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MASTER THESIS

COUNTEREXAMPLES IN ANALYSIS

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We certify that, as the jury, we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

COUNTEREXAMPLES IN ANALYSIS

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It is very important to make argument in mathematics and to understand it. Generally, we get help from the theorems proving another. However in some special cases we utilize counter examples. Counterexamples are used to indicate a predicate or a theory is wrong. We also say counterexamples are a kind of opponent proof. Anything that is meant to be described cannot be more illustrative than a good example. For this reason, we make use of counterexamples in mathematics. The aim of this thesis is to make some proofs with the help of counterexamples.

Key Words: counterexample, differentiation, functions and limit, sequences and series, proof of counterexamples

ÖZ

TÜRKÇE BAŞLIK

ANALIZDE TERS ÖRNEKLER

Karagözoğlu, Ecem
Yüksek Lisans, Uygulamalı Matematik
Danışman: Yrd.Doç. Dr. Ahmet YANTIR
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Matematikte ispat yapmak ve ispatı anlamak çok önemlidir. Genellikle, ispat yaparken teoremlerden yararlanırız. Fakat bazı özel durumlarda karşıt örnekler ya da örneklemeler kullanırız. Karşıt örnekler bir önermenin ya da bir teorinin yanlış olduğunu göstermek için kullanılır. Karşıt örnekler için bir tür karşıt ispatta diyebiliriz. Anlatılan ya da anlatılmak istenen herhangi bir şey iyi bir örnekten daha iyi olamaz. Bu sebeple, matematikte karşıt örnekler kullanarak yardım alırız. Bu tezin amacı, karşıt örnekler yardımıyla bazı ispatlar yapmaktır.

Anahtar Kelimeler: ters örnekler, türev, fonksiyon ve limit, diziler ve seriler, ters örnekler ispatı

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Ecem Karagözoğlu
İzmir, 2017

TEXT OF OATH

I declare and honestly confirm that my study, titled "COUNTEREXAMPLES IN ANALYSIS" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions. I declare, to the best of my knowledge and belief, that all content and ideas drawn directly or indirectly from external sources are indicated in the text and listed in the list of references.

Ecem Karagözoğlu

Signature

………………………………………

July 21, 2017

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SYMBOLS AND ABBREVIATIONS

ABBREVIATIONS:

$C[a,b]$	The set of continuous functions defined on $[a,b]$
$UC[a,b]$	The set of uniformly continuous defined on $[a,b]$
$AC[a,b]$	The set of absolutely continuous defined on $[a,b]$
$mesA$	Measure of A
$B(I)$	The set of bounded functions defined on I.
$C^1(I)$	The set of continuously differentiable functions defined on I.
$BV[a,b]$	The set of functions having bounded variation property on $[a,b]$.

SYMBOLS:

$\mathbb{R}$	Real numbers.
$\mathbb{Q}$	Rational numbers.
$\mathbb{N}$	Natural numbers.

CHAPTER 1
INTRODUCTION

Counterexamples play an important role in mathematics. They show that given mathematical statement (hypothesis, conjecture, proposition, etc.) is not correct.

They are efficient tool for mathematicians. If a mathematician seek for counterexample before trying to prove a conjecture or hypothesis, this process will provide to save time an effort for her/him. Here are list of examples for this advantage in the history of very famous prime numbers theory:

- In history of mathematics, listing the prime numbers and finding a formula for primes atracted many mathematicians. For a natural numbers of the form

  $$2^{2^n} + 1$$

  were beleieved to be prime number for many years until a counterexample was found. For $n = 5$, the number

  $$2^{2^5} + 1 = 4294967297$$

  is a composite number and product of 641 and 6700417.

- Another famous conjecture about prime numbers is Goldbach or Goldbach-Euler conjecture and it is still waiting to be solved. The statement of the conjecture is very simple: Every even number greater than 2 is the sum of two prime numbers. For example, $12 = 5 + 7, 20 = 3 + 17,$ and so on. No counterexample have been found for this conjecture up to the number $4 \text{ x } 10^{14}$. It is a 1 million $ problem.

  Another advantage for counterexamples is to understand the importance of the hypothesis of the theorems. For example for the existence and uniqueness of differentiation equation of the form

  $$y' = f(x, y)$$

$$y(x_0) = y_0$$

if the conditions are satistified it is guarateed that there exist a unique solution for a spesific region. However it does not mean that the solution is not unique if one of the condition fails to hold.

The aim of this thesis is to provide some counterexamples with solutions in limits, continuity, differentiation, sequences and series in order to clarify these concepts.

In chapter 2, we will give brief information about the concepts mentioned above and we prepare the reader for the next chapter.

Chapter 3 gives the counterexamples with solutions.

Finally we conclude the results of the thesis by conclusion chapter.

CHAPTER 2
PRELIMINARIES

In this chapter, we will introduce some preliminary definitions and theorems that will be used throughout the thesis. Definitions and theorems and make explanations with examples. Each definition and theorem will help us in using and proving in future chapters. These basic informations can be found in many mathematics textbooks. We use the sources (Olmsted, 2003; J.Appell) for the informations. In this chapter we do not give all the details, only definitions and required theorems are stated. For more information please see (Olmsted, 2003)

2.1. DEFINITIONS AND IMPORTANT THEOREMS

2.1.1. Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

$$\lim_{x \to x_0} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$$

Example

$$\lim_{x \to x_0} x = x_0$$

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$, implies $|x - x_0| < \varepsilon$. The implication will hold if δ equals ε or any smaller positive number. This proves that $\lim_{x \to x_0} x = x_0$.

2.1.2. The Sandwich Theorem

Sandwich theorem (or squeeze theorem) is used to find the limits of some complicated functions.

Let the function $f(x)$ be bounded by $g(x)$ and $h(x)$; i.e, $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Also suppose that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Example

Given that

$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$,

The limit $\lim_{x \to 0} u(x) = 1$, since $\lim_{x \to 0} 1 - \frac{x^2}{4} = \lim_{x \to 0} 1 + \frac{x^2}{2} = 1$, no matter how complicated u is.

2.1.3. Upper and Lower Limit of a Sequence

Let the least term h of a sequence be a term which is smaller than all but a finite number of the terms which are equal to h. Then h is called the lower limit of the sequence.

Let the greatest term H of a sequence be a term which is greater than all but a finite number of the terms which are equal to H. Then H is called the upper limit of the sequence.

The upper and lower limit of a sequence of real numbers $\{x_n\}$ (called also lim superior and lim inferior) can be defined in several ways and are denoted, respectively as

$\lim_{n\to\infty} sup\ x_n \qquad \lim_{n\to\infty} inf\ \ x_n$.

Example

If $x_n = (-1)^n$, then

$$\lim\inf x_n = -1 \quad and \quad \lim\sup x_n = 1.$$

If $x_n = n + (-1)^n$, then

$$\lim\inf x_n = 0 \quad and \quad \lim\sup x_n = \infty.$$

2.1.4. Continuity

- A function $f(x)$ is said to be continuous at $x = x_0$ if for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < 0$ implies $|f(x) - f(x_0)| < \varepsilon$.
- A function is said to be continuous on an interval I, if it is continuous at every point of the I. The set of continuous functions on $[a, b]$ is denoted by $C[a, b]$
- A function is said to be continuous function if it is continuous at each point of its domain.

A continuous function need not to be continuous on every interval.

For example, $y = {}^1\!/_x$ is not continuous , but it is continuous over its domain $(-\infty, 0) \cup (0, \infty)$.

2.1.5. The Intermediate Value Theorem

Let a, b be real numbers with $a < b$, and let f be a continuous function from $[a, b]$ to $\mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$. Then there is some number $c \in (a, b)$ such that $f(c) = 0$.

Corollary

Let f be a continuous function from some interval $[a, b]$ to $\mathbb{R}$, such that $f(a)$ and $f(b)$ have opposite signs. Then there is some number c between and b such that $f(c) = 0$.

2.1.6. Absolutely Continuity

A function $f: [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$$

for any finite collection $\{ [a_i, b_i] : 1 \leq i \leq N\}$ of non-overlapping subintervals $[a_i, b_i]$ of $[a, b]$ with

$$\sum_{i=1}^{N} |b_i - a_i| < \delta.$$

The set of absolutely continuous functions on $[a, b]$ is denoted by $AC[a, b]$.

On a compact set K of $\mathbb{R}$,

$$AC(K) \subseteq UC(K) = C(K)$$

where $UC(K)$ denotes the set of uniformly continuous functions on K.

Example

$$f(x) = \begin{cases} 0 & , \ x = 0 \\ x \sin x (1/x) & , \ x \neq 0 \end{cases}$$, continuous but not absolutely continuous.

2.1.7. The Derivative

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

Example

Let us use this definition to differentiate $f(x) = \frac{x}{x-1}$

Here we have $f(x) = \frac{x}{x-1}$ and $f(x+h) = \frac{x+h}{(x+h)-1}$,

so $f'(x) = \lim_{h\to 0}\frac{(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h} = \lim_{h\to 0}\frac{1}{h}.\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}$

$= \lim_{h\to 0}\frac{1}{h}.\frac{-h}{(x+h-1)(x-1)} = \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$.

2.1.8. Differentiable Function

A differentiable function of one variable is a function whose derivative exists at each point of its domain.

If $f:(a,b)\to R$ is differentiable at $c\in(a,b)$, then f is continuous at c.

Example

$f(x) = |x|$ continuous but not differentiable at $x = 0$.

2.1.9. Periodic Function

A function $f(x)$ is said to be periodic (or, when emphasizing the presence of a single period instead of multiple periods, singly periodic) with period p if

$$f(x) = f(x+np)$$

For $n = 1,2,\ldots$ For example, the sine function $\sin x$, illustrated above, is periodic with least period (often simply called the "period") 2π (as well as with period $-2\pi, 4\pi, 6\pi$, etc.).

2.1.10. Bounded Variation

(a) The function $\alpha: [a, b] \to R$ is said to be of bounded variation on $[a, b]$ if and only if there is a constant $M > 0$ such that

$$\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \leq M$$

for all partitions $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$.

(b) If $\alpha: [a, b] \to R$ is of bounded variation on $[a, b]$, then the total variation of α on $[a, b]$ is defined by

$V_\alpha\ (a, b) = \sup\{\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})||, P =$
$\{x_0, x_1, \dots, x_n\}$ is a part of $[a, b]\}$.

The set of functions obeying the rules of bounded variation is denoted by $BV[a, b]$.

Example

If $\alpha: [a, b] \to R$ is monotonically increasing, then for any partition $P =$
$\{x_0, x_1, \dots, x_n\}$ of $[a, b]$.

$$\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^{n} [\alpha(x_i) - \alpha(x_{i-1})] = \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a)$$

Thus α is of bounded variation and $V_f\ (a, b) = \alpha(b) - \alpha(a)$.

2.1.11. Luzin-n-Property

A function f continuous on an interval $[a, b]$.

For any set $E \subset [a, b]$ of measure mesE =0 , the image of this set, $f(E)$, also has measure zero.

- A function $f \not\equiv$ constant on $[a, b]$ such that $f'(x) = 0$ almost-everywhere on $[a, b]$ does not have the Luzin-n-property.
- If f does not have the Luzin-n-property, then on $[a, b]$ there is a perfect set P of measure zero such that $mesf(P) > 0$.
- An absolutely continuous function has the Luzin-N-property.
- If f has the Luzin-N-property and has bounded variation on $[a, b]$ then f is absolutely continuous on $[a, b]$.

- If f does not decrease on $[a, b]$ and f' is finite on $[a, b]$, then f has the Luzin-N-property.
- In order that $f(E)$ be measurable for every measurable set $E \subset [a, b]$ it is necessary and sufficient that f have the Luzin-N-property on $[a, b]$.
- A function f that has the Luzin-N-property has a derivative f' on the set for which any non-empty portion of it has positive measure.
- For any perfect nowhere-dense set $P \subset [a, b]$ there is a function f having the Luzin-N-property on $[a, b]$ and such that f' does not exist at any point of P. The concept of Luzin's N-property can be generalized to functions of several variables and functions of a more general nature, defined on measure spaces.

2.1.12. Nowhere Dense Set

A nowhere dense set in a topological space is a set whose closure has empty interior. In a very loose sense, it is a set whose elements are not tightly clustered (as defined by the topology on the space) anywhere.

Example

$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is nowhere dense in $\mathbb{R}$: although the points get arbitrarily close to 0, the closure of the set is $S \cup \{0\}$, which has empty interior.

2.1.13. Hamel Basis

A basis for the real numbers $\mathbb{R}$, considered as a vector space over the rational $\mathbb{Q}$, i.e., a set of real numbers $\{U_\alpha\}$ such that every real number β has a unique representation of the form

$$\beta = \sum_{i=1}^{n} r_i\, U_{\alpha i}$$

where r_i is rational and n depends on β.

The axiom of choice is equivalent to the statement: "Every vector space has a vector space basis," and this is the only justification for the existence of a Hamel basis.

CHAPTER 3
COUNTEREXAMPLES IN ANALYSIS

In this chapter, we will see lots of counterexamples and we try to resolve them step by step. Especially, we analyze counterexamples in functions and limit, differentiation, series and sequences.

3.1. Functions and Limit

In this section we consider some illustrative counterexamples about limit and functions. As we know, limit and functions are so important in mathematics (especially in calculus). The main concepts of mathematical analysis is based on functions and limit.

3.1.1. A nowhere continuous function whose absolute value is everywhere continuous.

$$f(x) = \begin{cases} 1 & if\ x\ is\ rational \\ -1 & if\ x\ is\ irrational \end{cases}$$

Let $x_0 \in R$. Since the interval $|x - x_0| < \delta$ involves both rational and irrational functions $|f(x) - f(x_0)| < \varepsilon$ does not hold $\forall x$ satisfying $|x - x_0| < \delta$; if we choose $\varepsilon < 2$.

Note that $|f(x)| = 1$ which is constant; hence continuous everywhere.

3.1.2. A bounded function having no relative extrema on a compact domain.

Let the compact domain be the closed interval $[0,1]$ and for $\in [0,1]$, define

$$f(x) \equiv \begin{cases} \dfrac{(-1)^n n}{n+1} & if\ x\ is\ rational, x = {}^{m}/_{n}\ in\ lowest\ terms, n > 0 \\ 0 & if\ x\ is\ irrational. \end{cases}$$

$$|f(x)| < 1 \Rightarrow f\ is\ bounded.$$

Choose compact domain: $[0,1]$.

For n even and $(x\ rational) f(x) = \frac{n}{n+1} < 1$

For n is odd $(x\ rational)\ f(x) = \frac{-n}{n+1} > -1$

$1 < f(x) < -1$ and there is no x such that $f(x) = 1, f(x) = -1$.

No relative extreme value.

3.1.3. A nonconstant periodic function without a smallest positive period.

$$f(x) = \begin{cases} 1 & if\ x\ is\ rational \\ -1 & if\ x\ is\ irrational \end{cases}$$

The periods of any real-valued function with domain R form an additive group (that is, the set of periods is closed with respect to subtraction).

- f is not constant
- f is periodic since the graph of f repeats itself for every interval.

Smallest possible period ≡ the difference between two conseqtive rationals which does not exist.

3.1.4. A function continuous and one-to-one on an interval and whose inverse is not continuous.

Our example in this case is a complex-valued function $z = f(x)$ of the real variable x, with continuity defined exactly as in the case of a real-valued function of a real variable, where the absolute value of the complex number $z = (a, b)$ is defined

$$|z| = |(a, b)| \equiv (a^2 + b^2)^{1/2}$$

Let the function $z = f(x)$ be defined:

$z = f(x) \equiv (cosx, sinx), 0 \leqq x < 2\pi.$

$$f: R \longrightarrow C$$

$$x \longmapsto z = f(x) \in C$$

$$f(x) = cosx + isinx$$

$|z| = 1\ compact.$ A continuous function converts a compact range into a compact range.

$(0{,}2\pi)\ not\ compact\ \Longrightarrow\ f^{-1}\ is\ not\ continuous.$

3.1.5. A function continuous at every irrational point and discontinuous at every rational point.

Let $f(x)$ be defined as follows:

If x is a rational number equal to m/n, where m and n are integers such that the fraction m/n is in lowest terms and $n > 0$, let $f(x)$ be defined to be equal to $1/n$; otherwise, if x is irrational, let $f(x) \equiv 0$, i.e

$$f(x) = \begin{cases} 1/n \ , & x \ rational \\ 0 \ , & x \ irrational \end{cases}$$

Let $x_0 \in \mathbb{Q}$

$$|x - x_0| < \delta \ \Rightarrow \left| f(x) - \frac{1}{n} \right| < \varepsilon$$

$$\left| x - \frac{m}{n} \right| < \delta \ \Longrightarrow \ \left| f(x) - \frac{1}{n} \right| < \varepsilon$$

There is always an irrational on interval $x_0 - \delta < x < x_0 + \delta$

$$\Rightarrow \ \left| \frac{1}{n} \right| < \varepsilon \ \Rightarrow n < \frac{1}{\varepsilon} \ \Rightarrow \ \frac{m}{n} < \varepsilon$$

3.1.6. A discontinuous linear function.

A function f on R into R is said to be linear if and only if $f(ax + by) = af(x) + bf(y)$ for all $x, y, a, b \ \in R$. A function that is linear and not continuous is very complicated indeed.

Construction of a discontinuous linear function can be achieved by use of a Hamel basis for the linear space of the real numbers R over the rational numbers $\mathbb{Q}$. The idea is that this process provides a set $S = \{r_\alpha\}$ of real numbers r_α such that every real number x is a unique linear combination of a finite number of members of S with rational coefficients p_α: $x = p_{a1}r_{a1} + \cdots + p_{ak}r_{ak}$. The function f can now be defined:

$$f(x) \equiv p_{a1} + \cdots + p_{ak}$$

$$x = 1.(x-1) + 2.2 + 3.(-1)$$

$$f(x) = 1 + 2 + 3 = 6$$

<u>*f is linear*</u>

$$f(x+y) = f(\sum p_{\alpha i} \, r_{\alpha i} + \sum q_{xi} \, r_{\beta i})$$

$$= f(\sum p_{\alpha i} \, r_{\alpha i} + q_{xi} r_{xi})$$

$$= f(\sum (p_{\alpha i} + q_{xi}) \, r_{xi})$$

$$= \sum (p_{\alpha i} + q_{\alpha i}) = \sum p_{\alpha i} + \sum q_{\alpha i} = f(x) + f(y)$$

Since $f(x) \in \mathbb{Q}$, then it does not attain any irrational number between q_1 and q_2.

$\Rightarrow f$ does not satisfy intermediate value property.

$\Rightarrow f$ is not continuous.

3.1.7. Functions $y = f(u), u \in R, and\ u = g(x), x \in R$, whose composite function $y = f(g(x))$ is everywhere continuous, and such that

$$\lim_{u \to b} f(u) = c, \lim_{x \to a} g(x) = b, lim f\big(g(x)\big) \neq c.$$

If

$$f(u) = \begin{cases} 0 & , \ u \neq 0 \\ 1 & , \ u = 0 \end{cases}$$

$g(x) \equiv 0$ for all $x \in R$.

$$\lim_{x \to 0} g(x) = 0, \quad \lim_{u \to 0} f(u) = 0$$

But

$$\lim_{x \to 0} f\big(g(x)\big) = \lim_{x \to 0} f(0) = \lim_{x \to 0} 1 = 1.$$

This counterexample becomes impossible in case the following condition is added:

$$x \neq a \Rightarrow g(x) \neq b.$$

3.1.8. Let $I = [0, 1]$ and $\alpha, \beta \in \mathbb{R}$, and let $f_{\alpha,\beta}: I \to \mathbb{R}$ be defined by

$$f_{\alpha,\beta}(x) := \begin{cases} x^{\alpha} \sin x^{\beta} & for \quad 0 < x \leq 1 \\ 0 & for \quad x = 0 \end{cases}$$

$f_{\alpha,\beta} \in C(I)$ holds precisely for $\alpha > 0$ and arbitrary β, $or\ \alpha \leq 0\ and\ \beta > -\alpha$.

- $\alpha > 0,\ \beta\ arbitrary$

$\lim_{x \to 0} f_{\alpha,\beta} = 0$

Let $\alpha > 0, \beta \in \mathbb{R}$. $\quad -1 \leq \sin x^{\beta} \leq 1$

$$\Rightarrow\ -x^{\alpha} \leq x^{\alpha} \sin x^{\beta} \leq x^{\alpha}$$

Since $\lim_{x \to 0} x^{\alpha} = 0$; by sandwich theorem;

$$\lim_{x \to 0} x^{\alpha}. \sin x^{\beta} = 0$$

$f_{\alpha,\beta}(x)$ is continuous at $x = 0$.

$\Rightarrow f_{\alpha,\beta} \in C(I)$.

- $\alpha \leq 0\ ;\ \beta > -\alpha$

Is $f_{\alpha,\beta}$ continuous at 0 ? ($\lim_{x \to 0} f_{\alpha,\beta}(x) = 0$)

for $x \neq 0$

$f_{\alpha,\beta}(x) = x^{\alpha}. \sin x^{\beta} \qquad , \qquad \beta > -\alpha \ \Rightarrow\ \beta = |\alpha| + \varepsilon \ , \varepsilon > 0$

$$= \frac{\sin x^{\beta}}{x^{|\alpha|}}$$

$$= \frac{\sin x^{|\alpha|+\varepsilon}.x^{\varepsilon}}{x^{|\alpha|}.x^{\varepsilon}} = x^{\varepsilon}.\frac{\sin x^{\beta}}{x^{\beta}}$$

$$\lim_{x\to 0} f_{\alpha,\beta}\,(x) = \lim_{x\to 0} x^{\varepsilon}.\frac{\sin x^{\beta}}{x^{\beta}}$$

$$= \underbrace{\lim_{x\to 0} x^{\varepsilon}}_{=0} \,.\, \underbrace{\lim_{x\to 0} \frac{x^{\beta}}{x^{\beta}}}_{=1} = 0.$$

$\Rightarrow$ $f_{\alpha,\beta}$ is continuous at $x = 0$; $\Rightarrow f \in C(I)$.

3.2. DIFFERENTIATION

We present some counterexamples about differentiation in this section. Differentiation in this part. differentiation is all about finding rates of change of one quantity compared to another. We need differentiation when the rate of change is not constant.

3.2.1. A differentiable function with a discontinuous derivative.

The function

$$f(x) \equiv \begin{cases} \quad x^2 sin\dfrac{1}{x} & if \;\; x \neq 0, \\ 0 \quad\quad if \; x = 0 \end{cases}$$

has as its derivative the function

$$f'(x) = \begin{cases} 2x\sin {}^{1}\!/_{x} - \cos {}^{1}\!/_{x} & if \; x \neq 0 \\ 0 & if \; x = 0 \end{cases}$$

$\lim\limits_{x\to 0} f'\,(x) =?$

$$-1 < \sin\frac{1}{x} < 1 \;\Rightarrow\; -2x < 2x\sin\frac{1}{x} < 2x \Rightarrow \underbrace{\lim_{x\to 0} -2x}_{=0} < \underbrace{\lim_{x\to 0} \ldots}_{=0} < \underbrace{\lim_{x\to 0} 2x}_{=0}$$

$-1 < \lim\limits_{x\to 0} \cos\frac{1}{x} < 1,$ which means

$\Rightarrow \lim\limits_{x\to 0} f'(x)$ does not exist. Therefore, $f'(x)$ is discontinuous at $x = 0$.

3.2.2. A differentiable function for which the law of the mean fails.

The complex-valued function of a real variable x,

$$f(x) \equiv \cos x + i\sin x\,,$$

is everywhere continuous and differentiable but there exist no $a, b, and\ \xi$ such that $a < \xi < b$ and $(\cos b + i \sin b) - (\cos a + i \sin a) = (-\sin \xi + i \cos \xi)\,(b - a)$.

By the law of mean, f must be continuous on $[a, b]$ and differentiable on (a, b).

Then, $\exists c(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$f(x) = \cos x + i \sin x$, continuous and differentiable

$$f'(x) = -\sin x + i \cos x$$

$$\frac{\cos b + i \sin b - \cos a + i \sin a}{b - a} = -\sin c + i \cos c\,. \qquad (1)$$

Assume that (1) is correct.

$\cos b + i \sin b - \cos a - i \sin a = (b - a).(-\sin c + i \cos c)$

$\Rightarrow (\cos b - \cos a) + i(\sin b - \sin a) = (b - a)(-\sin c + i \cos c)$

$(\cos b - \cos a)^2 + (\sin b - \sin a)^2 = (b - a)^2$

$$\Rightarrow \frac{(b-a)^2}{4} = \frac{\cos^2 b + \cos^2 a - 2cosacosb + \sin^2 b + \sin^2 a - 2sinbsina}{4}$$

$$= \frac{2(1 - (\cos a \cos b + \sin a \sin b))}{4}$$

$$= 2.\frac{1 - \cos(b - a)}{4} = \frac{1 - \cos(b - a)}{2} = \frac{1 - (1 - 2sin^2 \frac{b - a}{2})}{2} = sin^2 \frac{(b - a)}{2}.$$

$$\frac{(b - a)^2}{4} = sin^2 \left(\frac{b - a}{2}\right) \Rightarrow \left(\frac{b - a}{2}\right)^2 = sin^2 (\frac{b - a}{2}) \Rightarrow \frac{b - a}{2} = \sin \frac{b - a}{2}$$

From this equation, $\sin x = x$ but this is a contradiction.

Hence, no solution for $x > 0$.

3.2.3. Let $I = [0, 1]$ and $\alpha, \beta \in \mathbb{R}$, and let $f_{\alpha,\beta}: I \to \mathbb{R}$ be defined by

$$\boldsymbol{f_{\alpha,\beta}(x) := \begin{cases} x^{\alpha} \sin x^{\beta} & for \quad 0 < x \leq 1 \\ 0 & for \quad x = 0 \end{cases}}$$

$f_{\alpha,\beta} \in D^1(I)$ holds precisely for $\alpha > 1$ and arbitrary $\beta, or\ \alpha \leq 1\ and\ \beta \geq 1 - \alpha$.

- $> 1\ and\ \beta$ arbitrary

For $\neq 0$, $f_{\alpha.\beta}(x) \in D^1(I)$ this statement is trivial.

$$f'_{\alpha,\beta}(0) = \lim_{h \to 0^+} \frac{f_{\alpha,\beta}(0 + h) - \overbrace{f_{\alpha,\beta}(0)}^{=0}}{h}$$

$$= \lim_{h\to 0}\frac{f_{\alpha,\beta}(h)}{h} = \lim_{h\to 0}\frac{h^{\alpha}.\sin h^{\beta}}{h}$$

$$= \lim_{h\to 0^+} h^{\alpha-1}.\sin h^{\beta}$$

$\alpha > 1 \Rightarrow \ \alpha - 1 > 0 \Rightarrow h^{\alpha-1} > 0$

$$1 \le \sin h^{\beta} \le 1 \qquad ,\forall\beta$$

$$\Rightarrow h^{\alpha-1} \le h^{\alpha-1}.\sin h^{\beta} \le h^{\alpha-1}$$

$\lim_{h\to 0} h^{\alpha-1} = 0 \ \Rightarrow By\ sandwich\ theorem$

$$f'_{\alpha,\beta} = \lim_{h\to 0^+} h^{\alpha-1}\sin h^{\beta} = 0$$

$$\Rightarrow f_{\alpha,\beta} \in D^1(I).$$

- $\alpha \le 1\ and\ \beta \ge 1 - \alpha.$

$$f'(0) = \cdots = \lim_{h\to 0} h^{\alpha-1}.\sin h^{\beta}$$

$$\alpha \le 1 \ \Rightarrow \ \alpha - 1 \le 0$$

$$1 - \alpha \ge 0$$

$$\beta \ge 1 - \alpha \ \Rightarrow \ \beta = 1 - \alpha + \varepsilon;\ \varepsilon \ge 0$$

$f'(0) = \cdots = \ \lim_{h\to 0}\frac{\sin h^{\beta}}{h^{1-\alpha}}$

$= \lim_{h\to 0}\frac{\sin h^{\beta}.h^{\varepsilon}}{h^{1-\alpha}.h^{\varepsilon}}$

$= \lim_{h\to 0^+}\frac{\sin h^{\beta}.h^{\varepsilon}}{h^{\beta}}$

$= \underbrace{\lim_{h\to 0^+} h^{\varepsilon}}_{=0}.\underbrace{\lim_{h\to 0^+}\frac{\sin h^{\beta}}{h^{\beta}}}_{=1} = 0$

$\Rightarrow \ f'(0)$ exists $\Rightarrow f \in D^1(I)$ exists.

3.2.4. Let $\boldsymbol{I = [0, 1]}$ and $\boldsymbol{\alpha, \beta \in \mathbb{R}}$, and let $\boldsymbol{f_{\alpha,\beta}: I \to \mathbb{R}}$ be defined by

$$\boldsymbol{f_{\alpha,\beta}(x) := \begin{cases} x^{\alpha} \sin x^{\beta} & for \quad 0 < x \le 1 \\ 0 & for \quad x = 0 \end{cases}}$$

$\boldsymbol{f_{\alpha,\beta} \in B(I)}$ **holds precisely for arbitrary $\boldsymbol{\alpha}$ and $\boldsymbol{\beta \ge 1 - \alpha}$.**

a) Let $x \in (0,1] \Rightarrow f'_{\alpha,\beta}(x) = \alpha . x^{\alpha-1} . \sin x^{\beta} + x^{\alpha} . \cos x^{\beta} . \beta . x^{\beta-1}$

$\Rightarrow f'_{\alpha,\beta}(x) = \alpha . x^{\alpha-1} . \sin x^{\beta} + \beta . x^{\beta-1} . x^{\alpha} . \cos \beta$

$f'_{\alpha,\beta}(x) = \alpha . x^{\alpha-1} . \sin x^{\beta} + \beta . x^{\alpha+\beta-1} . \cos x^{\beta}$

- $\alpha - 1 \ge 0 \Rightarrow 0 < x^{\alpha-1} \le 1$ and $-1 < \sin x^{\beta} \le 1$

$$\Rightarrow \alpha . x^{\alpha-1} . \sin \beta \in B((0,1])$$

$$\underbrace{\alpha - 1}_{\Rightarrow 1-\alpha<0} \ge 0 \Rightarrow \beta + \alpha - 1 \ge 1 - \alpha$$

$$x^{\alpha+\beta-1} \le x^{1-\alpha} \in B(0,1) \Rightarrow x^{\alpha+\beta-1} \in B(I)$$

$\Rightarrow \beta . x^{\alpha+\beta-1} . \cos x^{\beta} \in B((0,1])$

$$\Rightarrow f'_{\alpha,\beta}(x) \in B\big((0,1]\big) \; for \; \alpha - 1 \ge 0. \qquad (1)$$

- $\alpha - 1 < 0 \Rightarrow \alpha < 1 \Rightarrow \beta \ge 0.$

$$f'_{\alpha,\beta}(x) = \alpha . \underbrace{x^{\alpha-1}}_{=1/x^{1-\alpha}} . \underbrace{\sin x^{\beta}}_{\in[-1,1]} + \beta . \underbrace{x^{\alpha+\beta-1}}_{\in(0,1]} . \underbrace{\cos x^{\beta}}_{\in[-1,1]}$$

$$\alpha + \beta - 1 \ge \alpha - 1 + 1 - \alpha = 0$$

$$\Rightarrow f'_{\alpha,\beta}(x) \in B\big((0,1]\big) \quad for \; \alpha - 1 < 0 \qquad (2)$$

$$\Rightarrow f'_{\alpha,\beta}(x) \in B\big((0,1]\big) \quad \forall \alpha \; and \; \beta > 1 - \alpha$$

b) Let $x = 0$.

$f'_{\alpha,\beta}(0) = \lim_{h\to 0} \frac{h^{\alpha} \sin h^{\beta} - 0}{h}$

$$= \lim_{h\to 0} \frac{\sin h^{\beta}}{h^{1-\alpha}} = \lim_{h\to 0} \frac{\sin h^{\beta}}{h^{\beta-\varepsilon}} = \lim_{h\to 0} \frac{\sin h^{\beta} . h^{\varepsilon}}{h^{\beta}} = 0$$

By (a) and (b)

$$f'_{\alpha,\beta}(x) \in B([0,1]) \quad \forall \alpha \; and \; \beta \ge 1 - \alpha$$

$$f_{\alpha,\beta} \in B^1([0,1])$$

3.2.5. Let $\boldsymbol{I} = [\mathbf{0}, \mathbf{1}]$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$, and let $\boldsymbol{f}_{\boldsymbol{\alpha},\boldsymbol{\beta}}: \boldsymbol{I} \to \mathbb{R}$ be defined by

$$\boldsymbol{f}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) := \begin{cases} \boldsymbol{x}^{\boldsymbol{\alpha}} \sin \boldsymbol{x}^{\boldsymbol{\beta}} & \boldsymbol{for} \quad \mathbf{0} < \boldsymbol{x} \leq \mathbf{1} \\ \mathbf{0} & \boldsymbol{for} \quad \boldsymbol{x} = \mathbf{0} \end{cases}$$

$\boldsymbol{f}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \boldsymbol{C}^{\mathbf{1}}(\boldsymbol{I})$ holds precisely for arbitrary $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} > \mathbf{1} - \boldsymbol{\alpha}$.

- $x \in (0,1]$.

$$f'_{\alpha,\beta} = \alpha . x^{\alpha-1} . \sin x^{\beta} + \beta . x^{\alpha+\beta-1} . \cos x^{\beta} \in \varepsilon((0,1])$$

$$\beta > 1 - \alpha \;\Rightarrow\; \alpha + \beta - 1 > 1 - \alpha + \alpha - 1 = 0.$$

- $f'_{\alpha,\beta}(0) = 0$.

Is $\lim\limits_{x\to 0} f'_{\alpha,\beta}(x) = f'_{\alpha,\beta}(0)$ correct ?

$$\lim_{x\to 0} \alpha . x^{\alpha-1} . \sin x^{\beta} + \beta . x^{\alpha+\beta-1} . \cos x^{\beta}$$

$$= \lim_{x\to 0} \alpha . \frac{\sin x^{\beta}}{x^{1-x}} + \underbrace{\beta \lim_{x\to 0} x^{\alpha+\beta-1} . \cos x^{\beta}}_{=0} = 0.$$

$f'_{\alpha,\beta}(x)$ is continuous $\forall x \in [0,1]$.

$\Rightarrow \; f_{\alpha,\beta}(x) \in C^{1}([0,1])$.

3.3. SERIES AND SEQUENCES

3.3.1. A divergent series whose general term approaches zero.

Let us choose the harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Clearly $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Now we will show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. There are several ways to show the divergence of this series such as integral test, and geometric way etc. However we will show this analitically.

Suppose that

$$\sum_{n=1}^{\infty} \frac{1}{n} = L$$

i.e,

$$L = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Clearly

$$L = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$= L + \frac{1}{2}$$

$\Rightarrow L > L + \frac{1}{2}$ which is a contradiction. Hence the series does not converge.

3.3.2. Bounded Divergent Sequences.

The simplest example of a bounded divergent sequence is possibly

$$0,1,0,1, \ldots$$

or $\{a_n\}$, where $a_n = 0$ if n is odd and $a_n = 1$ is n is even. Equivalently, $a_n = \frac{1}{2}(1 + (-1)^n)$.

Let $\{a_n\}$ be the sequence in R defined as $a_n = (-1)^n$.

It is clear that $\{a_n\}$ is bounded above by 1 and below by -1.

Note the following subsequences of $\{a_n\}$

- $\{a_n\}$ where n_r is the sequence defined as $n_r = 2r$
- $\{a_n\}$ where n_s is the sequence defined as $n_s = 2s + 1$

The first is $1,1,1, \ldots$ and the second is $-1, -1, -1, \ldots$

So, $\{a_n\}$ has two subsequences with different limits. Since any two subsequence of a sequence cannot converge to different limits, $\{a_n\}$ cannot be convergent.

3.3.3. A divergent sequence $\{a_n\}$ for which $\lim_{n\to\infty}(a_{n+p} - a_n) = 0$ for every positive integer p.

Let a_n be the nth partial sum of the harmonic series.

$$a_n \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Then $\{a_n\}$ is divergent, but for $p > 0$,

$$a_{n+p} - a_n = \frac{1}{n+1} + \cdots + \frac{1}{n+p} \leq \frac{p}{n+1} \to 0.$$

$$\lim_{n\to\infty}(a_{n+p} - a_n) = 0.$$

3.3.4. Sequences $\{a_n\}$ and $\{b_n\}$ such that $\liminf(a_n) + \liminf(b_n) < \liminf(a_n + b_n) < \liminf(a_n) + \limsup(b_n) < \limsup(a_n + b_n) < \limsup(a_n) + \limsup(b_n)$.

Let $\{a_n\}$ and $\{b_n\}$ be the sequences repeating in cycles of 4:

$\{a_n\} : 0, 1, 2, 1, 0, 1, 2, 1, \ldots \implies \liminf(a_n) = 0,\ \lim sup\,(a_n) = 2$

$\{b_n\} : 2, 1, 1, 0, 2, 1, 1, 0, \ldots \implies \liminf(b_n) = 0,\ \limsup(b_n) = 2$

Now, we sum up the expression

$a_n + b_n = 2, 2, 3, 1, 2, 2, 3, 1, \ldots$

$\liminf(a_n + b_n) = 1$

$\limsup(a_n + b_n) = 3$

Hence,

$0 < 1 < 0 + 2 = 2 < 3 < 4.$

CHAPTER 4
CONCLUSIONS

This thesis tries to emphasize the importance of counterexamples in mathematics. As in stated in Introduction chapter, counterexamples are very important tool to show that a given mathematical statement is incorrect.

The subjects are choosen from calculus in order to be more understandable. These examples also emphasisez the importance of premises of the theorems.

We aim to give very basic and illustrative examples to be clear. People who are interested in counterexamples in analysis may read the reference (Olmsted,2003) which is involves huge amount of examples in every branch of mathematical analysis.

In other branches of mathematics, such as algebra, topology etc., there are many sources about counterexamples.

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