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STOCHASTIC DIFFERENTIAL EQUATIONS and ITS APPLICATIONS

Emine TACAR

Thesis Advisor: Assoc. Prof. Dr. Şahlar MEHERREM

Department of Mathematics

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Jury Members: **Signature:**

Dr. Öğr. Üyesi Şahlar MEHERREM ....

Yaşar Üniversitesi

Prof. Dr. Şaban EREN ....

Yaşar Üniversitesi

Dr. Öğr. Üyesi Derya DURGUN ....

Celal Bayar Üniversitesi

Prof. Dr. Cüneyt GÜZELİŞ

Director of the Graduate School

ABSTRACT

STOCHASTIC DIFFERENTIAL EQUATIONS and ITS APPLICATIONS

TACAR, Emine

MSc in Department Of Mathematics
Supervisor: Assoc. Prof. Dr. Şahlar MEHERREM
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In this thesis, exact solution of the stochastic linear differential equations has been studied. It$\hat{o}$'s formula has been mainly considered as an exact solution method of linear SDEs. Moreover, structure of SDEs which contains It$\hat{o}$'s Product Rule and some properties about It$\hat{o}$'s integral has been explained. Proof of existence and uniqueness theorems of solutions based on SDEs has been briefly discussed. Furthermore, the exact solution for the general type scalar linear stochastic differential equations has been proved. In addition to examine studies, selected examples for SDEs showing how to use the It$\hat{o}$'s formula effectively have been solved in detail.

Keywords: It$\hat{o}$'s Formula, Stochastic Processes, Stochastic Differential Equations,

ÖZ

STOKASTİK DİFERANSİYEL DENKLEMLER VE UYGULAMALARI

TACAR, Emine

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Danışman: Doç. Dr. Şahlar MEHERREM

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Bu tezde, stokastik doğrusal diferansiyel denklemlerin tam çözümleri incelendi. SDE' lerin bir tam çözüm metodu olan It$\hat{o}$ formülü ağırlıklı olarak ele alındı. Ayrıca, It$\hat{o}$ Çarpım Kuralı ve It$\hat{o}$ integralinin bazı özelliklerini içeren SDE' lerin yapısı açıklandı. SDE' lerin çözümlerine yönelik varlık ve teklik teoremlerinin ispatları kısaca tartışıldı. Ayrıca, genel tipteki skaler stokastik doğrusal diferansiyel denklemlerin tam çözümü ispatlandı. İncelenen çalışmalara ek olarak, It$\hat{o}$ formülünün etkili olarak nasıl kullanıldığını gösteren SDE' lere yönelik seçilmiş örnekler detaylı olarak çözüldü.

Anahtar Kelimeler: It$\hat{o}$ Formülü, Stokastik Prosesler, Stokastik Diferansiyel Denklemler.

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Emine TACAR

İzmir, 2018

TEXT OF OATH

I declare and honestly confirm that my study titled "Stochastic Differential Equations and Its Applications", and presented as Master's Thesis has been written without applying to any assistance inconsistent with scientific ethics and traditions and all sources I have benefited from are listed in bibliography and I have benefited from these sources by means of making references. This master thesis has not been submitted elsewhere for examination purposes.

Emine TACAR

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SYMBOLS AND ABBREVIATIONS

Symbols	**Response**
A	Subset of power set
Ω	Sample space
σ	Algebra (or $\sigma -$ field)
$\mathcal{F}$	Subsets of Ω
A^c	Complement of A
P : $\mathcal{F} \rightarrow [0,1]$	Probability measure
P(A)	Probabaility of the event
$(\Omega, \mathcal{F})$	Measurable space
X(.)	Stochastic process
X : $\Omega \rightarrow \Omega'$	Measurable map
B(σ)	Borel σ - field
X : $\Omega \rightarrow R^n$	n dimensional random variable
$(\Omega, \mathcal{F}, P)$	Probability space
ODE	Ordinary differential equation
$W(.)$	Brownian motion (Wiener Process)
SDE	Stochastic differential equation
$\xi(.)$	White noise

1. INTRODUCTION

1.1 Historical Remarks

Stochastic differential equations have been widely used to model phenomena appearing in economics, finance, robotics and management science and it continues to be a vital role in many different fields of real world problems.

The exact solution of stochastic differential equations has been studied by many authors. Most of all, Kiyosi It$\hat{o}$, who leaded the field and made a breakthrough the theory of stochastic integration and SDEs, now regarded as the founder of It$\hat{o}$ Calculus, put forward his theory in 1944 (Ito, 1944, 1951). Probability theory and stochastic processes, Ito stochastic calculus and stochastic Taylor expansion are investigated in (Kloedan and Platen, 1999). Probability concepts, stochastic integration and the white noise limit are considered in (Gardiner, 1990). Solution methods of SDEs are discussed in (Gikhman and Skorochord, 2014).

Furthermore, finding the exact solution for SDEs is actually difficult to attain therefore, it will be very helpful to find a approximation solution. In this sence, numerical solutions of SDEs have been considered in many researches. Some of those can be given as follow. (Gard, 1987), (Higham, 2001), (Kloedan and Platen, 1992), (Milstein and Tretyakov, 2004) and (Schurz, 2002).
Deterministic and Stochastic systems constitute the two main categories of models used in science fields such as mathematics, physics, chemistry and social sciences. These frequently heard concepts can be clarified as follows:

Deterministic models are systems that have no randomness in determining the future state of systems. So for a well-modelled deterministic system, under the same conditions and for the same initial states, the system will always give the same result. For example, an object we drop from 100 meters above will repeat once every 4.5 seconds, and the rate at which it falls will be 44.1m/s. Or it is because of the development of weapons that can be shot at a target point several hundred miles away today. Deterministic systems are called systems that can be calculated at a time point and how they will behave in advance based on the dynamic model.

Stochastic models, on the other hand, are systems with randomness. This randomness can be at system parameters, at the input, at the moment of inertia.

Therefore, the output of the system will have a similarity to the similarity. In such systems, the same results may not be obtained even if the same experiment is repeated under the same conditions, but it is possible to calculate the range in which the results are or in what range, and what the likelihood of occurrence is. In the case of stochastic models, the future position of the system is not known but rather predicted and probable. We can give examples such as lotto withdrawal, dice throw, exchange rates, stock exchange systems.

The Brownian motion is the random movement of floating or suspended particles in a fluid. The mathematical model used to explain this movement. This model is also called the Wiener method. In 1827, when Scottish botanist Robert Brown examined pollen in watery environment under a microscope, he noticed that a number of small particles separated from the pollen were moving constantly. The premise was suspected that this movement was a vital source, but when it came to seeing particles made of inorganic materials making the same move, it was the result that it was not biological. The movement had already been recognized, but since most scientists learned of it from Brown's work, this phenomenon is referred to as the "Brownian movement". Explanation of what Brown sees, Albert Einstein (1879-1955), a German-born scientist in the 20th century, would wait until the causes of the motion of the particles were found.

This thesis is organized as follows. In section 2, some preliminaries and basic definitions about stochastic analysis are provided. The structure of SDEs are explained in section 3. The existence and uniqueness of solutions to SDEs are derived in section 4. Certain solution methods of SDEs and especially Ito's formula are mainly considered in section 5. Some illustrative examples are given by applying the Ito's formula in section 6. Finally, in section 7 gives the conclusion of the matter in hand.

2. BASIC CONSEPTS AND DEFINITIONS

2.1 σ-field

A σ-field onΩ provides the following features. If

(i) $\emptyset, \Omega \in$ Ƒ, $\Omega \neq \emptyset$, Ƒ $\subseteq 2^{\Omega}$

(ii) A $\in$ Ƒ, and $A^c \in$ Ƒ.

(iii) A_1,A_2, … $\in$ Ƒ,

then $\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in$ Ƒ

2.2 Measurable Space

Let , Ω be a any set and Ƒ be a set of events then the pair (Ω,Ƒ) is denoted as a measurable space.

2.3 Probability Measure

Let us define a map P : Ƒ $\rightarrow [0, 1]$.Then the map is called a probability measure if

(i) $P(\emptyset) = 0$, $P(\Omega) = 1$;

(ii) $A_i \in$ Ƒ, $A_i \cap A_j = \emptyset$, $i, j = 1,2, \ldots, i \neq j$

then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

A triple (Ω, Ƒ, P) is called a probability space provided Ω is any sets, and $A \in$ Ƒ is called an event, $P(A)$denotes the probability of the event, a $\sigma -$ algebra of subsets of Ω, then P is a probability measure on Ƒ.

2.4 Random Variables

Let (Ω, Ƒ) and (Ω', Ƒ$'$) be two measurable space and X: $\Omega \rightarrow \Omega'$ on Ƒ/ Ƒ$'$-measurable map.We call X on Ƒ/ Ƒ$'$-random variableor simply a random variable if there would be no confusion.It is called onƑ $-$ random variable when (Ω', Ƒ$'$)=($R^{m,}$, B(R^m)).Note that the notion of random variable is defined without probability measures.

Where, Ω is a topological space, then the smallest $\sigma -$ field containing all open sets of Ω is called the Borel $\sigma -$field of σ, denoted by $B(\sigma)$.

X: $\Omega \rightarrow \Omega'$, X^{-1}(Ƒ$'$) is a sub-σfield of Ƒ, which is called the $\sigma -$ field generated by X, denoted by σ(x). This is the smallest σ-field Ω under which X is measurable.

Also if $\{X_\theta, \theta \in \Theta\}$is a family of random variables fromΩ to Ω', then we denote by$\sigma(X_\theta, \theta \in \Theta) \triangleq \vee_{\theta\in\Theta} {X_\theta}^{-1}$(Ƒ$'$).

Where, V_αƑ$_\alpha \triangleq \sigma(\cup_{\alpha}$ Ƒ$_{\alpha}),\wedge_{\alpha}$ Ƒ$_\alpha \triangleq \cap_{\alpha}$ Ƒ$_\alpha$.

The smallest sub-σ-field of Ƒ under which all X_θ ($\theta \epsilon \Theta$) are measurable.

Let X,Y: $\Omega \rightarrow \Omega'$ be two random variables and G a σ-field on Ω. Then X is said to be independent of G if σ(X) is independent of $\mathcal{G}$, and X is said to beindependent of Y if$\sigma(X)$ and $\sigma(Y)$ are independent.

2.6 Stochastic Process

Let ($\Omega, \mathcal{F}$, P) a probability space. A family {X(t), t ∈ I} of random variables from ($\Omega, \mathcal{F}$, P) to R^n is called a stochastic process. A stochastic process can be denoted by $\{X(t)\}_{t\in T}$,$\{X_t\}_{t\in T}$, or simply X.

2.7 Hilbert Space of Random Variables

A complete inner product space is called Hilbert Space. Inner product on this space is defined as $< X, Y > = E < XY >$ and the norm of Hilbert space of random variables is written in the form: $\| X \|_{RV} = (E(|X|^2))^{\frac{1}{2}}$

3. STRUCTURE OF SDEs

3.1 Motivation

Let us consider the following (ODE) ordinary differential equation:

$$\text{(ODE)} \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)), \ \ (t > 0) \\ \\ \boldsymbol{x}(0) = x_0, \end{cases} \tag{1}$$

where $x_0 \in R^n$, $\boldsymbol{f} : R^n \longrightarrow R^n$, vector-valued function and the solution of the system denotes trajectory as $\boldsymbol{x}(.) : [0, \ \infty) \rightarrow R^n$.

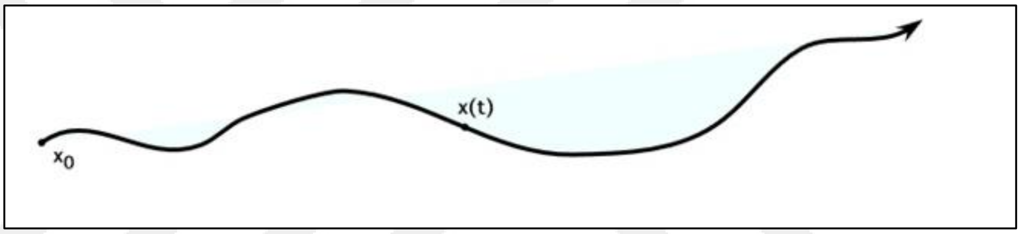

Figure 3.1 Trajectory of The ODE

The solution of the system at the time $t \geq 0$ is called "state" by **x**(t) and displayed the derivative as, $\dot{x}(t) := \frac{d}{dt} x(t)$.

However, considering many industrial problems, it is seen that the trajectories of the systems modeled with (ODE) as the result of experimental measurements do not behave as expected:

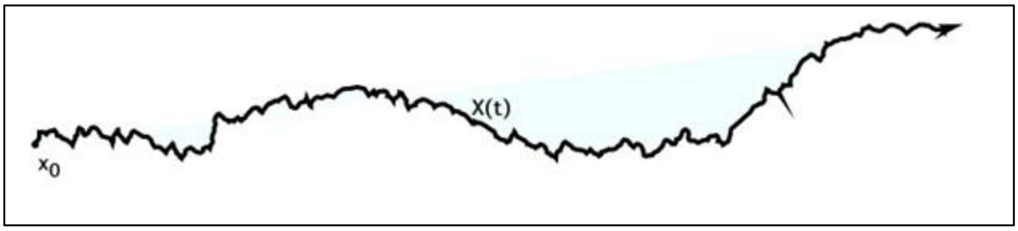

Figure 3.2 Sample Path of The Stochastic Differential Equation

Thus, ODE in (1) can be modified to determine the random effects on the system as follows:

$$\begin{cases} \dot{\boldsymbol{X}}(\mathbf{t}) = \boldsymbol{f}(\boldsymbol{X}(\mathbf{t})) + \boldsymbol{F}(\boldsymbol{X}(\mathbf{t}))\boldsymbol{\xi}(\mathbf{t}), \quad (\mathbf{t} > \mathbf{0}) \\ \boldsymbol{X}(\mathbf{0}) = \mathbf{x_0}. \end{cases} \tag{2}$$

Where , $F: \mathbb{R}^n \longrightarrow \mathbb{M}^{nxm}$, $\xi(t)$ ≔m-dimensional white noise.

Let us consider the equation (2) with the dimensions $m = n$, fixed point $x_0 = 0$ and the functions **f** = 0, and **F** = I. In this case, it turns out that the solution of (2) in this setting is Brownian motion or Wiener process (n-dimensional), with a notation $\boldsymbol{W}(.)$. Then, it can be shown the mathematical relation between white noise and Wiener process as$\dot{\boldsymbol{W}}(.) = \boldsymbol{\xi}(.)$. Due to the dot notation used for the derivative with respect to time t, we can also describe the following expression:

$$\frac{d\mathbf{X}(t)}{dt} = \boldsymbol{f}(\boldsymbol{X}(\mathrm{t}) + \boldsymbol{F}(\boldsymbol{X}(\mathrm{t}))\frac{\boldsymbol{dW}(\mathrm{t})}{\mathrm{dt}} \ .$$

Now, multiply the above equation by dt, and then we reach the following (SDE) stochastic differential equation:

$$(\mathrm{SDE})\begin{cases} \mathrm{d}\mathbf{X}(\mathrm{t}) = \mathbf{f}(\mathbf{X}(\mathrm{t}))\mathrm{dt} + \mathbf{F}(\mathbf{X}(\mathrm{t}))\mathrm{d}\mathbf{W}(\mathrm{t}) \\ \mathbf{X}(0) = \mathrm{x}_0. \end{cases} \tag{3}$$

As we will discussed in next chapters, the It$\hat{o}$ interpretation of SDE(3), i.e. $\mathbf{X}(.)$ satisfies the stochastic integral equation:

$$\boldsymbol{X}(t) = x_0 + \int_0^t \boldsymbol{f}(\boldsymbol{X}(s)ds + \int_0^t \boldsymbol{F}(\boldsymbol{X}(s))d\boldsymbol{W}(\mathrm{s}), \quad t > 0.$$

3.2 It$\hat{o}$'s Formula(Chain Rule)

The It$\hat{o}$'s formula is similar to the classic chain rule in Elementary Calculus.

Assuming that f is a differentiable function of a real variable x and take fixed x_0,

then it can be considered as $\Delta x = x - x_0$ with $\Delta f(x) = f(x) - f(x_0)$. By using Taylor expansion we get:

$$\Delta f(x) = f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O(\Delta x)^3.$$

If we use the differential dx instead of Δx , we have

$$\mathrm{d}f(x) = f'(x)\mathrm{d}x + \frac{1}{2}f''(x)(\mathrm{d}x)^2 + O(\mathrm{d}x)^3.$$

All terms of order $(dx)^2$ and higher are omitted, then basic calculus formula is obtained $\mathrm{d}f(x) = f'(x)\mathrm{d}x$.

Suppose next that $x = x(t)$, we get the differential form of the usual chain rule

$$\mathrm{d}f\big(x(t)\big) = f'\big(x(t)\big)\mathrm{d}x(t) = f'\big(x(t)\big)x'(t)dt. \quad (*)$$

Now, we plug into stochastic process X_t instead of the deterministic function $x(t)$. The relation between the differential function f and the process X_t is expressed as $F_t = f(X_t)$.

$$\mathrm{d}F_t = f'(X_t)\mathrm{d}X_t + \tfrac{1}{2}f''(X_t)(\mathrm{d}X_t)^2. \quad (**)$$

If a stochastic process X_t satisfies the following equality, it is called an Ito diffusion:

$$dX_t = \alpha(W_t,t)dt + \beta(W_t,t)dW_t.$$

Theorem 3.2.1 *Let X_t is an Itô diffusion and $F_t = f(X_t)$ then,*

$$\mathrm{d}F_t = [\alpha(W_t,t)f'(X_t) + \frac{\beta(W_t,t)^2}{2}f''(X_t)]dt + \beta(W_t,t)f'(X_t)\mathrm{d}W_t.$$

Proof:

We know the Itô diffusion as $\mathrm{d}F_t = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2$, then let us write

$$\begin{aligned}(dX_t)^2 &= (\alpha(W_t,t)dt + \beta(W_t,t)dW_t)^2 \\ &= \alpha(W_t,t)^2dt^2 + 2\alpha(W_t,t)\,\beta(W_t,t)dW_tdt + \beta(W_t,t)^2dW_t{}^2 \\ &= \beta(W_t,t)^2dt,\end{aligned}$$

where $dW_t{}^2 = dt$ and $dtdW_t = dt^2 = 0$. Substituting into (**) gives

$$\begin{aligned}\mathrm{d}F_t &= f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2 \\ &= f'(X_t)(\alpha(W_t,t)dt + \beta(W_t,t)dW_t) + \frac{1}{2}f''(X_t)\,\beta(W_t,t)^2dt \\ &= \left[\alpha(W_t,t)f'(X_t) + \tfrac{\beta(W_t,t)^2}{2}f''(X_t)\right]dt + \beta(W_t,t)f'(X_t)\,dW_t.\end{aligned}$$

Remark: If $X_t = W_t$, we obtain the following result:

$\mathrm{d}F_t = f'(W_t)\mathrm{d}W_t + \frac{1}{2}f''(W_t)dt.$

If we have time-dependent function $f = f(t,x)$, then the analog of (*) is submitted by

$$df(t,x) = \frac{\partial f(t,x)}{\partial t}dt + \frac{\partial f(t,x)}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial x^2}(dx)^2 + O(dx)^3 + O(dt)^2 .$$

Put $x = X_t$ then gives

$$df(t,X_t) = \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}(dX_t)^2.$$

In the case when X_t is an Ito diffusion, we get the following expression with additional term:

$$\mathrm{d}F_t = \left[\frac{\partial f(t,X_t)}{\partial t} + \alpha(W_t,t)\frac{\partial f(t,X_t)}{\partial x} + \frac{\beta(W_t,t)^2}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\right]dt$$

$$+ \beta(W_t,t)\frac{\partial f(t,X_t)}{\partial x}\, dW_t .$$

Moreover, when the process F_t depends on more than one Ito diffusions such that $F_t = f(t,X_t,Y_t)$, then an analogous formula to above yields to

$$\mathrm{d}F_t = \frac{\partial f(t,X_t,Y_t)}{\partial t}dt + \frac{\partial f(t,X_t,Y_t)}{\partial x}dX_t + \frac{\partial f(t,X_t,Y_t)}{\partial y}dY_t$$

$$+\frac{1}{2}\frac{\partial^2 f(t,X_t,Y_t)}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f(t,X_t,Y_t)}{\partial y^2}(dY_t)^2 + \frac{\partial^2 f(t,X_t,Y_t)}{\partial x\partial_y}dX_t dY_t .$$

Now let us consider SDE (3) with $n = 1$, $\boldsymbol{F} = I$ and $X(.)$ solves the following SDE:

$$d(X) = f(X)dt + dW. \tag{4}$$

Assuming that $\Psi : R^n \longrightarrow R^n$ is a smooth function, then we want to investigate which SDE is solved with $Y(t) := \Psi(\mathrm{X}(\mathrm{t}))$, $\mathrm{t} \geq 0$. If the equation (4) is observed, the following equation can be written:

$$dY = \Psi' dX = \Psi' f dt + \Psi' dW.$$

By chain rule, taking derivative with respect to x is represented as $\frac{d}{dx}$, so there is an error here. Because, in stochastic sense, $dW \approx (dt)^{\frac{1}{2}}$. If we evaluate dY which includes dt and $(dt)^{\frac{1}{2}}$ we have

$$dY = \Psi' dX + \frac{1}{2}\Psi''(dX)^2 + \cdots$$

$$dY = \Psi'(fdt + dW) + \frac{1}{2}\Psi''(fdt + dW)^2 + \cdots$$

$$dY = \left(\Psi' f + \tfrac{1}{2}\Psi''\right) dt + \Psi' dW + \{\text{Terms of order } (dt)^{\frac{3}{2}} \text{and higher}\}.$$

Since $(dW)^2 = dt$, and then we obtain,

$$dY = \left(\Psi' f + \frac{1}{2}\Psi''\right) dt + \Psi' dW.$$

Here, apart from the classical calculus, $\frac{1}{2}\Psi'' dt$ is also appeared in the equation as additional term.

3.3 Different Cases of It$\hat{o}$'s Formula

Theorem 3.2 *Let* $X(.)$ *be a solution of the following SDE*

$$dX_t = F\ dt + GdW_t\ .$$

For a integrable functions $F \in L^1$ *(0, T),* $G \in L^2$ *(0,T), and then for any function* $\Psi: R \times [0, T] \to R,\ \Psi$ *is continuous and its derivatives* $\Psi_t\ ,\Psi_x \Psi_{xx}$ *are exist and continuous.*

Proof:

Let us suppose that Y(t)$:=\ \Psi$(X(t), t).

Now, Y(t) corresponds to the following stochastic differential equation

$$\mathrm{dY}_t = \Psi_t dt + \Psi_x dX + \frac{1}{2}\Psi_{xx} G^2 \mathrm{dt}$$

$$d\mathrm{Y}_t = \left(\Psi_t + \Psi_x F + \frac{1}{2}\Psi_{xx} G^2\right) dt + \Psi_x$$

A) Now, take the special case Ψ (x) $= x^n$, where n changes on zero and one, and let provide that the below equality:

$$\mathrm{d}(X^n) = \mathrm{n}X^{n-1}\mathrm{dx} + \tfrac{1}{2}\mathrm{n(n-1)}X^{n-2}G^2\mathrm{dt}. \qquad (\blacksquare)$$

It is obvious that $n = 0,1, \ldots$ and for $n = 2$, related expression holds. Now we provide the above-stated formula for $n - 1$:

$$\mathrm{d}\,(X^{n-1}) = (n-1)X^{n-2}\mathrm{dX} + \tfrac{1}{2}\mathrm{(n-1)(n-2)}X^{n-3}G^2\mathrm{dt}$$

$$= \mathrm{(n-1)}X^{n-2}(\mathrm{Fdt + G\ dW}) + \tfrac{1}{2}\mathrm{(n-1)(n-2)}X^{n-3}G^2\mathrm{d}$$

Then we prove it for n:

Note that

$\mathrm{d}(X^n) = d(\mathrm{X}X^{n-1})$.

$= Xd(X^{n-1}) + X^{n-1}\mathrm{dX} + (n-1)X^{n-2}G^2\mathrm{dt}$

$= X((\mathrm{n}-1)X^{n-2}\mathrm{dX} + \frac{1}{2}(\mathrm{n}-1)(\mathrm{n}-2)X^{n-3}G^2\mathrm{dt}) + (\mathrm{n}-1)X^{n-2}G^2\mathrm{dt} + X^{n-1}\mathrm{dX}$

$= \mathrm{n}X^{n-2}\mathrm{dx} + \frac{1}{2}n(n-1)X^{n-2}G^2\mathrm{dt}$.

Because;

$(\mathrm{n}-1) + \frac{1}{2}(\mathrm{n}-1)(\mathrm{n}-2) = \frac{1}{2}n(\mathrm{n}-1)$. This proves (■).

B) Now let, $\Psi(\mathrm{x},\mathrm{t}) = \mathrm{p}(\mathrm{x})\mathrm{q}(\mathrm{t})$, where p and q are polynomials.

$d\big(\Psi(X,t)\big) = d(p(X)q) = p(X)dq + qdp(X)$

$= p(X)q'dt + q[p'(X)dX + \frac{1}{2}p''(X)G^2dt]$

$= \Psi_t dt + \Psi_x dX + \frac{1}{2}\Psi_{xx}G^2dt$

This computation affirms Itô's formula for $\Psi(\mathrm{x},\mathrm{t}) = \mathrm{p}(\mathrm{x})\mathrm{q}(\mathrm{t})$. Hence it is also true for each function Ψ getting the model :

$$\Psi(x,t) = \sum_{i=1}^{n} p^i(x)q^i(t).$$

Where p^i and $q^i$are polynomials. This means that the Itô formula applies to all polynomial functions which have the component variables x and t.

We now consider Ψ which is indicated in Itô's formula, then Ψ^n corresponds to sequence of polynomials such that;

$$\Psi^n \to \Psi \qquad \Psi_t^n \to \Psi_t$$

$$\Psi_x^n \to \Psi_x \qquad \Psi_{xx}^n \to \Psi_{xx},$$

Since $0 \le s \le T$, we can write the following in the sense of general Itô's formula by using limit via $s \to \infty$:

$$\Psi^n(\mathrm{X}(\mathrm{s}),\mathrm{s}) - \Psi^n(\mathrm{X}(0),0) = \int_0^s (\Psi_t^n + \Psi_x^n F + \frac{1}{2}\Psi_{xx}^n G^2)dt + \int_0^s \Psi_x^n \mathrm{GdW}$$

3.4 Generalized Itô's Formula

Let us now, briefly study the case where stochastic process X(.), drift and diffusion functions have m-components.

In this case, we supposed to have, $dX_t^i = \phi^i dt + \psi^i dW_t$with for$\phi^i \in L^1$(0,T), $\psi^i \in L^2$(0, T), and $i = 1, \dots, m$.

Let $\Psi: R^m \times [0,T] \to R$ be a continuous function and its partial derivatives are continuous such that $\frac{\partial \Psi}{\partial t}$, $\frac{\partial \Psi}{\partial x_i}$,$\frac{\partial^2 \Psi}{\partial x_i \partial x_j}$and $i = 1, \dots, m$, then we can state the following generalized Itô's formula:

$$d(\Psi(X^1, \dots, X^m, t) = \Psi_t dt + \sum_{i=1}^{m} \Psi_{x_i} dX^i + \frac{1}{2} \sum_{i,j=1}^{m} \Psi_{x_i x_j}^2 \phi^i \psi^i dt.$$

3.5 Multidimensional Itô's Formula

In this subsection we present the Itô's formula in the case of R^n- valued stochastic processes with the stochastic equation $d\boldsymbol{X_t} = \boldsymbol{\phi} dt + \boldsymbol{\psi} d\boldsymbol{W_t}$for some $\boldsymbol{\phi} \in L_n^1$(0,T), $\boldsymbol{\psi} \in L_{nxm}^2$(0, T).

Let $\Psi: [0,T] \times R^n \to R$ be a continuous function and its partial derivatives are continuous such that $\frac{\partial \Psi}{\partial t}$, $\frac{\partial \Psi}{\partial x_i}$,$\frac{\partial^2 \Psi}{\partial x_i \partial x_j}$and $i, j = 1, \dots, n$, then we can express the following multidimensional Itô's formula:

$$d(t, \Psi(\boldsymbol{X}(t)) = \Psi_t dt + \sum_{i=1}^{n} \Psi_{x_i} dX^i + \frac{1}{2} \sum_{i,j=1}^{n} \Psi_{x_i x_j}^2 \sum_{k=1}^{m} \psi^{ik} \psi^{jk} dt.$$

3.6 Itô Product Rule

In this subsection we show up how to drive Itô Product Rule by the following theorem.

Theorem 3.1 *Suppose that*

$$\begin{cases} dX = \phi_1 dt + \psi_1 dW \\ dY = \phi_2 dt + \psi_2 dW \end{cases} \quad (0 \le t \le T)$$

where,$\phi_i \in L^1$ *(0,T),* $\psi_i \in L^2$ *(0, T) , (i =1,2).*

Then;

d (XY)= $YdX + XdY + \psi_1 \psi_2 dt.$

Proof:

A) Let's choose $0 \leq r \leq T$. For simplicity we assume:

$X(0)= Y(0)=0$, $\phi_i(t) \equiv \phi_i$, $\psi_i(t) \equiv \psi_i$,

Here, ϕ_i, ψ_i are time-independent random variables,

$\mathcal{F}(0) -$ measurable random variables, then

$X_t= \phi_1 t + \psi_1 W_t$ and $Y_t= \phi_2 t + \psi_2 W_t$ $(t \geq 0)$.

Therefore,

$\int_0^r Y dX + X dY + \psi_1 \psi_2$

$= \int_0^r (X \ \phi_2 + Y \phi_1) dt + \int_0^r (X \psi_2 + Y \psi_1) Dw + \int_0^r \psi_1 \psi_2 dt$

$= \int_0^r (\phi_1 t + \psi_1 W) \ \phi_2 + (\phi_2 t + \psi_2 W) \ \phi_1 dt$

$+ \int_0^r \phi_1 t + \psi_1 W) \psi_2 + (\phi_2 t + \psi_2 W) \psi_1 dW \ + \psi_1 \psi_2 r$

$= \phi_1 \ \phi_2 r^2 + (\psi_1 \ \phi_2 + \psi_2 \phi_1)[\int_0^r W \ dt \ + \int_0^t t dW]$

$+ 2 \psi_1 \psi_2 \int_0^r W dW + \psi_1 \psi_2 r.$

According to above-stated Lemma, we evaluate $2 \int_0^r W dW = W^2(r) - r$

and $\int_0^r W dt + \int_0^r t dW = r W(r)$.

By using these properties, we obtain:

$\int_0^r Y dX + X dY + \psi_1 \psi_2 d_t$

$= \phi_1 \ \phi_2 r^2 + (\psi_1 \ \phi_2 + \psi_2 \phi_1) r W(r) + \psi_1 \psi_2 W^2(r)$

$= X(r) \ Y(r)$.

As a result, we have:

$\int_s^r Y dX = X(r) Y(r) - X(s) Y(s) - \int_s^r X d \, Y - \int_s^r \psi_1 \psi_2 dt.$

For the special case with $s = 0$, $X(0) = 0$, $Y(0) = 0$; φ_i and ψ_i are random variables. If this situation, $s \geq 0$, $X(s)$, $Y(s)$ are optional, at the same time φ_i and ψ_i are constant $\mathcal{F}(s)$- measurable random variables, then it can be proved in a similar way.

B) Let φ_i, ψ_i are step processes, then we apply phase A on $[t_{k,} t_{k+1})$ and using $\varphi_i \, and \ \psi_i$ constant random variables, then obtain integral equalities.

C) For the general case, we choose step processes as:

$\varphi_i{}^n \in L^1(0,T)$, $\psi_i{}^n \in L^2(0,T)$, with

$E(\int_0^T |\varphi_i{}^n - \varphi_i| dt) \to 0$ and

$E(\int_0^T | \psi_i{}^n - \psi_i|^2 dt) \to 0$

As n→ $\infty, i = 1, 2.$

Let us define the following equalities:

X^n(t) := $X(0)$ + $\int_0^t \varphi_1{}^n$ds + $\int_0^t \psi_1{}^n$dW ,

Y^n(t) := $Y(0)$ + $\int_0^t \varphi_2{}^n$ds + $\int_0^t \psi_2{}^n$dW

When we apply phase B to X^n(.), and Y^n(.) for (s,r) and use limits, where each the following formula:

X(r)Y(r) = X(s)Y(s) + $\int_s^r X\mathrm{d}Y + Y\mathrm{d}X + \psi_1\, \psi_2 dt.$

3.7 It$\hat{o}$ Integral

One of the most important stochastic integrals is called the Itô integral. In 1944, Japanese mathematician K. It$\hat{o}$ introduced this type of integral. Moreover, impression of diffusion processes originally motivated to the It$\hat{o}$ integral.

Itô integral $Y_t(B)$ (blue) of a Brownian motion B (red) with respect to itself, i.e., both the integrand and the integrator are Brownian. It comes in view:

$$Y_t(B)=(B^2-t)/2.$$

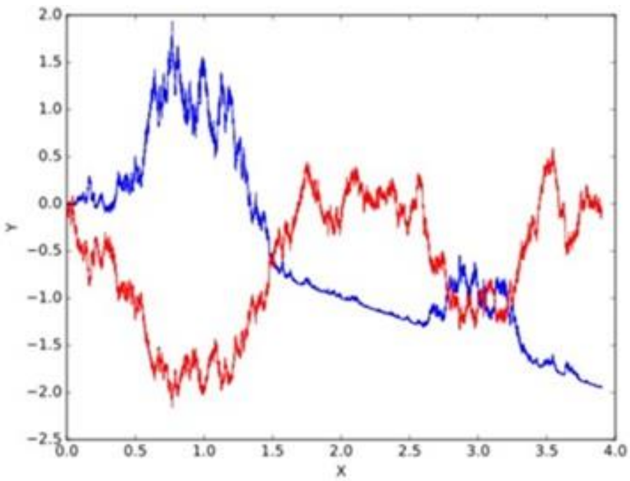

Figure 3.3 (Comparation of Itô integral and Brownian Motion)

Here are some important features of the Brownian movement. First, the movement is totally erratic, a random move. Under the microscope, the particles seem to move forward and backward, but this is very different from the vibrational motion we know. While there is no period in the Brownian motion, the particles can change their vibrational movement there is a movement with a definite period around a fixed point.

3.8 Stochastic Differential

In this subsection, we consider the following useful Lemmas for stochastic differential.

Lemma 3.8.1 $d\,({W_t}^2) = 2W_t dW_t + dt$.

Proof.

By using the product rule and the stochastic relation $d\,({W_t}^2) = dt$, gives

$d\,({W_t}^2) = W_t dW_t + W_t dW_t + dW_t dW_t = 2W_t dW_t + dt$.

Lemma 3.8.2 $d({W_t}^3) = 3{W_t}^2 dW_t + 3W_t dt$.

Proof.

Let us apply the product rule and use the previous lemma, then gives

$$\begin{aligned} d({W_t}^3) = d(W_t \,.{W_t}^2) &= W_t d({W_t}^2) + {W_t}^2 dW_t + d({W_t}^2) dW_t \\ &= W_t(2W_t dW_t + \mathrm{dt}) + {W_t}^2 dW_t + \mathrm{d}W_t(2W_t dW_t + \mathrm{dt}) \\ &= 2{W_t}^2 \mathrm{d}W_t + W_t dt + {W_t}^2 dW_t + 2W_t\,({dW_t}^2) + \mathrm{dt}dW_t \\ &= 3{W_t}^2 \mathrm{d}W_t + 3W_t \mathrm{dt}, \end{aligned}$$

where we used $({dW_t}^2) = dt$ and $dtdW_t = 0$.

Lemma 3.8.3 $d(tW_t) = W_t dt + t\mathrm{d}W_t$.

Proof.

By using the product rule and $dtdW_t = 0$, we have

$$\begin{aligned} d(\mathrm{t}W_t) &= W_t \mathrm{dt} + t\mathrm{d}W_t + dt\,\mathrm{d}W_t \\ &= W_t dt + t dW_t. \end{aligned}$$

3.9 Stochastic Calculus

Theorem 6.1.1 *For any $\tau < t$, then the following equality holds*

$d(\int_{\tau}^{t} f(s,W_s)dW_s) \ = \ f(t,W_t)dW_t.$

Proof.

Let us consider a stochastic process X_t whose increments satisfy the following equation:

$dX_t \ = \ f\ (t, W_t)dW_t.$

Integrating above equality between $\tau$and t yields

$\int_{\tau}^{t} dX_s \ = \int_{\tau}^{t} f(s, W_s)dW_s.$

The integral on the left side can be calculated by using the separation

$0 \ = \ t_0 < t_1 \ldots < t_{n-1} < t_n \ = t$, then

$\int_{\tau}^{t} dX_s \ = \lim_{n\to\infty} \sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j}) = X_t - \ X_\tau,$

Substituting them into previous integral equation gives:

$X_t = \ X_\tau + \int_{\tau}^{t} f(s, W_s)dW_s,$

and hence

$dX_t = \ d(\int_{\tau}^{t} f(s, W_s)\mathrm{d}W_s),$

since X_τ is a constant.

Lemma 3.9.1 $\int_0^t W_s \mathrm{d}W_s = \frac{W_t^2}{2} - \frac{t}{2}$

Proof.

Let $X_t \ = \int_0^t W_s \mathrm{d}W_s$ and $Y_t = \frac{{W_t}^2}{2} - \ \frac{t}{2}$ .From $I$to 's formula

$dY_t = d\ (\frac{{W_t}^2}{2}) - d(\frac{t}{2}) \ = \frac{1}{2}(2W_t\mathrm{d}W_t + \ dt) \ - \frac{1}{2}dt \ = W_t dW_t,$

and from the Theorem 6.1.1, we have

$dX_t \ = d(\int_0^t W_s\, dW_s) \ = W_t \mathrm{d}W_t$.

Hence dX_t =dY_t ,or $d(X_t - Y_t) = 0$. Since the process X_t - Y_t has zero increments, then $X_t - Y_t = \ c$, constant. Taking $t = 0$, yields

$c = X_0 - Y_0 \ = \int_0^0 W_s \mathrm{d}W_s - (\frac{{W_0}^2}{2} - \frac{0}{2}) = \ 0,$

and hence $c = 0$. It follows that $X_t \ = Y_t$, which verifies the desired relation.

Lemma 3.9.2 $\int_0^t sW_s\, dW_s = \frac{t}{2}(W_t^2 - \frac{t}{2}) - \frac{1}{2}\int_0^t W_s^2\, ds.$

Proof.

Consider the stochastic processes

$X_t = \int_0^t sW_s dW_s$, $Y_t = \frac{t}{2}(W_t^2$ - 1), and $Z_t = \frac{1}{2}\int_0^t W_s^2\, ds$.

Theorem 6.1.1 yields

$dX_t = tW_t dW_t$

$dZ_t = \frac{1}{2}W_t^2 dt.$

Applying It$\hat{o}$'s Formula, we get

$dY_t = d(\frac{t}{2}(W_t^2 - \frac{t}{2})) = \frac{1}{2}d(tW_t^2) - d(\frac{t^2}{4})$

$= \frac{1}{2}[(t + W_t^2)dt + 2tW_t dW_t] - \frac{1}{2}tdt$

$= \frac{1}{2}W_t^2 dt + tW_t dW_t.$

We can easily see that

$dX_t = dY_t - dZ_t$.

This implies $d(X_t - Y_t + Z_t) = 0$, i.e. $X_t - Y_t + Z_t = c$, constant.

Since $X_0 = Y_0 = Z_0 = 0$,

it means that $c = 0$.

Lemma 3.9.3 $\int_0^t (W_s^2 - s)\, dW_s = \frac{1}{3}W_t^3 - tW_t.$

Proof.

Let us consider the function $f(t,x) = \frac{1}{3}x^3 - tx$, and let $F_t = f(t, W_t)$.

Since f_t = - x, $f_x = x^2$ - t, and f_{xx} = 2x,

then It$\hat{o}$'s Formula provides

$dF_t = f_t \text{dt} + f_x\, \text{d}W_t + \frac{1}{2}f_{xx}(dW_t^2)$

$= -W_t \text{dt} + (W_t^2 - \text{t})\text{d}W_t + \frac{1}{2}2W_t \text{dt}$

$= (W_t^2 - \text{t})\text{d}W_t.$

From the Theorem 6.1.1 we get

$\int_0^t (W_s^2 - s)\, \text{d}W_s = \int_0^t dF_s = F_t - F_0 = F_t = \frac{1}{3}W_t^3 - tW_t.$

4. EXISTENCE AND UNIQUNESS SOLUTIONS FOR SDEs

The existence and uniqueness theorems are the most important main constituent in SDEs theory. For that reason, the following theorems are below-stated.

4.1 Theorem (Existence and Uniquness)

Consider the following stochastic differential equation

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0, \end{cases}$$

where $b(.)$ and $\sigma(.)$coefficients are continuous on$[0, T] \times R$and x_0 is a fixed random variable providing

1) $|b(t,x)| + \sigma|(t,x)| \leq M(1 + |x|); \quad x \in R,\ T \in [0, T]$

2) $|b(t,x) - b(t,y)| + \sigma|(t,x) - \sigma(t,y)| \leq K|x - y|; \quad x, y \in R$

with constants $M, K > 0$. Then X_t is a unique stochastic process, which is continuous, and providing the following:

$$E[\int_0^T X_t^2 dt] < \infty.$$

For the first condition in the sense of linear growthiness, $b(.)$ and $\sigma(.)$ arguments are upper-bonded with linear function in x. Beside this, the second one emphasize that the drift and volatility functions are Lipschitz in the meaning of second variable.

4.2 Uniqueness

Proof:

- Suppose that X_t and X_t^* be two solutions.
- Note that, we get the following equality:

$$X_t - X_t^* = \int_0^t [b(s, X(s)) - b(s, X^*(s))]ds$$

$$+\int_0^t [\sigma(s, X(s)) - \sigma(s, X^*(s))dW(s).$$

- By using Lipschitz condition, Hölder inequality and Gronwall's inequality, we can obtain the following inequality:

$$E(\sup_{0\leq s\leq t} |X(s) - X^*(s)|^2) \leq 2K(T+4) \int_0^t E(\sup_{0\leq l\leq s} |X(l) - X^*(l)|^2)ds.$$

- The Gronwall inequality then yields that

$$E\left(\sup_{0\leq t\leq T} |X(t) - X^*(t)|^2\right) = 0.$$

Hence, $X(t) = X^*(t)$ for all $0 \leq t \leq T$. ∎

4.3 Existence

Proof:

- Let $X_t = \int_0^t b(s, X(s))\, ds + \int_0^t [\sigma(s, X(s))dW(s)$,
- Note that

$$E\left|\int_0^t b(s, X(s))ds - \int_0^t b(s, X^*(s))\, ds\right|$$

$$+E\left|\int_0^t \sigma(s, X(s))dW(s) - \sigma(s, X^*(s))dW(s)\right|$$

$$\leq K(T+1)\int_0^T E|X_n(s) - X(s)|^2 ds \to 0.$$

- Thus we can let $n \to \infty$ in

$$X_n(t) = x_0 + \int_0^t b(s, X_{n-1}(s))ds + \int_0^t \sigma(s, X_{n-1}(s))dW(s),$$

- We obtain that

$$X_t = x_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s).$$

Consequently, we prove that the Picard iterations $X_n(t)$ converge to the unique solution X_t of the following equation:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$ ∎

5. SOLUTION METHODS OF SDEs

We can investigate stochastic differential equations under two main headings, such that linear SDEs and non-linear SDEs. Moreover, we can also classify linear SDEs into two branches such that scalar and vector-valued.

There are three effective methods to compute solution of SDEs:

- The first major method is based on the It$\hat{o}$'s Formula and has already been used for linear solutions of SDEs.
- The second one is numerical methods (which are essential for the analysis of random phenomena) to compute path-wise solutions of SDEs. The simplest effective computational methods for the approximation of SDEs are based on similar techniques for ODEs, but generalized to provide support for stochastic dynamics. The most commonly known method is the Euler-Maruyama Method,another useful method can be given as Milstein Method, Monte-Carlo Method, Taylor Method, Runge-Kutta Method, etc.
- The third one is based on partial differential equations, which is associated with the probability density function of the solution.

5.1 Linear Stochastic Differential Equations:

We start off the SDEs with the common case, the scalar linear SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X(0) = x_0. \end{cases}$$

Above-stated SDE for a one-dimensional stochastic process X_t is called a linear (scalar) SDE if and only if the functions $b(.)$ and $\sigma(.)$ are affine functions of X_t, $\in \mathbb{R}$.

$b(X_t, t) := A(t)X_t + \alpha(t)$,

$\sigma(X_t, t) := [B_1(t)X_t + \beta_1(t), \dots, B_m(t)X_t + \beta_m(t)]$,

where A(t), $\alpha(t) \in \mathbb{R}$, $W(t) \in \mathbb{R}^m$ is an m-dimensional Brownian Motion, and $B_i(t)$, $\beta_i(t) \in \mathbb{R}$, $i = 1, \dots, m$.

Thus, $b(t, X_t)$, $x_0 \in \mathbb{R}$ and$\sigma(t, X_t) \in \mathbb{R}^{1\times m}$

A stochastic vector-valued differential equation:

$$\begin{cases} d\boldsymbol{X}_t = b(t, \mathbf{X}_t)dt + \sigma(t, \mathbf{X}_t)d\mathbf{W}_t \\ \boldsymbol{X}(0) = \boldsymbol{x}_0. \end{cases}$$

$\mathbf{X}_t$ is called a linear SDE if the functions $b(t, \mathbf{X}_t)$, $\in \mathbb{R}^n$ and

$\sigma(\mathrm{t}, \mathbf{X}_t) \in \mathbb{R}^{n\times m}$are affine functions of $\mathbf{X}_t$ and thus,

$b(\mathrm{t}, \mathbf{X}_t) := A(t)\mathbf{X}_t + \alpha(t)$,

$\sigma(\mathrm{t}, \mathbf{X}_t) := [B_1(t)\mathbf{X}_t + \beta_1(t), \dots, B_m(t)\mathbf{X}_t + \beta_m(t)]$,

where A(t) $\in \mathbb{R}^{n\times n}$, $\alpha(t) \in \mathbb{R}^n$, $\mathbf{W}_t \in \mathbb{R}^m$ is an m- dimensional Brownian Motion, and $B_i(t) \in \mathbb{R}^{n\times n}$, $\beta_i(t) \in \mathbb{R}^n$.

Assume first **A**≡ A is constant.Then the solution of the following SDE:

$$\begin{cases} d\boldsymbol{X_t} = (\alpha(t) + A\boldsymbol{X_t})dt + \sum_{i=1}^{m} \beta_i(t)\, d\boldsymbol{W_t} \\ \boldsymbol{X}(0) = x_0 \end{cases}$$

$\mathbf{X}$(t)= $e^{At}\boldsymbol{X}_0$+$\int_0^t e^{A(t-s)}(\alpha(s)$ds+$\sum_{i=1}^{m} \beta_i(s)d\boldsymbol{W_t}$), where, $e^{At} := \sum_{k=0}^{\infty}\frac{A^k t^k}{k!}$.

For more general type, the solution of the following SDE:

$$\begin{cases} d\boldsymbol{X_t} = (\alpha(t) + A(t)\boldsymbol{X_t})dt + \sum_{i=1}^{m} \beta_i(t)\, d\boldsymbol{W_t} \\ \boldsymbol{X}(0) = x_0 \end{cases}$$

$\mathbf{X}$(t)= $\psi(t)(\boldsymbol{X_0}$+$\int_0^t \psi(s)^{-1}(\alpha(s)$ds+$\sum_{i=1}^{m} \beta_i(s)d\boldsymbol{W_t}$)).

In the following non-autonomous ODE system, the function $\psi(t)$, which is the fundamental matrix of the system, described as:

$$\frac{d\psi}{dt} = \mathbf{A}(t)\,\psi, \quad \psi(0) = I\ .$$

5.2 Solution of General Scalar Linear SDEs

As preparation for the study of the vector – valued cause, let us investigate first the case d=1 (with *m* arbitrary) of the equation

$$(*)\begin{cases} dX_t = A(t)X_t + \alpha(t))dt + \sum_{i=1}^{m}\big(B_i(t)X_t + \beta_i(t)\big)dW_t^i \\ X_{t_0} = c. \end{cases}$$

All the quantities in this equation (except $W_t \in R^m$) are scalar functions. Suppose that the coefficients $A(t), \alpha(t), B_i(t)$, and β_i(t) are measurable and bounded on the interval [t0,T], so that there always exists a unique solution X_i, which we shall now determine explicitly.

Equation (⁎) has the solution:

$$X_t = \psi_t(c + \int_{t_0}^{t} \psi_t^{-1}(\alpha(s) - \sum_{i=1}^{m} B_i(s)\beta_i(s))ds + \sum_{i=1}^{m} \int_{t_0}^{t} \psi_t^{-1}\beta_i(s)dW_t^i),$$

where

$$\psi_t = \exp(\int_{t_0}^{t} (A(s) - \sum_{i=1}^{m} B_i s^2/2)ds + \sum_{i=1}^{m} \int_{t_0}^{t} B_i(s)dW_t^i)$$

is the solution of the following homogeneous equation:

$$d\psi_t = A(t)\psi_t dt + \textstyle\sum_{i=1}^{m} B_i(t)\psi_t dW_t^i \,, \ \psi_{t_0} = 1.$$

Proof. Let us use Itô's theorem to show that this process has the stochastic differential (⁎). If we now set $\psi_{t_0} = \exp(Y_t)$ and

$$Z_t = c + \int_{t_0}^{t} e^{-Y_t}(\alpha(s) - \sum_{i=1}^{m} B_i(s)\beta_i(s))ds + \sum_{i=0}^{m} \int_{t_0}^{t} e^{-Y_t}\beta_i(s)dW_t^i,$$

we get $X_i = U(Y_t, Z_t)$ where U is defined by $U(x,y) = e^x y$.

Application of Itô's formula yields

$$dX_t = X_t dY_t + e^{Y_i}dZ_t + \frac{1}{2}\sum_{i=1}^{m} tr(\begin{matrix} X_t & \psi_t \\ \psi_t & 0 \end{matrix})(\quad)dt$$

$$= X_t(A(t) - \sum_{i=1}^{m} B_i(s)^2/2)dt + X_t(\sum_{i=1}^{m} B_i(t)dW_t^i)$$

$$+ (a(t) - \sum_{i=1}^{m} B_i(t)\beta_i(t))dt + \sum_{i=1}^{m} \beta_i(t)dW_t^i$$

$$+ \sum_{i=1}^{m} (X_t B_i \left(\frac{t)^2}{2} + B_i(t)\beta_i(t)\right) dt = \big(A(t)X_t + a(t)\big)dt$$

$$+ \sum_{i=1}^{m} \big(B_i(t)X_t + \beta_i(t)\big)dW_t^i.$$

6. APPLICATIONS

Example 1:

Consider the following stochastic scalar linear differential equation with $\mathbb{R}^1$- valued stochastic process $X_t$and one-dimensional Brownian motion W_t :

$$\begin{cases} dX_t = \lambda X_t dW_t \\ X(0) = 1, \end{cases}$$

where λ is a constant coefficient, $\lambda \in \mathbb{R}$.

Solution:

Let Y: = ln(X)= U(X(t),t)

$dY= U_t dt+ U_X dX +\frac{1}{2} U_{XX}\ G^2 dt$

$dY =0+\frac{1}{X}dX -\frac{1}{2X^2}\lambda^2 X^2 dt =\frac{1}{X}dx -\frac{\lambda^2}{2} dt$

$dY=\frac{1}{X} X.\lambda.dw-\frac{\lambda^2}{2} dt$

$$\int dY = \int \lambda dw - \frac{\lambda^2}{2} dt$$

$Y=\lambda w(t) - \frac{\lambda^2}{2} t + c$

$X= C_1 e^{\lambda w(t)- \frac{\lambda^2}{2}t} \Rightarrow X= C_1 e^{\lambda w(0)- \frac{\lambda^2}{2}.0} =1$

$\xRightarrow{w(0)=0} C_1 e^0=1 \Rightarrow C_1 = 1$

As a result, we have the solution as

$X(t)=e^{\lambda w(t)-\frac{\lambda^2}{2}t}$.

Example 2:

Consider the following stochastic scalar linear differential equation with $\mathbb{R}^1$- valued stochastic process $X_t$and one-dimensional Brownian motion W_t :

$$\begin{cases} dX_t = g(t)X_t dW_t \\ x(0) = 1, \end{cases}$$

where, g(t) is a continuous function (not a random variable).

Solution:

Let,Y:= ln(X)= U(X(t),t)

dy=$U_t$dt+U_XdX +$\frac{1}{2}U_{XX}G^2$dt

dY=0+$\frac{1}{X}$dX -$\frac{1}{2X^2}g^2X^2dt$

=$\frac{1}{X}$.g.Xdw-$\frac{1}{2}g^2$dt

$\int dY = \int(gdw - \frac{1}{2}g^2$dt)

Y=$\int_0^t gdw$- $\frac{1}{2}\int_0^t g^2$ ds

lnX =$\int_0^t gdw$- $\frac{1}{2}\int_0^t g^2$ ds

X(t)=$c_1$exp[$\int_0^t gdw$- $\frac{1}{2}\int_0^t g^2$ ds], c_1=1;

X(t)=exp[$-\frac{1}{2}\int_0^t g^2$ds+$\int_0^t gdw$]

Example 3:

Consider the following stochastic scalar linear differential equation with $\mathbb{R}^1$- valued stochastic process $X_t$and one-dimensional Brownian motion W_t :

$$\begin{cases} dX_t = f(t)X_t dt + g(t)X_t dW_t \\ x(0) = 1, \end{cases}$$

where f(t) and g(t) are continuous functions.

Solution:

Let, Y := ln(X)= U(X(t),t)

dY=$U_t$dt+ $U_X$dx +$\frac{1}{2}U_{XX}G^2$dt

dY=0+$\frac{1}{X}$dX -$\frac{1}{2X^2}G^2dt$

dY =$\frac{1}{X}$.[f.X.dt+ g.X.dw]-$\frac{1}{2X^2}g^2X^2dt$

$\int dY = \int(fdt + gdw - \frac{1}{2}g^2$dt)

Y=$\int_0^1(f - \frac{1}{2}g^2)$ds+$\int_0^t gdw$ = lnX

X(t)= $c_1$exp[$\int_0^t(f - \frac{1}{2}g^2)ds + \int_0^t gdw$], c_1=1;

X(t)= exp[$\int_0^t(f - \frac{1}{2}g^2)ds + \int_0^t gdw$].

Example 4:

Consider the following stochastic scalar linear differential equation with $\mathbb{R}^1$-valued stochastic process $X_t$and one-dimensional Brownian motion W_t :

$$\begin{cases} dX_t = f(t)X_t dt + g(t)X_t dW_t \\ \qquad X(0) = X_0. \end{cases}$$

where f(t) and g(t) are continuous functions.

Solution:

We will try to find a solution depart from previous example as a different approach by using product form of stochastic processes:

X(t)= X_1(t)X_2(t)

where, $$\begin{cases} dX_1 = g(t)X_1 dW \\ \quad X_1(0) = X_0, \end{cases} \qquad (\star)$$

and $$\begin{cases} dX_2 = A(t)dt + B(t)dW \\ \qquad X_2(0) = 1. \end{cases} \qquad (\star\star)$$

By using Ito product rule, we can write the following expressions:

dX =d$(X_1X_2) = X_1 dX_2 + X_2 dX_1 + g(t)X_1 B(t)dt$

dX= g(t)XdW+$(X_1 dX_2 + g(t)X_1 B(t)dt)$

dX = g(t)XdW+ $X_1(A(t)dt$+B(t)dW)+ g(t)X_1B(t)dt

dX = $(A(t)X_1 +$ g(t)B(t)X_1)dt+ [g(t) X +B(t)X_1]Dw.

When we choose, B(t) ≡ 0 $\quad and \quad A(t) \equiv X$(t)d(t) ,

then, equation (⋆⋆) transforms into simple non-random stochastic equation:

$$\begin{cases} dX_2 = f(t)\, X_2 dt \\ X_2(0) = 1 \end{cases} \Rightarrow \int \frac{dX_2}{X_2} = \int f(t)dt \Rightarrow \ln X_2 = \int_0^t f(s)ds$$

$$\Rightarrow X_2(t) = \exp\left(\int_0^t f(s)ds\right)$$

Now we try to solve (⋆) $\begin{cases} dX_1 = g(t)X_1 dW \\ X(0) = X \end{cases}$

Let Y:= $\ln\big(X_1(t)\big) = u(X_1(t), t)$

dY =utdt+uX_1d$X_1 + \frac{1}{2}$u$X_1 X_1 G^2$dt

dY =0+$\frac{1}{X_1}$d$X_1 - \frac{1}{2X_1{}^2}. g^2. X_1{}^2$dt

dY =$\frac{1}{X_1}. g. X_1 dW - \frac{1}{2} g^2 dt$

Y(t)= $c_0 + \int_0^t g(s)dw(s) - \frac{1}{2}\int_0^t g^2(s)ds = ln(X_1(t))$

$X_1(t) = x_0$exp[$\int_0^t g(s)dw(s) - \frac{1}{2}\int_0^t g^2$(s)ds]

By combining $X_1(t)$ and $X_2(t)$ in the product form, we can obtain the desired solution as follows:

X(t)= $x_0$exp[$\int_0^t (f(s) - \frac{1}{2} g^2$(s))ds+$\int_0^t g(s)dw(s)$]

Example 5:

Consider the following SDE, which is related with stock prices. Let ℙ(t) represents the price of a stock at time t and $\frac{d\mathbb{P}}{\mathbb{P}}$ is the relative change of price model of P(t):

$$\begin{cases} \frac{d\mathbb{P}}{\mathbb{P}} = \mu dt + \sigma dW \\ \mathbb{P}(0) = p_0, \end{cases}$$

where, $\mu > 0$ and σ are constant coefficients, called drift and volatility of the stock, respectively.

Solution:

Let's write the above SDE as

$d\mathbb{P}$=$\mu\mathbb{P}$dt + $\sigma\mathbb{P}$dw $(*)$

Let, Y := ln($\mathbb{P}$)= U($\mathbb{P}$ (t),t)

dY = $U_t$dt+$U_{\mathbb{P}}$d$\mathbb{P}$+$\frac{1}{2}U_{\ \mathbb{P}\mathbb{P}}G^2$dt

dY =0+$\frac{1}{\mathbb{P}}$d $\mathbb{P}$ -$\frac{1}{2\mathbb{P}^2}G^2dt$

dY =$\frac{1}{\mathbb{P}}$.[μ. $\mathbb{P}$.dt+ σ. $\mathbb{P}$.dw]-$\frac{1}{2\mathbb{P}^2}\sigma^2\mathbb{P}^2dt$

$\int dY = \int(\mu dt + \sigma dw - \frac{1}{2}\sigma^2$dt)

Y= $\mu t + \sigma w(t) - \frac{\sigma^2}{2}$t

Y = $ln\mathbb{P} \implies \mathbb{P}(\text{t}) = p_0\text{exp}\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)\right],\ p_0 > 0.$

Equation $(*)$ implies $\mathbb{P}(t) = p_0 + \int_0^t \mu\mathbb{P}ds + \int_0^t \sigma\mathbb{P}dW$,

then take expectation both side of above equality, we can easily observe that

$E\left(\int_0^t \sigma\mathbb{P}dW\right) = 0$ and $E\big(\mathbb{P}(t)\big) = p_0 + \int_0^t \mu E(\mathbb{P}(s))\,ds.$

Thus, $E\big(\mathbb{P}(t)\big) = p_0\text{exp}(\mu t)$ for $t \geq 0$.

Consequently, the expected value of the stock price is equivalent to the deterministic solution of (*) corresponding to $\sigma = 0$.

Example 6:

Consider the following SDE, which is called "Brownian Bridge":

$$\begin{cases} dB = -\frac{B}{1-t}dt + dW \\ B(0) = 0. \end{cases} \qquad (0 \leq t < 1)$$

Solution:

Let Y:= $\frac{B}{1-t} = U(B(t), t)$

dY = $U_t$dt+U_BdB +$\frac{1}{2}U_{BB}G^2$dt

dY=$\frac{B}{(1-t)^2}$dt+$\frac{1}{1-t}$.[-$\frac{B}{1-t}$dt+dw]=$\frac{dw}{1-t}$

$\int dY = \int \frac{dw}{1-t} \Longrightarrow Y = \int_0^t \frac{1}{1-s}$dw(s)+$c_0$

$\Longrightarrow \frac{B(t)}{1-t}$ = $\int_0^t \frac{1}{1-s}$dw(s)+ c_0

B(t)=(1-t)$\int_0^t \frac{1}{1-s}$ dw(s)+c_0, $c_0 = 0$

B(t)=(1-t)$\int_0^t \frac{1}{1-s}$ dw(s).

Example 7:

Consider the following linear SDE, which is called "Langevin Equation" with $\mathbb{R}^1$-valued stochastic process $X_t$and one-dimensional Brownian motion W_t:

$$\begin{cases} \mathrm{dX}_t = -\alpha X_t dt + \sigma dW_t \\ \mathrm{X}(0) = X_0 \end{cases}$$

,

where, α, σ and $X_0$are real constants.

Solution:

Let, $Y := e^{\alpha t}X(t) = U(X(t),t)$

dY = $U_t$dt+U_XdX +$\frac{1}{2}U_{XX}G^2$dt

dY= $\alpha e^{\alpha t}$X(t)dt+ $e^{\alpha t}dX + 0$

dY= $\alpha e^{\alpha t}$X(t)dt+$e^{\alpha t}$[-$\alpha X(t)dt + \sigma dw(t)$]

$\int dY = \int \sigma e^{\alpha t}$dw(t) $\Longrightarrow Y = \int_0^t \sigma e^{\alpha s}$dw(s)+$c_0$

$e^{\alpha t}$X(t)=$\int_0^t \sigma e^{\alpha s}$dw(s)+$c_0$

$e^{\alpha 0}$X(0)=$c_0 = X_0$

$e^{\alpha t}$X(t)= $X + \int_0^t e^{\alpha s}\, \sigma dw(s)$

X(t)= $X_0 e^{-\alpha t}$+$e^{-\alpha t} \int_0^t e^{\alpha s}\, \sigma dw(s)$

X(t)= $X e^{-\alpha t}$+ $\sigma \int_0^t e^{-\alpha(t-s)}\, dw(s)$

Example 8:

Consider the following linear SDE, which is called "Ornstein-Uhlenbeck Equation" associated with second order stochastic process $Y(t)$and white noise $\xi(t)$:

$$\begin{cases} \ddot{Y}(t) = -b\dot{Y}(t) + \sigma\xi(t) \\ Y(0) = Y_0, \qquad \dot{Y}(0) = Y_{1,} \end{cases}$$

where, b> 0 is the friction coefficient and σ is the diffusion coefficient.

Solution:

Let's get assumption X(t):= $\dot{Y}(t)$ the velocity process, satisfies the Langevin Equation:

$$\begin{cases} dX(t) = -bX(t) + \sigma dW(t) \\ X(0) = Y_1. \end{cases}$$

We know the above equation from Example 7, so we get X(t) solution as follow:

X(t)=$e^{-bt}.Y_1$+ $\sigma. \int_0^t e^{-b(t-s)}$ dW(s).

Take the integral on assumption, we get:

Y(t)= Y_0+$\int_0^t Xds,$ so we obtain the following:

Y(t)=Y_0 +$\int_0^t [\, e^{-bt}.Y_1 + \sigma. \int_0^t e^{-b(t-s)}$ dW(s)]dr

Example 9:

Verify that, X_t = Sin (W_t) with W_0 = a ∈ $(-\frac{\pi}{2}, \frac{\pi}{2})$ solves,

dX_t = -$\frac{1}{2}X_t$dt + $\sqrt{1 - {X_t}^2}$dW_t

Solution:

Let, $dX_t = d(\sin W_t) = \cos W_t dW_t - \frac{1}{2}\sin W_t\, dt$

$X_t = \int_0^t \cos W_t\, dW_s - \int_0^t \frac{1}{2} X_s ds$

Example 10:

Consider the following SDE, which is called "Cox-Ingesall-Ross (CIR) Interest Rate Model":

$dR(t)=(\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}\, dW(t),$

where α, β and σ are positive constants.

Solution:

Let, $R(t)=e^{-\beta t}U(t)$ with $U(0)=R(0)$,

$dR(t)= - \beta e^{-\beta t}U(t)dt + e^{-\beta t}dU(t),$

$dR(t) = -\beta R(t)dt + e^{-\beta t}dU(t),$

$e^{-\beta t}dU(t)= \alpha dt + \sigma\sqrt{R(t)}dW(t).$

Thus,

$U(t) - U(0)= \alpha \int_0^t e^{\beta s}ds+ \sigma \int_0^t e^{\beta s} \sqrt{R(s)}dW(s),$

$U(t)=R(0)+\frac{\alpha}{\beta}(e^{\beta t}-1)+ \sigma \int_0^t e^{\beta s}\sqrt{R(s)}dW(s),$

$R(t)= e^{-\beta t}R(0)+\frac{\alpha}{\beta}(1-e^{-\beta t})+ \sigma e^{-\beta t} \int_0^t e^{\beta s}\sqrt{R(s)}dW(s).$

Example 11:

Consider the following general type linear stochastic differential equation with $\mathbb{R}^1$-valued stochastic process $X_t$and one-dimensional Brownian motion W_t:

$$\begin{cases} dX = (c(t) + d(t)X)dt + (e(t) + f(t)X)dW \\ X(0) = X_0 \end{cases} \qquad (\dagger)$$

Solution:

We can use product form of stochastic processes as follow:

$X_t = \int_0^t \sqrt{1 - sin^2}\, W_s dW$ - $\frac{1}{2}\int_0^t X_s$ds

$X_t = \int_0^t \sqrt{1 - {X_s}^2} dW_s - \frac{1}{2}\int_0^t X_s$ds

$dX_t = -\frac{1}{2}X_t$dt + $\sqrt{1 - {X_t}^2}$ dW_t dt

Example 12:

Consider the following linear SDE, which is called "Vasicek Interest Rate Model":

dR(t)=($\alpha - \beta R(t))dt + \sigma dW(t)$,

where, R(t) interest rate process, α, β and σ are positive constants.

Solution:

Let, dR(t)=- $\beta R(t)dt$, $\alpha = 0,\ \sigma = 0$

R(t)=R(0)$e^{-\beta t}$.

Then, R(t)=$e^{-\beta t}.V(t)$ $with\ V(0) = R(0)$.

dR(t)= - $\beta e^{-\beta t}.V(t)dt + e^{-\beta t}.dV(t)$

dR(t)= - $\beta R(t)dt + e^{-\beta t}.dV(t)$, so

$e^{-\beta t}.dV(t)$= $\alpha dt + \sigma dW(t)$

V(t)-V(0)= $\alpha.\int_0^t e^{\beta s}$ds+ $\sigma \int_0^t e^{\beta s}$dW(s)

V(t)=R(0)+$\frac{\alpha}{\beta}$($e^{\beta t}$-1)+ $\sigma \int_0^t e^{\beta s}$dW(s)

Consequently, we get;

R(t)= $e^{-\beta t}R(0) + \frac{\alpha}{\beta}$(1-$e^{-\beta t}$) + $\sigma e^{-\beta t}\int_0^t e^{\beta s}$dW(s).

X(t) = X_1(t).X_2(t).

Now, main SDE (†) transforms into two Adjoint SDEs:

(Adjoint SDE-1): $\begin{cases} dX_1 = d(t)X_1 dt + f(t)X_1 dW \\ X_1(0) = 1, \end{cases}$

and

(Adjoint SDE-2): $\begin{cases} dX_2 = A(t)dt + B(t)dW \\ X_2(0) = X_0, \end{cases}$

By using Ito product rule,

dX(t) = $X_2 dX_1 + X_1\, dX_2$+ f(t)X_1B(t)dt,

dX(t) = $X_2[d(t)\, X_1$dt + f(t) $X_1 dW]$ + X_1[A(t)dt + B(t)dW] + f(t)X_1B(t)dt,

dX(t) = [Xd(t) + X_1A(t) + f(t)X_1B(t)]dt + [f(t)X + B(t)X_1]Dw,

f(t)X + B(t)X_1 = e(t) + f(t)X,

Here, B(t) := e(t).$(X_1(t))^{-1}$.

Xd(t) + X_1A(t) + f(t) X_1B(t) = C(t) + Xd(t),

X_1A(t) + f(t). $X_1 . e(t)(X_1)^{-1} = c(t)$,

X_1A(t) + f(t).$e(t) = c(t)$,

Here, A(t) := $[c(t) - f(t)e(t)] . (X_1(t))^{-1}$.

Now, we can solve the following Adjoint SDE-1:

$$\begin{cases} dX_1 = d(t)X_1 dt + f(t)X_1\, dW \\ X_1(0) = 1 \end{cases}$$

Let, Y: = ln(X) = U(X(t), t),

dY = $U_t$dt + UX_1dX_1 + $\frac{1}{2}$ U$X_1 X_1 G^2$dt,

dY = 0 + $\frac{1}{X_1}$dX_1 - $\frac{1}{2X_1{}^2}$. $f^2 X_1{}^2$dt,

dY = $\frac{1}{X_1}$[d(t) X_1dt + f(t) X_1dW] - $\frac{1}{2} f^2$dt,

7. CONCLUSION

In this thesis, stochastic linear differential equations and applications has been studied.Moreover , Ito's stochastic differential equations and some properties of Ito Formula and Ito product rule have been discussed. Ito's Formula has been displayed for finding exact solutions to certain stochastic differential equations. Also, Ito Lemma has been shown how it can be applied to solve linear stochastic differential equations analytically. Furthermore, existence and uniquness theorems has been explained for SDEs and exlicit solution of general scalar linear for SDEs has been proved. Then, several examples have been given to illustrate Ito's Formula in applications.

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