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AN EQUIVALENT DEFINITION OF LATTICE IMPLICATION ALGEBRAS

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ABSTRACT

AN EQUIVALENT DEFINITION OF LATTICE IMPLICATION ALGEBRAS

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In this thesis, we give an equivalent definition of a lattice implication algebra (LIA) via a reduced number of axioms, and hereby we think we have significantly simplified the widely-accepted definition of an LIA.

Key Words: Lattice, Boolean Algebras, Order – Reversing Involution, Lattice Implication Algebras.

ÖZ

KAFES İMPLİKASYON CEBİRLERİNİN DENK BİR TANIMI

Sertoğlu Berken
Yüksek Lisans, Mathematics
Danışman: Prof. Dr. Mehmet Terziler

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Bu tezde, aksiyomların sayısını indirgeyerek bir kafes implikasyon cebirinin denk bir tanımını veriyoruz ve bu şekilde bir LIA' nın her zaman kabul gören tanımını önemli ölçüde sadeleştirdiğimizi düşünüyoruz.

Anahtar Kelimeler: Kafesler, Boole Cebirleri, Sıra – Tersleyen Involüsyon, Kafes İmplikasyon Cebirleri.

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Also, I am grateful to my parents for being encouraging, loving and caring to me for my entire life.

Berken SERTOĞLU
İzmir, 2019

TEXT OF OATH

I announce and assure that my study, Master's Thesis and titled "An Equivalent Definition of Lattice Implication Algebras", has been written in accordance with ethics and tradition of science. Therefore, I declare that all content and ideas obtained directly or indirectly from other sources are shown in the text and listed in the reference list.

Berken SERTOĞLU
Signature
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September 6, 2019

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SYMBOLS AND ABBREVIATIONS

ABBREVIATIONS:

LIA	Lattice Implication Algebra
TI	Trellis Implicational

SYMBOLS:

$\leq$	less than or equals to
$=$	equals
$\mathcal{N}(G)$	normal group
$\in$	element
$\subseteq$	subset
$\{\ldots\}$	curly braces
$<$	less than
$\prec$	coverlug
a'	not a
$a \vee b$	a or b
$a \wedge b$	a and b
$\mapsto$	elements correspondence
$\preccurlyeq$	binary relation

1. INTRODUCTION

Reasoning on the classical two – valued logic is based on certainty. Natural extensions of this logic deal with undercertainties, vagueness, and fuzziness; this kind of logics are many – valued, i.e, non –classical, logics. Among them, the lattice implication algebra is a logic system with truth value in a lattice, lattice – valued logic, based on an implication algebra. That is why, a lattice implication algebra is an alternative logic for knowledge representation and reasoning; more precisely, it is a combination of an algebraic lattice and an implication algebra for which the first axiomatization is proposed in Xu Y. In this logic, the lattice is applied to define uncertainties, and particularly in comparability whereas the operation, →, of the implication algebra aims to demonstrate the way human reasons. For further information, see Xu, Y. Ruan D., Qin K ., Liu J.

Different but equivalent definitions of an LIA can be formulated; see, for instance, song Li – Xia. In this study, we give a new definition of an LIA with only four axioms equivalent to generally accepted axiomatization as in Jun Y. B.

BASIC CONCEPTES, LATTICE and BOOLEAN ALGEBRAS

This chapter includes concepts and structures that facilitate the readability of the thesis. The proofs of very few stated results have been given. ''Givant - Halmos Introduction to Boolean algebras'' can be consulted as the main source.

1.1 POSETS

Definition 1.1.1

Let S be a nonempty set and $\leq$ be a binary relation on S. Afterwards S is a *partially order set* or a *poset*, represented by $(S, \leq)$ if properties written below hold for any $s_1, s_2, s_3 \in S$:

(r) $\leq$ is *reflexive*: $s_1 \leq s_1$.

(a) $\leq$ is *antisymmetric*: $s_1 \leq s_2$ and $s_2 \leq s_1$ implies $s_1 = s_2$.

(t) $\leq$ is *transitive*: $s_1 \leq s_2$ and $s_2 \leq s_3$ implies $s_1 \leq s_3$.

Definition 1.1.2

Given a poset $(S, \leq)$. If $s_1 \leq s_2$ or $s_2 \leq s_1$ for any s_1, s_2 in S, then the elements s_1 and s_2 are said to be *comparable*. A poset that satisfies this condition is named a *linearly ordered set* or *totally ordered set*, and the relation $\leq$ a *linear* or *total order*, or *chain*.

Examples 1.1.3

(a) If $\leq$ is defined as the division relation and $\mathbb{Z}^+$ denotes the set of all nonnegative integers, then $(\mathbb{Z}^+, \leq)$ is a linearly ordered set. Note that $(\mathbb{Z}, \leq)$ is not even a poset since $\leq$ is not antisymmetric: For a, b in $\mathbb{Z}$, $a \leq b$ means $a \mid b$ and $b \leq a$ means $b \mid a$. But it does not follow that $a = b$; indeed, we have $2 \mid -2$ and $-2 \mid 2$, but -2 is not equal to 2.

(b) For the set of $\mathcal{N}(G)$ normal subgroups of the group G. If $\leq$ is defined as the inclusion relation $\subset$, then $(\mathcal{N}(G), \leq)$ is not a poset. For any $H \in \mathcal{N}(G)$, $H \leq H$ since we have $hH = Hh$ for all $h \in H$, so $\leq$ is reflexive. $\leq$ is also antisymmetric because for any H_1, H_2 in $\mathcal{N}(G)$, if $H_1 \leq H_2$ then $H \subset K$ and if $H_2 \leq H_1$ then $H_2 \subseteq H_1$, hence $H_1 = H_2$. That $\leq$ is not transitive is left as an exercise. Drawings of finite posets can be made under the name of Hasse diagram.

Definition 1.1.4

An element x in a poset $(S, \leq)$ *covers* an element $s_1 \in S$ if $s_1 < x$ and $s_1 < s_2 < x$ does not hold for any s_2 in S, and this is written as $s_1 \prec x$. A *Hasse diagram* of a poset $(S, \leq)$ displays the elements in this way when $s_1 \prec x$, s_1 and x are joined by an upward line from s_1 to x.

Definition 1.1.5

Given a poset $(S, \leq)$ and a subset A of S. Then $u \in S$ is an *upper bound* for A if $a \leq u$ for all $a \in A$. The element $v \in S$ is a *least upper bound* (lub) of A if v is an upper bound for A and $v \leq w$ for any upper bound w for A . Similarly, a lower bound and the greatest lower bound (glb) are described as A.

The lub of a set A is named as the *supremum*, denoted supA, and the glb of A is also called *infimum*, denoted inf A.

Proposition 1.1.6

Given a poset $(S, \leq)$ and a subset A of S. Then if it exists,

(a) Sup A is unique, and

(b) Inf A is unique.

Proof

We prove only part (a); part (b) is thrown in a similar way.
Suppose that A has two lubs, u_1 and u_2. Then, each of them is an upper bound for A. Since u_1 is an lub for A, we have $u_2 \leq u_1$. Similarly, u_2 is an lub for A, hence $u_1 \leq u_2$. Then, by antisymmetry of $\leq$, we get $u_1 = u_2$, and therefore, sup A is unique. □

For $S_1 = \{a, b\}$ and $S_2 = \{a, b, c\}$, the following are the Hasse Diagrams of the posets $(\mathcal{P}(S_1), \subseteq)$ and $(\mathcal{P}(S_2), \subseteq)$, respectively,

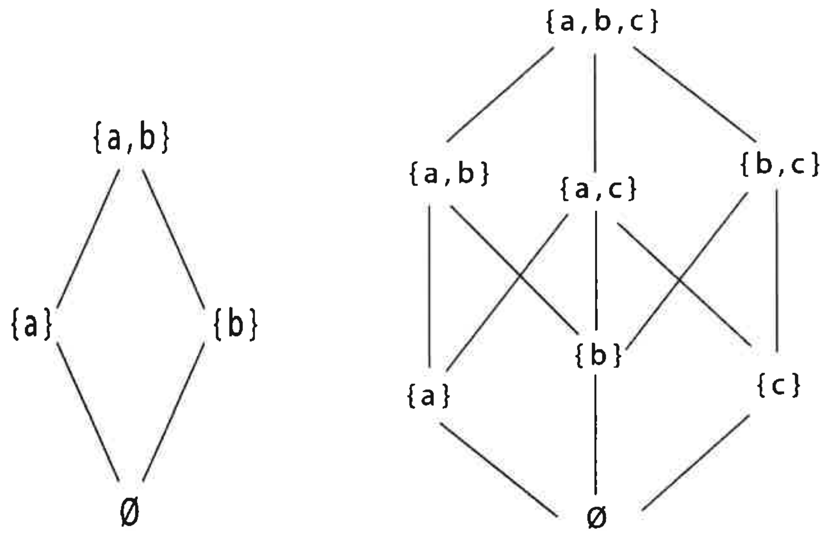

Figure 1.1

1.2 Lattices

A lattice can be defined in two ways: algebraically and relationally. Cleary, in a poset any two elements or a subset of the poset may not possesses a lub or glb. A lattice is a special poset.

Definition 1.2.1

A *lattice* is a poset $(\mathcal{L}, \leq)$ such that for any ℓ_1, ℓ_2 in L, sup $\{\ell_1, \ell_2\}$ and inf $\{\ell_1, \ell_2\}$ exist. Sup $\{\ell_1, \ell_2\}$ is denoted $\ell_1 \vee \ell_2$ and is read as ' ℓ_1 join ℓ_2 ' or ' ℓ_1 or ℓ_2'. Inf $\{\ell_1, \ell_2\}$ is denoted $\ell_1 \wedge \ell_2$ and is read as ' ℓ_1 meet ℓ_2 ' or ' ℓ_1 and ℓ_2 '. If each subset of $(\mathcal{L}, \leq)$ has a sup and an inf, then the lattice is *complete*.

Definition 1.2.2

Given a lattice $(\mathcal{L}, \leq)$.

(i) A *bottom* element of $\mathcal{L}$, denoted $\perp$, satisfies $1 \leq \ell_1$ for all $\ell_1 \in \mathcal{L}$. A bottom element is usually called a *zero*, denoted 0.

(ii) A *top* element of $\mathcal{L}$, denoted T, satisfies $\ell_1 \leq T$ for all $\ell_1 \in \mathcal{L}$. A top element is usually called a *unity*, denoted 1.

(iii) If an element $\ell_1' \in \mathcal{L}$ satisfies $\ell_1 \vee \ell_1' = 1$ and $\ell_1 \wedge \ell_1' = 0$, then it is called the *complement* of ℓ_1.

Proposition 1.2.3

Let $\mathcal{L} \times \mathcal{M} = \{ (\ell_1, m_1) \mid \ell_1 \in \mathcal{L}, b \in \mathcal{M} \}$ where $\mathcal{L}$ and $\mathcal{M}$ are lattices. If we define the relation $\leq$ on $\mathcal{L} \times \mathcal{M}$ by $(\ell_1, m_1) \leq (\ell_2, m_2)$ if and only if $\ell_1 \leq \ell_2$ in L and $m_1 \leq m_2$ in M, then $(\mathcal{L} \times \mathcal{M}, \leq)$ is a lattice.

Proof

That $\mathcal{L} \times \mathcal{M}$ is a poset can be easily established. Show that any two elements of $\mathcal{L} \times \mathcal{M}$ have sup and an inf.

Since $\mathcal{L}$ is a lattice, then $\ell_1 \vee \ell_2 \in \mathcal{L}$ and $\ell_1 \wedge \ell_2 \in L$ for any ℓ_1, ℓ_2 in $\mathcal{L}$. Similarly for $\mathcal{M}$: $m_1 \vee m_2 \in \mathcal{M}$ and $m_1 \wedge m_2 \in \mathcal{M}$ for any b, m_2 in $\mathcal{M}$. Then $(\ell_1, m_1) \vee (\ell_2, m_2) = (\ell_1 \vee \ell_2, m_1 \vee m_2) \in L \times M$ and $(\ell_1, m_1) \wedge (\ell_2, m_2) = (\ell_1 \wedge \ell_2, m_1 \wedge m_2) \in \mathcal{L} \times \mathcal{M}$; in other words, sup and inf exist.
Therefore, L x M is a lattice.

Theorem 1.2.4

Let $(\mathcal{L}, \leq)$ be a lattice. Then for any ℓ_1, ℓ_2, ℓ_3 in $\mathcal{L}$,

(i) $\ell_1 \vee \ell_2 = \ell_2 \vee \ell_1$ and $\ell_1 \wedge \ell_2 = \ell_2 \wedge \ell_1$ (Commutativity)

(ii) $\ell_1 \vee (\ell_2 \vee \ell_3) = (\ell_1 \vee \ell_2) \vee \ell_3$ and $\ell_1 \wedge (\ell_2 \wedge \ell_3) = (\ell_1 \wedge \ell_2) \wedge \ell_3$ (Associativity)

(iii) $\ell_1 \vee \ell_1 = \ell_1$ and $\ell_1 \wedge \ell_1 = \ell_1$ (Idempotenty)

(iv) $\ell_1 \vee (\ell_1 \wedge \ell_2) = \ell_1$ and $\ell_1 \wedge (\ell_1 \vee \ell_2) = \ell_1$ (Absorption)

Duality Principle 1.2.5

Any statement that is true for a lattice continues to be true if

(a) $\leq$ and $\geq$ are interchangeable throughout the statement, and

(b) $\vee$ and $\wedge$ are interchangeable throughout the statement.

(c) If the lattice has a zero 0 and a unity 1, then 0 and 1 are interchangeable throughout.

Note that the proof of Theorem 1.2.4 can be done by crossing between $\vee$ and $\wedge$, by the Duality Principle.

Theorem 1.2.6

Let $\mathcal{L}$ be a nonempty set and let $\vee$ and $\wedge$ have the properties of Theorem 1.2.4. If we define $\leq$ on $\mathcal{L}$ by $\ell_1 \leq \ell_2$ if and only if $\ell_1 \vee \ell_2 = \ell_2$ or $\ell_1 \wedge \ell_2 = \ell_1$, then $(\mathcal{L}, \leq)$ is a lattice. □

Definition 1.2.7

Let $(\mathcal{L}, \vee, \wedge)$ be a lattice. If for all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$,

$$\ell_1 \vee (\ell_2 \wedge \ell_3) = (\ell_1 \vee \ell_2) \wedge (\ell_1 \vee \ell_3)$$

or

$$\ell_1 \wedge (\ell_2 \vee \ell_3) = (\ell_1 \wedge \ell_2) \vee (\ell_1 \wedge \ell_3)$$

hold, then $\mathcal{L}$ is said to be a *distributive lattice.*

Proposition 1.2.8

A lattice $\mathcal{L}$ is distributive if and only if the operations $\vee$ and $\wedge$ are left – cancelable, i.e. if and only if for all $\ell_1, \ell_2, \ell_3 \in$ L, $\ell_1 \vee \ell_2 = \ell_1 \vee \ell_3$ and $\ell_1 \wedge \ell_2 = \ell_1 \wedge \ell_3$ implies $\ell_2 = \ell_3$.

Proof

Let $\mathcal{L}$ be a distributive lattice. It follows as:

$\ell_2 = \ell_2 \wedge (\ell_2 \vee \ell_1)$	(Absorption)
$= \ell_2 \wedge (\ell_3 \vee \ell_1)$	(Hypothesis)
$= \ell_2 \wedge (\ell_1 \vee \ell_3)$	($\vee$ is commutative)
$= (\ell_2 \wedge \ell_1) \vee (\ell_2 \wedge \ell_3)$	(Distribution)
$= (\ell_1 \wedge \ell_2) \vee (c \wedge \ell_2)$	(Commutativity)
$= (\ell_1 \wedge \ell_3) \vee (\ell_3 \wedge \ell_2)$	(Hypothesis)
$= (\ell_3 \wedge \ell_2) \vee (\ell_3 \wedge \ell_2)$	(Commutativity)
$= \ell_3 \wedge (\ell_1 \vee \ell_2)$	(Distributions)
$= \ell_3 \wedge (\ell_1 \vee \ell_3)$	(Hypothesis)
$= \ell_3 \wedge (\ell_3 \vee \ell_1)$	(Absorption)
$= \ell_3.$	

Thus, if $\mathcal{L}$ is distributive $\ell_1 \vee \ell_2 = \ell_1 \vee \ell_3$ and $\ell_1 \wedge \ell_2 = \ell_1 \wedge \ell_3$ imply $\ell_2 = \ell_3$.

Now, suppose that $\mathcal{L}$ is left – cancelable.

Start from $(\ell_1 \vee \ell_2) \wedge (\ell_1 \vee \ell_3)$ to arrive at $\ell_1 \vee (\ell_2 \wedge \ell_3)$.

$$
\begin{aligned}
(\ell_1 \vee \ell_2) \wedge (\ell_1 \vee \ell_3) &= (\ell_1 \vee \ell_3) \wedge (\ell_1 \vee \ell_3) && \text{(Hypothesis)}\\
&= \ell_1 \vee \ell_3 && \text{(Idempotent)}\\
&= \ell_1 \vee (\ell_3 \wedge \ell_3) && \text{(Since } \ell_2 = \ell_3 \text{)}\\
&= \ell_1 \vee (\ell_2 \wedge \ell_3). \; \square \; \square
\end{aligned}
$$

1.3 Boolean Algebras

Boolean algebras are one of the most commonly - used structures in mathematics; they constitute primary examples given in the branches of computer science, classical logic, topology etc.

Definition 1.3.1

A *Boolean Algebra* is a structure $(\mathcal{B}, \wedge, \vee, ', 0, 1)$ of type $< 2, 2, 1, 0, 0 >$ with the distinguished elements 0 and 1, called the bottom and top, meeting the axioms below for all $b_1, b_2 \in \mathcal{B}$:

(i) $\wedge$ and $\vee$ are commutative, associative and distributive over each other;

(ii) $b_1 \wedge (b_1 \vee b_2) = b_1$ and $b_1 \vee (b_1 \wedge b_2)$ (Absorption);

(iii) $b_1 \wedge 0 = 0$ and $b_1 \vee 1 = 1$;

(iv) $b_1 \vee b_1' = 1$ and $b_1 \wedge b_1' = 0$;

Examples 1.3.2

(a) Let $\mathcal{P}$ (S) denote the power set of the set S. Then if we interpret $\wedge$ as set – intersection $\cap$, $\vee$ as set – union $\cup$, $'$ as complementation , and 0 as $\emptyset$ and 1 as S, then $(\mathcal{P}(S), \cap, \cup, ^c, \emptyset, S)$ is clearly a Boolean algebra, called the *power set algebra* of S.

(b) Given a positive integer n , let $\mathcal{D}(n)$ denote the set of positive divisors of n. Assume that n is a product of *distinct* primes. On $\mathcal{D}(n)$, we can define the operations the greatest common divisor, gcd, and the least common multiple, lcm. Now, taking $\wedge$ as gcd and $\vee$ as lcm, distinguished elements as 1 for gcd and n for lcm, i . e., gcd (1 , x) = 1 and lcm (x , n) = n, and finally to each $x \in \mathcal{D}(n)$. We can create a complement n / x, therefore it is demonstrated that $(\mathcal{D}(n), \text{gcd}, \text{lcm}, ', 1, n)$ is a Boolean algebra.

Clearly, operations gcd and lcm are commutative.

Associativity

It suffices to verify it for $\vee$, for instance, by Duality Principle.

$$b_1 \vee (b_2 \vee b_3) = \text{lcm} (b_1 , \text{lcm} (b_2 , b_3))$$
$$= \text{lcm} (\text{lcm} (b_1 , b_2) , b_3)$$
$$= (b_1 \vee b_2) \vee b_3$$

Absorption

$b_1 \wedge (b_1 \vee b_2) = \gcd (b_1 , \text{lcm} (b_1 , b_2)) = b_1$ and $b_1 \vee (b_1 \wedge b_2) = \text{lcm} (b_1 , \gcd (b_1 , b_2)) = b_1$.

Complementation

$b_1 \wedge b_1' = \gcd (b_1 , b_1') = \gcd (b_1 , n/ b_1) = 1$ and $b_1 \vee b_1' = \text{lcm} (b_1 , b_1') = \text{lcm} (b_1 , n / b_1) = n$.

Distribution

We only verify that $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$. Now consider $b_1 \wedge (b_2 \vee b_3) = \gcd (b_1 , \text{lcm} (b_2 , b_3))$. We will try to make a proof by taking exponents on the prime factors of b_1, b_2, and b_3 into account. Let b_1 be the product of prime factors with exponents i, b_2 with j, and b_3 with k; note that i_1, i_2, i_3 can be 0 or 1. Then the exponent of a prime factor in $\gcd (b_1 , \text{lcm} (b_2 , b_3))$ is $\min (i_1 , \max (i_2 , i_3)) = \max (\min (i_1 , i_2), \min (i_1 , i_3))$, which equals the exponent of a prime factor in $\text{lcm} (\gcd (b_1 , b_2), \gcd (b_1 , b_3)) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$. Likewise, $(b_1 \vee (b_2 \wedge b_3)) = (b_1 \vee b_2) \wedge (b_1 \vee b_3)$ can be shown .

We now give several results, well – known, related to the distinguished elements and complementation.

Theorem 1.3.3

Let $\mathcal{B} = (\mathcal{B} , \wedge , \vee , ' , 0 , 1)$ be a Boolean algebra, then

(i) the 0 and 1 elements are unique;

(ii) the complement, b_1', of b_1 in B is unique;

(iii) $(b_1 \wedge b_2)' = b_1' \vee b_2'$ and $(b_1 \vee b_2)' = b_1' \wedge b_2'$

(iv) $((b_1)')' = b_1$

(v) $0' = 1$ and $1' = 0$.

Proof

(ii) (De Morgan's Laws). We only prove

$(b_1 \wedge b_2)' = b_1' \vee b_2'$, the other follows from Duality Principle.

We show that $(b_1 \wedge b_2) \wedge (b_1' \vee b_2') = 0$ and $(b_1 \wedge b_2) \wedge (b_1' \vee b_2') = 1$.

$(b_1 \wedge b) \wedge (b_1' \vee b_2')$

$= [(b_1 \wedge b_2) \wedge b_1'] \vee [(b_1 \wedge b_2) \wedge b_2']$ (Distribution)

$= [(b_1 \wedge b_1') \wedge b_2] \wedge [b_1 \wedge (b_2 \wedge b_2']$ (Commutative and Associativity)

$= (0 \wedge b_2) \vee (b_1 \wedge 0)$

$= (0 \vee 0)$

$= 0$

and

$(b_1 \wedge b_2) \vee (b_1' \vee b_2')$

$= [b_1 \vee (b_1' \vee b_2')] \wedge [b_2 \vee (b_1' \vee b_2')]$ (Distribution)

$= [(b_1 \vee b_1') \vee b_2'] \wedge [(b_2 \vee b_2') \vee b_1']$ (Associative and Commutative)

$= (1 \vee b_2') \wedge (1 \vee b_1')$

$= 1 \vee 1$

$= 1$.

Thus, $b_1' \vee b_2'$ is the complement of $b_1 \wedge b_2$ and by duality, $b_1' \wedge b_2'$ is the complement of $b_1 \vee b_2$. Proofs of the remaining parts are easily done. □

Lemma 1.3.4

Let $\mathcal{B}$ be a Boolean algebra. Then the following are equivalent:

(i) $\ell_1 \leq \ell_2$;

(ii) $\ell_1 \wedge \ell_2' = 0$;

(iii) $\ell_1' \vee \ell_2 = 1$.

Proof

(i) → (ii) If $\ell_1 \leq \ell_2$, then as a Boolean algebra is above all a lattice, it implies $\ell_1 \vee \ell_2 = \ell_2$. Then

$$\begin{aligned} \ell_1 \wedge \ell_2' &= \ell_1 \wedge (\ell_1 \vee \ell_2)' \\ &= \ell_1 \wedge (\ell_1' \wedge \ell_2') && \text{(De Morgan)} \\ &= (\ell_1 \wedge \ell_1') \wedge \ell_2' && \text{(Associativity)} \\ &= 0 \wedge \ell_2' \\ &= 0. \end{aligned}$$

(ii) → (iii) If $\ell_1 \wedge \ell_2' = 0$, then

$$\begin{aligned} \ell_1' \vee \ell_2 &= \ell_1' \vee (\ell_2')' && \text{(Th.1.3.3. (iv))} \\ &= (\ell_1 \wedge \ell_2')' && \text{(De Morgan)} \\ &= 0' \\ &= 1 \end{aligned}$$

(iii) → (i) If $\ell_1' \vee \ell_2 = 1$, then

$$\begin{aligned} \ell_1 &= \ell_1 \wedge 1 && \text{(} \mathcal{B} \text{ is a lattice with 1)} \\ &= \ell_1 \wedge (\ell_1' \vee \ell_2) && \text{(Hypothesis)} \\ &= (\ell_1 \wedge \ell_1') \vee (\ell_1 \wedge \ell_2) && \text{(Distribution)} \\ &= 0 \vee (\ell_1 \wedge \ell_2) \\ &= \ell_1 \wedge \ell_2 && \text{(} \mathcal{B} \text{ is a lattice with 0)} \end{aligned}$$

, which is equivalent to $\ell_1 \leq \ell_2$. Thus, we proved that (i) → (ii) → (iii) → (i). □

Theorem 1.3.5

Let $\mathcal{B}$ be a Boolean Algebra. Then the following conditions hold:

(i) $\ell_1 \leq \ell_2$ if and only if $\ell_2' \leq \ell_1'$;

(ii) $\ell_1 \leq \ell_2'$ if and only if $\ell_1 \wedge \ell_2 = 0$;

(iii) $\ell_1 \leq \ell_2$ if and only if $\ell_1' \vee \ell_2 = 1$.

Proof

(i) Assume $\ell_1 \leq \ell_2$. Then we have

$\ell_1 \wedge \ell_2' = 0$	(Lemma 1.3.4)
$\ell_2' \wedge \ell_1 = 0$	(Commutativitiy)
$\ell_2' \wedge (\ell_1')' = 0$	(Involution)
$\ell_2' \leq \ell_1'$	(Lemma 1.3.4)

Thus, $\ell_1 \leq \ell_2 \rightarrow \ell_2' \leq \ell_1'$.

Now assume that $\ell_2' \leq \ell_1'$. Then

$\ell_2' \wedge (\ell_1')' = 0$	(Lemma 1.3.4)
$\ell_2' \wedge \ell_1 = 0$	(Involution)
$\ell_1 \wedge \ell_2' = 0$	(Commutativity)
$\ell_1 \leq \ell_2$	(Lemma 1.3.4)

Thus (i) is proved.

(ii) If $\ell_1 \leq \ell_2'$, then $\ell_1 \wedge (\ell_2')' = 0$ by Lemma 1.3.4, which implies $\ell_1 \wedge \ell_2 = 0$ by involution. Now, assume that $\ell_1 \wedge \ell_2 = 0$. Then we have by involution $\ell_1 \wedge (\ell_2')' = 0$, which is $\ell_1 \leq \ell_2'$ by Lemma 1.3.4. Thus, (ii) is proved.

(iii) If $\ell_1 \leq \ell_2$, then by Lemma 1.3.4 (iii), $\ell_1' \vee \ell_2 = 1$. Now assume that $\ell_1' \vee \ell_2 = 1$. Then again by Lemma 1.3.4, we have $\ell_1 \leq \ell_2$. Therefore, (iii) also is proved.

Definition 1.3.6

An element x of a Boolean algebra $\mathcal{B}$ is said to be an *atom* if there does not exist $b_1 \in \mathcal{B}$ such that $0 < b_1 < x$. To put it in a different way, if x is an atom, then x covers only the zero element of $\mathcal{B}$.

The set of atoms of $\mathcal{B}$ is represented by At ($\mathcal{B}$). $\mathcal{B}$ is atomless if $\mathcal{B}$ contains no atoms, and atomic if there is an atom for each nonzero element of $\mathcal{B}$.
For any set S, the power set algebra $\mathcal{P}$ (S) is an atomic Boolean algebra; its atoms are the singletons. As we shall prove it below, every finite Boolean algebra is atomic. However, it is in general not easy to contract atomless Boolean algebras.

An example of this is the Boolean algebra F / ~ of equivalence classes of formulas of propositional logic when the set $\mathcal{P}$ of propositinal variables is infinite. For this, one can consult Cori, R. and Lascar, D. We conclude this chapter by giving Representation Theorem of Finite Boolean Algebras.

Theorem 1.3.7 [Representation Theorem for Finite Boolean Algebra]

Given a finite Boolean algebra $\mathcal{B}$, each element b in $\mathcal{B}$ can be expressed as

$$b = \bigvee_{x \in At(\mathcal{B}),\ x \leq b} x$$

where $\bigvee$ denotes the supremum of x.

Furthermore, this expression is unique: if $b = \bigvee_{x \in X} x$ for some subset X of At ($\mathcal{B}$) , then $X = \{ x \in At(\mathcal{B}) \mid x \leq b \}$.

Proof

Suppose that $\bigvee_{x \in X} x = y$. Then, clearly $y \leq b_1$.
Hence $b_1 = (b_1 \wedge y) \vee (b \vee y')$. From this observation, we will show that $b = y$, or $b \wedge y' = 0$. For this, then we have $b_1 = y \vee 0 = y$, and it is accomplished. If $a_1 \in At(\mathcal{B})$ satisfies $a_1 \leq b_1 \wedge y'$, then $a_1 \leq b_1$ and $a_1 \leq x'$, $x \in At(\mathcal{B})$ and $x \leq b$.

In particular, we have $a_1 \leq a_1'$, so $a_1 = a_1 \wedge a_1' = 0$, a contradiction. This shows that there does not exist an atom that is less than or equal to $b_1 \wedge y' = 0$.

Now we show that the expression for b is unique. Assume that $b = \bigvee_{x \in X} x$ for some subset X of $At(\mathcal{B})$. Then that $X \subseteq \{x \in At(\mathcal{B}) \mid x \leq b\}$ is clear. Now if $y \in \{x \in At(\mathcal{B}) \mid x \leq b\} \setminus X$, then we get

$$y = y \wedge b = y \wedge (\bigvee_{x \in X} x) = \bigvee_{x \in X}(y \wedge x) = \bigvee_{x \in X} \perp = 1$$

This is a contradiction, hence the uniqueness of the expression for b is established.

2. LATTICE IMPLICATION ALGEBRAS

Lattice – valued logics are non – classical, many – valued logics in which classical propositional value, truth value of a propositional variable, is given in a lattice. In order to investigate such systems Y. Xu suggested the concept of lattice implication algebras.

These algebras are the basis of the works carried out in the field of artificial intelligence, AI. In this chapter, we reproduce the description of the algebra and some of the results in order to facilitate the readability of the thesis.

2.1 Preliminaries

Following Y. Xu, we define a lattice implication algebra as a structure $(\mathcal{L}, \wedge, \vee, \rightarrow, ', 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$ such that

1. $(\mathcal{L}, \wedge, \vee, \rightarrow, ', 0, 1)$ is a bounded lattice;

2. A unary operator $'$ is an order – reversing involution;

$(\ell_1')' = \ell_1$ and if $\ell_1 \leq \ell_2$ then $\ell_2' \leq \ell_1'$;

3. A binary operation $\rightarrow$ on L, $(\ell_1, \ell_2) \longmapsto \ell_1 \ell_2$

(meant to represent , $\rightarrow$) satisfying the axioms below,

for all $\ell_1, \ell_2, \ell_3 \in L$,

(I_1) $\ell_1(\ell_2 \ell_3) = \ell_2(\ell_1 \ell_3)$,

(I_2) $\ell_1 \ell_1 = 1$,

(I_3) $\ell_1 \ell_2 = \ell_2' \ell_1'$

(I_4) $\ell_1 \ell_2 = \ell_2 \ell_1 = 1 \rightarrow \ell_1 = \ell_2$

(I_5) $(\ell_1 \ell_2) \ell_2 = (\ell_2 \ell_1) \ell_1$,

(L_1) $(\ell_1 \vee \ell_2) z = (\ell_1 z) \wedge (\ell_2 \ell_3)$

(L_2) $(\ell_1 \wedge \ell_2) \ell_3 = (\ell_1 \ell_3) \vee (\ell_2 \ell_3)$

In the lattice structure, we define a relation as $\preccurlyeq$ through

$\ell_1 \preccurlyeq \ell_2 \leftrightarrow \ell_1 \vee \ell_2 = \ell_2$ (or $\leftrightarrow$ $\ell_1 \wedge \ell_2 = \ell_1$).

The implication algebra is defined by:

$\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \ell_2 = 1$.

As we show in chapter 3, these two relations are partial orders. Concerning the relation $\leq$, we have the following result.

Proposition 2.1.1 (Kondo – Dudek)

Let $\mathcal{L}$ be a lattice implication algebra. Then for all $\ell_1, \ell_2, \ell_3 \in$ L, we have

$(\ell_1 \ell_2)((\ell_2 \ell_3)(\ell_1 \ell_3) = 1$.

Proof (See Kondo – Dudek).

Using this result, we can give another proof that $\leq$ is a partial order.

Proposition 2.1.2

Let $\mathcal{L}$ be a lattice implication algebra. Then, the relation $\leq$ is a partial order on $\mathcal{L}$.

Proof

$\leq$ is reflexive by (I_2). $\leq$ is antisymmetric by (I_4).

Now show that $\leq$ is transitive: Let $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_3$. Then from Proposition 2.1.1, we obtain

$1 = (\ell_1 \ell_2)((\ell_2 \ell_3)(\ell_1 \ell_3)) = 1(1(\ell_1 \ell_3)) = \ell_1 \ell_3$,

the last equality follows from proposition 2.1.3 below.

This implies that we have $\ell_1 \leq \ell_3$, i.e, $\leq$ is a partial order.

Proposition 2 .1.3 (Qin , Liu , Xu)

In a lattice implication algebra L, the following hold.

(1) $0\ell_1 = 1$, $1\ell_1 = \ell_1$, $\ell_1 1 = 1$,

(2) $\ell_1{}' = \ell_1 0$

(3) $\ell_1 \vee \ell_2 = (\ell_1 \ell_2)\ell_2$,

(4) $(\ell_1 \ell_2) \vee (\ell_2 \ell_3) = 1$,

(5) $((\ell_2 \ell_1)\ell_2{}')' = \ell_1 \wedge \ell_2 = ((\ell_1 \ell_2)\ell_1{}')'$,

(6) $\ell_1 = \ell_2 \rightarrow \ell_2 \ell_3 = \ell_1 \ell_3$ and $\ell_3 \ell_1 = \ell_3 \ell_2$,

(7) $\ell_1 \leq \ell_2 \rightarrow \ell_2 \ell_3 \leq \ell_1 \ell_3$ and $\ell_3 \ell_1 \leq \ell_3 \ell_2$,

(8) $\ell_3 \leq \ell_2 \ell_3 \leftrightarrow \ell_2 \leq \ell_3 \ell_1$.

Let's prove part (7) as an exercise. Suppose $\ell_1 \leq \ell_2$, that is, $\ell_1 \ell_2 = 1$.

From Proposition 2.1.1, there is $1 = \ell_1 \ell_2 \leq (\ell_2 \ell_3)(\ell_1 \ell_3)$, which implies $\ell_2 \ell_3 \leq \ell_1 \ell_3$.

For the second part, we also have

$$
\begin{aligned}
(\ell_3 \ell_1)(\ell_3 \ell_2) &= 1((\ell_3 \ell_1)(\ell_3 \ell_2)) \\
&= (\ell_1 \ell_2)((\ell_3 \ell_2)(\ell_3 \ell_2)) \\
&= (\ell_3 \ell_1)((\ell_1 \ell_2)(\ell_3 \ell_2)) \\
&= 1,
\end{aligned}
$$

implyings that $\ell_3 \ell_1 \leq \ell_3 \ell_2$.

We shall also prove part (3); after this, it is easily concluded that we have part (5).

From $\ell_1((\ell_1 \ell_2)\ell_2) = (\ell_1 \ell_2)(\ell_1 \ell_2) = 1$ and (I_5), there is $\ell_1 \leq (\ell_1 \ell_2)\ell_2$ and $\ell_2 \leq (\ell_1 \ell_2)\ell_2$. Thus, $(\ell_1 \ell_2)\ell_2$ is an upper bound for not only ℓ_1 but also ℓ_2, which indicates that it is the least upper bound. Let ℓ_3 be any element of L such that $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_3$.

Now from

$$
\begin{aligned}
((\ell_1 \ell_2)\ell_2)\ell_3 &= ((\ell_1 \ell_2)\ell_2)(1\ell_3) \\
&= ((\ell_1 \ell_2)\ell_2)((\ell_2 \ell_3)\ell_3) \\
&= ((\ell_1 \ell_2)\ell_2)((\ell_3 \ell_2)\ell_2)
\end{aligned}
$$

$\geq (\ell_3 \ell_2)(\ell_1 \ell_2)$
$\geq \ell_1 \ell_3 = 1,$

We get $(\ell_1 \ell_2)\ell_2 \leq \ell_3$. It means $(\ell_1 \ell_2)\ell_2$ is the supremum of $\{\ell_1, \ell_2\}$. Hence, $\ell_1 \vee \ell_2 = \sup\{\ell_1, \ell_2\} = (\ell_1 \ell_2)\ell_2$. □

Many other results on lattice implication algebras are abundant in the books Y. Xu, D. Ruan, K. Qin, J. Liu and almost all of these results are proved in great detail.

2.2 Order – Reversing Involution

We shall elaborate the concept of involution because some people seem to have missed essentiality of the concept. Let $(S, \leq)$ be a poset. Attached to the partial order $\leq$, there is its dual, $\leq^*$, which is defined by

$$s_1 \leq^* s_2 \leftrightarrow s_2 \leq s_1.$$

Obviously, enough, $\leq^*$ is also a partial order on S.

Definition 2.2.1

An order – reversing involution on $(S, \leq)$ is any isomorphism

$$f : (S, \leq) \rightarrow (S, \leq^*)$$

such that $f(f(s_1)) = s_1$, for each $s_1 \in S$.

Namely, f is an *involutive bijection* from S onto S such that

$$s_1 \leq s_2 \leftrightarrow f(s_1) \leq^* f(s_2) \leftrightarrow f(s_2) \leq f(s_1).$$

Examples 2.2.2

(a) Consider the following lattices :

$L_1 = \{0, a, 1\}$, $0 < a < 1$;

$L_2 = \{ 0, a, b, 1 \}$, $0 < a < 1$, $0 < b < 1$;

$L_3 = \{ 0, a, b, c, 1 \}$, $0 < a < c < 1$, $0 < b < c < 1$.

Then one can prove the following statements:

(i) There is one and only one order – reversing involution on L_1 which is simply $0 \to 1, a \to a, 1 \to 0$

(ii) There are two different order – reversing involutions on L_2,

$0 \mapsto 1, a \mapsto a, 1 \mapsto 0, b \mapsto b, 1 \mapsto 0,$

$0 \mapsto 1, a \mapsto a, 1 \mapsto 0, b \mapsto b, 1 \mapsto 0.$

(iii) There is no order – reversing involution, whatsoever, on L_3 because the posets $(L_3 , \leq)$ and $(L_3 , \leq^*)$ are simply not isomorphic .

All that can be seen on drawings.

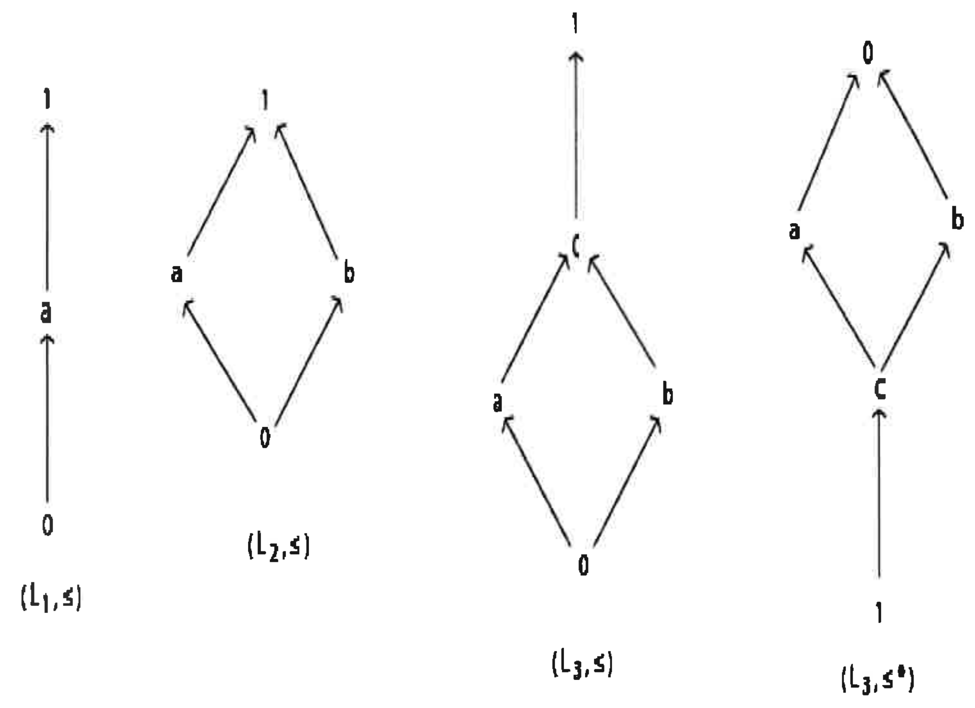

Figure 2.2

(b) The unit interval [0 ,1] of reals, as a poset, has one and only two order – reversing isomorphism, namely, $x \mapsto 1 - x$.

Thus, not all lattices have order – reversing involutions. According to cases, they may have only one, many and not at all.

2.3 Examples of Lattice Implication Algebras

There are abundant examples of lattice implication algebras in the literature; we will copy out some of them here.

(a) Let $(\mathcal{L} , \wedge , \vee , ' , 0 , 1)$ be a Boolean lattice. If $\ell_1 \rightarrow \ell_2$, $\ell_1 \ell_2$, is defined to be $\ell_1' \vee \ell_2$, then $\mathcal{L}$ becomes a lattice implication algebra.

(b) Luckasiewicz implication algebra on the unit interval [0 , 1] of reals is a lattice implication if the operations on [0 ,1] are defined as follows:

$\ell_1 \vee \ell_2 = \max \{ \ell_1 , \ell_2 \}$, $\ell_1 \wedge \ell_2 = \min \{ \ell_1 , \ell_2 \}$, $\ell_1' = 1 - \ell_1$,
$\ell_1 \rightarrow \ell_2 = \ell_1 \ell_2 = \min \{ 1 , 1 - \ell_1 + \ell_2 \}$

(c) Let $\mathcal{L} = \{ 0 , m_1 , m_2 , m_3 , m_4 , 1 \}$ and $0 < m_4 < m_1 < 1$, $0 < m_3 < m_2 < 1$.

Define the operations on $\mathcal{L}$ as below:

$\ell_1 \vee \ell_2 = (\ell_1 \ell_2) \ell_2$, $\ell_1 \wedge \ell_2 = ((\ell_1' \ell_2')')'$,

$0' = 1$, $m_1' = m_3$, $m_2' = m_4$, $m_3' = m_1$, $m_4' = m_2$, $1' = 0$

$\rightarrow$	0	m_1	m_2	m_3	m_4	1
0	1	1	1	1	1	1
m_1	m_3	1	m_2	m_3	m_2	1
m_2	m_4	m_1	1	m_2	m_1	1
m_3	m_1	m_1	1	1	m_1	m_2
m_4	m_2	1	1	m_2	1	1
1	0	m_1	m_2	m_3	m_4	1

Then $(L , \wedge , \vee , ' , 0 , 1)$ is a lattice implication algebra. This example is taken from Y. Xu, D. Ruan, K. Qiu, J. Liu.

However,

$L = \{ 0 , m_1 , m_2 , 1 \}$, $0 < m_1 < m_2 < 1 \}$, $0' = 1$, $m_1' = m_2$, $m_2' = m_1$,

$1' = 0$, $\ell_1 \vee \ell_2 = \max \{ \ell_1 , \ell_2 \}$, $\ell_1 \wedge \ell_2 = \min \{ \ell_1 , \ell_2 \}$

$\rightarrow$	0	m_1	m_2	1

0	1	1	1	1
m_1	m_2	1	m_2	1
m_2	m_1	m_1	1	1
1	0	m_1	m_2	1

is not a lattice implication algebra because the axioms $(L_1) - (L_5)$

are satisfied is $(L, ', \rightarrow, 0, 1)$ but (L_1) fails to hold, because we have

$(m_1 \vee m_2)\, m_2 = m_2\, m_2 = 1$ as well as $(m_1\, m_2) \wedge (m_2\, m_2) = m_2 \wedge 1 = m_2$, hence

$(L, \wedge, \vee, ', 0, 1)$ is not a lattice implication algebra.

3. REDUCED AXIOMATIZATION OF LATTICE IMPLICATION ALGEBRAS

Different but equivalent axiomatizations of lattice implication algebras can be formulated although the number of formulations is very little. In the last chapter of the thesis, we provide a new axiomatization of lattice implication algebras with only four axioms, and hereby we are of the opinion, we have applied significantly simplified and the commonly-accepted axiomatization of a lattice implication algebra.

3.1 Common Definition of Lattice Implication Algebras

From now on, we abbreviate a lattice implication algebra by LIA. The usual definition of an LIA is as follows.

Definition 3.1.1 (Y. Xu)

An LIA, $(L , \wedge , \vee , ' , 0 , 1)$ is a bounded lattice with order – reversing involution ' ' ' together with a binary relation on L, $(\ell_1 , \ell_2) \mapsto \ell_1 \ell_2$ (meant to represent $\rightarrow$) satisfying the following axioms:

(I_1) $\ell_1 (\ell_2 \ell_3) = \ell_2 (\ell_1 \ell_3)$,

(I_2) $\ell_1 \ell_1 = 1$,

(I_3) $\ell_1 \ell_2 = \ell_2' \ell_1'$,

(I_4) $\ell_1 \ell_2 = \ell_2 \ell_1 = 1 \rightarrow \ell_1 = \ell_2$,

(I_5) $(\ell_1 \ell_2) \ell_2 = (\ell_2 \ell_1) \ell_1$,

(L_1) $(\ell_1 \vee \ell_2) \ell_3 = (\ell_1 \ell_3) \wedge (\ell_2 \ell_3)$,

(L_2) $(\ell_1 \wedge \ell_2) \ell_3 = (\ell_1 \ell_3) \vee (\ell_2 \ell_3)$.

Note that (L_1) and (L_2) are equivalent to (L_3) and (L_4), respectively, as we shall prove below:

(L_3) $\ell_3 (\ell_1 \wedge \ell_2) = (\ell_3 \ell_1) \wedge (\ell_3 \ell_2)$,

(L_4) $\ell_3 (\ell_1 \vee \ell_2) = (\ell_3 \ell_1) \vee (\ell_3 \ell_2)$.

3.2 Two Order Relations on LIA

For any LIA, a binary relation is introduced, defined as follows:

$$\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \ell_2 = 1.$$

Proposition 3.2.1

The relation $\leq$ is a partial order.

Proof

That $\leq$ is reflexive follows from axioms (I_2). Assume that we have $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_1$. Then $\ell_1 \ell_2 = 1$ and $\ell_2 \ell_1 = 1$, hence $\ell_1 \ell_2 = \ell_2 \ell_1 = 1$, which implies $\ell_1 = \ell_2$ by axiom (I_4); thus $\leq$ is antisymmetric.

As to transitivity, let $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_3$; then $\ell_1 \ell_2 = 1$ and $\ell_2 \ell_3 = 1$. Now, $1 = \ell_2 \ell_3 = \ell_2 (\ell_1 \ell_2) \ell_3 = \ell_1 (\ell_2 \ell_2) \ell_3 = \ell_1 \ell_3$, so $\ell_1 \leq \ell_3$, and this completes the proof.

The other binary relation is the partial order of the lattice itself, which we denote $\preccurlyeq$, defined by

$$\ell_1 \preccurlyeq \ell_2 \leftrightarrow \ell_2 = \ell_1 \vee \ell_2.$$

That $\preccurlyeq$ is a partial order follows from lattice axioms. It turns out that these two partial orders coincide. Before we prove this very important property, we need to demonstrate some simple results.

Proposition 3.2.2

Let L be a lattice with an order – reversing involution ' ′ '. Then, we have

$$\ell_1' \wedge \ell_2' = (\ell_1 \vee \ell_2)' \quad \text{and} \quad (\ell_1' \wedge \ell_2')' = \ell_1 \vee \ell_2.$$

Proof

Since L is a lattice, $\ell_1' \wedge \ell_2'$ is the greatest lower bound, m of ℓ_1' and ℓ_2';

that is,

$\ell_3 \leq \text{m} \leftrightarrow \ell_1' \leq \ell_3, \ \ell_2' \leq \ell_3.$

The involution reverses orders , therefore

$m' \leq \ell_3' \leftrightarrow \ell_3' \leq \ell_1, \ell_3' \leq \ell_2$.

This last statement amounts to say that m′ is the least upper bound of ℓ_1 and ℓ_2, i.e ., $m' = \ell_1 \vee \ell_2$, or equivalently, $m = (\ell_1 \vee \ell_2)'$. Thus, the following formulas are obtained:

$\ell_1' \wedge \ell_2' = (\ell_1 \vee \ell_2)'$

$(\ell_1' \wedge \ell_2')' = \ell_1 \vee \ell_2$.

Likewise, there are the duals as well:

$\ell_1' \vee \ell_2' = (\ell_1 \vee \ell_2)'$

$(\ell_1' \vee \ell_2')' = \ell_1 \wedge \ell_2$.

Proposition 3.2.3

In an LIA, $(L_1) \leftrightarrow (L_3)$; a similar proof obtains $(L_2) \leftrightarrow (L_4)$.

Starting with (L_1), $(\ell_1 \vee \ell_2) \ell_3 = (\ell_1 \ell_3) \wedge (\ell_2 \ell_3)$, for each ℓ_1, ℓ_2, ℓ_3,

we get

$\ell_3' (\ell_1 \vee \ell_2)' = (\ell_3' \ell_1') \wedge (\ell_3' \ell_2')$ (by (I_3))

$\ell_3' (\ell_1' \wedge \ell_2') = (\ell_3' \ell_1') \wedge (\ell_3' \ell_2')$ (by Proposition 3.2.2)

for each $\ell_1', \ell_2', \ell_3'$, which yields $\ell_3 (\ell_1 \wedge \ell_2) = (\ell_3 \ell_1) \wedge (\ell_3 \ell_2)$.

This shows that we leave $(L_1) \rightarrow (L_3)$.

Proposition 3.2.3

In an LIA, $(L_1) \leftrightarrow (L_3)$ and $(L_2) \leftrightarrow (L_3)$ and $(L_2) \leftrightarrow (L_4)$.

Proof

We only prove $(L_1) \leftrightarrow (L_3)$; a similar proof obtains $(L_2) \leftrightarrow (L_3)$.

Starting with (L_1), $(\ell_1 \vee \ell_2)\ell_3 = (\ell_1 \ell_3) \wedge (\ell_2 \ell_3)$, for each ℓ_1, ℓ_2, ℓ_3, we get

$\ell_3'(\ell_1 \vee \ell_2)' = (\ell_3' \ell_1') \wedge (\ell_3' \ell_2')$ (by (I_3))

$\ell_3'(\ell_1' \wedge \ell_2') = (\ell_3' \ell_1') \wedge (\ell_3' \ell_2')$ (by Proposition 3.2.2)

For each $\ell_1', \ell_2', \ell_3'$, which yields

$\ell_3(\ell_1 \wedge \ell_2) = (\ell_3 \ell_1) \wedge (\ell_3 \ell_2)$.

This shows that we have $(L_1) \leftrightarrow (L_3)$.

Property In a LIA, the two partial orders $\leq$ and $\preccurlyeq$ coincide.

Proof

We show $\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \preccurlyeq \ell_2$. Recall $\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \ell_2 = 1$ and $\ell_1 \preccurlyeq \ell_2 \leftrightarrow \ell_2 = \ell_1 \vee \ell_2$.

On one side, we have

$\ell_1 \preccurlyeq \ell_2 \leftrightarrow \ell_2 = \ell_1 \vee \ell_2 \rightarrow \ell_1 \ell_2 = \ell_1(\ell_1 \vee \ell_2) \overset{L4}{\rightarrow} (\ell_1 \ell_1) \vee (\ell_1 \ell_2) \rightarrow \ell_1 \ell_2 = 1 \vee (\ell_1 \ell_2) \rightarrow \ell_1 \ell_2 = 1 \leftrightarrow \ell_1 \leq \ell_2$.

On the other side, we have

$\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \ell_2 = 1 \overset{L1}{\rightarrow} (\ell_1 \vee \ell_2)\ell_2 = (\ell_1 \ell_2) \wedge (\ell_2 \ell_{12}) = 1$

and

$\ell_2(\ell_1 \vee \ell_2) = (\ell_2 \ell_1) \vee (\ell_2 \ell_2) = 1 \rightarrow (\ell_1 \vee \ell_2)\ell_2 = 1$

and

$\ell_2(\ell_1 \vee \ell_2) = 1 \rightarrow \ell_2 = \ell_1 \vee \ell_2 \leftrightarrow \ell_1 \preccurlyeq \ell_2$.

That completes the proof.

3.3 An Equivalent Definition of an LIA

In this section, we offer an equivalent definition of on LIA which (we have found) is simpler and more natural. All results in this section are our results.

Definition 3.3.1

A 'trellis implicational', [briefly , aTI], is a bounded lattice (L , ∧ , ∨ , 0 , ≤) with an order – reversing involution ' ' ', $x \to x'$, and a multiplication $(\ell_1, \ell_2) \to \ell_1 \ell_2$ which satisfies the following axioms:

(A) $\ell_1 \leq \ell_2 \leftrightarrow \ell_1 \ell_2 = 1$

(B) $\ell_1 \ell_2 = \ell_2' \ell_1'$

(C) $\ell_1 (\ell_2 \ell_3) = \ell_2 (\ell_1 \ell_3)$

(D) $(\ell_1 \ell_2) \ell_2 = (\ell_2 \ell_1) \ell_1$.

We shall prove that it is equivalent to that given in Y. Xu. To see this, we have to show that (I_2), (I_4), (L_1), and (L_2) hold in any TI.

Observe that (I_2) and (I_4) are satisfied via (A). To prove that (L_1) and (L_2) also hold, we need an intermediate result, some of its parts were already stated in proposition 2.1.3.

Lemma 3.3.2

In any TI, the following always hold:

(1) $0 \ell_1 = 1$, $\ell_1 1 = 1$.

(2) $1 \ell_1 = \ell_1$, $\ell_1 0 = \ell_1'$

(3) $((\ell_1 \ell_2) \ell_2) = \ell_1 \ell_2$

(4) $\ell_1 \leq \ell_2 \to (\forall \ell_3) (\ell_2 \ell_3 \leq \ell_1 \ell_3)$,

[i.e., $\ell_1 \ell_2 = 1 \to (\forall \ell_3) ((\ell_2 \ell_3) (\ell_1 \ell_3)) = 1$]

(5) $(\ell_1 \vee \ell_2) \ell_3 \leq (\ell_1 \ell_3) \wedge (\ell_2 \ell_3)$.

(6) $(\ell_1 \vee \ell_2) = ((\ell_1 \ell_2) \ell_2)$.

Proof

(1) Follows from $0 \leq \ell_1 \leq 1$, for each x.

(2) Start with $\ell_1 (1 \ell_1) = 1 (\ell_1 \ell_1) = 11 = 1$, which proves that $1 = 11 = (\ell_1 1) = (1\,\ell_1)\,\ell_1$,

which proves that we $\ell_1 0 = \ell_1'$ follows from (B) (by involution),

since $\ell_1 0 = 0'\,\ell_1' = 1\,\ell_1' = \ell_1'$.

(3) Start with

$$((\ell_1 \ell_2)\ell_2)\ell_2 = (\ell_2(\ell_1\ell_2))(\ell_1\ell_2) \qquad \text{(by (D))}$$

$$= ((\ell_1(\ell_2\ell_2)(\ell_1\ell_2) \qquad \text{(by (C))}$$

$$= (\ell_1 1)(\ell_1\ell_2)$$

$$= 1\,(\ell_1\ell_2)$$

$$= \ell_1\ell_2.$$

(4) We have

$$(\ell_2\ell_3)(\ell_1\ell_3) = \ell_1((\ell_2\ell_3)\ell_3) \qquad \text{(by (C))}$$

$$= \ell_1((\ell_3\ell_2)\ell_2) \qquad \text{(by (D))}$$

$$= (\ell_3\ell_2)(\ell_1\ell_3) \qquad \text{(by (C))}$$

If $\ell_1\ell_2 = 1$, then $(\ell_2\ell_3)(\ell_1\ell_3) = (\ell_3\ell_2)(\ell_1\ell_2) = (\ell_3\ell_2)\,1 = 1$.

(5) We have $\ell_1 \le \ell_1 \vee \ell_2$ and $\ell_2 \le \ell_1 \vee \ell_2$, so $(\ell_1 \vee \ell_2)\ell_3 \le \ell_1\ell_3$ and $(\ell_1 \vee \ell_2)\ell_3 \le \ell_2\ell_3$, and $\ell_2 \le \ell_1 \vee \ell_2$, by (4), therefore $(\ell_1 \vee \ell_2)\ell_3 \le (\ell_1\ell_3) \wedge (\ell_2\ell_3)$.

(6) Proof consists of two part.

Part I

Firstly, $\ell_1 \vee \ell_2 \le (\ell_1\ell_2)\ell_2$ is proven.
Indeed, $\ell_1((\ell_1\ell_2)\ell_2) = (\ell_1\ell_2)(\ell_1\ell_2) = 1$ and $\ell_2((\ell_1\ell_2)\ell_2) = (\ell_1\ell_2)(\ell_2\ell_2) = 1$, by (C), therefore, $\ell_1 \le (\ell_1\ell_2)\ell_2$ and $\ell_2 \le (\ell_1\ell_2)\ell_2$ so that $\ell_1 \vee \ell_2 \le (\ell_1\ell_2)\ell_2$.

Part II

We then prove that $(\ell_1 \ell_2)\ell_2 \le \ell_1 \vee \ell_2$. Indeed, if $\ell_1 \le a$ and $\ell_2 \le a$, then $\ell_1 a = 1$ and $\ell_2 a = 1$, by (A). Then,

$$((\ell_1 \ell_2)\ell_2)a = ((\ell_1 \ell_2)\ell_2(1a)) = ((\ell_1 \ell_2)\ell_2)((\ell_2 a)a)$$
$$= ((\ell_1 \ell_2)\ell_2)((a\ell_2)\ell_2)$$
$$= ((a\ell_2)\ell_2)$$
$$= (a\ell_2)(((\ell_1 \ell_2)\ell_2)\ell_2)$$
$$= (a\ell_2)(\ell_1 \ell_2),$$

using (3) above. Using (4) above, we have $((\ell_1 \ell_2)\ell_2)a = 1$, since $xa = 1$. Therefore, $(\ell_1 \ell_2)\ell_2 \le a$, so that $(\ell_1 \ell_2)\ell_2 \le \ell_1 \vee \ell_2$.

We shall now prove that (L_1) and (L_2) hold in every TI. Here is a proof that (L_1) holds; one can produce a similar proof to show that (L_2) also holds.

Proof

We use Lemma 3.3.2 (5), (6) and the dual (6′)

$\ell_1 \wedge \ell_2 = ((\ell_1 \ell_2)\ell_1')'$.

We have $((\ell_1 \ell_3) \wedge (\ell_2 \ell_3)((\ell_1 \vee \ell_2)\ell_3)$

$= (((\ell_1 \ell_3)(\ell_2 \ell_3))((\ell_1 \ell_3)')'((\ell_1 \vee \ell_2)\ell_3)$ (by (6 ′))

$= ((\ell_2(\ell_1 \ell_3)\ell_3))(\ell_1 \ell_3)')'((\ell_1 \vee \ell_2)\ell_3)$

$= ((\ell_2(\ell_1 \vee \ell_3))(\ell_1 \ell_3)')'((\ell_1 \vee \ell_2)\ell_3)$ (by (6))

$= ((\ell_1 \vee \ell_2)\ell_3)'((\ell_2(\ell_1 \vee \ell_3)(\ell_1 \ell_3)')$ (by ‘ ′ ‘)

$= (\ell_2(\ell_1 \vee \ell_3))((((\ell_1 \vee \ell_2)\ell_3)'(\ell_1 \ell_3)')$ (by (C))

$= (\ell_2(\ell_1 \vee \ell_3))((\ell_1 \ell_3)(\ell_1 \vee \ell_2)((\ell_1 \ell_3)\ell_3)$ (by ‘ ′ ‘)
$= (\ell_2(\ell_1 \vee \ell_3))((\ell_1 \vee \ell_2)((\ell_1 \ell_3)\ell_3)$ (by (C))

$= (\ell_2(\ell_1 \vee \ell_3))((\ell_1 \vee \ell_2)(\ell_1 \vee \ell_3))$ (by (6)

$= (\ell_1 \vee \ell_2)((\ell_2(\ell_1 \vee \ell_3))(\ell_1 \vee \ell_3))$ (by (C)).

We have $(\ell_2(\ell_1 \vee \ell_3))(\ell_1 \vee \ell_3) = ((\ell_1 \vee \ell_3)\ell_2)\ell_2 = (\ell_1 \vee \ell_3) \vee \ell_2$ by (D) and (6), so $(\ell_1 \vee \ell_2)((\ell_2(\ell_1 \vee \ell_2))(\ell_1 \vee \ell_3) = (\ell_1 \vee \ell_2)(\ell_2 \vee \ell_1 \vee \ell_3) = 1$, since $(\ell_1 \vee \ell_2) \leq (\ell_2 \vee \ell_1 \vee \ell_3)$. Summing up, we have $(\ell_1 \ell_3) \wedge (\ell_2 \ell_3) \leq (\ell_1 \vee \ell_2) \vee \ell_3$. We also have $(\ell_1 \vee \ell_2)\ell_3 \leq (\ell_1 \ell_3) \wedge (\ell_2 \ell_3)$, by (5). This shows that (L_1) holds.

Proposition 3.3.3

The lattice of a TI is always distributive.

Proof

We obtain these equations from Lemma 3.3.2 (6)

$\ell_1 \vee \ell_2 = (\ell_1 \ell_2)\ell_2 = (\ell_2 \ell_1)\ell_1$.

We would like to show $\ell_1 \vee (\ell_2 \wedge \ell_3) = (\ell_1 \vee \ell_2) \wedge (\ell_1 \vee \ell_3)$.

Starting with $\ell_1 \vee (\ell_2 \wedge \ell_3) = (\ell_2 \wedge \ell_3) \vee \ell_1 = ((\ell_2 \wedge \ell_3)\ell_1)\ell_1$ (by Lemma 3.3.2 (6))

$= ((\ell_2 \ell_1) \vee \ell_3 \ell_1)\ell_1$ (by (L_2))

$= (\ell_2 \ell_1)\ell_1 \wedge ((\ell_3 \ell_1)\ell_1)$ (by (L_1))

$= ((\ell_1 \ell_2)\ell_2) \wedge ((\ell_1 \ell_3)\ell_3)$ (by (D))

$= (\ell_1 \vee \ell_2) \vee (\ell_1 \vee \ell_3)$.

CONCLUSION

Reducing the number of axioms, usually adopted for LIA, we have significantly simplified the definition of an LIA. At this stage, we could ask several questions.

Question 1

Can our definition be simplified further?

Question 2

Can each lattice with involution receive a TI structure?

Question 3

Are there lattices which can receive two different TI structures?

Answering the first of these questions can have very important consequences.

As to Question 2, this is possible for each Boolean lattice. Just set

$xy = x' \vee y$. Then axioms (A) – (D) are easily verified.

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