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MASTER THESIS

DYNAMIC INEQUALITIES ON TIME SCALES

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

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ABSTRACT

DYNAMIC INEQUALITIES ON TIME SCALES

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In recent times, the theory and applications of dynamic inequalities on time scales has attracted great interest. Although many results of differential equations are similar to the results of difference equations, there are also situations where they are different. The study of dynamic equations on time scales reveal these differences. The time scale theory was introduced by Stefan Hilger in his PhD thesis in order to unify discrete and continuous analysis. In the first part of the thesis, we give basic definitions about the calculus on time scales. The purpose of this thesis is to give basic properties of some dynamic inequalities which are Gronwall's, Hölder's, Minkowski's and Jensen's inequality with their proofs.

Keywords: Time Scale, Dynamic Inequalities, Gronwall's Inequality, Hölder's Inequality, Minkowski's Inequality, Jensen's Inequality.

ÖZET

ZAMAN SKALASINDA DİNAMİK EŞİTSİZLİKLER

Tuba TAÇYILDIZ
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Son zamanlarda zaman skalasındaki dinamik eşitsizliklerin teori ve uygulamaları büyük ilgi görmektedir. Diferensiyel denklemler ile ilgili birçok sonucun fark denklemleri ile benzer olmasına rağmen farklı olduğu durumlar da bulunmaktadır. Zaman skalasında dinamik denklemlerin çalışılması bu farklılıkları ortaya çıkarır. Zaman skalası teorisi, Stefan Hilger tarafından doktora tezinde ayrık ve sürekli analizi birleştirmek amacıyla tanıtılmıştır. Bu tezin ilk bölümünde zaman sklasında hesaplamalar ile ilgili bazı temel kavramlar verilmiştir. Bu tezin amacı Gronwall, Hölder, Minkowski ve Jensen eşitsizlikleri gibi bazı dinamik eşitsizliklerin ispatlarıyla birlikte bazı temel özelliklerinden bahsedilmiştir.

Anahtar sözcükler: Zaman Skalası, Dinamik Eşitsizlikler, Gronwall Eşitsizliği, Hölder Eşitsizliği, Minkowski Eşitsizliği, Jensen Eşitsizliği.

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Tuba TAÇYILDIZ
İzmir, 2016

TEXT OF OATH

I declare and honestly confirm that my study, titled "DYNAMIC INEQUALITIES ON TIME SCALES" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

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1 INTRODUCTION

Aulbach and Hilger unified and extended the differential equation, difference equation and quantum equation on time scale as dynamic equations. It provides unifying framework for difference, differential and quantum equations on discrete intervals with non-uniform size (q-numbers) and continuous intervals. Therefore, the notion of time scales can combine the continuous, discrete and q-discrete analysis. The consideration of differential equations and difference equations as a dynamic equation states the unification property of time scale while the consideration of a quantum equation as a dynamic equation states the extention property of time scale.

Inequalities are very practical part of mathematics. They give us an idea about the size of the quantities and provide an accurate estimate. In many areas of applied mathematics they also provide us a knowledge about the location of the things. One of the most practical part of the inequalities that it is usually far easier to satisfy assumptions involving inequalities that it is for those involving equations.

When derivatives and inequalities are combined, we refer to them as "differential inequalities" and they are vey useful in the analysis of solutions to nonlinear differential equations.

In mathematics, Gronwall's inequality (also called Gronwall's lemma or the Gronwall–Bellman inequality) is a method to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the corresponding differential or integral equation. This inequality has two main forms: a differential form and an integral form. For the latter there are several variants.

In order to obtain various estimates in the theory of ordinary and stochastic differential equations Gronwall's inequality (It is named for Thomas Hakon Gronwall (1877–1932).) is one of the most significant tools. In particular, it provides a comparison theorem that can be used to prove uniqueness of a solution (the Picard–Lindelöf theorem) to the initial value problem.

2 TIME SCALES CALCULUS

In this chapter of this dissertation, we state the concept of time scale, definitions forward and backward jump operators, graininess functions. Next by the aid of these concepts, we give the definitions and primary theorems of Δ- and ∇-derivatives and Δ- and ∇-integrals. We illustrate how the time scale derivatives and integrals differ from ordinary derivative and integral $\mathbb{R}$ and difference derivative and integral on $\mathbb{Z}$ by examples. Also the definition and some basic properties of exponential function on time scales are given.

2.1 Basic Definitions

After the concept of time scale is introduced by Hilger[11], the time scale calculus is rapidly developed by many scientists[1-3, 5-14]. The three important monographs about the time scale calculus and dynamic equations on time scales are written by Bohner and Peterson[6, 7, 14]. For more details about the concept of time scales are refer reader to the references above and the references therein.

Definition 2.1 [6] A *time scale* is an arbitrary nonempty closed subset of the real numbers. The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the naturel numbers $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$, $[0,5]\cup[6,7]$, $[0,1]\cup\mathbb{N}$ are examples of time scales. The rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (since they are not closed), the complex numbers $\mathbb{C}$ (not a subset of $\mathbb{R}$), and the open interval $(0,1)$ (not closed) are not time scales.

Throughout this thesis, a time scale will be denoted by the symbol $\mathbb{T}$.

Definition 2.2 [6] Let $\mathbb{T}$ be a time scale. The *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$$

in particular $\sigma(max\mathbb{T}) = max\mathbb{T}$, while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t) = sup\{s \in \mathbb{T}: s < t\}$$

in particular $\rho(min\mathbb{T}) = min\mathbb{T}$.

In order to get rid of pathological cases, it is assumed that $inf\emptyset = sup\mathbb{T}$ and $sup\emptyset = inf\mathbb{T}$.

Definition 2.3 [6] The *graininess function* μ: $\mathbb{T}\rightarrow[0,\infty)$ is defined by

$\mu(t) = \sigma(t) - t$.

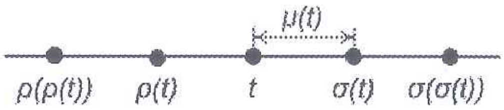

Figure 1.1 Basic operators on time scale

Definition 2.4 [6] Any point of a time scale is classified by means of jump operators. If $\sigma(t) > t$, t is *right-scattered*, while if $\rho(t) < t$, then t *is left-scattered*. If a point is both left and right scattered, it is called an *isolated point*. Also, if $t < sup\mathbb{T}$ and $\sigma(t)=t$, then t is called *right-dense*, and if $t > inf\mathbb{T}$ and $\rho(t)=t$, then t is called *left-dense*. If a point is both left and right dense, it is called a *dense point*.

Figure 1.2 Classifications of points

As illustrated by the Figure 1.2 above:

t_1 is dense point,

t_2 is left-dense and right-scattered point,

t_3 is isolated point,

t_4 is left-scattered and right dense point.

Example 2.5 For each of the following time scales $\mathbb{T}$, the illustration of jump operators and classification of the points are given as follows:

(i) $\mathbb{T} = \mathbb{R}$;

$$\sigma(t) = inf\{s \in \mathbb{R}: s > t\} = \inf(t, +\infty) = t,$$

$$\rho(t) = sup\{s \in \mathbb{R}: s < t\} = \sup(-\infty, t) = t,$$

$$\mu(t) = \sigma(t) - t = t - t = 0,$$

Hence every point $t \in \mathbb{R}$ is dense.

(ii) $\mathbb{T} = \mathbb{Z}$;

$$\sigma(t) = inf\{s \in \mathbb{Z}: s > t\} = inf\{t+1, t+2, \dots\} = t + 1,$$

$$\rho(t) = sup\{s \in \mathbb{Z}: s < t\} = sup\{t-1, t-2, \dots\} = t - 1,$$

$$\mu(t) = \sigma(t) - t = t + 1 - t = 1,$$

Hence every point $t \in \mathbb{Z}$ is isolated.

(iii) $\mathbb{T} = \{\frac{n}{2} : n \in \mathbb{N}_0\}$;

$$t = \frac{n}{2} \Rightarrow n = 2t \in \mathbb{N}_0,$$

$$\sigma(t) = inf\left\{\frac{2t+1}{2}, \frac{2t+2}{2}, \dots\right\} = \frac{2t+1}{2} = t + \frac{1}{2},$$

$$\rho(t) = sup\left\{\dots, \frac{2t-2}{2}, \frac{2t-1}{2}\right\} = \frac{2t-1}{2} = t - \frac{1}{2},$$

$$\mu(t) = \sigma(t) - t = t + \frac{1}{2} - t = \frac{1}{2},$$

Hence every point $t \in \mathbb{N}_0$ is isolated.

2.2 Differentiation

Definition 2.6 [6] Let $f: \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^k$

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - [\max \mathbb{T}]\,; & if\ max\mathbb{T}\ is\ left-scattered \\ \mathbb{T}\,; & otherwise \end{cases}$$

$$\mathbb{T}_k = \begin{cases} \mathbb{T} - [min\mathbb{T}]; & if\ min\mathbb{T}\ is\ right-scattered \\ \mathbb{T}\,; & otherwise \end{cases}$$

If there exist 'a' such that $\forall \varepsilon > 0$ there exist a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - a[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \quad ; \forall\, s \in U$$

then 'a' is called the Δ- derivative of f at the point t, and denoted by $\alpha = f^{\Delta}(t)$

$$\left|\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f^{\Delta}(t)\right| \leq \varepsilon$$

which implies $f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$.

We say that f is *delta differentiable* at each point t on $\mathbb{T}^k$.

Theorem 2.7 Assume $f: \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the followings:

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered (σ(t)>t), then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exist and is finite. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

Proof. [1,6,7]

Note that if $\mathbb{T}=\mathbb{R}$, then Theorem 2.7 (iii) yields that $f: \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff the limit exists. We know $\sigma(t)=t$ when $\mathbb{T}=\mathbb{R}$, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

where $f'(t)$ is the derivative used in standard calculus.

If $\mathbb{T}=\mathbb{Z}$, then Theorem 2.7 (ii) yields that $f: \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with $\sigma(t)=t+1$

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(t+1) - f(t)}{t + 1 - t} = f(t+1) - f(t) = \Delta f(t)$$

where Δ is *forward difference operator* used in difference calculus.

Example 2.8 For each of the followings let us find f^{Δ} using Theorem 1.8 for the function $f(t) = t^2$.

(i) $\mathbb{T}$ is arbitrary;

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s\to t} \frac{(\sigma(t))^2 - s^2}{\sigma(t) - s} = \sigma(t) + t$$

(ii) $\mathbb{T}$=$\mathbb{R}$;

$\sigma(t) = t, \ \forall t \in \mathbb{R}$. Hence by (i) $f^{\Delta}(t) = t + t = 2t$.

(iii) $\mathbb{T}$=$\mathbb{Z}$;

$\sigma(t) = t + 1 \ \forall t \in \mathbb{Z}$. Therefore

$$f^{\Delta}(t) = \sigma(t) + t = t + 1 + t = 2t + 1 = \Delta f(t)$$

(iv) $\mathbb{T} = \mathbb{N}_0^{1/2}$;

$\sigma(t) = \sqrt{t^2 + 1}$ which implies $f^{\Delta}(t) = \sigma(t) + t = \sqrt{t^2 + 1} + t$

(v) $\mathbb{T} = \frac{\mathbb{N}_0}{2}$;

$\sigma(t) = t + \frac{1}{2}$ gives us $f^{\Delta}(t) = \sigma(t) + t = t + \frac{1}{2} + t = 2t + \frac{1}{2}$.

Theorem 2.9 Assume f, g: $\mathbb{T}\to\mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then we have the followings:

(i) $f \mp g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f \mp g)^{\Delta}(t) = f^{\Delta}(t) \mp g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $f.g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f.g)^{\Delta}(t) = f^{\Delta}(t).g(t) + f(\sigma(t)).g^{\Delta}(t)$$

or

$$(f.g)^{\Delta}(t) = f(t).g^{\Delta}(t) + f^{\Delta}(t).g(\sigma(t)).$$

(iv) If $f(t).f(\sigma(t)) \neq 0$, then $1/f$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t).g(\sigma(t)) \neq 0$, then f/g is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Proof. We only prove (iii). For the proofs of other statements see[6].

$$(f.g)^{\Delta}(t) = \lim_{s\to t}\frac{(fg)(\sigma(t)) - (fg)(s)}{\sigma(t) - s}$$

$$= \lim_{s\to t}\frac{f(\sigma(t)).g(\sigma(t)) - f(s)g(s)}{\sigma(t) - s}$$

Adding and substracting the term $f(s)g(\sigma(t))$, we obtain

$$= \lim_{s\to t}\frac{f(\sigma(t))g(\sigma(t)) - f(s)g(\sigma(t)) + f(s)g(\sigma(t)) - f(s)g(s)}{\sigma(t) - s}$$

$$= \lim_{s\to t}\frac{g(\sigma(t))[f(\sigma(t)) - f(s)]}{\sigma(t) - s} + \lim_{s\to t}\frac{f(s)[g(\sigma(t)) - g(s)]}{\sigma(t) - s}$$

$$= g(\sigma(t))f^{\Delta}(t) + f(t)g^{\Delta}(t)$$

□

Remark 2.10 By adding and substracting the term $f(\sigma(t))g(s)$ in the proof, we obtain the other statements.

Example 2.11 We can use Theorem 2.9 (iii) to find the Δ-derivative of $h(t) = t^2e^t$ on the time scale $\mathbb{T} = \{q^n : n \in \mathbb{N}_0 \; ; q > 1\} \cup \{0\}$.

Let $f(t) = t^2$, $g(t) = e^t$. Then $h(t) = f(t).g(t)$.

By Theorem 2.9 (iii), we have

$$h^\Delta(t) = (fg)^\Delta(t) = f^\Delta(t)g(t) + f\big(\sigma(t)\big)g^\Delta(t)$$

By Example 2.8 (i) $f^\Delta(t) = \sigma(t) + t$ on an arbitrary time scale. Hence since $\sigma(t) = tq$ on $\mathbb{T} = q^{\mathbb{N}_0}$, then $f^\Delta(t) = tq + t$.

Similarly by definition of Δ-derivative (Definition 2.7)

$$g^\Delta(t) = \lim_{s\to t}\frac{g\big(\sigma(t)\big) - g(s)}{\sigma(t) - s} = \frac{e^{\sigma(t)} - e^t}{\sigma(t) - t} = \frac{e^{tq} - e^t}{tq - t}.$$

Hence

$$h^\Delta(t) = (tq + t).e^t + (tq)^2.\frac{e^{tq} - e^t}{tq - t}.$$

Example 2.12 We can derive the formula of second derivative of (fg): $(fg)^{\Delta\Delta}(t)$.

$$(fg)^{\Delta\Delta}(t) = [(fg)^\Delta(t)]^\Delta$$

$$= \left[f^\Delta(t)g(t) + f\big(\sigma(t)\big)g^\Delta(t)\right]^\Delta$$

$$= [f^\Delta(t)g(t)]^\Delta + [f(\sigma(t))g^\Delta(t)]^\Delta$$

$$= f^{\Delta\Delta}(t)g(t) + f^\Delta\big(\sigma(t)\big)g^\Delta(t) + \left[f\big(\sigma(t)\big)\right]^\Delta g^\Delta(t) + f\Big(\sigma\big(\sigma(t)\big)\Big)\,g^{\Delta\Delta}(t)$$

2.3 Integration

In this section we give a brief introduction for integration on time scales. For this purpose we first give preliminary definitions. The readers who are interested in detailed information about time scale integration can see [1,5-11].

Definition 2.13 [6,7] A function $f: \mathbb{T} \to \mathbb{R}$ is called *regulated* if

(i) $\lim_{s \to t^+} f(s)$ exist at all right-dense points in $\mathbb{T}$.

(ii) $\lim_{s \to t^-} f(s)$ exist at all left-dense points in $\mathbb{T}$.

Definition 2.14 [6,7] A function $f: \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points in $\mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exist for all left-dense points in $\mathbb{T}$. The set of all rd-continuous function is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$ or simply C_{rd}.

Theorem 2.15 Assume $f: \mathbb{T} \to \mathbb{R}$.

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If f is regulated or rd-continuous, then so is f^σ.

(v) Assume f is continuous. If $g: \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Proof. [6]

Definition 2.16 [6,7] A continuous function $f: \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with region D, if $D \subset \mathbb{T}^k, \mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, then f is differentiable at each $t \in D$.

Let us illustrate the above with an example.

Example 2.17 For each of the following determine if f is regulated on $\mathbb{T}$, if f is rd-continuous on $\mathbb{T}$, and if f is pre-differentiable. If f is pre-differentiable, find its region of differentiability D.

(i) The function f is defined on $\mathbb{T}$ and $\forall t \in \mathbb{T}$ is isolated.

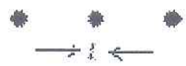

Figure 1.3 Isolated point

$\lim_{s \to t^-} f(s) = t$, $\lim_{s \to t^+} f(s) = t$ then any f defined on an isolated time scale is regulated.

Since right-dense points of $\mathbb{T}$ is $\emptyset$ and left-dense points of $\mathbb{T}$ is $\emptyset$, we can say that f is continuous on right-dense points of $\mathbb{T}$ and left-dense points of $\mathbb{T}$. Then *f* is rd-continuous.

(ii) Let $\mathbb{T} = \mathbb{R}$ and $f(t) = \begin{cases} 0; & \text{if } t = 0 \\ \frac{1}{t}; & t \in \mathbb{R} \setminus \{0\} \end{cases}$

$\lim_{s \to 0^+} f(s) = +\infty$, $\lim_{s \to 0^-} f(s) = -\infty$ limit does not exist then f is not regulated. Also f is not continuous at t=0 then f is not rd-continuous.

(iii) Let $\mathbb{T} = \mathbb{N}_0 \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ and $f(t) = \begin{cases} 0; & \text{if } t \in \mathbb{N} \\ t; & \text{otherwise} \end{cases}$

$\mathbb{T} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$ and right dense points of $\mathbb{T}$ is $\emptyset$, also left dense points of $\mathbb{T}$ is $\emptyset$. Then f is regulated. $\lim_{s \to t^-} f(s)$ exist for all points then f is rd-continuous.

Therorem 2.18 (Mean Value Theorem) Let $f, g: \mathbb{T} \to \mathbb{R}$ be both pre-differentiable on D. Then

$$|f^{\Delta}(t)| \leq g^{\Delta}(t) \qquad for\ all\ t \in D$$

implies

$$|f(s) - f(r)| \leq g(s) - g(r) \quad for\ all\ r, s \in \mathbb{T}, r \leq s.$$

Proof. [6]

Definition 2.19 [6,7] Assume that $f: \mathbb{T} \to \mathbb{R}$ is a regulated function. A function $F: \mathbb{T} \to \mathbb{R}$ with a region of pre-differentiation D such that $F^{\Delta} = f$ on D is called a *pre-antiderivative* of f on $\mathbb{T}$.

Definition 2.20 [6,7] Assume that $f: \mathbb{T} \to \mathbb{R}$ is a regulated function and F is a pre-antiderivative of f on $\mathbb{T}$. We define the *indefinite integral* of a regulated function f by

$$\int f(t)\Delta(t) = F(t) + c,$$

where $c \in \mathbb{R}$ is an arbitrary constant. We define the *Cauchy integral* by

$$\int_r^s f(t)\Delta(t) = F(s) - F(r) \quad for\ all\ r, s \in \mathbb{T}.$$

Definition 2.21 [6,7] Assume that $f: \mathbb{T} \to \mathbb{R}$ is a regulated function. A function $F: \mathbb{T} \to \mathbb{R}$ satisfying $F^{\Delta} = f$ on $\mathbb{T}^k$ is called an *antiderivative* of f on $\mathbb{T}$.

Example 2.22 The indefinite integral $\int 3^t\, \Delta t$ on $\mathbb{Z}$, can be found by means of anti-derivatives.

Let $f(t) = 3^t$. Then

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t) = 3^{t+1} - 3^t = 2.3^t$$

Hence $f(t) = \frac{f^{\Delta}(t)}{2}$. Then we get

$$\int f(t)\Delta t = \int \frac{(3^t)^{\Delta}}{2} = \frac{1}{2}\int (3^t)^{\Delta}\Delta t = \frac{3^t}{2} + c\,,$$

where c is an arbitrary constant.

Theorem 2.23 (Existence of Antiderivatives) Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \qquad for\, t \in \mathbb{T}$$

is an antiderivative of f.

Proof. [6,7].

Theorem 2.24 Assume that $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then we have the followings:

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t\,;$

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta(t)\,;$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t\,;$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t\,;$

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t\,;$

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t\,;$

(vii) $if\ |f(t)| \leq g(t)\ on\ [a,b), then\ \left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t\,.$

The formulas (v) and (vi) are known as integration by parts formulas on time scales.

Proof. [6,7]

The following result is a direct consequence of mean value theorem (Theorem 2.18).

Theorem 2.25 If $f^{\Delta} \geq 0$, then f is increasing.

Proof. [6,7] □

The following result is very useful tool for evaluation of definite integrals on time scales.

Theorem 2.26 Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}^k$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = (\sigma(t) - t)f(t)$$

$$= \mu(t)f(t)$$

where *f(t)* is lower bound.

Proof. [6]

Theorem 2.27 Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i) $If\ \mathbb{T} = \mathbb{R}, then\ \int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the right side of integral is *Riemann integral*.

(ii) $If\ [a, b]$ consist of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & if\ a < b \\ 0 & if\ a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & if\ a > b. \end{cases}$$

(iii) $If\ \mathbb{T} = h\mathbb{Z} = \{hk{:}\, k \in \mathbb{Z}\}, where\ h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & if a < b \\ 0 & if\ a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & if\ a > b. \end{cases}$$

(iv) $If\ \mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & if\ a < b \\ 0 & if\ a = b \\ -\sum_{t=b}^{a-1} f(t) & if\ a > b. \end{cases}$$

Proof. (ii) Let $[a,b] = \{t_1, t_2, \dots, t_n\}$, then

$$\int_a^b f(t)\Delta t = \int_{t_1}^{t_2} + \int_{t_2}^{t_3} + \cdots + \int_{t_{n-1}}^{t_n} \Delta t$$

$$= f(t_1)(t_2 - t_1) + f(t_2)(t_3 - t_2) + \cdots + f(t_{n-1})(t_n - t_{n-1})$$

$$= \sum_{i=1}^{n} f(t_i)(t_{i+1} - t_i)$$

$$= \sum_{t\in[a,b)} \mu(t)f(t).$$

(iii) Let a= hk and $k = \frac{a}{h}\ where\ k \in \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \int_{hk}^{h(k+1)} + \int_{k(k+1)}^{h(k+2)} + \cdots + \int_{h(k+n-1)}^{k(k+n)} .$$

$$= hf(hk) + hf(hk + h) + hf(hk + 2h) + \cdots + hf(hk + (n-1)h)$$

$$= h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(hk)$$

□

Example 2.28 We can evaluate $\int_0^t s\Delta s$ for $t \in \mathbb{T}, for\ \mathbb{T} = \mathbb{R},\ for\ \mathbb{T} = \mathbb{Z}$, and for $\mathbb{T} = [0,1] \cup [2,3]$.

i. $\int_0^t s\Delta s = (t-0)f(0) = t.0 = 0$ (by Theorem 2.26)

ii. $\mathbb{T}=\mathbb{R} \Rightarrow \int_0^t s\Delta s = \int_0^t sds = \frac{t^2}{2}$

iii. $\mathbb{T}=\mathbb{Z} \Rightarrow \int_0^t s\Delta s = \sum_{k=0}^{t-1} k = \frac{(t-1)t}{2}$

iv. $\mathbb{T} = [0,1] \cup [2,3]$

- $t = 0 \Rightarrow \int_0^t s\Delta s = \int_0^0 s\Delta s = 0$
- $0 < t \leq 1 \Rightarrow \int_0^t s\Delta s = \int_0^t sds = \frac{t^2}{2}$
- $t = 2 \Rightarrow \int_0^t s\Delta s = \int_0^1 sds + \int_1^2 s\Delta s = \frac{1}{2} + (2-1)1 = \frac{3}{2}$
- $2 < t \leq 3 \Rightarrow \int_0^t s\Delta s = \int_0^1 sds + \int_1^2 s\Delta s + \int_2^t sds = \frac{t^2}{2} - \frac{1}{2}$

Theorem 2.29 (Substitution) Assume $a, b \in \mathbb{T}, g \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is strictly increasing and $\widetilde{\mathbb{T}} := g(\mathbb{T})$ is a time scale, and $f \in C_{rd}(\widetilde{\mathbb{T}}, \mathbb{R})$. Then

$$\int_a^b (f \circ g)(t)\, g^{\Delta}(t)\Delta t = \int_{g(a)}^{g(b)} f(s)\tilde{\Delta} s\,.$$

Proof. [6]

2.4 Exponential Function on Time Scales

In this subsection, we give a brief information about the generalized Δ-exponential functions which we use in the rest of the thesis. In order to define the generalized Δ-exponential function in the following definitions.

Definition 2.30 [6,7] For $h > 0$, the Hilger's complex numbers is defined by

$$\mathbb{C}_h = \left\{z \in \mathbb{C}: z \neq -\frac{1}{h}\right\}.$$

For $h = 0$, we set $\mathbb{C}_0 = \mathbb{C}$.

Definition 2.31 [6,7] The *cylinder transformation* $\xi_h: \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} Log(1 + hz)$$

where

$$\mathbb{Z}_h = \left\{z \in \mathbb{C}: -\frac{\pi}{h} \leq Im(z) \leq \frac{\pi}{h}\right\}$$

For $h > 0$ and $\mathbb{Z}_0 = \mathbb{C}$. If $h = 0$, then we set $\xi_0(z) = z \; for \; z \in \mathbb{C}$.

Definition 2.32 [6,7] A function $p: \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided that

$$1 + \mu(t)p(t) \neq 0, t \in \mathbb{T}^k.$$

The set of all rd-continuous and regressive functions is denoted by $\mathcal{R}$.

Definition 2.33 [6,7] If $p \in \mathcal{R}$, then the first order linear dynamic equation

$$y^{\Delta}(t) = p(t)y(t)$$

is said to be regressive dynamic equation.

By the above definitions we are going to define the generalized Δ-exponential function.

Definition 2.34 [6,7] If $p \in \mathcal{R}$, then for all $s, t \in \mathbb{T}$ the *generalized Δ-exponential function* is defined by

$$e_p(t,s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right).$$

Theorem 2.35 If $t_0 \in \mathbb{T}$ and $y^{\Delta}(t) = p(t)y(t)$ is a regressive dynamic equation, then $e_p(.,t_0)$ is the unique solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(t_0) = 1.$$

Proof. [6] □

For $p, q \in \mathcal{R}$, let the binary operations $\oplus$ and $\ominus$ be defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),$$

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)},$$

$$(p \ominus q)(t) = (p \oplus (\ominus q))(t).$$

The generealized Δ-exponential function satisfies the following properties:

1. $e_0(t,s) = 1$ and $e_p(t,t) = 1$,
2. $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$,
3. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s),\ e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,
4. $e_p(t,s)e_p(s,r) = e_p(t,r),\ e_p(t,s)e_q(t,s) = e_{(p\oplus q)}(t,s)$,
5. $\frac{e_p(t,s)}{e_q(t,s)} = e_{(p\ominus q)}(t,s)$,

6. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Example 2.36 We are giving some examples about the generalized Δ-exponential function for different time scales.

- $\mathbb{T} = h\mathbb{Z}\ for\ h > 0$. If $\alpha \in \mathcal{R}$ is a constant, then

$$e_{\alpha}(t,0) = (1+\alpha h)^{t/h},$$

- $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. In this case,

$$e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})!\,.$$

More detaled infirmation on the exponential function on time scales can be found in [1,6,7] and references therein.

3 DYNAMIC INEQUALITIES

3.1 Gronwall's Inequality

Gronwall inequality which is given below is one of the most useful inequality that is used in the theory of differential equations [15].

Theorem 3.1 Let β and u are real valued continuos defined on I.

$u'(t) \leq \beta(t)u(t), t \in I^0$ then $u(t) \leq u(a)exp \int_a^t \beta(s)ds$.

In order to adapt the Gronwall's inequality to time scales, we need to give the following Comparison result:

Theorem 3.2 (Comparison Theorem) Let $y, f \in C_{rd}$ and $p \in \mathcal{R}^+$. ($\mathcal{R}^+$: regressive and C_{rd} functions). Then

$$y^\Delta(t) \leq p(t)y(t) + f(t), \forall t \in \mathbb{T}$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau .$$

Proof . By using the product rule for Δ-derivatives(Theorem 2.10), we obtain

$$[ye_{\ominus p}(., t_0]^\Delta(t) = y^\Delta(t)e_{\ominus p}(\sigma(\tau), t_0) + y(t)[e_{\ominus p}(t, t_0)]^\Delta$$

$$= y^\Delta(t)e_{\ominus p}(\sigma(t), t_0) + y(t)(\ominus p)(t)e_{\ominus p}(t, t_0)$$

$$= y^\Delta(t)e_{\ominus p}(\sigma(t), t_0) + y(t)(\ominus p)(t)\frac{e_{\ominus p}(\sigma(t), t_0)}{1 + \mu(t)(\ominus p)(t)}$$

$$= e_{\ominus p}(\sigma(t), t_0)\left[y^{\Delta}(t) + y(t)\frac{(\ominus p)(t)}{1 + \mu(t)(\ominus p)(t)}\right]$$

$$= e_{\ominus p}(\sigma(t), t_0)[y^{\Delta}(t) + y(t) \ominus (\ominus p(t))].$$

Integrating both side of the equation leads us

$$y(t)e_{\ominus p}(t, t_0) - y(t_0) = \int_{t_0}^{t} [y^{\Delta}(\tau) - y(\tau)p(\tau)]e_{\ominus p}(\sigma(\tau), t_0)\Delta\tau$$

$$\leq \int_{t_0}^{t} f(\tau)e_{\ominus p}(\sigma(\tau), t_0)\Delta\tau$$

$$= \int_{t_0}^{t} e_p(t_0, \sigma(\tau))f(\tau)\Delta\tau$$

Therefore

$$y(t)e_{\ominus p}(t, t_0) = y(t_0) + \int_{t_0}^{t} e_p(t_0, \sigma(t))f(\tau)\Delta\tau.$$

Multiplying both sides by $e_p(t, t_0)$ finishes the proof. □

The following two theorems are consequences of comparison theorem (Theorem 3.2).

Theorem 3.3 (Bernoulli Inequality) Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathbb{R}^+$. Then for all $t \geq s$;

$$e_\alpha(t, s) \geq 1 + \alpha(t - s).$$

Proof. Since $\alpha \in \mathcal{R}^+$, we have $e_\alpha(t, s) > 0$ *for all* $t, s \in \mathbb{T}$. Let $t \geq s$ and $y(t) = \alpha(t - s)$. Then

$$\alpha y(t) + \alpha = \alpha^2(t-s) + \alpha \geq \alpha = y^\Delta(t)$$

Since y(s) = 0, we have by Theorem 2.2

$$y(t) \leq \int_s^t e_p(t, \sigma(t))\alpha\Delta\tau = -1 + e_\alpha(t,s)$$

So that $e_\alpha(t,s) \geq 1 + \alpha(t-s)$ follows.

□

Gronwall's inequality on time scales is as follows:

Theorem 3.4 (Gronwall's Inequality on Time Scales) Let $y, f \in C_{rd}(\mathbb{T}, \mathbb{R}), p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R}), p \geq 0$. Then

$$y(t) \leq f(t) + \int_{t_0}^t y(\tau)p(\tau)\Delta\tau, \qquad \forall t \in \mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau, \qquad \forall t \in \mathbb{T}.$$

Proof. Define $z(t) = \int_{t_0}^t y(\tau)p(\tau)\Delta\tau$. Then

$$z^\Delta(t) = y(t)p(t) \leq [f(t) + z(t)]p(t)$$

$$z^\Delta(t) \leq p(t)z(t) + p(t)f(t)$$

Then by Comparison theorem (Theorem 3.2), we obtain

$$z(t) \leq z(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)p(\tau)\Delta\tau$$

$$z(t) \leq \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)p(\tau)\Delta\tau$$

Since $y(t) \leq f(t) + z(t)$

$$\leq f(t) + \int_{t_0}^{t} e_p\left(t,\sigma(t)\right)f(\tau)p(\tau)\Delta\tau.$$

□

Example 3.5 Let $\mathbb{T} = h\mathbb{Z} \cap [0,\infty)$. If y and f are functions defined on $\mathbb{T}$ and $\gamma > 0$ is constant such that

$$y(t) \leq f(t) + \gamma \sum_{\tau=0}^{\frac{t}{h}-1} y(\tau h), \qquad \forall t \in \mathbb{T}, then$$

$$y(t) \leq f(t) + \gamma \sum_{\tau=0}^{\frac{t}{h}-1} f(\tau h)(1+\gamma h)^{\frac{t-h(\tau+1)}{h}} \quad \forall t \in \mathbb{T}.$$

$$\mathbb{T} = \{0, h, 2h, 3h \ldots\} = \{hk : k \in \mathbb{N}_0\}$$

From previous theorem we have $y(t) \leq f(t) + z(t)$ then $z(t) = \gamma \sum_{\tau=0}^{\frac{t}{h}-1} y(\tau h)$

$$z(t) = \gamma \int_{t_0}^{t} y(t) \Rightarrow z^{\Delta}(t) = \gamma\, y(t) \leq \gamma[f(t) + z(t)]$$

$$y(t) \leq f(t) + \gamma \int_{t_0}^{t} y(t)$$

Corollary 3.6 Let $y \in C_{rd}(\mathbb{T}, \mathbb{R}), p \in \mathbb{R}^+, p \geq 0$ and $\alpha \in \mathbb{R}$. Then $y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau, \forall t \in \mathbb{T}$ implies $y(t) \leq \alpha e_p(t, t_0), \forall t \in \mathbb{T}$.

Proof. In basic Gronwall's inequality theorem let f(t)=α . Then by this theorem;

$$y(t) \leq \alpha + \int_{t_0}^{t} e_p\left(t, \sigma(t)\right)\alpha p(\tau)\Delta\tau$$

$$= \alpha\left[1 + \int_{t_0}^{t} e_p\left(t, \sigma(t)\right)p(\tau)\Delta\tau\right]$$

$$= \alpha\left[1 + (e_p(t, t_0) - e_p(t, t)\right]$$

$$= \alpha\left[1 + e_p(t, t_0) - 1\right]$$

$$y(t) \leq \alpha e_p(t, t_0)$$

□

Corollary 3.7 Let $y \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$. Then

$y(t) \leq \alpha + \beta(t - t_0) + \gamma \int_{t_0}^{t} y(\tau)\Delta\tau, \ \ \forall t \in \mathbb{T}$ implies

$y(t) \leq \left(\alpha + \frac{\beta}{\gamma}\right) e_\gamma(t, t_0) - \frac{\beta}{\gamma}, \forall t \in \mathbb{T}.$

Proof. In Gronwall's inequality theorem, let $f(t) = \alpha + \beta(t - t_0)$ and $p(\tau) = \gamma$. Then note that $w(t) = e_\gamma(t, t_0)$ we have $w^\Delta(t) = -\gamma e_\gamma(t, \sigma(\tau))$ by using exponential function rule.

By using these facts,

$$y(t) \leq f(t) + \int_{t_0}^{t} e_\gamma\left(t, \sigma(\tau)\right)\gamma f(\tau)\Delta\tau$$

$$= f(t)w(t) - \int_{t_0}^{t} w^\Delta(t)f(\tau)\Delta\tau$$

$$= f(t_0)w(t_0) + \int_{t_0}^{t} w\left(\sigma(\tau)\right)f^\Delta(\tau)\Delta\tau$$

$$= \alpha e_\gamma(t, t_0) + \int_{t_0}^{t} e_\gamma(t, \sigma(\tau))\beta\Delta\tau$$

$$= \alpha e_\gamma(t, t_0) + \frac{\beta}{\gamma}\int_{t_0}^{t} \gamma e_\gamma(t, \sigma(\tau))\Delta\tau$$

$$= \alpha e_\gamma(t, t_0) + \frac{\beta}{\gamma}\left[-e_\gamma(t, t) + e_\gamma(t, t_0)\right]$$

$$= \alpha e_\gamma(t, t_0) + \frac{\beta}{\gamma}(e_\gamma(t, t_0) - 1)$$

$$y(t) \leq e_\gamma(t, t_0)\left[\alpha + \frac{\beta}{\gamma}\right] - \frac{\beta}{\gamma}$$

□

Theorem 3.8 Let g: $\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ be a function with $g(t,x_1)\le g(t,x_2)\ \forall t\in\mathbb{T}$ whenever $x_1\le x_2$. Let v,w: $\mathbb{T}\to\mathbb{R}$ be differentiable with $v^{\Delta}(t)\le g\big(t,v(t)\big)$ and $w^{\Delta}(t)\ge g\big(t,w(t)\big)\ \forall t\in\mathbb{T}^k\setminus\{t_0\}$ where $t_0\in\mathbb{T}$. Then $v(t_0)<w(t_0)$ implies $v(t)<w(t),\forall t\ge t_0$.

Proof. [6]

3.2 Hölder's and Minkowski's Inequalities

The followings are one of the most famous inequalities of mathematical analysis.

Theorem 3.9 (Hölder's Inequality) Let $f,g:[a,b]\to\mathbb{R}$ where $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$. Then

$$\int_a^b |f(x)g(x)|dx \le \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}}\left[\int_a^b |g(x)|^q dx\right]^{\frac{1}{q}}$$

Theorem 3.10 (Hölder's Inequality on Time Scales) Let a,b $\in\mathbb{T}$. For rd-continuous functions $f,g:[a,b]\to\mathbb{R}$ we have

$$\int_a^b |f(t)g(t)|\Delta t \le \left\{\int_a^b |f(t)|^p\Delta t\right\}^{\frac{1}{p}}\left\{\int_a^b |g(t)|^q\Delta t\right\}^{\frac{1}{q}}$$

where $p>1$ and $q=\frac{p}{p-1}$.

Proof. For nonnegative real numbers α and β, the basic inequality

$\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\le\frac{\alpha}{p}+\frac{\beta}{q}$ holds. Then integrate both side of this inequality between a and b:

$$\int_a^b \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\Delta t \le \frac{1}{p}\int_a^b \alpha(t)\,\Delta t + \frac{1}{q}\int_a^b \beta(t)\,\Delta t$$

Apply

$$\alpha(t) = \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \Delta\tau}$$

$$\beta(t) = \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \Delta\tau}$$

then we have

$$\int_a^b \frac{f(t)}{\left(\int_a^b |f(\tau)|^p \Delta\tau\right)^{\frac{1}{p}}}\frac{g(t)}{\left(\int_a^b |g(\tau)|^q \Delta\tau\right)^{\frac{1}{q}}}\Delta t \le \frac{1}{p}\int_a^b \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \Delta\tau}\Delta t + \frac{1}{q}\int_a^b \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \Delta\tau}\Delta t$$

$$\frac{1}{\left(\int_a^b |f(\tau)|^p \Delta\tau\right)^{\frac{1}{p}}\left(\int_a^b |g(\tau)|^q\right)^{\frac{1}{q}}}\int_a^b f(t)g(t)\Delta t$$

$$\le \frac{1}{p}\int_a^b (|f(t)|^p \Delta t)^{1-\frac{1}{q}} + \frac{1}{q}\int_a^b (|g(t)|^q \Delta t)^{1-\frac{1}{p}}$$

$$= \frac{1}{p}\int_a^b (|f(t)|^p \Delta t)^{\frac{1}{p}} + \frac{1}{q}\int_a^b (|g(t)|^q \Delta t)^{\frac{1}{q}}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

□

The special case p= q =2 leads as to have the Cauchy Schwarz Inequality:

Suppose that f and g are continuous on $[a, b]$. Then

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt$$

The time scale version of Cauchy Scahwarz inequality is as follows:

Theorem 3.11 (Cauchy Schwarz Inequality) Let $a, b \in \mathbb{T}$. For rd-continuous $f, g: [a, b] \to \mathbb{R}$ we have

$$\int_a^b |f(t)g(t)|\Delta t \leq \sqrt{\int_a^b |f(t)|^2 \Delta t \int_a^b |g(t)|^2 \Delta t}$$

Theorem 3.12 (Minkowski's Inequality) Let $f, g: [a, b] \to \mathbb{R}\, with\, p > 1$ then Minkowski's inequality for integrals states that

$$\left[\int_a^b |f(x) + g(x)|^p dx\right]^{\frac{1}{p}} \leq \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^p dx\right]^{\frac{1}{p}}$$

The time scale version of Minkowski inequality is as follows:

Theorem 3.13 (Minkowski's Inequality on Time scales) Let $a, b \in \mathbb{T}$ and $p > 1$. For rd-continuous $f, g: [a, b] \to \mathbb{R}$ we have

$$\left(\int_a^b |(f+g)(t)|^p \Delta t\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p \Delta t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \Delta t\right)^{\frac{1}{p}}$$

Proof. Apply Hölder's inequality with $q = \frac{p}{p-1}$.

$$\int_a^b |(f+g)(t)|^p \Delta t = \int_a^b |(f+g)(t)|^{p-1} |(f+g)(t)| \Delta t$$

$$\leq \int_a^b |f(t)||(f+g)(t)|^{p-1} \Delta t + \int_a^b |g(t)||(f+g)(t)|^{p-1} \Delta t$$

$$\leq \left(\int_a^b |f(t)|^p \, \Delta t \right)^{\frac{1}{p}} \left(\int_a^b |(f+g)(t)|^{(p-1)q} \Delta t \right)^{\frac{1}{q}}$$

$$+ \left(\int_a^b |g(t)|^p \Delta t \right)^{\frac{1}{p}} \left(\int_a^b |(f+g)(t)|^{(p-1)q} \Delta t \right)^{\frac{1}{q}}$$

$$= \left(\int_a^b |(f+g)(t)|^p \Delta t \right)^{\frac{1}{q}} \left[\left(\int_a^b |f(t)|^p \, \Delta t \right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \Delta t \right)^{\frac{1}{p}} \right]$$

Dividing both side of the equation with $\left(\int_a^b |(f+g)(t)|^p \Delta t \right)^{\frac{1}{q}}$. Then we have

$$\left(\int_a^b |(f+g)(t)|^p \Delta t \right)^{1-\frac{1}{q}} \leq \left(\int_a^b |f(t)|^p \Delta t \right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \Delta t \right)^{\frac{1}{p}}$$

where $1 - \frac{1}{q} = p$.

□

3.3 Jensen's Inequality

Definition 3.14 (Convex Set) Let X be a vector space and C⊂X, C is said to be a convex set if

$$tx + (1 - t)y \in C , \forall x, y \in C \quad , t \in [0,1]$$

Definition 3.15 (Convex Function) Let *f*: $X \to X$ be a function. *f* is called convex if $\forall x_1, x_2 \in X, \forall t \in [0,1]$

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2).$$

Theorem 3.16 Suppose γ is convex function on real line and g is an integrable real valued function we have

$$\gamma\left(\int_a^b g(x)dx\right) \leq \int_a^b \gamma(g(x))dx$$

Theorem 3.17 (Jensen's Inequality) Let $a, b \in \mathbb{T}$ *and* $c, d \in \mathbb{R}$. If $g: [a, b] \to (c, d)$ is rd-continuous and $F: (c, d) \to \mathbb{R}$ is continuous and convex, then

$$F\left(\frac{\int_a^b g(t)\Delta t}{b - a}\right) \leq \frac{\int_a^b F(g(t))\Delta t}{b - a}$$

Proof. Let $x_0 \in (c, d)$. Then there exists $\beta \in \mathbb{R}$ with

$$F(x) - F(x_0) \geq \beta(x - x_0) , \forall x \in (c, d).$$

Since $g \in C_{rd}$, $x_0 = \frac{\int_a^b g(\tau)\Delta\tau}{b-a}$ is well-defined. Apply Bernoulli inequality with $x = g(t)$

$$F(g(t)) - F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b - a}\right) \geq \beta(g(t) - x_0)$$

And integrate from a to b to obtain:

$$\int_a^b [F(g(t))\Delta t] - (b-a)F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b-a}\right) = \int_a^b [F(g(t)) - F(x_0)]\,\Delta t$$

$$\geq \beta \int_a^b [g(t) - x_0]\,\Delta t$$

$$= \beta\left[\int_a^b g(t)\Delta t - x_0(b-a)\right]$$

Dividing both side with *(b-a)*

$$\frac{\int_a^b F(g(t))}{b-a} \geq \frac{(b-a)F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b-a}\right)}{b-a}$$

$$\frac{\int_a^b F(g(t))}{b-a} \geq F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b-a}\right)$$

□

Example 2.18 Let $\mathbb{T}=\mathbb{R}$. $F(x) = -logx$ is convex and continuous on $(0,\infty)$, so we apply Jensen's inequality with a=0 and b=1 to obtain

$$\log \int_0^1 g(t)\,dt \geq \int_0^1 \log g(t) dt.$$

From Jensen's inequality,

$$F\left(\frac{\int_a^b g(t)dt}{b-a}\right) \leq \frac{\int_a^b F(g(t))dt}{b-a}$$

$$-\log\left(\frac{\int_0^1 g(t)dt}{1-0}\right) \leq \int_0^1 -\log(g(t))dt$$

$$\log(\int_0^1 g(t)dt) \geq \int_0^1 \log\big(g(t)\big)dt.$$

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