EXAMINING PROSPECTIVE ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE ABOUT STUDENTS' MISTAKES RELATED TO FRACTIONS

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ABSTRACT

EXAMINING PROSPECTIVE ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE ABOUT STUDENTS' MISTAKES RELATED TO FRACTIONS

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The purpose of this study was to investigate the prospective teachers' knowledge of mistakes held by elementary students in fractions and their proposed strategies to overcome those mistakes. The data were collected from 149 prospective elementary mathematics teachers enrolled in the elementary mathematics education programs from a public university in Central Anatolian Region. Fraction Knowledge Questionnaire was used to accomplish the purpose of the study. The data collection tool included nine open ended questions, and each question had two sub-tasks. In this study, the items in the "Fraction Knowledge Questionnaire" were analyzed in-depth in order to reach a detailed description of prospective teachers' knowledge about students' mistakes on fractions.

The results of this study revealed that prospective elementary mathematics teachers mostly could identify the students' mistakes. However, although prospective teachers could notice the students' mistakes, they could give superficial reasons for these mistakes. Furthermore, verbal explanations, using area representation, using real life model, reviewing prior knowledge, teaching standard algorithm, asking guided questions, using simple examples, using counter examples, using drill and practice, making students aware of their mistakes, and increasing students' motivation were the suggested strategies by prospective teachers in order to overcome students' mistakes in fractions.

Keywords: Teacher Knowledge about Students' Mistakes, Prospective Elementary Mathematics Teachers, Students' Mistakes in Fractions

ÖZ

İLKÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ ÖĞRENCİLERİN KESİRLER KONUSUNDAKİ HATALARIYLA İLGİLİ BİLGİLERİ

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Bu çalışmanın amacı, ilköğretim matematik öğretmen adaylarının öğrencilerin kesirler konusundaki hatalarıyla ilgili bilgilerini ve bu hataları gidermek için önerdikleri yöntemleri incelemektir. Çalışmanın amacı doğrultusunda Kesir Bilgisi Anketi kullanılmıştır. Ölçme aracı dokuz açık uçlu sorudan oluşmaktadır ve her bir sorunun iki alt boyutu bulunmaktadır. Bu çalışmada, öğretmen adaylarının öğrencilerin kesirler konusundaki hatalarıyla ilgili bilgileri hakkında derinlemesine bilgi edinmek için, Kesir Bilgisi Anketindeki maddeler incelenmiştir.

Araştırmanın sonuçlarına göre, ilköğretim matematik öğretmen adaylarının çoğu öğrencilerin kesirler konusundaki hatalarının farkındadırlar. Ancak, öğretmen adayları öğrenci hatalarının farkında olmasına rağmen, öğrenci hatalarına yüzeysel sebepler bildirebilmişlerdir. Ayrıca, öğrencilerin hatalarını gidermek için; sözel açıklamalar, alan modelleri, günlük hayat örnekleri, ön bilgilerin tekrarı, standart çözümün öğretilmesi, yönlendirici soruların sorulması, kolay örneklerin kullanımı, karşıt örneklerin kullanımı, alıştırma ve uygulama yaptırma, öğrencileri hatalarının farkına vardırma ve öğrencinin motivasyonunu arttırma stratejilerini önermişlerdir.

Anahtar Kelimeler: Öğretmenlerin Öğrenci hataları hakkındaki bilgisi, İlköğretim Matematik Öğretmen Adayları, Öğrencilerin Kesirler Konusundaki Hataları

To my husband, Yılmaz
For his love, patience, and understanding
To my parents
For their care, support, and encouragement

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LIST OF ABBREVIATIONS

CCK: Common Content Knowledge

FKQ: Fraction Knowledge Questionnaire

KCS: Knowledge of Content and Students

KCT: Knowledge of Content and Teaching

NCTM: National Counsil of Teachers of Mathematics

PCK: Pedagogical Content Knowledge

SCK: Sprecialized Content Knowledge

SMK: Subject Matter Knowledge

CHAPTER I

INTRODUCTION

Researchers have generally agreed that there is a close relationship between teachers' knowledge and classroom instruction (Ball, 2000; Ball, Thames, & Phelps, 2008; Ma, 1999; Shulman, 1987). The National Council of Teachers of Mathematics [NCTM] (2000) pointed out that "effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (p.17). In addition, teachers need to understand what students know and need to learn and they should challenge and support students to learn the content well (NCTM, 2000). Thus, teachers who had a weak understanding of a subject could not have the knowledge that was necessary to teach this content (Ball, Thames, & Phelps, 2008).

NCTM (2000) emphasized that teachers had to facilitate students' procedural and conceptual understanding of the mathematics contents and connections among these contents. Teachers who made the rich interconnections within the knowledge would use the activities and strategies involving these connections to teach for understanding (McLaughlin, & Talbert, 1993). Moreover, Ball (1990) indicated that teachers' knowledge directly affected the students' understanding and achievement. Ball's (2005) findings supported the argument that when teachers' mathematical knowledge improved, students' mathematics achievement was also improved. On the other hand, teachers who had deficiencies in subject matter knowledge (SMK) passed their misunderstandings to their students (Even, 1990). Therefore, SMK of teachers had an important effect on students' conceptions (Tirosh, 2000).

The importance of the teachers' knowledge on students' learning directed researchers to investigate teachers' and prospective teachers' knowledge. Many studies focused on teachers' and prospective teachers' understandings of specific topics and knowledge of mathematics for teaching (Ball, 1988; Ball, 1990; Ball & McDiarmid, 1988; Even, 1989; Huang, Liu, & Lin, 2009; Owens, 1987; Post, Behr, Harel, & Lesh, 1988). These research studies investigated how teachers would think, understand, and teach specific mathematical knowledge and ideas. The results of these studies revealed that prospective and in-service teachers had limited content knowledge for teaching mathematics (Even & Tirosh, 1995; Hill, 2007; Işıksal, 2006; Türnüklü & Yeşildere, 2007). More specifically, the results of these studies showed that prospective teachers had weak conceptual understanding of the key concepts of multiplication and division of fractions (Işıksal, 2006), limited conceptions about functions (Even, 1993; Karahasan, 2010), and limited knowledge about undefined mathematical operations and ability to evaluate students' reasoning (Even & Tirosh, 1995). Thus, the researchers suggested examining the teachers' knowledge of reasons for students' learning difficulties in different subject areas in mathematics (Bingölbali, Akkoç, Özmantar, &Demir, 2011).

To make the subject matter understandable for students, teachers should know and understand the content of the curriculum, know students' preconceptions and misconceptions, know how to evaluate and respond to students' mistakes, and select appropriate representations (Ball, Hill, & Bass, 2005). Based on the tasks which was necessary for teachers stated by Ball et al. (2005), the prospective teachers' knowledge of students' mistakes and their proposed strategies in order to overcome these mistakes were questioned in this study.

1.1. Statement of Problem

Fractions could be considered as one of the most important areas that are mathematically rich, cognitively complicated, and difficult to teach in elementary school mathematics (Smith, 2002). Fractions had the basic concepts that students should understand the basic mathematics facts and algorithms to learn it. Its importance in the elementary and middle school mathematics would be due to its own structure and connections to other concepts in mathematics (Mundy, Schmidt, Leroi, Bates, & Joyner, 2006). Stated differently, fractions are the essential topic for mathematics curricula and textbooks. However, it is common for students to memorize the rote procedures, and then forget the memorized procedures after awhile, and therefore find fractions difficult to learn (Mack, 1990). Thus, fractions are one of the topics with which both teachers and students have some difficulties and misconceptions (Behr, Khoury, Harel, Post, & Lesh, 1997; Cramer, Post, & del Mas, 2002; Simon, 1993; Wearne & Kourne, 2000).

According to Tirosh (2000), students' mistakes in fractions could be organized in three categories namely; algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge. The erroneous computations are included in the algorithmically based mistakes. The mistakes caused by students' intuitions, such as 'multiplication always makes bigger, are included in the intuitively based mistakes. And lastly, the mistakes because of students' limited conceptions and inadequate knowledge related to operations are involved in the mistakes based on formal knowledge. Based on the students' mistakes in fractions, teachers need to be familiar with students' common conceptions and misconceptions about fractions in order to enhance the conceptions and to overcome these misconceptions.

Teachers' mathematical content knowledge has an important role in order to promote students' mathematical understanding (Even &Tirosh, 1995). A teacher who knows students' conceptual challenges could extend student thinking and modify or develop appropriate activities for students (Even & Tirosh, 1995). Tirosh (2000) suggested that teacher education programs familiarize prospective teachers with various, and sometimes erroneous, common types of cognitive processes and how they may lead to various ways of thinking. In addition, Tirosh's (2000) study showed that it was possible to learn the students' misconceptions and common mistakes during the process of a teacher education program. Therefore, since the instructional interventions help to enhance prospective teachers' knowledge, the details of prospective teachers' knowledge should be analyzed before the instructional interventions.

Many research studies have been carried out on prospective teachers' knowledge of specific topics including multiplication, division, equivalence, and number line in fractions (Işıksal, 2006; Izsak, 2008; Pesen, 2008; Tirosh, 2000). Those studies pointed out that the deficiencies of prospective teachers in mathematical content knowledge have influenced their future teachings. Moreover, those studies have implied that it is important to conduct studies which examine prospective teachers' understanding of children's knowledge in pre-service teacher education programs, since the mathematics teaching methods courses had an important role in improving prospective teachers' mathematical content knowledge for teaching (Seviş, 2008). Thus, this study aims to answer the following questions:

1. To what extend can prospective elementary mathematics teachers notice elementary students' mistakes in fractions?

    1.1. What is the nature of prospective elementary mathematics teachers' knowledge of reasons for students' mistakes based on the formal knowledge related to fractions?

    1.2. What is the nature of prospective elementary mathematics teachers' knowledge of reasons for students' algorithmically based mistakes related to fractions?

    1.3. What is the nature of prospective elementary mathematics teachers' knowledge of reasons for students' intuitively based mistakes related to fractions?

2. What kinds of strategies do prospective elementary mathematics teachers suggest to overcome those mistakes held by elementary students related to fractions?

    2.1. What kinds of strategies do prospective elementary mathematics teachers suggest to overcome elementary students' mistakes based on formal knowledge related to fractions?

    2.2. What kinds of strategies do prospective elementary mathematics teachers suggest to overcome elementary students' algorithmically based mistakes related to fractions?

    2.3. What kinds of strategies do prospective elementary mathematics teachers suggest to overcome elementary students' intuitively based mistakes related to fractions?

1.2. Definitions of Important Terms

The research question included the following terms which need to be defined:

Prospective elementary mathematics teachers

Prospective elementary mathematics teachers are the junior and senior students in elementary mathematics teacher education program. The prospective elementary mathematics teachers are the candidates to teach mathematics in primary and middle schools after their graduation.

Mathematical Knowledge for Teaching

Mathematical knowledge for teaching refers to "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students (Ball et al., 2008, p.399). In this study, the mathematical knowledge for teaching could be summarized as the knowledge needed in order to recognize a wrong answer, size up the nature of a mistake, especially an unfamiliar mistake, and prosposed strategies to overcome these mistakes. In this study, this knowledge was measured by the "Fraction Knowledge Questionnaire".

Fraction

Fraction refers to "the equal shares or equal-sized portions of a whole or unit" (Van de Walle, 2004, p.242).

Mathematical Mistake

Mathematical mistake refers to "the result of carelessness, misinterpretation of symbols or text; lack of relevant experience or knowledge related to that mathematical topic/ concept; a lack of awareness or inability to check the answer given; or the result of a misconception" (Drew, 2005, p.14).

Algorithmically based Mistakes

The algorithmically based mistakes refer to the errors in performing the algorithmic operations. Adding or subtracting both the numerator and denominator from each other in order to perform the addition or subtraction operation is an example of an algorithmically based mistake.

Intuitively based Mistakes

Intuitively based mistakes refer to the errors resulting from intuitions (Tirosh, 2000). For example, *division always makes smaller* is an example of an intuitively based mistake.

Mistakes based on Formal Knowledge

Difficulties based on formal knowledge refer to the "incorrect performance due to both limited conceptions and inadequate knowledge" (Tirosh, 2000, p. 7). Dividing a whole into unequal unit portions in order to show a fraction is an example of a mistake based on formal knowledge.

1.3. Significance of the Study

Fractions are one of the most important topics in the elementary mathematics curriculum and have multiple connections with other topics. In addition, it is conceptually one of the most complex topics for students and teachers (Wearne & Kourne, 2000). Students transfer their inadequate fraction concept knowledge into difficulties with fraction computation, decimal and percent concept, the use of fractions in measurement, and ratio and proportion concept (Van de Walle, 2007). Therefore, it could be indicated that teachers should promote the students' conceptual understanding of fractions.

Several research studies have investigated prospective teachers' mathematical content knowledge. The results of those previous research studies revealed that the prospective and in-service teachers had limited content knowledge for teaching mathematics (Even & Tirosh, 1995; Hill, 2007; Türnüklü & Yeşildere, 2007). More specifically, previous research studies concerning the prospective teachers' knowledge about fractions demonstrated that they had limited knowledge about this topic (Ball, 1990; Tirosh, 2000). The studies also revealed the relationship between teachers' mathematical content knowledge and pedagogical content knowledge

(Türnüklü, 2005). Moreover, Işıksal (2006) stated that prospective teachers' limited conceptions affected their knowledge of students' conceptions and reasons for these conceptions.

In order to teach for understanding, teachers have to have rich interconnections within the content knowledge and use the activities and strategies addressing these connections (Ball, 1990; Hill, Rowan, & Ball, 2005). Teachers who knew the subject and the ways of making it meaningful for students would have more mathematically competent students (Ball et al., 2008). Moreover, Simon and Blume (1994) stated that teachers, who are aware of the important conceptual connections for students, could improve their ability to interpret the students' reasoning. Conversely, Even (1990) mentioned that teachers who have misunderstandings and misconceptions passed these deficiencies to their students. These results were also consistent with the prospective teachers; since prospective elementary mathematics teachers will soon be teaching in the elementary classes.

In order to enhance students' learning, An, Kulm and Wu (2004) emphasized that teachers should focus on students' conceptual understanding rather than procedures or rules. Moreover, they stated that teachers might identify students' misconceptions correctly and eliminate such misconceptions. More specifically, teachers need to be able to determine the reasons for students' difficulties and mistakes in order to correct them effectively (Kılıç & Özdaş, 2010).

Teachers' knowledge about students' mathematical conceptions and misconceptions are important for teaching (Tirosh, 2000). However, previous research studies pointed out that prospective teachers' reasoning about student difficulties were inadequate (Ball, 1990; Even & Tirosh, 1995). In a research study, Bingölbali, Akkoç, Özmantar and Demir (2011) found that both pre-service and in-service teachers mostly attributed students' mathematical difficulties to student related causes and psychological causes, but not pedagogical causes. These deficiencies could be detected and overcome during the teacher education programs. Thus, it could be inferred that more attention should be given to teacher education programs and prospective teachers' knowledge of students' knowledge and mistakes.

Tirosh (2000) observed that it was possible to teach the knowledge of students' mathematical conceptions and difficulties during the teacher-preparation courses. Moreover, Işıksal (2006) stated that prospective teachers might develop better understanding of student thinking if they were provided with a chance to analyse elementary students' conceptions and mistakes during the courses. Therefore, if prospective teachers' mathematical content knowledge could be detected during teacher education, their teaching performance would improve during their teaching life (Johnson, 1998).

Review of literature shows that there have been a few studies investigating the prospective teachers' knowledge about students' mistakes and reasons for these mistakes. The current study aimed to investigate the prospective teachers' knowledge of mistakes in the fractions held by elementary students and their proposed strategies to overcome those mistakes. The findings of this study would have implications for the improvement of teacher education programs in terms of the prospective teachers' needs. In addition, by this study, the prospective teachers could have a chance to evaluate their knowledge about students' mistakes about fractions before they graduate from the program.

CHAPTER II

LITERATURE REVIEW

The purpose of this study was to investigate the prospective teachers' knowledge of mistakes held by elementary students in fractions and their proposed strategies to overcome those mistakes. In this chapter, the framework for mathematical content knowledge for teaching, related studies concerning the teachers' mathematical knowledge and students' mistakes with fractions, and the definition of misconceptions and mistakes are reviewed.

2.1. The Framework for Teachers' Mathematical Content Knowledge

"Teaching is one of the most common—and also one of the most complicated—human activities" (Ball & Forzani, 2010, p.43). Teachers' knowledge is an essential component in order to improve teaching. Many researchers have addressed the close relationship between teachers' knowledge and classroom instruction (Ball, 2000; Ball, Thames, & Phelps, 2008; Ma, 1999; Shulman, 1987). More specifically, effective mathematics classroom instruction requires teachers to know about students, their learning, and strategies for supporting that learning (NCTM, 2000). Therefore, many researchers in mathematics education focus on teachers' mathematical knowledge for teaching.

Several researchers defined teachers' knowledge with several components (An, Kulm, & Wu, 2004; Ball et al., 2008; Fennema & Franke, 1992; Shulman, 1987). In general, definitions and components are based on the Shulman's (1987) definitions and components of teachers' knowledge. Shulman's (1986) framework distinguished teachers' knowledge about different categories of knowledge; Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK), and curricular knowledge. Firstly, in Shulman's framework, SMK refers to "the amount and organization of knowledge per se in the mind of teacher" (p.9). In this domain, teachers are required to understand what the subject is, why this subject is worth knowing, and how it relates to other subjects both within and outside of the discipline (Shulman, 1986). Secondly, Shulman's PCK provided a basis for the SMK dimension for teaching. This category included a) the knowledge of the ways of representing and expressing the subject, b) an understanding of students' conceptions and preconceptions, and c) the knowledge of strategies to overcome the students' difficulties and misconceptions (Shulman, 1986). According to him, the pedagogical understanding of subject matter was at the core of the definition of PCK. Finally, curricular knowledge is the knowledge of an available program and instructional materials to teach a particular subject or topic. In addition curricular knowledge includes, knowing the content of the given course with the contents in other classes and with the topics studied in the same subject area during the preceding and the following years (Shulman, 1986). The above three categories of teachers' content knowledge are an indication of the knowledge of what they teach, how they teach, and the curriculum of the given subject and other related subjects. Moreover, these knowledge bases distinguish between a subject major and a subject teacher in a pedagogical way.

The second approach was An, Kulm, and Wu's (2004) PCK approach. An and his colleagues (2004) distinguished between three components of PCK that teachers need to have knowledge of content, knowledge of curriculum, and knowledge of teaching. Broad mathematics knowledge and specific mathematics content knowledge at the grade level being taught are the components of the knowledge of content. In addition, selecting and using suitable curriculum materials, fully understanding the goals and key ideas of textbooks and curricula are the key parts of the knowledge of curriculum. Lastly, knowledge of teaching includes knowing students' thinking, preparing instruction, and mastery of modes of delivering instructions (An et al., 2004). In the knowledge of teaching part of PCK, knowing students' thinking entails addressing students' misconceptions, engaging students in mathematics learning, building on students' mathematical ideas, and promoting students' thinking in mathematics. In An and his colleagues' categorization of knowledge of teaching was improved from content and curriculum knowledge. Based on the definitions above, An and his colleagues' categorization could be regarded as more detailed compared to Shulman's categorization. Moreover, all the components of the PCK had a relationship in this approach.

Another framework was Fennema and Franke's (1992) approach of mathematics teachers' knowledge. They determined the components of mathematics teachers' knowledge as; (1) Knowledge of mathematics, (2) Knowledge of mathematical representations, (3) Knowledge of students, and (4) Knowledge of teaching and decision making. Firstly, the knowledge of mathematics has two crucial components in its conceptual nature. These components are the nature of mathematics itself and the mental organization of teachers' knowledge. The knowledge of mathematics in Fennema and Franke's framework refers to the subject matter knowledge in Shulman's categorization. The second type of knowledge is the knowledge of mathematical representations. Fennema and Franke (1992) mentioned that mathematics was composed of many related abstractions, and therefore teachers had to translate those abstractions into another form in order to support learners to relate the topics to other topics that they already know. Otherwise, learners would not learn with understanding. The knowledge of mathematical representations in this framework was not separated from knowledge of mathematics. In addition to these two components, teachers' knowledge about their learners was the other component of teachers' knowledge structure. They emphasized the necessity of the knowledge of learners' thinking and how this knowledge should be used in teachers' decision making. The last component of teachers' knowledge is teachers' general knowledge of teaching and decision making. According to Fennema and Franke (1992), since teachers' knowledge, thoughts, and beliefs influenced their decisions, actions, and plans in the classroom, they had to have this type of knowledge. Fennema and Franke's (1992) components could be regarded as more detailed components of teacher knowledge compared to An et al.'s components. Moreover, as seen from the definitions, knowledge of mathematics and knowledge of mathematical representations could be related to the content knowledge in Shulman's categorizations, and knowledge of students and knowledge of teaching and decision making could be related to the pedagogical content knowledge in Shulman's categorizations.

Another framework was the Ball and colleagues' (2008) framework of teachers' mathematical knowledge for teaching. Ball and colleagues (2008) had an empirically developed approach for the content knowledge needed for mathematics teaching. They first defined mathematical knowledge for teaching, and they used teaching in their definition instead of teacher. They defined mathematical knowledge for teaching as the necessary mathematical knowledge in order to perform the work of teaching. Moreover, teaching referred to every thing that teachers do in the classroom in order to facilitate their students' understanding.

Ball and colleagues also defined mathematical knowledge for teaching as "a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, physics, accounting, or carpentry" (Ball, Hill, & Bass, 2005, p.17). Thus, teachers have to know more and different mathematics such as knowing the topic they teach, identifying the students' mathematical mistakes, evaluating reasons for these mistakes, assessing students' alternative responses, explaining the procedures, and knowing the strategic procedures (Ball, Thames, & Phelp, 2008).

In their framework, Ball and colleagues (2008) divided Shulman's (1986) subject matter knowledge into Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK), and his pedagogical content knowledge into knowledge of content and students, and knowledge of content and teaching. The relationship between Ball and colleagues' (2008) mathematical knowledge for teaching framework and Shulman's (1986) categorization of subject matter knowledge and pedagogical content knowledge are presented in the following diagram.

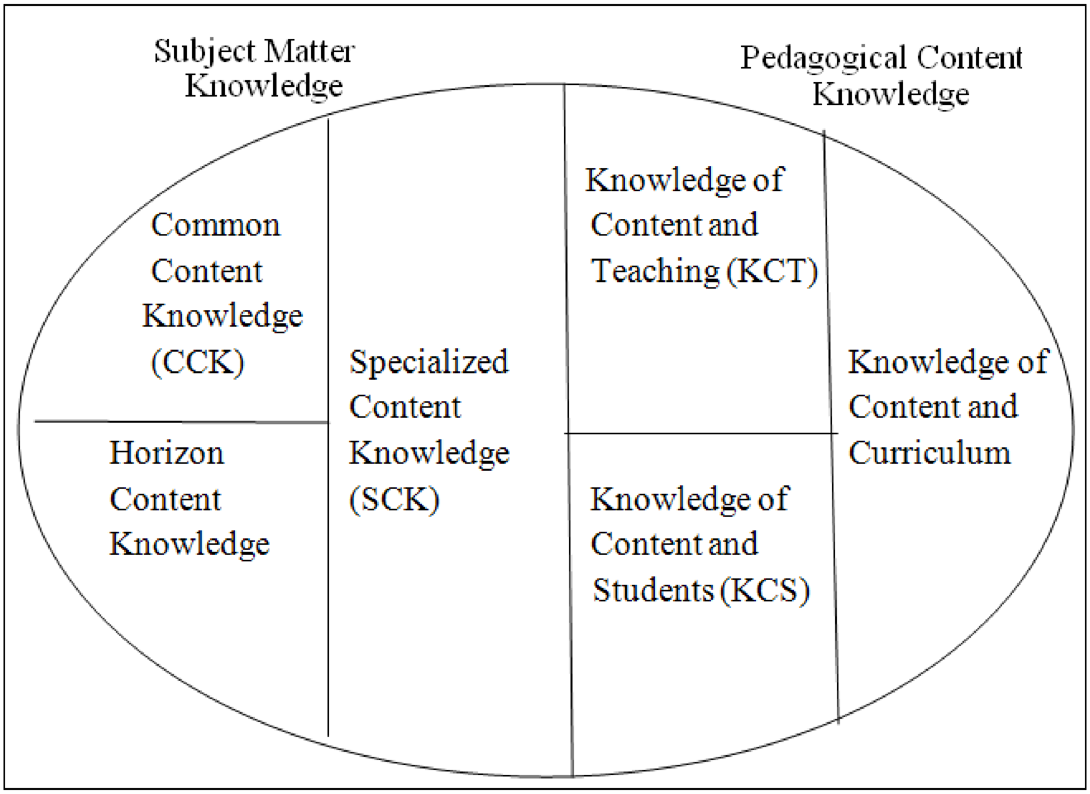


Figure 2.1. The comparison of Ball et al., and Shulman's categorization of mathematical content knowledge
Source: Ball, Thames, and Phelp (2008)

Common content knowledge and specialized content knowledge were two important components of mathematical content knowledge. Ball and colleagues (2008) defined CCK as "the mathematical knowledge and skill used in settings other than teaching" (p.399). According to them, CCK was the knowledge that any well educated adults needed to have. This knowledge base was not unique for a person who was teaching mathematics. More specifically, the knowledge of a number that lay between 1.1 and 1.11, the knowledge that a square is a rectangle and the knowledge that 0/7 is 0 were all examples for common content knowledge that any well educated person necessarily knew. Besides, Ball and colleagues defined SCK as "the mathematical knowledge and skill unique to teaching" (Ball, Thames, & Phelp, 2008, p.400). SCK was the knowledge that was only essential for teachers, since the knowledge beyond that being taught to students was necessary for teaching. For example, the understanding of the difference between take away and the comparison model of subtraction and between the measurement and the partitive model of division were not necessary for students; on the other hand, teaching entailed the understanding of these different interpretations (Ball et al., 2008). Therefore, the demands of teaching mathematics required the mathematical knowledge specialized to teaching.

Shulman's (1986) pedagogical content knowledge was subdivided into knowledge of content and students (KCS), and knowledge of content and teaching (KCT). Ball and her colleagues (2008) defined KCS as knowledge that combines knowing about students and knowing about mathematics. Knowledge of common student conceptions and misconceptions about mathematical content is entailed by KCS. They stated that teachers have to anticipate their students' thinking, which concepts they will find confusing, and which mistakes students tend to be most likely to make. In addition, KCT combines knowing about teaching and knowing about mathematics. For instance, designing instruction, selecting beginning examples, and deciding on appropriate and inappropriate representations are some teaching tasks required by KCT. Furthermore, Ball and her colleagues' four domains that were CCK, SCK, KCS, and KCT related to each other. Ball and her colleagues (2008, p.401) explained this intersection by stating that "recognizing a wrong answer is CCK, whereas sizing up the nature of an mistake, especially an unfamiliar mistake, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of SCK. In contrast, familiarity with common mistakes and deciding which of several mistakes students are most likely to make are examples of KCS" (p.401).

In this intersection, the knowledge bases overlap each other. For example, in order to determine the nature of mistake, teachers need to recognize the wrong answer. This means that the SCK base includes necessary knowledge for CCK base. Therefore, the four domains of the knowledge bases are interrelated to each other.

In this part of the literature review, the various definitions and categories of teachers' knowledge for teaching have been mentioned. In the next section, the related studies of the teachers and pre-service teachers' mathematical content knowledge for teaching will be presented.

2.2. Related Studies about Prospective Teachers' Mathematical Content Knowledge for Teaching

In recent years several studies have been conducted to investigate what mathematical knowledge is needed for teaching mathematics abroad and in Turkey (Ball, 1990; Bingölbali, Akkoç, Özmantar, & Demir, 2011; Even & Tirosh, 1995; Hill, Rowan, & Ball, 2005; Işıksal, 2006; Işıksal & Çakıroğlu, 2011; Karahasan, 2010; Tirosh, 2000; Türnüklü & Yeşildere, 2007; Türnüklü, 2005; Yeşildere & Akkoç, 2010; Izsak, 2008; Zembat, 2004; Zembat, 2007). In this part of the literature review, the previous studies on teachers' mathematical content knowledge for teaching will be discussed.

In order to teach effectively, teachers must be able to understand the topic, teach it, identify students' erroneous answers and interpret them, evaluate alternative algorithms, know underlying principles of the topic, understand meanings for terms and explanations for the topic, and consider the strategic examples to teach the topic (Ball, Thames, & Phelps, 2008). However, previous studies have shown that teachers' mathematical content knowledge for teaching is limited (Ball, 1990; Even & Tirosh, 1995; Simon, 1993; Tirosh, 2000; Türnüklü & Yeşildere, 2007; Türnüklü, 2005).

In a research study, Ball (1990) investigated prospective elementary and secondary teachers' understanding of division with fractions. Nineteen prospective elementary and secondary teachers participated in the study. Prospective teachers were asked division problems in three contexts. All prospective teachers except two correctly found the answers. However, Ball (1990) stated that most of the prospective teachers were not able to explain the rationales and meanings of the division, and only knew the partition meaning of the division which was an easier model for division with fractions. Moreover, she said that prospective teachers' knowledge of division depended mostly on memorization instead of conceptual understanding. Therefore Ball (1990) claimed that the teacher education programs could not provide adequate subject matter knowledge for teaching mathematics for understanding. Parallel to Ball's study, Even and Tirosh (1995) examined prospective teachers' subject matter knowledge of functions and undefined mathematical operations, and their understanding of students' ways of thinking. Researchers examined 162 prospective secondary mathematics teachers' knowledge about functions. Participants were in the last stage of their formal pre-service preparation at eight mid-western universities in the USA. Questionnaires and interviews were used to collect data. First, 152 participants completed an open-ended questionnaire, and 10 prospective teachers were interviewed. Moreover, 33 Israeli secondary mathematics teachers' conceptions of undefined operations were explored. Even and Tirosh (1995) stated that prospective teachers lacked understanding of undefined mathematical operations and could not provide an appropriate explanation for the reasons for undefined operations. Moreover, they declared that this inadequate knowledge of undefined operations affected their choices of responses to students' questions and explanations. They concluded that most of the prospective teachers evaluated students' answers only in terms of right or wrong and provided them with their own explanations for the right answer.

Hill, Rowan, and Ball (2005) examined whether and how teachers' mathematical knowledge of teaching affects students' achievement in mathematics. In this study, the sample included 1190 first graders, 1773 third graders, 334 first grade teachers, and 365 third grade teachers. The researchers used student assessments and parent interviews in order to collect data from students. A log which was a highly structured self-report instrument and an annual questionnaire were used to gather data from teachers. Hill et al. (2005) asserted that teachers' mathematical knowledge positively predicted student gains in mathematics in both first and third grades. In conclusion, the researchers suggested that further research was required on teaching, on students' learning, and on mathematical demands of high-quality instructions in order to understand the role of content knowledge in teaching. The results of Even and Tirosh's (1995) study were consistent with Hill, Rowan, and Ball's (2005) study. Researchers concluded that teachers' mathematical knowledge affected their responses to students, and depending on this, their students' achievement.

In another research study, Tirosh (2000) examined the development of prospective elementary teachers' own SMK of division of fractions and the nature of their awareness and possible reasons for common misconceptions held by children. Thirty female prospective elementary teachers in their second year of the 4-year teacher education program in an Israeli State Teachers' College participated in the study. All prospective elementary teachers completed a questionnaire and were interviewed to assess their SMK and PCK of rational numbers at the beginning of the academic year. During the entire academic year, all prospective teachers participated in a mathematics methods course designed to develop teachers' understanding of mathematical concepts, structures, and relations among rational number topics. The researcher was interested in two main issues that were knowing that and knowing why. In this study, knowledge about students' conceptions and ways of thinking about rational numbers refers to knowing that. Furthermore, knowledge about the reasons of these students' conceptions and understanding the reasons of students' specific responses were about knowing why. They administered a diagnostic questionnaire to measure SMK and PCK of rational numbers at the beginning of the course. The researcher claimed that in terms of knowing that, most prospective teachers had the knowledge of listing common incorrect responses to division expressions involving fractions. In terms of knowing why, he asserted that most prospective teachers thought that students' mistakes were the algorithmically based mistakes; only a few suggested both algorithmically based and intuitively based reasons of mistakes. Furthermore, the researcher stated that all but one prospective teacher wrote correct expressions that would solve the word problems and wrote common incorrect students' responses. In addition, with respect to knowing why, intuitive beliefs about multiplication and division, children's tendencies to attribute properties of natural number operations to fractions, children's tendencies to think that multiplication makes bigger and division makes smaller, algorithmically based mistake, and reading comprehension difficulties were declared reasons of incorrect responses. The researcher also investigated the prospective elementary teachers' understanding of division of fractions, and knowledge of common conceptions and misconceptions held by children. The researcher reported that prospective teachers considered $\frac{1}{4} \div \frac{3}{5} = \frac{1:3}{4:5}$ as an incorrect way of performing the division operation. The researcher designed an instructional intervention in order to improve prospective teachers' subject matter knowledge and pedagogical content knowledge in the case of division of fractions. As a result, the researcher stated that participants were aware of various reasons of incorrect responses after the intervention. In addition to this result, she declared that they interpreted and evaluated the students' mathematical conceptions, students' explanations of division with fractions, and their ways of thinking after the intervention. All of the above studies (Ball, 1990; Even & Tirosh, 1995; Hill, Rowan, & Ball, 2005) showed that prospective teachers had limited mathematical content knowledge. Furthermore, this limited knowledge affected their teaching and their students. However, Tirosh's (2000) study showed that the instructional interventions positively changed the prospective teachers' mathematical knowledge.

There were also studies about teachers' mathematical content knowledge for teaching and pedagogical content knowledge about specific mathematics topics in Turkey (Bingölbali, Akkoç, Özmantar, & Demir, 2011; Dede & Peker, 2007; Işıksal, 2006; Türnüklü & Yeşildere, 2007; Türnüklü, 2005; Zembat, 2007).

In a research study, Türnüklü (2005) determined the relationship between prospective elementary teachers' pedagogical content knowledge and mathematical content knowledge. The researcher developed a four-problem questionnaire and administered it to 45 prospective elementary teachers. Then, the researcher determined the relationship between the scores that were obtained from the questionnaire and the grades from the mathematics courses that they took during the undergraduate study. The researcher declared that there was a relationship between pedagogical content knowledge and mathematical content knowledge. Moreover, she stated that prospective teachers who had high grades from mathematics lessons could not always have sufficient pedagogical content knowledge. In addition to this, prospective teachers who had insufficient mathematics knowledge could not have acceptable pedagogical content knowledge. In other words, she concluded that having mathematical knowledge was important to teach mathematics but not sufficient.

Similarly, Türnüklü and Yeşildere (2007) conducted a study to determine the relationship between mathematical knowledge and pedagogical content knowledge. More specifically, they investigated how prospective teachers use their mathematical knowledge and pedagogical content knowledge in dealing with problems that involve assessing students' ways of thinking and ways of constructing mathematical knowledge. Data were collected from 45 senior prospective elementary mathematics teachers in Turkey. Researchers used four open-ended problems to determine prospective teachers' interpretations of students' misconceptions or misunderstanding of mathematical knowledge about fractions, decimal numbers, and integers. Researchers stated that prospective teachers did not have sufficient mathematical knowledge such as the exact connection between addition and subtraction, and understanding of reasons for students' misconceptions. Moreover, they claimed that prospective teachers preferred procedural and rule–based explanations, instead of encouraging students to discover the mathematical relations. To conclude, the researcher suggested educating teachers in terms of mathematical knowledge and pedagogical content knowledge. The findings of these two studies were compatible with each other.

In a research study, Işıksal (2006) investigated the prospective elementary mathematics teachers' subject matter knowledge, pedagogical content knowledge, and the relationship between SMK and PCK on multiplication and division of fractions. Data were collected through questionnaire and interviews from 17 senior prospective elementary mathematics teachers. The researcher stated that prospective teachers suggested various misconceptions and difficulties on multiplication and division of fractions that elementary students might have. Algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge of fractions, misunderstanding of the symbolism of fractions, and misunderstanding of the problems were reported as suggested students' misconceptions and difficulties. In addition, Işıksal (2006) declared that prospective teachers offered many strategies to overcome these misconceptions and difficulties. She grouped these strategies under three headings: strategies based on teaching methods, strategies based on formal knowledge of fractions, and strategies based on psychological constructs.

Furthermore, Işıksal (2006) found that prospective teachers could easily solve the basic questions about multiplication and division of fractions; however their interpretation and reasoning of key facts and principles on these topics were not conceptually deep. These findings of the study were consistent with Zembat's (2007) research study. Zembat (2007) conducted a study to determine prospective teachers' reasoning styles as they solve problems related to division of fractions. He stated that prospective elementary teachers had difficulties about the process of understanding the components of division of fractions although they made the computations easily with invert and multiply algorithms. Moreover, Işıksal (2006) stated that prospective teachers' limited conceptions influenced their knowledge of students' common conceptions and reasons of these conceptions.

In another research study, Bingölbali and colleagues (2011) conducted a study to determine the views of pre-service and in-service teachers in terms of the reasons of students' mathematical difficulties. Forty prospective mathematics, 15 in-service mathematics, and 15 in-service elementary teachers participated the study. Researchers used questionnaires in order to gather the data. They found that both pre-service and in-service teachers mostly attributed students' mathematical difficulties to student related causes, which were psychological causes. On the other hand, they declared that pre-service and in-service teachers did not pay much attention to pedagogical causes for students' mathematical difficulties. Therefore, researchers concluded that teachers' awareness of the reasons of students' mathematical difficulties was crucial and needed further attention.

In addition to above studies, Dede and Peker (2007) investigated the prospective elementary mathematics teachers' prediction skills of seventh and eighth grade students' misconceptions and difficulties about algebraic expressions and operations. Furthermore, this study examined the prospective teachers' solution strategies in order to overcome the students' difficulties about these topics. Fifty six 7th-grade, fourty three 8th-grades, 55 secondary and 65 elementary prospective mathematics teachers participated in the study. Data were collected by using 10 open-ended questions. The researchers declared that there were three categories of prospective teachers' prediction skills. They addressed these categories concurrent responses, unexpected responses, and unpredicted responses. Concurrent responses included the mistakes that were made by students and predicted by prospective teachers. Unexpected responses consisted of the mistakes that prospective teachers predicted, however students did not make. And finally, unpredicted responses involved the mistakes that students made, however prospective teachers could not predict. Moreover, the researchers stated that prospective teachers were not able to provide adequate solution strategies in order to overcome the students' mistakes and difficulties with algebraic expressions. More specifically, they said that their solution strategies mostly depended on the direct teaching methods.

The research studies as mentioned above showed that prospective teachers had limited mathematical content knowledge for teaching mathematics. More specifically, teachers and prospective teachers' knowledge about the reasons of students' mathematical difficulties were limited. However, teachers' inadequate knowledge affected their responses to students' questions and explanations and their evaluations of students' answers. Therefore, teachers needed to examine and evaluate students' misconceptions and mistakes. This study investigated the prospective teachers' knowledge about students' mistakes and reasons of these mistakes. More specifically, prospective teachers' knowledge about students' mistakes and reasons of these mistakes related to fractions was investigated. The next section focuses on the definitions of misconceptions and mistakes and the distinction between these two concepts.

2.3. The Definitions of Mathematical Mistakes and Misconceptions

In the literature review, the terms mistake and misconception were used in various ways. In order to explain these terms, this chapter discusses the definitions of mistake and misconception, and the relationship between mistake and misconception. In order to make the terms more clear, examples of students' mathematical mistakes and misconceptions from the literature are given.

There are various definitions for misconception in the literature. Researchers refer to misconception as preconception, alternative conceptions, and naive conceptions (Hammer, 1996). Misconception refers to “a student conception that produces a systematic pattern of error” (Smith, diSessa, & Roschelle, 1993, p.205). In another definition, misconception is defined as the observed differences between students’ concepts and consistent experts’ ideas (Zembat, 2007). Hammer (1996) indicates that misconceptions “(i) are strongly held, stable cognitive structures, (ii) differ from expert conceptions, (iii) affect in a fundamental sense how students understand natural phenomena and scientific explanations; and (iv) must be overcome, avoided, or eliminated for students to achieve expert understanding” (Hammer, 1996, p.1318). The misapplication of a rule, an over- or under-generalization, or an alternative conception of the situation could be examples of misconception (Drew, 2005).

In addition to misconception, mistake is also defined in various ways in the literature. A mistake is defined as “an error, slip, blunder, or inaccuracy and a deviation from accuracy” (Luneta & Makonye, 2010, p.35). According to Riccomini (2005), unsystematic mistakes are unintended and rare wrong answers. These wrong answers could be readily corrected by learners. On the other hand, systematic mistakes cause the repetition of wrong answers. These wrong answers are methodically constructed and produced across space and time. In addition to Riccomini’s categorization of mistakes, Tirosh (2000) divided the student mistakes into three categories; algorithmically based mistake, intuitively based mistake, and mistakes based on formal knowledge. Algorithmically based mistakes included mistakes in arithmetical operations. For example, the below multiplication operation was a typical student mistake. In this case, the student has not moved over 45 on the second line:

$$\begin{array}{r} 45 \\ \times\ \ 15 \\ \hline 225 \\ 45 \\ \hline 265 \end{array}$$

The second category of student mistakes was intuitively based mistakes. Students' ideas and beliefs about mathematical entities and the mental models which were used for representing mathematical concepts and operations (Even & Tirosh, 2008) were the reasons for these mistakes. 'Multiplication always makes bigger, and division always makes smaller' was an example for this type of mistake. Finally, the last category was mistakes based on formal knowledge. Formal knowledge referred to axioms, definitions, theorems, and proofs (Fischbein, 1994). Mistakes resulted from the inadequate knowledge related to this formal knowledge (Tirosh, 2000). To give an example of the mistakes related to formal knowledge, the thought that division was commutative, therefore $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$. Mistakes could be made for many reasons. Careless, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical topic/learning objective/concept, a lack of awareness or inability to check the answer given, or a misconception could be the reasons of the mistakes (Drew, 2005).

Despite the fact that mistakes and misconceptions were related to each other, they were different. Previous approaches concealed the fundamental conceptual deficiencies of student mistakes, just dividing the students' responses into two categories as correct and incorrect (Smith, diSessa, & Roschelle, 1993). However, researchers agreed that the mistakes were the results of the misconceptions (Drew, 2005; Zembat, 2008). Furthermore, misconceptions were intuitively reasonable to learners and could be strong for instruction designed to correct them (Smith, diSessa & Roschelle, 1993). Mistakes are visible such as in learner's written text or speech. However misconceptions are often hidden. Sometimes misconceptions could even be hidden in correct answers (Smith, diSessa & Roschelle, 1993), when correct answers are accidental.

In order to explain the relationship between mistakes and misconceptions in detail, the next section will describe the students' mathematical mistakes and misconceptions with examples.

2.3.1. An Example to Distinguish Elementary Students' Misconceptions and Mistakes in Mathematics

What follows is an example used to illustrate the differences between the mistake and the misconception. In elementary mathematics, students generally apply whole number rules to rational numbers and decimals. For example, in order to compare 3.4 and 3.371, students ignored the decimal points and took each number as a whole number, thus concluding that 3.371 was greater than 3.4 (Smith, diSessa, & Roschelle, 1993). In this example, the conclusion about the comparison of the numbers was a mistake, and student intuition about comparing decimals could be regarded as the misconception.

In another example, 6^{th}-grade students incorrectly found the unknown as 12 or 17 in an equation 8 + 4 = n + 5. In this case, 12 or 17 were the student mistakes. In this mistake, students only added the 8 and 4, then got the definite value, 12. In another case, students added all numbers and got 17 (Falkner, Levi, & Carpenter, 1999). In these cases, students could not understand that the equal sign meant the equivalence between two quantities. Thus, these misconceptions about the meaning of the equals sign caused the erroneous answer. Students interpreted the equal sign "=" as having to do something. Moreover, students only understood 8 + 4" as a computation process in stead of an expression that could be the representation of a certain amount and an object (Li & Li, 2008).

As the understanding from both above examples shows, the misconceptions were the reasons of students' mistakes. In the next section, previous studies about students' misconceptions and difficulties will be reviewed.

2.4. Related Studies about Students' Misconceptions and Mistakes in Fractions

In this part, some studies including student misconceptions and difficulties in fractions are presented. Several researchers have examined students' difficulties and common misconceptions about fractions (Aksu, 1997; Brown & Quinn, 2006; Haser & Ubuz, 2003; Mack, 1990; Pesen, 2008), difficulty in understanding the meaning of a part and a quantity (Haser & Ubuz, 2003), difficulty in understanding the basic concepts of fractions (Aksu, 1997), difficulty in changing fractions to a common denominator, computation mistakes, difficulty in changing numbers to improper fractions, lack of comprehension of processes involved (Guiler, 1945), interpreting the symbol “ab” as “a+b” (Tall & Thomas, 1991) were some of the findings of the above research studies.

In the following two research studies, students’ understanding of fractions and their error patterns in terms of fractions are investigated.

Mack (1990) investigated students’ development of their understanding about fractions during the instructions in terms of students’ ways of using informal knowledge and the influence of knowledge of the rote procedures. In her study, she used eight average sixth-grade students who had limited understanding about fractions. The instructional content was determined from the topics in the fraction chapters in traditional textbooks by researcher. Furthermore, students’ informal knowledge about fractions estimation with fractions was emphasized for instruction.

The researcher stated that all students came to the instruction with misconceptions and a rich store of informal knowledge about fractions. After instruction, she mentions that students invented alternative algorithms based on their formal knowledge. In addition, according to the researcher, all students’ informal knowledge allowed them to determine the units in real life problems; on the other hand, students had difficulty identifying the unit represented in symbolic and concrete form. She suggested that relating the fractions symbols and students’ informal knowledge was possible in meaningful ways.

In another research study, Brown and Quinn (2006) investigated error patterns of students in applying fraction concepts and performing operations on fractions in order to provide a reason about student common mistakes to the teachers. A 25-item questionnaire was administered to 143 elementary algebra classes. The researcher asserted that students were not sure about the correct process of the algorithm in the items that could be directly applyed to a concept. Moreover, they said that student operations on fractions were not connected to understanding the operations. For example, according to them, student responses "$\frac{1}{2}$ of $\frac{2}{3}$ was equal to $\frac{3}{4}$" showing that students did not understand the relative size of the fractions or $\frac{1}{2}$ as a multiplicative operator. Last of all, they concluded that students were deficient in experience with basic fraction concepts, and the results demonstrated students' lack of fluency with fraction computation.

These two research studies showed that students had limited understanding of fractions and fraction operations. There have also been several studies about students' conceptions and difficulties with fractions in Turkey. In the following three research studies, the researchers investigated the students' conceptions and misconceptions about fractions.

In Haser and Ubuz's (2003) study, students' conceptions of fractions in solving word-problems were investigated. In this study, 10 word-problems were administered to 122 fifth grade elementary students. The researchers stated that students did not understand the part-whole relationship, and were not aware of the resultant unit of an operation. Moreover, they said that students had misconceptions about the basic fraction knowledge. For example, students in the study were not aware of the fact that the numerator and the denominator of a fraction must be a natural number. In addition, according to researchers, students had difficulty performing the mixed number computations with more than one whole. Lastly, confusing the fractional parts and the whole, and choosing incorrect operations were reported as two findings of the analysis, when students could not understand the problem clearly. These results were similar to the results in the following research study.

Pesen (2007) determined third grade students' misconceptions underlying common mistakes with fractions. One hundred and thirteen students from 11 different elementary schools were administered a diagnostic test with 24 items. At the end of the analysis, the researcher said that students made common mistakes in dividing the whole into equal parts. Moreover, he stated that students had difficulty in dividing the circular shapes into equal portions when comparing to rectangular shapes, and they confused the place of the numerators and denominator, and exchanged them with each other. According to the researcher, students also had difficulty reading the fractional numbers, on the other hand, students were successful in writing the fraction numbers belonging to a model. To sum up, the part-whole relationship could be considered as the main difficulty of students in fractions as seen in the above studies (Haser & Ubuz, 2003; Pesen, 2007). These findings of studies on students' difficulties with fractions are summarized in the following paragraph.

Alacacı (2009) mentioned students' misconceptions and difficulties with fractions and the reasons of these misconceptions in his article. He stated that students had limited understanding of the whole concept, fraction concept, fraction comparison, units of the improper fractions, and fraction computations. The first difficulty of the students is related to the whole concept. Students think that two same fractional numbers always describe the same amount; however, by using different sized wholes, two fractions could refer to different-sized fractional parts. For example, in a question "Jale eats a half of a pizza, and Ayşe eats a half of a different pizza. Jale claims that she eats a greater amount of pizza than Ayşe. Ayşe claimes that both of them eat the same amount of pizza. Which one is right?". The analysis of the results of this question showed that most of the students had conceptual deficiencies about the whole concept. Only a quarter of the students replied that Jale had eaten more pizza than Ayşe. The second difficulty is related to fraction concept. Students had a lack of conceptual understanding that fractional parts are equal shares or equal sized portions. Thirdly, students had difficulties with fraction comparison. They considered that a bigger numerator and denominator cause the bigger fractions. The other difficulty was determining the unit of an improper fraction. Furthermore, students had difficulty in computing fractions, since students saw the numerator and denominator as different numbers. Last of all, intuitions about fractions, and language problems with fractions caused the difficulty with fractions (Alacacı, 2009).

In conclusion, the results of these studies showed that students had conceptual deficiencies, mistakes and misconceptions about fractions. In this research study, prospective elementary mathematics teachers' knowledge of mistakes on fractions held by elementary students was investigated. The prospective teachers were asked whether they were aware of the students' algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge about fractions. Moreover, their knowledge about the reasons of these mistakes was examined.

2.5. Summary of the Literature Review:

"Teaching is one of the most common—and also one of the most complicated—human activities" (Ball & Forzani, 2010, p.40). Teachers' knowledge is an essential component in order to improve teaching. Researchers examined teachers and prospective teachers' mathematical content knowledge for teaching. These studies' results showed that prospective teachers had limited mathematical content knowledge for teaching mathematics.

Fractions are one of the most important areas that are mathematically rich, cognitively complicated, and difficult to teach in elementary school mathematics (Smith, 2002). Several researchers have examined students' difficulties and common misconceptions about fractions (Aksu, 1997; Brown & Quinn, 2006; Haser&Ubuz, 2003; Mack, 1990; Pesen; 2008). However, results of these studies revealed that students had limited understanding of fractions and several misconceptions about fraction concepts. Thus, teachers need to be familiar with students' common conceptions and misconceptions on fractions in order to enhance the conceptions and to overcome the misconceptions.

According to Hill and Ball (2004), student learning might result not only from teachers' content knowledge but also from the relationship between teachers' knowledge of students, their learning, and strategies for improving that learning. Moreover, Ball and colleagues (2008) stated that mathematical knowledge for teaching included prospective teachers' knowledge and understanding of students' common mistakes and misconceptions, and also teachers' responses to students' erroneous answers. However, as stated above, there have been few research studies focusing on prospective teachers' knowledge of students' mistakes and the reasons of these mistakes. In particular, there are also very few investigations on prospective teachers' knowledge of students' mistakes in Turkey. Furthermore, in previous studies, researchers generally focused on only some issues related to fractions such as multiplication and division with fractions and definition of fractions. However, this study investigates students' mistakes in all fractions topics without focusing on a specific topic. Therefore, the aim of this research study was to investigate prospective elementary mathematics teachers' knowledge of mistakes held by elementary students in fractions and their proposed strategies to overcome these mistakes.

CHAPTER III

METHODOLOGY

In this chapter the research design, population and sample, data collection instrument, data collection procedure, analyses of data, and lastly the internal and external validity of the study were described.

3.1. Design of the Study

The purpose of this study was to investigate the prospective teachers' knowledge of mistakes held by elementary students related to fractions and prospective teachers' proposed strategies to overcome those mistakes held by elementary students. For these purposes cross-sectional survey design was used.

Cross-sectional survey design was defined by Fraenkel and Wallen (2005) as follows:

> A cross-sectional survey collects information from a sample that has been drawn from a predetermined population. Furthermore, the information is collected at just one point in time, although the time it takes to collect all of the data may take anywhere from a day to a few weeks or more. (p.398)

In the current study, data regarding prospective elementary mathematics teachers' knowledge about students' mistakes related to fractions were gathered one point in time through Fraction Knowledge Questionnaire (FKQ), therefore the design of the study could be considered as a cross sectional survey.

3.2. Population and Sample

In this study, all prospective elementary mathematics teachers enrolled in the elementary mathematics education programs in Turkish public universities were identified as a target population. The accessible population of this study was determined as all junior and senior prospective elementary mathematics teachers enrolled in elementary mathematics education programs in public universities in Central Anatolian Region.

Convenience sampling method was used to obtain a sample of this study. In convenient sampling method, researchers might use a certain group of people who were available for study (Fraenkel & Wallen, 2006). The junior and senior prospective elementary mathematics teachers from a public university in Central Anatolian Region constituted the sample of this study.

The sample of this study consists of 149 prospective elementary mathematics teachers. Demographics regarding gender and grade level are given in Table 3.1

Table 3.1
Number of Prospective Teachers Participated in This Study

Grade	Male	Female	Total
Junior	31 (35,6%)	56 (64,4%)	87
Senior	19 (30,6%)	43 (69,4%)	62
Total	52 (34,8%)	97 (65,2%)	149

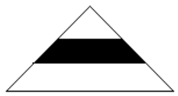
3.3. Data Collection Tool:

In order to understand the prospective teachers' knowledge of mistakes held by elementary students in the fractions and their proposed strategies to overcome those mistakes, Fraction Knowledge Questionnaire (FKQ) was developed based on the objectives of the fraction concepts of elementary mathematics curriculum and students' mistakes in these concepts. Most of the items were adapted from the findings of the studies investigating the students' mistakes related to fractions (Chang, 1997; Haser & Ubuz, 2003; Johnson, 1998; Mack, 1990; Pesen, 2007; Soylu & Soylu, 2005; Van de Walle, 2006). The third and seventh items were developed by the researcher. The focus of the questionnaire was prospective elementary mathematics teachers' knowledge about the students' mistakes related to fractions, the reasons of these mistakes, and their strategies in order to overcome the students' mistakes.

The data collection tool included nine open-ended questions, and each question had two sub-tasks. At the beginning of each item, students' erroneous answers to questions related to fractions were given. The first task of each question was "explain with reasons whether students' response or claim was acceptable or not". This sub-task was prepared to measure whether the prospective elementary mathematics teachers could identify the students' mistakes and their knowledge about reasons of these mistakes. The second task of each item was "If you were this students' teacher, how would you make an explanation to him/her?". The second sub-task of each item was prepared to measure the knowledge about strategies that prospective teachers would use to overcome students' mistakes. The prospective teachers were asked to suggest a solution for student's mistakes. The Turkish version of the questionnaire is given in Appendix A.

The first four items in the FKQ evaluated the prospective teachers' knowledge on students' mistakes based on formal knowledge. In the first item, prospective teachers were asked to interpret student's erroneous answer about area model of a fraction. This item was adapted from the Johnson's (1998) study. In this item, the triangle was separated into the number of parts designated by the denominator and with the number of parts specified by the numerator shaded. However, none of the parts were equivalent to each other. The item is given below in Figure 3.1:

Students were asked to shade $\frac{1}{3}$ of a triangle. Ayşe shaded the triangle this way (Ayşe was a fifth grade student).



a. Explain with reasons whether Ayşe's claim was acceptable or not.

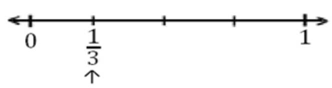
b. If you were Ayşe's teacher, how would you make an explanation to Ayşe?

Figure 3.1. The first item of the fraction knowledge questionnaire

The second item was about students' difficulty on partitioning the unit interval into equal parts. This item explored the prospective teachers' knowledge about student's erroneous measurement model. This item was adapted from the results of Pesen's (2008) study. The second item is given in Figure 3.2:

Gizem answered her teachers' question by drawing the number line below:

"Mark $\frac{1}{3}$ on the number line."



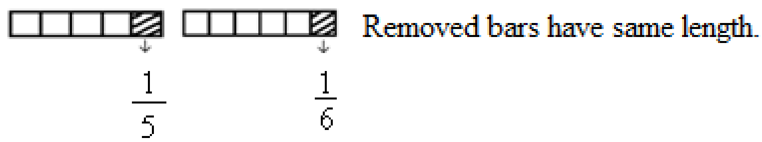
a. Explain with reasons whether Gizem's response was acceptable or not.

b. If you were Gizem's teacher, how would you make an explanation to Gizem?

Figure 3.2. The second item of the fraction knowledge questionnaire

In the third item, prospective teachers' reasoning on student's erroneous answer about comparison of fractions was investigated. This item was adopted from students' responses in Mack's (1990) study. In her study, the students focused on the number of missing parts rather than on the size of the fractions. Thus, in this item prospective teachers' were asked to identify this student mistake. The third item is given in Figure 3.3 below:

The teacher asked Mert "which fraction is the largest; $\frac{4}{5}$ or $\frac{5}{6}$?", and he said that "Two fractions are the same, because there is one piece missing from each."



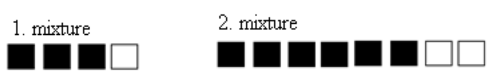
a. Explain with reasons whether Mert's response was acceptable or not.

b. If you were Mert's teacher, how would you make an explanation to Mert?

Figure 3.3. The third item of the fraction knowledge questionnaire

The fourth item was about student's erroneous answer to a proportional reasoning problem. This item was adapted from Chang's (1997) study. In this problem, prospective teachers evaluated student's erroneous multiplicative reasoning about mixtures. The fourth item is given in Figure 3.4:

"In the following figures, the black squares refer to the orange nectar, and the white ones refer to the water. Which mixture is denser?"



Arzu answered the above question as follows:

–The second mixture is denser because while 3 glasses of water were added into the first mixture, there are 6 glass of orange nectar in the second one.

a. Explain with reasons whether Arzu's response was acceptable or not.

b. If you were Arzu's teacher, how would you make an explanation to Arzu?

Figure 3.4. The forth item of the fraction knowledge questionnaire

The fifth, sixth, and seventh items examined the prospective teachers' responses to students' algorithmically based mistakes. In the fifth items, prospective teachers' were asked student's common erroneous answer about subtraction operation. In this item, students' common subtraction mistake of subtracting both the numerator and denominator from each other in subtraction operation were asked to prospective teachers. This item was adapted from the results of Soylu and Soylu's (2005) study. The fifth item is given in Figure 3.5:

> To teacher's subtraction question; $\frac{7}{8} - \frac{1}{4}$, Zehra responded as $\frac{7-1}{8-4} = \frac{6}{4}$, and Elif responsed as $\frac{7-2}{8} = \frac{5}{8}$. Both of them claim that their answers were true.
>
> a. Explain with reasons which response was acceptable.
>
> b. If you were Zehra and Elif's teacher, how would you make an explanation to them?

Figure 3.5. The fifth item of the fraction knowledge questionnaire

In the sixth item, prospective teachers' were asked about student's common mistake in adding fractions which was adding both numerators and denominators. This item was adapted from Van de Walle (2006). The sixth item is given below in Figure 3.6:

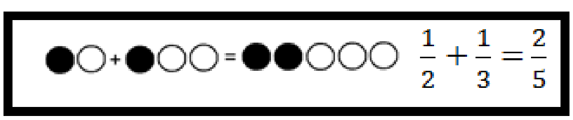
> Berk solved $\frac{1}{2} + \frac{1}{3}$ as follows;
>
> 
>
>
> When we add marbles, we get
>
> 
>
>
> a. Explain with reasons whether Berk's response was acceptable or not.
>
> b. If you were Berk's teacher, how would you make an explanation to Berk?

Figure 3.6. The sixth item of the fraction knowledge questionnaire

In the seventh item, prospective teachers were asked about student's mistake about multiplication of mixed numbers. This item demonstrated a student's erroneous answer about the multiplication of mixed numbers by using distributive property. In this question, the student could not use all partial products and found a wrong answer. This item was developed by the researcher. The seventh item is given in Figure 3.7:

Zehra found the result of $2\frac{1}{2} \times 1\frac{1}{3}$ operation by multiplying the whole parts and fractional parts of fractions separately. And she wrote it mathematically as follows:

$2\frac{1}{2} \times 1\frac{1}{3}$; $2\times1=2$ and $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. And she found the result $2\frac{1}{6}$.

a. Explain with reasons whether Zehra's response was acceptable or not.

b. If you were Zehra's teacher, how would you make an explanation to Zehra?

Figure 3.7. The seventh item of the fraction knowledge questionnaire

In the eighth item and ninth items, prospective teachers were asked to analyze the students' intuitively based mistakes. In the eighth item, there was a student's response about the overgeneralization of multiplication rule on natural numbers to rational numbers. This item was adapted from the results of Mack's (1990), and Haser and Ubuz's (2003) study. The eighth item is given in Figure 3.8:

The teacher found the result of $4 \times \frac{1}{16}$ operation $\frac{1}{4}$. Burçin made an objection to this result and she claimed that "Multiplication always makes bigger."

a. Explain with reasons whether Burçin's response was acceptable or not.

b. If you were Burçin's teacher, how would you make an explanation to Burçin?

Figure 3.8. The eighth item of the fraction knowledge questionnaire

Similarly, in the ninth item, student's erroneous claim was about the overgeneralization of division rules on natural numbers to rational numbers. This item was adapted from the results of Mack's (1990), and Haser and Ubuz's (2003) study. The ninth item is given in Figure 3.9:

Tülay said to her teacher that "Division always makes smaller, however I found the result of this operation $6 \div \frac{1}{3}=18$. I think I find a wrong result."

a. Explain with reasons whether Tülay's claim was acceptable or not.

b. If you were Tülay's teacher, how would you make an explanation to Tülay?

Figure 3.9. The ninth item of the fraction knowledge questionnaire

To sum up, Fraction Knowledge Questionnaire includes 9 open-ended questions adapted from the findings of Chang's (1997), Haser and Ubuz's (2003), Johnson's (1998), Mack's (1990), Pesen's (2007), Soylu and Soylu's (2005), and Van de Walle's (2006) studies about students' mistakes. The next section explained the pilot study of FKQ.

3.4. Pilot Study

In order to prepare the items in the questionnaire, the objectives of fractions concepts in the elementary mathematics curriculum were listed. Then the literature about the students' mistakes related to fractions was reviewed. The items in the FKQ were developed based on the fraction concepts in the elementary mathematics curriculum and findings of the studies related to students' mistakes about these concepts. Initially, there were sixteen items in the FKQ. Before the pilot study, two mathematics educators were asked to check the content of the items. The experts suggested decreasing the number of items in the questionnaire, because there were overlaps in the objectives and items measuring these objectives. Then, the number of the items was decreased from sixteen to fourteen. After the expert opinions, the FKQ were administered for pilot study.

The purpose of the pilot study was to ensure the validity and reliability of the instrument and to determine the possible problems of actual administration of the questionnaire. The FKQ with fourteen items was implemented to 67 prospective elementary mathematics teachers in a public university in Burdur. In terms of grade level, there were 39 junior and 27 senior prospective elementary mathematics teachers involved in the pilot study.

After the pilot study, the researcher made some changes on the items based on the dialogues with participants and responses of them to FKQ. The researcher noted prospective teachers' questions to clarify the items during the pilot study and also asked them about the clarity of the statements and figures in the items at the end of the pilot study. For example, based on the prospective teachers' responses to the first item, it appeared that indicating the student's grade in the item was important in order to correctly identify the mistake. Thus, in the last version of the FKQ, “Ayşe was the fifth grade student” was added to the first item. Moreover, prospective teachers asked questions about the length of the strips in the third item. In the first version of the FKQ, the strips’ length in the figure was not seen as they were same. Therefore, the researcher decided to add “two strips has equal length” next to the figure. Finally, after the administration of the pilot study, four of the fourteen items were excluded from the questionnaire as they measured the same objectives with two other items (third and fourth items in the last version of FKQ) and considering the time needed to respond to the questions. After the experts’ opinions and the pilot study, the Fraction Knowledge Questionnaire was finalized. Table 3.2 presents the table of specification of FKQ items. For the final version of the questionnaire, see Appendix A.

Table 3.2
Table of Specification for Questionnaire Items

TYPE OF MISTAKE	MISTAKES	Understanding students' mistakes	Understanding the reasons of students' mistakes	Suggesting strategies to overcome mistakes
Mistake based on Formal Knowledge	Lack of knowledge in addressing the fractions consisting of equal portions of a whole (Pesen, 2008)	Q1(a)	Q1(a)	Q1(b)
Mistake based on Formal Knowledge	Considering that fractions which has larger number is larger (Hart, 1980)	Q4(a)	Q4(a)	Q4(b)
Mistake based on Formal Knowledge	Lack of understanding about the relationship between a part and a whole (Haser & Ubuz, 2003; Pesen, 2008; Mack, 1990).	Q3(a)	Q3(a)	Q3(b)
Mistake based on Formal Knowledge	Difficulty in dividing a whole into equal parts on the number line (Pesen, 2008).	Q2(a)	Q2(a)	Q2(b)
Intuitively based mistakes	Conception that multiplication always makes bigger, and division always makes smaller (Haser & Ubuz, 2003; Mack, 1990; Mcleod & Newmarch, 2006).	Q8(a), Q9(a)	Q8(a), Q9(a)	Q8(b), Q9(b)
Algorithmic ally based mistake	Considering numerators and denominators as separate entities rather than as connected (Soylu & Soylu, 2005)	Q5(a), Q6(a)	Q5(a), Q6(a)	Q5(b), Q6(b)
Algorithmic ally based mistake	Difficulty in partial product method to perform the multiplication in mixed numbers	Q7 (a)	Q7 (a)	Q7 (b)

3.4.1 Validity and Reliability Issues:

Validity referred to the appropriate, correct, meaningful and useful inferences from the data. Therefore, the important point in a research study was using data to illustrate the guaranteed conclusion about the people on whom the data were collected (Fraenkel & Wallen, 2006). In this study, two mathematics educators in the Elementary Mathematics Education program at METU examined the test items in order to establish content validity of the data collection tool. Furthermore, the appropriateness of the language, adequacy of work space, and the clarity of directions and printing were checked and suggestions given by experts were taken into consideration in the revision of the questionnaire. These measures represented content related evidences of validity of the FKQ.

Reliability meant the consistency of the scores obtained from the data collection tool (Fraenkel & Wallen, 2006). In this study, scoring agreement method was used to establish the reliability. Scoring agreement required a satisfactory agreement of independent scorers in their scoring (Fraenkel & Wallen, 2006). During the analysis of the data, 149 participants' responses were analyzed with a second coder, who is a mathematics teacher. The comparison of the codes provided an evidence for inter-reliability. There was a %99 correlation between the codes and the coded sections in the beginning, and then it increased to %100 after the discussions. An example of the coding obtained by two coders is given in Appendix B.

3.5. Data Collection Procedure

The purpose of this study was to investigate the prospective teachers' knowledge of mistakes held by elementary students in fractions and their proposed strategies to overcome those mistakes. At the end of the fall semester of the academic year 2010-2011, the official permissions were gathered from the Middle East Technical University Human Subjects Ethics Committee. After the official permission was gathered, the researcher visited the university and explained the purpose and the procedure of the study to the department administrators. The researcher asked permission from the head of the Elementary Mathematics Education program to conduct the study.

The data was collected from junior and senior prospective elementary mathematics teachers enrolled in Elementary Mathematics Teacher education program during the spring semester of the academic year 2010-2011. In order to complete the questionnaire, 45 minutes were given to prospective teachers.

A schedule indicating the order of data collection is given in the Table 3.3.

Table 3.3
Time Schedule for Data Collection

Date	**Events**
September 2010 – November 2010	Development of instruments
December 2010 – January 2010	Pilot study of instruments and last version of data collection tool
January 2010 – February 2010	Data collection-Implementation

3.6. Data Analysis

In this study, the items in the “Fraction Knowledge Questionnaire” were analyzed in-depth in order to reach a detailed description of prospective teachers’ knowledge about students’ mistakes on fractions. More specifically, qualitative data obtained from “Fraction Knowledge Questionnaire” were read and then categories were formulated according to the prospective teachers’ responses. The frequencies and percentages of categories were gathered for each item.

The analysis of the responses given for the items in the FKQ was started first by coding these responses based on the concepts they referred to and grouping the coded responses in more comprehensive categories of knowledge of students’ mistakes. The categories were formed until the new categories could not be found. Then, the names of the categories were given based on the concepts in the literature and the researcher’s experience with the data. During the data analysis, researcher and the second coder tried to identify the categories among the participants’ responses (See Appendix B).

3.7. Assumptions and Limitations

In this section, the main assumptions and limitations of the research study were discussed. It was assumed that all of the participants answered the questions in fraction knowledge questionnaire in their full attention. In this study, the sampling procedure could be the limitation of the study. Participants were selected by using a non-random sampling. Moreover, the participants of the study were only the junior and senior prospective teachers. Therefore, the generalizability of the results of this study to the larger population would be limited. Additionally, because the Fraction Knowledge Questionnaire was administered to prospective teachers in their school routines, the convenience setting of the questionnaire might affect their responses. They might tend to give shorter and limited answers to questions. Thus, the responses of the prospective teachers would be limited.

3.8. Validity of the Study

In this section, the internal validity of the study and the external validity of the study were discussed.

3.8.1. Internal Validity of the Study

Internal validity of the study refers to the degree to which observed differences on dependent variable affected by the independent variable directly (Fraenkel & Wallen, 2006). Internal validity threats occur when the observed results are not related to dependent variable itself, but related to some unintended variables. For survey studies, the possible internal threats were location, mortality (loss of subject), and instrumentation (Fraenkel & Wallen, 2006).

Location might have an effect on the results of the study. Location threat was defined as “the particular locations in which data are collected, or in which an intervention is carried out, may create alternative explanations for results” (Fraenkel & Wallen, 2006, p. 172). In order to control this threat, the researcher administered all the questionnaires in the participants’ own classrooms. The researcher tried to keep all possible conditions the same for all participants. Thus, location threat was minimal in this study.

Mortality, in other words loss of subjects, is another threat to be considered in research studies. Fraenkel and Wallen (2006) stated that it was common to lose some of the participants no matter how the subjects of the study were selected. In order not to lose subject, the researcher selected the courses that all of the 3rd and 4th grade students took in the spring semester. Therefore, loss of subjects could not be a threat in this study.

The last threat for this study was instrumentation. Instrumentation could create a problem if the nature of the instrument or scoring procedure was changed in some way (Fraenkel & Wallen, 2006). This refers to the instrument decay. In this study, since the questionnaire included open-ended items, instrument permitted different interpretations of the results. In order to control instrument decay, the coding rubric was used. In addition, the second coder also analyzed data in order to minimize the changes in analysis procedure. Furthermore, since the researcher administered all questionnaires herself, the data collector characteristics were the same for all administrations. Therefore, the data collector characteristic was not a threat for this study. Data collector bias was the last issue for instrumentation threats. Fraenkel and Wallen (2006) stated that data collector or data scorer might unconsciously alter the results. In order to control this threat, all classes were allowed the same time on questionnaires and there were no interaction and communication between the researcher and participants during the administration of the questionnaire. Thus, data collector bias was not also a threat for this study.

3.8.2. External Validity of the Study

External validity refers to “the extent that the results of a study can be generalized from a sample to a population” (Fraenkel & Wallen, 2006, p.108). Both the population generalizability and ecological generalizability were taken into consideration in external validity.

The population generalizability was defined as “the degree to which a sample represents the population of interest” (Fraenkel & Wallen, 2006, p.104). In this study, 149 junior and senior prospective teachers enrolled in Elementary Mathematics Education program in a public university in Central Anatolia Region was the sample of this study. The accessible population was determined as all junior and senior prospective elementary mathematics teachers enrolled in public universities in the Central Anatolian Region. Since the sampling method of the study was the convenience sampling method, the generalizability of the study on the population would be limited. However, Fraenkel and Wallen (2006) stated that “the results of a study can be generalized to conditions or settings other than those that prevailed in a particular study” (p.108). This type of generalization was called ecological generalization. Therefore, the results could be generalized to the prospective teachers under the same conditions with the participants in this study. In other words, prospective elementary mathematics teachers could choose the elective courses in their teacher education among Mathematical Language, Technology Assisted Geometry Teaching, Computer Assisted Mathematics Teaching, Daily Life in Mathematics, Development of Algebra Thinking in Elementary School, Problem Solving in Mathematics, and Geometric Thinking and Its Development courses. The result of this study could be generalized to the prospective elementary mathematics teachers who took the same courses in the Elementary Mathematics Education program.

CHAPTER IV

RESULTS

The purpose of this study was to investigate prospective teachers' knowledge of mistakes held by elementary students in the fractions concepts and their proposed strategies to overcome those mistakes. This chapter summarizes the results of the study in two sections. In the first section, prospective teachers' awareness and knowledge of reasons of students' mistakes are explained in detail. In the second section, their strategies to overcome these mistakes are summarized.

4.1. Prospective Teachers' Knowledge of Students' Mistakes and the Reasons for these Mistakes

This section demonstrates the results obtained from the Fraction Knowledge Questionnaire (FKQ). The FKQ was administered to the prospective elementary mathematics teachers. The questions in the questionnaire could be grouped under four headings: noticing of students' mistakes, reasons for students' mistakes based on formal knowledge, reasons for students' algorithmically based mistakes, and reasons for students' intuitively based mistakes. The analysis of the questions categorized under four headings is given below.

4.1.1. Prospective Teachers' Knowledge of Noticing of Students' Mistakes

Prospective teachers' noticing of students' mistakes on fractions was mainly reflected by their responses to the first sub-dimension of the items. Results revealed that most of the prospective teachers could identify the students' erroneous answers. For example, these prospective teachers stated that the students' answers given in the items were not acceptable such as the area representation for $\frac{1}{3}$ in the first item. In this item, 82% of the junior prospective teachers and 79% of the senior prospective teachers said that the area model for $\frac{1}{3}$ was not acceptable. The other frequency and percentages of the prospective teachers who could identify the students' mistakes are shown in the Table 4.1 in terms of each item and grade level.

Table 4.1

Frequency and Percentages of Participants' Noticing

	Juniors				Seniors			
	Aware		Unaware		Aware		Unaware	
	n	%	n	%	n	%	n	%
Item 1	71	82.0	16	18.0	49	79.0	13	21.0
Item 2	81	93.0	6	7.0	61	98.0	1	2.0
Item 3	84	96.0	3	4.0	60	96.0	2	4.0
Item 4	81	93.0	6	7.0	62	100	0	0
Item 5	85	97.0	2	3.0	62	100	0	0
Item 6	82	94.0	5	6.0	60	96.0	2	4.0
Item 7	74	85.0	13	15.0	58	93.5	4	6.5
Item 8	78	89.0	9	11.0	59	95.0	3	5.0
Item 9	71	81.0	16	19.0	52	84.0	10	16.0

As shown in Table 4.1, most of the prospective teachers were familiar with the students' mistakes based on formal knowledge, algorithmically based mistakes, and intuitively based mistakes. All of the senior prospective teachers were aware of the students' mistakes in the fourth and the fifth items. Moreover, the fifth item was the item in which junior prospective teachers were aware of the mistakes the most. Additionally, although more senior prospective teachers identified the students' mistakes in each item, only in the first item, more juniors noticed the student's mistake.

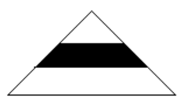
4.1.2. Prospective Teachers' Knowledge on Reasons for Mistakes Based on Formal Knowledge

Prospective teachers' knowledge on reasons of students' erroneous answers was examined to understand how they interpreted the students' mistakes based on formal knowledge of fractions.

First Item

In the first item, prospective teachers were asked to interpret a student's erroneous answer about the area model of a fraction. The item is given below:

[Students were asked to shade $\frac{1}{3}$ of a triangle. Ayşe shaded the triangle this way (Ayşe was a fifth grade student).



a. Explain with reasons whether Ayşe's claim was acceptable or not.]

In this item, the triangle was separated into the number of parts designated by the denominator and with the number of parts specified by the numerator shaded. However, none of the parts were equivalent to each other.

Analysis of participants' responses was presented in terms of prospective teachers' awareness of erroneous answers and their reasoning of students' erroneous answers about fractions. In other words, prospective teachers' decision refers to their response whether they are aware of the students' mistakes or not. Prospective teachers' decisions include the three categories namely; non-acceptable, acceptable, and conditionally acceptable. In addition, the evaluations of students' responses include the prospective teachers' reasons which they give for students' erroneous answers. The prospective teachers' answers which did not accept Ayşe's answer given in the questionnaire were presented under the non-acceptable category with subcategories of the necessity of equal parts, wrong reasoning and without reasoning. Prospective teachers' answers which accepted Ayşe's answer as the correct answer were coded under the acceptable sub-category. Similarity principle and without reasoning were these sub-categories for the acceptable category. If the prospective teachers accepted Ayşe's answer under a condition, these answers were grouped under the conditionally accepted category. "If Ayşe calculated the area of the triangle, and then divided the triangle into the equal parts, this answer could be acceptable" was an example for the conditionally accepted category. The frequencies and percentages of the reasons are given in Table 4.2.

Table 4.2

Frequency of Response Categories of the First Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	The equal parts principle	47	54	32	51,6
	Wrong Reasons	1	1,1		3,2
	Without Reasons	8	9,2	9	14,5
Acceptable	Similarity principle	4	4,5	2	3,2
	Without Reasons	9	10,3	5	8
Conditionally acceptable		18	20,6	12	19,2

As shown in Table 4.2, 47 junior prospective teachers and 32 senior prospective teachers stated rational reasons why the student's answer was wrong. In other words, more than half of prospective teachers claimed that since the whole must be divided into equal parts, the student's answer was not acceptable;

> Participant 17: "Student's drawing cannot be accepted, because the triangle, as a whole, it should be divided into three equal parts." [Bir bütün olan üçgenin 3 eş parçaya ayrılması gerektiği için öğrencinin gösterimi kabul edilemez.]

Participant 64: “The answer is not acceptable since $\frac{1}{3}$ fractional number means dividing the whole into three equal parts, and shading one of these three parts. In this triangle, the divided parts are not equal to each other. Shaded part is not also equal to other parts.” [Cevap kabul edilemez çünkü $\frac{1}{3}$ kesir sayısı demek bir kesri 3 tane eşit parçaya ayırıp 1 tane parçayı taramak anlamına gelmektedir bu üçgende ayrılan parçalar eşit parçalar değildir. Taranan parçada diğerlerine eşit değildir.]

Four junior and 2 senior prospective teachers accepted the student’s answer. They declared that when they calculated the area of triangle’s parts using the similarity ratio, they found that the shaded area was $\frac{1}{3}$ of the triangle. Thus, they accepted the student’s drawing as correct. For instance;

Participant 96: “The answer is correct regarding the similarity principle, $\frac{2}{3}=\frac{4S}{9S}$. Because the area of small triangle is S, the shaded region becomes 3S. Since the whole triangle is 9S, $\frac{3S}{9S} = \frac{1}{3}$.” [Doğrudur. Benzerlikten dolayı; $\frac{2}{3} = \frac{4S}{9S}$(Alanların karesine eşit). Üstteki küçük parça da S olduğundan dolayı taralı aln 3S’dir. Tamamı da 9S olduğuna göre $\frac{3S}{9S} = \frac{1}{3}$ olur.]

Participant 25: “The answer is correct. If we change the triangle into rectangle, it will be more obvious that the shade area is $\frac{1}{3}$ of triangle.” [Cevap doğru üçgeni bir dörtgene tamamladığımızda, taralı alanın $\frac{1}{3}$ olduğu daha net görülür.]

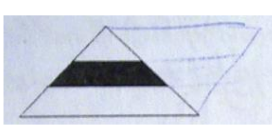


Figure 4.1. Participant 25’s drawing

Apart from these results, 18 junior and 12 senior prospective teachers conditionally accepted the student’s erroneous answer. They stated that because the length of the sides was not defined, they could not evaluate whether the student's answer was correct or not. Furthermore, they claimed that if student calculated the areas and found that the shaded region was $\frac{1}{3}$ of the whole triangle, then the answer would be correct. Otherwise, the randomly shaded region was not acceptable;

> Participant 56: "If she found the area and then shaded the part, the answer could be acceptable. In addition, if she divided the edges of the triangle into equal three parts and then shaded, his answer could also be acceptable. However, if she shaded the region randomly, the answer could not be acceptable." [Alan hesaplarını yapıp taramışsa kabul edilebilir. Ayrıca kenarları üçer eşit parçaya bölüp aradaki parçayı taramışsa oda olabilir. Ancak rastgele bir işlem yapmışsa kabul edilemez.]

> Participant 140: "Because the important thing is the area of the shaded part, first the area of the whole triangle should be measured. And then I measure the area of the shaded part. If it is $\frac{1}{3}$ of the whole triangle, then it is acceptable." [Önemli olan taralı bölgenin alanı olduğu için öncelikle tüm üçgenin alanı ölçülmeli, daha sonra taralı bölgenin alanını ölçerim. Eğer 3'te biriyse kabul edilebilir.]

Except from these results, one junior prospective teacher stated a wrong reasoning. This participant said that:

> Participant 39: "The shaded part refers to $\frac{1}{2}$ of the whole triangle. Student's answer is not acceptable." [Ayşe'nin taradığı kısım üçgenin $\frac{1}{2}$'lik kısmıdır. Cevabı kabul edilemez.]

Figure 4.2. Participant 39's drawing

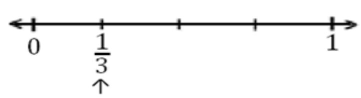
Lastly, 8 junior and 9 senior prospective teachers did not provide an explanation for the student's erroneous answer. However, these prospective teachers only claimed that this answer was not acceptable. On the other hand, 9 junior and 5 senior participants declared that the student's answer was acceptable; then again they did not give an explanation for the student's answer.

Second Item

In this section, the analysis of the second item is presented. The second item is as follows:

[Gizem answered her teacher's question by drawing the number line below:

"Mark $\frac{1}{3}$ on the number line."



a. Explain with reasons whether Gizem's response was acceptable or not.]

This item explored the prospective teachers' knowledge about the reasons for the student's erroneous understanding on a measurement model. In this item, there was one main category of prospective teachers' decisions. This category consisted of the responses which did not accept Gizem's answer as a correct solution to the problem. Moreover, prospective teachers' responses for the reason for Gizem's erroneous solution were grouped under four categories: wrong partitioning of the unit interval, inadequate knowledge of the part-whole relationship/number line/fraction, and without reasons. Without reasons category included the responses which did not give any reason for Gizem's erroneous answer. Table 4.3 shows the frequencies and percentages of the response categories for the second item.

Table 4.3

Frequency of Response Categories of the Second Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Wrong partitioning the unit interval	57	65,5	39	62,9
	Inadequate knowledge of the part-whole relationship/number line/fraction concept	18	20,6	22	35,3
	Without Reasons	12	13,7	1	1,6

In this item, 57 junior and 39 senior prospective teachers stated that the student marked three points and separated the unit interval into four congruent parts. This is because the student counted the points instead of the intervals.

> Participant 103: "The answer is incorrect. [0,1] interval is divided in four equal parts and represented the first hash mark. Because Gizem counts the three hash marks, she mistakenly marks $\frac{1}{4}$." [Cevap yanlıştır. [0,1] aralığı 4 eş parçaya ayrılmıştır ve ilk parçası işaretlenmiştir. $\frac{1}{4}$ noktası olması gerekirken Gizem işaretlediği 3 noktayı saydığı için hataya düşmüştür.]

> Participant 63: "The answer was wrong, since student used three hash marks and partitioned four equal regions instead of dividing it into three equal parts." [Cevap yanlıştır. Öğrenci 0-1 aralığını üç eşit parçaya bölmek yerine üç çizgi ile bölmüştür. 4 bölgeye ayrıldığı için cevap yanlıştır.]

In addition, 18 junior and 22 senior prospective teachers mentioned that the student's lack of knowledge about the part-whole relationship, fractions, and the number line led them to give wrong answer. For example;

> Participant 108: "The student does not understand how to represent a fraction on the number line. She does not know how to divide 0-1 interval in order to show $\frac{1}{3}$. Thus, her answer is not acceptable." [Öğrenci kesirleri sayı doğrusunda nasıl göstereceğini anlamamıştır. $\frac{1}{3}$' ü göstermek için 0–1 arasını nasıl böleceğini bilmiyor. Bu yüzden cevabı kabul edilemez.]

Correspondingly, one of the participants asserted the students' inadequate knowledge of the part-whole relationship as a reason for mistake in the second item.

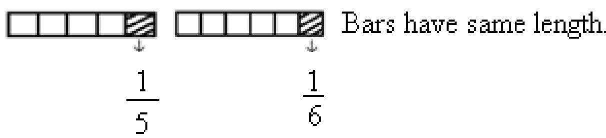
> Participant 43: "The student lacks knowledge on part-whole relationship. She does not know how to divide 0-1 interval. In other words, she does not understand how to divide a whole into [equal] portions, thus she makes a wrong division. Her answer is not acceptable." [Öğrencinin parça bütün konusunda eksiği var. 0 – 1 arasının nasıl bölünmesi gerektiğini bilmiyor. Yani 1 bütünün nasıl parçalara ayrılması gerektiğini anlamamış, bu yüzden yanlış bölmüş. Cevabı kabul edilemez.]

Finally, 20 junior prospective teachers and 10 senior prospective teachers claimed that the student's answer was not acceptable without giving any reasons.

Third Item

This part summarizes the results of analysis of the third item responses. This item is given below:

[The teacher asked Mert "which fraction is the largest; $\frac{4}{5}$ or $\frac{5}{6}$?", and he said that "because one piece was extracted from two, both sizes of the fractions are the same."]



a. Explain with reasons whether Mert’s response was acceptable or not.]

In this part, prospective teachers’ responses to students’ erroneous answer about the part-whole relationship are analyzed. In this item, none of the prospective teachers accepted Mert’s answer as a correct solution. Under the non-acceptable category, four different categories were formed based on the responses. Unequal unit fraction, unequal left over fractional parts, inadequate knowledge and without reasons were the categories of this item. Prospective teachers’ responses without giving any reasoning were categorized as the without reasons category.

Table 4.4

Frequency of Response Categories of the Third Item

Participants’ decisions/ Evaluation of the students’ response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Unequal unit fraction	52	59,8	48	77,4
	Unequal left over fractional parts	8	9,2	1	1,6
	Inadequate knowledge	15	17,2	3	4,9
	Without Reasons	12	13,8	10	16,1

Seventy five junior and 52 senior prospective teachers stated a valid reason for why the student’s answer was non-acceptable. The most common response to the reason for this mistake was the unequal unit fractions. Fifty two junior and 48 senior prospective teachers said that because the detached pieces had different lengths, two fractions were different.

> Participant 21: “The answer was not acceptable, because the detached parts from the same strip were different.” [Cevap kabul edilemez. Çünkü

cevaba göre aynı uzunluktaki çubuklardan çıkarılan bölümler birbirinden farklıdır.]

Participant 17: "Not acceptable, since $\frac{1}{5}$ is bigger and this piece is bigger, two fractions are different." [Kabul edilemez. Çünkü $\frac{1}{5}$ kesri daha büyüktür. Bu parça daha büyük olduğundan iki kesir farklıdır.]

Additionally, unequal left over fractional parts were the other reason for this item. Eight juniors and 1 senior mentioned that if the student had compared the left over pieces, s/he could see the difference in fractions.

Participant 4: "Not acceptable, because there are four pieces of $\frac{1}{5}$ in the first drawing, and five pieces of $\frac{1}{6}$ in the other drawing. Thus, the size of two fractions is not equal." [Kabul edilemez. Çünkü ilk şekilde $\frac{1}{5}$'lik parçadan dört tane, diğer şekilde $\frac{1}{6}$'lık parçadan beş tane kalmıştır. Bu yüzden iki kesrin büyüklükleri eşit değildir.]

Participant 24: "Mert's answer is not acceptable. The area of four pieces left from five was not equal to the area of five pieces left from six pieces." [Mert'in cevabı kabul edilemez. Beş parçadan kalan dördünün alanı ile altı parçadan kalan beşinin alanları aynı değildir.]

Fifteen junior and 3 senior prospective teachers stated students' inadequate knowledge as a reason. Nine juniors said that the student made this mistake because of inadequate knowledge of unit fractions, 4 juniors and 3 seniors said that it was because of inadequate knowledge of the part-whole relationship, and lastly 2 juniors addressed inadequate knowledge of fraction comparison. The following responses are three examples of the inadequate knowledge category:

Participant 31: "Mert's answer is not acceptable, because he does not understand the unit concept. He is not aware that different fractions has different units, thus he thinks that detached parts are the same. However, when these unit fractions are detached from the whole, the sizes of the left-over parts are not the same." [Mert'in cevabı da kabul edilemez. Çünkü Mert birim kesir kavramını anlamamış. Farklı kesirlerin farklı birim kesirleri olacağının farkında değil, bu yüzden çıkarılan parçaların aynı olduğunu düşünmüş. Ancak bu birim kesirler çıkarıldığı zaman kalan parçaların büyüklükleri aynı olmaz.]

Participant 44: "It is not acceptable. The student does not know that fraction is the relationship between the part and the whole." [Edilemez. Öğrenci kesirlerin parçanın bütünle ilişkisi olduğunu bilmiyor.]

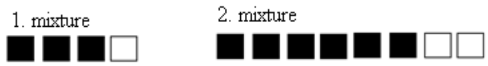
Participant 81: "It is not acceptable. He draws the correct figure however the sizes of the figures are different. He lacks knowledge on fraction comparison. I mean, he thinks that when one parts is detached from the whole, the left-over parts could be the same. He does not notice the number of left over parts. He even does not notice the total number of parts." [Kabul edilemez. Çizdiği şekiller doğru ama büyüklüğü eşit değil. Kesirlerde karşılaştırma konusunda eksikliği vardır demek ki. Yani bütünden bir parça çıkartınca eşit olur mantığı vardır, geride kalan parçaların sayısına bakılmamıştır. Hatta toplamda kaç parça oldukları da dikkate alınmamıştır.]

Last of all, 12 junior and 8 senior participants stated that the student's answer was not acceptable, and 2 seniors said that this answer was acceptable. However, none of these participants made any explanation for the reasons for this incorrect solution.

Fourth Item

In this item, prospective teachers' interpretation of a student's erroneous set model was asked. The following question was posed to participants.

"In the following figures, the black squares refer to the orange nectar, and the white ones refer to the water. Which mixture is denser?"



Arzu answered the above question as follows:

–The second mixture is denser because while three glasses of water were added into the first mixture, there are six glasses of orange nectar in the second one.

a. Explain with reasons whether Arzu's response was acceptable or not.

In this item, prospective teachers' responses were grouped under the one main category; non-acceptable. In addition, the non-acceptable category included five sub-categories about the reason for the students' erroneous explanation for this item. The sub-categories were ignorance of equal ratios, ignorance of the rate change, inadequate knowledge, without reasons, and wrong reasons. The frequencies and percentages of prospective teachers' responses to the forth item are given in Table 4.5.

Table 4.5
Frequency of Response Categories of the Fourth Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non- acceptable	Ignorance of equal ratios	44	50,5	40	64,5
	Ignorance of the rate change	15	17,2	2	3,2
	Inadequate formal knowledge	13	14,8	10	16,1
	Without Reasons	12	13,7	10	16,1
	Wrong reasons	3	3,4		

The participants had varying reasons for the student's wrong answer about the set model. Forty four junior and 40 senior prospective teachers stated that Arzu's explanation given in the questionnaire was not acceptable, because the density of the two mixtures given in this item were the same. One of the participants explained her/his reasoning as follows:

> Participant 20: "Arzu's answer is erroneous. Two mixtures have the same density. 1-cup of water was added for each 3-cup of orange juice. The rate of the orange juice could be considered in whole mixture. In the first mixture, the rate is $\frac{3}{4}$, and in the second mixture, the rate is $\frac{6}{8}$. Because $\frac{3}{4} = \frac{6}{8}$, the densities are same." [Arzu'nun cevabı hatalıdır. İki karışım eşit yoğunluktadır. Her üç bardak nektar için 1 su bardağı su eklenmiştir. Toplam karışımdaki portakal nektarı oranına bakılabilir. 1. Karışımda bu $\frac{3}{4}$ iken 2. Karışımda $\frac{6}{8}$'dir. $\frac{3}{4} = \frac{6}{8}$ olduğundan yoğunluklar eşittir.]

The second common finding was student's ignorance of the rate change. Fifteen junior and 2 senior prospective teachers stated that because the student ignored the amount of either water or orange juice, she gave a wrong answer to this question:

> Participant 22: "Arzu just takes the increased amount of orange juice into consideration; however she does not notice that the amount of water also increases. In this case, she makes a wrong judgment." [Arzu sadece eklenen portakal miktarının artmasıyla ilgilenmiş fakat; aynı zamanda eklenen su miktarının da artmış olduğunu fark etmemiştir. Bu durumda yanlış bir yargıya ulaşmıştır.]

> Participant 18: "Arzu does not take the added amount of water into consideration." [Arzu eklenen su miktarını dikkate almamıştır.]

Another finding of this item was the student's inadequate formal knowledge. Twelve junior and 9 senior participants declared that the student had weak conceptual understanding of the concepts related to the fractions. In more detail, they said that students had inadequate knowledge of common denominators, ratio, equivalence, and the part-whole relationship;

Participant 68: “Arzu lacks knowledge on the ratio rather than fractions. She does not consider the ratio between shaded parts and whole. She just considers the shaded parts.” [Arzu’nun kesirler konusundan ziyade oran-orantı konusunda bir eksiği vardır. Kesirler konusunda da taralı kısmın bütüne oranını almamış direk taralı kısımlara yönelmiştir.]

In order to give an example of the student’s inadequate knowledge of common denominators, Participant 14 said that “the student’s answer was not acceptable, since she didn’t know how to extend the denominator of the fraction” [Öğrencinin cevabı kabul edilmez, çünkü kesirlerde payda genişletmeyi bilmiyor].

Another example for inadequate knowledge on part-whole relationship is the following;

Participant 44: “If student had paid attention just to the ratio of water, she would say that the first mixture was denser than the other. She doesn’t relate to the whole” [Öğrenci sadece su oranlarını dikkate alsaydı 1. yoğun diyecekti. Bütünle ilişki kuramıyor].

In this item, besides the true reasonings, 12 junior and 10 senior prospective teachers responded to the item as not acceptable. However, these prospective teachers could not state any reasoning for this item. Furthermore, 3 junior participants stated wrong reasoning for the explanation of wrong student response. Three responses of prospective teachers with wrong reasoning are given below:

Participant 11: “Not acceptable, because the first mixture has 75% density, and is denser.” [Kabul edilemez. Çünkü birinci %75’lik bir karışımdır ve daha yoğundur.]

In this response, this prospective teacher wrongly calculated the ratio of the other mixture and said that the first mixture was denser. In the following response however, the participant stated that there were not enough data to calculate the rates and compare them.

> Participant 59: “It is not definite whether the response is acceptable or not, because only one variable is not enough to answer this question.” [Arzu’nun cevabının kabul edilip edilemeyeceği belli değil. Yalnız bir kritere bakarak bu sorunun cevabı yanıtlanamaz.]

In addition, 1 participant said that because the two wholes were different from each other, the comparison between them was not acceptable. The response is given below:

> Participant 86: “Because the major amounts are not same, student cannot find such a ratio. Thus, her answer is not acceptable.” [Ana miktarlar aynı olmadığından böyle bir oranlama yapılamaz. Bu yüzden kabul edilemez.]

In this part of the result section, the prospective teachers’ interpretations of the students’ mistakes based on their formal knowledge of fractions have been mentioned. In the next section, the results of prospective teachers’ responses to the reasons of algorithmically based mistakes will be indicated.

4.1.3. Prospective Teachers’ Knowledge on Reasons for Algorithmically Based Mistakes

In this part, the prospective teachers’ responses to students’ algorithmically based mistakes in addition, subtraction, and multiplication are summarized. The fifth, sixth and seventh items were as follows:

Fifth Item

In the fifth item, prospective teachers were asked about students’ common erroneous answer to subtraction operations. The fifth item is as follows:

[To teacher’s subtraction question; $\frac{7}{8} - \frac{1}{4}$, Zehra responded as $\frac{7-1}{8-4} = \frac{6}{4}$, and Elif responded as $\frac{7-2}{8} = \frac{5}{8}$. Both of them claimed that their answers were true.

a. Explain with reasons whether Zehra’s or Elif’s responses were acceptable.]

In this item, a common student algorithmic mistake that was subtracting both the numerator and denominator from each other while performing the subtraction operation was presented to prospective teachers.

The prospective teachers' responses to Zehra's algorithmic mistake about the subtraction operation were grouped under non-acceptable category and four sub-categories under the non-acceptable category. Non-acceptable was the main category. Moreover, inadequate knowledge, overgeneralization of rules on natural numbers, wrong reasons, and without reasons were four categories for the the prospective teachers' evaluations of students' algorithmic mistakes. The analyses of the responses are shown in Table 4.6.

Table 4.6
Frequency of Response Categories of the Fifth Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Inadequate knowledge	57	65,2	38	61,2
	Overgeneralization of rules on natural numbers	7	8	3	4,8
	Wrong Reasons	2	2,2		
	Without Reasons	21	24,1	21	33,8

Results revealed that 57 junior and 38 senior prospective teachers stated that this mistake stemmed from the student's inadequate formal knowledge. Most of the prospective teachers said that this student's mistake was caused by the student's inadequacy of knowledge of using common denominators. For example, prospective teachers emphasized using the common denominator to perform the subtraction operation:

> Participant 64: "Zehra's answer is wrong, but Elif's answer is correct, because Zehra performs the operation by subtracting the numbers from

each other in numerator and denominator. However, this operation is erroneous. Elif considers the common denominator rule and performs the operation with common denominator." [Zehra'nın ki yanlış Elif'inki doğrudur. Çünkü Zehra direk gördüğü sayıları işleme koyup paydan payı paydadan paydayı çıkarmıştır. Oysa bu işlem yanlıştır. Elif ise payda eşitleme kuralını göz önüne alarak ortak paydada işlem yapmıştır.]

Therefore, this prospective teacher mentioned that because the student did not know or consider the common denominator while performing the subtraction, she gave the wrong answer. In addition to the common denominator, prospective teachers considered the inadequate knowledge of unit fractions and the relationship between numerator and denominator as reasons of this mistake. The prospective teachers' responses in this category are as follows:

Participant 68: "Zehra lacks knowledge about using equal parts of a whole in order to perform the operations in fractions. Elif correctly performs the operation." [Zehra'nın kesirlerde işlem yaparken bir bütün aynı sayıda eş parçalara ayrılması ve bu takdirde işlem yapılması gerektiği konusunda bir eksiği vardır. Elif ise doğru yapmıştır.]

These participants stated that the whole should be divided into equal portions to perform the operation. Some of the other prospective teachers emphasized the inadequate knowledge of the relationship between numerator and denominator:

Participant 29: "Zehra does not understand the numerator and denominator in fraction concept. Elif's operation is correct, she understands the equality in fractions." [Zehra kesir kavramında pay ve payda kavramını anlayamamıştır. Elif'in işlemi doğrudur, denk kesir kavramını anlamıştır.]

In addition to this finding, 7 junior and 3 senior prospective teachers stated the student's overgeneralization of the natural number rules to fractions as a reason for this erroneous answer. They declared that Zehra used the rules of natural numbers in order to add the fractions:

Participant 86: “The student did not think that $\frac{7}{8}$ and $\frac{1}{4}$ were fractions, thus she performed the operations on numerator and denominator separately. She thought that she should perform the operations as if they were natural numbers.” [Öğrenci $\frac{7}{8}$ ve $\frac{1}{4}$ sayılarını kesir olarak düşünmemiş, bu yüzden pay ve payda üzerinde ayrı ayrı işlemler yapmış. Sanki doğal sayılarla işlem yapıyormuş gibi düşünmüş.]

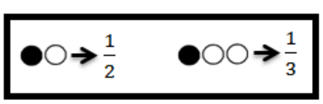
Participant 27: “Zehra’s answer cannot be acceptable. Zehra has a misconception. When she is performing the subtraction in fractions, she subtracts the numerator from the numerator and denominator from the denominator. Elif found the correct result. However, she should asked about how she found 2 in the operation $\frac{7-2}{8} = \frac{5}{8}$.” [Zehra’nın cevabı kabul edilemez. Zehrada bir kavram yanılgısı mevcut. Kesirleri birbirinden çıkarırken paydan payı, paydadan da paydayı çıkartmıştır. Elif sonucu doğru bulmuştur. Fakat $\frac{7-2}{8} = \frac{5}{8}$ işlemindeki 2’ye nasıl ulaştığı hakkında soru sorulmalı.]

Last of all, 21 junior and senior prospective teachers could not state any reasons as a reason of this erroneous answer.

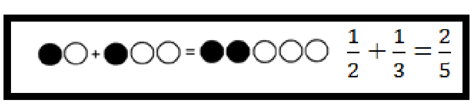
Sixth Item

In the sixth items, prospective teachers were presented with students’ common algorithmic mistake in adding fractions which was to add both numerators and denominators.

[Berk solved $\frac{1}{2} + \frac{1}{3}$ as follows;



When we add marbles, we get



a. Explain with reasons whether Berk's response was acceptable or not.]

In this item prospective teachers were asked about the reason of Berk's erroneous answer in adding fractions. There were one main category that was non-acceptable, , and four sub-categories of the non-acceptable category that were inadequate knowledge, overgeneralization of rules on natural numbers, without reasons, and wrong reasons. The categories and the frequencies are given Table 4.7.

Table 4.7

Frequency of Response Categories of the Sixth Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Inadequate knowledge	64	73,1	45	72,3
	Overgeneralization of rules on natural numbers	4	4,5	1	1,6
	Without Reasons	19	21,8	14	22,5
	Wrong Reasons			2	3,2

The analysis of the responses revealed that the most common reason for this erroneous answer was inadequate knowledge of formal knowledge. Sixty four junior and 45 senior prospective teachers stated the student had difficulty with adding the fractions because of the student's inadequate knowledge of common denominators, the part-whole relationship, and the relationship between numerator and denominator:

> Participant 41: "He lacks knowledge on addition and subtraction in fractions. Thus, he conducted the operation as given in the figure. He should equalize the denominators and then add the numerators." [Kesirlerde toplama ve çıkarma işlemlerinde bilgi eksiği vardır. Bu yüzden toplama işlemini yaparken şekildeki gibi bir yol izlemiştir. Payları toplamadan önce paydaları eşitlemeliydi.]

These prospective teachers mentioned that because the student did not understand the addition and subtraction operation in fractions, he could not carry out the operation. In the same way, one participant gave the following response:

> Participant 20: "Berk does not understand the addition operation in fractions. The student does not understand that $\frac{1}{2}$ refers to a half or $\frac{1}{3}$ is in the [0-1] interval. This solution is erroneous because it is perfomed like $\frac{\text{numerator+numerator}}{\text{denominator+denominator}}$." [Berk kesirlerde toplama işlemini anlamamıştır. Öğrenci $\frac{1}{2}$ kesrinin yarımı ifade ettiğini veya $\frac{1}{3}$ kesrinin [0,1] aralığında olduğunu kavrayamamıştır. Bu çözüm $\frac{\text{pay+pay}}{\text{payda+payda}}$ şeklinde olduğu için hatalıdır.]

This prospective teacher emphasized that the student lacked knowledge of the relationship between numerator and denominator. She mentioned that because the student did not notice that $\frac{1}{2}$ and $\frac{1}{3}$ was between 0 and 1, this student performed the operation in the wrong way. Besides, 1 prospective teacher stated that because the student used different wholes, he wrongly added the fractions. This response is given below:

> Participant 51: "The answer cannot be accepted, because the same wholes should be used in addition and subtraction in fractions. If he uses two marbles in the first fraction, he should also use two marbles in the second fraction. Or six marbles should be used in total." [Edilemez çünkü kesirlerde toplama, çıkarma işleminde aynı bütünler kullanılmalı. İlkinde iki bilye almışsa, ikincide iki bilye almalıydı. Ya da toplam altı bilye kullanılmalıydı.]

As well as these responses, prospective teachers declared the overgeneralization of rules in natural numbers to fractions as a reason of this mistake. For instance:

> Participant 86: "Berk does not notice the difference between addition in natural numbers and addition in fractions, and performs the operation as

in natural numbers. The solution cannot be accepted because it is erroneous." [Berk doğal sayılarla, kesirlerde toplamanın farkına varamamış ve işlemi doğal sayılarda işlem yapar gibi yapmıştır. Cevap yanlış olduğundan Kabul edilemez.]

These prospective teachers said that student's answer was incorrect, since the student considered the fractions like natural numbers and found the wrong result. Moreover, 19 junior and 14 senior prospective teachers claimed that student's solution was not acceptable; however, they could not state any reason of the student's erroneous answer.

Seventh Item

In the seventh item, prospective teachers were asked about a student's algorithmic mistake on multiplication of mixed numbers.

[Zehra found the result of $2\frac{1}{2} \times 1\frac{1}{3}$ by multiplying the whole parts and fractional parts of fractions separately. And she wrote it mathematically as follows:

$2\frac{1}{2} \times 1\frac{1}{3}$; 2×1=2 and $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. And she found the result $2\frac{1}{6}$.

a. Explain with reasons whether Zehra's response was acceptable or not.]

This item demonstrated Zehra's erroneous partial product method in order to multiply the mixed numbers. The student did not multiply all four partial products with each other. Therefore, she produced an erroneous result. Prospective teachers were asked the acceptability of this partial product method, and if it was unacceptable, what the reason was. Results were presented under one main category that was non-acceptable. Moreover, there were four sub-categories of non-acceptable that were inadequate knowledge, overgeneralization of addition rules on multiplication, wrong reasons and without reasons. Table 4.8 shows the frequencies and percentages of the response categories for reasons for the seventh item.

Table 4.8

Frequency of Response Categories of the Seventh Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Inadequate knowledge	43	49,2	36	58
	Overgeneralization of addition rules on multiplication	1	1,1	2	3,2
	Wrong Reasons	5	5,7	4	6,4
	Without Reasons	38	43,6	20	32,2

The analysis of the responses revealed that students' inadequate formal knowledge as the reason of this erroneous answer was the most common response in the seventh item. Forty three junior and 36 senior prospective teachers' responses were in the first category. They stated that this answer was not acceptable, since student lacked knowledge of mixed number operations, components of the mixed numbers, and the distributive property. The following response is an example of the inadequate knowledge of mixed number operations:

> Participant 31: "The answer is erroneous, because the application of multiplication in fractions is erroneous. She does not know the multiplication of mixed numbers." [Cevabı yanlıştır. Çünkü kesirlerde çarpmanın uygulanışı yanlış yapılmıştır. Tam sayılı kesirlerde çarpma konusunu tam bilmemektedir.]

The prospective teachers also stated that this algorithm could not be acceptable, since the students did not change the mixed numbers into improper fractions. They said that students should use the improper fractions in order to multiply them. For instance;

> Participant 17: "The answer cannot be accepted. The result is erroneous, because she does not change the mixed number into improper fraction. She should found $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}$. The operation cannot be performed like the

student performed." [Kabul edilemez. Bileşik kesre çevirmediği için cevap yanlış çıkmıştır. Çünkü $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}$ şeklinde bir sonuç bulunmalıdır. Öğrencinin yaptığı gibi bir işlem yapılamaz.]

In the same way, some of the prospective teachers said that student did not understand the mixed numbers and the components of the mixed numbers. For example;

Participant 57: "The answer cannot be accepted. She does not understand what the whole part means in mixed numbers." [Kabul edilemez. Tam sayılı kesrin tam kısmının ne anlama geldiğini anlayamamıştır.]

Participant 51: "The answer can not be accepted. The answer is $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}=\frac{10}{3}$. The student thinks $2\frac{1}{2}$ as $2\times\frac{1}{2}$ instead of $2+\frac{1}{2}$." [Edilemez. $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}=\frac{10}{3}$' dür cevap. Öğrenci $2\frac{1}{2}$ kesrini $2+\frac{1}{2}$ gibi değil, $2\times\frac{1}{2}$ şeklinde düşünmüştür.]

Furthermore, prospective teachers pointed out that the students' inadequate knowledge of the distributive property produced an incorrect solution for mixed number multiplication:

Participant 67: "The answer cannot be accepted. $2\frac{1}{2}$ is equal to $2+\frac{1}{2}$. Zehra does not understand the distributive property of addition on multiplication operation. The application of distributive property of addition on multiplication operation is missing when it is done like Zehra's operation. $(2+\frac{1}{2}) \times (1+\frac{1}{3})$]" [Cevap kabul edilemez. $2\frac{1}{2}=2+\frac{1}{2}$ dir. Ve Zehra toplamanın çarpma üzerinde dağılma özelliğini tam olarak anlamamıştır. Onun yaptığı şekilde toplamanın çarpma üzerinde dağılma özelliği eksik uygulanmış olur. $(2+\frac{1}{2}) \times (1+\frac{1}{3})$]

Secondly, prospective teachers stated the reason for Zehra's algorithmic mistake as the overgeneralization of addition rules on multiplication. These prospective teachers claimed that the student's operations were only acceptable while adding fractions:

> Participant 148: "Zehra used the operation which is valid for addition in multiplication. We cannot multiply and add only the wholes and fraction when the mixed number $2\frac{1}{2}$ is written in the form of $2+\frac{1}{2}$. However, Zehra considered the multiplication as addition and performed the operation. Her answer is not acceptable." [Zehra toplamada geçerli olan işlemi çarpma yaparken de kullanmıştır. $2\frac{1}{2}$ kesir sayısı $2+\frac{1}{2}$ olarak yazılıp sonra çarpmaya girdiğinde sadece tamlar ve kesirleri çarpıp toplayamayız. Ancak Zehra burada çarpmayı toplama gibi düşünüp işlem yapmıştır. Cevabı kabul edilemez.]

Finally, 38 junior and 20 senior prospective teachers could not state any reason to the given question that involved the incorrect mixed number multiplication.

In addition to the correct reasonings, five of the juniors and four of the seniors stated wrong reasoning in this item. For example, two of the prospective teachers stated that this way was acceptable since the result was correct and the student used the distributive property in multiplying fractions:

> Participant 2: "The answer can be acceptable; the multiplication of whole parts and fractional parts separately can be because of the commutative property of multiplication operation. [Cevap kabul edilebilir. Tam kısımların ve kesir kısımlarının ayrı ayrı yapılması çarpma işleminin değişme özelliğinden olabilir.]

Similar to this response,

> Participant 86: "Because this operation is multiplication and gives the correct result, the answer can be acceptable." [Bu işlem çarpma işlemi olduğundan ve yapılış doğru işleme ulaştırdığından kabul edilebilir.]

Correspondingly,

Participant 30: "It cannot be acceptable. Because the denominators of the fractions are not the same, we cannot multiply the whole parts with each other. If the denominators were the same, we could do such an operation." [Kabul edilemez. İki kesrinde paydaları eşit değil ki tam kısımları çarpabilelim. Paydalar eşit olsaydı böyle bir işlem yapabilirdik.]

These prospective teachers confused addition and multiplication and thought that the student should use the common denominator to multiply the fractions.

Participant 138: "The answer cannot be accepted because it is erroneous. He should multiply 2 halves and $\frac{1}{3}$. Because 2 halves is equal to 1, the result is $1\times\frac{1}{3}=\frac{1}{3}$."[Cevap kabul edilemez çünkü yanlıştır. 2 tane yarım ve $\frac{1}{3}$'ü çarpması gerekir. 2 tane yarım 1 olduğu için $1\times\frac{1}{3}=\frac{1}{3}$ eder.]

This prospective teacher wrongly interpreted the student's multiplication operation and found another wrong result. This participant confused the mixed number, two wholes and a half with two multiplied with half. This participant did not accept the student's wrong answer; however s/he gave a wrong reasoning for student's non-acceptable response.

4.1.4. Prospective Teachers' Knowledge on Reasons for Intuitively Based Mistakes

In this section, prospective teachers' knowledge on students' erroneous answers stemming from intuitions held about multiplication and division are investigated.

Eighth Item

The eighth item is given below:

[The teacher found the result of $4 \times \frac{1}{16}$ operation as $\frac{1}{4}$. Burçin made an objection to this result and she claimed that "The result is wrong, since multiplication always makes bigger."

a. Explain with reasons whether Burçin's response was acceptable or not.]

In this item, prospective teachers were asked to analyze student's response about the overgeneralization of multiplication rules on natural numbers to rational numbers. The analysis of the data showed that the responses of the prospective teachers could be grouped under two headings: non-acceptable and acceptable. The non-acceptable categorization consisted of six reasons for Burçin's mistake: overgeneralization of rules on natural numbers, inadequate knowledge, misinterpretation of the division operation, ignorance of the simplification of fractions, wrong reasons, and without reasons. The acceptable categorization included one sub-category which is inadequate knowledge. The categories and the frequencies of prospective teachers' reasons for Burçin's mistake are shown in Table 4.9.

Table 4.9

Frequency of Response Categories of the Eighth Item

Participants' decisions/ Evaluation of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Overgeneralization of rules on natural numbers	40	45,9	32	51,6
	Inadequate knowledge	11	12,6	8	12,9
	Misinterpretation of division operation	5	5,8	1	1,6
	Ignorance of simplification of fractions	2	2,3	1	1,6
	Wrong reasons			1	1,6
	Without Reasons	28	32,2	18	29
Acceptable	Inadequate knowledge			1	1,6

As it can be seen from Table 4.7, 40 junior and 32 senior prospective teachers stated that the overgeneralization of rules on natural numbers to rational numbers could be one of the reasons for the student's wrong answer. They claimed that the student's comment was not applicable in rational numbers. For example,

> Participant 20: "Burçin's answer is correct for natural numbers but not valid for fractions" [Burçin'in cevabı doğal sayılar için doğrudur ancak kesirler için geçerli değildir].

Similarly,

> Participant 69: "Burçin's answer is not acceptable, since students first learn the multiplication of natural numbers and reach such a generalization. And then, they considered this generalization for fractions and reach a wrong conclusion." [Burçin'in cevabı kabul edilemez. Çünkü önce doğal sayılarla çarpma işlemine başlıyoruz ve böyle bir genellemeye varıyoruz. Öğrencilerde bu genellemeyi kesirler içinde uyguluyor ve yanlış yoruma varıyorlar.]

Apart from the first finding, Burçin's inadequate formal knowledge was stated as the reason for her erroneous generalization. They said that because the student did not understand the concepts in fraction multiplication, she made a wrong generalization. For instance;

> Participant 39: "It is not acceptable. Burçin does not fully understand the fraction multiplication." [Kabul edilemez. Burçin kesirlerde çarpmayı tam anlamamış.]

Fraction comparison and equivalency was another area of inadequate formal knowledge, which was the reason for the student's mistake. These prospective teachers declared that since the student had weak understanding of fraction comparison, this conceptual deficiency leaded her to obstacles in her interpretation of the multiplication operation. For example;

> Participant 29: "Burçin's answer is erroneous. She does not understand the big and small concepts in fractions. In addition, she lacks knowledge on equality of fractions." [Burçin'in cevabı yanlıştır. Kesirlerde büyüklük küçüklük kavramını anlamamıştır. Denk kesir kavramında anlaşılmayan yerler vardır.]

> Participant 85: “Burçin does not understand the fractions. The fraction whose denominator is bigger is the smaller fractions.” [Burçin kesirler konusunu kavrayamamıştır. Paydası büyük olan küçüktür.]

However, this participant misunderstood the item, since s/he thought that the student was just comparing $\frac{1}{4}$ and $\frac{1}{16}$. However, in the item, the student’s claim suggested $\frac{1}{4}$ was smaller than 4.

The third finding of this item was the misinterpretation of the division operation. Prospective teachers asserted that multiplying a whole number with a fraction meant dividing the whole number with the denominator of the fractional number. For example,

> Participant 22: “Burçin just thinks the multiplication in terms of positive integers. However, she does not think that the numbers in multiplication decreases, because the multiplication of an integer and a rational number means the division of an integer to the number in denominator.” [Burçin çarpma işlemini sadece pozitif tam sayılar çerçevesinde düşünmüştür. Ama bir tam sayının rasyonel bir sayıyla çarpımı aslında paydadaki sayıya bölümü olacağından sayının küçüleceğini düşünememiştir.]

The last sub-category of the non-acceptable category was ignorance of simplification of fractions. To give an example,

> Participant 27: “The student found $4 \times \frac{1}{16} = \frac{4}{16}$. She ignored to simplify the fractions. Therefore, she might have thought that the result was erroneous.” [Öğrenci $4 \times \frac{1}{16} = \frac{4}{16}$ sonucuna ulaşmış ve sadeleştirme yapmayı göz ardı etmiş, bu yüzden sonucun yanlış olduğunu düşünmüş olabilir.]

Finally, 28 junior and 18 senior prospective teachers considered that Burçin’s explanations were not acceptable in multiplying fractions; however these prospective teachers did not state any reasoning as to why Burçin’s explanations were erroneous.

Ninth Item

The ninth item was as follows:

[Tülay said to her teacher that "Division always makes smaller, however I found the result of this operation $6 \div \frac{1}{3}$=18. I think I find the result wrong."

a. Explain with reasons whether Tülay's claim was acceptable or not.]

The student's erroneous answer in this item was related to the overgeneralization of division rules on natural numbers to rational numbers. There were two main categories: non-acceptable and no response and five sub-categories of non-acceptable category: overgeneralization of rules on natural numbers, inadequate knowledge, rote memorization, lack of self efficacy, and without reasons. The results of the analysis of the ninth item are given in Table 4.10.

Table 4.10

Frequency of Response Categories of the Ninth Item

Participants' decisions/ Evaluations of the students' response	Reasons	Juniors		Seniors	
		n	%	n	%
Non-acceptable	Overgeneralization of rules on natural numbers	31	35,6	22	35,4
	Inadequate formal knowledge	11	12,6	7	11,2
	Rote memorization	2	2,2		
	Lack of self efficacy	1	1,1		
	Without Reasons	35	40,1	25	40,2
No response		8	9,1	8	12,8

Results revealed that the most common response in the non-acceptable category was overgeneralization of rules on natural numbers. Thirty one juniors and 22 seniors claimed that because the rules on natural numbers were not applicable to rational numbers or fractions, Tülay made the wrong generalization and her explanation was not acceptable. To give an example;

> Participant 16: "She found the answer correctly, but her explanation wasn't accepted, because the result doesn't always smaller than the dividend. This is an operation in fractions." [Cevabı doğru bulmuş, fakat açıklaması kabul edilemez. Çünkü her zaman bölünenden küçük çıkmaz. İşlem kesirlerde yapılıyor.]

> Participant 23: "This claim is invalid in rational numbers. There are more pieces, since dividend is divided into small pieces." [Rasyonel sayılarda bu iddia geçerli değildir. Bölünen daha küçük parçalara bölündüğü için ortaya birçok parça çıkacaktır].

Additionally, 16 junior and 7 senior prospective teachers emphasized the student's inadequate formal knowledge. They stated that Tülay correctly found the result; however her explanation for the result of the division operation was incorrect because she did not know some of the concepts such as the meaning of the division by fractions and the definition of fraction. For example,

> Participant 57: "The student's explanation cannot be accepted. She does not understand the fractions." [Öğrencinin açıklaması kabul edilemez. Kesir konusunu anlamamıştır.]

One prospective teacher mentioned the student's inadequate knowledge on the meaning of division of fractions:

> Participant 46: "It cannot be accepted, because she found the number of $\frac{1}{3}$ units in the six whole. Thus, the result is bigger. The student does not know what the division in fractions means." [Kabul edilemez. Çünkü 6'nın içinde $\frac{1}{3}$'lük birimlerden kaç tane olduğu bulunmuştur, dolayısıyla daha büyük bir sonuç çıkmıştır. Öğrenci kesirlerde bölme yapmak ne demek bilmiyor.]

Other findings could be categorized under the categories of rote memorization and lack of self-efficacy. Two of the junior prospective teachers emphasized the rote memorization and they said that students found the correct result by chance. For example;

> Participant 22: “Tülay does not understand the division and perform the operation by rote-memorization. She inadvertently found the correct result.” [Tülay bölme işlemi anlamlandıramamış, ezbere bir yol izlemiştir. Sonucu doğru bulmuştur fakat bilmeyerek.]

> Participant 43: “Her answer is correct; however the student learns the division by rote-memorization instead of understanding its rationale.” [Cevabı doğrudur. Ama öğrenci bölmeyi öğrenirken mantığını kavramak yerine ezber yapmıştır.]

One of the junior prospective teacher indicated lack of self-efficacy as a reason for student’s claim:

> Participant 37: “The student suspected her result because she had lack of self-efficacy.” [Öğrencinin kendine güveni olmadığı için sonucundan şüphe etmiştir.]

Last of all, 35 junior and 23 senior prospective teachers stated that Tülay’s answer was not acceptable. However they did not give any reasons for student’s erroneous answer. On the other hand, two of the senior participants said that the student’s answer was acceptable, yet they also did not give any reasoning for their response. And finally, 8 junior and senior prospective teachers did not respond to this sub-dimension of the ninth item.

4.2. Prospective Teachers' Proposed Strategies to Overcome Students' Mistakes:

This section summarizes the results of prospective teachers' prosposed strategies to overcome the elementary students' mistakes in fractions. The second sub-dimension of the items investigated what prospective elementary mathematics teachers knew about how to evaluate and responded to students' mistakes. Prospective elementary mathematics teachers suggested various strategies, such as verbal explanations, using area representations, using real life model, reviewing prior knowledge, teaching standard algorithm, and asking guided questions. Table 4.12 summarizes the frequencies and percentages of junior and senior prospective teachers' proposed strategies to overcome students' mistakes on fractions addressed in the second sub-dimensions of the items given in the questionnaire.

Table 4.11

Type and Frequency of Proposed Strategies

		Verbal Explanation		**Using Area Representation**		**Using Real life Model**		**Reviewing Prior Knowledge**		**Teaching Standard Algorithm**		**Asking Guided Questions**	
	Item	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**
JUNIORS	**1**	36	41,4	20	23,0	6	6,9	11	12,6			9	10,3
	2	41	47,1	8	9,2	1	1,1	16	18,4			13	14,9
	3	23	26,4	12	13,8	28	32,2	11	12,6			7	8,0
	4	34	39,1	10	11,5	4	4,6	25	28,7			11	12,6
	5	19	21,8	13	14,9	6	6,9	19	21,8	20	23,0	5	5,7
	6	24	27,6	9	10,3	6	6,9	10	11,5	15	17,2	15	17,2
	7	6	6,9	5	5,7			12	13,8	13	14,9	34	39,1
	8	33	37,9	7	8,0	9	10,3	12	13,8			7	8,0
	9	38	43,7	16	18,4	5	5,7	4	4,6			3	3,4
SENIORS	**1**	32	51,6	9	14,5	2	3,2	8	12,9			5	8,1
	2	42	67,7	3	4,8	4	6,5					7	11,3
	3	22	35,5	10	16,1	16	25,8	5	8,1			7	11,3
	4	33	53,2	1	1,6	6	9,7	17	27,4			5	8,1
	5	38	61,3	3	4,8			5	8,1	8	12,9	2	3,2
	6	31	50,0	9	14,5	3	4,8	6	9,7	5	8,1	3	4,8
	7	34	54,8	3	4,8			4	6,5	11	17,7	2	3,2
	8	33	53,2	7	11,3			3	4,8			2	3,2
	9	36	58,1	4	6,5	1	1,6	3	4,8			1	1,6

Table 4.11 (continued).

Type and Frequency of Proposed Strategies

		Making Students Aware of their Mistake		**Using Simple Example**		**Using Drill and Practice**		**Providing Counter Example**		**Increase Student Motivation**		**No Strategy**		**Wrong Strategy**	
	Item	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**	**f**	**%**
JUNIORS	**1**											5	5,7		
	2											8	9,2		
	3											6	6,9		
	4											3	3,4		
	5									1	1,1	4	4,6		
	6					1	1,1					5	5,7	2	2,3
	7											16	18,4	1	1,1
	8	2	2,3	3	3,4	4	4,6	2	2,3			8	9,2		
	9	4	4,6			1	1,1					13	14,9	3	3,4
SENIORS	**1**											6	9,7		
	2											6	9,7		
	3			1	1,6							1	1,6		
	4														
	5					2	3,2					4	6,5		
	6					2	3,2					2	3,2	1	1,6
	7					2	3,2					6	9,7		
	8			9	14,5	1	1,6					7	11,3		
	9			5	8,1	1	1,6					11	17,7		

As shown in Table 4.12, prospective teachers mostly suggested verbal explanation in order to overcome the students' erroneous answers. Verbal explanation was the most common strategy for the first, second, fourth, sixth, eighth and ninth items, teaching standard algorithm was the most common strategy for the fifth item, using real life model was the most common strategy for the third item, and lastly asking guided questions was the most common strategy for the seventh item suggested by junior prospective teachers. Furthermore, using verbal explanations was the most common strategy for all items by senior prospective teachers. The detailed findings of the suggested strategies are given below.

Using Verbal Explanations

Using verbal explanations was one of the common strategies offered by prospective teachers. Half of the senior prospective teachers and 30% of the junior prospective teachers offered verbal explanations. Junior and senior prospective teachers stated that by using verbal explanations, they could explain the correct solution of the question. For example, for the third item,

> Participant 20: "I can say to Mert that $\frac{5}{5}=\frac{6}{6}=1$. However, $\frac{1}{6}$ and $\frac{1}{5}$ of the whole are not equal to each other. Thus, if we drop one of the parts out, the remaining parts will not be equal to each other." [Mert'e $\frac{5}{5}=\frac{6}{6}=1$'dir, ancak bir bütünün $\frac{1}{6}$'sı ile $\frac{1}{5}$'inin eşit olmadığını söylerdim. Bu nedenle bütünden farklı parçaları çıkarırsak sonucun eşit olmayacağını söylerdim.]

Similarly, Participant 56 also explained the reason for the erroneous answer in the second item and then explained the correct solution to the question:

> Participant 56: "I repeat the whole concept, and tell student to pay attention to the number of parts in the 0-1 interval on his number line. And then I explain that $\frac{1}{3}$ means the first point of three equal parts between 0 and 1 on the number line." [Bütün kavramını tekrar eder,

çizdiği sayı doğrusunda 0-1 arasında kaç parça olduğuna dikkat etmesi gerektiğini söyler ve $\frac{1}{3}$ kesrinin 0-1 aralığını 3 eşit parçaya böldükten sonraki ilk bölüm noktasını ifade ettiğini açıklardım.]

In addition to the above responses, one of the participants suggested how to explain the student's erroneous answer in the sixth item. S/he stated that the result of adding a half and one third should be bigger than the half; however in this solution it was smaller than the half. The prospective teacher's response is as follows:

> Participant 30: "The mistake on the sizes of fractions can be explained. This means that there is an addition of a half and one third; however the result of the student is smaller than the half. By these explanations, I can try to explain the student's mistake." [Yine büyüklük nicelikleri üzerinden hata yapıldığı anlatılabilir. Şöyle ki yarım büyüklük var ki bu yarım üzerine $\frac{1}{3}$ topluyoruz ama öğrencinin cevabı yarımdan küçük bir nicelik, cevabın yanlış olduğu anlatılmaya çalışılır.]

In the eighth and ninth items, prospective teachers generally stated that they could explain how students' generalizations were not applicable for the fractions:

> Participant 6: "I can tell the student that his explanation is correct for natural numbers, but there can be such a result of an operation with rational numbers." [Ona söylediği şeyin doğal sayılar arasında yapılan çarpma işlemlerinde doğru olduğunu fakat rasyonel ifadelerde böyle bir cevabın olabileceğini söylerdim.]

This prospective teacher explained that this generalization was correct for natural numbers; however it was not accepted for the multiplication of fractions. In the same way, another prospective teacher explained the correct generalization in dividing fractions in the ninth item:

> Participant 23: "It is normal to find the result smaller than $\frac{1}{3}$, because $\frac{1}{3}$ is smaller than one whole. I can tell student that we cannot make such an

overgeneralization in fractions." [$\frac{1}{3}$ kesri bir tamdan daha küçük olduğundan bölme işlemi sonunda bölünenden büyük bir sonuç çıkması doğaldır. Basit kesirlerde böyle bir genelleme yapamayacağımızı söylerdim.]

Using Area Representations

Using area representations was another strategy suggested by prospective teachers in order to overcome students' mistakes. For example, participant 81 proposed an alternative regular shape to deal with students' mistakes based on formal knowledge. This participant said that she could draw a rectangle to show how equal parts were formed and show the incorrect division of the triangle:

Participant 81: "Firstly, I want student to draw a rectangle. And then I want them to divide this rectangle into three equal parts. After students understand the division in the rectangle, they will understand the erroneous division of the triangle." [Oranları belli bir, örneğin dikdörtgen çizip; bunu 3 parçaya ayırmalarını isterim öncelikle. Dikdörtgendeki bölümü kavrattıktan sonra üçgendeki mantıksız bölümü anlamış olacaktır.]

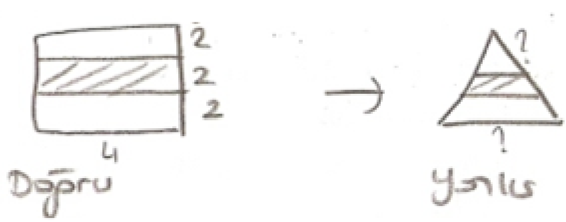


Figure 4.3. Participant's alternative regular shape

Furthermore, prospective teachers who offered alternative drawings of the triangle stated that if the triangle was horizontally divided, the parts of it would not be equal to each other. For instance:

Participant 96: "The triangle gets narrow from bottom to top. If the student divides the triangle using horizontal lines, none of the parts is equal to each other. The student should divide the triangle vertically and shade one of the parts." [Üçgen tabandan tepe noktasına doğru daralır. Üçgeni yatay şekilde parçalara ayırırsa hiçbir parça eşit olmaz. Üçgeni dikey şekilde bölümleyip bir parçasını taramalıdır.]

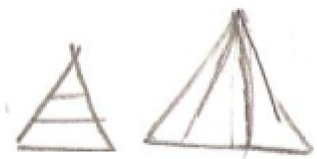


Figure 4.4. Participant's division of the triangle

The participants also suggested a rectangular representation to correctly perform the operations. For instance, one of the prospective teachers preferred to use the area of the rectangle to explain the multiplication of the mixed number:

> Participant 20: "I tell the student that she should change the mixed numbers into improper fractions. And then I want him to find the result of the multiplication from the area of the rectangle as represented below." [Kesirleri önce bileşik kesir haline getirmesini söylerdim. Daha sonra aşağıdaki şekilde olduğu gibi, sonucu dikdörtgenin alanından bulmasını isterdim.]

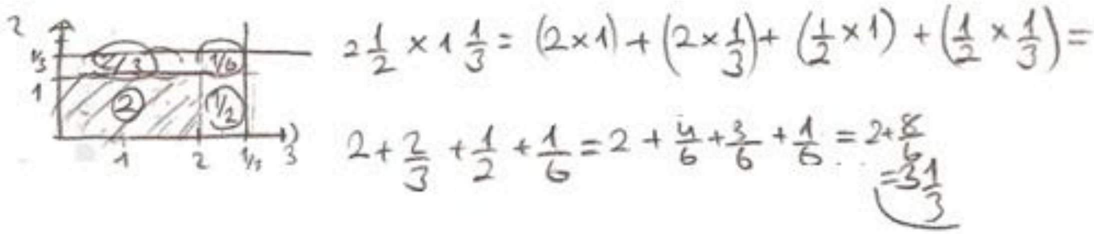


Figure 4.5. Participant's rectangular area representation of multiplication

This prospective teacher used this model to show four partial products in multiplication of two mixed numbers. Furthermore, this participant wrote all the products and then added them. Similarly, another prospective teacher suggested a rectangular representation in order to subtract fractions:

Participant 20: "I show the fraction subtraction to Zehra by using representation. For example, I tell her about the unit fraction and tell that two unit fractions should be the same. Therefore, the denominators should be the same." [Zehra için kesirleri modelleyerek çıkarma işlemini gösterirdim. Örneğin; birim kesir kavramından bahseder ve iki kesrin birim kesirlerinin eşit olması gerektiğini söylerim. Bu yüzden paydalarının eşit olması gerektiğini söylerim.]

Figure 4.6. Participant's representation of subtraction of two fractions

In this example, the prospective teacher drew three rectangles to model the fraction subtraction. S/he then conceptually explained the process of getting a common denominator while performing the subtraction operation.

Another prospective teacher suggested using two 6-equal partitioned boxes to help the student to add the fractions:

Participant 26: "I say the student to draw two rectangles divided into six equal parts. Firstly, take $\frac{1}{2}$ of the rectangle, and then take $\frac{1}{3}$ of the other rectangle, and then add them up. The student will notice that the result of the operation is $\frac{5}{6}$. He will understand that his operation is wrong." [6 parçaya ayrılmış iki tane dikdörtgen çizdirip, önce $\frac{1}{2}$'sini al ve sonra $\frac{1}{3}$'ünü al ve sonra topla derdim. Cevabın $\frac{5}{6}$ olduğunu görecektir. İşlemin yanlış olduğunu anlayacaktır.]

In addition to the above examples, prospective teachers stated that they could show the size of the parts by drawing figures. For example, one of the participants drew a pie chart in order to show how many three equal parts there were in six wholes. Her/his response was as follows:

Participant 20: "I ask Tülay how many $\frac{1}{3}$ fractions there were in the six. In the following figure, each shaded, dotted, and lined piece refers to $\frac{1}{3}$. In total, there are 18 pieces of $\frac{1}{3}$. I tell student that "the dividend does not have to be bigger than the divisor in fractions". In addition, I tell her that she can check her result by multiplying the fraction with each other, and if she finds $18\times\frac{1}{3}=6$, then her result is correct." [Tülay'a '6'nın için de kaç tane $\frac{1}{3}$ kesrinin olduğunu' sorardım. Aşağıdaki şekillerde taralı, noktalı ve düz parçaların her biri $\frac{1}{3}$'tür. Her bir parçada 3 tane $\frac{1}{3}$'lük dilim vardır. Toplamda 18 tane $\frac{1}{3}$'lük dilim vardır. Kesirlerde "bölünen bölme işleminin sonucundan büyük olmak zorunda değildir." derdim. Hata yapıp yapmadığını bölen ve bölümü çarparak da bulabileceğini söylerdim. $18\times\frac{1}{3}=6$ ulaşırsan cevap doğrudur derdim.]

Figure 4.7. Participant's pie chart modeling of multiplication of fractions

Another prospective teacher also used the area representation to show the same concept like the above participant:

Participant 29: "I emphasize that 6 should be divided into $\frac{1}{3}$ parts. I draw a representation in order to provide her understanding. I ask her the number of parts in this representation. I want student to explain what the division is. [6 sayısını $\frac{1}{3}$'lük parçalara ayırmam gerektiğini vurgularım. Bunu anlamasını sağlayacak şekli çizerim. Bu şekliden elde edilen parça sayısını sorarım. Bölme kavramının ne olduğunu açıklamasını beklerdim.]

Figure 4.8. Participant’s measurement modeling of fraction over whole number

In these two representations, prospective teachers showed each unit piece and put them together. They found out how many whole circles/rectangles are in 18 thirds. In addition, participant 41 used the common area model to show how the parts got smaller when two of the fractions were multiplied.

> Participant 41: “I make student multiply the fractions by using representation. She will see that the parts are getting smaller.” [Modelleme ile kesirlerde çarpmayı yaptırırdım. Parçaların küçüldüğünü görürdü.]

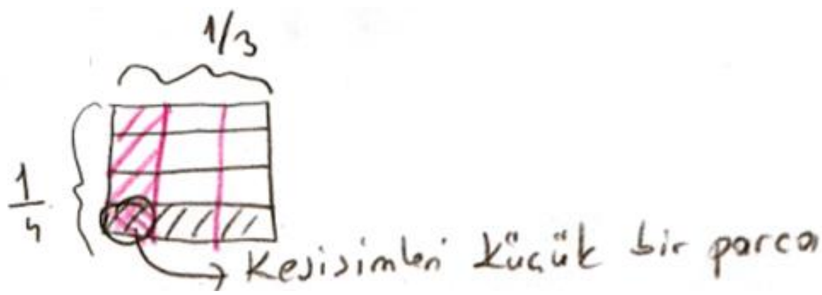


Figure 4.9. Participant’s common area model of multiplication

This prospective teacher used the drawing to determine the product $\frac{1}{4} \times \frac{1}{3}$ and explain how the parts got smaller when two of the fractions were multiplied. The participant drew three lines in one direction in order to show $\frac{1}{4}$ of a whole. And then took $\frac{1}{3}$ of it and showed the common region as the product.

Using Real Life Model

Using real life models was another strategy suggested by prospective teachers in order to overcome the students’ erroneous answers. For example, for the fourth item, the participant suggested the following strategy:

Participant 37: "I give student two pieces of paper in the shape of cake. I want him to divide 1 piece of paper into 4 and take 3 pieces, and other piece of paper into 8 and take 6 pieces. Later I want student to compare the areas of the detached pieces." [Ona iki tane pasta şeklinde kağıt verir birini 4 eşit parçaya bölüp 3'ünü almasını, diğerini de 8 eşit parçaya bölüp 6'sını almasını söylerdim. Daha sonra aldığı parçaların alanlarını karşılaştırmasını isterim.]

Similarly, one of other strategies for the third item was as follows:

Participant 51: "I take two equal length paper strings and divide each string into equal 5 and 6 pieces. By overlapping the pieces each other, I can show student that the pieces of 6-pieces string are shorter than the other pieces. I tell student that if the detached piece is shorter, then the left-over pieces are longer." [Aynı uzunluktaki iki kağıt şerit alıp birini 5 eş parçaya diğerini 6 eş parçaya ayırırım. 2 şeridin her bir parçasını üst üste koyarak 6 parçaya ayırdığım şeridin 1 parçasının daha küçük olduğunu gösteririm. Daha küçük parça çıkıyorsa kalan daha büyük derim.]

Prospective teachers additionally preferred to use real life models in order to produce meaningful uses of notations of algorithms, simplifiying the context of the problem, and increase student concept development in the ninth item:

Participant 30: "I want student to slice an apple into 2. In this case, student will see by this apple that when 1 is divided into $\frac{1}{2}$, the result is 2. Later, the topic will be taught to student by relating the operation and slicing the apple." [Öğrencilerden bir elmayı ikiye bölmesini isterim. Bu durumda bölünen 1 iken, bunun $\frac{1}{2}$'ye bölünmesi bölümü 2 yaptığını öğrenci bu elmayla görecektir. Sonrasında işlemle elmanın bölünmesi arasında bağ kurularak konu anlatılmaya çalışılır.]

In another example, one prospective teacher used real life model in order to explain the fraction comparison in the third item.

> Participant 81: "I try to explain the topic by using the real life model. I tell the student that eating four pieces of a 5-piece apple is not the same with eating five pieces of a 6-piece apple, because four pieces are ate in one apple while five pieces are ate in other apple. This means that $\frac{5}{6}$ is bigger than $\frac{4}{5}$." [Günlük hayattan örnek vererek açıklamaya çalışırım. Bir elmayı 5 parçaya ayırıp 4 parçasını yemekle, bir elmayı 6 parçaya ayırıp 5 parçasını yemenin aynı şey olmadığını; birinde 4 parça yemişken diğerinde 5 parça yediğini anlatırım. Dolayısıyla $\frac{5}{6}>\frac{4}{5}$ olur.]

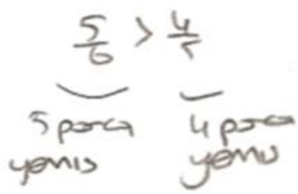


Figure 4.10. Participant 81's Equation

Reviewing Prior Knowledge

Prospective teachers also suggested to reviewing prior knowledge such as common denominators, fraction comparison, how to indicate the numbers on the number line, and how to set up the equation and ratio in order to overcome the students' erroneous answers. For instance, in the fourth item:

> Participant 56: "Firstly, I explain mixture ratio concept in order to evaluate the mixture's densities. The difference between densities should be determined by this mixture ratio." [Karışımların yoğunluklarını değerlendirebilmesi için ilk önce karışım oranı kavramını açıklarım. Yoğunluk farkının bu oranla belirlenmesi gerektiğini anlatırım.]

In the same way, prospective teachers suggested reviewing the common denominator principle in order to overcome the students' erroneous answers on fraction comparison. These participants pointed out the necessity of the common denominator to compare the fractions in the third item:

> Participant 17: “In order to compare the fractions, student should equalize the denominators of fractions. I mean, I tell her that she should write as $\frac{6}{30}$ and $\frac{5}{30}$. Therefore, I show that $\frac{6}{30}$ is bigger than $\frac{5}{30}$ by using the equality concept.” [Öğrencinin iki kesri karşılaştırabilmesi için paydaları ortak bir sayıda (eşit sayıda) yani $\frac{6}{30}$ ve $\frac{5}{30}$ şeklinde yazılması gerektiğini anlatırdım. Böylece denklik kavramından yararlanmasını sağlayarak $\frac{6}{30}$’un $\frac{5}{30}$’dan büyük olduğunu gösterirdim.]

Reaching the common denominator was also emphasized in operations with fractions in the fifth item as follows:

> Participant 23: “I tell student that we should equalize the denominators of fraction while performing the addition and subtraction operation. Later I show students that how the operations can be performed.” [Kesirlerde toplama ve çıkarma işlemleri yaparken paydası eşit olmayan kesirleri öncelikle ortak bir paydada buluşturmamız gerektiğini söylerdim. Sonrasında gerekli işlemlerin nasıl olacağını gösterirdim.]

Similar to above response,

> Participant 144: “I tell Zehra that, ‘we should pay attention that the same pieces are used while performing addition and subtraction. Since denominators indicate how many pieces there are in a whole, in order to make pieces equal, denominators must be equal. We learned how to expand denominators. If we multiply and divide the numerator and denominator with the same number, the value of the fraction does not change. Thus, first expand the denominators and then make the operation on the numerator and write the common denominator.’ [Zehra’ya, kesirlerde toplama-çıkarma yapılırken eşit parçaların olmasına dikkat

etmeliyiz. Paydalar, bütünün kaç parçaya bölündüğünü belirttiğine göre, eşit parçalar elde etmek için paydaların eşit olması şarttır. Kesirlerde genişletmeyi öğrenmiştik. Payı ve paydayı aynı sayıyla çarpar, bölersek kesrin değeri değişmez. O halde paydaları eşitleyecek şekilde genişletelim. Sonra paylar arasında işlem yapıp ortak paydayı yazalım, derdim.]

Teaching Standard Algorithm

Another suggested strategy was to teach the standard algorithm in order to overcome the students' erroneous answers. For example in the seventh item,

> Participant 77: "Because $2\frac{1}{2}$ means $2+\frac{1}{2}$ and $1\frac{1}{3}$ means $(1+\frac{1}{3})$, I tell student that multiplying the fractions in this form is wrong. Firstly, the mix numbers should be changed into improper fraction such as $\frac{5}{2}$ and $\frac{4}{3}$, and then the multiplication should be performed. I tell that the operation should be $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}=\frac{10}{3}$." [$2\frac{1}{2}=2+\frac{1}{2}$; $1\frac{1}{3}=(1+\frac{1}{3})$ demek olduğu için bunu direk çarpmanın yanlış olduğunu, önce kesri bileşik kesre çevirip $\frac{5}{2}$, $\frac{4}{3}$ gibi daha sonra çarpma işlemini uyguladığımızı anlatırdım. $\frac{5}{2}.\frac{4}{3}=\frac{20}{6}=\frac{10}{3}$ olduğu tahta da çözerek anlatırdım.]

Asking Guided Questions

Prospective teachers suggested asking guided questions about the basic concepts of the erroneous answers. For instance in the first item, one of the participants asked the definition of fractions in order to make students aware of the basic concept of the part-whole relationship:

> Participant 76: "I ask the meaning of fraction. Thus, she remembers that the fraction should be equal portions. And then I ask her to perform the operation again." [Kesrin tanımını sorardım. Böylece öğrenci kesrin eş parçalar olması gerektiğini hatırlardı. Daha sonra soruyu tekrar çözmesini isterdim.]

One prospective teacher preferred to help the student understand the equality of $\frac{3}{3}$ and 1 in the second item. The response about this strategy is given below:

> Participant 102: "I want student to explain $\frac{3}{3}$. I help student to understand $\frac{3}{3}$ = 1 by asking such questions; 'What do we get if we divide 3 by 3?' When he understands that $\frac{3}{3}$ = 1, he notices that he have wrongly marked the place of 1 on the number line." [$\frac{3}{3}$ kesrini açıklamasını isterim. 3'ü 3'e bölünce kaç elde ederiz? gibi sorularla $\frac{3}{3}$'ün 1'e eşit olduğunu bulmasına yardımcı olurum. Bu eşitliği anladığında 1'i yanlış yere işaretlediğini fark etmiş olacaktır.]

Making Students Aware of Their Mistakes

In addition to the above strategies, prospective teachers recommended making students aware of their mistakes and reaching the correct overgeneralization that division would not always make smaller. They stated that they could make students see their mistakes and make a correct generalization. For example in ninth item,

> Participant 2: "I want student to sequentially divide 6 by 1, 2, and 3. The results of this division are getting smaller. And then I want student to divide 6 by the numbers smaller than 1. At this time, the results are getting bigger. I help student to notice that in which numbers the results are getting smaller and bigger. Later, I help student to make a generalization." [Önce 6'yı 1'e bölmesini isterdim.Sonra 2'ye ve 3'e sonuçlar gitgide küçülecektir.Bu sefer 1'den küçük sayılara bölmesini isterdim.Bu defa sonuçlar büyüyecektir.Öğrencinin sonuçların hangi sayılarda büyüyüp hangilerinde küçüldüğünü görmesini sağlayıp, sonrasında bir genelleme yaptırmaya çalışırdım.]

Similarly,

> Participant 46: "I want student to find how many $\frac{1}{3}$ pieces there are in six wholes. I help student to make the correct generalization by showing the difference between $6 \div 3$ and $6 \div \frac{1}{3}$." [Ona böldüğü sayının yani $\frac{1}{3}$'ün, 6'nın içinde kaç tane olduğunu bulmasını isterdim. $6 \div 3$ ile $6 \div \frac{1}{3}$ arasındaki farktan yola çıkarak doğru genellemeyi yapmasını sağlarım.]

Using a Simple Example

Using a simple example was another strategy suggested by prospective teachers. The prospective teacher suggested performing a multiplication with simple multipliers in order to overcome the overgeneralization that multiplication always makes bigger:

> Participant 39: "I explain the topic using a simple example such as $2 \times \frac{1}{2}$. The result of this operation is $2 \times \frac{1}{2} = 1$, thus $1 < 2$. By this example, I think that the student will understand the question better." [Ona daha basit bir örnekten $2 \times \frac{1}{2}$ örneğinden yola çıkarak anlatırdım. $2 \times \frac{1}{2} = 1$, 1<2'tür.Buradan yola çıkarak, öğrencinin sorudaki örneği daha iyi anlayacağını düşünüyorum.]

Using Alternative Approaches: Drill and Practice, Providing Counter Examples, Increasing Students' Motivation

Furthermore, in order to overcome the students' erroneous answers, prospective teachers suggested using drill and practice.

> Participant 25: "I explain student that this generalization is valid for natural numbers, not for fractional numbers. And then I solve different questions such as $6 \div \frac{1}{2}$ and $10 \div \frac{1}{2}$. I show the student that her result is correct." [Bu genellemenin doğal sayılar için geçerli olduğunu kesirli sayılarda durumların değişebileceğini ifade ederdim. Ardından $6 \div \frac{1}{2}$, $10 \div \frac{1}{2}$

gibi çeşitli sorular çözerdim. Yaptığı işlemin sonucunun doğru olduğunu gösteririm.]

Additionally, prospective teachers stated that they could ask different questions to improve the students' ability to correctly perform operations:

> Participant 119: "Firstly, I explain student what her mistake is in the question. And then I ask them different examples. These examples are not only about subtraction but also addition, multiplication, and division. By this way, the student can understand all four operations." [Öğrenciye hatasının ne olduğunu açıklayıp, farklı örnekler çözdürürdüm. Sadece çıkarma ile ilgili değil toplama, çarpma ve bölmeyle ilgili alıştırmalarda yapardım. Bu sayede dört işlemi kavramış olurdu.]

Prospective teachers stated that they could prove that the problem was not correct by solving different examples. For example;

> Participant 43: "I explain student that dividing a fraction by $\frac{1}{2}$ means multiplying the fraction with 2. I also help student to understand the division by solving different examples." [Bir kesri $\frac{1}{2}$'ye bölmenin aslında kesri 2 ile çarpmak olduğunu gösterirdim. Başka örneklerle kavramasını sağlardım.]

These prospective teachers also did not accept the approach given in the questionnaire and tried to disprove it by using counter examples.

Additionally, prospective teachers suggested using counter examples in order to overcome the students' mistakes. They claimed that they could give a counter example to make students notice their mistakes:

> Participant 18: "I try to solve several counter examples in order to disprove the student's generalization. For example, $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$. I show student that the result of a multiplication can be bigger than the multipliers." [Öğrencinin genellemesini yanlışlayacak kadar çok tersi örnek

yaptırmaya çalışırdım. Örn: $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$. Sonucun iki sayıdanda küçük olabileceğini gösterirdim.]

Prospective teachers also suggested increasing students' motivation by verbally rewarding them. Their responses were as follows:

> Participant 68: "I congratulate the student in the class in order to lead his friends to perform such different solutions of the problem." [Öğrencinin çözümünün arkadaşlarına örnek olması için, onu sınıfın önünde tebrik ederdim.]

Strategies Used Wrong Common Knowledge

Apart from the above strategies, three of the junior participants and one senior prospective teacher suggested conceptually wrong strategies. Two of them used a wrong modeling of the common area for the multiplication of fractions.

> Participant 16: "I show and shade the fractions in a whole rectangle like in the following representation. And then I say that the common areas represent the result of the multiplication." [Aşağıdaki şekildeki gibi bir bütün üzerinde kesirleri gösterir ve tararım. Sonra ortak kesiştikleri kareler sonucu gösterir derim.]

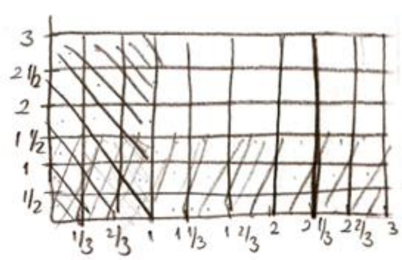


Figure 4.11. Participant's erroneous common area representation of multiplication of fractions

This prospective teacher tried to model the multiplication: $2\frac{1}{2} \times 1\frac{1}{3}$ by using the area approach and explained the result. However, since the participant could not correctly model the fractions in the square, s/he got the region that showed the product wrongly. Similarly, in the following example, another prospective teacher modeled the addition: $\frac{1}{2} + \frac{1}{3}$ using the area approach. However, s/he confused the models for addition with the common area approach used in modeling fraction multiplications, and showed an incorrect representation. The participant’s model is as follows:

Participant 38: “$\frac{5}{6}$ is represented.”

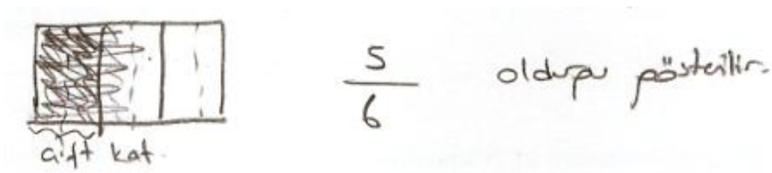


Figure 4.12. Participant’s erroneous common area representation of addition of fractions

These participants knew about the area representation of the addition and multiplication of fractions; however they could not correctly represent the area model for addition and multiplication operations. Moreover, one of the prospective teachers wrongly explained the division operation 6:$\frac{1}{3}$. S/he stated that:

> Participant 32: “Firstly, one whole divided into 6, and then each part divided into 3. Finally, we get 18 pieces. And 18 is the larger.” [Bir bütün 6’ya bölünüyor. 6’ya bölünen parçalardan her biri tekrar 3’e bölünüyor.Sonuçta yine 18 parça elde etmiş olduk, yani daha büyük oldu.]

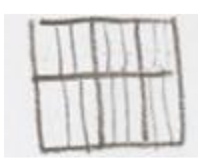


Figure 4.13. Participant’s erroneous representation of division of fractions

4.3. Summary of the Results

To investigate the prospective teachers' knowledge of mistakes held by elementary students in the fractions concept, reasons of these mistakes, and their proposed strategies to overcome those mistakes were the aim of this study. In the first part of this section, the results of prospective teachers' identification of the students' mistakes, and their reasons for these mistakes were given. As we understood from the findings of the study, prospective elementary mathematics teachers mostly could identify the students' mistakes. However, most of them could not give the underlying reasons for students' mistakes. In the first four items, prospective teachers evaluated the students' mistakes based on formal knowledge. In the first item, prospective teachers stated that because of the equal part principle, the student's answer was not acceptable. In the first item, other responses was not valid reasons for the student's erroneous answer. In the second item, the wrong partitioning of the unit interval and inadequate knowledge were stated as the reasons for the mistake. In the third item, prospective teachers declared the unequal unit fraction, unequal left over fractional parts, and inadequate formal knowledge as the reasons for the student's mistake. In the fourth item, participants stated that this mistake was due to ignorance of equivalency, ignorance of rate change, and inadequate formal knowledge. The fifth, sixth, and seventh items included the students' algorithmically based mistakes. Prospective teachers stated that inadequate formal knowledge and overgeneralization of rules of natural numbers were the reasons for students' mistakes. In the eighth and ninth items prospective teachers were asked about students' intuitively based mistakes. The results revealed that prospective teachers stated overgeneralization of rules on natural numbers and inadequate formal knowledge as the reasons for mistakes in both items. Moreover, they declared that ignorance of simplification of fractions and misinterpretation of the division operation were the reasons for the mistakes in the eighth item, and rote memorization and lack of self efficacy were the reasons for the mistakes in the ninth item.

In the second part of this section, the results of prospective teachers' proposed strategies to overcome the students' mistakes related to fractions were discussed. The results revealed that verbal explanations, using area representation, using real life model, reviewing prior knowledge, teaching standard algorithm, asking guided questions, using simple examples, using counter examples, using drill and practice, making students aware of their mistakes, and increasing students' motivation were the suggested strategies by prospective teachers in order to overcome students' mistakes.

CHAPTER V

DISCUSSION, IMPLICATIONS, AND RECOMMENDATIONS

The purpose of this study was to investigate the prospective teachers' knowledge of mistakes done by elementary students, the nature of reasons prospective teachers address for these mistakes and their proposed strategies to overcome these mistakes related to fractions. In this chapter, the findings of the present study will be discussed with references to the previous studies. Moreover, implications of the study and recommendations for further studies will be presented.

5.1. Prospective Teachers' Knowledge of Students' Mistakes and Reasons for these Mistakes

Findings of this study revealed that most of the prospective teachers could identify the students' erroneous answers. More specifically, most of the junior and senior prospective teachers were familiar with students' mistakes based on formal knowledge, algorithmically based mistakes, and intuitively based mistakes. This result was consistent with the result of Chick's (2010) study. In her study, most of the prospective teachers identified students' additive errors and explained possible reasons for these errors. The responses of the prospective teachers in the current study indicated that prospective teachers' knowledge in fractions was adequate to identify students' mistakes identified by Fraction Knowledge Questionnaire (FKQ). In other words, they realized students' mistakes in fractions. Although prospective teachers noticed the students' mistakes parallel to the previous studies (Erbaş, 2004), many could not account for students' mistakes.

Prospective teachers' knowledge of possible reasons for students' erroneous answers was also examined to understand how they reasoned the students' mistakes based on formal knowledge, algorithmically based mistakes, and intuitively based mistakes. Findings indicated that prospective teachers assessed students' mistakes based on formal knowledge in the first four items in the FKQ. In the first item, only half of them (54% of juniors and 51,6% of seniors) expressed the correct reasons for student's mistake about fraction concepts. These prospective teachers stated 'equal parts principle' as the reason for student's erroneous answer in this item. Other prospective teachers' reasons were not correct in this item. This could be because of their inadequate knowledge about fraction concepts that these prospective teachers did not consitder fractions as the equal shares of a whole. In the second, third, and fourth items, most of the prospective teachers stated the correct but the apparent reasons for students' mistakes. These prospective teachers focused on why the students' answers were erroneous and only examined the student's mistake in terms of superficial aspects. For example, in the second item, the student's mistake was due to the lack of knowledge of dividing the whole into equal pieces. However, prospective teachers only attributed this to student's wrong division of the unit interval. Only a few prospective teachers expressed the underlying reasons for mistakes in these items. These prospective teachers said that students' inadequate formal knowledge could be the reasons for these mistakes. It could be deduced from the present study that prospective teachers mostly attributed these mistakes to apparent reasons instead of underlying reasons. This could be because that they did not evaluate the influence of students' previous learning and prior knowledge on students' mathematical mistakes, only evaluated the process of the mistakes, and stated the apparent reasons of the erroneous results. Considering the Asquith, Stephens, Knuth and Alibali's (2007) study in which middle school mathematics teachers addressed students' lack of understanding as an obstacle to solve the algebra problems, the findings of the present study also showed that the lack of teaching experience might be the reason for prospective teachers' apperant reasons for students' mistakes.

Prospective teachers' reasons for students' algorithmically based mistakes were students' inadequate formal knowledge, as found in the previous studies (Bingölbali et al., 2011; Işıksal, 2006; Tirosh, 2000) and overgeneralization of rules. In this study, students' mistakes were subtracting both the numerator and denominator from each other while performing subtraction operation, adding both the numerator and denominator while performing addition operation, and multiplying only the whole parts and fractional parts in order to multiply the mixed numbers. Most of the prospective teachers stated that students' algorithmically based mistakes in addition and subtraction stemmed from students' inadequacy of knowledge of using a common denominator. This result was also consistent with the findings of Ward and Thomas's (2006) study. The finding of students' inadequacy of knowledge of using a common denominator indicated that prospective teachers mostly made students' mistakes depend on rule-based or procedural approach rather than students' understanding of the concepts involved as suggested in the literature (Erbaş, 2004; Ward & Thomas, 2006). To put differently, prospective teachers only evaluated the lack understanding of using common denominator instead of fundamental reasons of these mistakes in addition and subtraction. Therefore, it could be deduced from these results that because of prospective teachers' inclinations to rules or routines, they mostly regarded students' lack of knowledge of using common denominator as a reason for students' algorithmic mistakes.

In the current study, students' intuitively based mistakes were about the students' overgeneralization of multiplication and division rules on natural number to rational numbers. Overgeneralization of rules on natural numbers, inadequate formal knowledge, and misinterpretation of division operation, ignorance of simplification of fractions, rote memorization, and lack of self-efficacy were the stated reasons of these intuitively based mistakes. The most common reason (about 30%) for intuitively based mistakes was overgeneralization of rules on natural numbers. It could be deduced from this result that these prospective teachers were aware of the students' intuitively based mistakes, since they mostly attributed students' mistakes to overgeneralization of rules on natural numbers. This result was consistent with the results of Tirosh's (2000) study. Tirosh (2000) stated that prospective teachers noted to students' attributions of properties of operations with natural numbers to fractions after her interventions in order to improve prospective teachers' knowledge. Furthermore, she mentioned that prospective teachers who were unaware of students' tendencies attributed their mistakes to algorithmic or reading comprehension difficulties. The prospective teachers' responses under the categorizations of misinterpretation of division operation were also consistent with the results of Işıksal and Çakıroğlu's (2011) study. The researchers stated that the prospective teachers mentioned the misinterpretation of division and multiplication as in primitive models as a possible sources for student's intuitively based mistakes. In this study, even though there were some conceptual reasons for students' mistakes, more than half of the prospective teachers did not state any reason or any fundamental reason for students' intuitive mistakes. To state differently, in some cases, prospective teachers had some underlying reasons for why students would do these mistakes. On the other hand, prospective teachers mostly attributed students' mistakes to lack of understanding the multiplication and division, rote memorization, misinterpretation of the problem, or did not provide any reasons. It could be inferred from the current study's result that prospective teachers who concluded students' intuitively based mistakes as overgeneralization of rules might have adequate knowledge about students' intuitively based mistakes in fractions. Prospective teachers' knowledge about students' intuitively based mistakes also might have occurred because of the courses which prospective teachers took. The courses related to teaching mathematics might provide opportunities to improve the prospective teachers' knowledge of students' mistakes and reasons of these mistakes. Prospective teachers' limited knowledge about students' intuitions in fractions might have let them to give superficial reasons or not to give any reasons. This finding was consistent with the findings of Even and Tirosh (1995) in which teachers found the sources of students' conceptions and ways of thinking difficult to explain.

Analysis of this study revealed that prospective teachers also had some difficulties similar to the difficulties students held, since some prospective teachers provided wrong reasoning to students' erroneous answers. Some of the prospective teachers gave wrong reasoning to students' erroneous answers because of their incorrect calculation, wrong fraction comparison, confusion about the addition and multiplication, and thought about the inadequacy of data to solve the problem. In addition to these wrong reasons, some of the prospective teachers did not state any reasoning to students' erroneous answers. These findings were consistent with the studies in the literature (Ball, 1990; Graeber, Tirosh, & Glover, 1989; Tirosh, 2000; Ward & Thomas, 2006). In these studies, incorrect answers to division by fractions expressions, confusion of the multiplication and division by 2 and $\frac{1}{2}$, lack of understanding of common denominators, and difficulty in equivalent fractions were prospective teachers and teachers' difficulties in fractions. Thus, prospective teachers' inadequate knowledge of fractions might be the reason to not interpret the students' mistakes or to give a correct reasoning for them.

The present study showed that the lack of adequate curriculum knowledge might be the reasons behind prospective teachers' wrong reasoning. Although prospective teachers were informed about the grade level of the students, they provided responses without considering this knowledge and suggested incorrect reasoning. Another reason might be because that prospective teachers' inadequate experience in mathematics teaching led them to use the higher grade mathematics knowledge in teaching mathematics in lower grade mathematics classes.

5.2. Prospective Teachers' Strategies in order to Overcome Students' Mistakes

In the current study, prospective teachers suggested some strategies such as verbal explanations, using area representation, using real life examples, connecting prior knowledge, guided questions, using simple examples, using counter examples, using drill and practice, and increase student motivation. These strategies were consistent with An, Kulm and Wu's (2004) findings about strategies. Using students' life experiences, connecting the students' prior knowledge with the new topic, and pictorial representations were the common strategies in both studies. The findings in this study about prospective teachers' suggested strategies were consistent with Işıksal and Çakıroğlu's (2011) findings. The researchers stated that using multiple representations, using different methods of teaching and emphasizing practice were suggested by prospective teachers to overcome the misconceptions held by the elementary students. Furthermore, Chick (2010) investigated the prospective teachers' proposed strategies which might be useful in teaching. In her study, pictorial representation was one of the proposed strategies in dealing with the students' misconceptions. This finding was coherent with the using area representation in the current study. The variety of the suggested strategies could be because of prospective teachers' mathematics teaching method courses and elective courses that they took. Prospective teachers could choose the elective courses in their teacher education among Mathematical Language, Technology Assisted Geometry Teaching, Computer Assisted Mathematics Teaching, Daily Life in Mathematics, Development of Algebra Thinking in Elementary School, Problem Solving in Mathematics, and Geometric Thinking and Its Development courses. These courses might have promoted their knowledge about suggesting different strategies in order to overcome the students' mistakes. In other words, these elective courses and mathematics teaching method courses might have influenced prospective teachers' knowledge about teaching strategies in mathematics teaching.

Results of this study also revealed that verbal explanations were the most common strategy for the first, second, fourth, eighth and ninth items, connecting prior knowledge was the most common strategy for fifth, and sixth items, using real life examples was the most common strategy for the third item, and lastly asking guided questions was the most common strategy for the seventh item suggested by junior prospective teachers. Verbal explanations were the most common strategy for all items suggested by senior prospective teachers. There was no variation in senior prospective teachers' most common strategies in order to overcome the students' mistakes. This result was consistent with the results described in Chick's (2010) study. The researcher stated that teachers' repertoire of strategies to assist students was limited. Seniors' most common strategy for all items was verbal explanations as mentioned above. On the other hand, junior prospective teachers suggested different strategies for each item. At this point, since senior prospective teachers took more teaching method courses and elective courses than junior prospective teachers, it was expected that these courses might have positively affected the senior prospective teachers' knowledge about teaching strategies. However, few prospective teachers employed other strategies than verbal explanations. This could be because that the knowledge of junior prospective teachers might be more updated than the seniors' knowledge. The reason was that many prospective teachers took mathematics teaching methods courses and elective courses before or during the third grade of teacher education. Thus, the semester of the courses that prospective teachers took might have influenced on their knowledge of teaching strategies. Furthermore, the finding about the senior prospective teachers was supported by Watson, Beswick, and Brown's (2006) study with teachers. In their study, verbal explanations were found one of the proposed classroom strategy addressed by half of the teachers in order to deal with the fraction problems. Thus, senior prospective teachers who took the school experience course in order to observe the in-service teachers might watch over the teachers who mostly preferred verbal explanations in their classes. Thus, these observations might have affected senior prospective teachers' suggested strategy.

5.3. Recommendations and Implications

This study offers considerable information to mathematics teacher educators about prospective elementary mathematics teachers' knowledge of mistakes held by elementary students and their proposed strategies to overcome those mistakes related to fractions. Findings revealed that prospective teachers were aware of the students' mistakes related to fractions. However, prospective teachers mostly attributed these mistakes to apparent reasons instead of underlying reasons. In other words, although prospective teachers noticed students' mistakes, most of their reasons did not have a conceptual base. Therefore, teacher educators could take into consideration the improvement of the prospective teachers' knowledge of the reasons of the students' erroneous answers. More specifically, underlying resaons of students' mistakes should be discussed deeply in order to provide a conceptual understanding of students' mistakes. Furthermore, teacher educators might provide prospective teachers opportunities to discuss the meaning of concepts, relationships, common conceptions, and difficulties of elementary students in order to improve the prospective teachers' understanding of student thinking. Furthermore, as Işıksal (2006) suggested, prospective teachers should be provided chance to analyze the cases including students' conceptions, mistakes, and other thought processes in order to improve their knowledge of students' mistakes. Additionally, as Seviş (2008) suggested, the mathematics teaching method courses might be divided into several courses in terms of the subject areas in mathematics. In these courses, prospective teachers might have more opportunity to study each subject area; therefore they might improve their knowledge of the subjects in mathematics. Also, these courses should be designed to make prospective teachers aware of the students' mistakes in different subject areas of mathematics.

In addition to the reasons of students' mistakes, prospective teachers' strategies in order to overcome the students' mistakes were also an important issue. However, prospective teachers' knowledge of strategies could be accepted inadequate, because the variety of the strategies was limited. Most of the prospective teachers preferred verbal explanations in order to overcome students' mistakes. However, Zembat (2010) stated that since students' conceptions were strong and had an effect on students' views, direct teaching methods such as verbal explanations were not effective on students' conceptions, and they could cause other misconceptions. Based on these findings, teacher educators should plan and implement the tasks about teaching strategies using to overcome students' mistakes in their courses. Moreover, as Even (1999) suggested that prospective teachers should be familiar with research articles about students' intuitive, naive, and implicit ideas in learning mathematics, because participants' familiarity with such research articles would improve their formal and explicit knowledge.

This study suggested the importance of various research studies investigating the prospective teachers' reasoning for students' mistakes and their strategies to overcome these mistakes. Teacher education programs could make prospective teachers be familiar with students' common types of mistakes and sources of these mistakes. Since prospective teachers will soon be teaching in elementary classrooms, knowledge of students' mistaken thinking processes would help teachers to prepare their lessons and teach mathematics effectively. Moreover, teachers who were improved in knowledge about students' thinking would be more attentive to students' needs, and to promote students' understanding. Thus, prospective teachers' awareness and knowledge about students' mistakes should be developed.

This study explored the prospective teachers' knowledge about students' mistakes and their strategies in order to overcome these. However, there were still some unanswered questions. Further research was needed to find out how an intervention designed to enhance prospective teachers' knowledge of students' mistakes and their suggestions of strategies would affect their knowledge. Another research area could be related to investigating in-service mathematics teachers' knowledge of students' mistakes. Furthermore, it could be investigated whether teachers' teaching practices had an effect on the prospective teachers' knowledge of students' mistakes and their suggestions of strategies in order to overcome these mistakes and how teachers use students' mistakes in their teaching processes.

In addition to the above further research study areas, when and how prospective teachers' and in-service teachers' knowledge of students' mathematical learning and thinking develop, during pre-service education or during in-service practice (Even & Tirosh, 2008). Moreover, how prospective teachers' knowledge of reasons of students' mistakes affect their strategies to overcome these mistakes could also be investigated. Lastly, further studies might be done to explore prospective teachers' knowledge of students' mistakes in different subject areas of elementary mathematics.

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APPENDICES

APPENDIX A

Turkish Version of Fraction Knowledge Questionnaire

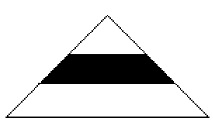
Değerli Katılımcı,

Bu anket ilköğretim matematik öğretmen adaylarının "kesirler konusundaki bilgilerini ölçmek amacıyla" Orta Doğu Teknik Üniversitesi Sosyal Bilimler Enstitüsü'nde yapmakta olduğum yüksek lisans tez çalışması kapsamında hazırlanmıştır. Elde edilecek veriler sadece bilimsel amaçlı olarak kullanılacak olup hiçbir kişi ya da kuruma açık tutulmayacaktır. Ankette yer alan hiçbir soruyu boş bırakmamanız soruların geçerli bir şekilde değerlendirilmesi açısından son derece önemlidir. Anket için ön görülen süre 45 dakikadır. Katılımınız için teşekkür ederim.

Arş. Gör. Deniz Eroğlu
İletişim: deroglu@mehmetakif.edu.tr

Cinsiyet: …………………… Bölümü:…………………………… Sınıfı:…........

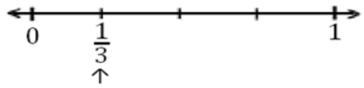
1. Öğretmen, öğrencilerden bir üçgenin $\frac{1}{3}$'ini taramalarını istemiştir. Ayşe üçgeni aşağıdaki gibi taramıştır. (Ayşe 5. Sınıf öğrencisidir).



a) Ayşe'nin cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Ayşe'nin öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

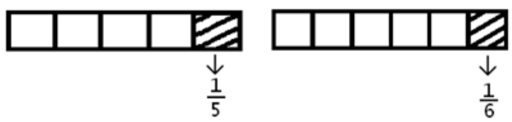
2. Öğretmenin "$\frac{1}{3}$ kesir sayısını sayı doğrusu üzerinde gösteriniz." sorusuna Gizem aşağıdaki şekli çizerek cevap vermiştir.



a) Gizem'in cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Gizem'in öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

3. Öğretmen Mert'e $\frac{4}{5}$ ve $\frac{5}{6}$ kesirlerinden hangisinin daha büyük olduğunu sorduğunda Mert aşağıdaki şekilleri göstererek;
ikisinden de birer parça çıkarıldığı için iki kesrin büyüklüğünün aynı olduğunu söylemiştir.



(Çubukların uzunlukları eşittir.)

a) Mert'in cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Mert'in öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

4. Öğretmenin "Aşağıdaki şekillerde taralı olan kareler portakal nektarını, beyaz kareler ise suyu temsil etmektedir. Hangi karışımın daha yoğun olduğunu açıklayınız."

sorusuna Arzu aşağıdaki cevabı vermiştir:

– Birinci karışıma 3 bardak portakal nektarı eklenirken, ikinci karışıma 6 bardak portakal nektarı eklenmiştir, dolayısıyla ikinci karışım daha yoğundur.

a) Arzu'nun cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

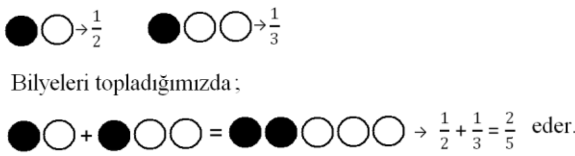
b) Siz Arzu'nun öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

5. Öğretmenin $\frac{7}{8} - \frac{1}{4}$ sorusuna, Zehra $\frac{7-1}{8-4} = \frac{6}{4}$ olarak, Elif ise aynı soruya $\frac{7-2}{8} = \frac{5}{8}$ olarak cevap vermiştir. İkisi de kendi yaptığı işlemin doğru olduğunu söylemiştir.

a) Zehra ve Elif'in cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Zehra ve Elif'in öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

6. Öğretmenin $\frac{1}{2}+\frac{1}{3}$ sorusunu Berk aşağıdaki gibi çözmüştür:



a) Berk'in cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Berk'in öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

7. Zehra, öğretmenine $2\frac{1}{2} \times 1\frac{1}{3}$ işleminin sonucunu birinci kesrin tam kısmıyla ikinci kesrin tam kısmını, pay ile payı, payda ile paydayı çarparak bulduğunu söylemiştir. Bunu matematiksel olarak;

$2\frac{1}{2} \times 1\frac{1}{3}$;   2×1=2 ve $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ şeklinde ifade etmiş ve sonucu $2\frac{1}{6}$ bulmuştur.

a) Zehra'nın cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Zehra'nın öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

8. Öğretmen $4 \times \frac{1}{16}$ işleminin sonucunu $\frac{1}{4}$ bulmuştur. Burçin ise öğretmeninin, sonucu yanlış bulduğunu iddia etmiş, sebebini de "Çarpma işleminin sonucu her zaman çarpanlardan daha büyüktür." diyerek açıklamıştır.

a) Burçin'in cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Burçin'in öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

9. Tülay, öğretmenine “Bölme işleminin sonucu her zaman bölünen sayıdan küçüktür ancak ben $6 \div \frac{1}{3}$ işleminin sonucunu 18 buluyorum, sanırım sonucu yanlış buluyorum” demiştir.

a) Tülay’ın cevabı kabul edilebilirliğini/edilemezliğini nedenleriyle açıklayınız.

b) Siz Tülay’ın öğretmeni olsaydınız, ona nasıl bir açıklama yapardınız?

APPENDIX B

An Example of Coding Obtained from Two Coders

		Non-acceptable			Acceptable		Conditionally Acceptable
Item No	**Std. No**	Equal parts principle	Wr. Rea.	W.out Rea.	Sim. Princ.	W.out Rea.	
1	**17**	Bir bütün olan üçgenin 3 eş parçaya ayrılması gerektiği için öğrencinin gösterimi kabul edilemez.					
	39		Ayşe'nin taradığı kısım üçgenin $\frac{1}{2}$'lik kısmıdır. Cevabı kabul edilemez.				

		Non-acceptable			Acceptable		Conditionally Acceptable
Item No	**Std. No**	Equal parts principle	Wr. Rea.	W.out Rea.	Sim. Princ.	W.out Rea.	
1	**56**						Alan hesaplarını yapıp taramışsa kabul edilebilir. Ayrıca kenarları 3’er eşit parçaya bölüp aradaki parçayı taramışsa oda olabilir. Ancak rastgele bir işlem yapmışsa kabul edilemez.
	96				Doğrudur. Benzerlikten dolayı; $\frac{2}{3} = \frac{4□}{9□}$(Alanların karesine eşit). Üstteki küçük parça da S olduğundan dolayı taralı aln 3S’dir. Tamamı da 9S olduğuna göre $\frac{3□}{9□} = \frac{1}{3}$ olur.		

APPENDIX C

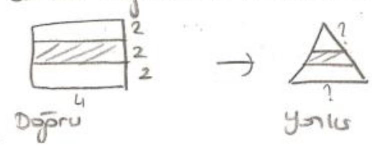
Examples of Prospective Teachers' Responses

Participant 56:

Bütün kavramını tekrar eder, yaptığı (yanlış) sayı doğrusu gösteriminin üzerinde sayı doğrusunda 0 - 1 arasında kaç parça olduğuna dikkat etmesi gerektiğini söyler ve 1/3 kesrinin 0-1 aralığını 3 eşit parçaya böldükten sonraki ilk bölüm noktasını ifade ettiğini açıklardım.

Participant 81:

Oranları belli bir, örneğin, dikdörtgen çizip; bunu 3 parçaya ayırmaları isterim önce. Onu kavrattıktan sonra üçgendeki mantıksızlık ortaya çıkmış olur



Participant 17:

Bir bütün olan üçgenin 3 eş parçaya ayrılması gerektiği için öğrencinin gösterimi kabul edilemez.

Participant 20:

Mert'e $\frac{5}{5} = \frac{6}{6} = 1$ 'dir Ancak bir bütünün $\frac{1}{6}$ sı ile $\frac{1}{5}$ 'inin eşit olmadığını söylerdim. Bu nedenle bütünden farklı parçaları çıkarırsak sonucun da eşit olmayacağını söylerdim.

APPENDIX D

TEZ FOTOKOPİSİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü ☐

Sosyal Bilimler Enstitüsü ☒

Uygulamalı Matematik Enstitüsü ☐

Enformatik Enstitüsü ☐

Deniz Bilimleri Enstitüsü ☐

YAZARIN

Soyadı : Eroğlu

Adı : Deniz

Bölümü : İlköğretim Fen ve Matematik Eğitimi

TEZİN ADI: Examining prospective elementary mathematics teachers' knowledge about students' mistakes related to fractions

TEZİN TÜRÜ : Yüksek Lisans ☒ Doktora ☐

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. ☒
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz. ☐

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: 01.02.2012