T.C.
UŞAK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

MATEMATİK ANABİLİM DALI

ON THE GENERALIZATION OF METRIC, ULTRAMETRIC AND ORDERED FIXED POINT THEOREMS

YÜKSEK LİSANS TEZİ

ELVAN GÖKTEKE

HAZİRAN 2018
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Elvan GÖKTEKE

METRİK, ULTRAMETRİK VE SIRALI UZAYLARDAKİ SABİT NOKTA TEOREMLERİNİN GENELLENMESİ ÜZERİNE

(Yüksek Lisans Tezi)

Elvan GÖKTEKE

UŞAK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

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ÖZET

Bu tezin amacı Banach sabit nokta teoremini, ultrametrik uzay ve sıralı uzaylardaki sabit nokta teoremlerini Kuhlmann ve Kuhlmann (2015) çalışmasında ortaya atılan bir yöntem ile ele almaktır. Sözü edilen bu yöntem metrik, ultrametrik veya sıra yerine bunlarla elde edilen yuvar uzaylarını dikkate alarak ortak genel bir sabit nokta teoremi vermiştir.

Altı bölümden oluşan bu tezin, birinci bölümü, giriş bölümü olup tezin içeriği hakkında bilgi verilmiştir. İkinci bölümde, metrik uzaylar ve sıralı uzaylar incelenmiştir. Üçüncü bölüm Banach sabit nokta teoreminin klasik ispatına ayrılmıştır. Dördüncü bölümde, ultrametrik uzaylar ve önemli örneklerinden p-sel metrik uzayları incelenmiş olup ultrametrik sabit nokta teoremi verilmiştir. Sıralı uzaylardaki sabit nokta teoremleri beşinci bölümde ele alınmıştır. Altıncı bölümde Kuhlmann ve Kuhlmann (2015) ışığında, üzerinde çalışılan uzay ister metrik, ultrametrik isterse de sıralı bir uzay olsun, bir sabit noktanın var olabilmesi için yeterli koşullar ele alınmıştır. Son olarak, tezde ele alınan metrik ve ultrametrik teoremlerinin ve tam yarı sıralı uzaylarda ifade edilen Bourbaki-Witt teoremlerinin hipotezlerinin genel sabit nokta teoreminin hipotezlerini sağladığı gösterilmiştir.

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ON THE GENERALIZATION OF METRİC, ULTRAMETRİC AND ORDERED FIXED POINT THEOREMS

(M.Sc. Thesis)

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ABSTRACT

This thesis aims to study a generalization of Banach Fixed Point Theorem in metric, ultrametric and ordered spaces following a recent paper of Kuhlmann and Kuhlmann (2015) which points out the common denominator of Banach's Fixed Point Theorem in metric, ultrametric and topological spaces. The general idea is to give a general fixed point theorem which works in all of these spaces by the structure of ball spaces and by the concept of spherical completeness.

This thesis includes six chapters. Chapter 1 introduces the literature and the objective of this thesis. Chapter 2 gives the preliminaries. In Chapter 3, Banach Fixed Point Theorem is given. In Chapter 4, we study the ultrametric spaces. Some examples including the remarkable p-adic spaces are given. Chapter 5 presents the ordered spaces and some fixed point theorems in these spaces. In Chapter 6, we give the general fixed point theorem due to Kuhlmann and Kuhlmann (2015) which combines various contents. Finally, we conclude with the proofs of Banach fixed point theorem, ultrametric fixed point theorem and as a our contribution to literature Bourbaki and Witt theorems given in ordered spaces by applying the general fixed point theorem.

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CHAPTER 1

INTRODUCTION

Fixed point theory deals with the maps defined on itself. Let X be a set and f be a map defined on X, a fixed point theorem states simply that under some conditions on X and f there exists a point $x \in X$ such that $f(x) = x$. This point is called a fixed point of f on X. Sometimes it is unique. It has various applications not only in analysis and topology but also in various domains such as physics, economics and informatics.

Banach Fixed Point Theorem provides a basis of fixed point theorems in different branches of analysis. It is a very important tool in the studies of metric spaces and its applications. Considering a metric space (X, d) and a mapping f defined from X to itself, Banach Fixed Point Theorem states that the sufficient conditions for having a unique fixed point on X are that (X, d) is complete and f is a contraction map. It is first studied and stated by Stefan Banach (1892-1945) in his book published in 1922 ([1]) since then the theorem has been referred as Banach Fixed Point Theorem. An iterative process producing approximations to the fixed point or the error bounds are given as the first corollaries of the theorem[1]. Banach Fixed Point Theorem has important applications in analysis itself. For instance, the theorem is used in order to obtain the iteration methods for solving linear equations. Also, it is noted in Banach's book that the existence and the uniqueness of the solutions of differential and integral equations while considering the function spaces was the first motivation of Banach Fixed Point Theorem.

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[1] We will not discuss the iterative process in this thesis. One may refer to [2].

Following Banach's work [1], in the 20th century, the fixed point theory was developped in divers contexts. The questions like the following ones has arised: What are the conditions on X and on f to have a fixed point? Is X a topological space? a metric space? an ultrametric space, a partially ordered space or a lattice? What are the conditions on f? Is f a continous map? a Lipschitz continuous? a contraction? Or what about if f is not necessarily continuous but only a progressive map? All of these questions lead the related litterature to classify the fixed point theorems.

Banach Fixed Point Theorem (or Contraction Mapping Theorem) is seen as the most classical result of the fixed point theory. In the important works of Bourbaki (1949) and Witt (1950) ([3] and [4]), the analogues of this classical result are given in the ordered spaces. [5], [6], [7], [8], [9] and [10] give the extensions of Banach Fixed Point Theorem in the case of ultrametric spaces. Recently, Kuhlmann and Kuhlmann (2015) ([11]) points out the common denominator of Banach's Fixed Point Theorem and its analogues in ultrametric and topological spaces. A general fixed point theorem is given in [11] which works in metric, ultrametric and topological spaces by the structure of *ball spaces* and by the concept of *spherical completeness*.

In this thesis, we study the fixed point theorems in three main topics. These three topics includes metrical, ultrametrical and ordered fixed point theorems. We first study the Banach Fixed Point Theorem. The classical result is given with an application on integral equations.[2] Next, we study the ultrametric spaces. Some examples, including the remarkable p-adic spaces, are given. We will give the proof of the ultrametric fixed point theorem following the ideas given in [11]. The essential contribution is this thesis is to give alternative proofs of the Bourbaki and Witt Theorems (first given in [3] and [4], considered recently in [14]) by applying the general fixed theorem of [11]. We prove two theorems of Bourbaki and Witt which are stated for a complete partial order set for a progressive and order preserving functions respectively.

This thesis is organized as follows. In Chapter 2, we give some preliminary results in order to present the basic definitions and tools of metrical, ultrametrical and ordered fixed point theory. Chapter 3 is devoted to the classical proof of Banach Fixed Point Theorem. In Chapter 4, we study the ultrametric spaces and we state a version of

[2] One may refer to [12] and [13] for the applications of Banach Fixed Point Theory in economics.

ultrametric fixed point theorem. Chapter 5 presents the ordered sets and a fixed point theorems in these sets. Then, in Chapter 6, we give the general fixed point theorem due to [11] which combines various contents.[3] Finally, concludes with the proofs of all the theorems given in Chapter 3,4 and 5 by using the general fixed point theorem.

[3] In this thesis, we do not give a proof this theorem. For a complete proof of the theorem, one may refer to [11].

CHAPTER 2

PRELIMINARIES

In this chapter, we give some important concepts and definitions which will be helpful for the rest of this thesis.

2.1 Metric Spaces

In this section, we will consider metric spaces. Metric spaces provides a base for the important problems in analysis.

Definition 2.1. *Let X be a nonempty set. Consider the distance map $d : X \times X \to \mathbb{R}_+$ such that the following axioms are satisfied:*

(M1) *For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$*

(M2) *For all $x, y \in X$, $d(x, y) = d(y, x)$*

(M3) *For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)*

We call such a distance map d, a metric on X and we call the couple (X, d) a metric space.

Example 2.2. *Let $X = \mathbb{R}$. Consider the map $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined as*

$$d(x, y) = |x - y|$$

The map d satisfies the conditions **(M1)-(M3)** for every $x, y \in \mathbb{R}$. Therefore, d is a metric (which is called the *usual metric*) on $\mathbb{R}$.

Example 2.3. *Consider the map $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ defined by*

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

where $x = (x_1, x_2),\ y = (y_1, y_2)$. $(\mathbb{R}^2, d)$ is a metric space since:

*We have $d(x, y) = 0$ if and only if $|x_1 - y_1| + |x_2 - y_2| = 0$. $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$ implies that $x_1 = y_1$ and $x_2 = y_2$. Hence, $x = y$ and **(M1)** is satisfied.*
We have

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y, x)\ for\ every\ x,\ y \in \mathbb{R}^2$$

*Therefore, **(M2)** is satisfied.*
Finally, we have

$$\begin{aligned} d(x, z) = |x_1 - z_1| + |x_2 - z_2| &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= d(x, y) + d(y, z)\ for\ every\ x, y, z \in \mathbb{R}^2 \end{aligned}$$

*Hence, **(M3)** is also satisfied.*

This metric d is also called the taxicab metric.

Example 2.4. *Consider the following set X which is defined as all bounded sequence of real numbers:*

$$X := \{\xi = (\xi_1, \xi_2, ...)|\ \ |\xi_i| < c_0,\ \xi_i \in \mathbb{R}, c_0 \in \mathbb{R}_+\}$$

Let $\xi, \xi^{'} \in X$, we have $\xi = (\xi_1, \xi_2, ...)$ and $\xi^{'} = (\xi_1^{'}, \xi_2^{'}, ...)$. Let us define the map

$$d : X \times X \to \mathbb{R}_+$$

with

$$d(\xi, \xi^{'}) = sup_{i \in \mathbb{N}}\{|\xi_i - \xi_i^{'}|\}$$

*The map d satisfies axioms **(M1)** and **(M2)**. Let us verify the axiom **(M3)**: For any $\xi, \xi^{'}, \xi^{''} \in X$, we have*

$$|\xi_i - \xi_i^{'}| = |\xi_i - \xi_j^{''} + \xi_j^{''} - \xi_i^{'}| \leq |\xi_i - \xi_j^{''}| + |\xi_j^{''} - \xi_i^{'}|$$

Hence,

$$d(\xi,\xi^{'}) = sup_{i\in\mathbb{N}}\{|\xi_i - \xi_i^{'}|\} \leq sup_{i,j\in\mathbb{N}}|\xi_i - \xi_j^{''}| + sup_{i,j\in\mathbb{N}}|\xi_j^{''} - \xi_i^{'}|$$

$$\implies d(\xi,\xi^{'}) \leq d(\xi,\xi^{''}) + d(\xi^{''},\xi^{'})$$

Thus (X,d) is a metric space which is usually denoted by ℓ_∞.

Example 2.5. *The usual metric defined on $\mathbb{R}^2$ can be generalized to as follows by the help of Cauchy-Schwarz Inequality on n-dimension. Let d is a map which is defined as:*

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + ... + (x_n-y_n)^2}$$

for every $x = (x_1,x_2,....,x_n)$, $y = (y_1,y_2,.....,y_n) \in \mathbb{R}^n$. $(\mathbb{R}^n,d)$ is a metric space.

Example 2.6. *Consider the map $d : X \times X \to \mathbb{R}_+$ given by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

*The axioms **(M1)** and **(M2)** are clearly satisfied by the map d. Let us show that **(M3)** is also satisfied for all $x,y,z \in X$:*

For $x \neq y$, we have three cases:
Case 1: If $z = x$: We have $d(x,z) = 0$ and $d(x,y) = d(y,z) = 1$ which implies $0 = d(x,z) \leq d(x,y) + d(y,z) = 2$.
Case 2: If $z = y$: In this case, $d(x,z) = d(x,y) = 1$ and $d(z,y) = 0$. We have $1 = d(x,z) \leq d(x,y) + d(y,z) = 1$.
Case 3: If $z \neq x$ and $z \neq y$: In this case $d(x,z) = d(x,y) = d(y,z) = 1$. We have $1 = d(x,z) \leq d(x,y) + d(y,z) = 2$.

For $x = y$, we have two cases:
Case 1: If $z = x$: In this case $d(x,z) = d(x,y) = d(y,z) = 0$. We have $0 = d(x,z) \leq d(x,y) + d(y,z) = 0$.
Case 2: If $z \neq x$: In this case $d(x,y) = 0$ and $d(x,z) = d(y,z) = 1$. We have $1 = d(x,z) \leq d(x,y) + d(y,z) = 1$.

Hence, (X, d) is a metric space which is called discrete metric space.

Example 2.7. *Let X be a set of all real-valued functions which $x, y, ...$ are continuous functions on closed interval $I = [a, b]$. Consider the map $d : X \times X \to \mathbb{R}_+$ defined as*

$$d(x, y) = max_{t \in I}|x(t) - y(t)|$$

*The map d satisfies the axioms **(M1)** and **(M2)**. Let us verfy **(M3)**: For every $x, y, z \in X$, we have*

$$|x(t) - y(t)| = |x(t) - z(t) + z(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$$

Hence,

$$d(x, y) = max_{t \in I}|x(t) - y(t)| \leq max_{t \in I}|x(t) - z(t)| + max_{t \in I}|z(t) - y(t)|$$

$$which\ implies\ d(x, y) \leq d(x, z) + d(z, y)\ for\ every\ x, y, z \in X$$

(X, d) is thus a metric space which is usually denoted by $C[a, b]$.

Example 2.8. *Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ be a map defined by*

$$d(x, y) = (x - y)^2$$

*We can immediately remark that for some values like $x = 5, y = 4$ and $z = 1$. **(M3)** is not satisfied:*

$$d(x, z) = 16 \nleq d(x, y) + d(y, z) = 10.$$

Thus, d is not a metric on $\mathbb{R}$.

Definition 2.9. *Let (X, d) be a metric space. Let Y be a nonempty subset of X. Let us define $d' := d|_{Y \times Y}$. The metric d' is said to be the induced metric by d on Y.*

Example 2.10. *The subsets $\mathbb{Q}$ and $[0, 1]$ of $\mathbb{R}$ are metric spaces with induced metrics on these sets by usual metric. Generally, all subsets on $\mathbb{R}$ are metric spaces with induced metrics on these intervals by usual metric.*

Definition 2.11. *Let R be a ring with unity element $1 = 1_R$. A function $N : R \to \mathbb{R}_+$ is called a norm on R if the followings hold:*

(N1) *For all $x \in R$, $N(x) = 0$ if and only if $x = 0$*

(N2) *For all* $x, y \in R$, $N(xy) = N(x)N(y)$

(N3) *For all* $x, y \in R$, $N(x + y) \leq N(x) + N(y)$

Let X *be a vector space.* (X, N) *is called a normed space if the norm* N *defined on* X*.*

Example 2.12. *Let* $X = \mathbb{R}^n$*. Consider the map* $N : \mathbb{R}^n \to \mathbb{R}_+$ *defined as*

$$N(x) = \|x\|_1 = |x_1| + |x_2| + ... + |x_n|$$

for any $x \in \mathbb{R}^n$*. The map satisfies the conditions* ***(N1)-(N3)****. Then* $(\mathbb{R}, N)$ *is a normed space.*

Proposition 2.13. *Every normed space is a metric space.*

Proof: Let (X, N) be a normed space. Consider the map $d : X \times X \to \mathbb{R}_{\geq 0}$ defined by

$$d(x, y) = N(x - y)$$

For **(M1)**: We have $d(x, y) = N(x - y) = 0$ if and only if $x = y$
For **(M2)**: We have
$d(x, y) = N(x - y) = N(-(y - x)) = N(-1)N(y - x) = N(y - x) = d(y, x)$
For **(M3)**: We have

$$\begin{aligned} d(x, y) = N(x - y) &= N(x - z + z - y) \\ &\leq N(x - z) + N(z - y) = d(x, z) + d(z, y) \end{aligned}$$

Therefore, a norm on X defines a metric d on X which is defined by

$$d(x, y) = N(x - y)$$

for any $x, y \in X$ and d is called the *metric induced by the norm.*

Example 2.14. *Consider the* (X, d) *discrete metric space. Let* $x \neq y$ *and* $|\alpha| \neq 0, 1$.

$$d(\alpha x, \alpha y) = d(\alpha(x, y)) = |\alpha| d(x, y) = |\alpha| \neq 1 = d(x, y)$$

Thus, discrete metric is not a norm. So, every metric space is not a normed space.

Definition 2.15. *Let (X, d) be a metric space. A sequence (x_n) defined on X is said to be convergent if there is an element x of X such that $\lim_{n\to\infty} d(x_n, x) = 0$. x is called the limit of (x_n) and we write $\lim_{n\to\infty} x_n = x$. We say that (x_n) converges to x. If (x_n) is not convergent, it is said to be divergent.*

Remark 2.16. The convergence of the sequence (x_n) is defined by the convergence of the real numbers sequence $a_n = d(x_n, x)$. That is, $\lim_{n\to\infty} x_n = x$ means that for an arbitrary $\varepsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that for all a_n with $n > N$ implies $|a_n - 0| < \epsilon$.

Example 2.17. *Let (x_n) be a sequence on the induced metric space $(0, 1) \subset \mathbb{R}$ by usual metric such that $(x_n) = (\frac{1}{n})$. Consider*

$$\lim_{n\to\infty} d(x_n, 0) = \lim_{n\to\infty} \left|\frac{1}{n} - 0\right| = 0$$

but $0 \notin (0, 1)$. Thus (x_n) is not convergent on $(0, 1)$.

Definition 2.18. *Let (X, d) be a metric space. A sequence (x_n) defined on X is said to be Cauchy sequence if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ is satisfied for all $m, n > N$.*

The metric space X is said to be complete if every Cauchy sequence defined on X does converge in X.

Example 2.19. *Let $(x_n) = \dfrac{n+3}{2n+1}$ be a sequence in $\mathbb{R}$.*
For any $\varepsilon > 0$, we can the choose $N > \dfrac{2-\varepsilon}{(2\varepsilon)}$ such that the following holds for all $m, n > N$:

$$\begin{aligned}
|x_n - x_m| &= \left|\frac{n+3}{2n+1} - \frac{m+3}{2m+1}\right| = 3\frac{|m-n|}{(2n+1)(2m+1)} \\
&< \frac{2m+2n}{(2n+1)(2m+1)} = \frac{(2m+1)+(2n+1)-2}{(2n+1)(2m+1)} \\
&= \frac{1}{(2m+1)} + \frac{1}{(2n+1)} - \frac{2}{(2n+1)(2m+1)} \\
&< \frac{1}{(2m+1)} + \frac{1}{(2n+1)} \\
|x_n - x_m| &< \frac{2}{(2N+1)} < \varepsilon
\end{aligned}$$

Thus, (x_n) is a Cauchy sequence in $\mathbb{R}$.

Example 2.20. *Consider the sequence* $(x_n) = (-1)^n$ *in* $\mathbb{R}$ *with usual metric. Since*

$$|x_{2n+1} - x_{2n}| = |(-1)^{2n+1} - (-1)^{2n}| = 2$$

for all $n \in \mathbb{N}$. *Therefore* $(-1)^n$ *is not a Cauchy sequence.*

Example 2.21. *Let* $d : X \times X \to \mathbb{R}_{\geq 0}$ *be the discrete metric. Let* (x_n) *be an arbitrary Cauchy sequence in* X.
Let $\varepsilon = \frac{1}{2}$. *There exists* $N > 0$ *such that for all* $n, m > N$, *we have:*

$$d(x_n, x_m) < \varepsilon = \frac{1}{2}.$$

Also, by the definition of discrete metric, since there are 0 *and* 1 *as distance we have:*

$$if \ d(x_n, x_m) = 0 < \frac{1}{2} \ then \ x_n = x_m.$$

(x_n) *is then a constant sequence, after an arbitrary natural number* N *which is* (x_n) *convergent in* X. *Hence, discrete metric space is a complete metric space.*

Example 2.22. $(\mathbb{Q}, d)$ *is a metric space with induced usual metric on* $\mathbb{R}$. *We will show that* $(\mathbb{Q}, d)$ *is not complete.*
Consider the rational sequence (x_n) *which is given by the following sequence:*

$$x_n = \begin{cases} x_0 & = 2 \\ x_{n+1} & = \dfrac{x_n}{2} + \dfrac{1}{x_n} \end{cases}$$

(x_n) *is a Cauchy sequence in* $\mathbb{Q}$ *but* (x_n) *converges to* $\sqrt{2} \notin \mathbb{Q}$. *Thus,* $\mathbb{Q}$ *is not complete.*

Definition 2.23. *Let* (X, d) *be a metric space which is not complete. Let* C_X *be the limit points of Cauchy sequences defined on* X. *The set* $\hat{X} = X \cup C_X$ *is then a complete metric space with respect to* $\hat{d} := d \mid_{X \cup C_X}$. *Thus a complete metric space is created by adding all possible limit points of Cauchy sequences defined on that metric space.* *(*$\hat{X}, \hat{d}$*) is called the completion of* (X, d).

Theorem 2.24. $\mathbb{R}$ *is a completion of* $\mathbb{Q}$ *with usual metric.*[1]

[1] We will see also [Chapter 4] that $\mathbb{Q}$ has an another completion with respect to p-adic metric. See also [15].

In the following, we will consider various definitions and examples of continuous functions.

Definition 2.25. *Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be continous at $x_0 \in X$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$d(x, x_0) < \delta \; implies \; d(f(x), f(x_0)) < \epsilon.$$

If f is continuous at each $x_0 \in X$ then we say that f is continuous on X.

Example 2.26. *Let us denote by d_X the usual induced metric on $\mathbb{R}_+^*$. Let us show that $f(x) = x^2$ is a continuous map on $\mathbb{R}_+^*$: Let $x_0 \in X \subset \mathbb{R}_+^*$. Assume $d(x, x_0) = |x - x_0| < \delta$. If we take $\delta := min(1, \varepsilon/2a)$ with $a = x_0 + 1$, we have $|x - x_0| < 1$. This implies:*

$$|x^2 - x_0^2| = |x + x_0||x - x_0| \leq 2a|x - x_0| < 2a\delta \leq 2a\frac{\varepsilon}{2a} = \varepsilon$$

Hence, f is continuous at $x = x_0$. But is it continous on whole $\mathbb{R}_+^*$? However, this x_0 is chosen arbitrary then f is continuous where $\mathbb{R}_+^*$.

Example 2.27. *The function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:*

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is not continuous at $x = 0$ with respect to the usual metric on $\mathbb{R}$, since,
For $x = 0$, choose $\delta > 0$. Put $x = \delta/2$. Then $|x - x_0| = \delta/2 < \delta$ but $|f(x) - f(x_0)| = |1 - 0| = 1 \geq \varepsilon$. Thus, f is not continuous at $x_0 = 0$.

Definition 2.28. *Let (X, d) be metric space. A function $f : X \to X$ is uniformly continuous on X if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for every $x, z \in X$ satisfying $d(x, z) < \delta$, we have $d(f(z), f(x)) < \epsilon$.*

Example 2.29. *Let us consider $X = (0, 4)$ with usual induced metric and f is uniformly continuous on X, since:*

$f(x) = x^2$. Let $\delta = \varepsilon/8$. $x_0 \in X$ with $0 < x_0 < 4$ and $0 < x < 4$ so $0 < x + x_0 < 8$. Assume $|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0| < 8\delta = \varepsilon.$$

Example 2.30. *Let us now consider $X = (0, \infty)$. The function $f(x) = x^2$ is not uniformly continuous on $X = (0, \infty)$. Since: There exists $\varepsilon > 0$ and for all $\delta > 0$ we can find $x_0 \in X$ and $x \in X$ such that $|x - x_0| < \delta$ and $|x^2 - x_0^2| \geq \varepsilon$.*

Let $\varepsilon = 1$. Choose $\delta > 0$. Let $x_0 = 1/\delta$ and $x = x_0 + \delta/2$. Then $|x - x_0| < \delta/2 < \delta$ but

$$|x^2 - x_0^2| = |(\frac{1}{\delta} + \frac{\delta}{2})^2 - (\frac{1}{\delta})^2| = 1 + \frac{\delta^2}{4} > 1 = \varepsilon.$$

Definition 2.31. *Let (X, d) be a metric space and $f : X \to X$ be a function. If there exist $\lambda > 0$ such that*

$$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$$

then f is said to be Lipschitz continuous. The smallest λ for which the inequality holds is called Lipschitz constant of f.

Definition 2.32. *If f is Lipschitz continuous with $\lambda = 1$ then f is said to be non-expansive, that is,*

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition 2.33. *Let (X, d) be a metric space. If, for all $x_1, x_2 \in X$ we have*

$$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

then f is called an isometry.

Example 2.34. *The function $f(x) = x + 1$ defined on $\mathbb{R}$ with usual metric d is an isometry.*

Definition 2.35. *Let (X, d) be a metric space. A map $f : X \to X$ is called a contraction map if there is λ with $0 \leq \lambda < 1$ such that for all $x, y \in X$*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

Proposition 2.36. *Let f be a contraction map on X. Then f is uniformly continuous on X.*

Proof: Given $\varepsilon > 0$ and let pick $\delta = \varepsilon$. Then if $d(x, y) < \delta$, we have

$$d(f(x), f(y)) \leq \lambda d(x, y) < \lambda \delta = \lambda \varepsilon < \varepsilon$$

which implies directly that f is uniformly continuous on X.

Corollary 2.37. *Let* (X, d) *be a metric space. A Lipschitz continuous map is uniformly continuous on* X*, hence continuous on* X*. The converse is not true in general.*

Example 2.38. *Let us show that* $\cos x$ *is a contraction on* $[0, \frac{\pi}{2}]$ *with the usual induced metric.*
By means of the Mean Value Theorem in differential calculus and the boundedness of sine function on $[0, \frac{\pi}{2}]$*, that is there exists* $c \in (0, \frac{\pi}{2})$ *such that*

$$|\cos x - \cos y| = |sinc||x - y| < |x - y| \textit{ for every } x, y \in [0, \frac{\pi}{2}] \textit{ and } c \in (0, \frac{\pi}{2})$$

Therefore, $\cos x$ *is a contraction mapping on* $[0, \frac{\pi}{2}]$.

2.2 Ordered Sets

Let X be a set. The pair $(X, \preccurlyeq)$ is called a *partially ordered set (poset)* if for all $x, y, z \in X$, the following conditions are satisfied:

(*i*) $x \preccurlyeq x$ (reflexivity)

(*ii*) If $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$ (anti-symmetry)

(*iii*) If $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ (transitivity)

Remark 2.39. *The following two conditions are equivalent:*

(a) $x \leq y \ \ and \ \ y \leq x \ \ implies \ \ x = y$

(b) $x \leq y, \ x \neq y \ \ implies \ \ y \nleq x$

*Suppose that statement **(a)** holds. Let* $x \leq y$ *and* $x \neq y$*. If* $y \leq x$ *by using definition **(a)** we would have* $x = y$ *but this is a contradiction because of the fact that* $x \neq y$*. On the other hand, suppose that the statement **(b)** holds. Let* $x \leq y$ *and* $y \leq x$ *then by using definition **(b)** we would have* $y \nleq x$*. But this is a contradiction. Hence* $x = y$*.*

Definition 2.40. *Let* X *be a partially ordered set. Let* $x, y \in X$*,* x *and* y *are said to be comparable if the relation "*$x \preccurlyeq y$ *or* $y \preccurlyeq x$*" is true.* X *said to be totally ordered if it is partially ordered and if any two elements of* X *is comparable. The pair* $(X, \preccurlyeq)$ *is called a totally ordered set (toset).*

Example 2.41. *Let $X = \mathbb{R}$. Consider the relation $\preccurlyeq$ defined by*

$$x \preccurlyeq y \Leftrightarrow x \leq y$$

*The relation of $\leq$ in real numbers satisfies the conditions **(i)**,**(ii)** and **(iii)**. Thus $(\mathbb{R}, \leq)$ is a partially ordered set. Moreover, for all $x, y \in \mathbb{R}$, we have either $x \leq y$ or $y \leq x$. Therefore $(\mathbb{R}, \leq)$ is a totally ordered set. In particular, we can choose any nonempty subset of $\mathbb{R}$ instead of $\mathbb{R}$. Thus, any subset of $\mathbb{R}$ is also a totally oredered set.*

Remark 2.42. *Every totally ordered set is a partially ordered set.*
Every subset of totally ordered set is totally ordered by the induced ordering.

Example 2.43. *Let Y be a set. All subsets of Y can be denoted as $X = \mathcal{P}(Y)$.*
We claim that X is a partially ordered set with the relation inclusion satisfying the following conditions:

(*i*) *For all $A \in X$ we have $A \subseteq A$*

(*ii*) *For all $A, B \in X$ if $A \subseteq B$ and $B \subseteq A$ we have $A = B$*

(*iii*) *For all $A, B, C \in X$ if $A \subseteq B$ and $B \subseteq C$ we have $A \subseteq C$*

Therefore, $(X, \subseteq)$ is a poset. If Y is empty set or it has exactly one element, it can be considered as totally ordered set with respect to the inclusion.

Let $A = \{a, b, c\}$. Let us draw the Hasse diagram of the set of all subsets which has a three elements of the set A, by inclusion order:

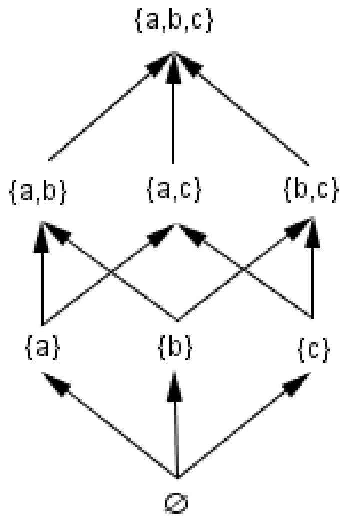

Figure 2.1: This figure is adapted from the [14].

A is a partially ordered set with the inclusion. However, $(P(A), \subseteq)$ is not a totally ordered set with inclusion since, sets on the same horizontal level don't share a precedence relationship. Some other pairs, such as $\{a\}$ and $\{b, c\}$, do not either.

Example 2.44. *Let $X = \mathbb{N}$. Consider the relation $\preccurlyeq$ defined by*

$$x \preccurlyeq y \Leftrightarrow x|y \in \mathbb{N}$$

Note as a convention that $0 \preccurlyeq 0$. We claim that $(\mathbb{N}, \preccurlyeq)$ is a partially ordered set:
(i) *For all $x \in \mathbb{N}$, we have $x|x = 1$, which implies $x \preccurlyeq x$.*
(ii) *Let $x \preccurlyeq y$ and $x \neq y$. $x|y$ means that $y = ax$ for some $a \in \mathbb{N}$. If $y|x$ we have $x = by$ for some $b \in \mathbb{N}$. Thus $x = by = bax$; so $a = 1$, $b = 1$. That is, $x = y$.*
(iii) *Let $x, y, z \in \mathbb{N}$. Let $x \preccurlyeq y$ and $y \preccurlyeq z$. We have $y = ax$ and $z = by$ for $a, b \in \mathbb{N}$. Therefore; $z = (ax)b = (ab)x$ which implies $x \preccurlyeq z$.*
Thus $(\mathbb{N}, \preccurlyeq)$ is a poset.
Since we can not have divisibility relation between an arbitrary natural numbers, $(\mathbb{N}, \preccurlyeq)$ is not a totally ordered set.

CHAPTER 3

BANACH FIXED POINT THEOREM

In this chapter, we will present Banach Fixed Point Theorem and we give its proof which is the primary source of the fixed point theory. We will also give some examples and applications.

Definition 3.1. *Let (X, d) be a metric space and $f : X \to X$ be a map. If there exists a point $x_0 \in X$ such that $f(x_0) = x_0$ then x_0 is called a fixed point of f on X.*

Example 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a map such that $f(x) = x^2$. Then f has two fixed points on $x = 0$ and $x = 1$.*
Remark that if $f : (0, 1) \to (0, 1)$ the domain of f is changed as $(0, 1)$ such that then f has no fixed point on this domain.

Example 3.3. *Let $a \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a map. Let $f(x) = x + a$. Then f has no fixed point.*

Theorem 3.4. ***(Banach Fixed Point Theorem)*** *Let (X, d) be a metric space with $X \neq \emptyset$. Suppose that (X, d) is complete and $f : X \to X$ is a contraction. Then f has a unique fixed point $x_0 \in X$.*

Proof: First, we will show the uniqueness part. We will consider the case that we have two distincts fixed points and end up with a contradiction by using property of contraction map. Secondly, we will show the existence part. The idea of the proof is we construct a Cauchy sequence by iteration in (X, d) and show that as it is Cauchy therefore converges in the complete metric space (X, d), that is, has a limit $x \in X$.

Then show that this x is the fixed point of f.

Uniqueness. Suppose that f has two fixed points $x_1, x_2 \in X$ such that $x_1 \neq x_2$, that is $f(x_1) = x_1$ and $f(x_2) = x_2$ on X. Then we have,

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$$

with $0 \leq \lambda < 1$ as f is a contraction. This is only possible if $d(x_1, x_2) = 0$, i.e., if $x_1 = x_2$. A contradiction.

Existence. Let us choose $x_0 \in X$ and define an iterated sequence $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$ such that by induction we will have:

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0)...$$

Now we can consider $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$ as follows

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \lambda d(x_n, x_{n-1})$$

since f is a contraction. We will have then:

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \lambda d(x_n, x_{n-1}) = \lambda d(f(x_{n-1}), f(x_{n-2}))$$

Again, using the fact that f is a contraction, we have:

$$d(x_n, x_{n-1}) = \lambda d(f(x_{n-1}), f(x_{n-2})) \leq \lambda^2 d(x_{n-1}, x_{n-2})...$$

Thus,

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0) = \lambda^n d(f(x_0), x_0) \ \ for \ \ n \in \mathbb{N}.$$

Hence for $n \in \mathbb{N}$ and $m \geq 1$:

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq \lambda^{n+m-1} d(f(x_0), x_0) + \lambda^{n+m-2} d(f(x_0), x_0) + ... + \lambda^n d(f(x_0), x_0) \\ &= (\lambda^{n+m-1} + \lambda^{n+m-2} + \lambda^{n+m-3} + ... + \lambda^n) d(f(x_0), x_0) \\ &= \lambda^n (1 + \lambda + \lambda^2 + ... + \lambda^{m-1}) d(f(x_0), x_0) \\ &= \lambda^n \cdot \frac{1 - \lambda^m}{1 - \lambda} d(f(x_0), x_0) \leq \frac{\lambda^n}{1 - \lambda} d(f(x_0), x_0) \end{aligned}$$

Since $\lambda < 1$, we can get the last expression as small as we want by choosing n large enough. Given $\epsilon > 0$, we can, in particular, find an N such that $\frac{\lambda^n}{1-\lambda} d(f(x_0), x_0) < \epsilon$. Thus for $n, n+m \geq N$, we have

$$d(x_{n+m}, x_n) \leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0) < \epsilon$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete $\{x_n\}$ converges to a point $x \in X$.

To prove that x is a fixed point that we are looking for we can just observe that we have $x_{n+1} = f(x_n)$ for all n, and taking limit as n goes to infinity we get $f(x) = x$.

x is a fixed point of f. By contraction of the Cauchy sequence with $x_{n+1} = f(x_n)$ for all n and by the fact that f is a contraction which implies f is uniformly continuous on X, we will have

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} f(x_n) \\ &= f(x) \end{aligned}$$

and

$$\lim_{n\to\infty} x_{n+1} = x$$

Thus $f(x) = x$. ■

Fredholm Integral Equation of Second Kind is one of the principal example for the application of Banach Fixed Point Theorem. One may refer to [2] for different applications in mathematics.

Example 3.5. *Let $K(x, y) : I \times I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continuous functions where $I = [a, b]$ for some a, b with $a < b$, let λ be a real number. Consider*

$$f(t) = \lambda \int_a^b K(t, s) f(s) ds + g(t) \tag{3.1}$$

$t \in [a, b]$. *This equation has a solution. We need to show that $\theta : C(I) \to C(I)$ is a contraction which is defined on the complete metric space $C(I)$ and by Banach Fixed Point Theorem θ will have a fixed point $f_0 \in C(I)$ such that $f_0 = \theta(f_0)$.*

$$d_{sup}(\theta(f), \theta(f_0)) = sup_{t \in I} |\theta(f)(t) - \theta(f_0)(t)|$$

For $f, f_0 \in C(I)$*, consider the following:*

$$\begin{aligned}
|\theta(f)(t) - \theta(f_0)(t)| &= |\lambda \int_a^b K(t,s) f(s) ds + g(t) - \lambda \int_a^b K(t,s) f_0(s) ds + g(t)| \\
&\leq |\lambda| \int_a^b |K(t,s)||f(s) - f_0(s)| ds \\
&\leq |\lambda| \; sup_{s \in I} |f(s) - f_0(s)| \; \int_a^b |K(t,s)| ds \\
&\leq |\lambda| \; sup_{s \in I} |f(s) - f_0(s)| \; max_{s,t \in I} |K(s,t)| (b-a)
\end{aligned}$$

Just for notation $\Gamma = |\lambda| \; max_{s,t \in I} |K(s,t)|(b-a)$*. Thus we have,*

$$d_{sup}(\theta(f), \theta(f_0)) \leq \Gamma \; sup_{s \in I} |f(s) - f_0(s)|$$

If $\Gamma < 1$ *then* θ *will be a contraction and Banach Fixed Point Theorem tells us that Fredholm Integral Equation of Second Kind has a solution.*

CHAPTER 4

ULTRAMETRIC SPACES

The aim of this section is to give a fixed point theorem which is an analogue of Banach Fixed Point Theorem for ultrametric spaces. We give first the definition of an ultrametric space. Then will give some related notions and some examples of ultra metric spaces. p-adic metric spaces will be investigated as the most important examples of ultrametric spaces.

Definition 4.1. *Let X be a non-empty set. Let Γ be a partially ordered set and $\Gamma_0 = \Gamma \cup \{0\}$. A map $u : X \times X \to \Gamma_0$ is called ultrametric if the followings hold:*

(U1) *For all $x, y, z \in X$, $u(x, y) = 0$ if and only if $x = y$*

(U2) *For all $x, y, z \in X$, $u(x, y) = u(y, x)$*

(U3) *For all $x, y, z \in X$, if $u(x, y) \leq r$ and $u(y, z) \leq r$ then $u(x, z) \leq r$*

*When we replace Γ in the third axiom **(U3)** by a totally ordered set, the axiom becomes:*

(U3$'$) *For all $x, y, z \in X$, $u(x, z) \leq max\{u(x, y), u(y, z)\}$ (Strong triangle inequality)*

(X, u) is then called an ultrametric space.

Remark 4.2. *A metric space (X, d) is an ultrametric space with $\Gamma = \mathbb{R}_+$ if it satisfies for each $x, y, z \in X$:*

$$d(x,z) \leq max\{d(x,y), d(y,z)\}$$

Example 4.3. *Discrete metric space is an ultrametric space.*

*Proof: Let X be a non-empty set. The axioms **(U1)** and **(U2)** is clearly satisfied by d. For $x, y, z \in X$ let us show **(U3)**:*

For $x \neq y$, we have three cases:

Case 1: If $z = x$: In this case $d(x,z) = 0$ and $d(x,y) = d(y,z) = 1$. We have $d(x,z) \leq max\{d(x,y), d(y,z)\}$.

Case 2: If $z = y$: In this case $d(x,z) = d(x,y) = 1$ and $d(z,y) = 0$. We have $d(x,z) \leq max\{d(x,y), d(y,z)\}$.

Case 3: If $z \neq x$ and $z \neq y$: In this case $d(x,z) = d(x,y) = d(y,z) = 1$. We have $d(x,z) \leq max\{d(x,y), d(y,z)\}$.

For $x = y$, we have two cases:

Case 1: If $z = x$: In this case $d(x,z) = d(x,y) = d(y,z) = 0$. We have $d(x,z) \leq max\{d(x,y), d(y,z)\}$.

Case 2: If $z \neq x$: In this case $d(x,y) = 0$ and $d(x,z) = d(y,z) = 1$. We have $d(x,z) \leq max\{d(x,y), d(y,z)\}$.

Remark 4.4. *We will denote discrete metric by $u_d : X \times X \to \{0,1\}$.*

Definition 4.5. *A norm is called ultranorm (or non-Archimedean) if **(N3)** is replaced with*

$$\textbf{(N3}'\textbf{)} N(x+y) \leq max\{N(x), N(y)\}$$

*for all $x, y \in R$. It can be strengthened if **(N3′)** is replaced with*

$$\textbf{(N3}''\textbf{)} N(x+y) = max\{N(x), N(y)\}$$

if $N(x) \neq N(y)$.

Remark 4.6. *A metric is an ultranorm(non-Archimedean) if it is induced by an ultranorm, since in that case;*

$$d(x,y) = N(x-y) = N(x-z+z-y) \leq max\{N(x-z), N(z-y)\} = max\{d(x,z), d(z,y)\}$$

One can say that the metric which is induced by an ultranorm is called an ultrametric space.

Example 4.7. *(Alphabetical metric) Let X be the set of words in alphabet of any language. Let $x, y \in X$ be two words of any length. Let n be the first letter where x and y differ. The alphabetical metric $u_a : X \times X \to [0, 1]$ is defined as:*

$$u_a(x,y) = \begin{cases} 0, & \textit{if } x = y \\ \\ \frac{1}{2^{n-1}}, & \textit{if } x \neq y \end{cases}$$

*Let us check the axioms of an ultra metric given in definition 4.1. The axioms **(U1)**-**(U2)** are satisfied trivially. We will verfy the axiom **(U3)**:*

If $x = z$ then $u_a(z, x) = 0 \leq max\{u_a(x, y), u_a(y, z)\}$. Similarly, for $y = z$ and $x = y$.

When $x \neq y$ and $y \neq z$ and $x \neq z$, assume that $u_a(x, y) = \frac{1}{2^{n-1}}$, $u_a(y, z) = \frac{1}{2^{l-1}}$ and $u_a(x, z) = \frac{1}{2^{m-1}}$.
Thus, for all $j < n,\ x_j = y_j$. For all $j < l,\ y_j = z_j$ and for all $j < m,\ x_j = z_j$. These facts imply that $x_j = z_j$ for all $j < min\{n, l\}$ and $m \geq min\{n, l\}$.

Hence,

$$\frac{1}{2^{m-1}} \leq max\{\frac{1}{2^{n-1}}, \frac{1}{2^{l-1}}\}$$

or equivalently

$$u_a(x, z) \leq max\{u_a(x, y), u_a(z, y)\}$$

Therefore (X, u_a) is an ultra-metric space.

4.1 p-adic Metric Spaces

In the following, we will give the most common example of ultrametric spaces, namely p-adic metric spaces. We give first the definition of p-adic order and then we will define p-adic metric and show that p-adic metric is an ultrametric.

Definition 4.8. *Let p be a prime. Let $R = \mathbb{Z}$. Let x be a non-zero integer $x \in \mathbb{Z}$, we define the p-adic order of x as follows:*

$$ord_p x := max\{r \mid p^r / x\} \geq 0$$

Let $R = \mathbb{Q}$. For $a/b \in \mathbb{Q}$, the p-adic order of a/b

$$ord_p \frac{a}{b} = ord_p a - ord_p b.$$

Let us write $y \in \mathbb{Q}$ in the form $y = \frac{a}{b} p^r$ where $r, a, b \in \mathbb{Z}$, $b \neq 0$. Here p divides neither a nor b then $ord_p y = r$.

Example 4.9. *Let $x = \dfrac{28}{26}$ be a rational number. Let us show the 2-adic order of x:*

$$\begin{aligned} ord_2 \frac{28}{16} &= ord_2 28 - ord_2 16 \\ &= ord_2(7 \cdot 2^2) - ord_2 2^4 \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

Convention 4.10. *We have then the following:*

(*i*) $ord_p 0 = \infty$

(*ii*) $ord_p p = 1$

(*iii*) $ord_p 1 = 0$

Proposition 4.11. *The p-adic order has the following properties:*

(*i*) *For all $x, y \in \mathbb{Q}$, $ord_p(xy) = ord_p x + ord_p y$;*

(*ii*) *For all $x, y \in \mathbb{Q}$, $ord_p(x + y) \geq min\{ord_p x, ord_p y\}$ with equality if $ord_p x \neq ord_p y$.*

Proof: Let x, y be non-zero rational numbers. We have $x = \frac{a}{b} p^r$ and $y = \frac{c}{d} p^s$.

(i)

$$ord_p x = max\{\ell \mid p^\ell / x\} = max\{\ell \mid p^\ell / \frac{a}{b} p^r\} = r$$
$$ord_p y = max\{m \mid p^m / x\} = max\{m \mid p^m / \frac{c}{d} p^s\} = s$$

However; $xy = \frac{ac}{bd} p^{r+s}$ and p divides neither ac nor bd, $bd \neq 0$. So we have:

$$ord_p(xy) = max\{k \mid p^k / xy\} = max\{k \mid p^k / \frac{ac}{bd} p^{r+s}\} = r + s = ord_p x + ord_p y.$$

(ii) The case where $x = 0$ and (or) $y = 0$ straightforward.

If $r = s$, we have:

$$x + y = p^r (\frac{a}{b} + \frac{c}{d}) = p^r \frac{(ad + bc)}{bd}.$$

Thus, $ord_p(x+y) \geq r = s$ since p does not divide neither b nor d so p does not divide bd.

If $r \neq s$, let $s > r$. We have:

$$x + y = p^r (\frac{a}{b} + p^{s-r} \frac{c}{d}) = p^r \frac{(ad + p^{s-r} cb)}{bd}.$$

Thus, $ord_p(x + y) = r = min\{ord_p x, ord_p y\}$ since p divides neither b nor d so p is not divide bd and also $s - r > 0$.

Definition 4.12. *Let p a prime. For $x \in \mathbb{Q}$, define the p-adic norm of x as follows:*

$$|x|_p = \begin{cases} p^{-ord_p x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Proposition 4.13. *The p-adic norm $|.|_p : \mathbb{Q} \to \mathbb{R}_+$ has the properties*

(N1) *For all $x \in \mathbb{Q}$, $|x|_p = 0$ if and only if $x = 0$*

(N2) *For all* $x, y \in \mathbb{Q}$, $|xy|_p = |x|_p|y|_p$

(N3$'$) *For all* $x, y \in \mathbb{Q}$, $|x+y|_p \leq max\{|x|_p, |y|_p\}$, *with equality if* $|x|_p \neq |y|_p$.

Hence, $|.|_p$ *is an ultranorm (non-Archimedean norm) on* $\mathbb{Q}$.

Proof: First property is trivial. We will therefore prove **(N2)** and **(N3)**.
For **(N2)**: If $xy = 0$ then $x = 0$ and (or) $y = 0$ then it is evident. Let $xy \neq 0$. Then

$$\begin{aligned} |xy|_p &= p^{-ord_p xy} = p^{-(ord_p x + ord_p y)} \\ &= p^{-ord_p x} p^{-ord_p y} \\ &= |x|_p |y|_p \end{aligned}$$

For **(N3$'$)**: Let $ord_p x \leq ord_p y$, that is, $max\{ord_p x, ord_p y\} = ord_p y$ and then $max\{|x|_p, |y|_p\} = |x|_p$. Moreover with the fact that $ord_p(x+y) \geq min\{ord_p x, ord_p y\} = ord_p x$, we will have

$$|x+y|_p = \frac{1}{p^{ord_p(x+y)}} \leq \frac{1}{p^{ord_p y}} = |y|_p \leq max\{|x|_p, |y|_p\}$$

If $ord_p x \neq ord_p y$, then $ord_p(x+y) = min\{ord_p x, ord_p y\} = ord_p x$ then we have the equality

$$\begin{aligned} |x+y|_p &= \frac{1}{p^{ord_p(x+y)}} = \frac{1}{p^{ord_p x}} \\ &= |x|_p \\ &= max\{|x|_p, |y|_p\} \end{aligned}$$

Example 4.14.

$$|75|_5 = |3.5^2|_5 = \frac{1}{5^2} = \frac{1}{25}$$

$$|\frac{2}{375}|_5 = |\frac{2}{3}.5^{-3}|_5 = 5^3 = 125$$

$$|3|_5 = |4|_5 = |7|_5 = |\frac{12}{5}|_5 = \frac{1}{5^0} = 1$$

Definition 4.15. *The map* $|.|_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_+$ *induced by p-adic norm(ultranorm) is called a p-adic metric.* $(\mathbb{Q}, |.|_p)$ *is then a p-adic metric space with* $d_p(x, y) = |x - y|_p$.

Example 4.16. *Let* $d(x, y) = |x - y|_3$.

$$d_3(3, 30) = |30 - 3|_3 = |27|_3 = |3^3|_3 = \frac{1}{3^3} = \frac{1}{27}$$
$$d_3(2, 3) = |3 - 2|_3 = |1|_3 = |3^0|_3 = \frac{1}{3^0} = 1$$

Proposition 4.17. *The p-adic metric is an ultrametric.*

Proof: **(U1)** and **(U2)** proporties are trivial. We will therefore prove **(U3)**. Let x, y be non-zero rational numbers. We have $x = \frac{a}{b}p^r$ and $y = \frac{c}{d}p^s$.
For **(U3)**: Since $x - z = (x - y) + (y - z)$,

$$ord_p(x - z) \geq min\{ord_p(x - y), ord_p(y - z)\}$$

We have

$$|x - z|_p = p^{-ord_p(x-z)} \leq max\{p^{-ord_p(x-y)}, p^{-ord_p(y-z)}\} = max\{|x - y|_p, |y - z|_p\}$$

Theorem 4.18. *Given a prime p and the p-adic metric, every rational number can be written in a unique way as a power series of the form*

$$\sum_{k=n}^{\infty} b_k p^k \ for some\ n \in \mathbb{Z},\ and\ b_k \in 0, 1, ..., p - 1\ for\ each\ k \geq n$$

We will not give a proof of Theorem 4.18. One may for a proof refer to [16] or [17]. This series is called the *p-adic expansion* of the number. If $z \in \mathbb{Z}_{\geq 0}$ has a finite expansion. For example; let us look to 3-adic expansion of 25:

$$25 = 1.3^0 + 2.3^1 + 2.3^2 + 0.3^3 + 0.3^4 + ...$$

Negative integers and many non-integers are represented by infinite series. Let us write the 3-adic expansion of -1:

Let (z_n) be a sequence such that $z_n = 2.3^0 + 2.3^1 + 2.3^2 + ... + 2.3^n$.

$$z_n = 2.(1 + 3 + ... + 3^{n-1} + 3^n) = 2.\frac{1 - 3^{n+1}}{1 - 3} = 3^{n+1} - 1$$

and

$$\lim_{n\to\infty} |z_n - (-1)|_3 = \lim_{n\to\infty} |3^{n+1} - 1 + 1|_3 = \lim_{n\to\infty} |3^{n+1}|_3 = 0$$

Hence,

$$-1 = 2.3^0 + 2.3^1 + 2.3^2 + ... + 2.3^n + ...$$

Similarly,

$$-2 = 1.3^0 + 2.3^1 + 2.3^2 + ... + 2.3^n + ...$$

$$-3 = 0.3^0 + 2.3^1 + 2.3^2 + ... + 2.3^n + ...$$

$$-4 = 2.3^0 + 1.3^1 + 2.3^2 + ... + 2.3^n + ...$$

The p-adic metric, for a fixed prime p, two points are *close* each other if their difference is divisible by a large positive power of p. In example 4.16., $d(3, 30) < d(2, 3)$. In the following, we will give a picture due to [18] of a 3-adic metric space $(\mathbb{Z}, d_3)$.[1]

Example 4.19. *(Picture of a p-adic metric space). Let us consider the metric space* $(\mathbb{Z}, d_p)$*. Let* $p = 3$*. In this example, we sketch a picture of 3-adic metric space* $\mathbb{Z}_3$*. The 3-adic metric between any two integers* x *and* y *can be at most* 1*, corresponding to* $ord_3(x-y) = 0$ *and can be closer with possible distances* $1/3, 1/3^2, 1/3^3$*, etc. Thus, this metric space can be traced as a rooted tree, where the root represent the least possible order* $ord_3 = 0$ *and highest possible distance* $1/3^0 = 1$*. There will be infinitely many order and distance levels.*
The 3-adic metric between any two integers x *and* y *is the height, as labeled in the distance column, to which one must climb in traversing a path from* x*'s leaf to* y*'s leaf.*

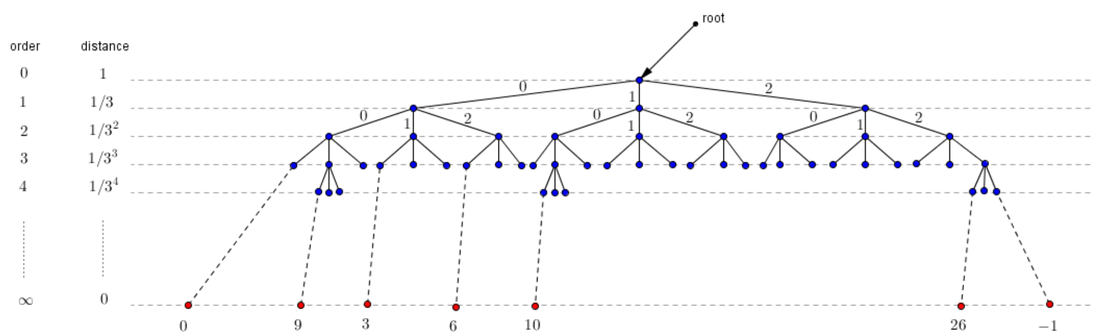

Figure 4.1: This figure is adapted from [16].

[1] In the literature, we may also cite [19] for a different picture of p-adic spaces.

4.2 Fixed Point Theorem for Ultra Metric Spaces

In this section, we first give some topological properties of an ultrametric space following [11], [20], [21] and [22].

Definition 4.20. *In a metric space, given a point $x_0 \in X$ and a real number $r > 0$, the set*
$B(x_0, r) = \{x \in X \mid d(x_0, x) \leq r\}$ *is a called a closed ball with center* x_0 *and radius* r.
In a ultrametric space, given a point $x_0 \in X$ and a real number $\alpha > 0$, the set
$B(x_0, \alpha) = \{x \in X \mid u(x_0, x) \leq \alpha\}$ *is a called a closed ball with center* x_0 *and radius* α.

Remark 4.21. *Let $x, y, z \in X$. Each three points x, y, z of any ultrametric space represent either vertices of an equilateral triangle or vertices of an isocales triangle with the unequal side being the shortest one. So each triangle in an ultrametric space is isocals or equalited. We will should show that $u(x, z) = u(x, y) = u(y, z)$ or $u(x, z) = u(x, y)$ (or $u(x, z) = u(y, z)$).*
When $u(x, y) = u(y, z)$ by the condition ***(U3)*** *we have*

(1) $u(x, z) \leq u(x, y)$ *or* $u(x, z) \leq u(y, z)$

In the case (1), we have two cases

(1a) $u(x, z) < u(x, y)$

(1b) $u(x, z) = u(x, y)$

If $u(x, y) \neq u(y, z)$ by the condition ***(U3)*** *we have*

(2) $u(x, z) \leq u(x, y)$ *or* $u(x, z) \leq u(y, z)$

In the case (2) we have two cases

(2a) $u(x, z) = u(x, y)$

(2b) $u(x,z) < u(x,y)$

But also $u(x,y) \leq max\{u(x,z), u(z,y)\}$*. If* $max\{u(x,z), u(z,y)\} = u(x,z)$ *or* $max\{u(x,z), u(z,y)\} = u(z,y)$*, therefore* $u(x,y) < u(x,z)$*. This case is a contradiction with (2b). So,* $u(x,z) = u(x,y)$*.*

Proposition 4.22. *Let* (X,u) *be an ultrametric space if* $y \in B(x,\alpha)$ *then* $B(x,\alpha) = B(y,\alpha)$*. That is, every element of a closed ball is also center of the ball. In that sense, the ultrametric can be called a democratic metric.*

Proof: By definition, $y \in B(x,\alpha)$ if and only if $u(x,y) < \alpha$. Let $z \in X$ such that $u(x,z) < \alpha$. We have $u(x,y) \leq max\{u(x,z), u(z,y)\} < \alpha$. This implies that $z \in B(y,\alpha)$ which shows that $B(x,\alpha) \subset B(y,\alpha)$. In a similar way, we have $B(x,\alpha) \supset B(y,\alpha)$. Thus $B(x,\alpha) = B(y,\alpha)$.

Proposition 4.23. *Let* (X,u) *be an ultrametric space. Every closed balls of* X *are either distinct or nested.*

Proof. Let $a \neq b$. Let $B(a,\alpha_1) = \{x \in |u(a,x) \leq \alpha_1\}$ and $B(b,\alpha_2) = \{x \in |u(b,x) \leq \alpha_2\}$ two closed balls in X with the property that $\alpha_1 \leq \alpha_2$. Therefore, there are two possibilities either $B(a,\alpha_1) \cap B(b,\alpha_2) = \emptyset$ or $B(a,\alpha_1) \subseteq B(b,\alpha_2)$. In particular, if $\alpha_1 < \alpha_2$ and if $a \in B(b,\alpha_2)$ then $B(a,\alpha_1) \subseteq B(b,\alpha_2)$.

Two closed balls in (X,u) are contained in each other, i.e., $B(a,\alpha_1) \cap B(b,\alpha_2)$ is non-empty then either $B(a,\alpha_1) \subseteq B(b,\alpha_2)$ or $B(a,\alpha_1) \supseteq B(b,\alpha_2)$.

Proposition 4.24. *Let* (X,u) *be an ultrametric space. Let* $B(a,\alpha_1)$ *and* $B(b,\alpha_2)$ *two closed balls in* X*. If* $B(a,\alpha_1) \subseteq B(b,\alpha_2)$ *and if* $b \notin B(a,\alpha_1)$ *then* $u(b,a) = u(b,z)$ *for each* $z \in B(a,\alpha_1)$*.*

Proof. Since $u(b,a) > \alpha_1$ and as $u(a,z) \leq \alpha_1$ for every $z \in B(a,\alpha_1)$, we have $u(b,a) > u(a,z)$. However, $u(b,z) = max\{u(b,a), u(a,z)\} = u(a,z)$.

Example 4.25. *Remember the discrete metric is an ultrametric. Let us define the closed balls in discrete metric. In particular, if* $\alpha < 1$ *then the closed ball* $B(x,\alpha) = \{x\}$ *and if* $\alpha \geq 1$ *then* $B(x,\alpha) = X$*. Moreover, if* $y \in B(x,\alpha)$ *then* $B(x,\alpha) = B(y,\alpha)$*.*

Definition 4.26. *Any collection of balls in B which is totally ordered by inclusion is called a nest of ball.*

Definition 4.27. (X, u) *is a spherically complete if every nest of balls has non-empty intersection.*

Definition 4.28. *Let* (X, u) *be an ultrametric space. A map* $f : X \to X$ *is called a contracting map such that for all* $x, y \in X$

$$u(f(x), f(y)) < u(x, y)$$

Definition 4.29. *Let* (X, u) *be an ultrametric space.* f *is contracting on orbits if*

$$u(f(x), f^2(x)) < u(x, y)$$

for all $x \in X$ *such that* $x \neq f(x)$.

Theorem 4.30. ***(Ultrametric Fixed Point)*** *Every non-expanding function on a spherically complete ultrametric space that is also contracting on orbits has a fixed point.*

We will give the proof of Theorem 4.30 in Chapter 6.

CHAPTER 5

ORDERED SPACES

In this section, we will give some fixed point theorems in the spaces, notably in totally ordered spaces and in complete partially ordered spaces. We had already defined the totally ordered set and partially ordered set in Chapter 2.

In the following, we define complete partially ordered space and lattices ([13], [14] and [23]).

Definition 5.1. *Let $(X, \preccurlyeq)$ be a partially ordered space. $(X, \preccurlyeq)$ is complete if every nonempty subspace S of X has a least upper bound in X.*

Definition 5.2. *A complete partially ordered set is a pointed complete partially ordered set if each of its subsets has a least element.*

Definition 5.3. *A partially ordered space $(X, \preccurlyeq)$ is a lattice if and only if every $S \subseteq X$ consisting exactly two elements has a least upper bound and a greatest lower bound.*

Example 5.4. *Let X be a cartesian square some of the natural numbers such that $X = \{(0,0), (0,1), (1,0), (1,1)\}$. X is a partially ordered set with $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. Every two-elements subset of X has least upper bound $(1, 1)$ and greatest lower bound $(0, 0)$ in X. So, X is a lattice.*

Definition 5.5. *A partially ordered space $(X, \preccurlyeq)$ is a complete lattice if and only if every $S \subseteq X$ has a least upper bound and greatest lower bound.*

Remark 5.6. *Every complete lattice is a bounded lattice.*

Definition 5.7. *Let X be a partially ordered set. An interval R in X is said to be a* *rectangle if it is defined as follows:*

$$R = \{x \in X \mid a \leq x \leq b, a \leq b \; and \; a, b \in X\}$$

Theorem 5.8. *If X is a complete lattice, then for any $a, b \in X$, the interval $R = \{x \in X \mid a \leq x \leq b\}$ is a complete lattice.*

Proof. Consider any subset S of R. We will prove that S has a least upper bound and a greatest lower bound.
Since X is a complete lattice than, there exists s^* in X such that $supS = s^*$. We will show that $s^* \in R$.
As b is an upper bound for R. Then, b is also an upper bound for $S \subseteq R$ such that $s^* \leq b$.
As a is a lower bound for R. Then, a is also an lower bound for $S \subseteq R$ such that $a \leq x \leq s^* \leq b$ for any x in S. Thus $s^* \in R$.
On the other hand, since X is a complete lattice then, there exists a $\bar{s}$ in X such that $infS = \bar{s}$. We will show that $\bar{s} \in R$.
As a is a lower bound for R, a is also a lower bound for $S \subseteq R$ such that $a \leq \bar{s}$.
As b is an upper bound for R, b is also an upper bound for $S \subseteq R$ such that $a \leq \bar{s} \leq y \leq b$ for any y in S. Therefore, $\bar{s} \in R$.
Hence, every $S \subseteq R$ has a least upper bound and a greatest lower bound, which means R is a complete lattice.

Definition 5.9. *Let $(X, \preccurlyeq)$ is ordered space. Let $f : X \to X$ be defined. If $f(x) \geq x$ where $x \in X$ then f is called progressive function.*

Definition 5.10. *Let $(A, \preccurlyeq)$ and $(B, \preccurlyeq)$ are ordered spaces. Let $f : A \to B$ be a function satisfying $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$ where $x, y \in A$ then f is called an order preserving function.*

5.1 Fixed Point Theorems for Ordered Spaces

In the following, we will state the fixed point theorems for an order preserving and progressive function which are defined in an ordered space.
The following theorem is important in the same that in order to have a fixed point

in totally ordered space $X = ([0,1], \leq)$, we will not need continuity, being order preserving will suffice.

Theorem 5.11. *Let $f : [0,1] \to [0,1]$ be an order preserving function. Then f has a fixed point.*

Proof. Let $A := \{x \in [0,1] \mid x \leq f(x)\}$. We will show that there exists α such that $supA = \alpha$ and then $f(\alpha) = \alpha$.
First observe that $0 \in A$ by definition of A and f on $[0,1]$. A is bounded since $A \subset \mathbb{R}$. There exists $\alpha \in [0,1]$ such that $\alpha = supA$ because A is nonempty and bounded. Therefore, for every $x \in A$, $x \leq \alpha$. Then we have

$$x \leq f(x) \leq f(\alpha).$$

Thus, $f(\alpha)$ is an upper bound for A such that

$$\alpha \leq f(\alpha) \ \textbf{(1)}.$$

By using (1) and as f is increasing, $f(\alpha) \leq f(f(\alpha))$. This implies that $f(\alpha) \in A$ by definition of A. Therefore,

$$f(\alpha) \leq \alpha = supA \ \textbf{(2)}.$$

By the **(1)** and **(2)**, we will have $f(\alpha) = \alpha$. Thus α is a fixed point of f.

Similarly, let $B := \{x \in [0,1] \mid x \geq f(x)\}$. We will show that there exists β such that $infB = \beta$ and then $f(\beta) = \beta$.

First, observe that $1 \in B$ by definition of f and f on $[0,1]$. B is bounded since $B \subset \mathbb{R}$. There exists $\beta \in [0,1]$ such that $\beta = infB$ because B is nonempty and bounded. Therefore, for every $x \in B$, $x \geq \beta$. Then we have

$$x \geq f(x) \geq f(\beta).$$

Thus, $f(\beta)$ is an lower bound for B such that

$$\beta \geq f(\beta) \ \textbf{(3)}.$$

By using (3) and as f is decreasing, $f(\beta) \geq f(f(\beta))$. This implies that $f(\beta) \in B$ by definition of B. Therefore,

$$f(\beta) \geq \beta = infB \ \textbf{(4)}.$$

Inequalities **(3)** and **(4)** show us that $f(\beta) = \beta$. Thus β is a fixed point of f, then the result follows. ■

Theorem 5.12. ***(Bourbaki-Witt 1)*** *Let* $(X, \leq)$ *be a complete partially ordered space. Then, every progressive function* $f : X \to X$ *has a fixed point on* X.

Theorem 5.13. ***(Bourbaki-Witt 2)*** *Let* $(X, \leq)$ *be a pointed complete partially ordered space. Then, every order preserving function* $f : X \to X$ *a fixed point on* X.

The proofs of Theorem 5.12 & Theorem 5.13 are given in Chapter 6.

CHAPTER 6

A GENERAL FIXED POINT THEOREM

In this chapter, we will give a general fixed point theorem given in [11]. This theorem offers a general idea on the fixed point theorem which is in fact working with ball spaces instead of metrics, ultrametrics, order or topology.
Our aim in this chapter is to prove the fixed point theorems that we gave in Chapter 3, 4, 5 by using this general fixed point theorem due to [11].
We have already presented the notion of "ball", "nest of balls", "spherically completeness" in Chapter 4 for ultrametric spaces. We will also present the analogue notions for metric spaces and ordered spaces.
We first define the ball spaces which will be common for all the spaces.

Definition 6.1. ***(Ball Space)*** *Let X be a space (with a metric, ultrametric or ordered space). Let $\mathbb{B}$ is a collection of closed balls with respect to the structure of the space. $(X, \mathbb{B})$ is the called a ball space.*

Then the following theorem stated in ball spaced notions will cover all concepts of metric, ultrametric and ordered spaces.

Theorem 6.2. ***(General Fixed Point Theorem)*** *Let $(X, \mathbb{B})$ be a spherically complete ball space. Let $f : X \to X$ be a function on a ball space $(X, \mathbb{B})$ satisfying the following conditions:*

(1) $f(B) \subset B$ *for all* $B \in \mathbb{B}$

(2) B *is a singleton or there exists a ball* $B' \subsetneqq B$

(3) *The intersection of any nest of balls is a singleton or contains a ball*

Then f has a fixed point.

We will not give a proof of the Theorem 6.2 in this thesis but refer to [11] for a proof. In the following, we will give the proofs of Theorem 3.4, Theorem 4.30, Theorem 5.11 and Theorem 5.12 by using the General Fixed Point Theorem.

6.1 Proof of Theorem 3.4

First of all, let us construct the ball space $(X, \mathbb{B})$ for the complete metric space X.
By the generalized triangle inequality axiom of a metric space and since f is a contraction with $0 < c < 1$, we have:

$$\begin{aligned} d(x, f(x)) &\leq d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^{i-1}(x), f^i(x)) \\ &\leq d(x, f(x))(1 + c + c^2 + ... + c^{i-1}) \\ &\leq d(x, f(x)) \sum_{i=0}^{\infty} c^i = \frac{d(x, f(x))}{1-c} \end{aligned}$$

Thus, we can choose the collection of the following balls in order to have a ball space $(X, \mathbb{B})$.

$$B_x := \{y \in X \mid d(x, y) \leq \frac{d(x, f(x))}{1-c}\}$$

As (X, d) is complete, $(X, \mathbb{B})$ is spherically complete.
By this choice of ball, it is clear that $f^i(x) \in B_x$ for all $i \geq 0$. In particular, $x \in B_x$.

We wish to show that $f(B_x) \subset B_x$ and $B_{f^k(x)} \subsetneq B_x$.

For $y \in B_x$,

$$\begin{aligned} d(x, f(y)) &\leq d(x, f(x)) + d(f(x), f(y)) \\ &\leq d(x, f(x)) + cd(x, y) \\ &\leq d(x, f(x)) + c\,\frac{d(x, f(x))}{1-c} \\ &= \frac{d(x, f(x))}{1-c} \end{aligned}$$

$f(y) \in B_x$ which will imply also $f(B_x) \subset B_x$ in the case of $x \neq f(x)$. (1) is satisfied.

Let $x \neq f(x)$. Let $z \in B_{f(x)}$, where

$$B_{f(x)} = \{z \in X \mid d(f(x), z) \leq \frac{d(f(x), f^2(x)}{1-c}\}$$

then we will have

$$d(f(x), z) \leq \frac{d(f(x), f^2(x))}{1-c} \leq c\,\frac{d(x, f(x))}{1-c} < \frac{d(x, f(x))}{1-c}$$

which means $z \in B_x$, hence $B_{f(x)} \subseteq B_x$.

We will prove also that for some $k > 1$, $B_{f^k(x)} \subseteq B_x$.

Since one can always find $k \geq 1$ for $c > 1$ such that

$$\frac{c^k}{1-c} < \frac{1}{2}$$

we will have

$$\frac{d(f^k(x), f^{k+1}(x))}{1-c} \leq \frac{c^k}{1-c}\,d(x, f(x)) < \frac{1}{2}\,d(x, f(x))$$

which implies that x and $f(x)$ can not lie in $B_{f^k(x)}$. Hence $B_{f^k(x)} \subsetneq B_x$.

Hence there is a ball $B' \subsetneq B$ for all $B \in \mathbb{B}$ or B is a singleton. (2) is satisfied.

Take a nest of balls $N = \{B_k := B_{f^k(x_0)}\}$ where $x_0 \in \cap N$.

Pick any $B_k \in N$, $k > 0$.

$$d(f^k(x), f(x_0) \leq c\,d(f^{k-1}(x), x_0) \tag{6.1}$$

using $x_0 \in \cap N \subseteq B_{f^k(x)}$

$$\begin{aligned} d(x_0, f(x_0)) &\leq d(x_0, f^k(x)) + d(f^k(x), f(x_0)) \\ &\leq d(x_0, f^k(x)) + c\ \ d(f^{k-1}(x), x_0))\ \ by\ \ (6.1) \\ &\leq \frac{d(f^k(x), f^{k+1}(x))}{1-c} + c\ \ \frac{d(f^{k-1}(x), f^k(x))}{1-c} \\ &\leq \frac{c^k d(x, f(x))}{1-c} + \frac{c\,c^{k-1} d(x, f(x))}{1-c} \\ &\leq 2\frac{c^k}{1-c} d(x, f(x)) \end{aligned}$$

Since, limit of right hand side is 0 as $k \to \infty$.

Thus, $x_0 = f(x_0)$, we have found a fixed point that means also $B_{x_0} \subseteq \cap N$, in fact $\cap N = x_0 = B_{x_0}$. **(3)** is satisfied.

Then, f has a fixed point.
Note that if we had two distinct fixed x_0 and y_0 points then $d(x_0, y_0) = d(f(x_0), f(y_0) \leq c \ : d(x_0, y_0)$. $0 < c < 1$ contradiction. ■

6.2 Proof of Theorem 4.30

First of all, let us construct the ball space $(X, \mathbb{B})$ for the spherically complete ultrametric space (X, u) with the following balls:

$$B_x := \{y \in X \mid u(x, y) \leq u(x, f(x))\}$$

(X, u) is spherically complete then every nest of these balls of also has non-empty intersection. By this choise of balls, it is clear that

$$x \in B_x \ \ as \ \ u(x, x) = 0 \leq u(x, f(x))$$

Let $y \in B_x$. Thus by the definition of balls,

$$u(x, y) \leq u(x, f(x)) \tag{6.2}$$

Since (X, u) is an ultrametric space $u(f(y), x) \leq max\{u(f(y), f(x)), u(f(x), x)\}$. Morever since f is non-expanding, $u(f(y), f(x)) \leq u(y, x)$ but by (6.2), we will have

$$u(f(y), f(x)) \leq u(y, x) = u(x, y) \leq u(x, f(x))$$

Hence $max\{u(f(y), f(x)), u(f(x), x)\} = u(f(x), x)$ and we will have $u(f(y), x) \leq u(f(x), x)$ that means by the definition of B_x that $f(y) \in B_x$. We have thus showed that if $y \in B_x$ then $f(y) \in B_x$, which can be written as $f(B_x) \subset B_x$. Condition **(1)** of General Theorem is satisfied.
If $z \in B_{y=f(x)}$,we will have

$$u(y, z) \leq u(y, f(y)) \leq u(x, f(x)) = u(x, y)$$

Also $u(z, x) \leq max\{u(z, y), u(y, x)\} = u(x, f(x))$ which implies $z \in B_x$.
Thus, we have proven that $B_y \subset B_x$, which is equal to $B_{f(x)} \subseteq B_x$. If $x \neq f(x)$, $B_{f(x)} \subsetneq B_x$. Thus **(2)** is satisfied.
Let N be a nest of balls B_x. If $z \in \cap N \neq \emptyset$ we have $B_z \subset \cap N$. Since $B_z \subseteq B_x$ for every $z \in B_x$ which is implied by **(1)** as $z \in B_x$ which means that $f(z) \in B_x$. If $B_z = \{z\}$ we have $\cup N = \{z\}$. **(3)** is satisfied.

6.3 Proof of Theorem 5.12

Let us choose the ball space $(X, \mathbb{B})$ for the complete partially ordered space. $(X, \mathbb{B})$ is a spherically complete space. We have :

$$B_x := \{[x, +\infty) \mid x \in X\}$$

We wish to show that $x \in B_x$ and $f(B_x) \subset B_x$.
Let $y \in B_x$. Since f progressive and by the definition of ball:

$$f(y) \geq y \geq x$$

Hence $f(y) \in B_x$ which will imply $f(B_x) \subset B_x$ for all $x \in X$ in the case of $x \neq f(x)$. **(1)** is satisfied.
If there does not exist a $y \in X$ such that $y > x$, $B_x = [x, +\infty) = \{x\}$. B_x is a singleton.
Otherwise, if $y \in X$ there exist such that $y > x$ which implies the existence of B' such that $B' = [y, +\infty) \subsetneq B_x = [x, +\infty)$. **(2)** is then satisfied.
Take a nest of balls of B_{x_i} $N = \{[x_i, +\infty)\}_{i \in I}$. As X is a complete partially ordered set such that there exist a $s = sup\{x_i \mid i \in I\}$. Therefore $\cap\{[x_i, +\infty)\}_{i \in I} = [s, +\infty)$ is a ball. **(3)** is satisfied.
By General Fixed Point Theorem, there exist a x_0 such that $f(x_0) = x_0$.

6.4 Proof of Theorem 5.13

Let us choice the ball space $(X, \mathbb{B})$ for the complete partially ordered space. $(X, \mathbb{B})$ is a spherically complete space. We have:

$$B_x := \{[x, +\infty) \mid f(x) \geq x\} \cup \{x\}$$

We wish to show that $y \geq x$. Since f order preserving function

$$f(y) \geq f(x) \geq x$$

Hence $f(y) \in B_x$ which will imply also $f(B_x) \subset B_x$ for all $x \in X$ in the case of $x \neq f(x)$. **(1)** is satisfied.

If $f(x) = x$. $B_x = \{x\}$ such that B_x is a singleton. Otherwise, if $f(x) > x$, $B^{'} = [f(x), +\infty) \subset B_x = [x, +\infty)$. **(2)** is satisfied.

Take a nest of balls $N = \{[x_i, +\infty)\}_{i \in I}$. As X is a complete partial order set such that there exist a $s = sup\{x_i \mid i \in I\}$. $s \geq x_i$ and since f order preserving $f(s) \geq f(x_i) \geq x_i$ by definition of B_x. Therefore $\cap[x_i, +\infty) = [s, +\infty) \in B_x$ is a ball. **(3)** is satisfied.

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