock level is non-decreasing while β is getting higher.

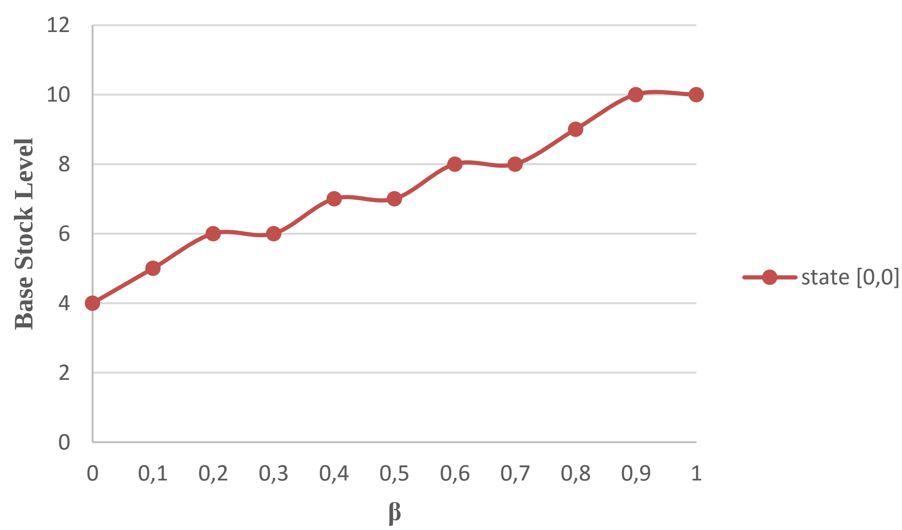

Figure 3.3 Effect of Visiting Probability on Base Stock Levels $(s = 1)$

Optimal rationing decisions are shown in Table 3.13 as well. In general, rationing decisions are getting more likely to reject the demand of class-two along with the β increment. Incoming demand of class-two is rejected in anticipation of a demand arrival from customer class-one because of the higher expected time to produce an item. Consider the production status [0,1], i.e. second phase is active, it is optimal to satisfy a demand if there are at least 4 items in the inventory when $\beta = 0.1$. On the other hand, it is optimal to reject if the inventory level is 4 when $\beta = 0.5$.

Table 3.13 Effect of Visiting Probability on Optimal Rationing Decisions ($s = 1$)

Optimal Rationing Decisions																																
State	$\beta = 0.0$								$\beta = 0.1$								$\beta = 0.3$								$\beta = 0.5$							
	Inventory Level								Inventory Level								Inventory Level								Inventory Level							
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
[0,0]	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	1	1	1
[0,1]	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	1	1	1
[1,0]	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	1	1	1
Costs	**9,6494**								**11,1131**								**13,448**								**15,4562**							
State	$\beta = 0.7$								$\beta = 0.8$								$\beta = 0.9$								$\beta = 1.0$							
	Inventory Level								Inventory Level								Inventory Level								Inventory Level							
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
[0,0]	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
[0,1]	0	0	0	0	0	1	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1
[1,0]	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
Costs	**17,4275**								**18,4469**								**19,554**								**20,7092**							

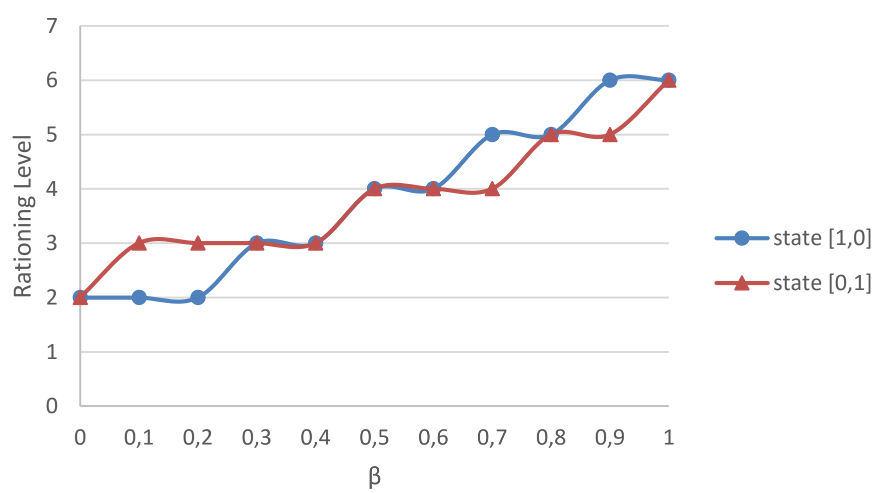

Figure 3.4 Effect of Visiting Probability on Rationing Levels ($s = 1$)

Figure 3.4 shows the rationing levels of the low prioritized class based on production status for given β values. As it is seen in the figure, it depends on the current status of the production. There is a non-decreasing trend in rationing levels in both states [1,0] and [0,1] but until the β value of 0.3, state [0,1] has higher rationing level than state [1,0]. However, state [1,0] has higher rationing level than the other at the higher β values. Although state [0,1] seems like closer to the inventory than the state [1,0], being that state may be disadvantageous because of higher rationing level. Change in β values affects the optimal rationing decisions either positively or negatively in terms of customer class-two. We observe the similar non-monotone behavior in multiple-channel cases.

Table 3.14 Effect of Visiting Probability on Optimal Production Decisions $s = 4$

Optimal Production Decisions																		
State $[x_1, x_2]$	$\beta = 0.1$						$\beta = 0.4$						$\beta = 0.7$					
	Inventory Level						Inventory Level						Inventory Level					
	0	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	4	3	1	0	0	0	4	4	2	0	0	0	4	4	3	1	0	0
[0,1]	3	2	0	0	0	0	3	3	1	0	0	0	3	3	2	0	0	0
[0,2]	2	2	0	0	0	0	2	2	0	0	0	0	2	2	1	0	0	0
[0,3]	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0
[0,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[1,0]	4	3	1	1	1	1	4	4	2	1	1	1	4	4	3	1	1	1
[1,1]	3	2	1	1	1	1	3	3	1	1	1	1	3	3	2	1	1	1
[1,2]	2	2	1	1	1	1	2	2	1	1	1	1	2	2	1	1	1	1
[1,3]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[2,0]	4	3	2	2	2	2	4	4	2	2	2	2	4	4	3	2	2	2
[2,1]	3	2	2	2	2	2	3	3	2	2	2	2	3	3	2	2	2	2
[2,2]	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
[3,0]	4	3	3	3	3	3	4	4	3	3	3	3	4	4	3	3	3	3
[3,1]	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
[4,0]	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

Additionally, we show the effect of visiting probability β on optimal policies by considering multiple production channel. Optimal production decisions with a parameter vector $(K, s, \mu_1, \mu_2, h, \lambda_1, \lambda_2, c_1, c_2) = (0{,}4{,}5{,}2.5{,}2{,}2{,}2{,}25{,}3)$ are given in Table 3.14 while β is increasing. In this case, production rate of second phase is as much as half of the production rate of first phase, thus it is more likely to visit the phase that has a lower production rate as β increases. As it is seen from the table, production decisions have non-decreasing trend.

When optimal rationing decisions are examined, it is reasonable to say that rationing levels are non-decreasing. Table 3.15 indicates that it is optimal to satisfy a second class demand in particular system states i.e. [0,0], [1,0], [2,0], [3,0], [4,0] earlier inventory levels other than remaining states. As β increases, being phase-one becomes less advantageous because of the chance of visiting phase-two.

Table 3.15 Effect of Visiting Probability on Optimal Rationing Decisions $s = 4$

Optimal Rationing Decisions																		
State $[x_1, x_2]$	$\beta = 0.1$						$\beta = 0.4$						$\beta = 0.7$					
	Inventory Level						Inventory Level						Inventory Level					
	0	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	0	1	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[0,1]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[0,2]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[0,3]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[0,4]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[1,0]	0	1	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[1,1]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[1,2]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[1,3]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[2,0]	0	1	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[2,1]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[2,2]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[3,0]	0	1	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[3,1]	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
[4,0]	0	1	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1

On the other hand, optimal rationing decisions are affected by Coxian parameters (μ_1, μ_2, β) jointly. The behavior of the optimal rationing policy is affected by relative values of production rates and visiting probability. We show the optimal rationing decisions by considering two cases with the common parameters such that $(K, s, h, \lambda_1, \lambda_2, c_1, c_2) = (0, 3, 2, 2, 2, 25, 3)$ in Table 3.16. We separate the cases based on the 2-phase Coxian parameters (μ_1, μ_2, β) where first case considers $(2, 5, 0.5)$ and second one considers $(5, 1, 0.3)$.

Table 3.16 Optimal Rationing Decisions $s = 3$

Optimal Rationing Decisions													
State	Case 1						State	Case 2					
	Inventory Level							Inventory Level					
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	0	0	0	1	1	1	[0,0]	0	0	1	1	1	1
[0,1]	0	0	1	1	1	1	[0,1]	0	0	1	1	1	1
[0,2]	0	0	1	1	1	1	[0,2]	0	0	0	1	1	1
[0,3]	0	0	1	1	1	1	[0,3]	0	0	0	1	1	1
[1,0]	0	0	0	1	1	1	[1,0]	0	0	1	1	1	1
[1,1]	0	0	1	1	1	1	[1,1]	0	0	1	1	1	1
[1,2]	0	0	1	1	1	1	[1,2]	0	0	0	1	1	1
[2,0]	0	0	0	1	1	1	[2,0]	0	0	1	1	1	1
[2,1]	0	0	1	1	1	1	[2,1]	0	0	1	1	1	1
[3,0]	0	0	0	1	1	1	[3,0]	0	0	1	1	1	1

It is more favorable to be in state $[1,2]$ than in state $[1,0]$ when we consider the rationing level of customer class-two in Case 1. It is optimal to satisfy a demand from second class in state $[1,2,2]$ since it is advantageous that there are two activated channel at stage-two, however it is optimal to reject it in state $[1,0,2]$. Consider the optimal rationing decisions in Case 2 with the same system states, it is optimal to reject a demand when current state is $[1,2,2]$ but it is optimal to satisfy the demand in state $[1,0,2]$. We obtain opposite decisions in the same system states when we change the Coxian parameters. In addition, it is seen that $u_{r_2}^{[1,0,2]} = 1$ and $u_{r_2}^{[0,3,2]} = 0$ in Case 2. Although all the available channels are active at stage-two, it is optimal to reject a demand and it is optimal to satisfy the demand when there is a single activated channel at stage-one. This is because production rate of second phase is lower than the first phase and there is a 0.3 visiting probability to phase-two. The rationale behind the decisions is the joint effect between production rates and visiting probability. This non-monotony forms a basis for the proposed rationing policy that is explained in Section 4.

4 PROPOSED RATIONING POLICY

In this section, we propose a rationing policy for the $M/Coxian-2/s$ make-to-stock queue. For a given parameter vector, any change in visiting probability β and production rates of stages μ_1, μ_2 affect the expected time to produce (ETTP) in two way. If there is a completed stage in production, ETTP either decreases or increases depending on the parameter vector. First instance may be explained trough the Erlangian processing times because Erlang distribution has an increasing failure rate distribution. However, second instance is not seen in typical Erlangian processing times because stage completion increases the ETTP. In the study of Lee and Hong (2003), it is analyzed an $M/Coxian-2/1$ model under static rationing, i.e. current status of the production is not considered.

By taking the parameter vector into consideration, we separate the states according to the current production status and compare the ETTP values. Let $E_{[x_1,x_2,x_3]}$ be the expected time to produce an item where the current state is $[x_1, x_2, x_3]$. Suppose a state that an item is completed stage-one and being processed at stage-two for a given inventory level, i.e. $[0,1,x_3]$, then expected time to produce for that item is obtained by $E_{[0,1,x_3]} = {}^{1}/_{\mu_2}$. On the other hand, suppose a state that an item is being processed at stage-one, i.e. $[1,0,x_3]$. At this time, ETTP is calculated by considering the probability of visiting stage-two and it is expressed by $E_{[1,0,x_3]} = {}^{1}/_{\mu_1} + \beta\, {}^{1}/_{\mu_2}$. We define a hierarchy between states $[0,1,x_3]$ and $[1,0,x_3]$. If $E_{[0,1,x_3]} < E_{[1,0,x_3]}$, then completing a production stage decreases the ETTP. On the contrary, ETTP increases if $E_{[0,1,x_3]} \geq E_{[1,0,x_3]}$ holds. Then, we obtain a ratio by

$$E_{[0,1,x_3]} < E_{[1,0,x_3]} \tag{4.1}$$

$${}^{1}/_{\mu_2} < {}^{1}/_{\mu_1} + \beta\, {}^{1}/_{\mu_2} \tag{4.2}$$

$$\mu_1 \frac{(1-\beta)}{\mu_2} < 1 \tag{4.3}$$

Hence, expected time to produce decreases if parameters β, μ_1, μ_2 satisfy the condition in (4.3). If the parameters satisfy the (4.6), then expected time to produce increases.

$$E_{[0,1,x_3]} \geq E_{[1,0,x_3]} \tag{4.4}$$

$$1/\mu_2 \geq 1/\mu_1 + \beta\, 1/\mu_2 \tag{4.5}$$

$$\mu_1 \frac{(1-\beta)}{\mu_2} \geq 1 \tag{4.6}$$

The proposed rationing policy is the policy that uses the information of current status of the production and separates the cases using a coefficient such that $\mu_1 \frac{(1-\beta)}{\mu_2}$. The proposed rationing policy is expressed by

$$if\ \mu_1 \frac{1-\beta}{\mu_2} < 1, \qquad u_{R_2} = \begin{cases} 1, & if \quad x_3 + a \cdot x_2 > R \\ 0, & \quad otherwise \end{cases} \tag{4.7}$$

$$if\ \mu_1 \frac{1-\beta}{\mu_2} \geq 1, \qquad u_{R_2} = \begin{cases} 1, & if \quad x_3 - a \cdot x_2 > R \\ 0, & \quad otherwise \end{cases} \tag{4.8}$$

where R is the rationing level of lower prioritized customer class (class-two) and $a \in [0, 1]$ is the age parameter which is the relative value of one outstanding order with respect to one unit of inventory. The value of a is upper bounded by 1 because an outstanding order cannot be more valuable than one unit of inventory. The values of R and a are obtained by searching and provided as input to the policy. The proposed rationing policy is developed based on the information of number of active channels at second production stage (x_2) and inventory level (x_3). It is not used the information of first production stage (x_1) because state [0,0] passes to the state [1,0] with an infinite rate and behavior of these states remains the same.

In Equation (4.7), it is summed up the relative value of the number of active channels at stage-two and the inventory level and if the sum is greater than a given rationing level, a demand of second class is satisfied, otherwise it is rejected. In Equation (4.8), the relative value of x_2 is subtracted from the inventory level and if the difference is greater than the predetermined rationing level, a demand from class-two is satisfied, else it is not satisfied.

Table 4.1 shows the performance of the proposed rationing policy against optimal rationing policy while number of channel is increasing with the predetermined system parameters $(K, \mu_1, \mu_2, \beta, h, \lambda_1, \lambda_2, c_1, c_2) = (0{,}2{,}5{,}0.5{,}2{,}2{,}2{,}25{,}3)$. As it is seen from the table, average system cost has a non-decreasing trend when the number of channel increases in both policies. For example, rationing level of class-two is computed as 1.3 and the relative value of an outstanding order is 0.2 when $s = 4$. Also, the percentage of the cost difference is 0,01.

Table 4.1 Performance of the Proposed Rationing Policy

Number of Channel s	Optimal Rationing Policy	Proposed Rationing Policy	Cost Difference %	Rationing Level R^*	Relative Value a^*
1	21,271	21,271	0,00%	6.0	1.0
2	12,004	12,004	0,00%	3.0	1.0
3	9,448	9,450	0,02%	2.0	0.1
4	8,438	8,438	0,01%	1.3	0.2
5	7,968	7,989	0,26%	1.1	0.3
6	7,763	7,788	0,32%	1.1	0.4
7	7,685	7,696	0,15%	1.2	0.5
8	7,663	7,663	0,00%	1.2	0.6
9	7,663	7,674	0,15%	1.1	0.7
10	7,663	7,673	0,14%	1.1	0.8

We consider two different policies in addition to the proposed rationing policy in order to compare their performances. Initially, a static rationing policy based on inventory level is considered to evaluate the value of dynamic rationing. When we set the age parameter to be zero $(a = 0)$, the policy imposes the static rationing. As it is, we expect the proposed rationing policy to perform at least a static policy. Then, we examine *First Come First Served Policy* (FCFS) in order to measure the value of rationing.

Rationing decisions of the policies are given in Table 4.2 for a given parameter vector $(K, s, \mu_1, \mu_2, \beta, h, \lambda_1, \lambda_2, c_1, c_2) = (0{,}3{,}2{,}5{,}0.5{,}2{,}2{,}2{,}25{,}3)$. The decisions of the proposed rationing policy is the same with the optimal one. There is a cost difference between the static and dynamic policy as it is seen from the table. The worst performance of the FCFS policy can be explained by power of the rationing. That means rationing reduces the average system cost. Moreover, Figure 4.1 shows the performance of the proposed rationing and FCFS policies with increasing number of channels. Although the difference between the performance of proposed rationing and

FCFS policy is really high in small number of channels, the difference is getting smaller as the number of channel increases. This is because the system increases production capacity and becomes more likely to satisfy a demand from customer class-two.

Table 4.2 Rationing Decisions of the Policies

State	Optimal Rationing Policy					Proposed Rationing Policy					Static Rationing Policy					FCFS Policy				
	Inventory Level					Inventory Level					Inventory Level					Inventory Level				
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*0*	*1*	*2*	*3*	*4*	*0*	*1*	*2*	*3*	*4*	*0*	*1*	*2*	*3*	*4*
[0,0]	0	0	0	1	1	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1
[0,1]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[0,2]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[0,3]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[1,0]	0	0	0	1	1	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1
[1,1]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[1,2]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[2,0]	0	0	0	1	1	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1
[2,1]	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	1	1	1	1
[3,0]	0	0	0	1	1	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1
Avg Cost	*9,448*					*9,450*					*9,598*					*11,010*				

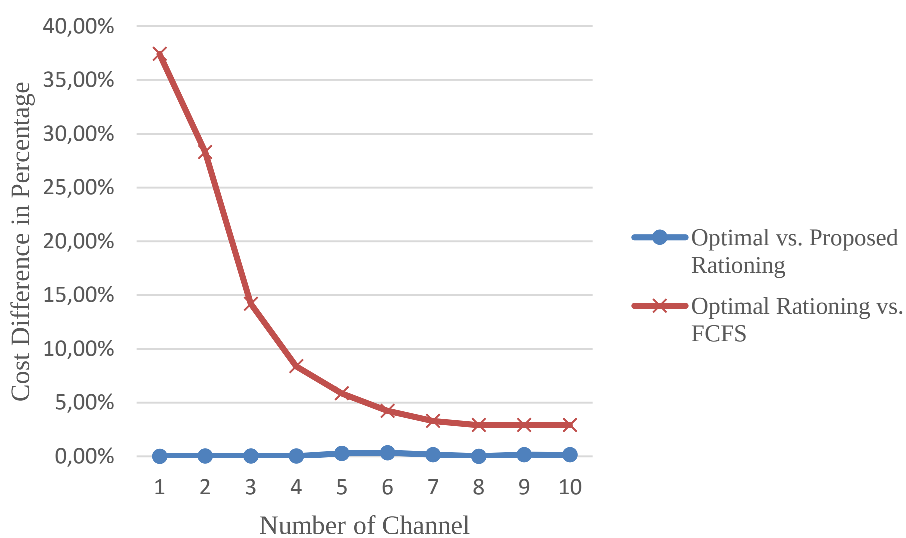

Figure 4.1 Performance of Proposed Rationing Policy and FCFS Policy

Figure 4.2 shows the performance of the policies while β is increasing with parameter vector $(K, s, \mu_1, \mu_2, h, \lambda_1, \lambda_2, c_1, c_2) = (0,3,2,5,2,2,2,25,3)$. As it is seen in the figure, cost difference between optimal rationing policy and FCFS policy increases as β increases. Proposed rationing policy performs well and the difference between optimal and static rationing policy is expected to be increased while s increases.

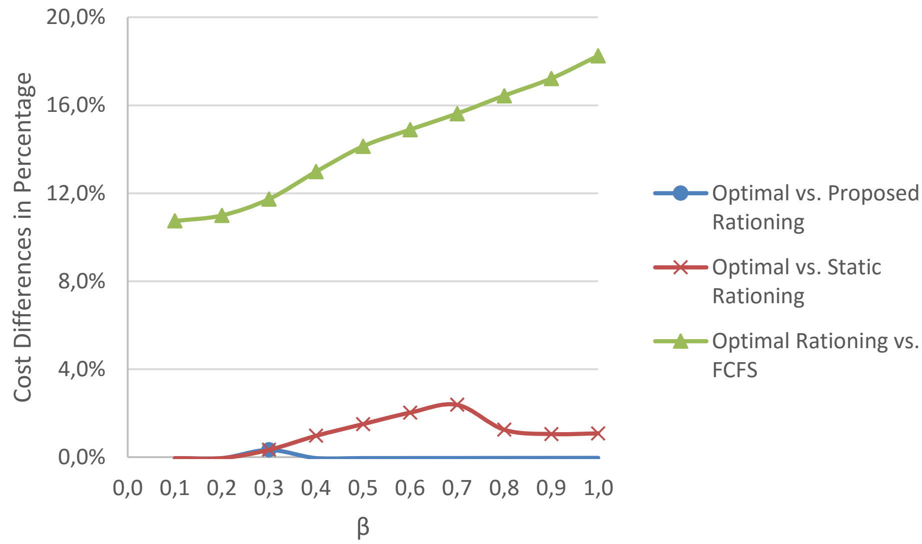

Figure 4.2 Performance of the Rationing Policies with Increasing β

Fixed start-up cost is not considered in aforementioned numerical analysis for the rationing policies. Then, we examine the effect of start-up cost in rationing decisions including optimal, proposed and FCFS policies. Table 4.3 includes the average cost differences between optimal and proposed rationing policies and optimal rationing and FCFS policies when K increases. As K increases, performance of the proposed rationing policy decreases but it performs among 2% worse than the optimal one. However, FCFS performs well with the increasing K.

Table 4.3 Effect of Start-up Cost on the Rationing Policies

K	Optimal vs Proposed Rationing Policy Cost Difference %			Optimal Rationing vs FCFS Policy Cost Difference %		
	β =0.1	β =0.5	β =0.9	β =0.1	β =0.5	β =0.9
0	0,77%	0,74%	0,81%	10,68%	14,71%	17,24%
3	0,43%	0,86%	1,64%	7,96%	10,29%	12,96%
6	1,36%	1,70%	1,90%	7,90%	9,07%	11,86%
9	1,85%	2,52%	2,01%	7,64%	9,78%	10,68%

5 ALTERNATIVE PRODUCTION POLICIES

In this section, we introduce the alternative production policies for the $M/Coxian-2/s$ make-to-stock queue. Since the behavior of the optimal production policy is highly dynamic/state dependent, it may not be practical. We propose alternative production policies where their performances are closer to the optimal production policy in a systematic way. It is well known that *two-critical-level policy* is optimal production policy in an $M/G/1$ production-inventory system with fixed cost. According to the study of Gavish and Graves (1981), it is optimal to produce if inventory level drops to a lower control level, and stop to produce if inventory level hits to an upper control level. In this context, there are two control parameters in proposed policies such that X^*, X^{**} where X^* is a lower and X^{**} is an upper critical levels. Proposed policies are defined in Section 5.1 and performance evaluation of the studies are expressed in Section 5.2.

5.1 Description of the Policies

At the very beginning, we start with a static production policy based on inventory level. Recalling the state definition, first two dimension keep track of the current status of the production and third dimension keeps the inventory level. Initial proposed policy does not consider the current production status but control the production with the information of the inventory level, it is called *Inventory Level Policy* (ILP). Production is triggered whenever the inventory level drops to X^* and continued until it reaches to X^{**}. When the inventory level hits to X^{**}, production channels are not activated and production period is truncated until inventory level drops to X^* again. Production decisions are expressed by

$$u^p(x_1, x_2, x_3) = \begin{cases} \max(\min(X^* - x_3, s - x_2), x_1) & x_3 \leq X^* \\ x_1 & X^* < x_3 < X^{**} \\ x_1 & x_3 \geq X^{**} \end{cases} \tag{5.1}$$

In the expression (5.1), production decision is bounded by the number of available channels and number of active channel at phase-one, since there is no order cancellation. ILP is also related to the base-stock policy where S is the base-stock level. When the control parameters of the ILP are equalized to the base-stock level i.e. $X^* =$

$X^{**} = S$, the policy is optimal production policy for the $M/Coxian-2/1$ as well as $M/M/1$ without start-up cost.

On the other hand, optimal production policy has a dynamic structure. Then, we include the information of the current production status to the inventory level. Thus we trigger the production based on the inventory position and call the *Inventory Position Policy* (IPP). Structure of the policy is given by

$$u^p(x_1,x_2,x_3) = \begin{cases} \max(\min(X^* - IP, s - x_2), x_1) & IP \leq X^* \\ x_1 & X^* < IP < X^{**} \\ x_1 & IP \geq X^{**} \end{cases} \tag{5.2}$$

where $IP = x_1 + x_2 + x_3$. ILP considers the total number of active servers, i.e. current production status, but not the current phase/stage status. Although the states $[2,1,x_3]$, $[1,2,x_3]$ and $[3,0,x_3]$ have the same inventory position, they may have different production decisions in the optimal policy. For this reason, we expect some fluctuations in the performance of ILP based on the system parameters.

Since the first stage of the production is visited certainly, we aim to analyze whether or not the information of stage-one affects the production decisions. Then, we eliminate the effect of stage-one and call the *Modified Inventory Position Policy* (MIPP). Production is controlled by

$$u^p(x_1,x_2,x_3) = \begin{cases} \max(\min(X^* - (x_2+x_3), s - x_2), x_1) & x_2 + x_3 \leq X^* \\ x_1 & X^* < x_2 + x_3 < X^{**} \\ x_1 & x_2 + x_3 \geq X^{**} \end{cases} \tag{5.3}$$

We also examine the effect of the age information on the production decisions with the policy defined above. Recalling the study of Ha (2000), *work storage level policy* (WSL) is optimal production policy for $M/E_k/1$ make-to-stock queue. WSL is defined as a function of number of completed production stages. An item that is being processed at k^{th} stage has $k-1$ completed stages and coefficient of a stage increases as the number of completed stage increases. MIIP is the policy that places between IPP and WSL.

Although an item passes to the following production stage with probability one in Erlangian processing times, the visiting probability takes value between zero and one in Coxian processing times. In order to use the age information, we adapt the WSL to our model. We may also obtain optimal production decisions for the $M/E_2/1$ by setting $\beta = 1$ and $s = 1$ in $M/Coxian - 2/s$ make-to-stock queue. In our setting, $WSL = 0x_1 + 1x_2 + 2x_3$ where the coefficients represent the completed number of stages. Production decision in WSL is the following:

$$u^p(x_1, x_2, x_3) = \begin{cases} \max(\min(X^* - (x_2 + 2x_3), s - x_2), x_1) & x_2 + 2x_3 \leq X^* \\ x_1 & X^* < x_2 + 2x_3 < X^{**} \\ x_1 & x_2 + 2x_3 \geq X^{**} \end{cases} \tag{5.4}$$

We expect the *WSL Policy* to perform well in some cases, but the production rates are assumed to be equal in WSL formulation. However, 2-phase Coxian random variables allow us to study different production rates at stage-one (μ_1) and stage-two (μ_2). Different production rates cause the different coefficients for the WSL. Based on this information, we repair the WSL by using the information of production rates μ_1, μ_2 and called *Modified Work Storage Level Policy* (MWSLP). The production decisions of the MWSLP is explained by

$$u^p(x_1, x_2, x_3) = \begin{cases} \max(\min(X^* - status_{MWSL}, s - x_2), x_1) & status_{MWSL} \leq X^* \\ x_1 & X^* < status_{MWSL} < X^{**} \\ x_1 & status_{MWSL} \geq X^{**} \end{cases} \tag{5.5}$$

where $status_{MWSL} = x_1 + \left(1 + \left\lceil \frac{\mu_h}{\mu_l} \right\rceil\right) x_2 + \left(2 + \left\lceil \frac{\mu_h}{\mu_l} \right\rceil\right) x_3$, $\mu_h = max(\mu_1, \mu_2)$ and $\mu_l = min(\mu_1, \mu_2)$. We consider the ratio of higher and lower production rates and round them up. $status_{MWSL}$ shows that coefficient of the completing first stage (x_1) is determined as one. Then, the coefficient of x_2 is calculated relatively. We sum the age of x_1 and the relative value of the production rates and obtain the age of x_2. Similarly, we sum the age of x_2 and the relative value of the inventory level where the age of an item that places in the inventory is one. For instance, according to the MWSL policy, the value of completing stage-one is one, completing the stage-two is two and the value of an item in the inventory is three when $\mu_1 = \mu_2$. We aim to catch the effect of different production rates on the production decisions with this policy.

5.2 Numerical Study: Performance Evaluation and Comparisons

This section includes numerical analysis of the proposed policies based on the average system cost criterion. Computational results are obtained via MATLAB program. In order to avoid the joint effect between production and rationing decisions, we provide a pure production environment. We equalize the lost sale costs of demand classes ($c_1 = c_2$), thus the inventory is controlled by a FCFS policy. First of all, a parameter vector is chosen such that $(K, s, h, \lambda_1, \lambda_2, c_1, c_2) = (0,3,2,3,4,4,4)$ for the analysis.

Table 5.1 Average Costs of the Production Policies $s = 3, K = 0$

μ_1	μ_2	β	OP	ILP	IPP	MIPP	WSLP	MWSLP
3	0.3	0.1	14,19	14,39	14,26	14,35	14,25	20,71
		0.4	21,64	21,69	21,67	21,69	21,66	24,94
		0.7	23,91	23,92	23,92	23,94	23,93	25,89
		1	24,98	25	24,99	25,01	25,02	26,32
	1.5	0.1	9,34	9,4	9,68	9,46	9,39	9,37
		0.4	12,47	12,53	12,53	12,49	12,74	12,76
		0.7	15,18	15,18	15,39	15,24	15,25	15,47
		1	17,32	17,37	17,46	17,36	17,36	17,47
	3	0.1	8,75	8,78	9,31	8,81	8,82	8,77
		0.4	10,12	10,13	10,26	10,18	10,14	10,15
		0.7	11,49	11,5	11,78	11,53	11,57	11,56
		1	12,91	12,96	12,96	13,03	12,95	12,92
	6	0.1	8,48	8,51	9,11	8,52	8,66	8,48
		0.4	9,08	9,11	9,36	9,1	9,14	9,08
		0.7	9,64	9,69	10,01	9,73	9,84	9,64
		1	10,21	10,24	10,38	10,24	10,26	10,22
	60	0.1	8,28	8,32	8,73	8,32	8,43	10,92
		0.4	8,33	8,44	8,76	8,45	8,42	11,08
		0.7	8,38	8,55	8,77	8,58	8,42	11,23
		1	8,41	8,54	8,82	8,47	8,43	11,39

Table 5.1 shows the average cost of the production policies where OP is the *optimal production policy*, ILP is an *inventory level policy*, IPP is an *inventory position policy*, MIPP is a *modified inventory position policy*, WSLP is a *work storage level policy* and MWSLP is a *modified work storage level policy*. For given μ_1, μ_2 values, the average system cost of production policies increases, as β increases. The reason

behind the increment is the effect of visiting probability β on the expected processing times, in other words higher β value causes higher expected time to produce an item. On the other hand, while μ_2/μ_1 is increasing, expected system cost of the optimal production policy is non-increasing for any given β value. In addition, since expected time to produce an item decreases as the production rate of the second stage increases, then expected cost of the optimal production policy decreases.

Table 5.2 Performances of the Alternative Production Policies $s = 3, K = 0$

μ_1	μ_2	β	OP vs. ILP	OP vs. IPP	OP vs. MIPP	OP vs. WSLP	OP vs. MWSLP
3	0.3	0.1	1,39%	0,49%	1,11%	0,42%	31,48%
		0.4	0,23%	0,14%	0,23%	0,09%	13,23%
		0.7	0,04%	0,04%	0,13%	0,08%	7,65%
		1	0,08%	0,04%	0,12%	0,16%	5,09%
	1.5	0.1	0,64%	3,51%	1,27%	0,53%	0,32%
		0.4	0,48%	0,48%	0,16%	2,12%	2,27%
		0.7	0,00%	1,36%	0,39%	0,46%	1,87%
		1	0,29%	0,80%	0,23%	0,23%	0,86%
	3	0.1	0,34%	6,02%	0,68%	0,79%	0,23%
		0.4	0,10%	1,36%	0,59%	0,20%	0,30%
		0.7	0,09%	2,46%	0,35%	0,69%	0,61%
		1	0,39%	0,39%	0,92%	0,31%	0,08%
	6	0.1	0,35%	6,92%	0,47%	2,08%	0,00%
		0.4	0,33%	2,99%	0,22%	0,66%	0,00%
		0.7	0,52%	3,70%	0,92%	2,03%	0,00%
		1	0,29%	1,64%	0,29%	0,49%	0,10%
	60	0.1	0,48%	5,15%	0,48%	1,78%	24,18%
		0.4	1,30%	4,91%	1,42%	1,07%	24,82%
		0.7	1,99%	4,45%	2,33%	0,48%	25,38%
		1	1,52%	4,65%	0,71%	0,24%	26,16%

Table 5.2 shows the performances of the alternative production policies with respect to the optimal production policy. When μ_2 is relatively smaller than μ_1, performances of the ILP, IPP, MIPP and WSLP are closer to the OP for any β values. In this case, being at stage-two is risky because of the lower production rate and policy decisions are to continue producing in order to avoid the risk. However, IPP worsens with the increasing μ_2. The optimal production policy uses the current status of the production because optimal decision depends on not only μ_1 but also μ_2, but IPP considers the total number of active channel instead of stage-based information. On the

other hand, WSLP performs well where the moderate values of s are considered. MWSLP performs well except for the extreme values of μ_2/μ_1. In general, ILP performs well for any μ_1, μ_2, β values in the table.

Table 5.3 represents the performances of the production policies when the number of channels is increased to 5 ($s = 5$). IPP is mainly affected by the increasing number of channels among the alternative policies because of the lack of the stage information. As the s increases, upper bound of the production decision increases and the information in the system state is also increases. WSLP conserves its performance but performs worse than the case where $s = 3$. Nonetheless, MWSLP is the best performed production policy within the moderate cases of μ_2/μ_1. This is because MWSLP uses the relative values of the production rates of stages.

Table 5.3 Performances of the Alternative Production Policies $s = 5, K = 0$

μ_1	μ_2	β	OP vs. ILP	OP vs. IPP	OP vs. MIPP	OP vs. WSLP	OP vs. MWSLP
3	0.3	0.1	4,15%	6,79%	5,77%	2,57%	35,62%
		0.4	0,34%	0,17%	0,34%	0,34%	19,30%
		0.7	0,05%	0,23%	0,19%	0,37%	11,55%
		1	0,09%	0,04%	0,35%	0,35%	7,40%
	1.5	0.1	4,35%	17,52%	8,11%	2,95%	1,49%
		0.4	1,90%	6,58%	2,98%	1,57%	0,56%
		0.7	0,57%	1,80%	0,86%	1,05%	0,00%
		1	0,58%	0,67%	0,67%	0,91%	0,00%
	3	0.1	5,47%	19,16%	8,52%	3,27%	1,39%
		0.4	3,25%	12,33%	4,08%	2,02%	1,90%
		0.7	1,06%	7,36%	2,09%	1,06%	1,17%
		1	1,10%	5,16%	0,66%	0,66%	0,77%
	6	0.1	6,56%	20,05%	7,92%	3,46%	1,13%
		0.4	5,19%	15,90%	5,56%	2,93%	1,35%
		0.7	5,22%	12,80%	3,78%	1,55%	0,52%
		1	6,27%	13,79%	5,69%	4,39%	4,15%
	60	0.1	7,93%	21,35%	8,18%	3,93%	9,75%
		0.4	8,42%	21,35%	9,51%	4,06%	11,15%
		0.7	7,58%	18,43%	6,59%	1,56%	11,35%
		1	7,79%	17,30%	4,51%	0,71%	12,53%

A fixed start-up cost is incurred when a production channel is activated. Production decisions of the policies are expressed in Table 5.4 where the parameter vector is $(K, s, \mu_1, \mu_2, \beta, h, \lambda_1, \lambda_2, c_1, c_2) = (6{,}3{,}3{,}6{,}0.1{,}2{,}3{,}4{,}4{,}4)$. It is optimal to activate 2 channels in the current state [0,0,0] as it is seen from the table. Instead of activating 3 channels, 2 channels are activated and production is continued with 2 channels in order to avoid the start-up cost. Although the production decision at state [0,0,0] is to activate 3 channels in all the alternative policies, production decisions of the MWSLP are closer to the optimal production decisions with respect to other alternative production policies.

Table 5.4 Production Decisions of the Policies $s = 3, K = 6$

$[x_1, x_2]$	OP						ILP						IPP					
	Inventory Level						Inventory Level						Inventory Level					
	0	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	2	2	2	1	1	0	3	2	1	0	0	0	3	2	1	0	0	0
[0,1]	1	1	0	0	0	0	2	2	1	0	0	0	2	1	0	0	0	0
[0,2]	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0
[0,3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[1,0]	2	2	2	1	1	1	3	2	1	1	1	1	2	1	1	1	1	1
[1,1]	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1
[1,2]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[2,0]	2	2	2	2	2	2	3	2	2	2	2	2	2	2	2	2	2	2
[2,1]	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
[3,0]	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
Avg Cost	*12,02*						*13,61*						*13,17*					

$[x_1, x_2]$	MIPP						WSLP						MWSLP					
	Inventory Level						Inventory Level						Inventory Level					
	0	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*	*0*	*1*	*2*	*3*	*4*	*5*
[0,0]	3	2	1	0	0	0	3	2	0	0	0	0	3	0	0	0	0	0
[0,1]	2	1	0	0	0	0	2	1	0	0	0	0	1	0	0	0	0	0
[0,2]	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
[0,3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[1,0]	3	2	1	1	1	1	3	2	1	1	1	1	2	1	1	1	1	1
[1,1]	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1
[1,2]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[2,0]	3	2	2	2	2	2	3	2	2	2	2	2	2	2	2	2	2	2
[2,1]	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
[3,0]	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
Avg Cost	*13,58*						*13,95*						*12,45*					

The detailed performance results of the policies are given in Table 5.5. The effect of start-up cost (K) makes the alternative policies worsen, especially in lower μ_1/μ_2 values. MWSL Policy may be tractable in moderate values of μ_1/μ_2, on the other hand ILP, IPP, MIPP and WSLP are applicable in higher μ_1/μ_2 values.

Table 5.5 Performances of the Alternative Production Policies $s = 3, K = 6$

μ_1	μ_2	β	OP vs. ILP	OP vs. IPP	OP vs. MIPP	OP vs. WSLP	OP vs. MWSLP
3	0.3	0.1	1,48%	1,73%	1,48%	1,60%	33,99%
		0.4	0,23%	0,37%	0,00%	0,51%	14,72%
		0.7	5,47%	5,94%	5,43%	5,78%	17,79%
		1	0,16%	0,16%	0,16%	0,16%	9,72%
	1.5	0.1	7,15%	5,44%	6,61%	7,95%	0,92%
		0.4	4,23%	3,65%	3,79%	5,09%	2,31%
		0.7	0,13%	0,00%	0,07%	1,36%	0,07%
		1	0,57%	0,06%	0,40%	0,35%	0,06%
	3	0.1	10,02%	8,55%	9,69%	9,76%	1,05%
		0.4	23,66%	25,36%	23,77%	14,52%	17,19%
		0.7	2,45%	2,13%	2,37%	2,61%	2,37%
		1	0,15%	0,46%	0,15%	4,14%	0,77%
	6	0.1	11,68%	8,73%	11,49%	13,84%	3,45%
		0.4	32,78%	30,58%	33,28%	18,51%	5,66%
		0.7	33,43%	35,33%	33,04%	17,99%	10,55%
		1	15,83%	16,15%	15,71%	10,70%	8,82%
	15	0.1	12,23%	9,29%	23,51%	13,82%	17,58%
		0.4	32,22%	28,10%	32,76%	24,53%	12,14%
		0.7	45,87%	39,19%	48,64%	25,20%	10,77%
		1	57,31%	48,57%	62,16%	39,78%	6,47%

Fixed start-up cost has an important effect on the optimality criterion. Either underestimated or overestimated optimal decisions may worsen the performance criterion. In case of underestimation in the production decision, an incoming demand may be rejected in possibility of stock-out. On the other hand, an over production decision causes the higher start-up costs, higher holding and production costs as well. Since the structure of such systems is highly dynamic, performance of the alternative policies varies from region to region. In general, proposed policies mostly perform well in moderate values of production channels. IPP is chiefly affected by increasing number of channels. In that case, it is worse to use the information of active channels partially than not to use, i.e. ILP. Referring to the moderate number of channels, MWLSP dominates the other policies when production rates of stages are closer to each other, WSLP obtains closer average cost with respect to the optimal one.

6 RENEWAL ANALYSIS

Since we consider a make-to-stock production system with fixed start-up cost, *two-critical-level policy* is optimal production policy for such a system. We call the critical numbers (X^{*}, X^{**}) in our studies. When fixed cost is set to be zero, the relation between critical levels become equal and that critical level represents the base-stock level, optimal production decision, for $M/Coxian-2/1$ without fixed cost. We obtain expected system cost using MDP during numerical studies but it is also possible to obtain the cost via steady-state probabilities or renewal reward analysis. The steady-state probabilities are computable for a single channel system; however multiple production channels require more complex state transitions although it still has Markov property. As we mention in Chapter 2, Lee and Srinivasan (1989, 1991) is our main contribution for the renewal analysis (RA) that allows us to solve sub-systems instead of whole system. Renewal reward theorem considers a regeneration point and each cycle until regeneration point is called sub-system. Since the horizon is infinite, accumulated system cost converges to a value then it is enough to calculate the cost for only one cycle and the ratio of accumulated cost and cycle length gives the expected system cost. In the following section, analysis of $M/Coxian-2/1$ make-to-stock queue is expressed.

6.1 Analysis of the Optimal Production Policy for M/Coxian-2/1

We first start to analyze two-phase Coxian processing times with single production channel, fixed start-up cost, lost sale cost and two demand classes under static rationing. Upper critical value (X^{**}) is chosen as the regeneration point. Production is triggered whenever inventory position (IP) drops to X^{*} and continues until it reaches to X^{**} and it is called production period. Besides, non-production period starts when IP reaches to X^{**} and ends when IP drops to X^{*} for the first time. Let

K = fixed start-up cost

h = holding cost

c_1, c_2 = lost sale costs of customer class 1 and 2

R = rationing level of customer class 2, $R \in [0, X^{**}]$

$C_N(X^*, X^{**}, R)$ = expected cost during a non-production period

$C_P(X^*, X^{**}, R)$ = expected cost during a production period

$L_N(X^*, X^{**}, R)$ = expected length of a non-production period

$L_P(X^*, X^{**}, R)$ = expected length of a production period

$AC(X^*, X^{**}, R)$ = expected system cost per unit time

Since production is triggered for once, one fixed start-up cost is charged in each cycle. Then, expected cost per unit time with given control values X^* and X^{**} and R is obtained by

$$AC(X^*, X^{**}, R) = \frac{C_N(X^*, X^{**}, R) + C_P(X^*, X^{**}, R) + K}{L_N(X^*, X^{**}, R) + L_P(X^*, X^{**}, R)} \quad (6.1)$$

By definition of the rationing operator, incoming demand of a customer class is rejected if the inventory position is less than or equal to rationing level of that class, otherwise it is satisfied. Expected system cost accumulates depending on the rationing level of second demand class. Basically, second class rationing level (R) is either between X^* and X^{**} or less than X^*. In case R is between X^* and X^{**}, the only rejection region for the second customer class is between R and X^*. Figure 6.1 shows the accumulated demand rates in both cases.

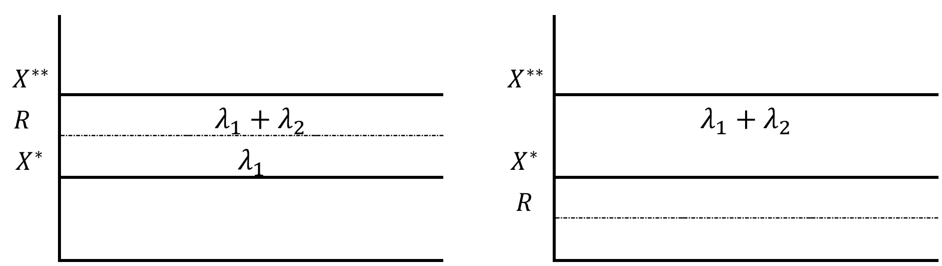

Figure 6.1 Demand Rates According to Critical Levels

Let $g_{x,x-1}$ be the expected cost when inventory level drops from x to $x - 1$ for the first time during a non-production period. The expected cost of the non-production period with given control values is specified by

$$C_N(X^*, X^{**}, R) = \sum_{x=X^*+1}^{X^{**}} g_{x,x-1} \quad (6.2)$$

where

$$g_{x,x-1} = \frac{hx}{\lambda_1+\lambda_2}, \forall_{x\in[1,R)} \quad (6.3)$$

$$g_{x,x-1} = \frac{hx}{\lambda_1 + \lambda_2}, \forall_{x\in[R+1,X^{**}]} \quad (6.4)$$

$$g_{x,x-1} = \frac{hx}{\lambda_1}, \forall_{x\in[X^*+1,R]} \quad (6.5)$$

Since the cost of the non-production period accumulates based on the position of R, then length of the non-production period is obtained in a similar way. Let L_n be the expected length during the non-production period. Total expected length of a non-production period is expressed by

$$L_N = \sum_{n=X^*+1}^{X^{**}} L_n \quad (6.6)$$

where

$$L_n = \frac{X^{**} - X^*}{\lambda_1 + \lambda_2}, \forall_{x\in[1,R)} \quad (6.7)$$

$$L_n = \frac{X^{**} - X^*}{\lambda_1 + \lambda_2}, \forall_{x\in[R+1,X^{**}]} \quad (6.8)$$

$$L_n = \frac{\mathrm{R} - X^*}{\lambda_1}, \forall_{x\in[X^*+1,R]} \quad (6.9)$$

In order to recall the system state definition (x_1, x_2, x_3), x_1 and x_2 denote the number of active production channel in phase-one and phase-two and x_3 denotes the inventory level. By reason of single channel, either first or second phase of the production becomes busy during a production period. Let $f_{(1,x),(1,x+1)}$ be the expected first passage cost from x unit inventory to $x + 1$ for the first time when first stage of the production is operational. Similarly, $f_{(2,x),(1,x+1)}$ is the expected first passage cost from x unit inventory to $x+1$ for the first time when second stage of the production is operational. Without loss of generality, $f_{(n,x),(n,x)} = 0, \forall_{n\in\{0,1,2\}}, \forall_{x\geq 0}$. Thus the expected cost of the production period with given control values is the following expression:

$$C_P(X^*, X^{**}, R) = \sum_{x=X^*}^{X^{**}-2} f_{(1,x),(1,x+1)} + f_{(1,X^{**}-1),(0,X^{**})} \tag{6.10}$$

Whenever production is triggered, it is started in phase-one. It is processed with rate μ_1 in phase-one, either it is passed to stage-two with probability β and processed with rate μ_2 or bypassed with probability $1-\beta$. At the end, the finished item places in the inventory and the following production is started in phase-one. Demand rates of customer class-one and two are λ_1 and λ_2 respectively and total demand is denoted by λ. R is the critical level for the expected first passage cost calculation, thus the rejection criterion for the incoming demand of class-two depends on its rationing level. Figure 6.2 is an explanatory example of state transitions in production period with given control values.

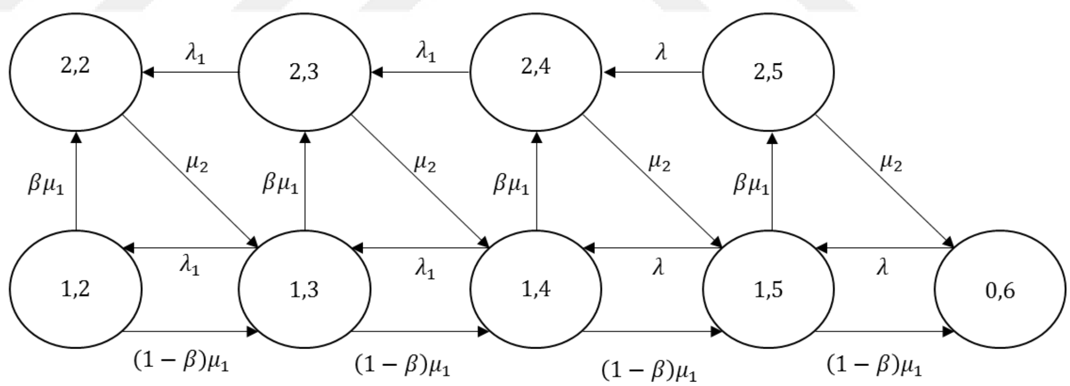

Figure 6.2 Example of State Transition Diagram ($X^* = 2, X^{**} = 6, R = 4$)

Based on the consideration above, expected first passage costs are obtained by

$$f_{(1,x),(1,x+1)} = \frac{hx}{\lambda+\mu_1} + \frac{\mu_1}{\lambda+\mu_1}\left[\beta f_{(2,x),(1,x+1)} + (1-\beta) f_{(1,x+1),(1,x+1)}\right] + \frac{\lambda_1}{\lambda+\mu_1} f_{(1,x-1),(1,x+1)} + \frac{\lambda_2}{\lambda+\mu_1}\left(c_2 + f_{(1,x),(1,x+1)}\right), \forall_{x>0, x\leq R} \tag{6.11}$$

$$f_{(2,x),(1,x+1)} = \frac{hx}{\lambda + \mu_2} + f_{(1,x+1),(1,x+1)} + \frac{\lambda_1}{\lambda + \mu_2} f_{(2,x-1),(1,x+1)} + \frac{\lambda_2}{\lambda + \mu_2}\left(c_2 + f_{(2,x),(1,x+1)}\right), \forall_{x>0, x\le R} \quad (6.12)$$

Equations (6.11) and (6.12) are valid for all positive inventory levels and when the inventory level is less than or equal to R. Recall that demand of class-two is rejected in this situation. During the production period, holding cost and lost sale cost are charged. In phase-one, completed production either passes to phase-two or leaves the system. In case of leaving the system with probability $1-\beta$, there is no accumulated cost because of the self-transition. Similarly, production completion at phase-one causes the self-transition in terms of expected first passage cost. If the class-one demand is occurred, it is satisfied and inventory level is decreased by one unit. If the class-two demand is occurred, it is rejected and system state remains the same. Equation (6.13) and (6.14) are valid for all inventory level that is greater than R. This condition allows us to satisfy incoming demand of both customer classes.

$$f_{(1,x),(1,x+1)} = \frac{hx}{\mu_1} + \frac{\mu_1}{\lambda + \mu_1}\left[\beta f_{(2,x),(1,x+1)} + (1-\beta) f_{(1,x+1),(1,x+1)}\right] + \frac{\lambda}{\lambda + \mu_1} f_{(1,x-1),(1,x+1)}, \forall_{x>R} \quad (6.13)$$

$$f_{(2,x),(1,x+1)} = \frac{hx}{\lambda + \mu_2} + f_{(1,x+1),(1,x+1)} + \frac{\lambda}{\lambda + \mu_1} f_{(2,x-1),(1,x+1)}, \forall_{x>R} \quad (6.14)$$

When there is no on-hand inventory, demand of any class is rejected and only lost sale costs accumulate. At the boundary, expected first passage cost is obtained by

$$f_{(1,0),(1,1)} = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\mu_1} + \frac{\beta(\lambda_1 c_1 + \lambda_2 c_2)}{\mu_2} \quad (6.15)$$

$$f_{(2,0),(1,1)} = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\mu_2} \quad (6.16)$$

When we consider the relationship between equations (6.15) and (6.16), we may define $f_{(1,0),(1,1)}$ as a function of $f_{(2,0),(1,1)}$. In addition to this, any $f_{(1,x),(1,x+1)}$ is related to $f_{(2,x),(1,1+1)}$ in general. Using this information, we obtain recursive expressions in twofold: *i*. in case x is less than or equal to R, *ii*. in case x is greater than R with the boundaries defined in (6.17) and (6.18).

$$
\begin{aligned}
f_{(1,x),(1,x+1)} = & \frac{h}{\mu_1}\sum_{i=1}^{x} i\left(\frac{\lambda_1}{\mu_1}\right)^{x-i} + \frac{\lambda_2 c_2}{\mu_1}\sum_{i=1}^{x}\left(\frac{\lambda_1}{\mu_1}\right)^{x-i} + \left(\frac{\lambda_1}{\mu_1}\right)^{x} f_{(1,0),(1,1)} \\
& + \beta\sum_{i=1}^{x}\left(\frac{\lambda_1}{\mu_1}\right)^{x-i}\left[\frac{h}{\lambda_1+\mu_2}\sum_{j=1}^{i} j\left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i-j}\right. \\
& + \frac{\lambda_2 c_2}{\lambda_1+\mu_2}\sum_{j=1}^{i}\left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i-j} + \left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i} f_{(2,0),(1,1)} \\
& \left. + \sum_{j=1}^{i}\left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i-j+1} f_{(1,j),(1,j+1)}\right], \forall_{x>0, x\leq R}
\end{aligned} \tag{6.17}
$$

$$
\begin{aligned}
f_{(1,x),(1,x+1)} = & \frac{h}{\mu_1}\sum_{i=1}^{x} i\left(\frac{\lambda_1}{\mu_1}\right)^{x-i} + \left(\frac{\lambda_1}{\mu_1}\right)^{x} f_{(1,0),(1,1)} \\
& + \beta\sum_{i=1}^{x}\left(\frac{\lambda_1}{\mu_1}\right)^{x-i}\left[\frac{h}{\lambda_1+\mu_2}\sum_{j=1}^{i} j\left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i-j}\right. \\
& + \left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i} f_{(2,0),(1,1)} \\
& \left. + \sum_{j=1}^{i}\left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^{i-j+1} f_{(1,j),(1,j+1)}\right], \forall_{x>R}
\end{aligned} \tag{6.18}
$$

For the aim of expected length of a production period, first passage time analysis (Solberg (2008)) is conducted. The analysis provides expected first passage time from any state to another with given transition rates. Let m_{ij} be the expected total time to achieve state *j* for the first time starting from the state *i*. In our case, we aim to calculate $m_{X^*,X^{**}}$ for the production period. Let λ_{ij} be the transition rate from state *i* to *j*. The mean first passage times m_{ij} in a continuous time Markov process should satisfy the following equation:

$$0 = 1 + \sum_{k\neq j} \lambda_{ik} m_{kj} \tag{6.19}$$

where $i \neq j$. Let Λ be the transition rate matrix from any state *i* to *j* to be traveled. Then, j^{th} row and j^{th} column of the Λ is replaced with zero and the intersection of the j^{th} row and j^{th} column is set to be one. Let λ_j^+ be the modified transition matrix. The modifications prevent to have singular matrix, then matrix becomes invertible. In equations (6.20) and (6.21), m_j array is a column vector that holds the first passage times from any state i to j where i^{th} state is included the i^{th} element of the vector.

$$0 = 1 + \sum_{k \neq j} \lambda_j^{+} m_j \tag{6.20}$$

$$m_j = \left(-\lambda_j^{+}\right)^{-1} 1 \tag{6.21}$$

We hereby obtain expected cost via RA using MATLAB with the aforementioned algorithms. We also conduct MDP analysis based on value iteration algorithm, then we compare the computational effort of these algorithms for given system parameters. Table 6.1 shows the CPU times of MDP and RA in two cases: *i.* $K = 0$, *ii.* $K = 6$ with a single demand class consideration. As it is seen, RA is not affected by start-up cost as MDP is.

Table 6.1 CPU Times of MDP and RA – Single Demand Class

X^*	X^{**}	CPU Time			
		$K = 0$		$K = 6$	
		MDP	RA	MDP	RA
1	16	8.66	0.69	15.02	0.72
3	16	8.69	0.68	11.1	0.77
5	16	8.96	0.73	11.34	0.70
7	16	8.86	0.68	11.39	0.70
9	16	8.71	0.68	11.47	0.70
11	16	8.67	0.71	11.79	0.70
13	16	8.65	0.69	11.83	0.70
15	16	8.68	0.69	11.96	0.72

When a rationing decision is also considered, computation time of MDP increases rapidly. Table 6.2 indicates the computational efforts of the algorithms for given parameters. When the rationing decision is included to the algorithms in addition to the production decision, a remarkable difference between computation time of MDP and RA is obtained.

It is observed that the computation time is significantly reduced. And this reduction will be more pronounced as number of production channel increases.

Table 6.2 CPU Times of MDP and RA – Two Demand Classes

R	X^{*}	X^{**}	CPU Time			
			$K=0$		$K=6$	
			MDP	RA	MDP	RA
2	3	16	47.61	1.64	53.52	1.60
2	5	16	47.97	1.62	49.85	1.60
2	7	16	48.53	1.63	50.46	1.63
2	9	16	49.11	1.60	51.20	1.61
2	11	16	49.97	1.59	50.91	1.63
2	13	16	50.56	1.60	51.20	1.61
2	15	16	50.79	1.61	51.08	1.65

7 CONCLUSION AND FUTURE WORK

This thesis considers a production-inventory system in a make-to-stock environment with multiple identical production channels, fixed start-up costs, several demand classes and lost sale costs. Production times are assumed to be 2-phase Coxian random variables that allow us to embed rework operations into production process. Demands of customer classes arrive according to independent Poisson processes and demand classes are differentiated based on their lost sale costs. The system is modeled as an $M/Coxian-2/s$ make-to-stock queue and dynamic programming formulation is developed under the average system cost criterion.

Phase-type production time consideration with multiple channels, several customer classes and start-up costs is an extensive study in the literature. $M/Coxian-2/s$ is a direct extension of $M/M/s$ make-to-stock queue. Additionally, 2-phase Coxian processing times extends the 2-stage Erlangian processing times by offering different production rates to the stages, i.e. $M/Coxian-2/s$ generalizes the $M/E_2/s$ make-to-stock queue.

We first characterize the optimal production and rationing policies by means of numerical analyses and show that optimal policy structure is highly dynamic. In consideration of applicability of the policies, we propose a dynamic rationing policy whose performance is close to the performance of the optimal policy. We then work on alternative production policies utilizing the dynamic information that the system state vector carries. We proposed different ways of assessing the value of inventory level and the number of active servers at both stages. Furthermore, we conduct a renewal analysis for the $M/Coxian-2/1$ with two demand classes and obtain the optimal average system cost. When we compare the renewal analysis with value iteration algorithm, we observe that the computation time is significantly reduced.

The renewal analysis may be easily extended to the *n*-many customer classes, since the current one is based on two classes. This analysis can be also modified to assess the performance of better performing dynamic rationing policies. Additionally, a desirable extension of our study may be n-phase Coxian processing time consideration where each production stage has its own rework operation.

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CURRICULUM VITEA

Özgün Öztürk was born in İzmir. Öztürk received her undergraduate Industrial Engineering degree from Yaşar University in 2014. In the same year, Öztürk started working in Industrial Engineering Department as a Research Assistant. Her research interests are production-inventory control, stochastic processes and stochastic modeling.

APPENDIX 1 QUESTIONAIRE

The detailed production and rationing decisions when *s* increases:

(s=3)															
Optimal Production Decisions								Optimal Rationing Decisions							
state	*Inventory Level*							state	*Inventory Level*						
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	3	3	3	3	1	0	0	[0,0]	0	0	0	1	1	1	1
[0,1]	2	2	2	1	0	0	0	[0,1]	0	0	1	1	1	1	1
[0,2]	1	1	1	0	0	0	0	[0,2]	0	0	1	1	1	1	1
[0,3]	0	0	0	0	0	0	0	[0,3]	0	0	1	1	1	1	1
[1,0]	3	3	3	3	1	1	1	[1,0]	0	0	0	1	1	1	1
[1,1]	2	2	2	1	1	1	1	[1,1]	0	0	1	1	1	1	1
[1,2]	1	1	1	1	1	1	1	[1,2]	0	0	1	1	1	1	1
[2,0]	3	3	3	3	2	2	2	[2,0]	0	0	0	1	1	1	1
[2,1]	2	2	2	2	2	2	2	[2,1]	0	0	1	1	1	1	1
[3,0]	3	3	3	3	3	3	3	[3,0]	0	0	0	1	1	1	1
(s=6)															
state	*Inventory Level*							state	*Inventory Level*						
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	6	6	3	1	0	0	0	[0,0]	0	0	1	1	1	1	1
[0,1]	5	4	1	0	0	0	0	[0,1]	0	0	1	1	1	1	1
[0,2]	4	2	0	0	0	0	0	[0,2]	0	0	1	1	1	1	1
[0,3]	3	0	0	0	0	0	0	[0,3]	0	0	1	1	1	1	1
[0,4]	1	0	0	0	0	0	0	[0,4]	0	1	1	1	1	1	1
[0,5]	0	0	0	0	0	0	0	[0,5]	0	1	1	1	1	1	1
[0,6]	0	0	0	0	0	0	0	[0,6]	0	1	1	1	1	1	1
[1,0]	6	6	3	1	1	1	1	[1,0]	0	0	1	1	1	1	1
[1,1]	5	4	1	1	1	1	1	[1,1]	0	0	1	1	1	1	1
[1,2]	4	2	1	1	1	1	1	[1,2]	0	0	1	1	1	1	1
[1,3]	3	1	1	1	1	1	1	[1,3]	0	1	1	1	1	1	1
[1,4]	1	1	1	1	1	1	1	[1,4]	0	1	1	1	1	1	1
[1,5]	1	1	1	1	1	1	1	[1,5]	0	1	1	1	1	1	1
[2,0]	6	6	3	2	2	2	2	[2,0]	0	0	1	1	1	1	1
[2,1]	5	4	2	2	2	2	2	[2,1]	0	0	1	1	1	1	1
[2,2]	4	2	2	2	2	2	2	[2,2]	0	0	1	1	1	1	1
[2,3]	3	2	2	2	2	2	2	[2,3]	0	1	1	1	1	1	1
[2,4]	2	2	2	2	2	2	2	[2,4]	0	1	1	1	1	1	1
[3,0]	6	6	3	3	3	3	3	[3,0]	0	0	1	1	1	1	1
[3,1]	5	4	3	3	3	3	3	[3,1]	0	0	1	1	1	1	1

[3,2]	4	3	3	3	3	3	3	[3,2]	0	0	1	1	1	1	1
[3,3]	3	3	3	3	3	3	3	[3,3]	0	1	1	1	1	1	1
[4,0]	6	6	4	4	4	4	4	[4,0]	0	0	1	1	1	1	1
[4,1]	5	4	4	4	4	4	4	[4,1]	0	0	1	1	1	1	1
[4,2]	4	4	4	4	4	4	4	[4,2]	0	0	1	1	1	1	1
[5,0]	6	6	5	5	5	5	5	[5,0]	0	0	1	1	1	1	1
[5,1]	5	5	5	5	5	5	5	[5,1]	0	0	1	1	1	1	1
[6,0]	6	6	6	6	6	6	6	[6,0]	0	0	1	1	1	1	1
(s=8)															
state	*Inventory Level*							state	*Inventory Level*						
$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	8	5	3	1	0	0	0	[0,0]	0	0	1	1	1	1	1
[0,1]	6	4	1	0	0	0	0	[0,1]	0	0	1	1	1	1	1
[0,2]	5	2	0	0	0	0	0	[0,2]	0	0	1	1	1	1	1
[0,3]	3	0	0	0	0	0	0	[0,3]	0	0	1	1	1	1	1
[0,4]	1	0	0	0	0	0	0	[0,4]	0	1	1	1	1	1	1
[0,5]	0	0	0	0	0	0	0	[0,5]	0	1	1	1	1	1	1
[0,6]	0	0	0	0	0	0	0	[0,6]	0	1	1	1	1	1	1
[0,7]	0	0	0	0	0	0	0	[0,7]	0	1	1	1	1	1	1
[0,8]	0	0	0	0	0	0	0	[0,8]	0	1	1	1	1	1	1
[1,0]	8	5	3	1	1	1	1	[1,0]	0	0	1	1	1	1	1
[1,1]	6	4	1	1	1	1	1	[1,1]	0	0	1	1	1	1	1
[1,2]	5	2	1	1	1	1	1	[1,2]	0	0	1	1	1	1	1
[1,3]	3	1	1	1	1	1	1	[1,3]	0	1	1	1	1	1	1
[1,4]	1	1	1	1	1	1	1	[1,4]	0	1	1	1	1	1	1
[1,5]	1	1	1	1	1	1	1	[1,5]	0	1	1	1	1	1	1
[1,6]	1	1	1	1	1	1	1	[1,6]	0	1	1	1	1	1	1
[1,7]	1	1	1	1	1	1	1	[1,7]	0	1	1	1	1	1	1
[2,0]	8	5	3	2	2	2	2	[2,0]	0	0	1	1	1	1	1
[2,1]	6	4	2	2	2	2	2	[2,1]	0	0	1	1	1	1	1
[2,2]	5	2	2	2	2	2	2	[2,2]	0	0	1	1	1	1	1
[2,3]	3	2	2	2	2	2	2	[2,3]	0	1	1	1	1	1	1
[2,4]	2	2	2	2	2	2	2	[2,4]	0	1	1	1	1	1	1
[2,5]	2	2	2	2	2	2	2	[2,5]	0	1	1	1	1	1	1
[2,6]	2	2	2	2	2	2	2	[2,6]	0	1	1	1	1	1	1
[3,0]	8	5	3	3	3	3	3	[3,0]	0	0	1	1	1	1	1
[3,1]	6	4	3	3	3	3	3	[3,1]	0	0	1	1	1	1	1
[3,2]	5	3	3	3	3	3	3	[3,2]	0	0	1	1	1	1	1
[3,3]	3	3	3	3	3	3	3	[3,3]	0	1	1	1	1	1	1
[3,4]	3	3	3	3	3	3	3	[3,4]	0	1	1	1	1	1	1
[3,5]	3	3	3	3	3	3	3	[3,5]	0	1	1	1	1	1	1
[4,0]	8	5	4	4	4	4	4	[4,0]	0	0	1	1	1	1	1
[4,1]	6	4	4	4	4	4	4	[4,1]	0	0	1	1	1	1	1
[4,2]	5	4	4	4	4	4	4	[4,2]	0	0	1	1	1	1	1
[4,3]	4	4	4	4	4	4	4	[4,3]	0	1	1	1	1	1	1

[4,4]	4	4	4	4	4	4	4	[4,4]	0	1	1	1	1	1	1
[5,0]	8	5	5	5	5	5	5	[5,0]	0	0	1	1	1	1	1
[5,1]	6	5	5	5	5	5	5	[5,1]	0	0	1	1	1	1	1
[5,2]	5	5	5	5	5	5	5	[5,2]	0	1	1	1	1	1	1
[5,3]	5	5	5	5	5	5	5	[5,3]	0	1	1	1	1	1	1
[6,0]	8	6	6	6	6	6	6	[6,0]	0	0	1	1	1	1	1
[6,1]	6	6	6	6	6	6	6	[6,1]	0	0	1	1	1	1	1
[6,2]	6	6	6	6	6	6	6	[6,2]	0	1	1	1	1	1	1
[7,0]	8	7	7	7	7	7	7	[7,0]	0	0	1	1	1	1	1
[7,1]	7	7	7	7	7	7	7	[7,1]	0	1	1	1	1	1	1
[8,0]	8	8	8	8	8	8	8	[8,0]	0	0	1	1	1	1	1

(s=9)

state	*Inventory Level*							state	*Inventory Level*						
$[x_1, x_2]$	0	*1*	*2*	*3*	*4*	*5*	*6*	$[x_1, x_2]$	*0*	*1*	*2*	*3*	*4*	*5*	*6*
[0,0]	8	5	3	1	0	0	0	[0,0]	0	0	1	1	1	1	1
[0,1]	6	4	1	0	0	0	0	[0,1]	0	0	1	1	1	1	1
[0,2]	5	2	0	0	0	0	0	[0,2]	0	0	1	1	1	1	1
[0,3]	3	0	0	0	0	0	0	[0,3]	0	0	1	1	1	1	1
[0,4]	1	0	0	0	0	0	0	[0,4]	0	1	1	1	1	1	1
[0,5]	0	0	0	0	0	0	0	[0,5]	0	1	1	1	1	1	1
[0,6]	0	0	0	0	0	0	0	[0,6]	0	1	1	1	1	1	1
[0,7]	0	0	0	0	0	0	0	[0,7]	0	1	1	1	1	1	1
[0,8]	0	0	0	0	0	0	0	[0,8]	0	1	1	1	1	1	1
[0,9]	0	0	0	0	0	0	0	[0,9]	0	1	1	1	1	1	1
[1,0]	8	5	3	1	1	1	1	[1,0]	0	0	1	1	1	1	1
[1,1]	6	4	1	1	1	1	1	[1,1]	0	0	1	1	1	1	1
[1,2]	5	2	1	1	1	1	1	[1,2]	0	0	1	1	1	1	1
[1,3]	3	1	1	1	1	1	1	[1,3]	0	1	1	1	1	1	1
[1,4]	1	1	1	1	1	1	1	[1,4]	0	1	1	1	1	1	1
[1,5]	1	1	1	1	1	1	1	[1,5]	0	1	1	1	1	1	1
[1,6]	1	1	1	1	1	1	1	[1,6]	0	1	1	1	1	1	1
[1,7]	1	1	1	1	1	1	1	[1,7]	0	1	1	1	1	1	1
[1,8]	1	1	1	1	1	1	1	[1,8]	0	1	1	1	1	1	1
[2,0]	8	5	3	2	2	2	2	[2,0]	0	0	1	1	1	1	1
[2,1]	6	4	2	2	2	2	2	[2,1]	0	0	1	1	1	1	1
[2,2]	5	2	2	2	2	2	2	[2,2]	0	0	1	1	1	1	1
[2,3]	3	2	2	2	2	2	2	[2,3]	0	1	1	1	1	1	1
[2,4]	2	2	2	2	2	2	2	[2,4]	0	1	1	1	1	1	1
[2,5]	2	2	2	2	2	2	2	[2,5]	0	1	1	1	1	1	1
[2,6]	2	2	2	2	2	2	2	[2,6]	0	1	1	1	1	1	1
[2,7]	2	2	2	2	2	2	2	[2,7]	0	1	1	1	1	1	1
[3,0]	8	5	3	3	3	3	3	[3,0]	0	0	1	1	1	1	1
[3,1]	6	4	3	3	3	3	3	[3,1]	0	0	1	1	1	1	1
[3,2]	5	3	3	3	3	3	3	[3,2]	0	0	1	1	1	1	1
[3,3]	3	3	3	3	3	3	3	[3,3]	0	1	1	1	1	1	1

[3,4]	3	3	3	3	3	3	3	[3,4]	0	1	1	1	1	1	1
[3,5]	3	3	3	3	3	3	3	[3,5]	0	1	1	1	1	1	1
[3,6]	3	3	3	3	3	3	3	[3,6]	0	1	1	1	1	1	1
[4,0]	8	5	4	4	4	4	4	[4,0]	0	0	1	1	1	1	1
[4,1]	6	4	4	4	4	4	4	[4,1]	0	0	1	1	1	1	1
[4,2]	5	4	4	4	4	4	4	[4,2]	0	0	1	1	1	1	1
[4,3]	4	4	4	4	4	4	4	[4,3]	0	1	1	1	1	1	1
[4,4]	4	4	4	4	4	4	4	[4,4]	0	1	1	1	1	1	1
[4,5]	4	4	4	4	4	4	4	[4,5]	0	1	1	1	1	1	1
[5,0]	8	5	5	5	5	5	5	[5,0]	0	0	1	1	1	1	1
[5,1]	6	5	5	5	5	5	5	[5,1]	0	0	1	1	1	1	1
[5,2]	5	5	5	5	5	5	5	[5,2]	0	1	1	1	1	1	1
[5,3]	5	5	5	5	5	5	5	[5,3]	0	1	1	1	1	1	1
[5,4]	5	5	5	5	5	5	5	[5,4]	0	1	1	1	1	1	1
[6,0]	8	6	6	6	6	6	6	[6,0]	0	0	1	1	1	1	1
[6,1]	6	6	6	6	6	6	6	[6,1]	0	0	1	1	1	1	1
[6,2]	6	6	6	6	6	6	6	[6,2]	0	1	1	1	1	1	1
[6,3]	6	6	6	6	6	6	6	[6,3]	0	1	1	1	1	1	1
[7,0]	8	7	7	7	7	7	7	[7,0]	0	0	1	1	1	1	1
[7,1]	7	7	7	7	7	7	7	[7,1]	0	1	1	1	1	1	1
[7,2]	7	7	7	7	7	7	7	[7,2]	0	1	1	1	1	1	1
[8,0]	8	8	8	8	8	8	8	[8,0]	0	0	1	1	1	1	1
[8,1]	8	8	8	8	8	8	8	[8,1]	0	1	1	1	1	1	1
[9,0]	9	9	9	9	9	9	9	[9,0]	0	0	1	1	1	1	1