AN INVESTIGATION OF PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS' ARGUMENTATION, PROOF, AND GEOMETRIC CONSTRUCTION PROCESSES IN THE CONTEXT OF COGNITIVE UNITY

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ABSTRACT

AN INVESTIGATION OF PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS' ARGUMENTATION, PROOF, AND GEOMETRIC CONSTRUCTION PROCESSES IN THE CONTEXT OF COGNITIVE UNITY

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The first purpose of this study is to investigate how prospective middle school mathematics teachers' argumentation process while producing conjectures in the cognitive unity based activities relates to the proving process of conjectures they produced. The second purpose is to examine the global argumentation structures emerged while producing conjectures, the components of argumentation, and the functions of the rebuttal component. The third purpose is to investigate the approaches offered to perform geometric constructions asked in the cognitive unity based activities and to what extent they could perform geometric constructions correctly while using compass-straightedge and GeoGebra. The last purpose is to scrutinize the conjectures produced during argumentation and whether they could present valid proofs for the recently produced conjectures. To that end, the data were collected from junior prospective middle school mathematics teachers in the 2016-2017 academic year. The data sources are video recordings and audio recordings of the cognitive unity based activities, documents, GeoGebra files, field notes, and focus group interviews. The findings presented that conjecture production process relates to proving in both positive and negative aspects. Regarding argumentation, the mono structures and the hybrid structures emerged, new components of argumentation were offered, and the eight functions of the rebuttal component were reported. Both compass-straightedge group and GeoGebra group presented at least one valid approach for geometric constructions embedded in the activities. Lastly, it was seen that groups could not conduct valid proof for all statements asked in the activities.

Keywords: Cognitive Unity, Argumentation, Proof, Geometric Construction, Prospective Middle School Mathematics Teachers

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ BİLİŞSEL BÜTÜNLÜK BAĞLAMINDA ARGÜMANTASYON, İSPAT VE GEOMETRİK İNŞA SÜREÇLERİNİN İNCELENMESİ

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Bu çalışmanın ilk amacı ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerde varsayım oluşturma aşamasındaki argümantasyon süreçlerinin, oluşturdukları varsayımları ispatlama süreçleriyle nasıl bir ilişkisi olduğunu incelemektir. İkinci amaç ise öğretmen adaylarının varsayım oluşturma sürecindeki global argümantasyon yapılarını, argümantasyon bileşenlerini ve çürüten bileşeninin işlevlerini araştırmaktır. Üçüncü amaç ise öğretmen adaylarının bilişsel bütünlük temelli etkinliklerdeki geometrik inşalar için önerdikleri yaklaşımları ve pergel-çizgeç ve GeoGebra kullanırken ne derece doğru geometrik inşa yapabildiklerini incelemektir. Çalışmanın son amacı öğretmen adaylarının argümantasyon sırasında oluşturdukları varsayımları belirlemek ve bu varsayımlara geçerli ispat yapıp yapamadıklarını araştırmaktır. Bu amaçlar bağlamında, 2016-2017 akademik yılında 3. sınıf ortaokul matematik öğretmen adaylarından veri toplanmıştır. Veri kaynakları bilişsel bütünlük temelli etkinliklerin video ve ses kayıtları, dokümanlar, GeoGebra dosyaları, alan notları ve odak grup görüşmeleridir. Bulgular varsayım oluşturma sürecinin hem olumlu hem de olumsuz açıdan ispat ile ilgili olduğunu göstermektedir. Argümantasyonla ilgili olarak, mono ve hibrit yapılar ortaya çıkmıştır, yeni argümantasyon bileşenleri önerilmiştir ve çürüten bileşeninin sekiz işlevi belirlenmiştir. Hem pergel-çizgeç hem de GeoGebra gruplarının etkinliklerdeki geometrik inşalar için en az bir geçerli yaklaşım ortaya koyduğu gözlemlenmiştir. Son olarak, grupların etkinliklerde ispatı istenen tüm önermeler için geçerli ispat sunamadıkları görülmüştür.

Anahtar Kelimeler: Bilişsel Bütünlük, Argümantasyon, İspat, Geometrik İnşa, Ortaokul Matematik Öğretmen Adayları

To my family

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

NCTM	National Council of Teachers of Mathematics
ERME	European Society for Research in Mathematics Education
MoNE	Ministry of National Education
CSG	Compass-straightedge Group
GG	GeoGebra Group
D	Data
W	Warrant
C	Conclusion or claim
B	Backing
R	Rebuttal
Q	Qualifier
C/D	Conclusion/Data
TC	Target conclusion
CH	Challenger
G	Guidance
O	Objection
AS	Argumentation Stream
DGS	Dynamic Geometry Software
CoHE	Council of Higher Education
ECTS	European Credit Transfer System
GPA	General point average
GC	Geometric construction
CTA	Construction type A
CTB	Construction type B
CTC	Construction type C
METU	Middle East Technical University
D→C	Connection between data and conclusion
A	Approach

CHAPTER 1

INTRODUCTION

The studies conducted related to mathematical proof have gained a new impetus in the recent decades (Arzarello, Micheletti, Olivero, & Robutti, 1998; Arzarello & Sabena, 2011; Stylianides, 2019; Stylianides, Bieda, & Morselli, 2016; Stylianides & Stylianides, 2017) and this resulted in a variety of research strands in mathematics education (Boero, Douek, Morselli, & Pedemonte, 2010; Mariotti, 2006). Compared to other disciplines, as the distinctive property of mathematics, proof is the needed mechanism for the verification and establishment of results (Conner, Singletary, Smith, Wagner, & Francisco, 2014b). According to the National Council of Teachers of Mathematics (NCTM, 2000), mathematical proof is “a formal way of expressing particular kinds of reasoning and justification” (p.56). The significance of proof in both mathematics and mathematics education is an issue that many studies have come to agree (Arzarello & Sabena, 2011; Edwards, 1997; Ellis, Bieda, & Knuth, 2012; Hanna, 2018; Komatsu, 2016; Mariotti, 2006; Mariotti, Durand-Guerrier, & Stylianides, 2018; Mejia-Ramos & Inglis, 2009; NCTM, 2000; Stylianides, 2019; Stylianides & Stylianides, 2017; Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2009). According to Ellis et al. (2012), proof is “part and parcel of doing mathematics and should be regular and ongoing part of learning of mathematics” (p.8). The proving process elucidates and influences a broader mathematical context than it was intended (Hanna, 2018). By dealing with the proving tasks, students may be provided the opportunity to develop deep learning, have a good grasp of mathematics, and appreciate the efficiency of proof (Stylianides, 2019; Tsamir et al., 2009).

According to Hanna (2014), “argumentation, reasoning, and proof are concepts with ill-defined boundaries” (p.404) because of the various usages of these concepts in mathematics. The review of the related literature also presented that proof, reasoning, and argumentation are the themes which are associated with each other in a way where their combinations are generally stated and focused (e.g., Boero, Fenaroli, & Guala, 2018; Boero, Garuti, & Mariotti, 1996; Conner, Singletary, Smith, Wagner, & Francisco, 2014a; Mariotti et al., 2018; NCTM, 2000; Pedemonte, 2007a, 2018a; Reid & Knipping, 2010; Reiss, Heinze, Renkl, & Groß, 2008; Stylianides, 2008; Stylianides, Stylianides, & Shilling-Traina, 2013; Toulmin, Rieke, & Janik, 1984; Tsamir et al., 2009). Accompanying proof, reasoning and argumentation constituted the concepts taken into consideration in the mathematics education research in the last decades from various aspects (Reiss et al., 2008). It can be stated that the direct and indirect relations and the overlapping constituents of the mentioned concepts in the mathematical domain are well documented in the studies. Before elaborating on the relations of these three concepts, reasoning and argumentation in mathematics education were explained briefly at this point.

In a general sense, reasoning was explained as “the central activity of presenting the reasons in support of a claim, so as to show how those reasons succeed in giving strength to the claim” (Toulmin et al., 1984, p.14). From the standpoint of mathematics, reasoning was defined as “the ability to think coherently and logically and draw inferences or conclusions from mathematical facts known or assumed” (Mansi, 2003, p.9). Reasoning refers to how the concepts are thought so that a variety of types of reasoning in mathematics such as geometric, deductive, inductive, and probabilistic can be seen (Gfeller, 2004). The importance of mathematical reasoning was underlined by Lannin, Ellis, and Elliot (2011) in a way that it is “the essence of mathematical activity and that without mathematical reasoning there is no mathematics” (p.7). Developing reasoning in mathematics, being able to present deductive arguments, and putting forth conjectures are critical activities for students since they underpin their progress in gaining the new insights and affect their following experiences positively (NCTM, 2000).

The critical stance of argumentation in the development of mathematical thinking and also in learning and teaching mathematical concepts was underlined in many studies (Fiallo & Gutiérrez, 2017; Knipping, 2008; Krummheuer, 1995; Mariotti & Goizueta, 2018; Metaxas, Potari, & Zachariades, 2016; Reid & Knipping, 2010). There is a growing research base that focuses on the argumentation process of students as well as mathematics teachers and mathematicians while working on mathematical concepts (Inglis, Mejia-Ramos, & Simpson, 2007). Argumentation was described as "the whole activity of making claims, challenging them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms, and so on" (Toulmin et al., 1984, p.14). Conner et al. (2014a) emphasized the importance of comprehending, recognizing, and conducting arguments in mathematics. Moreover, in the construction of knowledge, argumentation is attained as having a pivotal role (Cramer, 2011; Metaxas et al., 2016; Schwarz, Neumann, Gil, & Ilya, 2003).

1.1. Relations among the Concepts of Argumentation, Reasoning, and Proof

When stayed out of the direct relevance of proof, the first binary combination of the mentioned terms, which is the association of reasoning and argumentation constituted the base of some studies. Reasoning is a fundamental constituent of argumentation (Conner et al., 2014a) and also of learning and doing mathematics (Conner et al., 2014b). Toulmin et al. (1984) presented the association between the constructs of argument and reasoning by declaring arguments as "trains of reasoning" (p. 12). A classroom environment in which students feel free to share their ideas and reasoning with others is an effective factor for making students to modify or consolidate their justifications, learn to consider the opinions of both their own and others' from a criticizing perspective, and contribute to the reasoning process of others (NCTM, 2000). When this case is aligned with the definition of argumentation, it is seen that there are lots of overlapping issues with reasoning. Reasoning in mathematics does not just cover formal and logical actions in which the conclusions are produced based on theorems and definitions preceded by the given premises, it also covers the argumentation process where the validity of conclusions or statements are explained by simply giving reasons irrespective of whether these reasons hold a mathematical perspective (Pedemonte & Balacheff, 2016). Reasoning and argumentation have pivotal importance in mathematics; hence, the objectives of the mathematics course involve promoting students' ability in terms of reasoning and argumentation in a coherent manner (Reiss et al., 2008).

Compared to the studies which refer to the relevance of reasoning and argumentation, it was more common to observe studies which contextualize reasoning and proof as a pair across various grades. As Hanna (2014) declared, a vast majority of studies in the related literature mentions them as one entity which is signified as "reasoning and proof". Using the tasks comprised of reasoning and proof is a fundamental element of doing mathematics and takes place at the core of the development of mathematical sense starting from the elementary grades (Stylianides, 2008; Stylianides et al., 2013). Moreover, NCTM (2000) also framed reasoning and proof among the five process standards of mathematics as well as problem solving, communication, representation, and connection. It was also stated that reasoning and proof pave the way for getting a solid grasp of the mathematical concepts and phenomena (NCTM, 2000). Due to the demands of the reforms in mathematics education, reasoning and proof were involved in various levels of school mathematics in many countries around the world (Hanna, 2014; NCTM, 2000; Tsamir et al., 2009). For example, reasoning and proof were pointed out as the entailments in the guidelines of the mathematics curricula (Boero et al., 2018) or added as a section in mathematics curricula by focusing on the significance of exploration, producing conjectures, and argumentation as well (Yevdokimov, 2006).

Another term occasionally associated with proof in the research is argumentation which is one of the themes at the heart of the mathematics education research as well as mathematics research (Conner, 2017; Mariotti et al., 2018). As a matter of fact, a working group named as "argumentation and proof" has been organized by the European Society for Research in Mathematics Education (ERME) since the third conference conducted in 2003 (Reid & Knipping, 2010) which provides a substantial contribution to the growing body of related literature. Argumentation was described briefly as "precursor to proof" (Conner et al., 2014a, p.403). That is to say, the proving process is regarded as closely associated with the argumentation process (Boero et al., 1996; Pedemonte, 2007a, 2018a). Due to the significance of proof and argumentation in different grades of mathematics education, researchers keep working on the various questions pertaining to these issues (Cirillo et al., 2015; Mejia-Ramos & Inglis, 2009; Yevdokimov, 2006). Accordingly, the present study was mainly configured on argumentation and proof. In this respect, how argumentation and proof were inspected in the literature is reviewed thoroughly at this point.

While depicting argumentation and mathematical proof, Boero (1999) used the expression "a complex, productive, unavoidable relationship" within the title of his article. As expected, different conceptions pertaining to argumentation and proof have caused the multiplicity of approaches followed in the research studies. The approaches regarding argumentation and proof are subsumed under three categories. Firstly, there are studies which did not point out a clear cut difference between argumentation and proof (Pedemonte, 2007a). For example, the phrase "empirical proof" mentioned in some studies (e.g., Fiallo & Gutiérrez, 2017; Harel & Sowder, 1998, 2009; Maher, 2009) was presented as an example for the case that the word proof was also employed for not only deductive but also empirical cases by Pedemonte (2007a). Besides, the presence of studies investigating not only the argumentation process of the generation of conjecture but also the argumentation process of its proof put forward the issue that proving can be considered as an argumentation process in its own right (Pedemonte, 2007a).

In the second approach utilized in the studies regarding argumentation and proof, on the other hand, the presence of a distance between these notions in mathematics was cited by some researchers (e.g, Antonini & Mariotti, 2008; Boero et al., 1996; Douek, 1998, 2010; Fiallo & Gutiérrez, 2017; Garuti, Boero, Lemut, & Mariotti, 1996; Mariotti, Bartolini-Bussi, Boero, Ferri, & Garuti, 1997; Pedemonte, 2002b, 2007a, 2007b; Reid & Knipping, 2010) by means of the translation of the studies conducted by Duval (1991, 1992-1993, 1995a). The distance between argumentation and proof was studied by Duval thoroughly without ignoring the similarities between them such as linguistics and propositions (Pedemonte, 2002b). In other words, Duval (1995a, as cited in Pedemonte, 2007a) asserted the existence of a structural gap between argumentation and proof although a similar syntactic is used in both phases. The referred distance between argumentation and proof was also labeled as the cognitive rupture in some studies (Antonini & Mariotti, 2008; Fiallo & Gutiérrez, 2017). Similarly, the mentioned issue was expressed via the distinction or the difference between argumentative reasoning required to product the conjecture and deductive reasoning needed to validate it (Boero et al., 1996; Mariotti et al., 1997; Pedemonte, 2007a). More precisely, Duval (1991, as cited in Douek, 1998) described the mentioned difference as follows;

> Deductive thinking does not work like argumentation. However these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proof (p.125).

In more detail, it was stated that argumentation is composed of any rhetoric case deployed to convince and show whether the statement is true or false. On the other hand, proof was described in a more formal manner by referring to the theoretical validation of a statement followed by logical concatenations (Duval, 1992-1993, 1995a, as cited in Antonini & Mariotti, 2008). Besides, based on the translations, the mentioned cognitive rupture was pointed out as a possible reason regarding students' difficulties in comprehending and conducting the deductive proofs (Duval, 1991, as cited in Fiallo & Gutiérrez, 2017). To sum up, from this point of view, argumentation and proof processes are subject to a distance, particularly in the mathematical domain.

The last approach related to argumentation and proof takes into account and even supports the perspective in the secondly stated approach but also touches upon the subject from an educational perspective (Mariotti, 2006; Pedemonte, 2007a). In more detail, this approach emphasized the existence of continuity between argumentation, which refers to the process of conjecture production distinctively, and proving process of the recently produced conjecture. In other words, the issue focused is the relations of conjecturing phase to proving (Pedemonte, 2007a). In this respect, Garuti et al. (1996) introduced the concept of cognitive unity of theorems. Of the mentioned approaches regarding argumentation and proof, this study was framed within the theoretical construct of cognitive unity.

Up to this point, the concepts of argumentation, reasoning, and proof were explained by focusing on each one separately and the relations among them. Based on the review, it can be stated that the research about the possible implications of the relations between argumentation and proof for mathematics education and which approaches related to these concepts would be more helpful in terms of teaching proof should be considered. For example, the difference in the usages of these terms from the perspective of Duval and the perspective of the Italian group of researchers and also the related didactical implications might be focused since there is the need for more research in this area to present a clear picture (Reid & Knipping, 2010).

1.2. Cognitive Unity

By taking cognitive unity into consideration, Balacheff (1999) offered an analogy pertaining to the relation between argumentation and proof as “argumentation is to a conjecture what mathematical proof is to a theorem” (p.5). In other words, Pedemonte (2007b) expressed the base of the cognitive unity as “what is in play is the relationship between conjecturing and looking for a proof” (p.644). When cognitive unity is established, it refers to the fact that the section involving the argumentation by virtue of conjecturing reinforces the proving section (Pedemonte, 2007a). Cognitive unity emphasizes the possible use of the elements of the construction process of conjectures during the proving process (Garuti et al., 1996). In more detail, Garuti et al. (1996) anticipated that students could be successful in producing theorems if an environment with the properties given below was arranged.

> - during the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices;
> - during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statement according to a logical chain (p.113).

In that study, Garuti et al. (1996) used a task having the following stages; setting the problem, producing conjectures, discussing conjectures, arranging statements, preparing proof, proving that the condition is both sufficient and necessary, and final discussion with an individual report about the whole activity. Based on the findings of their study with the application of such a task, they concluded that their hypothesis regarding cognitive continuity constitutes a potential case. In addition, they depicted the fundamental points noticed in the analysis. The argumentation process conducted by students while reaching the statements would be beneficial for them since they had the opportunities to examine various alternative cases, follow a progressive nature while detecting the statements, and verify the veracity of the conjectures they came up with. It was also seen that offering correct conjectures is critical since students who stated incorrect conjectures needed to go back and reconstruct the correct ones to be able to present an acceptable proof. Another issue noticed in their study was that a poor argumentation process while producing the statements ended up with the deficient arguments during the proving process. This consequence was interpreted as the solid evidence regarding the connection between these processes.

From a broader perspective, proof construction process involves the collocation of a series of tasks which are “experience, insight, reasons, constructions, and arguments, gained through an exploration of a specific situation” (Sinclair, Pimm, Skelin, & Zbiek, 2012a, p.48). Nevertheless, it is not usually the case for students since they are not the attendants of the whole process of proving mentioned above. Thus, students are directly entering the proving process without furnishing their curiosity and inquiry related to the concept at stake (Sinclair et al., 2012a). Besides, since students are generally asked to understand and then replicate the proofs of the statements, which are not produced by students, in a way that is presented by the instructors, the concept of cognitive unity provides an alternative way for teaching and learning proof related concepts. When necessary conditions are set and the critical points are taken into consideration attentively, the tasks through which students deal with the proof of a statement they produced by means of an argumentation process constitute a significant potential to improve their ability to prove (Garuti et al., 1996). The key entailments of such a task were exemplified as follows; an open problem should be worked on by supplying a didactic environment in the classroom, the conjecture or statement are reached throughout an argumentation process in which the problem is aimed to solve, and the task is better to involve the group and classroom discussions which prepare to the proving stage of the task (Garuti et al., 1996). The aim of introducing cognitive unity is also to offer a theoretical construct for the cognitive gap between empirical and deductive reasoning (Leung & Lopez-Real, 2002). According to Antonini and Mariotti (2008), as a theoretical construct, cognitive unity covers the possibility of relationship and rapture between argumentation and proof. Besides, paying attention to cognitive unity in a study might open up the opportunity to probe the relationship between argumentation and proof with the help of analogies as well as considering their differences and to consider cognitive and epistemological factors and the individual and cultural aspects regarding mathematics.

As it is seen, in the scope of cognitive unity, argumentation deflected the focus on the production of conjectures which brings about another strand of research for mathematics education, namely, the relationship between the conjecturing phase and the proving phase. The process of conjecture production can be called as an argumentation activity (Pedemonte, 2007b). For example, Pedemonte (2007a) investigated the possible continuity and distance between argumentation and proof in a structural manner by using Toulmin's model. Thus, by incorporating cognitive unity with Toulmin's model, how the conjecture production process supports or hinders the following proof process can be examined (Komatsu, 2016; Pedemonte, 2002b). It was observed that the use of the argumentation model of Toulmin for the examination of the argumentation process is a recurring theme among the studies. Although how this model is used and adapted to the studies depends on some issues such as the discipline, the participants, and the idea focused distinctively during the analysis, Toulmin's model is an applicative tool in determination and comparison of different kinds of arguments (Mariotti et al., 2018). Therefore, it was deduced that Toulmin's model of argumentation can be considered as a proper tool while investigating the argumentation process in the conjecture production phase of the present study.

In the next section, how the argumentation, reasoning, and proof take place in mathematics curricula in Turkey will be examined.

1.3. Argumentation, Reasoning, and Proof in Mathematics Curricula in Turkey

As Schoenfeld (1994) stated, proof is "not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels" (p.76). Similarly, NCTM (2000) emphasized the importance reasoning of proof in mathematics education as follows; "reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (p.56). This call was taken up by a number of studies which underscored the importance of integration of reasoning and proof in mathematics education as early as elementary grades (Stylianides, 2007a, 2007b; Stylianides et al., 2013; Stylianides & Stylianides, 2009; Yackel & Hanna, 2003). Moreover, argumentation was underlined as "essential to teaching and learning of proof" (Reid & Knipping, 2010, p. 153). As a consequence of the significance of the mentioned terms, which are argumentation, reasoning, and proof, in mathematics education, the mathematics curricula of different levels in many countries directly or indirectly increased the emphasis on these issues (Boero et al., 2018; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004; Cramer, 2011; Ellis et al., 2012; Hanna, 2018; Mariotti, 2006; NCTM, 2000; Stylianides, 2019; Tsamir et al., 2009; Yevdokimov, 2006). This movement also echoed in the status of these concepts in mathematics curricula in Turkey. There have been some revisions and renewals in both the general perspectives and the mathematical content such as learning areas and objectives in the mathematics curricula of all levels by the Ministry of National Education (MoNE) in Turkey. The last three revised mathematics curricula of middle school level and secondary school level in Turkey were examined in terms of the mentioned concepts at this point.

To begin with, the middle school mathematics curricula presented in 2009, 2013, and 2018 by MoNE were taken into consideration. The common statements regarding reasoning in the general objectives of mathematics education given in Mathematics Curricula for grades 6-8 (MoNE, 2009), for grades 5-8 (MoNE, 2013), and for grades 5-8 (MoNE, 2018) are that students will be able to express their mathematical thinking and reasoning in a problem solving process and to use mathematical terminology and language properly while expressing their ideas. Additionally, Mathematics Curriculum for grades 5-8 (MoNE, 2018) attached the case that students will be able to detect the deficiencies and gaps in mathematical reasoning of others. Moreover, in all abovementioned mathematics curricula, reasoning, as a process standard, was not devoted to any particular learning area or objective and it was recommended to incorporate reasoning as well as other standards throughout all possible mathematical concepts. It was observed that the terms at stake other than reasoning which are argument, argumentation, and proof were not explicitly mentioned in Mathematics Curricula for grades 5-8 (MoNE, 2013) and for grades 5-8 (MoNE, 2018). However, in Mathematics Curriculum for grades 6-8 (MoNE, 2009), the term proof is only mentioned as an example for the project assignment which covers the proof of Pythagorean theorem without expanding into details.

It was also seen that Mathematics Curricula for grades 6-8 (MoNE, 2009) and for grades 5-8 (MoNE, 2013) have more points in common compared to recent one with respect to the focused notions. Both curricula listed reasoning among the mathematical process standards and underlined that mathematics education aims to develop students' competencies in reasoning as well as problem solving, communication, and connections. However, in the current curriculum, it is not an issue pointed out in such a direct manner. Besides, in these two curricula, it was stated that students' reasoning in terms of mathematics can be promoted by providing them environments in which different mathematical cases can be examined and the awareness of the importance of reasoning can be gained by students. Another issue noticed is related to the roles cast both to students and to teachers throughout the curricula. That is to say, according to Mathematics Curriculum for grades 6-8 (MoNE, 2009), among the roles of students, active participation, questioning, inquiring, thinking, discussing, explaining the ideas, and collaboration are involved whilst the roles of teachers include guidance, supporting motivation, leading students to question, inquire, think, and discuss, and preparing activities appropriate to activate the mentioned skills via classroom discussions. In a similar way, in Mathematics Curriculum for grades 5-8 (MoNE, 2013), mathematics teachers are given the role of arranging classrooms where students feel free to explain their ideas written or verbally, discuss, and improve their communication skills. The aforementioned roles of both students and teachers in these mathematics curricula can be seen among the elements of a collective argumentation process although the term argumentation was not written apparently.

As stated, the last three secondary school mathematics curricula in Turkey, which were prevailed in 2011, 2013, and 2018 by MoNE, were also examined with respect to the focused concepts. Initially, some points kept nearly the same throughout the revisions were explained. The first notion expressed directly in all secondary school mathematics curricula is proof. In more detail, proof is introduced in the scope of learning area logic in all curricula but the year levels were changed with revisions. That is, logic was the subject of the 9^{th} grade (MoNE, 2011), then it was taken to the 11^{th} grade (MoNE, 2013), and finally it was shifted back to the 9^{th} grade in the current curriculum (MoNE, 2018). The proof-related objectives cover the explanation of the terms proposition, definition, axiom, theorem, proof, and quantifiers, the applications of basic rules of logic, and also conducting proofs by using proof methods. Except for Mathematics Curriculum for 9-12 grades (MoNE, 2018), there are occasions related to proof other than logic chapter such as proofs of some statements in various learning areas. For example, the proof of $\log_a(b.c)=\log_a b+\log_a c$ was mentioned in the logarithm subject given under the algebra area at the 11^{th} grade (MoNE, 2009) and the proofs of the law of cosine and the law of sinus were mentioned under the geometry area of the 9^{th} grade (MoNE, 2013). In addition, all mentioned curricula underlined the case that mathematics education covers more than supplying learning environments applicable in terms of facilitating the conceptual understanding; therefore, it should provide cases which encourage students to enhance their reasoning by virtue of daily life problems.

Similar to the situation in the middle school mathematics curricula mentioned previously, secondary school mathematics curricula in 2011 and 2013 have more common points compared to the last one presented in 2018. Regarding reasoning, Mathematics Curriculum for grades 9-12 (MoNE, 2011) stated the necessity of developing the process standards, namely, reasoning, problem solving, connection, communication, and modeling while Mathematics Curriculum for grades 9-12 (MoNE, 2013) listed reasoning among the process standards of mathematics along with connection and communication. However, Mathematics Curriculum for grades 9-12 (MoNE, 2018) did not specify such a list of process standards explicitly. Another mutually stated issue in both Mathematics Curricula for 9-12 grades in 2011 and 2013 was related to the rote teaching, which was generally utilized in mathematics classrooms. It was focused on how the rote teaching can be better regulated. To offer a rote teaching environment to students in mathematics might cause to frame an atmosphere in which neither the relations among the mathematical concepts nor the association of the concepts with the daily life can be arranged. Such a process was represented with this flow; “Definition $\rightarrow$ Theorem $\rightarrow$ Proof $\rightarrow$ Applications and Test” (MoNE, 2011, p.6; MoNE, 2013, p.1). Instead of the classrooms following the given flow, both curricula underscored the importance of preparing mathematics classrooms in which students have the opportunity to reach a formal mathematical structure arisen from an informal case by being an active participant in the examination process. This expectancy was represented with this sequence; “Problem $\rightarrow$ Discovery $\rightarrow$ Hypothesis $\rightarrow$ Verification $\rightarrow$ Generalization $\rightarrow$ Inference” (MoNE, 2011, p.6). In a similar vein, Mathematics Curriculum for 9-12 grades (MoNE, 2013) added association as a component of this learning sequence by placing it in this way; “Problem $\rightarrow$ Discovery $\rightarrow$ Hypothesis $\rightarrow$ Verification $\rightarrow$ Generalization $\rightarrow$ Association $\rightarrow$ Inference” (MoNE, 2013, p.1).

Any of the stated secondary school mathematics curricula applied in Turkey does not cover the terms argument and argumentation, but it can be stated that the curricula in 2011 and 2013 involve some sentences which presented connotation for argumentation. For example, it was stated that students should be supported in a way that they can produce mathematical ideas on their own, be responsible for their success and failure, discover, produce conjectures, explain their reasoning, discuss on their ideas, defend their ideas based on the rationale warrants, develop their critical thinking, detect incorrect ideas of others, notice that mathematics is more than memorization of procedure, and conduct the relationships and inferences regarding the daily life occasions based on their mathematical thinking (MoNE, 2011, 2013).

All in all, it was seen that the focus of mentioned mathematics curricula of middle school was on reasoning directly and on argumentation indirectly but not on proof. On the other hand, the focus on proof was increased in all mathematics curricula of secondary school. Another point to note is the resemblance of the emphasized process standards in mathematics curricula in Turkey with the process standards stated by NCTM (2000) which are problem solving, reasoning and proof, communication, representation, and connection. According to NCTM (2000), students at the end of the secondary school should be competent at understanding and conducting formal proofs and be able to see the necessity of conducting arguments. Likewise, it can be inferred that the mathematics curricula of secondary school in Turkey employed a similar stance. As seen, the concepts reasoning, proof, and argumentation constituted an intertwined nature in the abovementioned mathematics curricula of different levels.

1.4. Geometry and Geometric Constructions

It is a notable issue that there is neither a domain of mathematics nor a particular mathematics-related course where proving is stated as a particularly appropriate activity. That is, proving can be used effectively in mathematics teaching when it is set properly regardless of the domain (Ellis et al., 2012). In this study, the context selected to work on the cognitive unity construct is geometry since geometry is "a rich source of opportunities for developing notions of proof" (Jones, 2002, p.125) and also it was stated that "establishing geometric knowledge calls for reasoning" (NCTM, 2000, p.7). Moreover, geometry is a proper area to develop reasoning and justification skills of students (NCTM, 2000) and provides the opportunity for conjecture production and exploration (Gillis, 2005). In this manner, many researchers used geometry tasks and problems in their studies to assess students' mathematical reasoning (Mansi, 2003). In some curricula, geometry subject in secondary school level is the first place where proof is taken into account formally and how the proof is introduced affects students' learning of other subjects of mathematics such as algebra, trigonometry, statistics, and precalculus (Ellis et al., 2012). That is, geometry is a domain of mathematics used at the beginning of the issue of how proof should be learned (Pedemonte, 2007b). When students are engaged in an environment in which geometry is presented by aiming to stimulate their curiosity and interest, to encourage them to explore and discuss, and to justify their reasoning during argumentation, students might develop their learning, get the significance of proving, and have a positive attitude towards mathematics (Jones, 2002). How theorems are attained and proofs are conducted in a deductive manner are among the main issues striven when the subject is Euclidean geometry (Leung & Lopez-Real, 2002). Moreover, in Euclidean geometry, the validity of a statement is verified and justified by considering the entailments of the formal axiomatic system and following the deductive way with the contribution of the already known axioms and theorems. Therefore, it resulted in a case that deductive reasoning constitutes a major part of the teaching and learning process of concepts in Euclidean geometry (Leung & Lopez-Real, 2002). To prepare the occasions in which students might move between the practical and theoretical aspects of geometry is a critical challenge in terms of mathematics education (Fujita, Jones, & Kunimune, 2010). Besides, students should be offered environments in which they can see the differences between experimental and deductive approaches in geometry domain (Kunimune, Fujita, & Jones, 2009).

The mentioned studies of Italian researchers (e.g., Boero et al., 1996; Garuti et al., 1996; Garuti, Boero, & Lemut, 1998; Mariotti, 2001; Mariotti, et al., 1997) declared that applying the open problems is an effective way in learning proof since cognitive unity between argumentation while producing conjecture and proving can be noticed in such a case (Pedemonte, 2011). Similarly, it was stated that the open problem has an important potential in terms of proceeding proving, leading to being curious about the problem, providing a clearly noticeable argumentation process (Baccaglini-Frank & Mariotti, 2010), and calling for the production of conjectures as the initiative for proving (Baccaglini-Frank, 2010; Pedemonte, 2007b). Moreover, Baccaglini-Frank and Mariotti (2010) described two phases in a conjecturing open problem, namely, the conjecturing phase and the proving phase. In the former phase, students are expected to attend in the production process of a statement, which is called as the conjecture by Baccaglini-Frank and Mariotti (2010), by means of exploring the given task. In the latter phase, students are expected to deal with the proof of the conjecture they produced. According to Mogetta, Olivero, and Jones (1999), the open problems in geometry domain have the following properties.

> - the statement is short, and does not suggest any particular solution methods or the solution itself. It usually consists of a simple description of a configuration and a generic request of a statement about relationships between elements of the configuration or properties of the configuration.
> - the questions are expressed in the form “which configuration does…assume when…?” “which relationship can you find between…?” “What kind of figure can…be transformed into?". These requests are different from traditional closed expressions such as "prove that…", which present students with an already established result (p.91-92).

While engaging in a task which demands a conjecture, it was aimed that the solution process is not obvious so that an argumentation process is anticipated during the production of conjecture (Pedemonte, 2002b). In this respect, by considering the properties of the open problems listed above, geometric construction was selected as the geometric context situated in the conjecturing phase of the activities. That is, the participants were aimed to be involved in an argumentation process in which they are searching for the production of conjectures while dealing with geometric constructions related mainly to triangle and circle. It was noticed in the literature review that Baccaglini-Frank (2010) used step-by-step construction problems related to quadrilaterals, in which the instructions were present, so as to examine the effect of dynamic geometry programs to the conjecture production process. However, the content of the geometric construction tasks was planned to be different from the ones Baccaglini-Frank (2010) presented. That is to say, to keep all entailments of the description of the open problems and to avoid leading students while working on the construction, the tasks were planned in a way that the steps of geometric construction were not given to the participants.

The geometric construction process mainly covers empirical reasoning by using some tools and it is very difficult to consider the theoretical side thoroughly at the same time (Mariotti et al., 1997). Thus, it can be stated that the main issue in an activity which considers the geometric construction in the scope of cognitive unity is moving from empirical aspect to theoretical aspect and progressing from a generic aspect to formal proving (Mariotti, 1995; Mariotti et al., 1997). Geometric constructions by using various tools could facilitate students' ability to understand geometric connections and help them to reason on geometric generalizations in which more abstract reasoning is required (Arıcı, 2012). Regardless of the tool used, the process of solving geometric construction problems has several benefits for students (Djorić & Janičić, 2004). Through geometric constructions, students might have the chance to improve their logical thinking skills since they analyze the properties of geometric figures in the construction process. As Cheung (2011) stated, geometric construction might be used to develop students' ability in geometric proofs and increase their interest in learning geometry. Similarly, Fujita et al. (2010) presented that geometric constructions encourage students to study on proving statements, to produce some conjectures, and to develop their ability related to argumentation, reasoning, and proof. Moreover, to use different tools such as compass-straightedge and GeoGebra might cause some differences in students' argumentation process while working on geometric constructions. This case brings to surface another issue remaining to be addressed, which is how students evaluate the validity of their construction attempts.

As a concluding remark, some issues which led to the determination of the purposes of this study were mentioned herein. It is difficult to employ an effective method while teaching proof of a statement regardless of the mathematical domain. It was underlined that "there are a relatively small number of research studies that have developed promising classroom-based interventions to address important issues of the teaching and learning of proof" (Stylianides, Stylianides, & Weber, 2017, p. 258). In this respect, cognitive unity which was stated "to bring about a smooth approach to theorems in school" (Garuti et al., 1998, p.345) might be taken into consideration to inspect proving process in undergraduate level within the context of geometry. Since geometry in undergraduate level is anticipated to have a more formal stance, students need to develop reasoning originated from axioms and postulates rather than the intuitive and informal grounds (Hollebrands, Conner, & Smith, 2010). Besides, as stated before, to examine a setting based on cognitive unity thoroughly, the literature review persistently presented the use of argumentation model of Toulmin. However, due to some criticisms related to Toulmin's model of argumentation stated in the literature, it is also needed to have a critical look at the application of it and to carry out close scrutiny of the roles of the components of argumentation. To conclude, by underpinning the theoretical construct of cognitive unity and in accordance with the main concerns mentioned up to this point, the relation of prospective middle school mathematics teachers' argumentation during conjecture production to proving process, the argumentation structures emerged while producing conjectures by virtue of geometric constructions, their solution approaches in geometric constructions, and their proving process of the produced conjectures were determined as the main issues to be focused on in this study.

In what follows, the purposes of the present study which are followed by the research questions are clarified thoroughly.

1.5. Purpose of the Study and Research Questions

The purpose of the study is multifaceted. The first purpose of the study is to investigate how prospective middle school mathematics teachers' argumentation process while producing conjectures is related to the proving process of the conjectures they produced. As seen, this purpose refers to the underlying context of cognitive unity, which is also the first facet. The second purpose is to depict the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities. In line with the global argumentation structures, the components of argumentation emerged, and the functions of rebuttals are also aimed to be investigated. These purposes are the ones categorized in the scope of the second facet, which is mainly related to the argumentation concept. Along with the third purpose of the study, the third facet, which is related to the geometric construction, came into the play. The purpose at this point is to investigate the approaches prospective middle school mathematics teachers offered to be able to perform geometric constructions asked in the cognitive unity based activities. Meanwhile, how they evaluated the validity of their approaches and to what extent they can perform geometric constructions correctly while using compass-straightedge and GeoGebra are also sought to be described herein. By means of the last purpose of the study, it is time for the last facet which is related to proof. Basically, the last purpose of the study covers to investigate the conjectures prospective middle school mathematics teachers produced during the argumentation process and to what extent they can conduct valid proof for the conjectures.

Based on these purposes, the research questions were organized and revised throughout the study. Before listing the research questions, the rationale behind the organization of them was explained at this point. Since the main aim of the overall study is not to compare the practices of compass-straightedge group and GeoGebra group in the cognitive unity based activities, all research questions were designed in a way that they did not focus on the groups separately at the beginning. In more detail, as mentioned, the main concept of the first research question is related to the underlying context of cognitive unity. It was expected that the findings of the groups related to the first research question would not be different. The data analysis supported this situation; hence, the findings were not divided with respect to the groups, they were presented in the findings section as a whole, and the first research question was kept the same. Similar to the first one, the second research question and the sub-questions were aimed to be handled as a whole. Since the concept in the second research questions and the sub-questions is argumentation, it was expected that there would be no need to handle the findings of the groups separately. After the data analysis, it was seen that there was not a direct relevance of the group properties on the findings in terms of the second research question. Thus, it was decided to consider the findings regarding the argumentation concept in the second research question as a whole. However, it was not the case in the third research question. That is, some revisions were needed in the third research question due to the nature of geometric construction. After the analysis of the data in terms of the third research question, it was seen that the use of different tools directly affected the evaluation process of the validity of the approaches offered by participants to perform geometric constructions. Therefore, the sub-questions were revised and to what extent prospective middle school mathematics teachers perform geometric constructions correctly was taken into consideration separately in terms of groups. Finally, the concept focused in the fourth research question is proof. Based on the data analysis, the related findings were presented without focusing on the groups separately.

All in all, considering these purposes, the following research questions are aimed to be addressed throughout the study:

1. How does prospective middle school mathematics teachers' argumentation process while producing conjectures in the cognitive unity based activities relate to the proving process of the conjectures they produced?

2. What are the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

    2.1. What are the components of global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

2.2. What are the functions of rebuttals situated in the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

3. What are the approaches offered by prospective middle school mathematics teachers to perform geometric constructions in the cognitive unity based activities?

3.1. How do prospective middle school mathematics teachers evaluate the validity of the approaches they offered for geometric constructions?

3.2. To what extent do prospective middle school mathematics teachers perform geometric constructions correctly while using compass-straightedge?

3.3. To what extent do prospective middle school mathematics teachers perform geometric constructions correctly while using GeoGebra?

4. What are the arguments offered by prospective middle school mathematics teachers as the proof of the conjectures they produced?

4.1. What are the conjectures that prospective middle school mathematics teachers produced during the argumentation process?

4.2. To what extent do prospective middle school mathematics teachers conduct valid proof for the conjectures they produced during the argumentation process?

In line with these research questions, the rationale of the study is explained in the following section.

1.6. Significance of the Study

It is one of the highlighted issues that studies conducted in mathematics education have lack of implications in terms of promoting students' learning of mathematical concepts in classroom practices (Stylianides & Stylianides 2013, 2018). As expected, proof is one of these issues. Since it was strongly suggested that the focus of the proof related research should be shifted from the ones which document the problematic issues students have while engaging in proving to the ones which offer the methods to overcome the problems such as the suggestions related to the classroom-based intervention (Stylianides & Stylianides, 2017; Stylianides & Stylianides, 2018). In spite of the fact that there is a consensus about the difficulties encountered by students related to proof based on a great number of studies, the scarcity of the research which offers the design of intervention to overcome such difficulties was also noticed (Stylianides et al., 2016; Stylianides et al., 2017; Stylianides & Stylianides, 2017; Stylianides & Stylianides, 2018). Although there are studies related to classroom actions pertaining to proof which present notable results, there is still a need for studies in this area due to the lots of relatively unexplored contexts (Reid & Knipping, 2010; Stylianides et al., 2016; Stylianides & Stylianides, 2018). Regarding this issue, cognitive unity might be considered as a promising method to develop students' learning of proof related concepts, to deal with the negative cases encountered in the process, and to facilitate their conceptual understanding of the concepts as followed by the formal proof.

The main background to the present study is provided via the theoretical construct of cognitive unity, which was presented by Garuti et al. (1996). According to Fiallo and Gutiérrez (2017), cognitive unity covers the case that the argumentation process while producing a conjecture assists the proving process of it regardless of the fact that whether the type of the argument presented as proof is empirical or deductive. At least, the hypothesis of cognitive unity asserts that circumstance (Pedemonte, 2007b). Instead of taking into consideration argumentation and proof in a separate manner, focusing on both of them might provide the opportunity to consider a broader range of issues regarding proving (Stylianides et al., 2016). As Baccaglini-Frank (2010) stated, "passing from the development of a conjecture to the construction of a proof is a delicate process" (p.3), to investigate how these processes relate to each other is one of the aims of this study. To find out the rationale, affordances, and constraints of the argumentation process with respect to proof might be helpful to enhance the methods used while teaching proof (Reid & Knipping, 2010). Thus, it was expected that this study might provide new insights to the cognitive unity concept by considering the presence or absence of cognitive unity in-depth and casting light on not only positive but also negative instances faced pertaining to cognitive unity.

When argumentation phase of a task involves conjecture production and proof as a product, to determine the point in which the proof started is critical from the perspective of students. Besides, such a task would be helpful for students to interpret the relation and difference between argumentation and proof as well as to distinguish conjecture from theorem (Pedemonte, 2007a). Since cognitive unity was posited as a starting point of this study, it was aimed to arrange an environment that the participants will be producing their conjectures and then proving them. In this respect, by considering the basis and features of cognitive unity, activities within the context of geometry, which will be called as the cognitive unity based activities henceforth, were prepared so as to examine the mentioned issues. More specifically, the cognitive unity based activities were designed in a way that they involved two sections. The first section covers the argumentation process in which prospective middle school mathematics teachers are expected to reach at least one conjecture. The context of this section of the activities which provides an environment for collective argumentation was decided to be geometric construction. In other words, the participants were expected to produce conjectures while they were dealing with geometric constructions. Then, in the second section, the participants were given one of the conjectures they produced recently and asked to prove. Moreover, it is an inevitable case that some students present an empirical point of view and do not grasp the reasoning to conduct a deductive proof (Chazan, 1993). In this respect, to apply the activities with two stages like the cognitive unity based activities in this study might be helpful for students to fulfill the need and the outset of the proving phase and develop an intuitive approach.

Students should be in a learning environment in which they can produce conjectures, explain their reasoning with their own sentences, use materials, representations, and notation properly throughout the mentioned process, work collaboratively with others when exploring possible conjectures, learn to listen to the ideas and reasons of others and discuss about them. In this process, teachers should guide students in order to lead them to check their conjectures in different context and settings (NCTM, 2000). According to Ellis et al. (2012), the proving process is a broad construct; hence, it is attributed to various activities leading to writing a proof such as producing conjectures, looking for generalizations, considering examples and counterexamples, and searching for similarities among cases. This study paid regard to the importance of such elements of the proof-aimed process. In this respect, it can be inferred that this study also presents prospective middle school mathematics teachers the opportunity to experience such a learning environment from different aspects. As being a participant in the cognitive unity based activities, prospective middle school mathematics teachers might look from students' perspective. For example, they might see the importance of conjecturing phase and collective argumentation in their learning process of geometry. Secondly, they might examine the implementation of the activities from the perspective of mathematics teachers. For instance, they might notice the importance of guidance of teachers, especially when students get stuck during argumentation. Besides, mathematics teachers have a critical role in raising students' awareness and promoting their ideas regarding the necessity of proof (Stylianides et al., 2016). Providing a classroom in which students are led to engage in both empirical verification and deductive proofs is critical in terms of the development of justification capability of students as well as helping them to notice the importance and difference of the argumentation processes in the mentioned stages (Chazan, 1993). Since prospective middle school mathematics teachers are regarded as the mathematics teachers of near future, their active participation in the cognitive unity based activities might enlighten them about the mentioned concerns and help them to capture the essence of cognitive unity in terms of educational implications.

As a last word regarding cognitive unity, it was noticed that there is not a commonly accepted or used framework which is particularly designed to analyze the cognitive continuity between conjecture production and proof (Pedemonte, 2007a). Despite the increasing interest on proof related research, any study which focused distinctively on the construct of cognitive unity in undergraduate level accompanied by the geometry domain of mathematics was not found in the accessible literature. Moreover, argumentation process outlined in cognitive unity was provided with the inclusion of the geometric constructions by means of using different tools which are compass-straightedge and GeoGebra. From this point of view, it can be inferred that this study has the potential to contribute to the growing body of literature.

Another remark of this study is argumentation in mathematics education. Regarding argumentation, it was emphasized that "there is still work to be done" (Reid & Knipping, 2010, p.164). Grounded on the mentioned descriptions and studies, it can be inferred that both the process of conjecture production and the process of proving it involve argumentation (Pedemonte & Buchbinder, 2011). Based on the initial definition of cognitive unity and the approach of Pedemonte which refers argumentation as “the process of conjecturing” (Reid & Knipping, 2010, p.163), argumentation process was devoted to the conjecturing phase in this study and the global argumentation structures of this phase were constructed while addressing the second research question. In this phase, the participants were expected to work as a group during the cognitive unity based activities, which was called as collective argumentation. It was pointed out in a recent study conducted by Tekin-Dede (2018) that the term collective argumentation is not directly involved in neither the mathematics curricula nor the research conducted as related to argumentation in Turkey; however, there are studies which probe the term argumentation as related to mathematics education. In this manner, this study might serve for the collective argumentation research base, especially in the national context. Besides, as mentioned before, mathematics curricula in Turkey do not refer to the term argumentation literally. By means of the courses in the teacher education program, prospective teachers became familiar with the terms reasoning and proof within the context of mathematics, but argumentation is not presented in these courses distinctively. Within the scope of the study, prospective middle school mathematics teachers were informed about the meaning of argumentation by virtue of the teaching sessions arranged before the application of activities and also participated in an argumentation process through the activities. Thus, the findings might be considered as a reference document to examine the efficiency of argumentation in mathematics education.

Studies devoted to argumentation bring out another aspect of this study, which is global argumentation structures. Toulmin’s model of argumentation with some needed modifications is an effective tool to examine the overall structure of argumentation (e.g., Erkek, 2017; Erkek & Işıksal-Bostan, 2019; Knipping, 2008; Knipping & Reid, 2013, 2015, 2019; Reid & Knipping, 2010). In other words, Toulmin’s model is an effective tool in the description of one step of an argument so that it can be used to point out separate arguments situated in a comprehensive argumentation process. However, there is a need to consider more than this model to be able to portray the overall structure of argumentation (Knipping, 2008; Reid &

Knipping, 2010). For example, the argumentation in a mathematics classroom or the argumentation of a small group while working on an activity might cover a long period and have a complex nature. In such cases, the global argumentation structure cannot be provided by one argumentation step only. Thus, the global argumentation structures function to present the picture of the entirety of the argumentation process and to unfold the individual arguments by means of the features of components in the layout (Reid & Knipping, 2010). The representation of the global structure of an argumentation process would be effective in noticing the patterns involved in argumentation process (Knipping, 2008) and in getting a grasp of the whole discussion which is a solid movement before understanding the parts of argumentation. Moreover, the global argumentation structures might offer insights into the issues that how students are moving during argumentation, how the components influence the flow of the arguments, and how argumentation might be enriched. For example, by inspecting a global argumentation structure which has a complex and wide structure and also involves many parallel argumentation streams, rebuttals, and backward movements, it might be inferred that the general concept of the argumentation might be challenging, there might be an evolving process due to backward and forward movements, and the arguers are not sharing similar ideas due to the rebuttals. Besides, the review of the related literature presented that the global structure of argumentation is a relatively less studied concept in the mathematical domain so that further research about this issue is needed. All in all, since this study aimed to display the overall argumentation structures of groups in the conjecture production process while endeavoring on geometric constructions by using compass-straightedge and GeoGebra, it can be stated that it might present valuable feedback for interpreting the nature of argumentation emerged in conjecture production. Moreover, it might provide a reference point for other studies since it presents a classification for the global argumentation structures in a broader stance.

Toulmin's model presents the opportunity for investigating the characteristics of a particular component such as warrant and backing by elaborating in-depth (e.g., Inglis et al., 2007; Nardi, Biza, & Zachariades, 2012; Verheij, 2005). Krummheuer (1995) used the narrower version of Toulmin's model, which involves data, warrant, claim, and backing components, and some other researchers also followed the same approach. On the other hand, it was highlighted the importance of counting in all components of Toulmin's model while examining the whole range of the argumentation. Using the restricted form of Toulmin's model causes to downplay the functions of the other two components, which are qualifier and rebuttal (Inglis et al., 2007). In this respect, this study concentrated on using all components constituted in the argumentation model of Toulmin, but the findings presented the need for components more than the six components mentioned above. Thus, the extra components, which were set out based on the findings, might provide valuable feedback to other studies which plan to employ Toulmin's model. In this manner, it can be stated that this study also called the attention to the possibility of the presence of some other components in argumentation depending on the context that argumentation takes place.

As expected, since the participants in an argumentation do not have an affirmative stance all the time, the presence of some rejections is inevitable. Among the components of argumentation, rebuttal serves for the mentioned notion. Toulmin et al. (1984) described rebuttal as follows; "it applies wherever a general presumption is set aside in the light of certain exceptional facts" (p.96). It was observed that arguers move backward and forward during argumentation. For example, noticing a counterexample might be an incidence for moving backward in the argumentation (Stylianides et al., 2013). When counterexamples are stated in the argument, they would act as rebuttal since they consider the exceptional cases that argumentation is not valid anymore (Pedemonte & Buchbinder, 2011). Walton (2009) described refutation as "a rebuttal that is successful in carrying out its aim" (p.4). Besides, there is the scarcity of studies focused distinctively on the nature of rebuttal in the accessible literature compared to some other components such as warrant and backing. Since counterexamples can refute a conjecture and also cause modifications in the conjectures (Sinclair et al., 2012a), proposing a rebuttal requires higher-order thinking skills and it has a high potential to be regarded as a difficult task during argumentation by student (Lin & Mintzes, 2010), it can be inferred that how prospective middle school mathematics teachers come up with a rebuttal during the argumentation process has a critical role in terms of the conjectures or statements they depicted.

It is a widely mentioned issue in the literature that subject matter knowledge is a fundamental constituent of teacher knowledge (Ball, 1990; Ball & McDiarmid, 1990; Ball, Thames, & Phelps, 2008; Cochran, DeRuiter, & King, 1993; Fennema & Franke, 1992; Shulman, 1986, 1987). In a similar vein, Fernandez (2005) underlined the importance of the content knowledge as follows; “it is hard to imagine teachers engaging their students in deep and productive conversations about mathematics without themselves having a strong grasp of the content that they are trying to discuss” (p. 266). Therefore, mathematics teachers should have the necessary content knowledge as the key entailment of teaching as well as be aware of how students can relate the concept to other ideas and extend the idea to new circumstances. For example, mathematics teachers should know about the algebraic proof of a statement so that they can benefit from the knowledge, ideas or principles coming from the formal proof while guiding students to find and interpret related examples and to develop their reasoning in a more solid base when such a formal proof transcends the level of students (Lannin et al., 2011). In this manner, the idea that mathematics teachers, especially the ones teaching in middle school, do not have to learn more advanced mathematics concepts since they are not teaching such concepts to students can be considered as an underestimation and limitation for mathematics education.

The importance of developing an inclusive mathematical understanding for mathematics teachers was highlighted by Sinclair, Pimm, Skelin, and Zbiek (2012b) as “rich mathematical understanding guides teachers’ decisions in much of their work, such as choosing the tasks for a lesson, posing questions, selecting materials, ordering topics and ideas over time, assessing the quality of students’ work, and devising ways to challenge and support their thinking” (p.2). Moreover, mathematics teachers’ knowledge should be more than what they will teach and students are aimed to learn. The reason behind this idea is that mathematics teachers are required to know various representations, models, and technologies, to relate them with the curriculum entailments properly, and to interpret which mathematical issues should be emphasized while teaching so as to present students effective learning environments.

Besides, mathematics teachers should be aware of the potential pitfalls of students peculiar to the subject matters, know about the actions which might be used to help students avoid having the pitfalls, and be ready to address the unexpected situations in the classrooms (Ellis et al., 2012).

As mentioned, the context of this study in terms of the mathematical domain is geometry. Comprehension of geometry is a requirement for students both in middle school level and in secondary school level since it is needed for their success at these levels and their future learning experiences (Sinclair et al., 2012a, 2012b). Since the transition of students from the intuitive aspect to the formal aspect in geometry concepts is not a basic and self-induced process (Leung & Lopez-Real, 2002), some extra conditions such as the guidance of an instructor are needed. The raw material for making students to develop a permanent comprehension of geometry is mathematics teachers' understanding of the concepts deeply. In other words, since mathematics teachers are expected to teach the concepts of geometry to students, they also need to be competent in geometry as a subject matter and the ways of teaching geometry effectively. Moreover, mathematics teachers should have the logic behind the geometry concepts to be able to detect and notice the origin of students' unexpected questions, answers, and comments during teaching and be aware of the possible concepts that students might relate to the geometry concepts with respect to their year level and background knowledge (Sinclair et al., 2012a, 2012b).

At this point, due to the mentioned issues, the focus diverges from the competencies of mathematics teachers to teacher education programs. There is an agreement about the case that the courses in teacher education programs are not quite adequate to prepare prospective teachers for future teaching experiences (Black & Halliwell, 2000). Moreover, Veenman (1984) underlined that there should be clear justifications and reasons to blame teacher education programs related to beginning teachers' problems since it could not be the source of all possible problems of them. However, in terms of the development of content knowledge, teacher education programs have a critical role. In this respect, since the participants of the current study are prospective middle school mathematics teachers, it is expected that this study could provide information about their knowledge level in geometry, their correct inferences and wrong interpretations, and how the content of the undergraduate courses involving geometry should be revised. Moreover, this study provides an environment that prospective middle school mathematics teachers might carry out self-assessment, notice their deficiencies regarding geometry, and take precautions to deal with these issues before becoming in-service teachers.

The last issue mentioned related to the rationale of this study is geometric constructions inserted in the cognitive unity based activities. Geometric construction is a convenient tool when the discovery of geometry is aimed (Kostovskii, 1961). Duval (1998) defined construction as a cognitive process of using tools and stated construction as a path for writing proof. Similarly, Battista and Clements (1995) mentioned about the positive effect of construction on proof. When students study with construction before formal proofs, it might be helpful for students to make conjectures. Since prospective middle school mathematics teachers can be considered as mathematics teachers of the future, to what extent they are capable of performing geometric constructions is an important subject for investigation. As the literature review showed that the number of studies related to geometric construction involving the use of compass-straightedge is limited, this study might contribute to the related literature. Since the study also involves the use of GeoGebra in geometric constructions, the results might help to see whether prospective middle school mathematics teachers can use GeoGebra effectively while studying with geometric constructions. The aim of including two settings in terms of the tool is to examine whether there exists a change of state in geometric constructions and argumentation process depending on the tools. Moreover, it was aimed to present the criteria which can be used in further studies to determine whether a given geometric figure can be accepted as a geometric construction when using both compass-straightedge and GeoGebra. Thus, this study might contribute to the literature by offering different step-by-step frameworks for both of the mentioned tools which can be used to control whether a geometric figure can be constituted as a geometric construction and by calling attention into the significance of geometric construction in terms of the development of proving in geometry.

1.7. Definitions of Important Terms

To facilitate the readability of the following chapters, to avoid the ambiguity, and to depict a clear picture of the study with respect to its purposes, the constitutive and operational definitions of the key terms are presented in this section.

Cognitive Unity

Garuti et al. (1998) defined the cognitive unity of a theorem as "the continuity existing between the production of a conjecture and the possible construction of its proof" (p.345). According to Fiallo and Gutiérrez (2017), "there is cognitive unity when the argumentations used during the conjecturing phase help students to construct a proof, either empirical or deductive" (p.149). Instead of using the term cognitive rupture, the terminology "the presence or absence of cognitive unity" as stated by Pedemonte (2002b, p.72) was decided to be used in this study. By following the given definitions, the term cognitive unity was considered as a theoretical construct on which the activities of the study were based.

Cognitive unity based activity

Based on the mentioned definitions of cognitive unity, activities involving two sections were designed. In more detail, the first section demands the production of conjectures while the second section asks the formal proof of one of the recently produced conjectures.

Argumentation

Argumentation was defined as "a process in which a logically connected mathematical discourse is developed" (Smith, 2010, p.2) and also identified as "sharing, explaining, and justifying of mathematical ideas" (Cross, 2009, p. 908). The description of argumentation held in many studies is "the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false" (Stylianides et al., 2016, p.316). In this study, the argumentation process was devoted to the conjecturing phase in the cognitive unity based activities, and it was analyzed by using the revised version of the argumentation model of Toulmin (1958, 2003).

Global argumentation structure

Knipping (2008) produced a structure, which represents the whole argumentation process, and called it as 'global argument' or 'argumentation structure'. In this study, as the intersection of these terms, the term 'global argumentation structure' was used to represent the entire argumentation process of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities.

Components of argumentation

According to Toulmin (1958, 2003), components of arguments are data, claims, warrants, backings, qualifiers, and rebuttals. The definitions of six components in Toulmin's model utilized in the related studies were examined so as to determine the operational definitions for the present study. The details of the cited studies regarding the definitions of components can be found in the next chapter (See Tables 2.1-2.6).

Data component of argumentation

By combining a variety of definitions of data presented in the literature (See Table 2.1) and focusing on using a comprehensive definition for the data component, the following definition of data was formed. Data was defined as some form of facts, evidence, statements, undoubted statements, specific piece of information or general information, and methods or mathematical relationships which function as the foundation, basis, ground, and inference of the claims/conclusions/argument, and also support, justify, and lead to the claims/conclusions/argument.

Warrant component of argumentation

Similar to the determination of the definition of data, the operational definition of warrant was arranged by considering the definitions of warrant given in studies (See Table 2.2). As a result, warrant was defined as a general statement, a principle, a definition, a rule, an example or an analogy which acts as bridges between data and claim/conclusion, functions as rule of inference that authorizes the legitimacy of the step from data to claim/conclusion, justifies the relationship/connection between data and claim/conclusion, explains how data leads to the claim/conclusion, and provides more evidence to clarify an argument.

Conclusion component of argumentation

The terms claim and conclusion were used as having the same meaning in some studies (e.g., Rasmussen & Stephen, 2008; Stephan & Rasmussen, 2002; Toulmin, 2003). On the other hand, according to the studies of Knipping (2008) and Knipping and Reid (2013, 2015), the term claim was used when data and warrant were not provided, and the term conclusion was used in the case that data and warrant were provided. This perspective was also utilized in the present study. Moreover, definitions of claim/conclusion stated in the literature (See Table 2.3) were considered. In terms of this study, it can be stated that it is more appropriate to arrange an operational definition for particularly conclusion. Thus, conclusion was defined as the statement being argued, established, justified, and inferred from data and also the assertion put forward for general acceptance or basic convictions.

Backing component of argumentation

The definitions of backing presented in the literature (See Table 2.4) were taken into consideration. Upon these definitions, backing was defined as the statements which support warrants, describe the validity of warrants, and explain why warrants have the authority.

Rebuttal component of argumentation

To be able to constitute an operational definition of rebuttal, descriptions of rebuttal were examined via the review of the literature (See Table 2.5). In light of the definitions, rebuttal was defined as conditions/circumstances/exceptions under which conclusion/claim would not hold, the warrants would not be valid, and all exceptions regarding the argument.

Qualifier component of argumentation

The definitions of qualifier given in the studies were examined (See Table 2.6) and operational definition was aimed to be determined. In this respect, qualifiers express the degree of confidence and the certainty of claim/conclusion and describe the strength of argument/claim/conclusion as determined by warrant.

Conjecture

Sinclair et al. (2012a) defined conjecture as “a proposition that has not been proved or disproved” (p.48). In line with the meaning of cognitive unity, Pedemonte

(2002b) expressed that "the conjecture can be transformed into a valid statement if a proof justifying it is produced" (p.71). In the present study, the propositions that prospective middle school mathematics teachers produced during the argumentation process while working on geometric constructions were accepted as conjectures.

Proof

Stylianides (2007b) defined proof as a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
> 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community;
> 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

In this study, proof refers to stating appropriate methods, definitions, and mathematical language by following logical steps to verify the conjectures produced within the context of geometry.

Geometric construction

Geometric construction was defined by Djorić and Janičić (2004) as "a sequence of specific, primitive construction steps" (p.71). According to Smart (1998), geometric construction was used for three cases which are "to describe the geometric problem to be solved", "to describe the process of solving the problem", and "to describe the completed drawing that results from solving the problem" (p.164). In this study, as a combination of all cases, geometric construction refers to the formation of geometric figures by using different tools which are compass-straightedge and GeoGebra.

Compass-straightedge

Compass is a tool used to draw a circle with some given properties such as center and another point on it. Straightedge is described as a tool which can be used to draw a line but cannot be used for measurement since it does not have a scale marked on it (Djorić & Janičić, 2004, Leonard, Lewis, Liu, & Tokarsky, 2014; Stillwell, 2005; Stupel & Ben-Chaim, 2013). In this study, compass-straightedge was used while prospective middle school mathematics teachers were endeavoring on geometric constructions in the cognitive unity based activities.

Compass-straightedge group

The group of prospective middle school mathematics teachers who used compass-straightedge while working on geometric constructions was called as compass-straightedge group, which was abbreviated as CSG.

GeoGebra

GeoGebra is "dynamic mathematics software for all levels of education that brings together geometry, algebra, spreadsheets, graphing, statistics and calculus in one easy-to-use package" (GeoGebra, 2019, para.1). In this study, GeoGebra was selected to use while prospective middle school mathematics teachers are endeavoring on geometric constructions in the cognitive unity based activities. Moreover, depending on the context of the cognitive unity based activities, some tools of GeoGebra which present the aimed construction with a few clicks were restricted in the given GeoGebra files directly or in a stepwise manner.

GeoGebra group

The group of prospective middle school mathematics teachers who used GeoGebra while working on geometric constructions was called as GeoGebra group, which was abbreviated as GG.

Prospective middle school mathematics teachers

Prospective middle school mathematics teachers are students enrolled in the Elementary Mathematics Teacher Education program. As participants, junior prospective middle school mathematics teachers, which corresponds to students in the third year level of a four-year teacher education program, in a state university in Ankara were selected by using purposeful sampling.

The following chapter is devoted to the review of the literature to depict the concepts underlined in this chapter thoroughly.

CHAPTER 2

LITERATURE REVIEW

The first purpose of the study is to investigate how prospective middle school mathematics teachers' argumentation process while producing conjectures is related to the proving process of the conjectures they produced. The second purpose is to examine the global argumentation structures that emerged while producing conjectures in the cognitive unity based activities, the components of these argumentation structures, and the functions of rebuttals. As the third purpose, the approaches offered to perform geometric constructions in the cognitive unity based activities and to what extent these approaches resulted in geometric constructions were investigated. The last purpose of the study is to examine the conjectures produced during the argumentation process and to what extent they can conduct valid proof for the conjectures.

Before embarking on describing the methodological details of the current study, this chapter elaborates the focused fundamental concepts by giving reference to the related literature. To provide the necessary background of the concepts, five sections, which are related to cognitive unity, argumentation, reasoning and proof, geometry, and geometric constructions, were proposed. Subsequent to these sections, the chapter ends with a summary of the related concepts.

2.1. Cognitive Unity

Through the review of literature, it was noticed that there is a body of research which is mainly or partially concentrated on cognitive unity by pursuing different purposes (e.g., Antonini & Mariotti, 2008; Arzarello et al., 1998; Baccaglini-Frank & Antonini, 2016; Boero, 2017; Boero et al., 2010; Boero et al., 1996; Fiallo & Gutiérrez, 2017; Fujita et al., 2010; Garuti et al., 1998; Garuti et al., 1996; Leung & Lopez-Real, 2002; Mariotti et al., 1997; Pedemonte, 2007a, 2007b, 2008; Pedemonte & Buchbinder, 2011). Moreover, it was seen that there are many studies citing or giving credit to cognitive unity without framing the study mainly on it (e.g., Arzarello & Sabena, 2011; Baccaglini-Frank, 2010; Conner et al., 2014b; Douek, 1999, 2007; Gfeller, 2004; Komatsu, 2016; Mariotti, 2004, 2006; Nardi & Knuth, 2017; Pedemonte, 2018a, 2018b; Reid, 2018; Sinclair et al., 2017; Soldano & Arzarello, 2017; Stylianides et al., 2016; Yan, Mason, & Hanna, 2017; Zazkis & Villanueva, 2016). As stated previously, cognitive unity, which dates back nearly twenty years, was offered to determine the proper conditions in terms of providing a smooth approach to proving (Boero, 2017). Moreover, Pedemonte (2002b) presented an analogy regarding the cognitive unity; "the processes used to construct a conjecture and its validation: argumentation and proof" (p.71).

As mentioned in the previous chapter, cognitive unity of theorems was introduced by Garuti et al. (1996) by aiming to present a hypothesis about the cognitive process while producing conjectures and conducting their proofs. In more detail, they carried out experimental research with 8th grade students and then came up with the idea about the presence of a possible cognitive continuum between the production process and the proving process of a statement. Actually, it was an extension of an earlier study which was conducted by Boero, Chiappini, Garuti, and Sibilla (1995). In that study, it was reported that students in middle school grades showed a direct coherence between the document they formed while producing statements and the proof of them. Besides, it was observed that the process in which students involved in the formulation of statements related to elementary arithmetic helped them during the proving stage since they already had the idea about the validity of the statement. At the end of the study of Boero et al. (1995), without dilating upon the results, they presented a suggestion for the further studies and called the attention to the issue that when a statement is produced by a student, the proof of it might evolve from the textual development process of the statement. Subsequent to this study, the first mentioned study of Garuti et al. (1996) was conducted and the concept of cognitive unity of theorems was presented.

In the following year, Mariotti et al. (1997) approached to the teaching of geometry theorems in classrooms by focusing on two constructs, namely, cognitive unity and mathematical theorem. Besides, by studying with different levels of students, they considered and examined various contexts such as the role of the teacher during classroom discussions, the place of the dynamic exploration with respect to geometry theorems, and the theorems as a combination of statement, proof, and theory. In more detail, they focused on the use of Cabri by aiming to present a deductive aspect while learning geometry. By means of geometric construction, it was expected that students would work in an environment where they set up a system by considering the geometric facts and axioms deductively. Moreover, it was reported that discussion is an essential component of the activities engaged in classrooms. The progress of discussion was represented by Mariotti et al. (1997) as in the following figure.

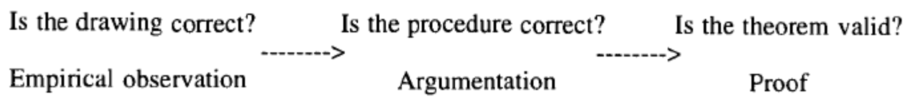


Figure 2. 1. Development of discussion (Mariotti, et al., 1997, p.191)

As it can be inferred from Figure 2.1, the first aim of the discussion was reported as to specify the epistemic value of the statement and the second aim of the discussion was stated as to decide whether the statement is correct and what the status of the statement within the mathematical concept is. Besides, by moving on the aims of discussion, the importance of teachers in terms of guidance was emphasized by Mariotti et al. (1997).

Garuti et al. (1998) conducted another study to investigate whether cognitive unity can be regarded as a tool to interpret the difficulties encountered while proving a directly given statement. In that study, cognitive unity of a theorem was defined as "the continuity existing between the production of a conjecture and the possible construction of its proof" (Garuti et al., 1998, p.345). Based on the findings, the role of cognitive unity of a theorem was attributed to being a pointer and a predictor regarding students' difficulties while proving a given statement. Moreover, there are some other studies which associated cognitive unity with the possible reasons of students' success or failure in proving (e.g., Fiallo & Gutiérrez, 2017; Garuti et al., 1996; Pedemonte, 2007a). Consequently, Garuti et al. (1998) came up with the idea that the connection between conjecture production and proof might be used as an approach while teaching but also underlined the necessity to consider the incontrovertible differences between them. Besides, it was warned that students should learn to transform the given statement properly in order to create continuity between the exploration process of the statement and the proving process of it (Garuti et al., 1998; Mariotti et al., 1997). Indeed, when the difficulties encountered in teaching and learning proofs in various domains of mathematics are taken into consideration, it seems that the plausible didactic implications of cognitive unity concept have critical importance.

The pioneer studies conducted related to cognitive unity presented evidence for the case that when students are involved in an argumentation process to produce a conjecture, the proof of the recently stated conjecture becomes a more accessible case for them (Boero et al., 1996; Boero et al., 2010; Garuti et al., 1996; Garuti et al., 1998). Moreover, it was seen that personal arguments of students at the statement production stage were kept in the same way at the proving stage and they were apt to resume the reasoning type, use similar expressions, and pursue the similar steps (Boero et al., 1996; Garuti et al., 1996; Mariotti et al., 1997). In more detail, Boero, Garuti, and Lemut (1999) focused on the generation phase of the conditionality of statements and the linkage to their proving phase. As a conclusion, they stated the presence of a significant link between the mentioned phases of the same task as follows; "frequently the same mental exploration which leads to the conjecture is re-started by the student with entirely different functions during their proving process" (p.142). In other words, cognitive unity refers to the fact that students might use the underpinnings of the argumentation process in which the conjectures are produced while working on proving the previously stated conjecture (Pedemonte, 2007a). Similarly, Pedemonte and Balacheff (2016) exemplified the continuity between argumentation and proof in the cases where proofs were arranged with the help of the mathematical elements such as formulas and computations used during the argumentation in which the conjecture was presented. By combining Toulmin's model of argumentation (1958, 2003) and the perspective of Peirce (1956), Conner et al. (2014b) prepared a framework and showed the examples supporting cognitive unity in conjunction with the types of reasoning.

On the other hand, Garuti et al. (1998) also mentioned the gap between the process in which the statement is explored and the process in which the related statement is proved. While working on geometry, the differences in the semiotic registers of both argumentation and proof also increase the difficulty in terms of moving among the mentioned processes and support the idea regarding the gap between them (Barrier, Mathé, & Durand-Guerrier, 2009). Bearing this in mind, Garuti et al. (1998) took the road of the following hypothesis in their study "the greater is the gap between the exploration needed to appropriate the statement and the proving process, the greater is the difficulty of the proving process" (p.347). Whilst Garuti et al. (1998) used the term gap, Fiallo and Gutiérrez (2017) used the expression of cognitive rupture for the same issue. While investigating cognitive unity and rapture, Fiallo and Gutiérrez (2017) deployed the categorization of Pedemonte (2005) regarding the analysis of conjecturing and proving and translated it as follows;

> - *Structural analysis:* refers to the link between the structures of statements used in argumentations and in proofs. There is structural cognitive unity when statements used in the argumentation are also used in the proof. Otherwise, there is structural cognitive rupture.
> - *Referential analysis:* refers to the systems of reference used in argumentations and in proofs, that is, the systems of signs (drawings, calculations, algebraic expressions, etc.) and systems of knowledge (definitions, theorems, etc.) used. There is referential cognitive unity when some systems of signs or knowledge are used both in the argumentation and the proof. Otherwise, there is referential cognitive rupture. (p.149)

That is to say, structural analysis aims to examine whether the statements employed in the argumentation are also used in the proof. In such a case, it is called as structural cognitive unity. In the case of the absence of such a link between the statements, it is called as structural cognitive rupture (Pedemonte, 2005, as cited in Fiallo & Gutiérrez, 2017). Moreover, if the ways of inference have the same structure such as deduction, induction, and abduction in both argumentation and proof, it presents the structural continuity between them. For example, if some abductive steps of argumentation are used in proving, it refers to structural continuity (Boero et al., 2010). In referential analysis, it is scrutinized whether the systems of signs such as drawings and notations and the systems of knowledge such as theorems and definitions are used in both argumentation and proof processes. When used in both processes, the presence of referential cognitive unity is stated; otherwise, it refers to referential cognitive rupture (Boero et al., 2010; Pedemonte, 2005, as cited in Fiallo & Gutiérrez, 2017).

Based upon this categorization which was accompanied with the cK¢ model presented by Balacheff and Margolinas (2005), argumentation model of Toulmin (1958, 2003), and types of proof proposed by Marrades and Gutiérrez (2001), Fiallo and Gutiérrez (2017) unfolded four cases which were entitled as empirical cognitive unity, deductive cognitive unity, referential rupture and empirical structural unity, and structural rupture and referential unity. For example, when the case stated as having empirical cognitive unity was examined, it was seen that there is both empirical referential unity and empirical structural unity. Since the group used some notions such as backing, representation, operators, and controls in both production of conjecture and proving, there was empirical referential unity. Since they also had an empirical stance while explaining conjecture and proving attempts, there was empirical structural unity. As another example, in the case coded as referential rupture and empirical structural unity, the group worked empirically in both phases so that the presence of empirical structural unity was stated. Referential rupture was added since the group used new operators and controls while trying to present a deductive proof although the product was still a generic proof (Fiallo & Gutiérrez, 2017).

While framing such an analysis structure, the aim of Fiallo and Gutiérrez (2017) was to investigate the concepts of cognitive unity and cognitive rapture by approaching from two aspects. The first aspect was to examine how the presence or absence of a connection between the process of conjecture production and the process of its proof affects the type and structure of proof, such as deductive or empirical. The second aspect was to inspect whether the types of proofs progress by means of the intervention conducted in the study. As derived from the findings of the study of Fiallo and Gutiérrez (2017), cognitive unity might have an effect on the types of proof such as deductive or empirical. After the intervention in which the participants worked on conjecture-and-proof problems by using Cabri, Fiallo and Gutiérrez (2017) reported the development of students in terms of proving since they offered proofs coded as naïve empiricism at the beginning and then they started to offer the ones coded as deductive proofs.

2.1.1. Concepts related to Cognitive Unity

Another issue noticed in the literature review is that there are some other terms which have a similar nature with cognitive unity. For example, Fiallo and Gutiérrez (2017) used the term "conjecture-and-proof problems" (p.146). In more detail, such problems, which were expressed as a substantial factor while learning how to prove, involve two sections. In the former section, a conjecture is produced, and then the latter section covers the proof of it. It was seen that there are studies which have focused on students' reasoning process in both sections of the mentioned type of problems. Depending on the case, sections of these problems do not have to be sequential and some entwined occasions might be seen (Fiallo & Gutiérrez, 2017). Similarly, Arzarello et al. (1998) observed that students in both middle and high school levels presented two modalities which are "exploring/selecting a conjecture" and "concatenating sentences logically" (p.28) while working on the geometry tasks aiming to produce conjecture and prove. Besides, they used the term "process of exploration-conjecturing-proving" for such tasks. Although not labeled it as cognitive unity, Yevdokimov (2006) set a learning environment in which secondary school students were involved in a conjecturing process leading to refutations or proofs related to geometry concepts and concluded that many students are successful in carrying out both conjecturing and proving task when presented appropriately. It was also underlined that the utilization of a constructivist approach while helping students to produce conjectures is an effective way to develop students' learning of geometry concepts.

In addition, Edwards (1997) set forth the phrase "the territory before proof" as a metaphor to imply the set of possible precursors to proof which involves the process of discussing, thinking, and working on the statement at stake to provide support for the validity. This phrase does not ensure the existence of formal proof but refers to the basis for the extensive comprehension regarding the intended proof. In more detail, Edwards (1997) asserted five elements involved in the territory before proof. As the first element, it covers noticing or establishing process of any rule, invariance, or pattern by means of problem solving and exploration. The second element is the description of the regularity, rule, or pattern regardless of whether it is conducted informally or formally. To this end, students might describe what they have found verbally and by using notations, diagrams, figures, or pictures. The third one refers to the production of conjectures as well as the anticipation of the potential of pattern for generalization. As the fourth element, inductive reasoning, which refers to the control of whether the pattern or statement holds for the particular instances, was presented. The final element was stated as deductive reasoning by implying deductive proof and the presentation of why the reached generalization is valid. Since coming up with generalization regarding the issue based on the analysis of some particular cases constitutes the prominent element of the territory before proof (Edwards, 1997), it can be considered that there is a resemblance between the term territory before proof and the reasoning behind cognitive unity.

In addition to cognitive unity, another concept named as structural continuity was also uttered. When argumentation and proof have the same structure such as deductive, inductive, and abductive, it was coded as a structural continuity (Pedemonte, 2007a, 2011; Pedemonte & Buchbinder, 2011). By considering both structural and cognitive unity, Pedemonte (2007a) investigated the relation between argumentation and proof by asking open-ended problems to students in 12th and 13th grades in Italy and France in geometry domain. Based on the findings, Pedemonte (2007a) drew attention to the structural distance between argumentation and proof. Meanwhile, Pedemonte (2007a) offered a way to analyze structural continuity by means of combining Toulmin’s model with the cK¢ model of Balacheff and Margolinas (2005). Then, Pedemonte (2008) examined the same issue for algebra and concluded that structural distance between argumentation and proof was not the reason of the students’ difficulties, which is not the case when the domain was geometry in the previous study. Moreover, it was stated that generic examples work like a bridge between argumentation and proof so that both structural and cognitive unity turned out to be noticeable and the mentioned types of unities that are constituted between argumentation and proof facilitated the proving process (Pedemonte & Buchbinder, 2011).

2.1.2. Conjecture Production Process in Cognitive Unity

As seen, all of the mentioned cognitive unity related studies gave point to the process of conjecture production. According to Lannin et al. (2011), conjecturing process can be described as follows; "conjecturing involves reasoning about mathematical relationships to develop statements that are tentatively thought to be true but are not known to be true. These statements are called conjectures" (p.13). In line with the theorem definition of Mariotti et al. (1997) which unfolded that it is composed of statement, proof, and mathematical theory, Pedemonte (2007a) came up with the definition of conjecture given as follows; "a triplet: a statement, an argumentation and a system of conceptions" (p.28). During the argumentation emerged while producing conjectures, students are provided the opportunity to examine the ways which can be used to support the conjecture, notice a variety of issues pertaining to the conjecture, and come up with different conjectures through the given task (Antonini & Mariotti, 2008). The conjecturing process involves the presentation of examples or counterexamples related to the context and also it affects both defining the related concepts and proving process (Sinclair et al., 2012a). The sequential nature of the conjecturing process through the proving process is not quite simple since the problem solving phase encountered in conjecturing should encompass and promote the entailments of the theoretical system involved in the proofs such as definitions, axioms, and theorems (Pedemonte & Balacheff, 2016). In addition to leading to the production of statements, argumentation process, which involves a reasoning effort, leads students to consider and explore different circumstances and also reason the plausibility of the discussed statements (Mariotti et al., 1997). Moreover, the properties of the task in which students are expected to produce conjectures are also important. The tasks starting with the expression "prove that…" are not suitable since they did not present an opportunity to reach a conjecture since it was already given so that the unity becomes off the table. To avoid this, the task should involve some steps such as exploration, conjecture production, and reexploration (Garuti et al., 1998). Another point underlined in the studies is the need for the dynamic exploration of the given problem while producing conjectures and proving (Boero et al., 1996; Mariotti et al., 1997).

Based on functions, two types of argumentation were described, namely, constructive argumentation and structurant argumentation (Pedemonte, 2002a, as cited in Fiallo & Gutiérrez, 2017; Pedemonte, 2007a). The former one refers to the argumentation when it contributes to the production of conjecture and the latter one aims to justify a conjecture which has been already stated. According to Pedemonte and Buchbinder (2011), to offer examples in constructive argumentation and structurant argumentation might be a beneficial act in terms of promoting the proving process. Moreover, according to Pedemonte (2007b), in some cases, structurant argumentation might be an effective source to decrease the cognitive gap between argumentation and proof by means of facilitating the referential continuity between them.

Until this point, the studies related to cognitive unity were reviewed. As mentioned, the starting point of the present study is the construct of cognitive unity. Accordingly, the first aim of the study is to investigate how being involved in an argumentation which covers conjecture production process relates to the proof of the conjectures recently produced. Then, the focus of the study diverged from the cognitive unity to its main components, namely, argumentation and proof. In the following sections, the studies regarding argumentation and proof will be mentioned, respectively. What comes next is the literature review of argumentation with respect to the purposes of the study and its place in mathematics education.

2.2. Argumentation in Mathematics Education

Argumentation is a concept difficult to define in the context of mathematics education even though it is a frequently seen construct in mathematics classrooms (Duval, 1990, as cited in Reid & Knipping, 2010; Pedemonte, 2007a). Basically, Cross (2009) identified argumentation as “sharing, explaining, and justifying of mathematical ideas” (p. 908) by focusing on the actions taken during argumentation. According to Reid and Knipping (2010), argumentation can be described as the strategies, methods, and techniques used to discuss and address a mathematical claim. In a more comprehensive manner, Fiallo and Gutiérrez (2017) defined argumentation as “a discourse consisting of a sequence of verbal statements based on mathematical elements (definitions or properties, results of experiments, observations, etc.), organized with the aim of explaining how a conjecture was identified or convincing that it is plausible" (p.147). As seen, the lastly given two definitions touch upon not only the mathematical elements of argumentation by uttering some terms such as strategy and definition but also the functions of it by referring to some terms such as convince and discuss. As seen, there is not a consensus on neither the meaning of the term argumentation because of the diversity of its usages (Reid & Knipping, 2010) nor the characteristics of arguments (Pedemonte, 2007a). Moreover, as Toulmin et al. (1984) stated, convincing others about anything by proposing a claim and then justifying it is one of the goals of arguments. Another goal of the argument might be to find answers to the problems in which clear answers or solutions are not initially known. While the former is called as the advocacy which refers to the type of reasoning established to support the claim, the latter is called as the inquiry which is also another type of reasoning arranged to lead to the discovery.

Based on the social construction of knowledge, discussion conducted during a classroom activity has a critical role in learning (Mariotti et al., 1997). When students were involved in social interaction, it was seen that they started to get in charge of their own learning by being active and productive (Balacheff, 1999). Collaboration both with peers and with the experts is mentioned as a facilitator for promoting the conceptual understandings of students due to the numerous benefits to the overall structure of the instruction. Among these benefits, focusing on the content in a more thorough way, evoking the previous knowledge by means of argumentation, discussing alternative aspects of the concepts, proffering more than one solution for the problem, developing problem solving skills, increasing the quality of the discourse, and supporting higher level thinking of students can be listed (O'Donnell, 2006; Salomon & Globerson, 1989).

Conner et al. (2014a) explained the collective argumentation as follows, "participating in discussions in a distinctively mathematical way can be framed as collective argumentation, where collective argumentation involves multiple people arriving at a conclusion, often by consensus" (p.401). Brown (2017) remarked that collective argumentation "has the potential to create communicative spaces in the classroom where students have regular opportunities to 'represent', 'compare', 'explain', 'justify', 'agree' about and 'validate' their ideas" (p.186). Building on the findings of the studies in the literature, Brown (2017) summarized some conclusions; collective argumentation enhances the quality of mathematics education, helps students to deal with higher-order thinking skills, encourages the productive talks in the classroom and also it can be used effectively in mathematics classrooms. Moreover, according to the findings of the study of Brown (2017) which investigated the affordances and constraints of applying collective argumentation in terms of mathematics education, teachers might utilize collective argumentation to develop students' engagement with mathematics from cognitive, behavioral, and emotional aspects since students would have the chance to express and justify their ideas as well as compare with the ideas of others. After all, it can be inferred that teachers undertake a crucial role while managing a collective argumentation.

How teachers can support the collective argumentation in a mathematics classroom was framed by Conner et al. (2014a) under three headings which are to provide some components of the argument, to pose questions so as to unfold the parts of the argument, and to utilize some other promotive actions. In more detail, the first one is also called as the direct contributions since the teacher directly presented a component of the argumentation. Since who is contributing to the argumentation such as teacher or students or collaboratively was also focused in their study while preparing the argumentation structures, the first category regarding the support of teacher was apparent after the analysis. The second category regarding the support of teachers covers asking questions in a way that it demands action, not just an interrogative manner. In addition, the types of questions that can be asked were classified as to request a factual answer, a method, an idea, elaboration, and evaluation. As seen, the third one is a more inclusive category and it involves the kind of support to the collective argumentation in which neither a direct contribution to the argument as a component nor the mentioned type of questions were present. The sub-categories were listed as "directing, promoting, evaluating, informing, and repeating actions" (p.420). Particularly, directing aims to lead students' attention to the argument, promoting aims to lead exploration, evaluating is related to the assessment of the cases mathematically, informing aims to present necessary information in the argumentation, and repeating functions as a restatement what has been declared in the argumentation.

2.2.1. Toulmin's Model of Argumentation

According to the review of related literature, argumentation model of Toulmin (1958, 2003) is used to examine the arguments by pursuing various purposes in different disciplines since it was designed to be applicable to any discipline (Knipping, 2008; Knipping & Reid, 2015). Owing to this field-independent stance, Toulmin's model was devoted attention by a variety of disciplines such as medicine and computer science (Pedemonte & Balacheff, 2016). When the case is to examine argumentation related issues, Toulmin's model is a prominent tool to be used (Boero et al., 2010; Pedemonte, 2007a; Smith, 2010). Put differently, it was also noticed that argumentation model of Toulmin is one of the most used frameworks while analyzing argumentation (e.g., Barrier et al., 2009; Boero et al., 2010; Erkek, 2017; Erkek & Işıksal-Bostan, 2019; Fujita et al., 2010; Hollebrands et al., 2010; Krummheuer 1995; Metaxas et al., 2016; Pedemonte, 2007a; Pedemonte & Balacheff, 2016). Besides, it was seen that there are studies in which some adaptations to Toulmin's model were conducted depending on the context of studies (e.g., Bench-Capon, 1998; Conner et al., 2014a; Knipping, 2008; Reid & Knipping, 2010; Verheij, 2005).

More precisely, it can be used to examine the argumentation during both conjecturing and proving phases (Pedemonte & Buchbinder, 2011). While the argumentation model of Toulmin can be utilized to examine and analyze the arguments ranging from the exploratory ones to more deductive ones, the Habermas' construct can be used to investigate the proof process with respect to the epistemic, teleological, and communicative components (Boero et al., 2010). It was also observed in the studies of mathematics education that Toulmin's model is considered as a useful tool both in studying on formal and informal arguments in classroom (e.g., Conner et al., 2014a; Krummheuer 1995; Knipping 2002, 2008) and each student's proving process individually (e.g., Hollebrands et al., 2010; Inglis et al., 2007; Pedemonte, 2002a). The application of Toulmin's model in the scope of mathematics education was started with the study of Krummheuer (1995) which examined the argument in a mathematics classroom (Inglis et al., 2007; Metaxas et al., 2016). Then, it was used in some other strands in mathematics such as the kinds of reasoning (Conner et al., 2014b), argumentation process in undergraduate mathematics course while constructing definitions (Ubuz, Dinçer, & Bülbül, 2012), types of warrants such as inductive, deductive, and structural-intuitive (Inglis et al., 2007), teachers' arguments (Metaxas et al., 2016), the enrichment of argumentation model of Toulmin by aligned with the cK¢ model while investigating students' learning during argumentation (Pedemonte, 2007a; Pedemonte & Balacheff, 2016), the role of examples while proving process (Pedemonte & Buchbinder, 2011), the presence of cognitive unity or cognitive rupture related to trigonometry concept while using a dynamic geometry software by offering the conjunction with the cK¢ model (Fiallo & Gutiérrez,2017), to investigate the structures of argumentation (Erkek & Işıksal-Bostan, 2019; Knipping, 2003, 2004, 2008; Knipping & Reid, 2013, 2015; Reid & Knipping, 2010; Pedemonte, 2002b), the warrant component (Freeman, 2005; Nardi et al., 2012; Walter & Johnson, 2007), and the features of rebuttal (Verheij, 2005).

The basic form of argumentation model of Toulmin (1958, 2003) involves three components which are data (D), warrant (W), and claim (C), each of which has a different role throughout the argument (Fukawa-Connelly, 2014; Inglis et al., 2007; Metaxas et al., 2016). The meaning of data component is the justification of the claim, warrant is the statement which is used for connecting data with claim, and claim is defined as the statement of the speaker. Basic argumentation model of Toulmin (2003) was presented in the figure given below.

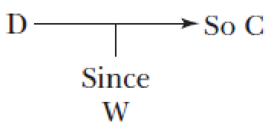


Figure 2. 2. Basic argumentation model of Toulmin (2003, p.92)

Toulmin (2003) stated that additional explanation might be required for argumentation model since there are various types of warrants and also warrants may have different amount of effect on the justification of conclusion. When the features of different arguments were considered, the model of argumentation becomes more complex. In this respect, in addition to data (D), claim (C), and warrant (W) in the basic argumentation model, qualifier (Q), rebuttal (R), and backing (B) components were involved in the model of argumentation as given in Figure 2.3.

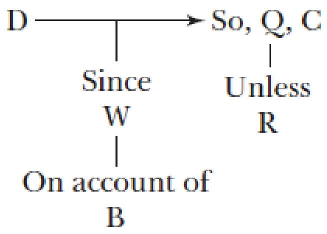


Figure 2. 3. Argumentation model of Toulmin (2003, p.97)

Moreover, Toulmin et al. (1984) listed four elements of an explicit argument as claim, ground, which is a term meeting the properties of the data component in the previously mentioned structure, warrant, and backing. Then, rebuttal and qualifier were also added and the final diagram regarding the argument was arranged as in Figure 2.4.

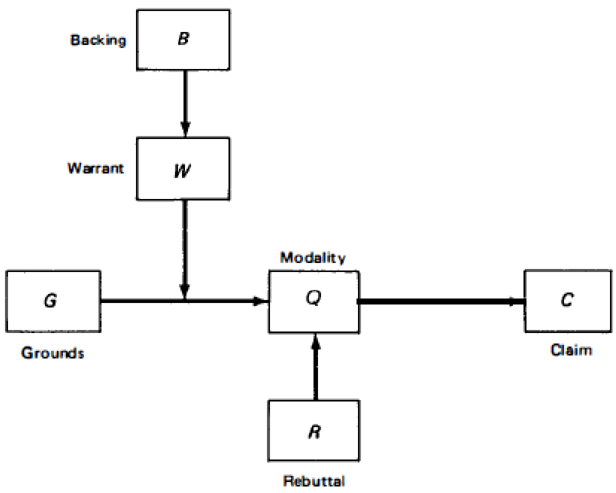


Figure 2. 4. Basic analytical diagram of an argument (Toulmin et al., 1984, p.98).

Based on Toulmin’s model (1958, 2003), Conner et al. (2014b) prepared the diagram of Toulmin’s argumentation model with descriptions of the components.

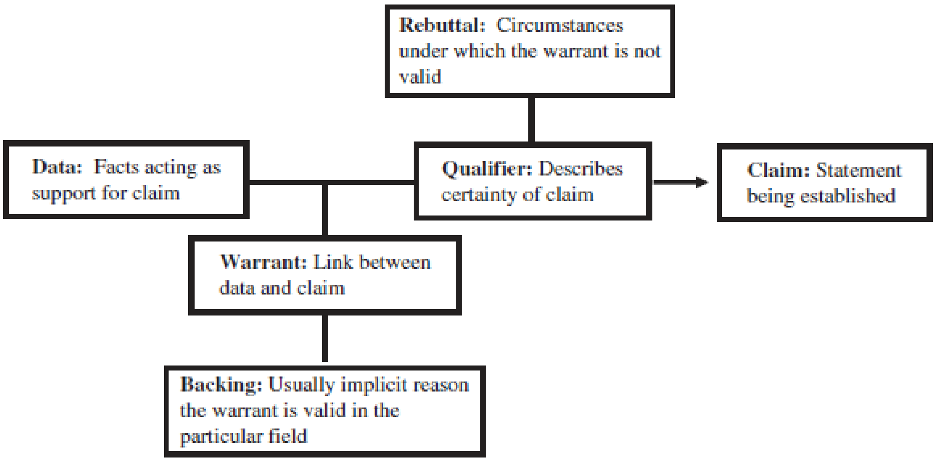


Figure 2. 5. Argumentation diagram based on Toulmin (2003) with descriptions of components (Conner et al., 2014b, p.184)

Another representation of the same model in the study of Boero et al. (2010) was displayed as given below.

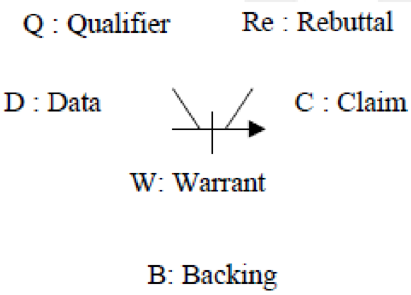


Figure 2. 6. Argumentation model of Toulmin (Boero et al., 2010, p.3)

As can be inferred from the structures displayed in previous figures, although the presentations of the model are quite similar, some different points were noticeable. For example, the locations of the rebuttal and qualifier components in figures are different. In this respect, it can be stated that it was common to observe studies offering different representations although they were based on the argumentation model of Toulmin (1958, 2003). Boero et al. (2010) presented an example for the application of this model by means of the answer of a student to question related to algebra. The question was “What can you say about $-a^2$ if a is an integer number different from 0?

Is it a positive or a negative number?" (p.3). The answer of the student with the addition of related components in parenthesis was presented below.

> $-a^2$ is a negative number (*claim*) because the square of each number is a positive number, but with minus it becomes a negative number (*warrant*)… unless the square is made for the whole number and the minus… in this case $-a^2$ is a positive number (*rebuttal*). No… this is impossible because $-a^2$ is different than the square of $(-a)^2$ (p.3-4)

Based on the analysis, the following representation of the answer was arranged by Boero et al. (2010).

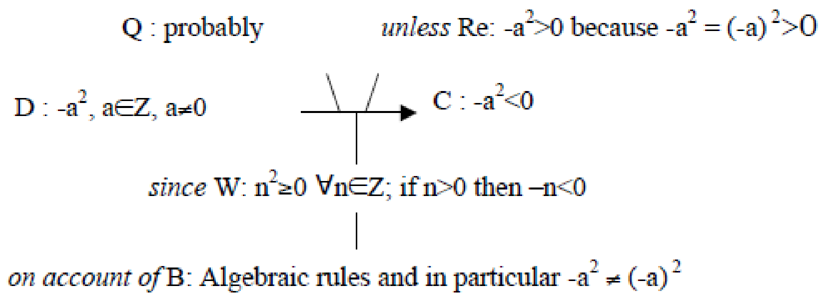


Figure 2. 7. The example for argumentation model of Toulmin (Boero et al., 2010, p.4)

While reviewing the related literature, it was noticed that the difficulty of distinguishing data from warrant in practice was stated in some studies (e.g., Erduran, 2007; Knipping, 2008). As Knipping and Reid (2013) stated, Toulmin also admitted that data and warrant might be difficult to differ depending on the context. Toulmin (1958) explained the distinction between the functions of data and warrant respectively as "in one situation to convey a piece of information, in another to authorise a step in an argument" (p.99). Similarly, van Eemeren, Grootendorst, and Kruiger (1984, as cited in Hitchcock, 2003) declared that the main difference between data and warrant is their functions in the argument and stated the function of data as "providing the basis of the claim" (p.205) and the function of warrant as "justifying the step from this basis to the claim" (p.205). Thus, to examine the definitions of data and warrant in the literature in detail, the method of Brinkerhoff (2007) which outlined the components of Toulmin's model by focusing separately on their forms and functions in the argument was used. That is, the definitions of data and warrant were handled in two sections as form and function while the definitions of the remaining four components (claim, backing, rebuttal, and qualifier) were examined without a partition. First of all, the definitions of the data component given in the studies were presented in Table 2.1.

Table 2. 1

Definitions of data stated in the studies

	Statements in the definitions of data
Form of data	Facts (Brinkerhoff, 2007; Conner et al., 2014a, 2014b; Erduran, Simon, & Osborne, 2004; Hollebrands et al., 2010; Knipping, 2008; Knipping & Reid, 2013, 2015; Metaxas et al., 2016; Toulmin, 1958, 2003; Verheij, 2005)
	Evidence (Hollebrands et al., 2010; Metaxas et al., 2016; Nardi et al., 2012; Stephan & Rasmussen, 2002)
	A specific piece of information (Brinkerhoff, 2007)
	General information (Hollebrands et al., 2010)
	Method or mathematical relationships (Rasmussen & Stephen, 2008)
	Undoubted statements (Krummheuer, 1995, 2007)
	Statements (Van Ness & Maher, 2018)
Function of data	Foundation of the claim/argument (Brinkerhoff, 2007; Fukawa-Connelly, 2014; Nardi et al., 2012; Metaxas et al., 2016; Toulmin, 1958, 2003)
	The basis of the claim (Hitchcock, 2003)
	Conclusion is grounded (Krummheuer, 1995)
	The inference of the claim/ argument (Krummheuer, 2007; Van Ness & Maher, 2018)
	Support for claim (Conner et al., 2014a, 2014b; Erduran et al., 2004; Knipping & Reid, 2013; Metaxas et al., 2016; Verheij, 2005)
	Lead to the conclusions (Rasmussen & Stephen, 2008)
	Justify the claim/conclusion (Boero et al., 2010; Hollebrands et al., 2010; Knipping, 2008; Knipping & Reid, 2015; Pedemonte, 2007a, 2008)

Based on all definitions given in Table 2.1, a comprehensive definition for the data component, which was given in the previous chapter, was arranged. In general terms, it can be seen that there is a variety in the concepts situated in the form section in Table 2.1 while the function section focused the same notion of data which is the basis for the conclusion even though it was stated with different phrases. Similar to the data component, the definitions of warrant given in some studies were listed below.

Table 2. 2

Definitions of warrant stated in the studies

	Statements in the definitions of warrant
Form of warrant	A definition, a rule, an example, or an analogy (Nardi et al., 2012)
	A general statement or principle (Brinkerhoff, 2007; Conner et al., 2014b)
	A principle or a rule (Boero et al., 2010; Pedemonte, 2007a, 2008)
Function of warrant	Justify the relationship/connection between data and conclusion (Boero et al., 2010; Fukawa-Connelly, 2014; Nardi et al., 2012; Pedemonte, 2007a, 2008)
	Explain how the data leads to the claim (Brinkerhoff, 2007; Rasmussen & Stephen, 2008; Stephan & Rasmussen, 2002)
	Provide link between/connect data and claim (Conner et al., 2014a, 2014b; Erduran et al., 2004; Hollebrands et al., 2010; Rasmussen & Stephen, 2008)
	Rule of inference that authorizes/give reason to legitimacy of the step from the data to the claim (Boero et al, 2010; Brinkerhoff, 2007; Knipping, 2008; Knipping & Reid, 2013, 2015; Krummheuer, 1995; Metaxas et al., 2016; Pedemonte, 2007a, 2008; Toulmin, 1958, 2003; Verheij, 2005)
	Provide more evidence to clarify an argument (Van Ness & Maher, 2018)
	Act as bridges between data and claim (Boero et al., 2010; Knipping, 2008; Knipping & Reid, 2013, 2015; Pedemonte, 2007a, 2008; Toulmin, 1958, 2003)

Grounded on the definitions in Table 2.2, it can be inferred that warrant is any statement which justifies the connection between data and conclusion. As the next component, definitions of claim/conclusion stated in the literature were listed in Table 2.3.

Table 2. 3

Definitions of claim/conclusion stated in the studies

Statements in the definitions of claim/conclusion
Statements being argued (Metaxas et al., 2016)
Statements being established (Brinkerhoff, 2007; Conner et al., 2014a, 2014b; Krummheuer, 1995)
Statements being justified (Brinkerhoff, 2007; Hollebrands et al., 2010)
Statements being inferred from data (Van Ness & Maher, 2018)
An assertion put forward publicly for general acceptance or basic convictions (Erduran et al., 2004)
The statement of the speaker (Boero et al., 2010; Pedemonte, 2007a, 2008)

According to Van Ness and Maher (2018), claim is a conclusion which can be declared before or after the data in the flow of an argument. In this perspective, a claim might be either a solution to a problem or a mathematical statement needed to be clarified (Rasmussen & Stephen, 2008). It was noticed that Toulmin (2003) used the terms claim and conclusion as having the same meaning. Similarly, claim was explained as conclusion in the studies of Stephan and Rasmussen (2002) and Rasmussen and Stephen (2008). However, according to the studies of Knipping (2008) and Knipping and Reid (2013, 2015), the term claim was used when data and warrant were not provided and the term conclusion was used in the case that data and warrant were provided.

Since data, warrant, and claim are the components of basic argumentation model of Toulmin, the studies using Toulmin's model generally involve those components. However, it was seen that many studies do not cover backing, rebuttal, and qualifier or not mention about them in detail. Thus, the definitions stated in the studies for the latter ones are not covered as much as the definitions of the former ones. The definitions of backing presented in the literature were stated in Table 2.4.

Table 2. 4

Definitions of backing stated in the studies

Statements in the definitions of backing
Support the warrant by suggesting why it is valid/ by stating an additional information, further evidence, justifications or reasons (Boero et al., 2010; Erduran et al., 2004; Fukawa-Connelly, 2014; Hollebrands et al., 2010; Krummheuer, 1995; Metaxas et al., 2016; Nardi et al., 2012; Pedemonte, 2008)
Support the warrant if it is in doubt (Brinkerhoff, 2007; Knipping, 2008; Knipping & Reid, 2013, 2015; Van Ness & Maher, 2018)
Explain why the warrant has authority (Boero et al., 2010; Brinkerhoff, 2007; Krummheuer, 1995, 2007; Pedemonte, 2008; Rasmussen & Stephen, 2008; Stephan & Rasmussen, 2002)
Implicit reason the warrant is valid in a particular field (Conner et al., 2014a; Verheij, 2005)

As seen from Table 2.4, the common ground of all definitions of backing is to support warrant. However, the instances where backing is uttered have minor differences such as explaining the authority of warrant and in the case that warrant is in doubt. The descriptions of rebuttal were examined via the review of the literature and summarized in Table 2.5 given below.

Table 2. 5

Definitions of rebuttal stated in the studies

Statements in the definitions of rebuttal
Conditions under which conclusion/claim would not hold, exceptions/potential refutations of the conclusion (Erduran et al., 2004; Fukawa-Connelly, 2014; Nardi et al., 2012)
Conditions/circumstances under which the warrants would not be valid, exceptions to the applicability of the warrant/the rule as warrant (Boero et al., 2010; Conner et al., 2014a, 2014b; Hollebrands et al., 2010; Metaxas et al., 2016; Pedemonte, 2008)
Conditions of exception for the argument (Erduran et al., 2004; Toulmin, 1958; Verheij, 2005)

In the light of the definitions of rebuttal in Table 2.5, it was summarized that rebuttal represents the statements which weaken the overall stance of the argument. For example, when rebuttals are inserted as an exception regarding the statement in the warrant, the force of the warrant would be weakened (Boero et al., 2010; Pedemonte, 2008). As the final component, the definitions of qualifier in the studies were stated in Table 2.6.

Table 2. 6

Definitions of qualifier stated in the studies

Statements in the definitions of qualifier
Express the degree of confidence of the conclusion (Erduran et al., 2004; Fukawa-Connelly, 2014; Nardi et al., 2012)
Describe the certainty of claim (Conner et al., 2014a, 2014b)
Express the strength of the argument/the claim as determined by the warrant (Boero et al., 2010; Hollebrands et al., 2010; Metaxas et al., 2016; Pedemonte, 2008; Verheij, 2005)

The qualifier or modal qualifier may be represented implicitly or explicitly by stating a word such as certainly or probably in an argument (Hollebrands et al., 2010; Metaxas et al., 2016). Moreover, Inglis et al. (2007) underlined the importance of the modal qualifiers in the arguments in terms of presenting proper justification for the conclusion. The following phrase can be seen as the indication for the need of the qualifier component in the scope of the argumentation model of Toulmin; “the restricted form of Toulmin’s (1958) scheme used by earlier researchers to model mathematical argumentation constrains us to think only in terms of arguments with absolute conclusions” (Inglis et al., 2007, p. 17). Pedemonte and Buchbinder (2011) also emphasized the necessity of such a component while examining the argument. On the other hand, compared to the basic three components of the argument, the additional components in Toulmin’s model which are qualifiers, rebuttals, and backings are considered as not affecting the type of the reasoning. For example, the qualifier ‘certainly’ in an inductive structure of an argument does not alter the nature of the reasoning in the argument (Conner et al., 2014b).

2.2.2. Studies regarding Toulmin’s Model of Argumentation

As stated previously, there are some studies which did not use Toulmin’s model directly and conducted some modifications in the light of the purposes and context of their study. Since the arguments are considerably complicated processes, Toulmin’s model of argumentation may require some adaptations (Conner et al., 2014a). For example, Conner et al. (2014b) used the argumentation model of Toulmin and reflected the issue that who stated the content of components in the structure as shown in Figure 2.8 given below.

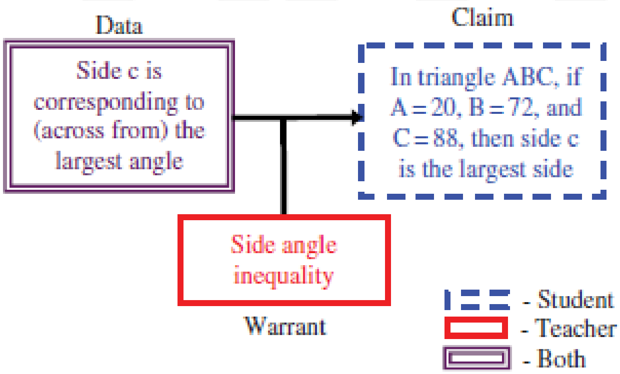


Figure 2. 8. Modified diagram of an argument from the study of Conner et al. (2014b, pp.185)

As seen from Figure 2.8, Conner et al. (2014b) focused on who stated the sentences in the argumentation process and used different colors to represent. That is, the component offered by a student was represented with the blue dashed line, the component stated by the teacher was displayed with a single red line, and the component given by both students and teacher was represented by a double purple line. The colors used herein were not attributed to a particular component so that a data component with the blue dashed lines can be seen in the case that it is stated by a student. By offering such a differentiation, Conner et al. (2014b) provided the opportunity to examine the roles of students and teacher in argumentation.

Although Toulmin (2003) did not focus on the categorization regarding rebuttals, Verheij (2005) elaborated on the nature and structure of rebuttals and introduced five types of rebuttal. According to Verheij (2005), argumentation model of Toulmin covers five cases which can be argued against;

> 1. The data D
> 2. The claim C
> 3. The warrant W
> 4. The associated conditional 'If D, then C' that expresses the bridge from datum to claim.
> 5. The associated conditional 'If W, then if D, then C' that expresses the bridge between warrant and the previous associated conditional (Verheij, 2005, p.360)

Moreover, Verheij (2005) named each idea against these five cases as a type of rebuttal and represented them via the following layouts of arguments, respectively.

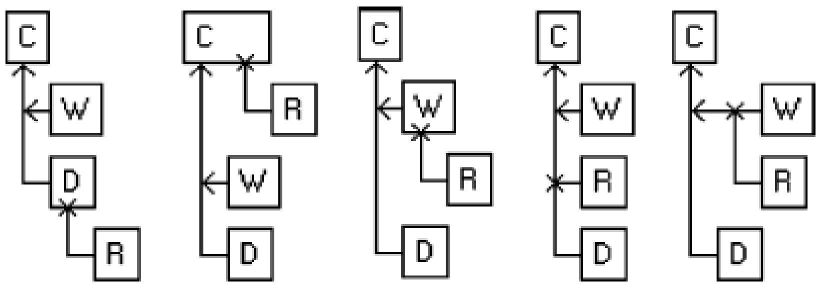


Figure 2. 9. Types of rebuttals stated by Verheij (2005, p.360)

As displayed in Figure 2.9, the first three types of rebuttal provided by Verheij (2005) are quite clear since they directly affect a specific component of argumentation. Particularly, the first type of rebuttal can be explained as statements against data. The second type of rebuttal is any statement argued against claim (or conclusion depending on the argumentation stream) and it corresponds to the defeat of conclusion which was also mentioned as a function of rebuttal in the explanation of Toulmin (2003). The third type of rebuttal refers to the statements against warrant and it corresponds to the defeat of warrant authority characteristics of rebuttal as stated by Toulmin (2003). However, the fourth and the fifth type of rebuttals can be regarded as more confusing compared to previous ones. The fourth type of rebuttal is a statement proposed against "the connection between data and claim" (Verheij, 2005, p.361). The fifth type of rebuttal was described as "an attack against the warrant's applicability" (Verheij, 2005, p.361). That is, warrant functions as a bridge between data and claim, but warrant cannot provide the justification role in the existence of the fifth type of rebuttal. Moreover, similar to the second and the third types of rebuttals, the last type of rebuttal which is the defeat of warrant applicability was also mentioned in the description of rebuttal stated by Toulmin (2003). According to Verheij (2005), the remaining two types of rebuttals (the first and the fourth one) were not covered through the description of rebuttal of Toulmin (2003).

To be more precise, how Verheij (2005) instantiated these five types of rebuttal were presented by using the commonly cited example of Toulmin (2003) regarding the components of argumentation model. Before presenting the rebuttal examples of Verheij (2005), the example of Toulmin (2003) which was given to explain the layout of the argument was presented in Figure 2.10. As it is seen, this is a reflection of Toulmin's argumentation model given in Figure 2.3 previously.

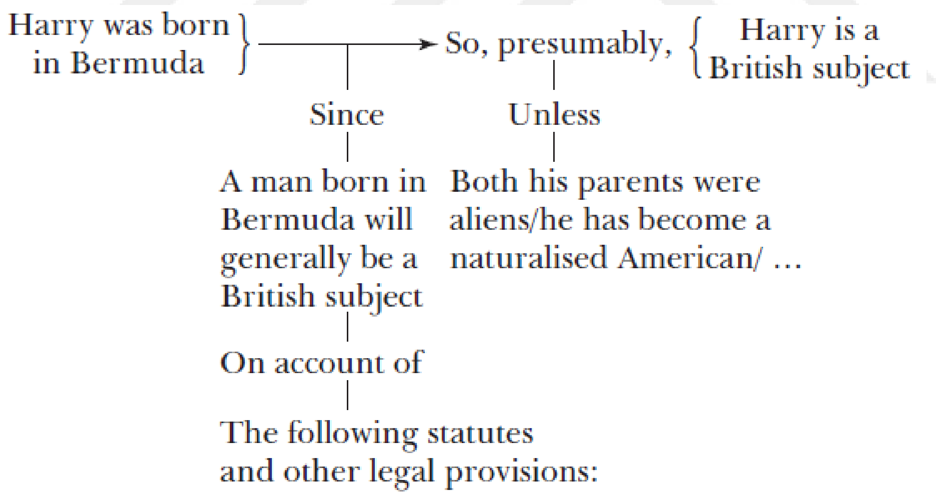


Figure 2. 10. Toulmin's example (2003, p.97)

The phrases Verheij (2005, p.360-361) expressed as the examples of rebuttal types based on the example of Toulmin in Figure 2.10 were summarized in the following table.

Table 2. 7

Examples for types of rebuttal stated by Verheij (2005, p.360-361)

Case	**Example of Toulmin**	**Example of Verheij for rebuttal types**
Data	Harry was born in Bermuda	Harry was born in London
Claim	Harry is a British subject	Harry has become a naturalized American
Warrant	A man born in Bermuda will generally be a British subject	Those born in Bermuda are normally French
The connection between data and claim	The connection between Harry being born in Bermuda and Harry being a British subject.	Harry has become a naturalized American
The warrant's applicability	The applicability of the warrant that a man born in Bermuda will generally be a British subject	Harry's parents both being aliens

Another study which modified Toulmin's model was conducted by Bench-Capon (1998). The modified version of the model of Toulmin, which was used to investigate the implementation of a dialogue game, was given in Figure 2.11.

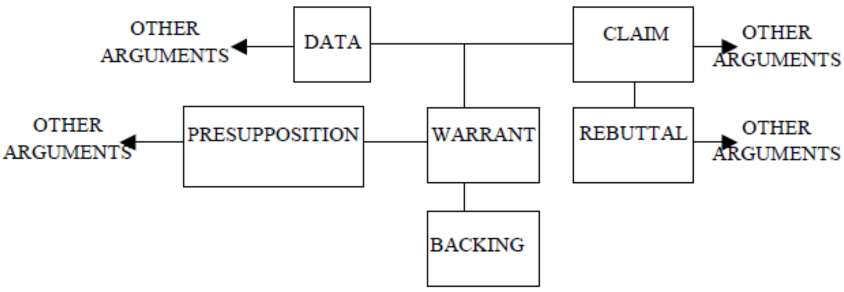


Figure 2. 11. Revised argumentation model (Bench-Capon, 1998, p.7)

As can be seen from Figure 2.11, Bench-Capon (1998) excluded the qualifier component whilst offered a new component which was called as presupposition. In more detail, the details of the presupposition component were put forward as "supposed to represent propositions assumed to be true in the context, and so which do not need to be discussed but which can be made explicit if required" (p.7). Besides, as represented with the arrows pointing other arguments in Figure 2.11, the linkages among arguments were considered by Bench-Capon (1998). In other words, the claim of an argument might function as the data of another argument (Bench-Capon, 1998).

In a similar vein, the idea that the conclusion of an argument may be the data of the following argument was taken into consideration in other studies (e.g., Conner et al., 2014a; Knipping, 2004, 2008; Krummheuer, 1995, 2007). For example, Knipping (2008) offered to use a component called as data/conclusion so as to represent the phrases which are both conclusion of an argument and the data of the next one. In this respect, Knipping (2008) referred that the data/conclusion component might be considered as an indication for the transition to a new argument. Similar to the data/conclusion component, Conner et al. (2014a) noticed that some statements function in favor of two components. Thus, they labeled some statements as data/claim and warrant/claim while constructing the structure of argumentation. Besides, Knipping (2008) employed a new component entitled as target conclusion which was described as "the final conclusion of the argumentation" (p.434). Another point worth to mention herein is that Knipping (2008) used some components of Toulmin's model by combining in the schematics representation of argumentation. For example, they used a circle to represent conclusion or data and also used a diamond to represent warrant or backings, which will be presented in detail in the following pages. This application was followed by some other studies which are based on the model of Knipping (2008) such as the studies conducted by Erkek and Işıksal-Bostan (2019), Knipping and Reid (2013, 2015), and Reid and Knipping (2010).

Another study which used Toulmin's model was conducted by Conner et al. (2014b) to investigate the reasoning process and particularly types of reasoning in collective argumentation. To this end, Conner et al. (2014b) offered a model by combining the perspectives of Reid and Knipping (2010) and Toulmin (1958, 2003). In more detail, Reid and Knipping (2010) described different types of reasoning which are inductive reasoning, deductive reasoning, reasoning by analogy, and abductive reasoning by reinterpreting the study of Peirce (1956). To differentiate the mentioned types of reasoning, Reid and Knipping (2010) suggested to examine how case, rule, and result were used and to decide about the type of reasoning based on the relations between them. According to the definition of Reid and Knipping (2010), case is “a specific observation that a condition holds” (p.83). For example, the statement “a square is a rectangle” is a case where being a rectangle is the condition. Rule is “a general proposition that states that if one condition occurs, then another one will also occur” (Reid & Knipping, 2010, p.83). For example, the statement “a rectangle is a quadrilateral” is a rule and the conditions being a rectangle and being a quadrilateral are associated. Result is “a specific observation, similar to a case, but referring to a condition that depends on another one linked to it by a rule” (Reid & Knipping, 2010, p.83). Thus, the statement “a square is a quadrilateral” is the result for the mentioned example. It is also an example of deductive reasoning since a rule and a case led to a result. However, in inductive reasoning, a case and a result imply a rule. In a similar way, in abductive reasoning, a rule and a result imply to a case. In reasoning by analogy, to state something about an unfamiliar situation, a familiar situation is used. Unlike the first three types of reasoning, a case or a rule may be linked to another case or another rule by analogy (Reid & Knipping 2010). Thus, Conner et al. (2014b) adapted types of reasoning stated by Reid and Knipping (2010) to the argumentation model of Toulmin (1958, 2003) as presented in Figure 2.12.

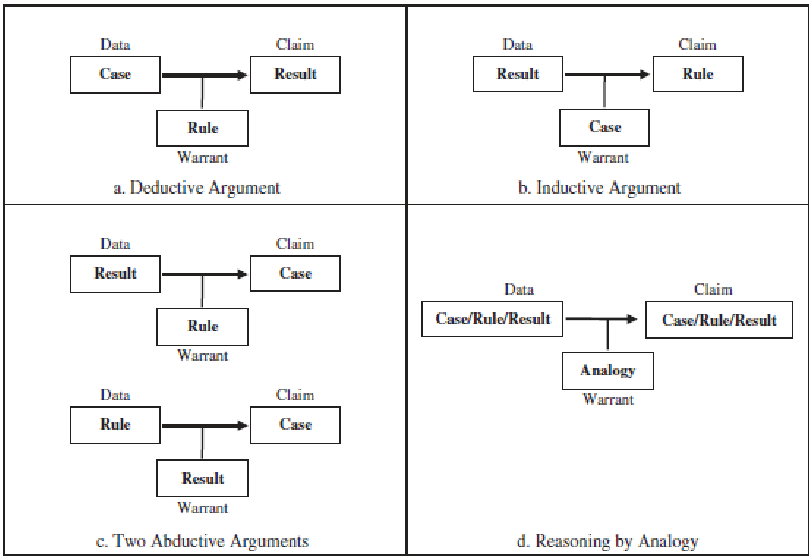


Figure 2. 12. Types of reasoning (Conner et al., 2014b, p.186)

As seen in Figure 2.12, for each type of reasoning, the core components of Toulmin's model were associated with the concepts of case, rule, and result in the study of Reid and Knipping (2010). For example, in deductive reasoning, data is a case, warrant is a rule, and claim is a result. Moreover, Conner et al. (2014b) used only core components since they concluded that the remaining components of Toulmin's model which are qualifier, rebuttal and backings do not have an effect on the type of reasoning. Conner et al. (2014b) developed this structure to identify types of reasoning in the episodes of a collective argumentation and analyzed how it works in a 9th grade mathematics course in which students were studying about triangles, quadrilaterals, and polygons.

The argumentation model of Toulmin (1958, 2003) has been subject to some criticisms (Conner et al., 2014b; Mariotti et al., 2018; Pedemonte & Balacheff, 2016). For example, it was criticized since it was frequently used to examine the arguments which are deductive in nature. The reason behind this situation was stated as the descriptions of the warrant component. In the case that warrant is explained by using the terms rule, principle, definition, algorithm or formula functioning as the bridge between data and claim, it seems that the argument takes a stand in a deductive way. However, all of the arguments in mathematics are not necessarily deductive (Conner et al., 2014b; Inglis et al., 2007). Moreover, the use of the reduced version of Toulmin's model might lead to considering the argumentation as covering the absolute conclusions only (Inglis et al., 2007). To determine students' difficulties in proof originated from the devoid of structural continuity between argumentation and proof, Toulmin's model might be effectively used. However, this model was attained as limited to investigate the reasons of such difficulties and the possible ways to overcome them; hence, further analysis with different tools was needed (Mariotti et al., 2018).

Some criticisms which are comparatively pertinent from a mathematical aspect were listed by Pedemonte and Balacheff (2016). One of them is that the knowledge base of the arguers is occasionally disregarded in the structure of argument. Another criticism is that warrants are ambiguous in some cases when the rule used is not described explicitly. They also mentioned the case that Toulmin's model has some deficiencies in terms of explaining the complex nature of the knowledge system and proffered the integration of the cK¢ model into Toulmin's model to deal with this idea.

Another issue related to Toulmin's model is that it unfolds the structure of an argument by means of presenting the locations of components. Although it was stated in some studies that Toulmin's model does not describe the quality of argument, an in-depth investigation of the content and nature of particular components might be explanatory. For example, regarding the quality of arguments, the examination of warrant and backing components can be conducted. Besides, it was also asserted that an argumentation schema might be used to examine the quality of the argument since the type of argumentation schema can be considered as an indicator regarding the rationale of the argument (Metaxas et al., 2016).

By considering such criticisms of Toulmin's model given in the literature, it was decided to use the adapted versions of Toulmin's model in this study. For example, the components involved in Toulmin's model were reconstructed since the structure of complex argumentation process was focused. Moreover, the functions of rebuttal were examined by using the study of Verheij (2005) as the base. By utilizing the adapted version of the mentioned model, the global argumentation structures were also taken into consideration in the study. In the next section, the argumentation structures in the literature were reviewed.

2.2.3. Argumentation Structures

The relations between data and conclusion in a simple argument are immediate and direct, but advanced mathematics contents involve complicated arguments (Fukawa-Connelly, 2014). Moreover, the notions global argument and local argument reflect the relationship between a claim aimed to prove and the necessary sub-proofs of it. Fukawa-Connelly (2014) mentioned about the proof of Lagrange's theorem to exemplify this situation. While proving Lagrange's theorem, some lemmas are also required to prove. The proof of each lemma might be called as a local argument while the proof of Lagrange's theorem is considered as the global argument. In other words, each sub-proof corresponds to local arguments while the whole proof process corresponds to the global argument. Conner et al. (2014a) called that preparing the diagram of an argument, which was named as argumentation structure by Reid and Knipping (2010), works like a sieve. In more detail, argumentation structures center upon the discussions distinctively related to mathematics and do not cover any other unrelated activity occurred within the classroom or group. By this way, the investigation of a particular issue in the argumentation might be examined in a more detailed way such as the examination of how the teacher can support the collective argumentation by Conner et al. (2014a).

The components of an argument are taken into consideration based on a single argument. However, any argument might be the first step of another argument and the series of arguments might continue in such a way, that is, the presence of a single argument is not mostly the case. In addition, the arguments do not have to proceed in a linear manner. There might be reasoning which moves backward and forward in the ongoing argumentation process (Toulmin et al., 1984). Due to the complicated nature of some arguments, any statement coded as claim might be the data of another sub-argument (Conner et al., 2014a).

Regarding global argumentation structures, Reid and Knipping conducted a series of studies which are Knipping (2003, 2004, 2008), Knipping and Reid (2013; 2015, 2019), and Reid and Knipping (2010). Toulmin's model of argumentation constituted the foundation of the mentioned studies (Reid & Knipping, 2010). In these studies, Toulmin's model was used while analyzing distinct argumentation streams in a proving discourse in a classroom, but the overall structure of arguments could not be deduced by using this model. Therefore, Toulmin's argumentation model was extended to display the argumentation structure of a classroom discussion as a whole and a schematic representation was proposed. In this manner, Knipping (2008) presented a three-stage process for the analysis of argumentation structure in classrooms:

> - reconstructing the sequencing and meaning of classroom talk (including identifying episodes and interpreting the transcripts);
> - analyzing arguments and argumentation structures (reconstructing steps of local arguments and short sequences of steps which form ‘‘streams’’; reconstructing the global structure); and
> - comparing local argumentations and comparing global argumentation structures, and revealing their rationale (p. 431)

Based on this three-stage process, Knipping (2003, 2008) arranged and then classified the global argumentation structures as the source-structure and the reservoir-structure. Then, Reid and Knipping (2010) added two types of global argumentation structures which are the spiral-structure and the gathering-structure and summarized four types of global argumentation structures by citing examples and comparisons. Knipping and Reid (2013, 2015) also explained and instantiated the source-structure and the spiral-structure in their subsequent studies. In addition, based on this classification, Erkek (2017) investigated the nature of argumentation structures of prospective middle school mathematics teachers during geometry tasks and also Erkek and Işıksal-Bostan (2019) centered upon this issue by considering a technology enhanced environment particularly. In both of the last mentioned two studies, there was no argumentation structure which can be labeled under the gathering-structure, but there were argumentation structures which fit into the remaining three structures. Moreover, Erkek (2017) added two types of argumentation structures which are the line-structure and the independent-structure since some of the argumentation structures emerged in the study did not fall to any type of argumentation structures stated by Reid and Knipping (2010). To sum up, six types of global argumentation structures stated in the literature, which are the source-structure, the spiral-structure, the reservoir-structure, the gathering-structure, the line-structure, and the independent arguments-structure, were discussed below.

2.2.3.1. Source-structure

As the first type of global argumentation structure, the characteristics of the source-structure were explained by Reid and Knipping (2010) as follows:

> - Argumentation streams that do not connect to the main structure.
> - Parallel arguments for the same conclusion.
> - Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream.
> - The presence of refutations in the argumentation structure (p.180-181)

In addition, it was stated that the lack of explicitly stated data or warrants is a frequent situation in the source-structure. Moreover, Reid and Knipping (2010) described it metaphorically as "arguments and ideas arise from a variety of origins, like water welling up from many springs" (p.180) and specified the discriminating characteristics of the source-structure with the term "funnelling effect" (p.181). More specifically, there are parallel argumentation streams originated from different sources at the beginning section of the argumentation schema and then argumentation is funneled towards one argumentation stream covering the final conclusion through the end of the argumentation schema. As an example, an overall argumentation structure coded as the source-structure from the study of Reid and Knipping (2010) was presented in Figure 2.13.

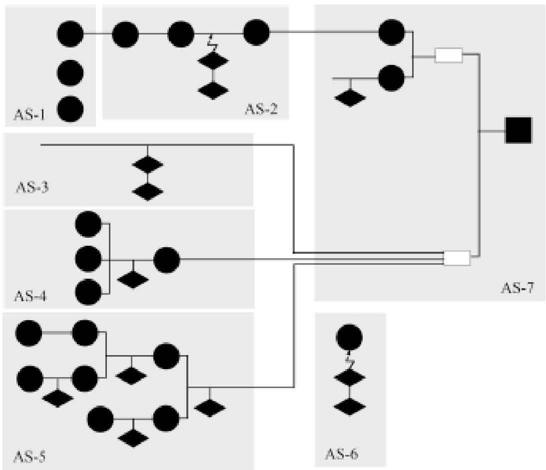


Figure 2. 13. The source-structure from the study of Reid and Knipping (2010, p.182)

The schema given in Figure 2.13 represents the argumentation process of a 9th grade classroom in Germany while dealing with the proof of the Pythagorean Theorem. In more detail, AS-6 is an argumentation stream which is not connected to the main structure and involves refutations. AS-3, AS-4, and AS-5 are parallel arguments leading to the same conclusion. There is more than one datum which is also the conclusion of another argumentation stream in AS-5 and AS-7 (Reid & Knipping, 2010). These features fulfill the conditions for being labeled as the source-structure.

2.2.3.2. Spiral-structure

Another type of structure Reid and Knipping (2010) pointed out is the spiral-structure and it has the same four characteristics with the source-structure, which was stated previously. The lack of explicitly stated data or warrant is also seen in this type but not as common as the source-structure. The issue which distinguishes them is where parallel arguments are located in the layout of argumentation. In the source-structure, there are parallel arguments at the beginning, and then it funnels through one stream, which leads to the final conclusion. On the other hand, in the spiral-structure, parallel arguments reach the final conclusion repeatedly via different methods (Reid & Knipping, 2010). An argumentation structure named as the spiral-structure from the study of Reid and Knipping (2010) was explained below.

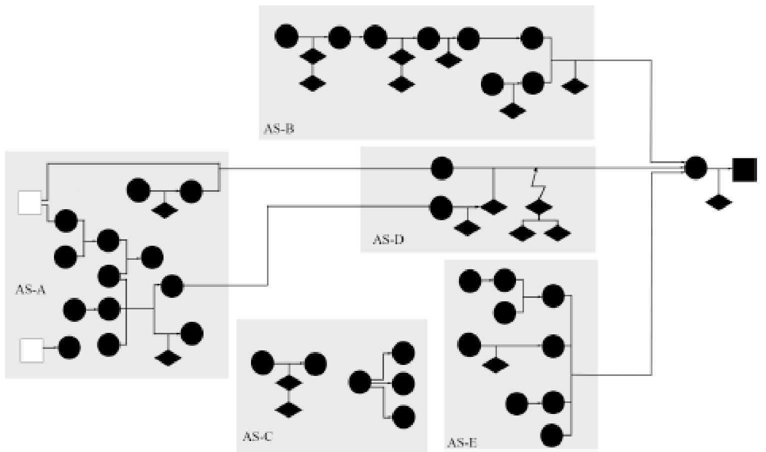


Figure 2. 14. The spiral-structure from the study of Reid and Knipping (2010, p.188)

The example of the spiral-structure in Figure 2.14 is from a 9th grade classroom in Canada and covers the argumentation process while they were discussing why two diagonals which bisect each other perpendicularly refer to a rhombus. As seen from Figure 2.14, AS-C is an argumentation stream which is separate from the main structure. AS-B, AS-D, and AS-E are parallel argumentation streams reaching the same conclusion. AS-A, AS-B, and AS-E meet the characteristics related to involving more than one datum as conclusions of another argumentation stream. Lastly, it can be seen that a refutation is present within AS-D (Reid & Knipping, 2010). As seen, these are the features needed to classify an argument under the spiral-structure.

2.2.3.3. Reservoir-structure

The third type of argumentation structure outlined in the studies of Knipping (2003, 2008) and Reid and Knipping (2010) is the reservoir-structure. According to Reid and Knipping (2010), there are intermediate target conclusions which divide the overall argumentation into distinct and independent parts in the reservoir-structure. The target conclusions which represent the transition from the first part to next part of the argumentation were described metaphorically as reservoirs that "hold and purify water before allowing it to flow on the next stage" (Reid & Knipping, 2010, p.185). Most of the characteristics listed for the source-structure and the spiral-structure are not present for the reservoir structure. For example, while there are not refutations in the reservoir-structure, argumentation streams which involve more than one datum as the conclusions of another argumentation can be seen in the reservoir-structure. The key characteristics of this type of structure is that reasoning in the argumentation sometimes flows backward and then forward to have further support for the desired conclusion by data (Knipping, 2008; Reid & Knipping, 2010). The following argumentation schema from the study of Reid and Kipping (2010) serves as an example for the reservoir-structure.

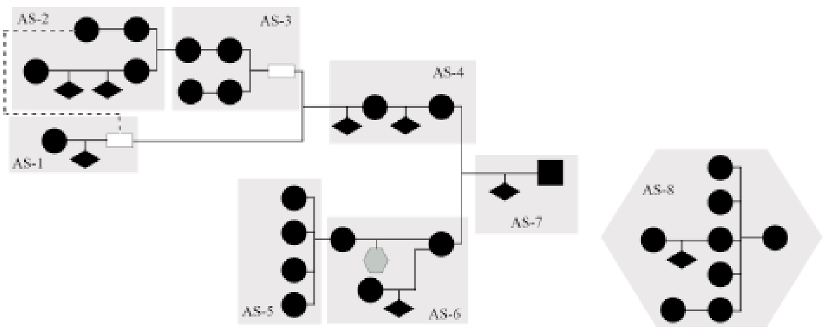


Figure 2. 15. The reservoir-structure from the study of Reid and Knipping (2010, p.186)

The example given for the reservoir-structure in Figure 2.15 is originated from a French classroom involving level 4 students which correspond to 13-14 ages while they were working on the proof of the Pythagorean Theorem. As can be seen in Figure

2.15, AS-1, AS-2, and AS-3 constituted a reservoir and also AS-5, AS-6, and AS-7 formed another closed structure in the following process of the argumentation. While the reservoir in the first part involves backward reasoning which was shown with the dashed line, the second part has a structure flowing forward (Reid & Knipping, 2010).

2.2.3.4. Gathering-structure

The last type of structure entitled by Reid and Knipping (2010) is the gathering-structure. As the name suggests, a large amount of data was gathered to reach several conclusions. Moreover, the conclusions are not evident beforehand, and similarly, all data are not given in advance and new data are presented when required. Unlike the source-structure and the spiral-structure, there are not parallel arguments for the same conclusion and separate streams from the main structure. Different from the reservoir-structure, the gathering-structure involves refutations while not covering reasoning backward. Reid and Knipping (2010) summarized it as “the class moves along, gathering interesting information as it goes” (p.189). An example of the gathering-structure was presented in Figure 2.16, which was taken from the study of Reid and Knipping (2010).

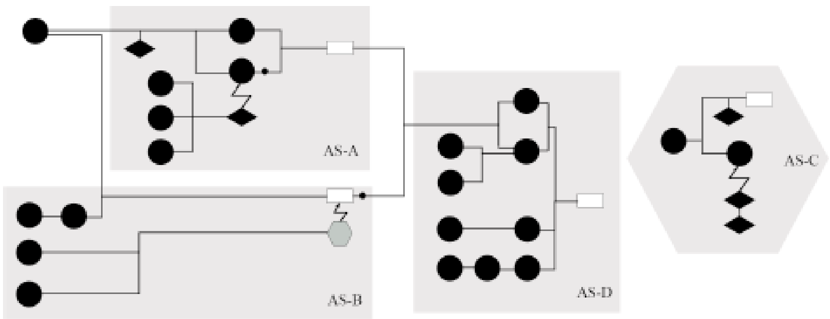


Figure 2. 16. The gathering-structure from the study of Reid and Knipping (2010, p.190)

Argumentation structure in Figure 2.16 is from a 9th grade Canadian classroom when students were discussing the conditions related to forming a triangle according to given lengths of three sides. Through AS-A, students reached a conclusion that given lengths for three sides form a unique triangle. In AS-B, an argumentation process which was both supporting and refuting the conjecture emerged. However, in AS-C, this conjecture was refuted by virtue of the additional data and this resulted as the final conclusion for AS-B. By using the additional data and combining the conclusions of AS-A and AS-B, a conclusion was formed in AS-D (Reid & Knipping, 2010).

2.2.3.5. Line-structure and Independent Arguments-structure

After all, four types of global argumentation structures offered in the studies of Knipping and Reid were outlined. The remaining two argumentation structures stated in the studies of Erkek (2017) and Erkek and Işıksal-Bostan (2019) were also reviewed. As the fifth type of structure, the line-structure was offered. In this type of argumentation structure, the argumentation flows like a line and results in the target conclusion. In the process, the transitions in the argumentation are conducted by means of the claim/data components which are generally confronted. There are explicitly stated warrants and also refutations, but these refutations do not cause the ending of the argumentation. Moreover, unlike the reservoir-structure, reasoning backward and forward is not one of the characteristics of the line-structure. Different from the source-structure and the spiral-structure, parallel argumentation streams are not seen in the line-structure (Erkek & Işıksal-Bostan, 2019). In the following figure, an example of the line-structure was presented.

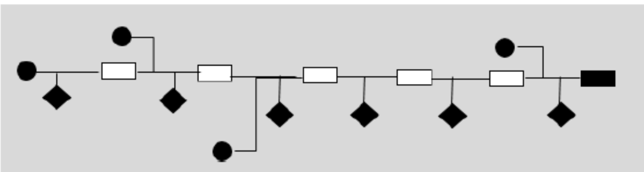


Figure 2. 17. The line-structure from the study of Erkek and Işıksal-Bostan (2019, p.621)

The last argumentation structure is the independent arguments-structure. In the studies of Erkek (2017) and Erkek and Işıksal-Bostan (2019), this type of argumentation schemas were formed in the cases where students stated an idea about the problem without being able to solve it, thought aloud, refuted themselves after suggesting an idea, stated an idea irrelevant to the previous ones, or tried to solve the different parts of the problem without finishing the first one. Figure 2.18 presents an example for the independent arguments-structure.

Figure 2. 18. The independent arguments-structure from the study of Erkek and Işıksal-Bostan (2019, p.621)

In this study, the global argumentation structures of prospective middle school mathematics teachers were investigated. In this manner, the mentioned types of argumentation structures presented in the literature were taken into consideration while classifying the global argumentation structures emerged. Besides, some modifications were conducted which were described in detail in the following chapter.

2.3. Reasoning and Proof in Mathematics Education

Reasoning is a fundamental component of both learning and doing mathematics (Conner, 2012; Conner et al., 2014a; Yackel & Hanna, 2003). However, there is not an agreement about the definition of reasoning (Galotti, 1989; Walton, 1990; Yackel & Hanna, 2003). Since the boundaries regarding the use of the term reasoning are not clear, some terms such as thinking, decision making, and problem solving were sometimes used as a substitute of reasoning (Galotti, 1989). One of the reasons behind the difficulty of defining reasoning is that it covers many mathematical practices (Conner et al., 2014a). According to Mansi (2003), reasoning is “the ability to think coherently and logically and draw inferences or conclusions from mathematical facts known or assumed” (p.9). Similarly, NCTM (2009) described reasoning as “the process of drawing conclusions on the basis of evidence or stated assumptions” (p. 4). As seen, these definitions focused on the drawing conclusions in a logical manner. Moreover, mathematical reasoning is the main area but also a challenging issue in terms of both teaching and learning. To be able to help students to improve their reasoning capacities, mathematics teachers should understand the meaning and function of mathematical reasoning in detail. In this manner, mathematics teachers' task is not finished when they know about reasoning. They should interpret the relations among the mathematical concepts, know about the reasons behind why a particular statement is true or false, understand the reasoning of students, and be ready to defeat or justify their students' ideas whenever needed (Lannin et al., 2011).

Proving involves a dual nature, one of which is to convince and the other one is to explain (Fukawa-Connelly, 2014; Hersh, 1993). Reid and Knipping (2010) summarized the roles of proof under seven categories, which are verification, explanation, exploration/discovery, systemization, communication, getting theorem credits, and other roles of proof. Proof in mathematics domain was described by Edwards (1997) as follows; "the set of processes involved in translating intuitions or generalizations into assertions of certainty, expressed in language which is unambiguous, precise, and accepted within a community of mathematicians" (p.190). Stylianides (2007b) described the relevance of argumentation to proof as follows;

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
> 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

As seen, Stylianides (2007b) described proof directly as an argument and drew attention to the modes of argumentation and how they are represented. Proving was described as "a mathematical task in which the prover is provided with some initial information (e.g., assumptions, axioms, definitions) and is asked to apply rules of inference (e.g., recall previously established facts, apply theorems) until a desired conclusion is deduced" (Weber, 2005, p. 352). In this regard, it might be stated that the functions undertaken by the term proof are stemmed from or directly related to the definitions of it.

Not only proving theorems is an important base of mathematics but also disproving conjectures. While refuting a statement, what is conducted is to present an example which is applicable to the stated hypotheses but this does not satisfy the conclusion part (Joshi, 2015). Besides, to disprove a conjecture, one counterexample is sufficient whilst an example is not sufficient to prove that conjecture is true (Sinclair et al., 2012a). Joshi (2015) explained the differences of the mentioned concepts as;

> The big difference between proving theorems and constructing counterexamples is specificity. A counterexample is just a single solitary example that rules out conjecture being true. This contrasts with proving a theorem where we are trying to prove many cases at once. (p.173).

However, it was also observed that some students developed a skeptic stance regarding the insurance of deductive proof and the possibility of the presence of a counterexample (Chazan, 1993). This issue brings the question of whether a proof and a counterexample related to a statement can coexist (Stylianides & Al-Murani, 2010). As an answer, Ellis et al. (2012) underscored that “once a statement has been proved, finding a counterexample is not possible” (p.9).

Although the significance of proof and proof-related concepts in mathematics education is underlined in the research and paid attention by involving them in the mathematics curricula of different countries around the world, there is a widespread result among the research pertaining to proof that not only students at different levels but also mathematics teachers have difficulty in conducting proof involved tasks (Ellis et al., 2012; Healy & Hoyles, 1998, 2000; Jones, 2002; Mejia-Ramos & Inglis, 2009; Moore, 1994; NCTM, 2000; Reid & Knipping, 2010; Reiss et al., 2008; Reiss, Klieme, & Heinze, 2001; Stylianides & Stylianides, 2017). Even mathematicians who worked on the proving process of many theorems might struggle in conducting the proof of a new statement and spend plenty of time to complete. Thus, it is a quite expected situation that students have difficulty in such a demanding task throughout school mathematics (Ellis et al., 2012).

Although many suggestions to overcome these difficulties in proof were depicted in the studies, the persistent struggles of students were an inevitable part of the mentioned research area. Moore (1994) summed up the cases which constitute the potential to lead students to have difficulty in proving as follows; methods of proof and logic, use of mathematical language, how students perceive the nature and meaning of proof, students' conceptual understanding, and problem solving. In addition, drawing on the findings of the study conducted with undergraduate students in both mathematics and mathematics education, Moore (1994) underlined the major difficulties encountered which are deficiency in the formation and applications of definitions of mathematical concepts, giving proper examples, inadequate concept image and intuitive understanding regarding the concepts, accurate use of the notation and language in mathematics, and structuring the beginning of a proof. Research in the literature presented that some students in secondary school and undergraduate level accept empirical arguments such as examples and measurements as qualified enough and convincing while validating or trying to prove a statement (Jahnke, 2007, 2008). This situation might be originated from the fact that the epistemological meaning of the concept of proof could not be completely understood by students as well as the deficiency of their mathematical competence (Jahnke, 2007). Another factor causing students to have difficulty in proof might be the discrepancy between the pragmatic and cognitive stance of argumentation and the theoretical nature of proof (Antonini & Mariotti, 2008). Besides, the research in the related domain showed that students did not even see why proof is needed in any case and could not see the difference among the notions argument, verification, and proof (Jones, 2002). In a similar vein, another reason for students' struggles in proving might be the fact that they cannot grasp the difference between mathematical proof and ordinary argumentation (Douek, 1998). Therefore, many students might be failing to see the proof as necessary due to the presence of the argumentation which supports and justifies the statement (Pedemonte, 2002b).

To help students to experience the deductive reasoning during teaching is a difficult issue (Jones, 2002). With this purpose, Stylianides and Stylianides (2018) explicated three characteristics which should be considered while arranging interventions to deal with students' difficulties related to proof in the classrooms. These characteristics are listed as follows; "an explanatory theoretical framework", "a narrow and well-defined scope", and "an appropriate mechanism to trigger and support conceptual change" (p.103). These characteristics were also referred as to be applicable in any area of mathematics. While introducing students with proving, open problems which demand the conjecture production might be used (Pedemonte, 2007a). To be involved in the argumentation process in which the production of conjectures is aimed is more effective in developing students' understanding of proof rather than simply reading written proofs (Fujita et al., 2010; Mariotti, 2000). According to Ellis et al. (2012), to improve students' proving potential, the structure of the tasks planned to be engaged in the classroom is an important step. How such tasks might be modified is listed as follows;

> 1. Have students investigate mathematics through exploratory, pattern-generating activates that motivate them to make conjectures.
> 2. Reduce scaffolding that leads students step-by-step through a proof, instead making students responsible for reasoning through a proof. (Ellis et al., 2012, p.78).

Using microworlds in the exploration phase of a proof related task might promote the path intending to reach a proof by facilitating students' move from informal search to formal structure (Edwards, 1997).

What should be counted as a proof is a difficult issue to set forth clearly due to the numerous perspectives and uses of this term in the scope of both mathematics and mathematics education (Stylianides, 2019; Weber & Czocher, 2019). Ko and Knuth (2013) presented a classification regarding the validity of the arguments. It was seen that four categories were formed in their study which are valid, invalid with a structural error, invalid with a content-based line-by-line error, and invalid with a structural error and a content-based line-by-line error. In addition, Ko and Knuth (2013) described some indications of the flaw in the arguments for the invalid ones. In the cases coded as invalid with a structural error, the examples given as related to the flaw of the argument were stated as follows; the first sentence of it involves the assumption of the conclusion tried to be reached actually, the presence of ten cases for a true statement, and displaying a graph without clearing up the case in detail by aiming to refute a false statement. As seen, such examples violate the basic criteria needed in a formal proof. Some indications were also stated another invalid argument category named as covering a content-based line-by-line error. For example, it was expressed that one step of the argument was not derived from the previous assertion, the rationale of another step is actually incorrect, and the line has an algebraic error. According to Ko and Knuth (2013), the mentioned two cases resulting in an invalid argument might be seen simultaneously in the arguments. Such cases were categorized as invalid arguments with a structural error and a content-based line-by-line error.

In addition to the mentioned categories regarding the validity of an argument, Ko and Knuth (2013) also presented the strategies for the participants of their study, who were mathematics majors, to use in the validation of the arguments. In this respect, two main strategies were mentioned which are “the examination of the arguments’ structure and line-by-line checking” (p.25). The sub-sections of the line-by-line checking strategy are listed as informal deductive reasoning, example-based reasoning, experience-based reasoning, and example-based and informal deductive reasoning. How Ko and Knuth (2013) defined the mentioned strategies was presented in the following figure.

Strategies	Description
Examination of the argument's structure	Individuals examine the logical structure used in the argument.
Line-by-line checking	Individuals judge the argument by checking step-by-step.
Informal deductive reasoning	Individuals' explanations are based on (partially) correct definitions, properties, theorems, or axioms that cannot constitute an acceptable argument, which do not constitute a rigorous argument.
Example-based reasoning	Individuals rely on numbers or diagrams.
Experience-based reasoning	Individuals' explanations are based on similar arguments they had seen in the past with little mathematical reasoning.
Informal deductive and example-based reasoning	Individuals provide both informal explanations and numbers or diagrams.

Figure 2. 19. Strategies for validating proofs and counterexamples (Ko & Knuth, 2013, p.27)

According to Ko and Knuth (2013), the aim of the line-by-line checking is to examine whether “all steps were presented legitimately and/or followed logically from previous assertions, which may or may not be accepted as a valid argument” (p.27). Similarly, Alcock and Weber (2005) mentioned the line-by-line checking in the validation of arguments. In more detail, they presented an invalid argument which has the flaw in terms of content, not in the form of the arguments. Although the given invalid arguments involve some minor errors at the beginning lines such as the insufficient definition of a variable and the absence of the required restrictions for ensuring the correctness of an assertion, the main flaw was at the last line of the argument which asserts a statement is not true for all cases. They also put emphasis on the case that the warrant should be true. Otherwise, when the warrant is false, the argument is accepted as invalid even though the data and conclusion are correct. After all, Weber and Alcock (2005) suggested a framework to determine the validity of the proofs. In more detail, they mentioned the issue that line-by-line verification of a proof, associated the structure of proof with the argumentation model of Toulmin, and highlighted the importance of the warrant in terms of the validity of the proof.

Selden and Selden (2003) mentioned the line-by-line analysis as a kind of textual analysis and presented examples for the validation process. Moreover, the cases which affect neither the correctness of the next expression nor affect the whole structure of the arguments in terms of its validity were named as the extraneous errors (Selden & Selden, 2003). In more detail, Selden and Selden define the errors having the following properties as the extraneous errors "Because they do not affect the correctness of [3] *(the next line in the argument),* they cannot affect whether or not the argument is a proof. Such errors are nevertheless undesirable because they can make arguments confusing and more difficult to validate" (p.14).

Bleiler, Thompson, and Krajčevski (2014) listed a rubric for the evaluation of the validity of arguments. The best case scenario is that a valid mathematical proof has the highest score in the rubric. The other case is that the arguments could be accepted as valid proof in the case that some minor mistakes are corrected. On the other hand, unsuccessful answers involve two types. In the first type, the argument is started properly but contains major conceptual errors resulting such arguments to be labeled as invalid. The worst case of the given arguments is that any meaningful and effective progress is not presented in the arguments. As expected, such cases were also labeled as invalid. As seen, from the broadest aspect, the arguments were evaluated as either valid or invalid in the study of Bleiler et al. (2014). Since Bleiler et al. (2014) focused on both the mode of argumentation and the mode of argumentation representation, they emphasized that to examine the global structure of an argument is important as well as the local line-by-line analysis. In a similar vein, it was found out that the participants in the study of Weber (2008) examined the structure of the arguments as the first step and then continued with the line-by-line examination process.

In the analysis of the proof-related data of this study, the mentioned studies were combined and the guidelines of the evaluation process of the validity of participants' arguments were arranged. As seen, in the last three sections, the concepts of cognitive unity and its main elements which are argumentation and proof were tried to be explained by taking mainly the studies related to the research questions of the present study into consideration. Since the mathematical domain in which these concepts were examined is geometry, the following section was arranged to be related to geometry.

2.4. Geometry

Geometry has been counted as one of the major concepts in mathematics since the beginning of the recorded history of mathematics (Albrecht, 1952; Bayerthal, 1988; Cantürk-Günhan, 2014; Duatepe, 2004; Duval, 1998; Gökbulut & Ubuz, 2013; Jones, 2002; Mariotti, 1995; Napitupulu, 2001; Sinclair & Bruce, 2015; Stupel, Sigler, & Tal, 2018). To accept geometry as a set of definitions, concepts, axioms or theorems can be regarded as oversimplification since it is primarily about "describing relationships and reasoning" (NCTM, 2000, p.47). In this respect, geometry is defined as "a complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imaged spatial environments" (Battista, 2007, p.843). Geometry is described as the most intuitive, concrete, and real life-related component of mathematics (Mammana & Villani, 1998). Sinclair et al. (2012b) emphasized the importance of geometry by declaring it as a broad area in its own right and also underlining the effects of geometric perspective on the understanding of other concepts involved in mathematics.

Geometry is a fundamental area of school mathematics (Clements, 2003; Clements & Battista, 1992; Duatepe, 2004; Erbaş & Yenmez, 2011; Fidan & Türnüklü, 2010; Köse, 2008; Mariotti, 1995; Sinclair et al., 2012a, 2012b; Tan-Şişman, 2010;

Utley, 2004). NCTM (2000) listed geometry as one of the content standards for school mathematics as well as number and operations, algebra, measurement, and data analysis and probability. The objectives of geometry teaching include learning the properties of geometric figures and concepts and the relationships among them, developing geometric thinking, reasoning and justification skills, enhancing the proof of the geometric statements, learning different representations in geometry, and comprehending the relations of geometry with other strands of mathematics (NCTM, 2000). Through studying geometry concepts, students can develop their abilities related to visualization, guessing, reasoning, justification, proving, conjecturing, critical thinking, and problem solving, comprehend other areas of mathematics, explore the space, improve their spatial intuition about the real world, and obtain the knowledge required to study advanced mathematics (Battista, 2007; Fidan & Türnüklü, 2010; Jones, 2002; Jones, Fujita, & Kunimune, 2012; Köse, 2008; NCTM, 2000; Olkun, Sinoplu, & Deryakulu, 2005; Suydam, 1985; Utley, 2004). Since geometry involves many abstract concepts and relationships among them, the use of concrete materials and real life examples should be integrated into the teaching of geometry in order to develop students' geometric reasoning (Köse, 2008). There is a variety of tools which can be used as pertaining to geometry such as paper, grid paper, ruler, parallel ruler, compass, straightedge, carpenter square, pantograph, link-ages, blocks, proportional dividers, flipbooks, diagrams, screen images, and human bodies, origami, paper folding, T-square and triangles (Albrecht, 1952; Arıcı & Aslan-Tutak, 2015; Sinclair et al., 2012b; Smart, 1998).

In addition to the place of geometry in both mathematics and mathematics education, geometry is an appealing area of mathematics for many people since it comprises interesting problems and theorems, is applicable in terms of various approaches, has a key importance in some other areas such as architecture and design, and provokes visual, intuitive, and aesthetic senses (Jones, 2002; Köse, 2008). Due to the substantial benefits and applications of geometry, the majority of mathematics curriculum around the world is comprised of geometry concepts to varying degrees (Arıcı & Aslan-Tutak, 2015; Jones, 2002). As is the case with geometry, it presents the call for conducting studies pertaining to geometry, and as expected, the call has been taken up by many researchers. Thus, there is a growing literature base related to the various concepts of geometry approached by a number of contexts.

As regards geometry, there are various frameworks which focus on different aspects such as cognitive processes of geometry and development stages of geometric thinking. For example, Usiskin (1987) approached geometry based upon four dimensions; drawing, visualizing, and construction of the geometric figure, considering the physical world from the spatial aspects, representing nonvisual geometric concepts and the connections among them, and working on it as a formal mathematical system. Jones (1998) examined three main theoretical frameworks regarding geometric thinking, which can be used as the resources to uncover and describe how geometric reasoning of students was developed, and also underscored the complexity of geometry in a cognitive manner. The frameworks discussed in the study of Jones (1998) are the geometric thinking model of van Hiele, the theory of figural concepts stated by Fischbein, and cognitive model of geometry presented by Duval. It was noticed in the literature that geometric thinking model of van Hiele, which encompasses five levels of geometric thinking, has been used in many studies (Osmanoğlu, 2019) and reported as providing a proper description regarding the development of geometric thinking (Battista, 2007; Clements & Battista, 1992). Fuys, Geddes, and Tischer (1988) summarized these levels by translating from the work of van Hiele as follows:

> Level 0: The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.
> Level 1: The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).
> Level 2: The student logically interrelates previously discovered properties/ rules by giving or following informal arguments.
> Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems.
> Level 4: The student establishes theorems in different postulational systems and analyzes/compares these systems (p.5).

The term figural concept was introduced by Fishbein (1993). More specifically, Fishbein (1993) asserted that the images and concepts in geometry constitute different mental entities; therefore, another construct named as figural concept was propounded since it has both conceptual and figural aspects. Based on this idea, figural concept was described as “a mental construct characterized by all the properties of concepts (generality, essentiality, abstraction, ideality), but which at the same time preserves figural properties (shape, distances, positions)” (Fishbein, 1993, p.150). Since enhancing the relations of both figural and conceptual aspects in a given case is difficult, it can be considered as a fragile situation for students. The harmony between the mentioned aspects, which does not refer to an absolute equivalence, presents the ideal situations (Fischbein, 1993; Fischbein & Nachlieli, 1998). In addition, it was stressed that the interactional and dialectic relationship between figural and conceptual aspects is an important factor while framing the geometric reasoning and it can be considered as a theoretical tool while examining the process in geometric problems (Jones, 1998; Mariotti, 1995).

From the perspective of Duval (1995b, 1998), two frameworks related to geometry were presented. Firstly, Duval (1995b) queried how geometric figures function while solving a geometry problem or comprehending a geometrical situation and questioned why a geometric figure could not be used heuristically in every case. Thus, Duval (1995b) came up with four cognitive apprehensions pertaining to the analysis of a geometric figure, namely, perceptual apprehension, sequential apprehension, discursive apprehension, and operative apprehension. The structure comprised of these four apprehensions was called as an analytic resource to examine the semiotics of drawings in geometry by Jones (1998). In more detail, perceptual apprehension refers to the identification of a geometric figure at first glance and it may cover entitling what have been recognized and noticing some sub-figures involved in it. Sequential apprehension is about how students perceive the construction process of a geometric figure and technical constraints, especially in terms of the tools used and the mathematical properties considered during the construction. Discursive apprehension is about the presentation of some mathematical features of a drawing by giving a speech about it since all features cannot be derived via perceptual apprehension. Some other properties of drawing might be reached based on the given properties in the speech. Finally, operative apprehension implies the modification of geometric figures both mentally and physically in order to have an idea regarding the solution of the problem. Any drawing can be processed as a geometric figure if it prompts perceptual apprehension aligned with at least one of the remaining three apprehensions (Duval, 1995b). Based on this framework, Deliyianni, Elia, Gagatsis, Monoyiou, and Panaoura (2010) conducted a study with 1086 students in primary and secondary school levels to examine the roles of three apprehensions except sequential apprehension since the curriculum which the participants were subject to did not emphasize the figure construction and suggested that all apprehensions should be paid attention in all school levels.

As another issue related to geometry, Duval (1998) focused on the cognitive processes by declaring the presence of three components, which are visualization, construction, and reasoning. Construction covers the configuration of geometric figures by using tools; reasoning is related to the thinking process to prove, to explain, and to extend knowledge; and visualization is about the representation for illustration of a statement, for exploration of a complex case heuristically, and for verification of it subjectively. The aforementioned cognitive processes are considerably connected to each other and the association among them is an essential factor for being proficient in geometry (Duval, 1998). The underlying cognitive interactions in geometry concepts were explained by Duval (1998) as given in Figure 2.20.

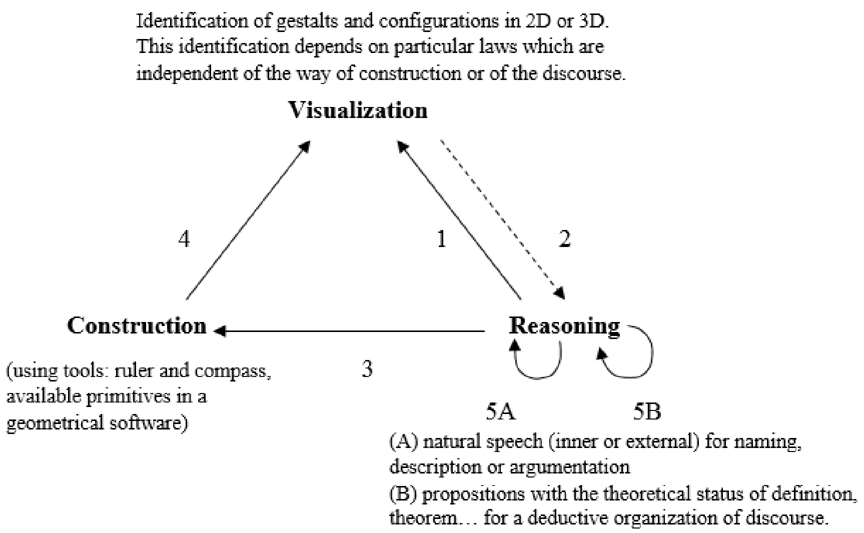


Figure 2. 20. The underlying cognitive interactions involved in geometry (Duval, 1998, p.38)

In Figure 2.20, the arrows were used to represent how a cognitive process can support another one in any geometric task. For example, arrow 2 was constructed as a dotted line since visualization does not always help reasoning. Arrows 5A and 5B indicate that reasoning can be developed independently of visualization or construction processes. 5A involves the use of inner or external natural speech for description or argumentation, and 5B contains the use of theoretical propositions to construct a deductive organization of discourse. Longer circuits in Figure 2.20 can be used to explain the relationship between the cognitive processes. For instance, 4-2-5A or 5B can represent a way of describing the construction order, and 2-5B-3 refers to finding a construction order for the given geometric figure (Duval, 1998).

In addition to the arrows representing interactions in Figure 2.20, different interactions might be observed between the mentioned components of geometry. In more detail, the use of new technologies and tools in geometry teaching, the domain and level of the geometric concepts being studied, and the structure of different geometry activities might cause to include additional arrows to explain the relationships in cognitive processes. For example, in an appropriate activity, construction might directly affect the geometric reasoning of students which may be considered as an additional arrow for Figure 2.20. By combining the schema of Duval (1998) regarding the cognitive processes of geometry given in Figure 2.20 and the notion soft construction stated by Healy (2000), Or (2013) put forward a model for designing tasks in dynamic geometry environments which was presented in Figure 2.21.

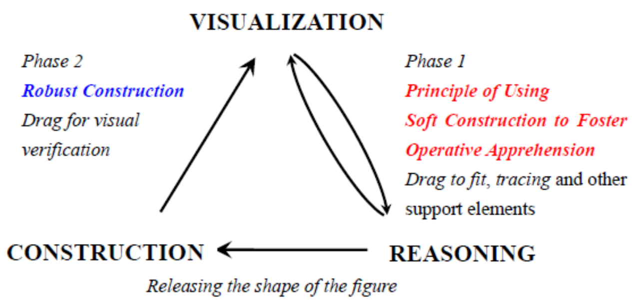


Figure 2. 21. Task design model in dynamic geometry environments stated by Or (2013, p.211)

While intending to foster students with respect to conducting the robust constructions, Healy (2000) brought the concept of robust construction by observing students' behaviors in the tasks where they used Cabri. More specifically, Healy (2000) defined the soft construction as "in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student" (p.107). Or (2013) assigned different roles to robust and soft constructions while aiming to promote operative apprehension. As it can be tracked in Figure 2.21, in phase 1, there is reciprocation among the components visualization and reasoning while attending the soft constructions to foster operative apprehension. At this phase, the dragging has the key importance by means of getting an insight into the geometric concepts empirically. After the clarification of the solution in phase 1, the task is moved to phase 2. In phase 2, students are asked to perform a robust construction in the previously worked task in order to justify and verify the idea they came up with in phase 1. At this phase, the dragging functions to verify the idea in terms of producing a robust construction. As seen, the connections among the mentioned three components of geometry might vary depending on the context that they involved in.

What comes next are the details of the geometric construction, the status of geometric construction in middle school mathematics curricula in Turkey, and tools used in the process.

2.5. Geometric Construction

Geometric construction is described as a problem in which a requested geometric figure is formed by following the given data and by using some particular instruments such as compass and straightedge (Albrecht, 1952). In a similar vein, no matter which tool or tools (compass-straightedge or dynamic geometry software) are used in the construction process, geometric constructions are also defined as "valid solutions of construction problems" (Stylianides & Stylianides, 2005, p.32). The purpose of the geometric constructions was expressed as not only to construct the desired geometric figure by pursuing particular rules and strategies via using compass and straightedge but also to suggest a solution for the given problem (Erduran & Yeşildere, 2010;

Karakuş, 2014). Smart (1998) listed four steps endeavored while solving a construction problem with compass and straightedge as follows.

> 1. Analysis. In this step, the solver assumes that the construction has been performed, then analyzes the completed picture of the solution to find the needed connections between the unknown elements in the figure and the given facts in the original problem.
> 2. Construction. The result of this step is the drawing itself, made with straightedge and compass and showing the construction marks.
> 3. Proof. It is necessary to prove that the figure constructed is actually the required figure.
> 4. Discussion. The number of possible solutions and the conditions for any possible solution are explained in this step. (p.168)

According to Schreck, Mathis, and Narboux (2012), solving geometric construction problems covers finding geometric figures which supply the requisite information. In the solution process of a geometric construction problem, the input is presented as a literal statement and the output is expected as any solution for the given case. As it could be seen, these descriptions approach to geometric constructions as the problem situations or the correct solutions offered for the problems which demand the construction of a geometric figure.

On the other hand, there are some other explanations which mention constructions without referring to problem solving process distinctively. For example, Lim (1997) defined constructions as “standard procedures for constructing geometrical entities such as angle bisectors using compasses and straightedges only” (p. 138). Similarly, geometric construction was defined by Djorić and Janičić (2004) as “a sequence of specific, primitive construction steps” (p.71). The primitive construction steps in this definition, also called as elementary constructions, were listed as follows; the construction of a line and a line segment between two points, the construction of a circle when its center and another point on it are given, the construction of the intersection point of two lines, and the construction of the intersection points between a line and a circle. In the aforementioned primitive constructions, straightedge and compass were stated as the tools used (Djorić & Janičić, 2004; Janičić, 2006, 2010). Smart (1998) also presented some examples for geometric constructions such as the construction of a congruent line segment and a congruent angle, the construction of the perpendicular bisector of a line segment, and the construction of a parallel line passing through a given point under the name of basic construction. As seen, the geometric construction examples stated by Smart (1998) seemed to be more complex compared to the ones stated by Janičić (2006, 2010) and also Djorić and Janičić (2004). Thus, the term geometric construction covers both primitive and high level constructions (Djorić & Janičić, 2004; Janičić, 2006, 2010). Through combining a set of primitive constructions, more comprehensive and complex constructions such as right angle construction and angle bisector construction can be performed (Djorić & Janičić, 2004; Janičić, 2006, 2010). In other words, basic geometric constructions specified in Euclidean geometry are used to perform more challenging and multifaceted geometric constructions (Karakuş, 2014; Smart, 1998).

Although geometric construction is the most commonly used term, it was seen that the following terms "compass constructions", "ruler and compass constructions", "compass and straightedge constructions", and "Euclidean constructions" were also used in the studies instead of the term geometric construction (Cheung, 2011). For example, Lim (1997) used the term compass constructions for the construction of geometric figures by using both compass and straightedge. Since geometric construction has a major place in Euclid's Elements, they are also called as Euclidean constructions in some studies (e.g., Aichele, 1982; Presmeg, Barrett, & McCrone, 2007).

Geometric construction is a component of geometry since Euclid's Elements and the ancient Greeks which correspond to thousands of years (Awtrey, 2013; Cheung, 2011; Djorić & Janičić, 2004; Fujita et al., 2010; Janičić, 2010; Kuzle, 2013; Sarhangi, 2007; Smart, 1998; Stillwell, 2005; Stupel, Oxman, & Sigler, 2014). The Elements which was written by Euclid around 300BC has been acknowledged as a preeminent among Greek mathematical texts and a highly used mathematical work in history of mathematics (Erduran & Yeşildere, 2010; Fitzpatrick, 2008; Hansen, 1998; Hartshorne, 2000; Kuzle, 2013; Stillwell, 2005). In the Elements, a constructive approach regarding geometry was pursued by Euclid. In other words, many propositions given in the book are not exactly theorems, but they are like construction problems (Cheung, 2011; Hartshorne, 2000). The five postulates of Euclid were listed as follows:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles (Heath, 1956, p.154-155).

As seen, Euclid made assumptions concerning geometric constructions. According to Euclid, a line can be drawn between any two points and a circle can be drawn in the case that the center and the radius are given. In more detail, postulates 1 and 2 indicate that straightedge is used for drawing lines in the constructions and postulate 3 involves the use of the compass for measurement. Thus, it can be understood that Euclid separated the function of drawing straight lines from the function of measurement. By the use of these lines and circles, all propositions Euclid stated were built (Stillwell, 2005). In this manner, by considering the issues that Euclidean geometry is based on Euclid's Elements (Napitupulu, 2001) and the Elements covers many geometric constructions, it can be stated that geometric construction has critical importance in Euclidean geometry (Karakuş, 2014; Smart, 1998). Moreover, it can be inferred that mathematicians studying Euclidean geometry are also interested in problems related to geometric constructions (Erduran & Yeşildere, 2010).

The Greeks determined a set of limits about the use of tools in geometric constructions (Sarhangi, 2007). In more detail, the Greeks used only compass and straightedge in constructing geometric figures. For example, the Greeks were able to construct many regular polygons such as square, pentagon, hexagon, and decagon by using compass and straightedge (Albrecht, 1952). Moreover, Kostovskii (1961) explained the history of geometric constructions with using compass and straightedge. For example, in 1797, Lorenzo Mascheroni, who was an Italian mathematician, proved that all constructions that can be done by compass and straightedge could also be conducted by using only a compass. However, in 1928, Hjelmslev, who was a Danish mathematician, found a book that had been written by Mohr in 1672, and noticed that this book also involves the proof of the idea of Mascheroni about using only a compass.

Furthermore, in 1833, Jacob Steiner, a Swiss geometer, showed that all geometric figures constructed using compass and straightedge could be constructed by using straightedge only if a circle and its center were given.

According to Stupel and Ben-Chaim (2013), the popularity of geometric constructions is considered as related to the four famous constructions which are doubling the cube, trisecting the angle, squaring the circle, and inscribing a regular heptagon in a circle suggested in the ancient Greeks. Similarly, Albrecht (1952) and Robertson (1986) mentioned the history of three of these problems except for inscribing a regular heptagon in a circle. In more detail, doubling the cube means that a cube is given and then the construction of a cube with the twice the volume of the given cube is asked. In trisecting the angle, an angle is given, and then the construction of the angle corresponding to the one-third of the given angle is asked. In the squaring the circle, a circle is given, and then the construction of a square with the same area of the given circle is asked (Albrecht, 1952; Baragar, 2002; Stupel & Ben-Chaim, 2013). The inscribing a regular heptagon in a circle refers to the construction of a regular seven-sided polygon into a circle (Stupel & Ben-Chaim, 2013). In the time of the ancient Greeks, the mentioned constructions were accepted as impossible to construct, but the impossibility of them was not proved. Thus, many mathematicians have been interested in these constructions throughout the centuries (Stupel & Ben-Chaim, 2013). For example, the statement "it is impossible to trisect an arbitrary angle" (p.151) has been accepted as a challenge by many mathematicians throughout history. The solutions offered for this problem were either not correct or did not follow the rules of construction. As an alternative method, it has been proven that it is possible to trisect an angle using a compass and a straightedge which is notched in two places. This method is called "Archimedes' trisection algorithm" (Baragar, 2002). There is a misunderstanding related to the impossibility of these famous construction problems. It was generally omitted that the notion of impossibility was attributed to the case that compass and straightedge are used (Albrecht, 1952; Baragar, 2002). The aforementioned constructions were showed to be impossible to construct with compass and straightedge many years later, namely in the 19th century (Stupel & Ben-Chaim, 2013).

In addition to the fact that geometric construction is one of the enduring and popular concepts throughout the history of mathematics (Karakuş, 2014; Sarhangi, 2007; Stupel et al., 2014), it is also considered as an important issue in mathematics teaching of the present (Djorić & Janičić, 2004; Kuzle, 2013; Stupel & Ben-Chaim, 2013). According to NCTM (2000), geometric constructions can spur on students to "draw and construct representations of two- and three-dimensional geometric objects using a variety of tools" (p. 308). Since reaching accurate conclusions, organizing a strict structure, and adopting a rigorous language are needed while working on geometric construction problems, it was accepted that geometric constructions constitute a proper field for training students (Djorić & Janičić, 2004). Moreover, Stupel and Ben-Chaim (2013) stressed the importance of construction in geometry teaching as follows: "trying to learn geometry without geometric construction is like trying to learn chemistry or biology without laboratories" (p.9-10). Moreover, Duval (1998) and Usiskin (1987) also underlined the importance of construction in geometry by classifying it as a cognitive process and a dimension of geometry, respectively. Therefore, it can be concluded that geometric construction is a fundamental part of geometry (Cheung, 2011), closely linked to the main objectives of geometry teaching (Djorić & Janičić, 2004; Weigand & Ludwig, 2009, as cited in Kuzle, 2013), and one of the important concepts of both mathematics and mathematics education (Janičić, 2010; Kostovskii, 1961; Sarhangi, 2007).

Regardless of the tools used, integration of constructions into geometry teaching brings some benefits for students (Stupel & Ben-Chaim, 2013). While dealing with construction problems, students have the opportunity to gain more insight of the geometric figures and reach the properties by themselves (Napitupulu, 2001); hence, geometric constructions can be used as a basis while teaching properties of geometric notions (Stupel & Ben-Chaim, 2013). Additionally, geometric constructions allow students to form new solution strategies, encourage them to be creative and think, and expand their comprehension of mathematics (Stupel & Ben-Chaim, 2013). Since the first step of construction is difficult to detect and decide, students require using their mathematical skills (Erduran & Yeşildere, 2010; Karakuş, 2014). Having knowledge about geometric constructions enlarges the background needed to analyze relationships in mathematical concepts (Aichele, 1982). Moreover, geometric constructions constitute an effective tool for geometric investigation, exploration, and discovery (Napitupulu, 2001; Pandisico, 2002). Through exploring the properties of a geometric figure in the geometric construction process, students can widen their reasoning and logical thinking (Djorić & Janičić, 2004), improve prediction abilities (Cheung, 2011), and develop their geometric thinking (Köse, Tanışlı, Erdoğan, & Ada, 2012). Similarly, geometric constructions via using different manipulatives can help students to develop their visualization skills. By the visualization process, students' reasoning abilities regarding geometry, which is needed for conceiving more abstract concepts, can be improved (Arıcı, 2012). Besides, geometric constructions constitute a motivation factor for students (Djorić & Janičić, 2004). In consequence of these points, it can be stated that geometric constructions have immense importance in geometry education (Djorić & Janičić, 2004; Kuzle, 2013; Napitupulu, 2001).

Erduran and Yeşildere (2010) investigated middle school mathematics teachers' use of compass and straightedge in geometric constructions and student-teacher-tool interactions. The results of the study showed that all teachers used a teacher-centered approach during the courses. In other words, teachers simply gave the instructions for construction and asked the students to follow them. Therefore, students could not understand the logic behind the given constructions. In this respect, Erduran and Yeşildere (2010) suggested that a student-oriented approach is more effective than a teacher-oriented approach when using compass and straightedge in activities related to geometric constructions. Furthermore, the way teachers used construction activities was not consistent with the idea of Cherowitzo (2006). According to Cherowitzo (2006), students can analyze the construction and see how it works when they use compass and straightedge. Similarly, Shryock (1995) investigated the effects of two different instructional formats on the college students' performance in geometric construction and concluded that students' task performance in geometric construction was directly related to the instruction format. To help students develop their geometric thinking, they should be encouraged to discover and explore the geometric concepts rather than simply giving information about the concepts (Fidan & Türnüklü, 2010). Similarly, in the geometric construction activities, students should be encouraged to be active, evaluate, make presumptions, and discuss their ideas (Lim, 1997). Moreover, the literature review showed that studies on geometric constructions involve the use of dynamic geometry programs, compass and straightedge as instruction tools. As mentioned, when using these tools, students have the chance to discover and explore geometric concepts (Laborde, Kynigos, Hollebrands, & Strässer, 2006; Napitupulu, 2001). Therefore, teachers might have a constructivist approach while teaching through geometric construction activities.

Geometric construction is seen as a challenging activity due to the need to justify the logic of steps in the process (Sarhangi, 2007). Moreover, since complex geometric constructions provide students a challenging environment, it can be stated that geometric constructions also provide students opportunity to develop a deeper point of view towards geometry, to improve their thinking and reasoning abilities (Stupel & Ben-Chaim, 2013), and to apply not only the previous knowledge about geometry but also higher order thinking skills (Lim, 1997). According to Sanders (1998), geometric construction "can reinforce proof and lends visual clarity to many geometric relationships" (p. 554). Since geometric construction lead students to consider various strategies and to find out the necessary steps, these activities may help to develop their proof writing abilities and can be used as a step for them while passing to formal proof (Arıcı, 2012; Battista & Clements, 1995; Cheung, 2011; Schreck et al., 2012). In more detail, Battista and Clements (1995) stated that students could work on making and testing conjectures in the construction process before they engage in a formal proof. With a careful design of teaching, geometric constructions may be used to support students in terms of producing conjectures and constructing proofs (Fujita et al., 2010).

2.5.1. Geometric Constructions in Mathematics Curricula in Turkey

To determine the status of geometric construction in middle school mathematics curricula in Turkey and to track the changes pertaining to geometric construction among the revised versions of mathematics curricula, the latest three middle school mathematics curricula, which span nearly a decade, were taken into consideration herein. Firstly, three mathematics curricula were examined in terms of the objectives covering construction and drawing in geometry, and then the tables involving such objectives were formed. At this section, each of these tables was presented as the first step and the comparisons of them were considered later on. First of all, the objectives related to geometric construction situated in Mathematics Curriculum for grades 6-8 (MoNE, 2009) were presented as in Table 2.8.

Table 2. 8

Objectives related to the geometric construction from Mathematics Curriculum for grades 6-8 (MoNE, 2009)

Grade	Objectives
Grade 6	- to construct a line segment congruent to a given line segment - to construct an angle congruent to a given angle and divide an angle into two equal angles - to construct polygons - to construct the image of a geometric figure formed via translation
Grade 7	- to construct the line which is perpendicular to a given line passing from a point on the given line or not on the given line - to construct the perpendicular bisector of a line segment - to construct the line parallel to a given line - to determine the relative positions of three lines in a plane and construct them - to determine the properties of the circle and construct the circle - to determine the basic elements of a circular cylinder, construct, and draw the net of it
Grade 8	- to draw the triangle when the lengths of the necessary number of elements are given - to construct the median, the perpendicular bisector of the sides, the angle bisector, and the altitudes of a triangle - to construct, determine the basic elements, and draw the net of a prism, a pyramid, and a cone - to determine the basic elements and construct a sphere - to determine and construct the intersection of a plane and geometric figure -to determine and construct the translational symmetry of figures

As displayed in Table 2.8, each grade level in Mathematics Curriculum for grades 6-8 (MoNE, 2009) involves objectives related to geometric construction. The verb construct was generally used while the verb draw was also used in a few objectives as well. The basic constructions such as the construction of congruent line segment and angle were involved in grade 6. Although the curriculum covers an objective related to the construction of the polygons in grade 6, it was noticed that some necessary basic constructions such as the construction of parallel and perpendicular lines were situated in grade 7. In addition, the construction related to the circle was presented in grade 7. Since students are expected to be familiar with the basic constructions in the grades 6 and 7, there are objectives related to the constructions of the median, perpendicular bisector, and altitudes of a triangle in grade 8. Moreover, as seen, the verb construct was referred while working on the circular cylinder, prism, pyramid, and cone. As different from other mathematics curricula, this one used the verb construct in conjunction with the concepts of translation, translational symmetry of figures, sphere, the relative positions of three lines in a plane, and the intersection of a plane and geometric figure.

According to Mathematics Curriculum for grades 5-8 (MoNE, 2013), the construction-related objectives were presented in Table 2.9 given below.

Table 2. 9

Objectives related to the geometric construction from Mathematics Curriculum for grades 5-8 (MoNE, 2013)

Grade	Objectives
Grade 5	- to draw line segments congruent to a given line segment on a graph paper or dot paper - to construct parallel line segments to a given line segment on a graph paper or dot paper - to draw rectangle, parallelogram, rhombus, and trapezoid on a graph paper or dot paper - to draw the net of rectangular prism
Grade 6	- to draw the line which is perpendicular to a given line passing from a point on the given line or not on the given line - to determine center and radius by drawing a circle

Table 2. 9 (continued)

Grade 7	- to draw an angle congruent to a given angle - to draw the angle bisector of a given angle
Grade 8	- to construct the median, the angle bisector, and the altitude of a triangle - to draw the triangle when the lengths of the necessary number of elements are given - to construct, determine the basic elements, and draw the net of a right prism, a right circular cylinder, a right pyramid, and a right cone

After regulations, grade 5 was involved in the middle school so that Table 2.9, which was prepared based on Mathematics Curriculum for grades 5-8 (MoNE, 2013), covers grade 5 as well. When Tables 2.8 and 2.9 are compared, many similarities can be seen. For example, the fundamental constructions such as the constructions of angle bisector, parallel lines, and perpendicular lines were also involved in Mathematics Curriculum for grades 5-8 (MoNE, 2013). As a difference, the first issue noticed is the addition of the expression “a graph paper or dot paper” into some objectives. This situation might also result in the use of the verb draw in more objectives compared to the verb construct. Compared to Mathematics Curriculum for grades 6-8 (MoNE, 2009), it can be seen that the year levels of lots of objectives were changed in Mathematics Curriculum for grades 5-8 (MoNE, 2013). For example, the constructions of congruent angle and angle bisector were in grade 6 and then moved to grade 7 in this mathematics curriculum. As a new issue, to draw rectangle, parallelogram, rhombus, and trapezoid on a graph paper or dot paper was distinctively given as an objective in grade 5.

Lastly, the current one, which is Mathematics Curriculum for grades 5-8 (MoNE, 2018), was examined with respect to geometric construction and the following table was prepared.

Table 2. 10

Objectives related to the geometric construction from Mathematics Curriculum for grades 5-8 (MoNE, 2018)

Grade	Objectives
Grade 5	- to draw line segments congruent to a given line segment - to draw the line which is perpendicular to a given line passing from a point on the given line or not on the given line - to construct parallel line segments to a given line segment - to name, construct, and determine basic elements of polygons - to draw rectangle, parallelogram, rhombus, and trapezoid
Grade 6	- to draw an angle congruent to a given angle
Grade 7	- to draw the angle bisector of a given angle
Grade 8	- to construct the median, the angle bisector, and the altitude of a triangle - to draw the triangle when the lengths of the necessary number of elements are given - to construct, determine the basic elements, and draw the net of a right prism, a right circular cylinder, a right pyramid, and a right cone

When the recent three tables are compared, it can be seen that there is a decrease in the number of geometric construction related objectives. Moreover, it can be stated that Mathematics Curricula in 2013 and 2018 have more common points compared to the first one. However, the expression “a graph paper or dot paper” was not directly given in the objectives of the current curriculum. In a similar vein, there are objectives which the grade they belong to were changed. For example, the objective related to the construction of perpendicular line was moved from grade 6 to grade 5.

All in all, it was concluded that there are many construction-related objectives in the mentioned mathematics curricula although some changes were conducted and the degree of emphasis on the verb construct was decreased with the revisions.

Moreover, prospective middle school mathematics teachers should be competent at the concept of geometric construction and follow the related changes conducted in mathematics curricula. In this manner, the approaches offered by prospective middle school mathematics teachers to perform geometric constructions were aimed to be investigated in this study.

2.5.2. Tools Used in Geometric Construction

As stated before, Duval (1998) explained construction as a cognitive process of geometry in which some particular tools are used. Although compass and straightedge are the tools widely used pertaining to geometric construction throughout the history of mathematics (Kuzle, 2013; Pandisico, 2002), there are some other tools which can be used in construction process (Gibb, 1982; Pandisico, 2002; Robertson, 1986; Schreck et al., 2012). For example, Mira (for constructing the image of a line reflection), a three-by-five-inch card (for drawing the right angle and transfer length), and two-edged straightedge (for drawing parallel lines) can be listed among the tools available to use in construction process (Kuzle, 2013; Pandisico, 2002; Robertson, 1986; Serra, 2003). In addition to the physical tools such as compass and straightedge, virtual tools such as Geometer's Sketchpad and GeoGebra can be used in geometric constructions (Arıcı, 2012). Regarding geometry, a great deal of software can be listed. Many of these tools are based on Euclidean geometry and propose entities applicable to geometric constructions (Janičić, 2010). Dynamic geometry software such as Cinderella, Cabri, and Geometer's Sketchpad are considered as effective tools in terms of working interactively and preparing animations (Janičić, 2010). In addition to these tools, using paper folding was recommended for performing constructions in geometry (Arıcı, 2012; Coad, 2006). Since each tool enhances various mathematical ideas and has some weaknesses and strengths, which tool is more feasible and applicable in geometric construction should be decided by the user (Pandisico, 2002). Setting restrictions or rules about the type of the tools used in construction is also important since the possible and impossible constructions with the given tool are needed to be considered in order to perform more complex constructions (Stupel & Ben-Chaim, 2013).

When geometric construction is stated, many people who are familiar with geometry associate it with the use of compass and straightedge as tools (Albrecht, 1952; Schreck et al., 2012). Since Euclid's Elements involves the construction of geometric figures by using compass and straightedge (Fitzpatrick, 2008), both of them have a long history in geometry as construction tools (Kuzle, 2013). Therefore, to understand Euclidean geometry, it is necessary to comprehend the scope of compass and straightedge constructions (Stillwell, 2005). In the case that the rules of the ancient Greek mathematics are followed in geometric constructions, compass and straightedge are generally allowed to be used as tools (Sarhangi, 2007; Stupel & Ben-Chaim, 2013). To put it simply, compass is used to draw circular arcs and straightedge is used to draw line segments (Pandisico, 2002). In more detail, compass is described as a tool used to draw a circle with some given properties such as center and another point on it. Straightedge is described as a tool which can be used to draw a line but cannot be used for measurement since it does not have a scale marked on it. Therefore, straightedge is different from ruler (Djorić & Janičić, 2004, Leonard et al., 2014; Petersen, 1927; Sarhangi, 2007; Stillwell, 2005; Stupel & Ben-Chaim, 2013).

Regarding the geometric constructions performed via compass and straightedge, there are five basic constructions which are conducted repeatedly. These five methods are forming a line through two points, forming a circle through a center point with another point, forming a point as the intersection of two nonparallel lines, forming one or two points as the intersections of a line and a circle, and forming one or two points as the intersections of two circles (Stupel & Ben-Chaim, 2013). As mentioned before, more complex geometric constructions can be performed based on basic ones (Djorić & Janičić, 2004; Janičić, 2006, 2010; Karakuş, 2014). For example, eight geometric constructions which are the construction of a line segment and an angle congruent to the givens, the construction of the midpoint of a given line segment, the construction of the angle bisector, the construction of a line perpendicular to a given line at a point on the given line and from a point not on the given line, the construction of a line parallel to a given line, and the construction of line tangent to a given circle at a specified point on the circle explained in the book of Alexander and Koeberlein (2011) can be performed based on such basic constructions. Moreover, the case where students are allowed to use compass and straightedge only in geometric constructions might help them to understand many geometric concepts and lead them to think about how the intended geometric figure can be constructed (Karakuş, 2014). For example, in the construction of a perpendicular line to a given line through a point on it, students may consider two possible solutions which are determining a line segment whose perpendicular bisector passes through the given point on the line and selecting two points to form a 180° angle with the given point on the line and then finding the angle bisector of the formed angle. Both of these methods end up with the same construction steps (Lim, 1997). Due to the restrictions of using compass and straightedge only, geometric construction activities which can be used during teaching in mathematics course propose students a challenging environment and practice for the development of their problem solving skills (Aichele, 1982; Lim, 1997).

Mathematicians also tried to perform constructions by using compass only or straightedge only. According to Aichele (1982), straightedge was not enough to handle Euclidean constructions while compass can be used only in the case where required points were given. Similarly, Kostovskii (1961) studied on the constructions done by using compass only and presented that certain constructions such as the construction of a symmetrical point of a given point with respect to a given line and dividing a circumference into six equal sections can be performed by using compass only and without using the ruler. On the other hand, Stupel and Ben-Chaim (2013) suggested that if a construction can be performed by using compass and straightedge, it can also be performed by a compass only or straightedge only. In such a construction, a circle and its center are provided when using straightedge alone and the straight line is defined with a pair of points while using compass alone.

The development of technology affected many areas in life and the education system is inevitably one of the areas which approved the effects of technology. Mathematics is a domain suitable for the integration of many technological tools such as dynamic geometry software (DGS) (Stupel et al., 2018). Accordingly, there has been an increase in the availability of technology in mathematics education in the last quarter of this century (Zbiek, Heid, Blum, & Dick, 2007). The development of some tools such as GeoGebra, Cabri, Geometer's Sketchpad, and Fathom paved the way for arranging "the dynamic and interactive mathematics learning environments" in classrooms (Martinovic & Karadag, 2012, p.41). By development of the technological tools for mathematics education, these tools started to reach more students in mathematics teaching (Kondratieva, 2013) and the diversification in teaching mathematics concepts came into play. For example, this has led to a change in the role of constructions in geometry and also a new aspect for geometric constructions was formed (Djorić & Janičić, 2004; Kılıç, 2013; Stylianides & Stylianides, 2005). Since DGS has become one of the important and commonly used components in teaching of geometry (Heid, 2005; Hoyles & Noss, 2003; Myers, 2001), using these programs in geometric constructions can be considered as an expected situation.

Although there are different types of DGS, all are designed to model Euclidean geometry and support constructions in geometry (Hoyles & Noss, 2003). For example, Geometer's Sketchpad and Cabri, which have been mostly used DGS in the last decade, can be considered as interactive geometric construction tools (Heid, 2005). Olive (2002) stated the effect of DGS on mathematics teaching as follows; "dynamic geometry turns mathematics into a laboratory science rather than the game of mental gymnastics, dominated by computation and symbolic manipulation" (p.17). Mathematics educators generally consider DGS as tools used to support school mathematics in a way that students can take part in exploration and investigation process (Kostovskii, 1961; Ruthven, Hennessy, & Deaney, 2008). Therefore, following the development of DGS, mathematics educators devised a pedagogical approach and used them "in creating experimental environments where collaborative learning and student exploration are encouraged" (Chazan & Yerushalmy, 1998, p. 72). Thus, by the use of DGS in education, both teachers and students are involved in an environment where they can study about geometry dynamically (Stupel & Ben-Chaim, 2013).

Geometric construction is a suitable concept for applying interactive teaching which can be supported by software (Djorić & Janičić, 2004). Moreover, DGS is useful to help and motivate mathematics teachers to integrate geometric constructions into mathematics teaching (Stupel & Ben-Chaim, 2013) and to produce mathematical illustrations (Janičić, 2010). When DGS is used in a construction, the geometric figure is built in an interactive way and calculations or numbers on the program change in accordance with the movement of a free object belonging to the geometric figure (Schreck et al., 2012). In DGS, students can manipulate the geometric figures directly by using dragging item and when they transform the geometric figures, the corresponding measurements and calculations change relatively (Gerretson, 1998; Hoyles & Noss, 2003). While employing DGS, the users begin to construct geometric figure by dealing with several points, form more complicated figures based on the existing structure, and check the changes in all geometric construction by dragging the movable points (Janičić, 2010). The dragging property of DGS has a great influence on the justification process since it provides the opportunity to check whether the properties of geometric figure remain constant under dragging (Healy & Hoyles, 2002; Mariotti, 2001). In more detail, by using dragging in DGS, students can see how the properties are changing, try various scenarios for the given case, search whether the idea always works or it is a one-off example, realize the key points without explaining them, and be more involved in the lesson (Ruthven et al., 2008). It was stated that the multi-functional structure and critical role of the dragging feature in DGS were not anticipated at the beginning, so the complex nature of the dragging feature was not available by default. However, the dragging was progressively structured and became the delineative feature of DGS (Arzarello, Olivero, Paola, & Robutti, 2002; Ruthven et al., 2008). Moreover, DGS helps students observe changes in a geometric construction, hypothesize, check their hypotheses by dragging, generalize, analyze, and understand abstract structures (Marrades & Gutierrez, 2001). DGS has a critical place among the didactic software growing in the last years since they provided students a field for developing geometrical reasoning and understanding (Mariotti, 2001). Moreover, DGS provides the opportunity to students to conduct and manipulate geometric figures, facilitate their exploration and understanding of abstract geometric concepts (Janičić, 2010), activate the interest on geometric construction concept (Mariotti, 2001), and help them to define the geometry concepts correctly (Kılıç, 2013). According to the study of Ruthven et al. (2008), DGS helps students to work with geometric figures fast, easily, and accurately, keeps them away from the distraction in the lesson which comes with the workload of drawing, supports teachers to organize a classroom environment in which teacher has the role of a guide in the discovery and understanding process of students.

According to the review of the related literature, there is a disagreement about the support of DGS to theoretical thinking and proof (Chazan, 1993, Mariotti, 2001). According to Chazan (1993), some properties of DGS such as dragging and measure might lead students to misinterpret the role of the proof. Ruthven et al. (2008) also emphasized the possible negative consequence of using DGS for students. When a technological program is used for validation of results and for measurement and observation of geometry concepts by both teachers and students, it might lead to accepting geometry with an experimental point of view rather than deductive (Kaiser, 2002; Ruthven et al., 2008) and also students might use it to confirm already known statements empirically (Hölzl, 2001). It was stated that DGS support the visualization in geometry concepts, but it does not offer formal proofs for students. Thus, students should also be taught formal proofs in geometry related subjects (Stupel & Ben-Chaim, 2013). Although it was not denied that there might be some unintended effects of DGS by some researchers, it was also stressed that the appropriate and effective use of DGS in the classroom could change this case (Mariotti, 2001).

By considering the possible effects of using different tools in geometric construction process integrated into the cognitive unity based activities on argumentation and proof, it was decided to arrange different settings. Thus, both compass-straightedge and GeoGebra were utilized in this study.

2.6. Summary of Literature Review

In line with the purposes of the study, the related literature was reviewed in this chapter to provide the background of the concepts mentioned throughout the study. According to the review of the literature, proof was taken into consideration as a critical issue in both mathematics and mathematics education (Conner et al., 2014b; Edwards, 1997; Ellis et al., 2012; Hanna, 2014, 2018; Jones, 2002; Komatsu, 2016; Mariotti et al., 2018; Mejia-Ramos & Inglis, 2009; NCTM, 2000; Stylianides, 2019; Stylianides & Stylianides, 2017). However, a concern declared as related to proof is the lack of the studies conducted by aiming to offer suggestions regarding the classroom-based intervention to utilize while learning and teaching proof related concepts and to propose the approaches to promote students' learning of proof and to overcome their difficulties in proving (Stylianides et al., 2016; Stylianides & Stylianides, 2013, 2018). In this respect, cognitive unity was approached as a promising method to promote learning of proof in this study. Thus, the construct of cognitive unity was focused as the first issue in this chapter. It was followed by the review of the literature related to argumentation and proof in mathematics education since these two concepts constitute the main components of cognitive unity. As related to argumentation, the studies conducted related to Toulmin's model of argumentation and the global argumentation structures were inspected thoroughly. Since the open problems are regarded as effective in terms of the development of proving (Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010; Pedemonte, 2007b), it was decided to involve the open problems in the study. By considering the properties of the open problems, geometric construction was selected as the geometric context situated in the activities. Since geometric constructions has the potential to provide the opportunity to work on making and testing conjectures before dealing with a formal proof (Battista & Clements, 1995) and support students in terms of producing conjectures and constructing proofs (Fujita et al., 2010), it can be inferred that geometric construction can be used as a step for students while passing to formal proof (Arıcı, 2012; Battista & Clements, 1995; Cheung, 2011; Schreck et al., 2012). Thus, as the last remark, the literature related to the concepts of geometry and geometric construction were reviewed in this chapter.

CHAPTER 3

METHODOLOGY

The first purpose of the study is to investigate how prospective middle school mathematics teachers' argumentation while producing conjectures relates to proving within the context of geometry. The second purpose of the study is to examine the global argumentation structures of prospective middle school mathematics teachers emerged while producing conjectures by means of the cognitive unity based activities related to geometric construction. The components of argumentation and the functions of rebuttal were also searched within this purpose. As the third purpose, prospective middle school mathematics teachers' approaches to perform geometric constructions while using compass-straightedge and GeoGebra were investigated. The final purpose is to examine the conjectures produced by prospective middle school mathematics teachers and whether they can conduct valid proofs for these conjectures.

For the ease of reference, the research questions guided this study were restated below;

1. How does prospective middle school mathematics teachers' argumentation process while producing conjectures in the cognitive unity based activities relate to the proving process of the conjectures they produced?

2. What are the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

    2.1. What are the components of global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

    2.2. What are the functions of rebuttals situated in the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities?

3. What are the approaches offered by prospective middle school mathematics teachers to perform geometric constructions in the cognitive unity based activities?

3.1. How do prospective middle school mathematics teachers evaluate the validity of the approaches they offered for geometric constructions?

3.2. To what extent do prospective middle school mathematics teachers perform geometric constructions correctly while using compass-straightedge?

3.3. To what extent do prospective middle school mathematics teachers perform geometric constructions correctly while using GeoGebra?

4. What are the arguments offered by prospective middle school mathematics teachers as the proof of the conjectures they produced?

4.1. What are the conjectures that prospective middle school mathematics teachers produced during the argumentation process?

4.2. To what extent do prospective middle school mathematics teachers conduct valid proof for the conjectures they produced during the argumentation process?

In accordance with the listed research questions, the methodological approach of the study was arranged and explained in this chapter. To that end, this chapter includes the headings of research design, context and participants of the study, data collection procedure, analysis of data, trustworthiness of the study, role of the researcher, and ethical considerations.

3.1. Research Design

In light of the purposes of the study, it was needed to get an in-depth understanding of prospective middle school mathematics teachers' practices as a group while working on the cognitive unity based activities. Put differently, to be able to address the research questions, it was critical to gain a clear understanding of the process that prospective middle school mathematics teachers had been involved in the cognitive unity based activities in which they endeavored on the conjecture production and proof processes within the context of geometry. In this respect, a qualitative research design was decided to utilize since a great number of interconnected and explanatory practices were taken into consideration to be able to interpret properly and get a solid grasp of the issues at stake in such designs (Denzin & Lincoln, 2017). Besides, qualitative research also aims to enhance a multifaceted and in-depth understanding regarding the central phenomenon of the study and to explore the contexts in which the related problems or issues emerge (Bogdan & Biklen, 2007; Creswell, 2007, 2012). In more detail, Denzin and Lincoln (2017) defined qualitative research as given below;

> *Qualitative research* is a situated activity that locates the observer in the world. Qualitative research consists of a set of interpretive, material practices that make the world visible. These practices transform the world. They turn the world into a series of representations, including field notes, interviews, conversations, photographs, recordings, and memos to the self. At this level, qualitative research involves an interpretive, naturalistic approach to the world. This means that qualitative researchers study things in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them (p.43).

Basically, Lapan, Quartaroli, and Riemer (2012) described two dimensions of qualitative research which are “the *interpretive* perspective, which focuses on uncovering participants’ views, and a *critical* perspective, which builds on the interpretive perspective but also examines ways in which power is embedded in social settings” (p.16). Qualitative researchers pay attention to the voice of the participants, their experiences, and how they interpret their experiences (Creswell, 2007; Merriam, 2009).

Qualitative research may be considered as the best option to analyze a phenomenon in the case where the researchers do not know the variables, so the exploration is required (Creswell, 2012). In this respect, although the related literature presents information regarding the main issues of this study which are cognitive unity, argumentation, proof, and geometric construction, a detailed investigation regarding the issues in their own right and the relations and interactions among them is required in this study by learning more from the participants. Moreover, Merriam (2009) signified the four fundamental characteristics of qualitative research. Firstly, in qualitative research, the process, meaning, and understanding are mainly focused. Secondly, the researcher is also regarded as an instrument in both data collection and analysis. Similarly, Bogdan and Biklen (2007) also highlighted that the researcher is a main instrument and the setting is the data source as well. The third characteristics is that qualitative research possesses an inductive process. Lastly, it brings out a thoroughly described product. Based on these characteristics, qualitative research is the best design to answer all research questions stated in this study. Another issue that Merriam (2009) pointed is that "the design of a qualitative study is *emergent and flexible*, responsive to changing conditions of the study in progress" (p.16). Since the study was revised whenever necessary during the ongoing data collection and data analysis processes, it is more suitable to the nature of qualitative research. All in all, to arrange the whole picture of prospective middle school mathematics teachers' practices in the cognitive unity based activities with respect to the purposes of the study, a qualitative research design was required.

In the review of the related literature, it was seen that researchers proposed various categorizations for qualitative research methods (e.g., Creswell, 2007, 2014; Merriam, 2009). The classification of Creswell (2007, 2014) subsumes five types of qualitative research approaches, namely, narrative research, phenomenology, grounded theory, ethnography, and case study. Merriam (2009) mentioned seven approaches in qualitative research which are basic qualitative research, critical qualitative research, phenomenology, grounded theory, ethnography, narrative analysis, and qualitative case study. After deciding to proceed with qualitative research design, the question of which type of qualitative research approach suits best to the purposes of this study came to the fore. At this point, that question shifted the focus to case study research.

3.1.1. Case Study Research

In this study, case study research which is one of the approaches of qualitative research (Creswell, 2007; Merriam, 2009; Stake, 1995; Yin, 2003, 2014) was employed to address research questions in detail. According to Merriam (2009), case study is "an intensive, holistic description and analysis of a single entity, phenomenon, or social unit" (p.46). Similarly, Creswell (2007) stated that case study research involves "an issue explored through one or more cases within a bounded system" (p. 73). More specifically, Creswell (2007) touched upon some notions while explaining case study research. Firstly, the researchers focus on a bounded system or more than one bounded system, each of which corresponds to a case. Secondly, multiple sources such as documents, interviews, observations, and reports are used to carry out a deep data collection process. Lastly, researchers frame the report based on the description of the case and case-based themes. Since case study research is utilized when the aim of the study is to conduct an in-depth investigation of a complex phenomenon, it has potential to present valuable inferences about the focused aspects of the study for the stakeholders (Moore, Lapan, & Quartaroli, 2012). In this respect, since the purpose of this study is to investigate prospective middle school mathematics teachers' practices in cognitive unity based activities thoroughly by considering some aspects such as argumentation, proof, and geometric construction particularly, it was decided that case study research is suitable to the purposes of the study.

In the literature, there are different classifications regarding case study designs based on some criteria such as the size of the case, the interest of the researcher, and the intent to utilize this design (Creswell, 2007; Merriam, 2009). For example, Stake (1995) mentioned three types of case study, which are intrinsic, instrumental, and collective. In more detail, when a particular case is given to work on, the reason for focusing on that case is not to learn about other cases or to search for a general aspect, but just to learn about that specific case and the researcher has "an intrinsic interest in the case" (Stake, 1995, p.3); hence, such studies might be entitled as intrinsic case study. That is, in intrinsic case study, the case itself is in the center since it symbolizes a unique case (Creswell, 2007). In instrumental case study, the aim is to get a general understanding and learn about the problem by examining the case, and also it was stated that "an issue question is of more interest to the researcher than is the case" (Stake, 1995, p.18). To this end, the researchers might select one case to focus on the issue (Creswell, 2007). Alternatively, the researchers might feel the need to select more than one case to learn about the focused concern. Such a study was named as collective case study (Creswell, 2007; Stake, 1995). On the other hand, Yin (2003) classified case studies as explanatory, exploratory, and descriptive. In explanatory case study, the purpose is to explain causes of relationships; in exploratory case study, developing cases and hypotheses for further investigation is aimed; and in descriptive case study, the aim is to explain and define a phenomenon within its own settings.

In case study research, it is critical to determine what would be constituted as case. In a general sense, the case is “a phenomenon of some sort occurring in a bounded context” (Miles & Huberman, 1994, p.25). What can be counted as a case covers many options such as “an instance, incident, or unit of something and can be anything- a person, an organization, an event, a decision, an action, a location like a neighborhood, or a nation-state” (Denzin & Lincoln, 2017, p.600). The definitions regarding the case study design given in the literature (e.g., Creswell, 2009; Merriam, 2009; Moore et al., 2012; Yin, 2003, 2014) have many consistent points. As underscored in many studies, the concept of being bounded is a main issue regarding the determination process of the case. For example, Merriam (2009) defined case study design as “an in-depth description and analysis of a bounded system” (p.40). In this manner, it can be considered that there are two cases in the current study which are compass-straightedge group’s practices in the cognitive unity based activities and GeoGebra group’s practices in the cognitive unity based activities. More precisely, it can be stated that the research design of this study is a multiple-case embedded design based on the classification of Yin (2014), which was presented in Figure 3.1 given below.

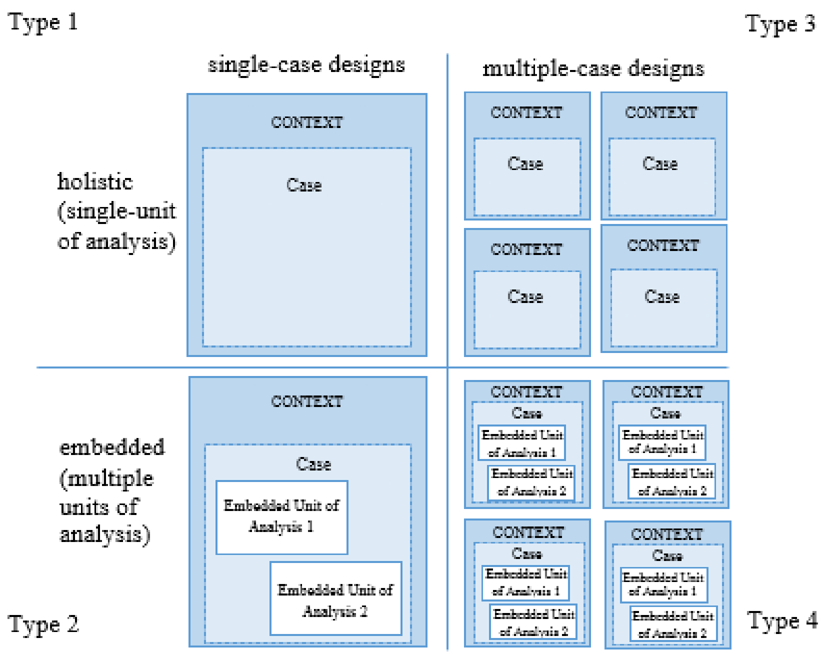


Figure 3. 1. Basic types of designs for case studies (Yin, 2014, p.50)

Yin (2014) arranged types of case study research designs by means of the 2x2 matrix given in Figure 3.1. The matrix in Figure 3.1 introduces four basic case study designs which were organized with respect to two issues, which are the number of the cases and the number of the units of analysis. In more detail, the single case designs cover Type 1 and Type 2, which refer to single-case holistic design and single-case embedded design, respectively. The multiple case designs involve Type 3 and Type 4, which refer to multiple-case holistic design and multiple-case embedded design, respectively. Since the study does not aim to examine the overall nature of each groups' practices in the cognitive unity based activities only and there are main issues considered in the research questions, it could not be coded as multiple-case holistic design. The mentioned issues, which are cognitive unity, argumentation, proof, and geometric construction constituted the embedded units of analysis of the study. In accordance with the basic types of designs for case study in Figure 3.1, how the present study can be modeled was presented in Figure 3.2.

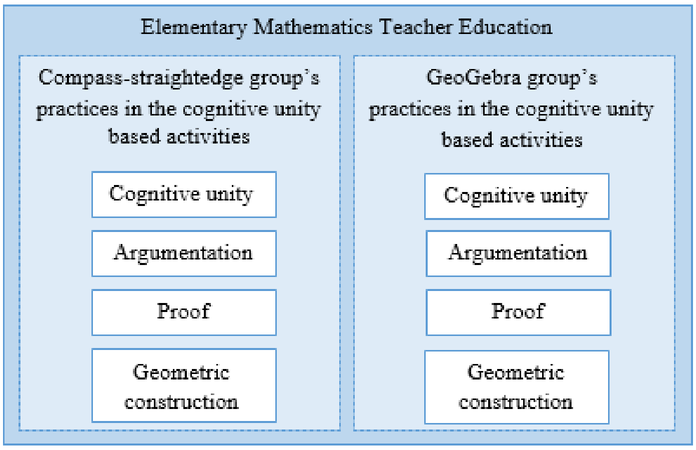


Figure 3. 2. Multiple-case embedded design in this study

After the explanation of the research design, the context of the study and how the participants were determined were explicated in the next section.

3.2. Context and Participants of the Study

A detailed description of the case and the setting are presented in the case study research (Creswell, 2007; Yin, 2003). To have a broader and clearer picture of the study, the context and the participants were explained in detail in this section. First of all, the teacher education program in which the participants enrolled was explained and then the sample selection procedure and the characteristics of the participants were described.

3.2.1. Context of the Study

This study was conducted with prospective middle school mathematics teachers and the program in which the participants registered was depicted at this point. The program at stake is Elementary Mathematics Teacher Education situated under the Department of Mathematics and Science Education in the Faculty of Education. It is a four-year undergraduate program and can be seen in both state universities and private universities in Turkey. In more detail, according to the data taken from the Council of Higher Education (CoHE, 2019), 70 state universities and 8 private universities in Turkey involve Elementary Mathematics Teacher Education program. Upon graduating from this program, prospective middle school mathematics teachers get the qualification to work as mathematics teachers in middle schools whose grades span from 5 to 8 in Turkey. The graduates of this program have to take the Public Personnel Selection Examination, which is an exam conducted once in a year, to be recruited as mathematics teachers in public middle schools. However, for the occupations in private middle schools, to take the mentioned exam is not an obligatory situation.

A varying degree of differences can be seen among the courses in Elementary Mathematics Teacher Education programs across different universities in Turkey. The programs were entitled to conduct some revisions in the courses such as changing a course from must to elective or vice versa and arranging the semester of the courses offered as long as the necessary justification were reported. Thus, the case that a must course of the program in a university may not be among the courses offered in the same program in a different university might be seen. However, types of courses in this program across universities are outlined in a similar way. That is, Elementary Mathematics Teacher Education program offers four types of must courses which are mathematics courses such as Calculus I and II, mathematics education courses such as Methods of Teaching Mathematics I and II, pedagogy courses such as Guidance and Classroom Management, and common compulsory courses such as Ataturk's Principles and History of Revolutions I and Basic English I. Besides, prospective middle school mathematics teachers have to take some elective courses to be able to complete the necessary credits for graduation. To that end, this program also offers some elective courses which are generally related to mathematics and mathematics education. However, with the revision of teacher education programs conducted by CoHE (2018), the courses that would be offered in Elementary Mathematics Teacher Education program were determined in a stricter and standardized manner.

More specifically, the participants are junior prospective middle school mathematics teachers enrolled in Elementary Mathematics Teachers Education program in a state university in Ankara. In the selected university, Elementary Mathematics Teachers Education program has started to admit students since the 1998-1999 academic year which refers to the case that the subject program has a 20-year old experience in teacher education. Moreover, according to the comparison of the university placement test scores of students registered in Elementary Mathematics Teacher Education program, the one in the selected university is always listed in the top five universities. Moreover, the lists of both must and elective courses in Elementary Mathematics Education program of the selected university, which the participants were subject to, were presented in Appendix A. To be able to graduate from the program, prospective middle school mathematics teachers have to complete 240 credits of European Credit Transfer and Accumulation System (ECTS). Since this program involves 37 must courses which correspond to 177 ECTS, they have to take elective courses which bring at least 63 ECTS. Among 37 must courses that the program offers, 9 of them are mathematics courses, 13 of them are mathematics education courses, 7 of them are pedagogy courses, and 8 of them are common compulsory courses. Moreover, as can be seen in Appendix A, Teaching of Geometry Concepts within which the data collection of the main study was conducted, is an elective course offered in the program of the selected university. What comes next will be the details related to the participants.

3.2.2. Participants of the Study

To determine the participants, purposeful sampling in which researchers select the participants who have the highest potential to provide necessary and rich data so as to have an in-depth understanding (Creswell, 2007, 2012; Frankel & Wallen, 2005; Merriam, 2009; Patton, 2002) was utilized. According to Merriam (2009), purposeful sampling is "based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (p.77). In a similar vein, regarding purposeful sampling, Patton (2002) emphasized the importance of "selecting *information-rich cases* for study in depth" (p.46). Instead of purposeful sampling, the terms purposive and judgment sampling were also used (Patton, 2002).

How the criteria for selecting the participants were determined and the selection process were described step by step as follows. The first criterion focused to determine the participants was the accessibility of them for the researcher since it was planned that the researcher would spend the plenty of time with participants during the data collection process to be able to conduct an in-depth investigation. Thus, prospective middle school mathematics teachers in a state university in Ankara were selected as the first step. Secondly, seniors were selected since they were expected to have the highest potential for gathering the detailed information in terms of the basis of the study by comparing the year levels in the program. Since Elementary Mathematics Teacher Education program in the selected university involves all mathematics courses and also geometry and technology related courses in the first three years (See Appendix A), it was anticipated that seniors would have more experience in the main concepts of the study. However, the pilot study conducted with senior prospective middle school mathematics teachers changed this criterion. As it turns out, seniors might not be the most suitable year level in the program. In more detail, at the beginning of the pilot study, to arrange a schedule with seniors was a challenging issue. Since the majority of them were taking two courses only in that term which are Teaching Practice and Turkish Educational System and School Management, they were not at the campus as much as the previous terms. It was not possible to conduct the pilot study within the scope of any course since the majority of them finished all necessary elective courses, there was not a proper course to embed the activities at the time of the pilot study, and also the number of activities aimed to apply in the pilot study was comparatively high. Thus, it was decided to conduct the pilot study with seniors in their free time. Moreover, nearly all of the participants in the pilot study were taking the extra course for the Public Personnel Selection Examination. In this respect, it was not possible to arrange a schedule to apply all twelve activities to all groups. Thus, two activities were applied to each group in the pilot study, the details of which were explained in the following sections.

Due to these issues, it was doubted that seniors in the main study might not volunteer to participate in the data collection process in a longer period. Thus, at the end of each application of the pilot study, senior prospective middle school mathematics teachers were asked about any suggestions about the activities, the degree of difficulty of activities, whether they prefer to participate in such an activity as being a junior or a senior, which year level (junior or senior) they suggest for the main study by considering workload of the activities and their examination concerns to find occupation. Except for one GeoGebra group out of 14 groups in the pilot study declared that they would prefer to participate in such a study in the third year of the program due to their occupation concerns. After the analysis of the pilot study, by discussing with the advisor of the study, the criterion regarding involving seniors was changed and junior prospective middle school mathematics teachers were decided as the participants of the main study. Moreover, for the main study, it was decided to embed the cognitive unity based activities into an elective course and to apply one activity per week. Among the elective courses, Teaching of Geometry Concepts was selected as the most proper one.

During the fall semester of the 2016-2017 academic year, which is also the semester the main study was conducted, the number of junior prospective middle school mathematics teachers in the selected university was 73. The first section involves 38 juniors while the second section involves 35 juniors. Of all the juniors in the program, the participants were also selected by following some criteria. Before the pilot study, it was planned to work with voluntary prospective middle school teachers. After the pilot study, it was seen that prospective teachers who do not have the high GPAs and the relatively high grades in some courses regarding proof and geometry such as Discrete Mathematics and Geometry had difficulty in suggesting ideas, following the collective argumentation, and being an active participant especially in proving process. By aiming to avoid the isolation of any participant during the activities involved in the course and by considering the difficulty level of the activities and the need for the presence of the argumentation in the groups, it was decided to involve the juniors who have the highest success in the program. In this respect, the grades of junior prospective teachers in some related courses and the GPAs were listed. The mentioned courses were selected according to their relevance to the content of the study. Since this study mainly covers the concepts of geometry and proof, the courses involving them were taken into consideration, namely, Discrete Mathematics and Geometry. Then, it was considered that there are other mathematics courses which also cover proof and some geometric concepts in some instances which are General Mathematics, Calculus I, and Calculus II. Moreover, since the study involves the use of GeoGebra, the participants needed to be good at using computer, so their grades in the course Computer I were also involved in the selection process. By listing the grades of the mentioned courses and the GPAs, junior prospective teachers who have the highest success in general terms were determined for each section separately.

Having decided the rationale for selecting the participants, the question of what should be the number of the participants in each section of Teaching of Geometry Concepts course came into the play. Since there are nearly 70 students in each year level of Elementary Mathematics Teacher Education program in the selected university, majority of the courses were offered as two sections so as to present more effective instruction and to decrease the disadvantages of a crowded classroom. Except for the extra circumstances, prospective middle school mathematics teachers in each section were together throughout all courses. In this manner, the same number of participants was aimed to select from each section.

As mentioned, collective argumentation was described as "multiple people working together to establish a claim" (Conner et al., 2014b, p.184). That is, it should be multiple but not as much as to cause some participants to stay out of the argumentation. Depending on the results of the pilot study, it was considered that the number of participants in groups would be either 3 or 4. Therefore, it was decided to involve 7 junior prospective middle school mathematics teachers from each section to form one group of 3 participants and one group of 4 participants. By taking into consideration of their grades of the determined courses and their state of being voluntary, 14 junior prospective middle school teachers who took Teaching of Geometry Concepts as an elective course were determined. Table 3.1 presents information about them such as gender and general point average (GPA) and also the setting in terms of the study such as which section and group they belong to.

Table 3. 1

The characteristics of junior prospective middle school teachers who took Teaching of Geometry Concepts course

Section	Group	Participants	Gender	GPA-1*	GPA-2**
Section 1	Compass-straightedge Group 1 (CSG)	Filiz	Female	3.55	3.62
		Gizem	Female	3.39	3.50
		Bahar	Female	3.35	3.42
	Compass-straightedge Group 2	Selcen	Female	3.66	3.70
		Ahmet	Male	3.37	3.44
		Fulya	Female	3.23	3.29
		Volkan	Male	3.19	3.27
Section 2	GeoGebra Group 1 (GG)	Güler	Female	3.42	3.46
		Zuhal	Female	3.42	3.47
		Berna	Female	3.31	3.34
	GeoGebra Group 2	Sema	Female	3.63	3.70
		Candan	Female	3.42	3.48
		Kerime	Female	3.40	3.44
		Hande	Female	3.18	3.26

*GPA-1 refers to the GPAs at the beginning of the 2016-2017 fall semester

**GPA-2 refers to the GPAs at the end of the 2016-2017 fall semester

As seen in Table 3.1, there are 12 females and 2 males among 14 prospective middle school mathematics teachers who took Teaching of Geometry Concepts course. The column coded as GPA-1 in Table 3.1 indicates their GPAs out of 4.00 at the beginning of the 2016-2017 fall semester. That is, the column of GPA-1 in Table 3.1 can be considered as an indication of the success levels of them based on their grades of all courses which they took in the previous four semesters in the program and these are the GPAs used while selecting the participants. All juniors in the course have a GPA above 3.00. There are 3 prospective teachers who have the GPAs in the range of 3.00-3.24, 8 prospective teachers with the GPAs ranged in 3.25-3.49, and lastly 3 prospective teachers had the GPAs falling between 3.50 and 3.75. Moreover, the column coded as GPA-2 in Table 3.1, which indicates the GPAs at the end of the 2016-2017 fall semester, was given in order to present that all prospective teachers who took this course had increasing success in the program. It was noticed that the GPAs of all participants get above 3.25 at the end of the mentioned semester. To avoid unveiling the identities of the participants, the pseudonyms were determined for each of them.

Although there are 14 prospective middle school mathematics teachers in Teaching of Geometry Concepts course as four groups and it was aimed to involve the data coming from all groups, it was noticed during the analysis that it was not feasible to present the analysis of all data with respect to the research questions aimed to be answered. To this end, the data of all groups for Activity 1 were analyzed in detail as the first step. It was seen that the groups involving four prospective teachers have many instances that they could not keep all of them active in the argumentation all the time and they worked in pairs to discuss some ideas and omitted to inform others about their idea. On the other hand, the groups involving three prospective teachers kept their coherence in working as a group and there was no instance that any member of the group was not aware of the issue argued. Thus, in this study, it was decided to present the findings of the groups involving three prospective teachers, namely, compass-straightedge group 1 and GeoGebra group 1 in Table 3.1. After all, it can be stated that the participants of the study are 6 junior prospective middle school mathematics teachers in a state university in Ankara.

Henceforth, prospective middle school mathematics teachers in compass-straightedge group 1 will be called as compass-straightedge group (CSG) directly and the ones in GeoGebra group 1 will be called as GeoGebra group (GG). In this respect, as it can be deduced from Table 3.1, the participants in CSG and GG are females and their GPA-1s are above 3.25. Since the findings presented in the following chapter were gathered from the six prospective middle school mathematics teachers, the course grades considered while selecting them were presented distinctively in Table 3.2 to provide more information related to them which might be used while interpreting the findings.

Table 3. 2

The grades of the participants in some related courses

		Course Grades*					
Group	**Participants**	General Mathematics	Computer I	Discrete Mathematics	Geometry	Calculus I	Calculus II
CSG	Filiz	A3	A1	B3	A3	A3	A3
	Gizem	A2	A2	B2	B1	A1	A1
	Bahar	C2	A1	C1	A3	B1	C1
GG	Güler	A3	A1	B2	A2	B3	A3
	Zuhal	A3	A1	C1	A1	A1	B1
	Berna	B2	A1	C1	A2	B1	A3

* The coefficients and the point ranges were presented as follows;
A1-4.00 (95-100 points), A2-3.75 (90-94 points), A3-3.50 (85-89),
B1-3.25 (80-84), B2-3.00 (75-79), B3-2.75 (70-74), and C1-2.50 (65-69)

As seen, the majority of them had high grades in the courses which were regarded as completely or partially related to geometry, proof, and technology. As mentioned, among these courses, Discrete Mathematics and Geometry are the ones directly related to proof and geometry, respectively. In this manner, their grades in these courses might provide the interpretable information to the readers with respect to the findings. Moreover, to have more information about participants of the study, the other courses they took in the program might be considered (See Appendix A).

3.3. Data Collection Procedure

As Merriam (2009) stated, case study design does not particularly favor any data collection method, but it was also inevitable that some tools are more commonly used compared to others. In data collection, multiple sources which are audio and video recordings of groups during the cognitive unity based activities, groups' written works and GeoGebra files submitted at the end of each activity, field notes, and focus group interviews were utilized. The data collection process of the present study which lasted nearly one and a half year was presented in Table 3.3 in detail.

Table 3. 3

Time schedule of data collection process

Date	**Activity**
January 2016- February 2016	Preparation of the activities
March 2016	Experts' opinion regarding activities and purposes of the study
April 2016- June 2016	Pilot study
July 2016- August 2016	Analysis of data obtained from the pilot study
October 2016	Preparation of the content of Teaching of Geometry Concepts course in which data were collected
September 3-4 2016- December 26-27, 2016	Teaching of Geometry Concepts course (13 weeks)
January 2017- February 2017	Preliminary analysis of the data obtained from the main study, preparation of interview questions
March 2017- April 2017	Focus group interviews

As seen in Table 3.3, the first step of the data collection process was the preparation of the activities. After getting the opinions of experts regarding activities and refining the activities based on the suggestions and corrections of experts, the pilot study was conducted. Then, the data gathered from the pilot study were analyzed. Depending on this analysis and also considering the purposes of the study, the data collection process of the main study was scheduled. More specifically, it was planned to collect the data from the participants via four cognitive unity based activities in Teaching of Geometry Concepts course. With this aim, the content of Teaching of Geometry Concepts was framed. Moreover, the majority of the data of the main study was collected through Teaching of Geometry Concepts course which involves one introduction week, four teaching session weeks, and eight activity weeks. Moreover, during the course period, the researcher took field notes both during and after the application of the activities. After the preliminary analysis of the data collected in the main study, focus group interviews were conducted regarding the cognitive unity based activities. Thus, the data collection process for this study was completed.

The steps of the aforementioned process were explained in detail in the following parts.

3.3.1. Preparation of the Activities

Since the starting point of the study was the construct of cognitive unity, the activities based on it were aimed to be used to collect the data. In this respect, it was planned that each activity comprises a conjecture production phase by means of working on geometric constructions and a sequent proof phase. At the beginning, twelve activities were prepared by the researcher via searching geometry, geometric construction, and proof related literature (e.g., Aarts, 2008; Alexander & Koeberlein, 2011; Bottema, 2008; Altshiller-Court, 1952; Coxeter & Greitzer, 1967; Fitzpatrick, 2008; Gutenmacher & Vasilyev, 2004; Hajja & Martini, 2013; Honsberger, 1995; Leonard et al., 2014; Morris & Morris, 2009; Serra, 2003; Smart, 1998; Stylianides & Stylianides, 2005; Velleman, 2006; Venema, 2012, 2013). The contents of these activities were presented briefly as in Table 3.4.

Table 3. 4

Activities before the pilot study

Activities	**Content**
Activity 1. Circumcircle of the triangle Activity 2. Incircle of the triangle Activity 3. Orthocenter of the triangle Activity 4. Euler line Activity 5. Miquel point Activity 6. Simpson line	Triangle and circle related activities
Activity 7. Center of the circle	Circle related activity
Activity 8. Deltoid Activity 9. Rhombus Activity 10. Isosceles trapezoid	Quadrilateral related activities
Activity 11. Square inside a circle Activity 12. Circumscribed quadrilateral	Quadrilateral and circle related activities

As stated in Table 3.4, the first six of these activities (Activities 1, 2, 3, 4, 5, and 6) were related to triangle and circle, Activity 7 was related to circle, three activities (Activities 8, 9, and 10) were about quadrilaterals, and the last two activities (Activities 11 and 12) were about both quadrilateral and circle. All of these activities were aimed to be based on the construct of cognitive unity. That is, in the activities, the participants were expected to reach a conjecture or conjectures by means of geometric constructions and then they were asked to prove one of the conjectures they proposed previously. However, after the opinions of experts and the pilot study, four of them, which were concluded to be the most appropriate to the logic of the cognitive unity, were determined and administered in the main study to collect data.

These activities were submitted to the experts in both mathematics education and mathematics. Three experts in mathematics education, who studied about geometry, geometric construction, dynamic geometry software, and proof before, were asked to evaluate the activities in terms of the appropriateness of content of activities to the purposes of the study and the level of the participants, the validity of the given possible construction methods and proof methods for each activity, the potential of restrictions of some tools in GeoGebra, the usage of mathematical terms, and the clarity of the statements. In addition, two doctoral students in mathematics education who were also involved in the studies related to geometry were asked to check these activities by pursuing the same criteria. Moreover, two experts in mathematics were requested to evaluate whether the activities planned based on the cognitive unity fulfill this condition or not, the appropriateness of the statements asked to prove in terms of the level of participants and the relevance to the whole structure and content of the activity, and the validity of the given possible proof methods. According to the corrections and suggestions of these experts, activities were readjusted and organized for the administration of the pilot study.

3.3.2. Pilot Study

Pilot study is a critical step of a qualitative study (Kim, 2011) since it might provide guidance in many aspects such as the concerns of researchers and the applicability of the methods used in the study, help to deepen the understanding of concepts and theory focused in the study, and give insight about the perspectives of the participants (Maxwell, 2013). According to Light, Singer, and Willett (1990), the pilot study is not a waste of time and effort in any occasion and it is needed especially in the case that there are points requiring clarification. The purposes of the pilot study is to determine the points which may cause problems in the actual administration, to detect and develop nonworking points in the activities, to check which activities were appropriate to the level of prospective middle school mathematics teachers, to decide the duration of each section of activities, to arrange the possible guided questions to be used in the cognitive unity based activities for the main study, to determine the number of prospective teachers for each group in the main study, and to select which activities would be more proper to the general purposes of the study.

The pilot study was conducted with senior prospective middle school mathematics teachers enrolled in a state university in Ankara during the spring semester of the 2015-2016 academic year. The participants of the pilot study were selected by convenience sampling. The number of senior prospective middle school mathematics teachers in the selected university was 67 at the time of the pilot study. All prospective teachers were asked whether they wanted to participate in the study and 46 of them responded positively for the participation. To form an environment in which prospective teachers could feel comfortable and free to discuss and study the activities effectively, groups in the pilot study were arranged carefully. By considering prospective teachers' convenient time periods, their characteristics, and relationships among them, groups were formed, and a pilot study schedule was established. According to the schedule of the pilot study, there were 14 groups comprised of 46 prospective middle school mathematics teachers as 40 females and 6 males. Moreover, 10 groups involved 3 prospective teachers and 4 groups involved 4 prospective teachers. Since geometric constructions in the activities were aimed to be conducted by using compass-straightedge and GeoGebra, 7 groups were assigned to the use of compass-straightedge and the remaining 7 groups were assigned to the use of GeoGebra randomly. Six groups among 7 groups which used compass-straightedge worked on two different activities, so twelve activities were administrated in the pilot study. The remaining 1 group was organized as a substitute or a back-up group. Therefore, this last group worked on two activities which were considered as superficially conducted by other groups. Similarly, 6 groups among 7 groups which used GeoGebra conducted two activities and 1 group conducted relatively unclearly worked two activities again.

Each group worked on the activities at different times and in two different classrooms which are a computer laboratory and a classroom. For the groups using compass-straightedge, worksheets on which necessary information about the activity was written and compass-straightedge packs were distributed to each member. Thus, all members of groups have their own worksheets and tools. For the groups using GeoGebra, worksheets were given to every group member, but only one computer was supplied. All groups in the pilot study were asked to share their ideas with the group, discuss about them, and submit their documents as a group, not individually. In the case that group members disagree about a method or an idea, they were asked to write about it on their worksheets. However, such a disagreement was not reported on the worksheets. Moreover, in the pilot study, GeoGebra groups were also asked whether they would prefer to use one computer as a group or use computers individually. All groups preferred to use one computer since they were thinking about the activities as a group. Groups' practices in the activities and the following focus group interviews were audio and video recorded.

After the pilot study, the data were analyzed and possible revisions and problematic points in the activities were listed. By discussing with the advisor of the study regarding the revisions and purposes of the study, activities were rearranged and the data collection tools for the main study were determined. The revisions conducted after the pilot study were explained as follows.

As mentioned before, the participants were decided to be junior prospective middle school mathematics teachers instead of seniors by considering the suggestions of participants in the pilot study and senior prospective teachers' concerns related to occupation due to being in the last year of the program. Since juniors did not take a course related to GeoGebra in the first two years of the program, the need for a teaching session regarding GeoGebra was noticed. In addition to this, it was observed that the majority of seniors in the pilot study did not write their ideas in detail and passed some points quickly due to the time constraints. Thus, it was decided to conduct the main study by integrating into an elective course, which is Teaching of Geometry Concepts. By doing so, it was aimed to apply one activity at a time, to make the participants allocate the necessary time to the activities, and to add some teaching sessions before the application of the activities. Another issue decided after the pilot study is that GeoGebra group would be using one computer in a face-to-face setting to enhance collective argumentation.

Among twelve activities, it was noticed that some of them were more systematic and coherent in terms of relating construction and proof sections. That is, four activities were found out to be more appropriate to call as the cognitive unity based activity. Thus, the main focus of the study was directed on these activities and it was decided that the data of the main study would be collected by means of these four activities. How the activities were evaluated based on the pilot study and the feedbacks of the experts were explained herein. First of all, the first two activities in Table 3.4 which are about circumcircle of a triangle and incircle of a triangle were combined under one activity since all groups in the pilot study constructed incircle while trying to find an approach to construct circumcircle. Since it was noticed that these two activities might overlap for the groups in the main study like the ones in the pilot study, the activity related to the circumcircle of a triangle was decided to be kept and the other one was excluded. Moreover, it was decided that if the argumentation process of groups in the main study does not lead them to construct incircle while trying to construct circumcircle, the construction of incircle would be embedded into the activity by guiding prospective teachers so that they can experience both constructions. Regarding cognitive unity, it was found out that both of them were proper since the participants could produce the main conjecture related to the content of the activity just by working on the construction. That is, when they found a valid approach to perform the asked geometric construction correctly, they would also reach to the main conjecture. Since both of these activities were concluded as proper to the construct of cognitive unity and also combined as one activity, the first cognitive unity based activity to be used in the main study was determined.

It was concluded that activities 3, 4, and 5 in Table 3.4 also resulted in an expected way. That is, all of these activities were also proper to be called as the cognitive unity based activity. However, activity 6 in Table 3.4, Simpson line activity, was one of the most difficult activities. Both compass-straightedge and GeoGebra groups in the pilot study failed to construct the asked geometric figure, reach to the conjecture, and prove the given statement. Thus, it was decided that it is better not to include this activity in the main study. After all, among the six triangle and circle related activities in Table 3.4, it was decided that activities 2 and 6 would not be involved in the main study and the remaining four activities (activities 1, 3, 4, and 5 in Table 3.4) were quite proper to call as the cognitive unity based activity; hence, it was decided to involve them in the main study.

According to the pilot study and the feedbacks of the experts, it was seen that the next six activities given in Table 3.4 involve some problems in terms of calling them as the cognitive unity based activity. In other words, there was not a solid coherence between the construction and proof sections in these activities. It turned out that there was not at least one general conjecture deduced from the construction section which has the potential to be asked for proof. The conjectures the participants produced could not go beyond the ones used in the construction and they could not produce a comprehensive conjecture to be asked in the proof section. Moreover, the experts warned about this issue for the last six activities before the pilot study, but more information about the application of activities from the pilot study was needed. When the results of the pilot study were combined with the feedbacks of the experts, it was decided that these activities were not proper to be called as the cognitive unity based activities due to the nature of them. However, since this study was applied in Teaching of Geometry Concepts course, it was decided that some of these six activities might be utilized in this course. In more detail, 2 of them (activities 7 and 11 in Table 3.4) were decided to be used as orientation activities before the cognitive unity based activities. Moreover, 2 of them (activities 9 and 10 in Table 3.4) were decided to be used as break activities since the participants in the pilot study were quite successful in these activities, developed their self-efficacy regarding the activities, and presented a positive stance towards the construction and proof. However, the last 2 of them (activities 8 and 12 in Table 3.4) were omitted. In more detail, the groups in the pilot study who conducted activity 8 were able to construct the deltoid and prove the given statement quickly and correctly. Therefore, it was seen that there were not comprehensive argumentation processes in the groups of the pilot study. Thus, activity 8 was decided not to be included in any section of the course. On the other hand, activity 12 which is about the circumscribed quadrilateral has a difficult construction section for the groups of the pilot study. Since they could not be clear about what should be constructed in this activity, it was also decided that activity 12 would not be involved in any section of the course. All in all, the last six activities in Table 3.4 are not applicable in terms of cognitive unity. Thus, some of these activities were planned to be either orientation activities for the beginning of the course or break activities interspersed among cognitive unity based ones throughout the course.

To conclude, after the pilot study, eight activities were prepared to be handled in the Teaching of Geometry Concepts course, but the data were conducted by means of the four cognitive unity based activities.

3.3.3. Teaching of Geometry Concepts Course

It can be stated that the revision process of the activities and the preparation of the content of Teaching of Geometry Concepts course were carried out simultaneously. Teaching of Geometry Concepts is an elective and a three-hour per week course offered in the Elementary Mathematics Teacher Education program in the university and it was selected for the main study. Although there is not a prerequisite for this course, it is generally offered for junior or senior prospective middle school mathematics teachers due to the workload and the content of the course. For the data collection of the current study, Teaching of Geometry Concepts course was opened in the fall semester of the 2016-2017 academic year for junior prospective middle school mathematics teachers. It was also offered as two sections since the study involves the use of two different tools which are compass-straightedge and GeoGebra during the activities. Thus, section 1 of the course was offered on Mondays at 9.00-12.00 a.m. for 7 junior prospective middle school mathematics teachers who fulfill the criteria of the study and similarly section 2 of the course was offered on Tuesdays at 9.00-12.0 a.m. for 7 junior prospective middle school mathematics teachers. Moreover, all prospective teachers attended to the course every week. In the case that one of them could not come to the course, the course was taken to an alternative time in the same week. Thus, participation of prospective teachers to the course in full percentage was supported. The details of this course were summarized as in Table 3.5.

Table 3. 5

The content of Teaching of Geometry Concepts course

Course dates	**Content of the week**
The 1st week (September 3-4, 2016)	Introduction of the course, determination of the groups
The 2nd week (September 10-11, 2016)	Basic geometric constructions by using compass-straightedge
The 3rd week (September 17-18, 2016)	GeoGebra
The 4th week (September 24-25, 2016)	Basic geometric constructions by using GeoGebra

Table 3. 5 (continued)

The 5th week (September 31- November 1, 2016)	Reasoning, argumentation, and proof (definitions, methods, proof in geometry)
The 6th week (November 7-8, 2016)	Orientation activity 1 (Center of the circle)
The 7th week (November 14-15, 2016)	Orientation activity 2 (Square inside of the circle)
The 8th week (November 21-22, 2016)	Activity 1- Cognitive unity based activity 1 (Circumcircle of the triangle)
The 9th week (November 28-29, 2016)	Activity 2- Cognitive unity based activity 2 (Orthocenter of the triangle)
The 10th week (December 5-6, 2016)	Break activity 1 (Rhombus)
The 11th week (December 12-13, 2016)	Activity 3- Cognitive unity based activity 3 (Euler line)
The 12th week (December 19-20, 2016)	Break activity 2 (Isosceles trapezoid)
The 13th week (December 26-27, 2016)	Activity 4- Cognitive unity based activity 4 (Miquel theorem)

The content of the course was prepared in a way that the application of the main study was engaged in it. In the registration week of the spring semester of the 2016-2017 academic year, the determined 14 participants were informed about the study, asked to add the course and to attend the first week of the course in the case that they wanted to be involved in it. All of the selected participants came to the first meeting of the course which was planned to be as an introduction to both the course and the study. After the information about the course layout, one prospective teacher in the first section gave up taking the course and also participating in the study. Therefore, the researcher got in contact with the next person in the list of possible participants for section 1 and informed him about the course and study in a similar way. He decided to be a participant and took the course during the add-drop week.

As indicated in Table 3.5, this course is made up of one introduction week, four teaching session weeks, and eight activity weeks, four of which were the cognitive unity based activities. In the first week, the aim of the study and the content of the course were explained and syllabus of the course was distributed. After finalizing who the participants are in each section, the teaching sessions for four weeks were conducted. More specifically, basic geometric constructions by using compass-straightedge were the subject of the second week of the course. The basic geometric constructions were given to the participants with the order presented in Table 3.6.

Table 3. 6

Basic geometric constructions from Alexander and Koeberlein (2011)

	Geometric constructions
GC1	To construct a segment congruent to a given segment (p.16)
GC2	To construct an angle congruent to a given angle (p.30)
GC3	To construct the midpoint M of a given line segment AB (p.16)
GC4	To construct the angle bisector of a given angle (p.30)
GC5	To construct the line perpendicular to a given line at a specified point on the given line (p.49)
GC6	To construct the line that is perpendicular to a given line from a point not on the given line (p.72)
GC7	To construct the line parallel to a given line from a point not on that line (p.90)
GC8	To construct a tangent to a circle at a point on the circle (p.310)

As presented in Table 3.6, eight basic geometric constructions in the book of Alexander and Koeberlein (2011) were given to prospective teachers and they were aked to work individually at first. After each geometric construction, participants shared their ideas with others in the classroom, showed their construction idea on the board, and discussed whether their approaches could be labeled as construction or not. The basic geometric constructions were given to the participants with the order presented in Table 3.6. At the end of the second week, a handout explaining one known approach for each basic construction was sent to prospective teachers via e-mail and they were asked to read the handout until the next week.

In the third week of the course, GeoGebra was introduced to the students. Since participants took Computer I only as a course regarding technology in the first year of the program and did not take a course related to the use of technology in mathematics education such as the course Computer Assisted Mathematics Education which they will take in the spring semester of the third year of the program, GeoGebra was not a familiar program for them at the time of the data collection. Therefore, GeoGebra was explained to both of the sections and some activities involving GeoGebra were prepared to make them practice working with GeoGebra. As the preparation for the next week, prospective teachers were asked to work on the properties of GeoGebra until the following week.

In the fourth week of the course, each section worked on the same basic geometric constructions they performed with compass-straightedge by using GeoGebra. Similar to the compass-straightedge week, prospective teachers were asked to work on constructions individually, try to use different tools of GeoGebra for performing the same constructions, explain their ideas to others, and check whether their methods represent a valid construction or not by dragging. Likewise, a handout involving how each basic construction can be performed by using GeoGebra was sent to prospective teachers via e-mail at the end of that week and assigned as the reading until the next week. Moreover, these two weeks of the course in which GeoGebra was used were conducted in a computer laboratory for each section.

In the last week of teaching sessions, the concept of the course was planned as argumentation, reasoning, and proof. Prospective teachers were informed by distributing handouts and using a presentation about argumentation, reasoning, and proof in mathematics education, definitions of these concepts in the literature, methods of proof such as direct proof, proof by contradiction, proof by contrapositive, counterexample, and induction, proof schemes, notation and formal writing in proofs, proof in geometry, and examples of proof regarding geometry concepts. Moreover, at the end of this week, which section of the course would be working with compass-straightedge and GeoGebra was determined. Since all prospective teachers in section 1 were willing to use compass-straightedge and similarly all prospective teachers in section 2 agreed to use GeoGebra while dealing with activities, this issue was handled easily. As stated, each section involved 7 prospective middle school mathematics teachers. Prospective teachers in each section were also asked to form two groups involving 3 and 4 participants at the end of this week. Thus, it can be stated that two groups in each section were determined by prospective teachers before starting to work with activities.

As seen from Table 3.5, Teaching of Geometry Concepts course involves eight weeks for activities after four teaching session weeks situated at the beginning of the course. The cognitive unity based activities were not presented sequentially in Teaching of Geometry Concepts course. Firstly, two orientation activities were included to help prospective middle school mathematics teachers to become aware of the structure of the activities, to be familiar with the geometric construction and proof sections in the activities, to get used to working as a group and being recorded by camera, and to see what they are expected to do in terms of the study before dealing with the cognitive unity based activities. In addition to two orientation activities and four cognitive unity based activities, two break activities were planned to be added to the course. Since the cognitive unity based activities are related to triangles and circles and involve both construction and proof phases which are comparatively difficult, it was aimed to add two quadrilateral related and relatively easy activities to the course as break activities. By considering the analysis of groups and the suggestions of groups of the pilot study, two break activities were interspersed among cognitive unity based activities in order to create a course environment in which prospective middle school mathematics teachers can see that they are able to conduct these activities properly so that they will not be discouraged and not give up because of the difficulty of other activities. Moreover, since all cognitive unity based activities are related to triangles and circles, to deal with quadrilaterals in these two activities throughout the course might prevent them being bored and make the course and the study more interesting and non-monotonic for them.

3.3.4. Data Collection Sources

Qualitative studies involve many data sources such as documents, observation, interview, video, drawing, and newspaper. According to the purposes of the study, the researcher might use one of such sources or a combination of them for triangulation (Strauss & Corbin, 2008). In this study, the data were collected through audio and video recordings of groups during the cognitive unity based activities, groups' written works and GeoGebra files submitted at the end of each activity, field notes, and focus group interviews. In this section, each of the mentioned data collection sources is explained.

3.3.4.1. Cognitive Unity Based Activities

As mentioned, the data of the study were collected by means of four cognitive unity based activities, which were presented in Appendix B. All of these activities were related to triangle and circle and they cover two worksheets due to the dual nature of them. In the first section, the participants were expected to produce conjecture by virtue of geometric constructions. For this purpose, the worksheet A was employed in each activity. In addition to the worksheet A, the pairs of compass-straightedge were distributed to the participants of CSG and GeoGebra files were given to GG to use in this step. In the second section, the participants were expected to prove one of the conjectures they recently produced. For this purpose, the worksheet B was distributed in each activity. In this step, CSG and GG were allowed to use the compass-straightedge and GeoGebra, respectively, in any case they wanted. The practices of groups in the cognitive unity based activities were videotaped and audio recorded.

The first cognitive unity based activity, which will be called simply as Activity 1 henceforth, was prepared by the researcher based on the review of the literature (e.g., Alexander & Koeberlein, 2011; Gutenmacher & Vasilyev, 2004; Leonard et al., 2014; Serra, 2003). The worksheet A in Activity 1 was displayed below.

ACTIVITY 1

Worksheet A

Construct the circle passing through the vertices of the given ΔABC by using compass-straightedge/GeoGebra.

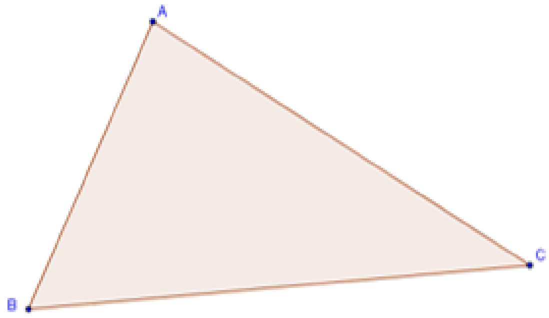


Figure 3. 3. Worksheet A in the cognitive unity based activity 1 (Activity 1)

As displayed in Figure 3.3, the first section of Activity 1 is related to the construction of circle passing through all vertices of a given acute triangle. CSG was asked to construct by using compass-straightedge. On the other hand, GG was asked to construct by using the given two GeoGebra files in the given order. The first GeoGebra file covers two restricted tools which are ‘circle through three points’ and ‘circumcircular arc’. These tools were removed from the toolbar since GG would construct the asked geometric figure with one or a few clicks by using these tools. The second GeoGebra file involves three more restricted tools which are ‘midpoint or center’, ‘perpendicular line’, and ‘perpendicular bisector’. GG was asked to work on the second GeoGebra file after they found a valid approach for geometric construction. Due to the absence of these extra tools, it was expected that the construction would be more challenging for GG and different approaches would be offered. Upon finishing the geometric construction section and producing some conjectures involving the target ones, it was the break time of the course. After the break, the worksheet B of Activity 1, which was given below, was distributed.

Worksheet B

Prove the statement given below.

The perpendicular bisectors of the sides of a triangle are concurrent and this point is the circumcenter of the triangle.

Figure 3. 4. Worksheet B in the cognitive unity based activity 1 (Activity 1)

For the second section of the activities, one conjecture which is directly related to the concept of the activity was selected. The conjectures which were expected to be produced and planned to be asked for proof in case of existence are stated as follows; “the perpendicular bisectors of the sides of a triangle are concurrent” and “the point of concurrency of the perpendicular bisectors of the sides of a triangle is the circumcenter”. Since both of the groups reached these conjectures, the worksheet B in Figure 3.4 was given to both groups.

The second cognitive unity based activity, which will be called simply as Activity 2 henceforth, was prepared by the researcher based on the review of the literature (e.g., Aarts, 2008; Alexander & Koeberlein, 2011; Bottema, 2008; Altshiller-Court, 1952; Gutenmacher & Vasilyev, 2004; Hajja & Martini, 2013; Leonard et al., 2014). The worksheet A in Activity 2 was presented in Figure 3.5.

ACTIVITY 2

Worksheet A

Construct the altitudes and the orthocenters (if exist) of the given ΔDEF , ΔABC , and ΔKLM by using compass-straightedge/GeoGebra.

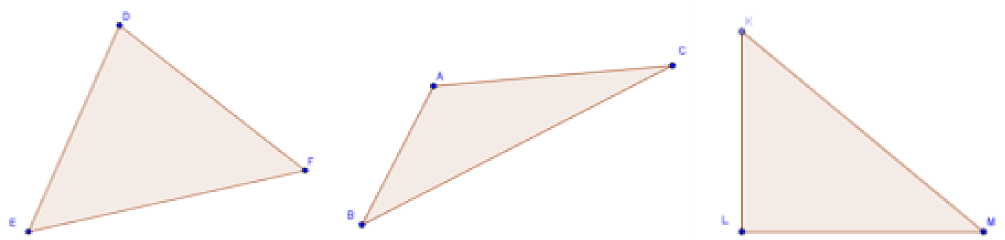


Figure 3. 5. Worksheet A in the cognitive unity based activity 2 (Activity 2)

Although all three triangles were presented in the same page in Figure 3.5, each triangle was given in a separate page. The first section of Activity 2 is related to the construction of the altitudes and the orthocenters of the given acute, obtuse, and right triangles. As expected, CSG worked on it by using compass-straightedge while GG worked on it by using GeoGebra. Three GeoGebra files, each of which involves one type of triangle, were given to GG, and the tool ‘perpendicular bisector’ was removed from all GeoGebra files since they would directly construct the altitudes by using this tool. After this section, it was time to distribute the worksheet B of Activity 2, which was presented in Figure 3.6.

Worksheet B

Prove the statement given below.

The altitudes of a triangle are concurrent.

Figure 3. 6. Worksheet B in the cognitive unity based activity 2 (Activity 2)

The possible conjectures that the groups might reach and also the ones asked for the proof in the second section of Activity 2 were listed beforehand. The more inclusive and general conjectures such as “the altitudes of a triangle are concurrent” and “each triangle has an orthocenter” were also among the expected ones. Since both groups were able to produce them, the worksheet B in Figure 3.6 was employed for both groups.

The third cognitive unity based activity, which will be called as Activity 3 henceforth, was prepared by the researcher upon reviewing the literature (e.g., Alexander & Koeberlein, 2011; Altshiller-Court, 1952; Leonard et al., 2014; Venema, 2013). The worksheet A of Activity 3 can be seen in Figure 3.7.

ACTIVITY 3

Worksheet A

Construct the orthocenter, the circumcenter, and the centroid of the given ΔABC by using compass-straightedge/GeoGebra.

Examine the connection/relationship among these points.

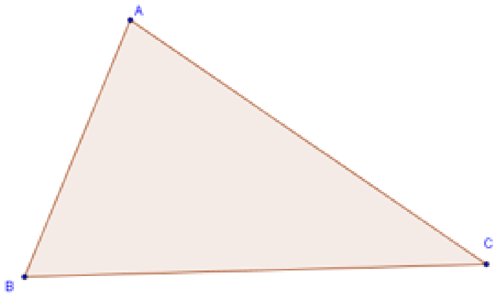


Figure 3. 7. Worksheet A in the cognitive unity based activity 3 (Activity 3)

The geometric construction phase of Activity 3 has some common points with the first two activities. The construction of the circumcenter of a triangle is the issue of Activity 1 and the construction of the orthocenter is the issue of Activity 2. In addition to these points, Activity 3 asked for the construction of the centroid of the given triangle. Regarding GeoGebra files, two GeoGebra files covering the same triangle were given to GG. The default toolbar was kept in the first GeoGebra file, but three tools which are 'midpoint or center', 'perpendicular line', and 'perpendicular bisector' were restricted in the second GeoGebra file. As different from the previous activities, Activity 3 asked to examine the connections among these points which led them to the target conjectures. Subsequent to this section and the break, the second phase of Activity 3 was started. In this respect, the worksheet B of Activity 3 was given below at this point.

Worksheet B

Prove the statement given below.

The circumcenter, the orthocenter, and the centroid of a triangle are collinear.

Figure 3. 8. Worksheet B in the cognitive unity based activity 3 (Activity 3)

The conjectures which were expected and would be represented with target conclusion in Activity 3 were “the circumcenter, the orthocenter, and the centroid of a triangle are collinear” and “the distance from the centroid to the orthocenter is twice of the distance from the centroid to the circumcenter”. However, both groups came up with the first one and did not notice the second one. Thus, the groups were asked to prove the statement presented in Figure 3.8 in Activity 3.

The fourth cognitive unity based activity, which will be called as Activity 4 henceforth, was prepared by the researcher based on the review of the literature (e.g., Aarts, 2008; Alexander & Koeberlein, 2011; Honsberger, 1995; Leonard et al., 2014; Venema, 2013). The worksheet A in Activity 4 was displayed in Figure 3.9.

ACTIVITY 4

Worksheet A

Mark random points X, Y, and Z on the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of the given ΔABC respectively.

Construct the first circle passing through the points A, X, and Z, second circle passing through the points B, Y, and Z, and the third circle passing through the points C, Z, and Y by using compass-straightedge/GeoGebra.

Examine the connection/relationship among these circles.

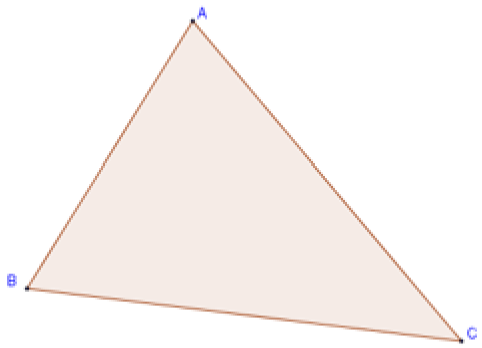


Figure 3. 9. Worksheet A in the cognitive unity based activity 4 (Activity 4)

Although it was not presented explicitly in the activities, the geometric constructions in Activities 1 and 4 have a common point, but the overall aims of them were quite different. In more detail, the construction of circumcircle was asked in both activities in an implicit manner. As can be seen in Figure 3.9, groups were asked to determine one point randomly from each side of the given ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point Z is on $\overline{CA}$. Then, they were asked to construct three circles, each of which is passing through one vertex and two points marked on the adjacent sides. For this activity, one GeoGebra file was given to GG and the tools 'circle through three points' and 'circumcircular arc' were removed from the toolbar. After this section, it was time to distribute the worksheet B of Activity 4, which was indicated in Figure 3.10.

Worksheet B

Prove the statement given below.

Suppose that the point X, Y, and Z are placed at random on the sides of ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point Z is on $\overline{CA}$. Then, in every case, the circles AXZ, BXY, and CYZ are concurrent.

Figure 3. 10. Worksheet B in the cognitive unity based activity 4 (Activity 4)

In Activity 3, it was expected that the participants might produce various types of conjectures. Among the expected ones, both groups produced two conjectures which were considered as the most likely to be stated. The mentioned conjectures were "three circles are concurrent" and "the new triangle formed by referring the centers of the circles as the vertices of it is similar to the given triangle". Since the second one would be more difficult in terms of proving for the participants, it was decided that the first conjecture was given to both groups to be proved. Thus, the worksheet B in Figure 3.10 was distributed to both groups.

3.3.4.2. Documents and GeoGebra files

Both CSG and GG worked on the given worksheets in the cognitive unity based activities. These worksheets were counted as the documents of the study. Since the groups' written works are quite important in terms of understanding the details of both geometric construction and proof, these documents constituted an important data source of the study. In addition, GG saved the GeoGebra files that they thought as covering valid approaches for geometric constructions and submitted them at the end of the activities. Both the worksheets and GeoGebra files were also critical in focus group interviews since they were actively examined by groups during the interviews and the researcher used them while determining the interview questions.

3.3.4.3. Field Notes

According to Bogdan and Biklen (2007), field notes are “the written account of what the researcher hears, sees, experiences, and thinks in the course of collecting and reflecting on the data in a qualitative study” (p.118-119). Although the researcher was the participant-observer during the activities, it was paid attention to take field notes during the activities as much as possible and the field notes were edited right after the activities.

3.3.4.4. Focus Group Interviews

In a qualitative study, interviews might serve as the main technique to collect data or they are combined with some other data sources such as document analysis and observation (Bogdan & Biklen, 2007). It is the second case in the present study since there are multiple data sources. Interview is one of the basic sources of information in case studies (Merriam, 1998; Yin, 2003). In this study, focus group interviews with CSG and GG were conducted. Patton (2002) described focus group interview as “an interview with a small group of people on a specific topic” (p.385). In focus group interviews, the participants do not have to agree on any issue. It involves considering the responses of others and commenting on them; thus the interaction in the group is expected (Merriam, 2009; Patton, 2002). Since the participants worked as a group and constituted a collective argumentation during the cognitive unity based activities, it was planned to conduct focus group interviews regarding the activities after the preliminary analysis of the data collected during Teaching of Geometry Concepts course. All focus group interviews were videotaped and audio recorded as well. Since the questions in the interviews depend on the activities and groups’ practices in these activities, there are both common and different questions in the interviews. An example for the interview questions for both groups in Activity 1 was presented in Appendix C.

3.4. Data Analysis

After collecting data via the mentioned sources, the verbatim transcriptions of audio recordings of groups were carried out as the first step. Then, these transcripts were converted to clearer and more applicable versions for the analysis by watching the video recordings of group discussions. More specifically, the raw versions of the transcripts were edited in terms of writing the explanation in parentheses for unclear utterances, complex drawings, and deictic terms regarding geometric figures by analogizing to video recordings, documents, GeoGebra files, and field notes. Thus, all videotaped and audio recorded data from cognitive unity based activities and focus group interviews concerning these activities were transcribed attentively and prepared for the analysis. Moreover, at the beginning of the analysis process regarding each research question, related transcripts were read and examined simultaneously with groups' documents and GeoGebra files, margin notes were taken, and relevant video recordings were watched when required for clarification by the researcher. Hence, the researcher became familiar with the data which is needed for a proper and accurate analysis of the data.

3.4.1. Data Analysis of the First Research Question

To answer this research question, MAXQDA which is one of the qualitative analysis software was used in the coding process. Before starting to analyze the data via MAXQDA, the preliminary analysis was taken into account such as taking margin notes while reading the transcripts of group discussion. As Elo and Kyngäs (2008) stated, content analysis might be utilized with both quantitative and qualitative data. Hsieh and Shannon (2005) described qualitative content analysis as "a research method for subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns" (p.1278). Moreover, content analysis might cover a deductive approach or an inductive approach

(Cho & Lee, 2014; Elo & Kyngäs, 2008; Moretti et al., 2011). When the studies in the literature could not provide enough knowledge about the phenomenon focused and it was needed to derive the codes from the data collected, the inductive approach can be utilized (Elo & Kyngäs, 2008). It can be stated that it is the case for the analysis of the data to address the first research question. To this end, the stages of inductive content analysis which are "open coding, creating categories and abstraction" (Elo & Kyngäs, 2008, p.109) were followed. In each cognitive unity based activity, both argumentation while producing conjecture and proving process of groups were examined in detail and compared, and then the related codes and themes were arranged.

3.4.2. Data Analysis of the Second Research Question

While addressing the second research question which is about the investigation of the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities, it can be stated that a combination of the frameworks outlined in some studies was used. Specifically, it can be stated that three-stage process of Knipping (2008) was followed during the analysis of data, but some adaptations to three-stage process were conducted by considering the forenamed purposes of the present study. In addition, while constructing the global argumentation structures, the study of Verheij (2005) was used in terms of the schematic representation of the rebuttals. The components which rebuttals were stated against were specifically pointed by using an arrow in a similar way Verheij (2005) used (See Figure 2.9). However, during the analysis of data obtained from both the pilot study and the main study, it was noticed that there some gaps and discrepancies between the data of the study and the mentioned argumentation related frameworks. Therefore, regarding the layout of arguments in this study, some adaptations were arranged by considering the context of the activities, the nature of the givens and questions in the activities, the number of the participants in the groups, the role of the instructor who was also the researcher, and the nature of the oppositions declared during the argumentation.

The study of Knipping (2008) is an extension of Toulmin's model to investigate the whole argumentation process in a classroom. Although the main ideas and components of Toulmin’s model were retained, some reconstructions for stating the global argumentation structures, which were explained below, were carried out in the present study. As stated before, there are three components in the basic argumentation model proposed by Toulmin (1958, 2003), namely, data (D), warrant (W), and claim (C). Then, three more components which are qualifier (Q), rebuttal (R), and backing (B) were added to the model (Toulmin, 2003). Based on the definitions of these six components of Toulmin’s model given in the literature, which were presented in the literature review chapter in detail (See Tables 2.1-2.6), the operational definitions for the analysis of the present study were determined, which can also be seen in the definition section of the introduction chapter. Moreover, a difficulty encountered in the analysis led to the addition of a sub-question in the study. There are some statements which do not fit any component of Toulmin’s model, but they are also affecting the flow of the argument. Such statements were decided to be included as new components later on. In this regard, two components which are data/conclusion and target conclusion from the study of Knipping (2008) were included with some adaptations. Since the conclusion of an argument may be the data of the following argument (Conner et al., 2014a; Krummheuer, 2007; Knipping, 2004, 2008), such statements in the argumentation stream were labeled as conclusion/data (C/D) in this study. As it is seen, conclusion/data was preferred to be used instead of data/conclusion as Knipping (2008) used since it was considered that data should be the second term in the component since this component provides the data for the following argumentation process. In addition, Knipping (2008) used the term target conclusion for all conclusions in the global argumentation structure except the ones labeled as data/conclusion. However, target conclusion (TC) was used with a different meaning in this study due to the context of the activities. It was used for the conjectures prospective middle school mathematics teachers produced which are particularly relevant to the concept of the activity, so they can be asked for proof.

In the present study, some new auxiliary components were also inserted to the global argumentation structures, namely, guidance (G), challenger (CH), and objection (O). When the expressions of the instructor were not directly fit into any one of the main six components, the need for a new component for such statements emerged.

Thus, the expressions of the instructor which present some clues related to the activity and affect the flow and direction of the argument were marked as guidance. Moreover, it was noticed that some statements of both the participants and the instructor were not directly proper to be coded as rebuttal, but they somehow interfered with the flow of the discussion. Such statements were coded as challenger or objection. In more detail, the characteristics of the statements categorized as challenger were presented as follows; this kind of statements basically challenge ideas by leading others to think for a while, causing others to have question marks or to hesitate, and putting a different case and point of view on surface regarding the concept of the activity but without aiming to defeat the argument like the rebuttal component. For example, in Activity 1, one participant stated that "what happens to the circumcenter when the given is an obtuse triangle" and then the rest of the group started to think about this extra case related to the activity. As seen, this statement directly affected the flow of the argumentation and did not cover the purpose of refutation. When the arguers state an objection throughout the discussion without giving the reasoning behind their opposition, this kind of statements was coded as objection during the analysis. For example, in Activity 1, a participant stated that "I think, it is not true, what you drew is incorrect" without explaining the reasoning and caused other participants to explain her method in order to convince. Moreover, Knipping (2003, 2004) stated the term objection in the schematic representation of global argumentation structures and organized it with a different function although it was not directly stated as a component of argumentation model.

At this point, all components (data, warrant, conclusion, backing, rebuttal, qualifier, conclusion/data, target conclusion, guidance, challenger, and objection) used in the formation of the global argumentation structures were clarified. After that, each component presented in the group discussions was identified from the transcripts and the global argumentation structures were organized. Moreover, the global argumentation structures of CSG and GG for a particular activity were generated consecutively and then the researcher passed to the analysis of another activity. By this way, it was aimed that all argumentation structures for one activity were conducted coherently; hence, extra conditions can be detected. How the global argumentation structures were organized in this study was explained in detail as follows.

Previously stated, Knipping (2008) used three-stage process to analyze the argumentation during proof in classrooms. Three-stage process was presented by descending into details as given in Figure 3.11.

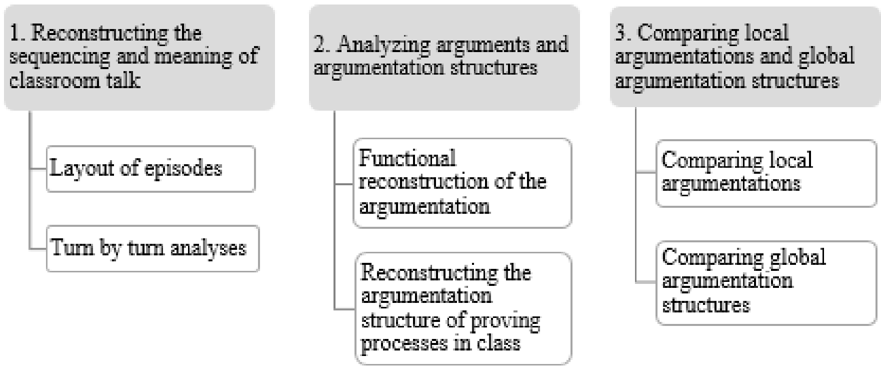


Figure 3. 11. The summary of three-stage process offered by Knipping (2008)

As seen from Figure 3.11, the first stage involves two sub-stages which are layout of episodes and turn by turn analysis. That is, as the first step, discussion in the classroom is arranged into the episodes in which a particular topic was discussed. The specification of episodes may help to discover an overview of the argumentation and to analyze in an easier way since the flow and the streams of argumentation become more visible. Since argumentation process involves a group of students, teacher and the interaction among them, the meanings emerged in this interaction process should be reconstructed and incomplete expressions and deictic terms should be made clear via conducting turn by turn analysis (Knipping, 2008). In this study, the first stage was carried out with minor adaptations. Since the argumentation process of groups was aimed to be investigated in this study, the episodes of the groups were determined in such a way that the discussions about the desired geometric construction and the production of conjectures related to the concept of the activity were focused on. All transcriptions were conducted thoroughly and read a few times during the process so as not to fail to notice any interaction in the argumentation. Moreover, the scope of the study is not the whole classroom. That is, it aims to analyze the argumentation structures of groups separately. Therefore, classroom emphasis in this stage was revised as using the term group.

The second stage in Figure 3.11 mainly involves the reconstruction of local arguments, argumentation streams, and the global argumentation structures consecutively. For this purpose, it covers two sub-stages which are the functional reconstruction of the argumentation and reconstructing the argumentation structure of proving processes in class. During the functional reconstruction of the argumentation sub-stage, the statements are identified and classified as components of argumentation depending on their functions in the argumentation process. At this point, it was suggested to start with detecting conclusions in the argumentation. After the formation of argumentation steps and argumentation streams, the global structure of argumentation is constructed. Then, the schematic representation of the whole argumentation process is organized. In schematic representation, all elements of the argumentation are represented with different items such as rectangles, circles, and diamonds to make the global argumentation structure easier to interpret and analyze (Knipping, 2008). An example of the schematic representation from the study of Knipping (2008) was presented below.

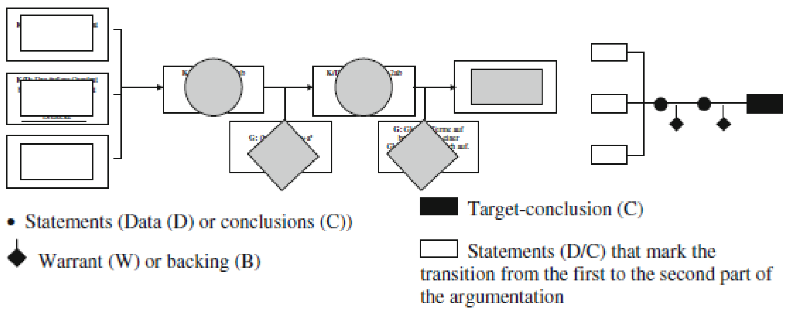


Figure 3. 12. The schematic representation of the argumentation structure (Knipping, 2008, p.435)

In the current study, it can be stated that this stage was also followed with some revisions. The proving and the classroom emphases in these sub-stages were removed while applying the framework. The components of argumentation used in the local arguments in the study of Knipping (2008) were not adequate. Therefore, the number of components was increased and eleven components (data, warrant, conclusion, backing, rebuttal, qualifier, conclusion/data, target conclusion, guidance, challenger, and objection) were utilized. Correspondingly, the schematic representation in reconstructing the argumentation structure was also revised in a way presented in Figure 3.13.

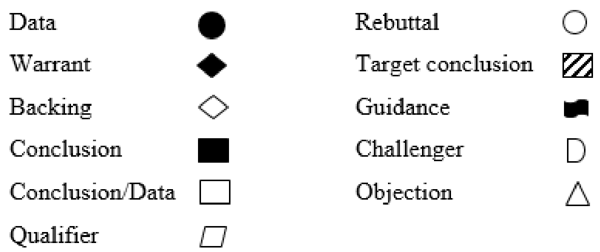


Figure 3. 13. The items used in the schematic representation in the present study

The last stage in Figure 3.11 involves two sub-stages which are comparing local argumentations and comparing global argumentation structures. In the comparison of local arguments, Knipping (2008) analyzed types of warrants with respect to the field of justification applied which are conceptual argumentation and visual argumentation. As the last sub-stage of all process, Knipping (2003, 2008) classified the global argumentation structures and explained two types of argumentation structures which are the source-structure and the reservoir-structure. In this study, the comparison regarding the local arguments was not conducted during the analysis since the purpose of this study centered upon the global argumentation structures. Regarding the comparison of the global argumentation structures, the related studies in the literature were examined and new types of argumentation structures were constituted based on the data of the present study. What was conducted in the last sub-stage of three-stage process in this study was explained in detail below.

In the present study, as the first step of entitling the global argumentation structures, the characteristics of all types of argumentation structures proposed in the mentioned studies (e.g., Erkek, 2017; Erkek & Işıksal-Bostan, 2019; Knipping 2003, 2004, 2008; Knipping & Reid, 2013, 2015, 2019; Reid & Knipping, 2010) were examined and also examples and excerpts from the argumentation process given in these studies were reviewed thoroughly. When the argumentation schemas formed in the current study and six types of global argumentation structures in the literature were compared, it was seen that there are some controversial points in terms of the applicability of existing types. Thus, some revisions were needed to be conducted related to the existing types and some new types of global argumentation structures were introduced to be able to classify the global argumentations which emerged in this study.

In general terms, the global argumentation structures formed in this study were categorized under two headings which are the mono structures and the hybrid structures. In more detail, after the analysis of the characteristics of the global argumentation structures in this study, four types of structure which are the line-structure, the reservoir-structure, the funneling-structure, and the branching-structure were arranged. Depending on the activity, some global arguments can be described by means of one of the mentioned four types of structures. Such global argumentation structures involve one argumentation block or more than one argumentation blocks which can be classified under the same type of structure. When the global argumentation structure can be coded with one type of structure, it was categorized under the mono structures. On the other hand, since global arguments involve the whole discussion of groups during conjecturing, each of which lasts at least one hour, some of them appeared to have a more extensive structure. It was observed that some global arguments involve more than one argumentation block which can be coded with different types of mono structures. Since such global argumentation structures are composite of more than one type of mono structure, they were coded under the hybrid structures.

The first type of argumentation structure is the line-structure which was presented in the studies of Erkek (2017) and Erkek and Işıksal-Bostan (2019). However, the characteristics of the line-structure were not directly applicable in this study, so some revisions were conducted. In the current study, the argumentation streams in which argumentation flows linearly and ends with either a conclusion or a target conclusion without linking and spreading to other streams were classified as the line-structure. Since such type of structures involves one argumentation stream, they look like a line which also means that parallel argumentation streams are not seen in the line-structure. Moreover, the feature that the transitions are conducted with the component claim/data was not accepted directly. Regarding this characteristics, it was accepted that the component claim/data with the function of transition could be involved in the argumentation stream, but it is not a necessary condition. Moreover, some other characteristics of the line-structure stated by Erkek (2017) such as the presence of the rebuttals and warrants and the absence of the reasoning backward were not focused in this study since they were accepted as issues independent from the classification of the global argumentation structures.

The next two types of argumentation structures which are the reservoir-structure and the funneling-structure were originated from the study of Reid and Knipping (2010). Firstly, the reservoir-structure introduced by Reid and Knipping (2010) was used with some revisions in this study. There are intermediate target conclusions which divide the overall argumentation into self-contained parts in the reservoir-structure. The target conclusions which represent the transition from the first part to the next part of the argumentation were described metaphorically as reservoirs that "hold and purify water before allowing it to flow on the next stage" (Reid & Knipping, 2010, p.185). This feature was the one focused in this study while using the term reservoir-structure. As mentioned, the reasoning backward and then forward in the argumentation was not kept as a necessary characteristics of the reservoir-structure. To sum up, without accepting the absence of rebuttals and the reasoning backward as prerequisites and focusing only on the intermediate target conclusion which functions like a reservoir, the reservoir-structure was utilized as a type of global argumentation structure in this study.

As the third type of global argumentation structure, it can be stated that the funneling-structure got inspired from the source-structure in the study of Reid and Knipping (2010). In the source-structure, there are multiple origins for parallel argumentation streams at the beginning of schema, but then a funneling effect is presented in the structure. In other words, parallel argumentation streams are funneled through one argumentation stream which reaches to the conclusion. However, multiple sources were not present at the same time with funneling characteristics in the global argumentations of the present study, but there is a funneling effect for parallel argumentation structures which arise from one origin. As seen, although the term source-structure was not quite proper for this study because of the lack of a variety of origins, it was also seen that the funneling characteristics had a correspondence in this study. Therefore, without following the remaining characteristics of the source-structure and focusing only on the funneling characteristics, a new type of argumentation structure was presented for the analysis of this study. All in all, in the funneling-structure, the beginning section of the schema is about the roots and it might be single-rooted or multiple-rooted. In the intermediate section of the schema, there are parallel argumentation streams which are reaching multiple conclusions. At the last section of the schema, the parallel argumentation streams are funneling towards one argumentation stream covering conclusion.

The final type of global argumentation structure is the branching-structure. As indicated via the name of it, this structure involves argumentation streams spreading from one root or multiple roots like the branches of a tree. The number of the roots determines the sub-categories of the structure which are the single-rooted branching-structure and the multiple-rooted branching-structure. The section named as root might contain only one component or an argumentation stream. The global arguments labeled under this type involve parallel argumentation streams reaching multiple conclusions.

By combining the mentioned four types of mono structures, various types of global argumentation structures which are classified under the hybrid structures can be produced. For example, three hybrid structures which are the reservoir-funneling-structure, the line-branching-structure, and the line-reservoir-branching-structure were presented in the findings chapter of the study.

The last study used as a base while addressing the second research question was the one conducted by Verheij (2005). The cases which the rebuttals were located against were listed and it was pointed out that there were eight cases which were argued against in the present study. These cases which rebuttals stated against were data, warrant, conclusion, target conclusion, conclusion/data, backing, challenger, and the bridge from data to claim which corresponds to 'if D, then C'. Based on this, the functions of rebuttals were described.

3.4.3. Data Analysis of the Third Research Question

In the analysis of the third research question, all approaches discussed by groups while searching a way to perform the geometric constructions asked in the activities were listed in tables. After this step, how groups comment on the validity of their approaches were focused by virtue of the transcripts of video recordings and interviews. Since the criteria for being regarded as a geometric construction while using compass-straightedge and GeoGebra differ, the data analysis procedures were explained separately. The geometric figures presented by CSG were accepted as constructions in the case that the criteria presented in Table 3.7 were fulfilled. The following criteria list was prepared by the researcher by reviewing the related literature and conducting revisions suggested by the experts.

Table 3. 7

Criteria list related to geometric construction used for CSG

Criteria	
C1	Geometric figure presented by CSG is proper to the construction asked in the activity
C2	Geometric figure was constructed by using compass-straightedge only
C3	Compass was used properly/correctly
C4	Straightedge was used properly/correctly
C5	Inferences in construction process were mathematically correct
C6	Explanations in construction process were mathematically correct

As seen from Table 3.7, the first criterion for accepting a geometric figure as construction is about being proper to the construction asked in the activity. The second criterion is whether the tools used in the geometric construction process were only compass and straightedge. CSG was expected not to use any other tool during the activities, so this issue was set as a criterion in the list. It was followed by two criteria about whether compass and straightedge were used properly. For example, if straightedge was used by means of measurement by a prospective middle school mathematics teacher, it cannot be admitted as the correct use of straightedge. The last two criteria are interrelated since they describe the need for mathematically correct inferences and explanation while searching an approach for geometric construction. As an example, the construction of a tangent to a given circle from an external point can be presented at this point. When the participants infer that they can directly determine the point of tangency of the circle by just looking at it, it can be stated that they violate the last two criteria. They should follow some construction steps to find the point of tendency definitely. Based on these criteria, each approach and geometric figure presented by GG along with four activities were analyzed.

The geometric figures presented by GG were evaluated based on the following diagram. This diagram was prepared by considering the studies conducted about construction and dynamic geometry programs (e.g., Arzarello et al., 2002; Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010; Hölzl, 1996; Jones, 2002; Köse, Uygan, & Özen, 2012; Laborde, 1995; Mariotti, 2001; Stylianides & Stylianides, 2005) and the cases participants of this study encountered during geometric constructions. Among all the mentioned studies, the study of Stylianides and Stylianides (2005) was mainly used to structure the diagram. In more detail, Stylianides and Stylianides (2005) stated two criteria for validation of construction problem solutions in a dynamic geometry environment which are the drag test criterion and the compatibility criterion. In the drag test criterion, the solution of the given construction problem is considered as valid if it keeps properties while dragging. In the compatibility criterion, the solution of the construction problem is accepted as valid if its steps are appropriate to construction restrictions. The term 'construction restriction' refers to using properties and tools of dynamic geometry software as if using compass and straightedge in the construction (Stylianides & Stylianides, 2005).

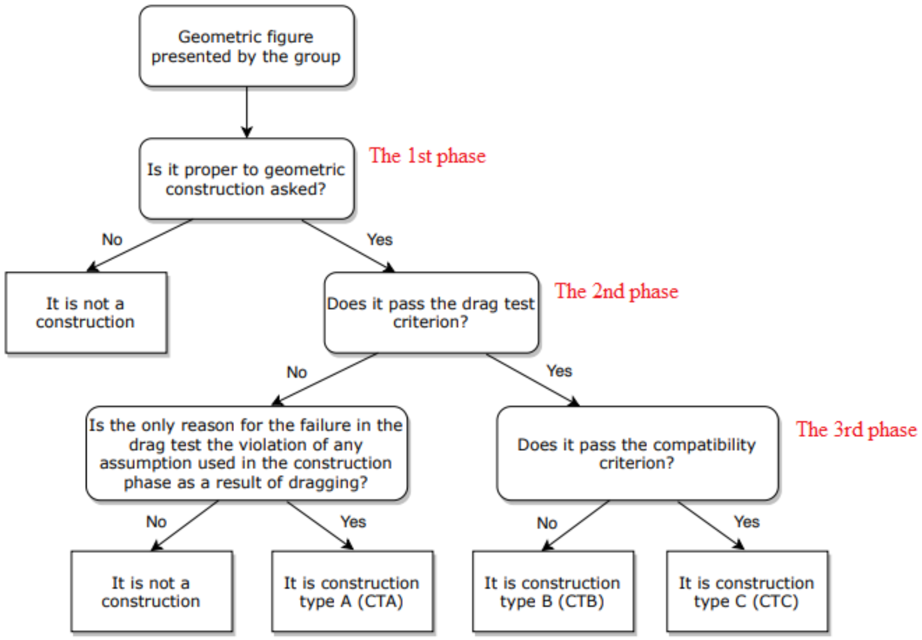


Figure 3. 14. Diagram related to geometric construction used for GG

As indicated in Figure 3.14, the diagram involves three phases which involve questions used to determine whether the approach offered by GG is valid or the geometric figure presented by GG can be labeled as a construction. The first phase covers one question which asks whether the geometric figure presented by GG is proper to the geometric construction asked. In the case that the answer is no, then it is not a construction. If the answer is yes, the diagram leads to the second question which is about the drag test criterion. When the geometric figure fails in the drag test, it is checked whether the only reason for the failure in the drag test is the violation of any assumption used in the construction phase as a result of dragging. If the answer is no, then the figure is regarded as not a construction and also the approach is accepted as invalid. On the other hand, in the case that the mentioned issue is the only reason for the failure in the drag test, then the geometric figure is accepted as valid and coded as construction type A (CTA). When the geometric figure passes the drag test criterion, it is checked for the compatibility criterion since the drag test is a necessary but not sufficient condition for compatibility criterion (Stylianides & Stylianides, 2005). In both cases, the geometric figure is coded as a geometric construction and also the approach used as a valid one. The question related to the compatibility criterion is used to determine the type of construction. The one which also passes the compatibility criterion is coded as construction type C (CTC). The geometric figure which only passes the drag test criterion but fails in the compatibility criterion is coded as construction type B (CTB).

3.4.4. Data Analysis of the Fourth Research Question

In a general sense, groups' arguments were accepted as proof if they apply mathematical and logical rules correctly and deduce the desired conclusion. In more detail, while analyzing the arguments of groups, two steps were utilized. As the first step, the overall structure of the argument was examined similar to the studies of Bleiler et al. (2014), Ko and Knuth (2013), and Weber (2008). Then, as the second step, the line-by-line analysis was conducted. The review of the related literature presented that the line-by-line analysis was a commonly used method related to the validity of the arguments (e.g., Alcock & Weber, 2005; Bleiler et al., 2014; Ko & Knuth, 2013; Selden & Selden, 2003; Weber, 2008). By taking into consideration of the classification of Ko and Knuth (2013) regarding the validity of the arguments, the arguments of groups were evaluated. Ko and Knuth (2013) presented four categories in their study which are valid, invalid with a structural error, invalid with a content-based line-by-line error, and invalid with a structural error and a content-based line-by-line error. Although this classification was not directly utilized, it can be stated that a similar approach was used. The arguments of CSG and GG were coded as invalid argument with warrant error, invalid argument with structural error, and valid proof. Moreover, the cases which affect neither the correctness of the next expression nor the whole structure of the arguments in terms of its validity were named as the extraneous errors (Selden & Selden, 2003).

3.5. Trustworthiness of the Study

The validity and reliability are critical issues which should be considered throughout the study such as while collecting and analyzing the data, reporting the findings, and interpreting them (Merriam, 2009; Patton, 2002). It was considered that the descriptions of validity and reliability in quantitative research might not be adequate and completely applicable to the perspective of qualitative research (Golafshani, 2003). The mentioned discussion turned into the involvement of "substituting new terms for words such as validity and reliability to reflect interpretivist conceptions" (Seale, 1999, p.465) which refers to the qualitative approach via the expression interpretivist conceptions. In this respect, Guba (1981) and Lincoln and Guba (1985) focused on the trustworthiness of the study and offered to use the terms credibility, transferability, dependability, and confirmability instead of internal validity, external validity, reliability, and objectivity, respectively.

Credibility was suggested for the trustworthiness of the study by Lincoln and Guba (1985) as referring to internal validity. Merriam (2009) defined internal validity as "Internal validity deals with the question of how research findings match reality. How congruent are the findings with reality?" (p.213). Besides, Merriam (2009) suggested some techniques to support the credibility of the study, which are triangulation, member checks (respondent validation), adequate engagement in data collection, researchers' position (reflexivity), and peer examination (peer review). Among these strategies, triangulation and peer examination were employed in this study. As related to triangulation, "researchers make use of multiple and different sources, methods, investigators, and theories to provide corroborating evidence" (Creswell, 2012, p.208). In this study, by collecting the data via different sources such as the video recordings of the activities, documents, GeoGebra files, field notes, and focus group interviews, it was aimed to learn more about the concepts focused and increase the credibility. Secondly, peer examination covers "asking a colleague to scan some of the raw data and assess whether the findings are plausible based on the data" (Merriam, 2009, p.220). In this study, a doctorate student in mathematics education was the second coder in the data analysis process. Moreover, depending on the context, some other experts were requested to code the data of the study. As related to argumentation domain of the study, two experts in mathematics education who studied on Toulmin's model before contributed to the study as second coders. As related to proof, an expert in mathematics was the second coder of the related data. Another strategy used related to credibility is prolonged engagement between the researcher and the participants (Lincoln & Guba, 1985; Shenton, 2004). Since the study was conducted within an elective course and then focus group interviews were conducted, it can be stated the engagement process helps the researcher to understand the site in detail, notice the possible distortions, and build trust with the participants (Lincoln & Guba, 1985).

According to Lincoln and Guba (1985), how external validity in quantitative studies is established is different from transferability in qualitative studies. Since the generalization is not aimed in the qualitative studies (Merriam, 2009), it was suggested to present a rich description to enhance the transferability to other settings (Lincoln & Guba, 1985; Merriam, 2009; Shenton, 2004). To ensure transferability, a thick and detailed description of the study was presented. The context and participants of the study, data collection procedure, and data analysis were explained in detail.

As stated before, Lincoln and Guba (1985) suggested the term dependability instead of using reliability. To ensure the dependability, Shenton (2009) suggested that "the processes within the study should be reported in detail, thereby enabling a future researcher to repeat the work, if not necessarily to gain the same results" (p.71). In this respect, Merriam (2009) suggested some strategies to increase dependability which are audit trial, triangulation, investigator's position, and peer examination. The last three were also suggested to increase credibility by Merriam (2009). In this study, all findings were discussed with a doctoral student in mathematics education, who conducted studies related to geometry, with respect to the purposes of the study. The inconsistent points in the analyses of two coders were discussed and the consensus was reached.

As the last issue related to the trustworthiness of the study, Lincoln and Guba (1985) offered the term confirmability instead of objectivity in quantitative research. For confirmability, Guba (1981) suggested triangulation and practicing reflexivity. Similarly, Shenton (2004) offered triangulation to weaken investigator bias, rich descriptions of the methodological details of the study, and description of the assumptions of the researcher. In this study, to ensure confirmability, triangulation and detailed description were utilized. Each of these issues was explained above while referring to other criteria of the trustworthiness. Besides, to support confirmability, the excerpts taken from the recordings of the cognitive unity based activities were presented.

3.6. Role of the Researcher

As previously stated, the data of the main study was mostly gathered during an elective course, Teaching of Geometry Concepts. During the registration week of the courses, 14 prospective middle school mathematics teachers, which were determined based on the selection criteria, were informed about both the course and the study. The participation in the study was not compulsory. I was careful about this issue, so I tried to set a communication with the prospective teachers in a way that they feel free to decide about adding the mentioned course. Teaching of Geometry Concepts course in which I participated as the instructor, involves two main parts which are the teaching sessions which lasted four weeks and the activities which lasted eight weeks, but four of which are the cognitive unity based activities. At the first four weeks, I was more active since these weeks can be regarded as a kind of preparation for the following activities. However, it does not mean that the prospective teachers are passive in the teaching sessions. Moreover, during the data collection process, it was aimed to make the participants of both the pilot study and the main study feel comfortable to share their ideas with others in groups. They were encouraged to think out loud and feel comfortable to declare the points they noticed during the classroom discussions at the end of the activities and during the teaching sessions at the beginning of the course.

Since I work as a research assistant in the Elementary Mathematics Teacher Education program in which the participants were enrolled, I had known the participants since their first term in the program. Actually, I was quite familiar with the participants at the beginning of their fifth semester in the program which corresponds to the beginning of the data collection process. In their third and fourth semesters, I was also attending Calculus I and Calculus II courses, each of which was offered as six hours in a week. Besides, Calculus I course was set up in a way that four hours of the course were for lecture and the remaining two hours were for recitation in each week and I was the instructor of the recitation hours of this course. In Calculus II course, I continued to attend, but there were not the recitation hours in this course.

However, prospective middle school teachers in that course kept asking questions related to Calculus in the office hours. Thus, it can be stated that I was getting in touch with these prospective teachers while they were in their second year in the program both during the mentioned courses and the office hours out of the courses. During this period, I had the chance to learn more about the whole group in that year level in the program and, as expected, they also learned more about me. Since we developed a fine relationship and any unsettling case did not happen during the process, this experience might have affected both sides positively. Moreover, after their fifth semester in the program in which the data were collected, I assisted another course they took in the sixth semester which is Community Service. I would truly say that we were able to keep the mutual positive attitude with not only the participants but also the whole group they belong to. Besides, since I was familiar with the participants since their first term in the program, this situation helped me to interpret the findings more properly.

Moreover, at the beginning week of the mentioned course, junior prospective middle school mathematics teachers were also informed that it would be better for the sake of the study if they attend to all weeks of the course and the time of the course would be changed in any extra circumstance. Based on the positive relationship between the researcher and the juniors, they behaved in a sensitive way so that all juniors who took the course attended to all weeks of the course.

It was stated that the researcher in qualitative research allocates a great amount of time to be in the natural setting of the study and keeps direct contact with the participants (Merriam, 2009). As expected, I also followed such a procedure. As mentioned, the researcher is an instrument in the data collection and analysis process of the study (Merriam, 2009). Since I was the instructor of the course Teaching of Geometry Concepts in which the administration of the cognitive unity based activities took place, I arranged the content and the ongoing changes related to the course. When needed, I asked guided questions and gave some clues during the application of the activities to enhance the collective argumentation.

3.7. Ethical Considerations

Before administration of the study, the official permissions were taken from Applied Ethics Research Center in Middle East Technical University (METU). The approval of Human Subjects Ethics Committee is presented in Appendix D. At the beginning of both the pilot study and the main study, the informed consent form was given to all participants to be signed even though they were informed about the study and they declared their voluntariness beforehand. That is, when they were asked to participate in the study during the sampling process, they were briefly informed about the content and the process of the study. By means of the consent form in the main study, all junior prospective middle school mathematics teachers who took Teaching of Geometry Concepts course were re-informed about the following issues; the overall content of the course, the general purposes of the study, who is conducting the research, how data would be collected, the camera and audio recorders to be used, what are the expectations from the participants, how they would be involved in the study, their being free to quit the participation to the study at any time they want, the fact that the data obtained in the process would be used in the scientific publications, their identities being kept confidential during all processes of the study, and the precaution that they should not be talking about the content of the course with others in order to avoid noticing the details of the activities before the application with the other section. It can be stated that any of the participants were not forced to participate in the study. For example, as mentioned, one of the juniors determined by following the selection criteria gave up taking the elective course in the first week of the semester. Then, the next junior prospective middle school mathematics teacher in the list of that section was re-asked whether he wanted to take the elective course and participate in the study. Besides, at the beginning of the mentioned course, they were asked about the video and audio recordings during the course and none of them stated a negative idea related to that. It was explained that the study does not involve any harmful or risky situation and deception for the participants.

Due to the presentation of findings in a detailed and rich manner in qualitative research, the researchers should be careful about the identities of the participants and take precautions to ensure confidentiality (Mertens, 2012). The anonymity of participants should be considered (Creswell, 2007; Miles & Huberman, 1994). To decrease the risk of identifiability, more detailed but irrelevant issues to the study such as the type of high school they graduated, age, and the experience regarding teaching were not presented. Instead of using real names of the juniors in Teaching of Geometry Concepts course, pseudonyms were used to ensure confidentiality. By doing so, it turned out to be that the guidelines stated by Bogdan and Biklen (2007) to help the qualitative researchers to take the ethical concerns in a study into account were followed as well.

In the following chapter, which was arranged by considering the order of research questions, the findings will be presented in detail.

CHAPTER 4

FINDINGS

In the preceding chapter, the methodological details of the current study were explained in detail. This chapter covers the findings of the study under four main sections, each of which aims to address one research question. In more detail, the first section, which aims to answer the first research question, covers the findings regarding how prospective middle school mathematics teachers' argumentation while producing conjectures relates to proving process. The second section, which is about the second research question and its sub-questions, presents the global argumentation structures emerged in the conjecture production process of the cognitive unity based activities, the components of these global argumentation structures, and the functions of the rebuttal component. In the third section, the concept of geometric construction situated in the cognitive unity based activities was looked in a closer way. That is to say, the approaches which compass-straightedge group and GeoGebra group presented for the geometric constructions and whether they could perform valid constructions or not were reported. These findings account for the third research question and its sub-questions. In the last section, which aims to answer the fourth research question, the conjectures produced by groups and whether they could offer valid proofs for some particular conjectures they recently produced were explained. After all, the findings presented in this chapter by pursuing the mentioned layout are based on the data corpus obtained from one compass-straightedge group (CSG) and one GeoGebra group (GG).

4.1. Relation of Argumentation while Producing Conjectures to Proving

Since the construct of cognitive unity was counted as the starting point of the study, the first research question aims to probe the cognitive unity dimension particularly. As Fiallo and Gutiérrez (2017) stated, "there is cognitive unity when the argumentations used during the conjecturing phase help students to construct a proof, either empirical or deductive" (p.149). Without focusing on the mentioned possible help of argumentation to proof only, the first research question aimed to examine how groups' argumentation process while producing conjectures by means of geometric constructions relates to proving process of conjectures they recently stated. In this respect, this section aims to provide the evidence to answer the first research question of the present study.

All data sources in the study, which are video recordings and audio recordings of the cognitive unity based activities, documents, GeoGebra files, field notes, and focus group interviews, were deployed while addressing this research question. By following the guidelines of inductive content analysis, which are open coding, arranging categories, and abstraction (Elo & Kyngäs, 2008), the aspects of how prospective middle school mathematics teachers' argumentation while producing conjectures relates to proof within the context of geometry were determined, as presented in Table 4.1. Moreover, since it was seen that the use of different tools in geometric constructions did not present a clear difference in the codes and themes arranged during the analysis, the findings were reported by not focusing on the groups separately.

Table 4. 1

The aspects of how argumentation relates to proof

Aspects	**Codes**
Positive aspects of argumentation process before proving	Positive affective occasions Arrangement of knowledge related to the content of the activity Visual aspect The veracity of the statement asked to prove
Negative aspects of argumentation process before proving	Negative affective occasions Confusion related to the difference between conjecturing and proving

As it can be seen in Table 4.1, it was pointed out that prospective middle school mathematics teachers' argumentation while producing conjectures relates to the proof of the recently produced conjectures in terms of two aspects, which are positive aspects and negative aspects. In more detail, the positive aspects of being involved in an argumentation process before proving might be examined via four codes, which are positive affective occasions, arrangement of knowledge related to the content of the activity, visual aspect, and the veracity of the statement asked to prove. On the other hand, the negative aspects of participating in an argumentation process before proving might be inspected via two codes, which are negative affective occasions and confusion related to the difference between conjecturing and proving. To present an in-depth description of the aspects and the related codes given in Table 4.1, the examples were given as follows.

4.1.1. Positive Aspects

To begin with, the positive aspects of argumentation process before proving were instantiated. The first issue mentioned in this aspect is the positive affective occasions noticed. This code involves the cases that the participants were motivated when they offered a working idea, having a positive attitude towards the whole activity, improving the self-efficacy regarding conjecturing and proof, and getting used to working collaboratively. To exemplify the mentioned positive affective occasions, some extracts were given below.

The first example is related to the increasing motivation of Zuhal (Z) and Berna (B), who are the members of GG, towards conducting the whole activity when they offered effective methods and ideas in Activity 1.

Activity 1, GG (while working on geometric construction)

Z I hope, they intersect *(She was working on the construction of the perpendicular bisectors of the sides of the triangle, and she hoped that their intersection would work in terms of the aimed construction)*

…

Z I worked, it worked *(She was calling for the attention of others in GG)*

…

Z I want to find something else *(She referred to finding another approach for performing the aimed construction)*

Activity 1, GG (when starting to work on the restricted GeoGebra file)

B Okay, let’s look at what we have *(She referred to the tools of GeoGebra).* We can find something for this one too. We have found for the first file *(She was* *stating that they could find a valid approach for performing the geometric construction by using the second GeoGebra file)*

Activity 1, GG (at the beginning of proving)

Z I think, we are proving now what we have done recently.

B Exactly

Z If we can prove, we can answer why we said that. Let's do *(She referred to conducting proof)*

Before explaining the extracts, it would be better to describe how these extracts were presented. The first line gives information about the activity, group, and the instances that the excerpts were taken. Then, the letters stated on the left stand for the abbreviations of the participants' pseudonyms. The statements given in the parentheses as in italics were added by the researcher as the explanations to make the expressions clearer. The triple dots presented on the left side refer to the presence of some other conversations which are not directly related to the issue aimed to present.

As seen above, three occasions related to the positive stance of the participants of GG in Activity 1 were presented. The first occasion is from the period that Zuhal was working on the construction of the circle passing through the vertices of the given triangle by using the first GeoGebra file. They tried some approaches such as finding the intersection of the angle bisectors, but they could not come up with a correct approach. Then, Zuhal noticed that they did not try the tool 'perpendicular bisector' and started to work on it by hoping that it would work. As deduced from the video recordings, she was quite excited when her idea worked, and she immediately explained it to the others in the group. After they have worked on the mentioned approach collectively, she kept her enthusiasm and declared that she also wanted to find other methods. The second occasion given above is from the instance that GG started to work on the second GeoGebra file, which has extra restricted tools. As soon as they opened the second GeoGebra file, Berna started to look at the tools in the toolbar and aimed to determine the possible ones for construction. Besides, she attempted to motivate the group by declaring that they found a valid approach by using the first GeoGebra file, and they were capable of finding another one with the second GeoGebra file. As seen, she had the self-efficacy regarding their performance in the activity. Then, they started to go over the possible tools. The last occasion is from the beginning of proving section of GG in Activity 1. As seen, since Zuhal stated that they were asked to prove what they have reached recently, it can be stated that both Zuhal and Berna directly noticed the underlying issue of cognitive unity. Similar to the first section of the activity, Zuhal was also the kick-start of the group in the proving section. Aligned with the video recordings, it can be stated that she presented her high motivation for proving by uttering that “let’s do”. Moreover, it can be inferred that she might be feeling more confident in terms of proving since they were asked to prove what they had found recently.

As another example of the positive affective occasions, some sentences of Bahar (B), who is a member of CSG, in Activity 2 were given below.

Activity 2, CSG (while working on geometric construction)

B I think, this is a very logical method *(She referred to the approach she used while constructing the altitudes of the triangle. It is the basic construction used to draw the line that is perpendicular to a given line from a point not on the given line)*

…

B I have done it so perfectly that we should present it as a good example, I think *(She was proud of what she have conducted with the mentioned approach)*

Activity 2, CSG (while working on proving)

B If I can show it, this would be like a complete proof *(She was trying to prove the statement that the altitudes of a triangle are concurrent by trying to set up congruence by means of angles. At this point, she is trying to show the equality of two angles)*

…

B I found many fine equalities and equations. I am trying to connect them right now.

The first excerpt given above is from the construction section in Activity 2, and it presents how Bahar was motivated since she offered a working idea and improved her self-efficacy regarding the whole activity. In more detail, the first part of the excerpt involves the sentences of Bahar from the geometric construction attempt in Activity 2, in which the construction of the altitudes and the orthocenters (if they exist) of the given triangles were asked. While constructing the altitudes of the given triangles, Bahar used one of the basic construction approaches which they worked on during the teaching sessions at the beginning of the course. That is, she regarded the construction of the altitude of a triangle as the construction of the line that is perpendicular to a given line from a point not on the given line. As seen, in the first sentence, she appreciated the logic underlying this approach. Moreover, by means of the second sentence given, it can be stated that she presented her pleasure for finding the approach, and this situation increased her self-efficacy regarding the whole activity. The second part of the excerpt given above is from the proving section of the same activity. As seen from the video recordings, Bahar did not stop working on proving attempts. Moreover, as deduced from the sentences, she kept her beliefs regarding the case that their argument would reach a valid proof and did not step back.

The second code given in the positive aspects of argumentation before proving is the arrangement of knowledge related to the content of the activity. More precisely, it covers the following cases; the participants evoked their previous knowledge through the general and auxiliary ideas in the conjecture production phase, they consolidated their knowledge based on the conjectures they produced, they reached conclusions by checking the validity of some statements via performing geometric constructions, they promoted their geometric knowledge with discussion, they prompted the knowledge about geometric concepts from previous activities, and they discussed the proper usage of notations, terms, and expressions during geometric construction and clarified the issues before proof in some activities.

The following excerpt, which was taken from the argumentation of CSG while dealing with the geometric construction asked in Activity 3, serves as an example for the arrangement of knowledge related to three points situated in a triangle before proving section.

Activity 3, CSG (while working on construction of centroid and orthocenter)

F We found the centroid before, didn't we? Is it the intersection of the angle bisectors? *(Actually, they did not work on the construction of the centroid before by means of the activities)*

B Yes, it *(the centroid)* is the intersection of the angle bisectors.

G The angle bisectors or the medians? *(She pointed out the hesitation)*

B It *(the centroid)* is not related to the medians.

F We used the medians while constructing the orthocenter, didn't we?

G I think, it *(the median)* is for the centroid.

B The angle bisectors or the medians? Hmm *(She agreed on the hesitation)*

F Let's try.

…

F The intersection of the angle bisectors is not the centroid.

B Then, it *(the centroid)* is the intersection of the medians.

…

F We used the perpendicular bisectors of the sides *(of the triangle)* to find the orthocenter, but then we noticed ımm…

B We noticed that what we have found is not the orthocenter.

…

B The orthocenter is not the intersection of the perpendicular bisectors of the sides.

G It is the altitudes *(She referred to the fact that the intersection of the altitudes of a triangle is the orthocenter)*

The given dialogue parts emerged while CSG was working on the construction of the centroid and the orthocenter of a given triangle. Unfortunately, it shows the fact that prospective middle school mathematics teachers had some deficiencies in even basic concepts of geometry. They could not be sure the meaning of the centroid and the orthocenter of a triangle and how they could construct them for a while. In more detail, Filiz (F) and Bahar (B) thought that the centroid of the triangle could be constructed by finding the intersection points of the angle bisectors. However, Gizem (G) hesitated this idea and offered that the centroid can be constructed by finding the intersection of the medians. Thus, they decided to try each idea and concluded that they should work on the medians to construct the centroid. While working on the constructions asked in Activity 3, they noticed that they conducted a mistake while trying to construct the orthocenter. That is to say, they concluded that the orthocenter is the intersection of the altitudes of a triangle, not the intersection of the perpendicular bisectors of the sides. All in all, after an exploration and trial-and-error process, they noticed their incorrect interpretations and reached the correct knowledge regarding the mentioned points. In addition to these two points, they were asked to construct the circumcenter of the given triangle, but they did not present an incorrect case related to it. Thus, CSG evoked and consolidated the knowledge related to the mentioned two points before passing to proof section. When the case that CSG was asked to prove that the circumcenter, the orthocenter, and the centroid of a triangle are collinear without being involved in such an argumentation process was considered, it might be stated that they would start the proof without having the necessary background knowledge. The argumentation process before proof helped them to arrange the geometry knowledge related to the activity.

The third issue given in Table 4.1 under the positive aspects of argumentation process before proving is the visual aspect. Since the groups worked on geometric constructions preceded by proof, this process increased their awareness of the geometric objects visually and helped them to notice some key drawings which might be used during proving. Moreover, it might be stated that to be involved in such an argumentation led the participants to present the properly formed geometric figures while writing the related proofs. As usual, to give examples for this code, two instances noticed were explained as follows.

The following dialog serves as evidence for the visual aspect. It has parts from both conjecture production and proof sections of GG in Activity 4. All sentences are also related to the cyclic quadrilaterals. As mentioned, Activity 4 asked to construct three circles, each of which was passing from one vertex of the given triangle and two randomly placed points on the adjacent sides of that vertex, and search for relations among them. Then, Activity 4 asked to prove that the mentioned three circles are concurrent.

Activity 4, GG (while searching for the connection among the circles)

B   Do I open what we have done? *(She was opening the GeoGebra file they worked on)*

Z   Look, for example, A, X, this point *(the point Y in (b) of Figure 4.1*), and this *(the point D in (b) of Figure 4.1.)* constitute a cyclic quadrilateral.

…

Z   For example, the sum of this angle and this angle is 180 *(She referred to the case that the sum of $\angle XBY$ and $\angle XDY$ given in (b) of Figure 4.1 is 180^{o})*

B   Which angles?

Z   According to the property of cyclic quadrilateral, what is the sum of these angles? (*the sum of $\angle XBY$ and $\angle XDY$ given in (b) of Figure 4.1)*

B   The sum of these two *(angles)* is 180.

…

B   Yes, it reached to 180 *(She calculated the sum of the mentioned angles by using GeoGebra)*

…

G   One minute. This is our figure, there is one cyclic quadrilateral here *(the cyclic quadrilateral XBYD in (b) of Figure 4.1)*, one cyclic quadrilateral here *(the cyclic quadrilateral ZDYC in (b) of Figure 4.1)*, and one cyclic quadrilateral is here *(the cyclic quadrilateral XDZA in (b) of Figure 4.1)*.
These intersect here *(showing the point D in (b) of Figure 4.1)*

B   We have found this.

Activity 4, GG (while working on proving)

Z If we use the chords *(As saying the chords, she referred to using* $\overline{XD}$, $\overline{YD}$, *and* $\overline{ZD}$ *given in Figure 4.2)*

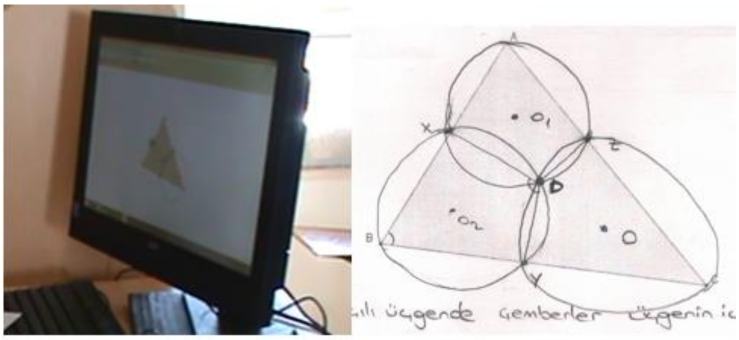
…

G Do you say that we draw chords like this? From this point? *(She was pointing on the point D and also* $\overline{XD}$, $\overline{YD}$, *and* $\overline{ZD}$ *in Figure 4.2)*

Z We saw that this *(the point D in Figure 4.2)* is the point of the cyclic quadrilaterals of two circles.

B Then, we will say that it is also for the third one *(They were searching a way for using them in the proof)*

The first part displayed above involves the sentences from the period that GG was searching for the connection among the circles in Activity 4. As seen, Berna (B) and Zuhal (Z) were focusing on the GeoGebra file to search for a relationship between the mentioned circles. A screenshot from this moment was given in (a) of Figure 4.1. At that point, Zuhal noticed a cyclic quadrilateral which was composed of the points A, X, D, and Y as in (b) of Figure 4.1. Moreover, what is indicated in (b) of Figure 4.1 is the geometric figure in the worksheet of GG used while working on geometric construction. That is, after GG worked on the intended geometric figure by using GeoGebra, they explained what they have done in the given worksheet, and the geometric figure given through (b) of Figure 4.1 is the one in that worksheet. After that, they discussed whether the sum of the opposite angles in a cyclic quadrilateral is 180º. By using the tool of GeoGebra, Berna calculated the sum of the mentioned angles. As the final issue related to the cyclic quadrilaterals in the geometric construction section, Güler (G) noticed the presence of three cyclic quadrilaterals while using GeoGebra. Actually, the screenshot in (a) of Figure 4.1 also shows three cyclic quadrilaterals as well. While explaining this to the others in the group, Güler drew the cyclic quadrilaterals and named each vertex of them, which can be seen in (b) of Figure 4.1. All in all, GG noticed three cyclic quadrilaterals, which are XBYD, ZDYC, and XDZA in (b) of Figure 4.1, before passing to the proving section in Activity 4.



(a) (b)

Figure 4. 1. Cyclic quadrilaterals GG formed during geometric construction in Activity 4

The second part of the excerpt given above is from the proving process of GG in Activity 4. As seen, while searching a way to start the proof, Zuhal offered to use the chords. By saying the chords, she referred to using $\overline{XD}$, $\overline{YD}$, and $\overline{ZD}$ in Figure 4.2. Actually, Figure 4.2 was given to present how GG used the idea related to the cyclic quadrilaterals coming from the argumentation section while proving. The geometric figure in 4.2 was taken from the argument that GG submitted as proof in Activity 4, the details of which will be explained in the followings section of this chapter. Besides, at the end of the excerpt, GG mentioned a way to use the cyclic quadrilaterals in proof. In more detail, they mentioned accepting that two cyclic quadrilaterals intersect at the point D and then inferring to the presence of the third cyclic quadrilateral which will bring the issue to drawing the third circle passing through the point D. From Figure 4.2, it can be observed that they focused on this idea while proving.

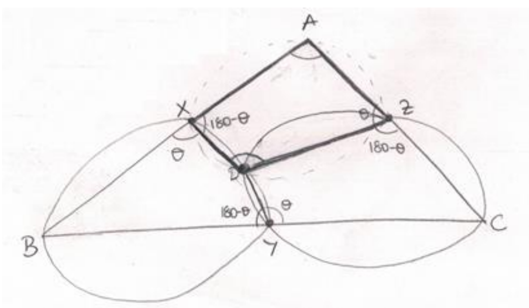


Figure 4. 2. Cyclic quadrilaterals GG formed during proof in Activity 4

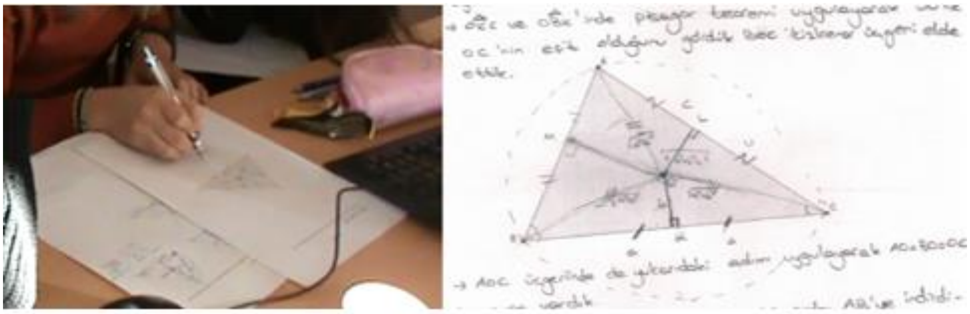
The second example of the visual aspect is from the beginning of the proof section of GG in Activity 1. It instantiated the attempt of GG to use the recently constructed geometric figures while proving.

Activity 1, GG (at the beginning of proving)

B Let’s open what we have done. We saved them, didn’t we? *(She referred to opening the GeoGebra files worked and saved during geometric construction)*

Z Yes, we might use them.

As seen, Berna (B) opened the GeoGebra files before starting to prove, and Zuhal (Z) agreed with her since she also saw the possibility of using them while proving. After a discussion regarding how they could prove the statement which lasted nearly 40 minutes, Berna took the empty worksheet A. As mentioned, the cognitive unity based activities involve two worksheets. The worksheet A was given for the conjecture production process by means of geometric constructions, and the worksheet B was given for the proof of conjectures they produced. Then, Berna made the triangles in the worksheet A and the GeoGebra file similar by dragging. She marked the circumcenter of the triangle printed on the worksheet A more accurately by checking the location form the GeoGebra file. After that, all participants of GG started work on the triangle on the worksheet A, as can be seen in (a) of Figure 4.3. Thus, they kept working on that worksheet and submitted the argument as proof which was written on triangle printed on the worksheet given for geometric construction section of the activity, as presented in (b) of Figure 4.3. It can be inferred that GG preferred to use the triangle that they were familiar via geometric construction while proving the related statement in the activity.



(a) (b)

Figure 4. 3. Cases from the proof process of GG in Activity 1

The last code of the positive aspects is the veracity of the statement asked to prove. Actually, this code might be considered as the summary of the following cases; the participants were sure about the validity of the statement which was asked to prove, they did not consider finding a counterexample since they were sure that the statement is true, and using tools supported their status of being sure about the veracity of the statement asked to prove. To provide an example, the excerpts taken from both argumentation and proof processes of CSG in Activity 3 were stated below.

Activity 3, CSG (at the end of conjecture production)

F I did not know that they *(the orthocenter, the circumcenter, and the centroid of a triangle)* are collinear.

B Me too.

G I did not know too.

Activity 3, CSG (at the beginning of proving)

B Here, can we do this? By accepting that two of them *(two of the points)* are collinear, can we say that if this and this are collinear, then this one is also collinear? *(She considered to accept that two of the mentioned points as collinear and then try to show that the third one is also collinear)*

…

F I am thinking about whether I can arrange any congruence among three lines.

B Three points.

F Does the congruence work again for this one? *(Since they arranged congruence in the previous activities, she also searched for any congruence for this proof)*

…

F If I accept that these are not collinear, then it turns out that they are collinear. However, when I accept that they are not collinear, how can I show that they are collinear? *(She referred to using the proof by contradiction)*

The first part of the excerpt indicated that they did not know that the orthocenter, the centroid, and the circumcenter of a triangle are collinear. All of the participants of CSG learned it at the end of the conjecture production section of the activity. Thus, at this point, it can be implied that CSG produced the conjecture related to the collinearity of the mentioned three points so that they were sure about the veracity of the statement asked to prove due to being involved in such an argumentation process.

The second part of the excerpt given above is from the beginning of proving process of CSG in Activity 3. It covers three ideas to start to prove. As the first idea, Bahar (B) stated whether they could use to accept that two of the mentioned points as collinear and then try to show that the third one is also collinear as a method for proof. Secondly, Filiz (F) offered to look for a congruence which might be used in the development of proof. In the last idea, Filiz referred to using proof by contradiction without stating the name literally. That is, she mentioned accepting that the orthocenter, the centroid, and the circumcenter of a triangle are not collinear and trying to reach the conclusion of the collinearity of them. All of the ideas mentioned for proof aim to prove that the statement is true. It can be stated that they did not consider to show that the statement is false since they did not present any evidence for the presence of a counterexample or the need to find a counterexample. Since they produced the statement aimed to prove by means of geometric construction embedded in the argumentation process, they were sure about the veracity of the statement asked to prove. The issues originated from the argumentation might affect the method of proof they focused.

4.1.2. Negative Aspects

Having explained the details of the positive aspects, it is time to continue with the detailed examination of the negative aspects of being involved in argumentation before proof. Similar to the positive aspects explained, the first code of negative aspect is negative affective cases. Particularly, the participants were sometimes bored during the first sections of the activities, and it reflected on the proof process, they presented evidence for low self-efficacy and frustration when they got stuck, and they constituted the negative attitude towards the whole activity. It might be stated that the mentioned cases affected the proving sections of the cognitive unity based activities in a negative manner.

The first example is from both argumentation and proving process of GG in Activity 1. It covers some negative sentences of Berna (B) and Güler (G).

Activity 1, GG (while working on geometric construction)

B	Dash it, it did not work *(An approach she tried to perform the intended geometric construction did not work)*

…

B	We drew this. Is this might be related to the one inside? *(She was talking about the approach focused)*
I could not think.

G I did not have any idea too. I get stuck. I noticed that I am not good at this concept of geometry. I do not know about it much.

Activity 1, GG (at the beginning of proving)

G We have barely found it *(the statement asked to prove).* How can we prove it now?

In the first part of the extract, Berna gets frustrated since a method she believed in much did not result in a geometric construction. After that, they kept looking for an approach to perform the geometric figure asked in Activity 1, which is the circumcircle of a given triangle. However, there were moments that they got stuck and presented low self-efficacy in terms of conducting the whole activity. Güler declared that she was not good at the concept of the activity. Although they found a working approach and concluded that the intersection of the perpendicular bisectors of the sides of a triangle is the circumcenter, which was the aimed conjecture in the activity, they still had a negative stance towards the activity. As seen in the second part of the extract given above, Güler said out loud her concern related their capability of conducting the proof of the given statement.

As the second example, the practice of CSG in Activity 4 was presented based on the field notes particularly. The field notes pointed out the overall negative stance of CSG in Activity 4 so that this issue was paid attention during the analysis. It was seen that many negative sentences of CSG were listed in this code. Some sentences uttered by the participants of CSG during both conjecture production and proving phases in Activity 4 were displayed below.

Activity 4, CSG (while working on producing conjecture)

B I could not understand what kind of a relationship we might find.

…

B I have always found the same thing at the end, then my psychological status is failing.

...

B I think, I could not draw it. Yes, I could not. Here we go, I am drawing it wrong again.

…

I You are so calm today. Do you have a problem?

F We could not produce anything today.

I What do you think about the relations of circles?

F I wish, we could think of something related to them.

Activity 4, CSG (while working on proving)

B How am I supposed to prove this?

…

G You will complete it to 180. This one is α, then plus β, it becomes 360. However, I could not prove it.

As usual, the first to be considered is the extract taken from the argumentation process of CSG in Activity 1. The first issue is related to the negative attitude of Bahar (B) while searching for a relationship among the circles. She declared that she neither understood what she was supposed to find nor drew the intended figure correctly. Since she kept finding the same result at the end of different attempts, she directly stated that she was affected from this situation negatively. Since the instructor noticed the overall negative status of CSG, she asked whether they have a problem. As the answer, Filiz (F) stated that they could not produce today. Then, by aiming to enhance the argumentation, the instructor asked their ideas related to the connections of circles. Another negative sentence was given by Filiz since she signified they could not find yet. However, it might be inferred that the way of saying this presents low self-efficacy related to the whole activity. Although they were able to produce the aimed conjectures by performing the constructions correctly, their status of being bored during the conjecture production process was maintained in the proving section of Activity 4. As seen, the second part of the extract is from the proving process of CSG. As soon as the worksheets were distributed, Bahar complaint about the proof. That is, she declared "how am I supposed to prove this" with a frustrated intonation. Similarly, in the followings of the proof process, Gizem (G) stated that she could not prove it a few times.

Lastly, the confusion related to the difference between conjecturing and proving was listed as a code among the negative aspects of argumentation process involved before proving. According to the analysis, this code was rarely seen in the data collected. To provide an example, the extracts from the proving process of CSG in Activity 1 were given below.

Activity 1, CSG (at the beginning of proving)

B What do I prove here? Isn't it already obvious?

…

I You will present a formal proof for the statement you have presented in the last section.

G We have to produce the congruent angles, triangles, etc. here. What can we do?

Activity 1, CSG (while working on proving)

B What if we transfer the angles? How do we transfer the angles?

F You do not have to conduct something by using compass while proving.

…

B Okay, I think that we perform geometric construction during proving. I sometimes get confused.

The first part of the excerpt is from the beginning of the proving process of CSG in Activity 1. As indicated in the first sentence, Bahar (B) could not see the difference between conjecturing and proving. Since CSG produced the statement that the intersection of the perpendicular bisectors of the sides of a triangle is the circumcenter by virtue of geometric construction in Activity 1, she could not see the need for proving. That is, according to her, they had already presented that statement. After this idea of Bahar, both the instructor and Filiz (F) tried to show the point she missed briefly. After that, in the following moments of proving process, Bahar presented another confusing point for her, which is the second part of the extract. She attempted to transfer the angle while trying to prove. Again, Filiz warned her by stating that there is no need to use compass while proving. Thus, Berna noticed what she was confused about.

Up to this point, the findings related to the first research question were reported. By doing so, two main concepts in the concept of cognitive unity, which are argumentation and proof, were taken into consideration together during the analysis. In the following section, the argumentation dimension of cognitive unity was particularly focused and the related findings were documented thoroughly.

4.2. Global Argumentation Structures

In this section, the global argumentation structures of groups involving prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities were explained. In addition, the components that emerged in these global argumentation structures and the function of the rebuttal component were also focused distinctively.

From the perspective of cognitive unity, the argumentation process while producing the conjecture has a critical importance for being able to conduct proof of it (Baccaglini-Frank, 2010; Boero et al., 1996; Fiallo & Gutiérrez, 2017; Garuti et al., 1996; Garuti et al., 1998; Pedemonte & Buchbinder, 2011). In this study, the critical argumentation process, in which the conjectures are produced, corresponds to the discussion of groups during the geometric construction section of the cognitive unity based activities. In this manner, argumentation processes of prospective middle school mathematics teachers while producing conjectures were examined thoroughly, and the global argumentation structures were formed in order to answer the second research question.

In line with this purpose, firstly, the global argumentation structures of compass-straightedge group (CSG) and GeoGebra group (GG) were formed for each activity. Since one global argumentation structure was organized from the whole discussion in conjecture production per group, two groups which are CSG and GG were focused on, and there are four activities, eight global argumentation structures in total were arranged based on the data gathered. To document the nature of the global argumentation structures and to explain how they were formed, a starting sub-section which explains the content of the global argumentation structures was presented at first. After this, types of global argumentation structures were expanded on under the sub-headings of mono structures and hybrid structures. Then, the components of argumentation that emerged in the study were reported by giving examples. Thus, the first sub-question of the second research question regarding the components of argumentation was aimed to address. Finally, the findings deduced as related to the functions of rebuttal were reported by presenting related extracts so as to address the second sub-question of the second research question.

4.2.1. Content of the Global Argumentation Structures

In this sub-section, the content of the global argumentation structures was elaborated by means of explaining the formulation process pursued and presenting some examples. It was anticipated that such a section displaying the findings related to the content of the global argumentation structures might be of benefit to make the subsequent sections more interpretable. In this manner, without dilating upon the analysis process, some main issues regarding the global argumentation structures were mentioned. Firstly, some terms peculiar to the global argumentation structures such as argumentation block and argumentation stream were presented by giving instances from the data. Then, the frame used while arranging the global argumentation structures was focused. Afterward, the theme of reasoning backward in the global argumentation structures was described in detail.

As stated, there are some terms frequently used while working with the global argumentation structures. At this point, it would be better to explain these terms to make the next sections easier to follow. The terms 'local argument', 'global argument', and 'argumentation stream' were utilized by Knipping (2008). In more detail, Knipping (2008) used Toulmin's model to reconstruct one step of an argument which was called as 'local argument' or 'argumentation step'. The terms 'argumentation step' and 'local argument' were used as having the same manner in the description of Knipping (2008). By combining these argumentation steps, Knipping (2008) produced a structure which represents the whole argumentation process and called it as 'global argument' or 'argumentation structure'. Thus, global arguments may be considerably complex compared to local arguments (Knipping, 2008). In the present study, as the intersection of the terms 'global argument' and 'argumentation structure', the term 'global argumentation structure' was used to present the entire argumentation process of a group in one activity. To present the correspondence of the specified terms in this study clearly, the global argumentation structure of CSG in Activity 1 was presented in Figure 4.4 as an example. In this respect, the examples for the 'argumentation step' or 'local argument' were pointed out with red through AS-3 in Figure 4.4. Similarly, the correspondence of the term 'global argumentation structure' in the study was framed as green in Figure 4.4.

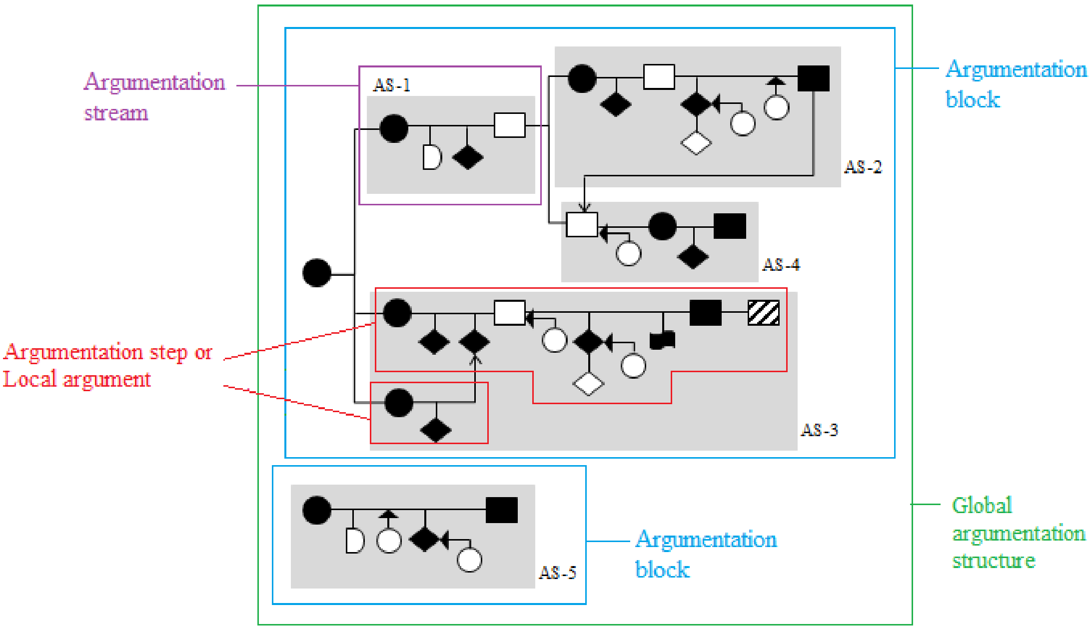


Figure 4. 4. The correspondence of the terms used in the global argumentation structures

According to Knipping (2008), the term 'argumentation stream' refers to "a chain of argumentation steps by which a target conclusion is justified" (p.434). In this study, the term 'argumentation stream', which was abbreviated as AS, was used in three cases. The first one is that when a particular concept-oriented argumentation process which reaches a conclusion or conclusion/data or a target conclusion such as AS-1, AS-2, and AS-3 in Figure 4.4, it was named as an argumentation stream. As the second case, an argumentation stream may cover just a series of related argumentation steps without the presence of a conclusion. For example, there is such an argumentation stream in the global argumentation structure of GG in Activity 4 which will be explained later in detail (See Figure 4.10 or Appendix E). AS-2 involves a flow of ideas offered for the construction of three circles. Since the ideas in AS-2 were stated consecutively and refuted or given up even without a refutation, they did not end with a conclusion component. In other words, the series of related ideas which do not involve a conclusion was also presented within one argumentation stream. Thirdly, the combination of an argumentation step reaching a conclusion and another argumentation step which is directly related to the first one was marked as an argumentation stream such as AS-3 in Figure 4.4. Moreover, as seen, AS-1 in Figure 4.4 was marked as purple to point out what argumentation stream refers to. Other than AS-1, the global argumentation structure of CSG given in Figure 4.4 covers four more argumentation streams, each of which was framed with a gray-filled background and labeled with the corresponding abbreviation such as AS-1 and AS-2. In addition to these terms, the term 'argumentation block' was introduced in this study. Argumentation block may be one argumentation stream or a combination of more than one and connected argumentation streams. In other words, when the global argumentation structure has a piecewise structure, each of these independent pieces was accepted as one argumentation block. In Figure 4.4, two argumentation blocks were framed with blue. As seen, the first argumentation block covers AS-1, AS-2, AS-3, and AS-4 while the second argumentation block covers only one argumentation stream AS-5.

After the terms, some general points related to the formulation of the global argumentation structures were also explained. In more detail, some issues regarding the location and nature of the argumentation streams such as how argumentation streams were connected, in which cases a new argumentation stream was presented as parallel to the previous argumentation streams, under which conditions a new stream was shown as if it was originated from another argumentation stream, and when a stream ends and a new stream starts were identified step by step. While explaining these points, the examples from the global argumentation structure of CSG in Activity 1 which was presented in the last figure (See Figure 4.4) were also deployed so as to be clearer. Firstly, if groups were passing to a new idea or approach in the argumentation and this new issue was related to another argumentation stream, they were represented schematically as connected. In other words, in the case that groups started to work a new idea which they noticed as a result of the previous stream or component, such argumentation streams were presented as if they are originated from the previous one. For example, AS-2 in Figure 4.4 was originated from the ideas noticed through AS-1 so that AS-2 was represented schematically connected to AS-1. Moreover, AS-1 and AS-3 arose from the data presented at the beginning so that both of them were represented as developed based on that data. Secondly, when a new case which is not completely relevant to the previous argumentation stream was started to be discussed, this stream was placed as parallel to the previous ones. As an example, AS-1 and AS-3 can be considered at this point. Since AS-3 was not directly related to AS-1, AS-3 was situated as a parallel argumentation stream and did not follow AS-1 and AS-2 in a linear manner. Lastly, the set of argumentation components was labeled as an argumentation stream when the unity of the discussed concept was provided. A comparatively irrelevant issue coming afterward and a new issue distinctively focused were represented in another argumentation stream. For example, CSG discussed how the centroid of the given triangle could be constructed by assuming that it would give the circumcenter through AS-1 in Figure 4.4. Then, CSG tried to find the centroid by constructing the angle bisectors in AS-2. Since this idea did not work, they continued with finding the intersection of the medians in AS-4 to find the circumcenter. As seen, each of the mentioned argumentation streams has integrity in terms of content so that they were constituted as different argumentation streams.

Another issue to note is the enumeration of the argumentation streams in the global argumentation structures. The order of presence of conclusions was the key issue in the enumeration. The argumentation stream involving the first idea discussed may not be labeled as AS-1. In the case that group stopped working on the first idea for a while and continued with the second idea and also presented a conclusion for it, the argumentation stream involving the second idea is named as AS-1. Therefore, in some global argumentation structures, it was seen that there is not an occurrence order in the argumentation streams. In the absence of a conclusion in the argumentation stream, the enumeration order was determined according to the finishing time of the discussed issue in the argumentation stream. For example, in Figure 4.4, it can be seen that AS-4 took place above AS-3 depending on their starting and finishing occasions.

When the reasoning backward was observed during argumentation, it was signified with a dotted line with the one-way arrow to point out the direction explicitly in the global argumentation structures. It was seen that the reasoning backward was configured not only from one argumentation stream to another one but also within one argumentation stream. The purposes of the reasoning backward are to provide further justification, to support a previous statement regardless of the component that statement was coded such as the warrant and conclusion, and to present the link from a latter component of argumentation to the former one. Moreover, the presence of a reasoning backward between argumentation blocks was not accepted as a factor for assuming the argumentation blocks as connected. That is to say, a dotted line between the argumentation streams situated in different argumentation blocks does not cause the connection of the argumentation blocks. For example, there are three argumentation blocks in the global argumentation structure formed from the discussion of GG in Activity 4 (See Figure 4.10 or Appendix E). In this global argumentation structure, there are two dotted lines representing the reasoning backward, one which is from the second argumentation block and to the first argumentation block and other one is from the third argumentation block to the first one. As it is seen, the presence of the reasoning backward is not an obstacle for keeping the argumentation blocks as separate in the current study.

Thus far, the content of the global argumentation structures was presented. In the section that follows, types of global argumentation structures emerged were explained in detail.

4.2.2. Types of Global Argumentation Structures

After the formation of these structures, the appropriateness of the existing types of global argumentation structures in the literature was examined in order to entitle the ones in the present study. As stated before, six types of global argumentation structures which are the source-structure, the spiral-structure, the reservoir-structure, the gathering-structure, the line-structure, and the independent arguments-structure were found out in the accessible literature. In this study, three of the mentioned types which are the source-structure, the reservoir-structure, and the line-structure were utilized with some adaptations. In more detail, the source-structure and the reservoir-structure which were stated by Reid and Knipping (2010) and also the line-structure which was pointed out by Erkek (2017) were not directly suitable to categorize the global argumentation structures emerged in this study. Therefore, some characteristics of the reservoir-structure and the line-structure were revised, and it can be stated that some basic and visual characteristics of these two structures were mainly focused on.

Eventually, the terms the reservoir-structure and the line-structure were directly employed. Although the frame of the source-structure was not applicable, it was seen that one characteristics of it was usable for the global argumentation structures presented in the study. That is, one visual feature of the source-structure regarding the funneling effect was focused. Thus, instead of using the source-structure, a new type of structure was offered, namely, the funneling-structure. Another new type of global argumentation structure presented in the current study was entitled as the branching-structure.

The aforementioned four types of global argumentation structures which are the line-structure, the reservoir-structure, the funneling-structure, and the branching-structure represent the mono structures for this study. In other words, if the global argumentation structure is comprised of one main argumentation block which can be categorized by means of one of these four types of structures, it was categorized under the heading of mono structures. Moreover, in the case that the global argumentation structure has more than one argumentation block, each of which was categorized into the same type of mono structure, such argumentation structures were also classified under the heading of mono structures. On the other hand, if global argumentation structure is piecewise which refers to the presence of more than one argumentation block, and these argumentation blocks fall into different types of mono structures, such piecewise structures were classified under the heading of hybrid structures. In short, it can be stated that types of global argumentation structures emerged in the study were mainly divided into two sections, namely, the mono structures and the hybrid structures.

The characteristics of the types of global argumentation structures utilized in the study were explained thoroughly in the data analysis section. In this section, to document the correspondence of these structure types in the study, each type of structure was reported by giving examples and explaining the argumentation process. First of all, types of global argumentation structures emerged from the discussions of both CSG and GG in each activity were presented in the following table.

Table 4. 2

Types of global argumentation structures

Types of global argumentation structures		CSG	GG
Mono structures	The reservoir-structure	Activity 3	-
	The funneling-structure	Activity 2	Activity 2
	The branching-structure	Activity 4	Activity 1
Hybrid structures	The reservoir-funneling-structure	-	Activity 3
	The line-branching-structure	Activity 1	-
	The line-reservoir-branching-structure	-	Activity 4

Due to the number of types of global argumentation structures arranged, there is the possibility of having four categories under the title of mono structures. However, a global argumentation structure which can be categorized under the line-structure only was not observed, and the line-structure was observed as a piece of the hybrid structures. In this respect, as seen in Table 4.2, three types of mono structures which are the reservoir-structure, the funneling-structure, and the branching-structure were seen in the scope of the study. Moreover, without focusing on which group the mono structures existed, it can be stated that the mono structures were seen in every activity. Since there are four types of mono structures, the possible hybrid structures involve binary combinations, triple combinations, and the one involving four of them. Thus, it can be stated that eleven types of combinations (six from binary combinations, four from triple combinations, and one involving all four of them) are possible for the hybrid structures. However, as presented in Table 4.2, three types of hybrid structures which are the reservoir-funneling-structure, the line-branching-structure, and the line-reservoir-branching-structure emerged in the current study. After all, it was observed that three of four global argumentations of CSG have the mono structures and one global argumentation has the hybrid structure. Two global argumentations of GG have the mono structures and the remaining two global argumentations have the hybrid structures.

In the following headings, each type of global argumentation structure presented in Table 4.2 was explained in detail. Specifically, the findings deduced from this categorization were presented in terms of the mono structures and the hybrid structures.

4.2.2.1. Mono structures

As given in Table 4.2, three types of mono structures among four possible cases were seen in the global argumentation structures of this study. More specifically, the global argumentation structure of CSG in Activity 3 was labeled as the reservoir-structure. In Activity 2, the global argumentation structures of both CSG and GG were entitled as the funneling-structure. Finally, the global argumentation structure of CSG in Activity 4 and the global argumentation structure of GG in Activity 1 were categorized as the branching-structure. As mentioned, the line-structure was not seen as a mono structure in any of the global argumentation structures. Moreover, the related cognitive unity based activities were explained briefly at the beginning part while instantiating the global argumentation structures, but all activities can be seen in Appendix B. The findings related to the mono structures were presented under the following sub-headings.

4.2.2.1.1. The reservoir-structure

Only one global argumentation structure which was gathered from the argumentation process of CSG in Activity 3, which was presented in Figure 4.5 below, was classified as the reservoir-structure. In Activity 3, prospective middle school mathematics teachers were given ΔABC, which is an acute triangle, and asked to construct the circumcenter, the orthocenter, and the centroid by using compass-straightedge or GeoGebra, and then search for possible connections or relationships among these three points (See Appendix B).

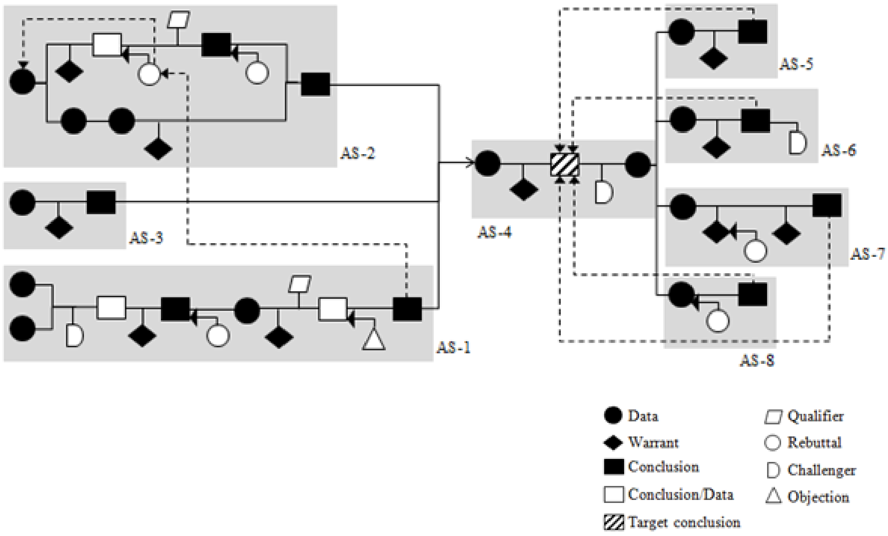


Figure 4. 5. The reservoir-structure example from CSG in Activity 3

According to the revised description of the reservoir-structure arranged for the study, the global argumentation structure in Figure 4.5 was labeled as the reservoir-structure. The overall argumentation structure of CSG in Activity 3 involves two self-contained parts which is one of the features of the reservoir-structure. The presence of an intermediate target conclusion in AS-4, which represents the transition point from the first part to the second part throughout the flow of the argumentation, can be seen in Figure 4.5. Target conclusion in AS-4 seems to hold all ideas coming from AS-1, AS-2, and AS-3 which constitute the first part and then lead the argumentation to move to the next part which is comprised of AS-5, AS-6, AS-7, and AS-8. As an expected but not a requisite feature of this type of structure, the reasoning backward was presented by virtue of the dotted line with the one-way arrow. The reasoning backward was seen twice in the first part, one of which was from AS-1 to AS-2, and the other one occurred within AS-2. In the second part, the reasoning backward was observed four times. In more detail, the conclusions of AS-5, AS-6, AS-7, and AS-8 led the argumentation move to target conclusion in AS-4. Moreover, one of the revisions conducted regarding the reservoir-structure which was explained in the data analysis of the methodology chapter is about the presence of the refutations. Since all global argumentation structures set out in the study involve the rebuttal component which refers to the refutation, the characteristics of the reservoir-structure about the absence of refutation stated by Reid and Knipping (2010) was omitted. As illustrated in Figure 4.5, there are five rebuttals stated against four different types of components of the argument. Based on these points, it was concluded that the global argumentation structure displayed in Figure 4.5 fulfills the entailments to be classified as the reservoir-structure for the present study.

After the explanation about the entitling process of the global argumentation structure of CSG in Activity 3, a more detailed explanation about the enumerated argumentation streams was presented below. As stated previously, Activity 3 involves the construction of the orthocenter, the circumcenter, and the centroid of ΔABC which corresponds to the first part of the global argumentation structure. Specifically, one of the participants in CSG tried to construct the orthocenter in AS-2 (See Figure 4.39) while another participant focused on the construction of the circumcenter in AS-3 (See Figure 4.39). They were familiar with the construction of the circumcenter and the orthocenter from the previous cognitive unity based activities. However, the construction of the centroid of a triangle was quite new for them, and another participant started to work on it which was represented through AS-1 (See Figure 4.39). Although CSG started to work on the construction of the orthocenter initially, the argumentation stream referring to this process was coded as AS-2. Therefore, at this point, it might be of benefit to restate the structure followed in the enumeration of the argumentation streams. The presence order of conclusions was focused on while enumerating argumentation streams. Thus, although CSG started to study about the construction of the centroid as their third idea, the first conclusion they reached is about the centroid which refers to the conclusion in AS-1. Therefore, the first idea that the group started to work with may not be labeled as AS-1 depending on the presence of conclusion.

Another issue noticed in this global argumentation structure is that a mistake conducted during the construction of the orthocenter caused confusion for the group. In AS-2, one of the participants in CSG constructed the perpendicular bisectors of the sides of ΔABC, found the point of concurrency of them, and assumed this point as the orthocenter of ΔABC although she actually found the circumcenter of ΔABC by using this method. Meanwhile, another participant in CSG was working on the construction of the perpendicular bisectors of the sides of ΔABC in order to find the circumcenter in AS-3. Since they observed that they were about to find the same point, they thought that the orthocenter and the circumcenter of ΔABC correspond to the same point. Then, they anticipated that the centroid of ΔABC might also be the same point. However, the one who was working on the construction of the centroid concluded that the point concurrency of the medians of a triangle is the centroid and also showed others that the centroid appeared at a different point than the point they expected, and this process was represented in AS-1. As a result of this conclusion in AS-1, the reasoning backward was conducted since further justification was needed. The initial reflection of the reasoning backward, which was shown by a dotted line with the one-way arrow from AS-1 to AS-2 in Figure 4.2, was seen as a rebuttal in AS-2. Thus, CSG noticed that what they constructed with a purpose of finding the orthocenter is actually the circumcenter and they should construct the altitudes of ΔABC and determine the point of concurrency of them to be able to find the orthocenter. This situation made the group move to the data component at the beginning of AS-2 where they aimed to construct the orthocenter of ΔABC and this moving backward was also presented by the dotted line with the one-way arrow in AS-2. Then, as the step of reasoning forward, CSG found an approach to draw the altitudes which was represented via the second argumentation step in AS-2. When CSG stated the final conclusion in AS-2, they were aware of the difference between the construction processes of the orthocenter and the circumcenter of a triangle. After that, they retraced over the construction approach of the circumcenter and stated that the intersection of the perpendicular bisectors of the sides of a triangle gives the circumcenter as presented through AS-3.

During the entire construction process, CSG came up with five approaches for the construction of the aimed three points. At this section, these approaches were mentioned briefly by focusing on the flow of the argumentation throughout the streams. The detailed explanations of the ideas CSG put forward with the aim of construction will be presented while addressing the third research question in a further section by referencing the use of tools and the geometric figures they formed. However, for ease of following, the figures presented in the next sections of this chapter were given in the parentheses when they are related to what is mentioned in this section.

After the constructions of three points asked in the activity, CSG started to look for the connections/relationships among these points by giving justification through their drawings, which took place in AS-4. Thus, they produced target conclusion that the orthocenter, the circumcenter, and the centroid are collinear. At this point, a challenger was presented whether their conjecture is true for all types of triangles which led them to check the statement they reached for different types of triangles. In AS-5, CSG checked the collinearity for an equilateral triangle and concluded that three points coincide. Therefore, they hesitated whether this situation makes their conjecture false for the equilateral triangle, but then they decided that the statement is still valid since the coincidence of three points is not a counter argument for being at the same line. Then, they mentioned about the validity of target conclusion. Thus, the conclusion of AS-5 was linked to target conclusion in AS-4 to support it. In a similar vein, AS-6 represents the argumentation process during the validation check of the statement reached for an obtuse triangle, AS-7 covers the same discussion for a right triangle, and AS-8 involves the same checking process for an isosceles triangle. By linking the statements at the conclusion components of AS-6, AS-7, and AS-8 to their conjecture in AS-4 which was coded as target conclusion, CSG supported the fact that the mentioned three points are collinear.

As seen, the global argumentation structure presented in Figure 4.5, which was categorized under the reservoir-structure, includes two parts. The first part of argumentation is about the construction of the intended geometric concepts in the activity and also involves five approaches tried with the aim of construction, each of which will be explained later in detail. After reaching to target conclusion, argumentation moved to the second part which covers the validity checking process of the conjecture produced.

4.2.2.1.2. The funneling-structure

The global argumentation structures prepared from the discussion of both CSG and GG in Activity 2 were categorized as the funneling-structure. In Activity 2, three different types of triangles which are an acute triangle (ΔDEF), an obtuse triangle (ΔABC), and a right triangle (ΔKLM) were given to groups and then they were asked to construct the altitudes and the orthocenter in case of existence for the given triangles by using compass-straightedge or GeoGebra (See Appendix B). As an example for the funneling-structure, the overall argumentation structure of CSG during Activity 2, which was presented in Figure 4.6, was explained in this section. Other global argumentation structure coded under the funneling-structure can be seen in Appendix E.

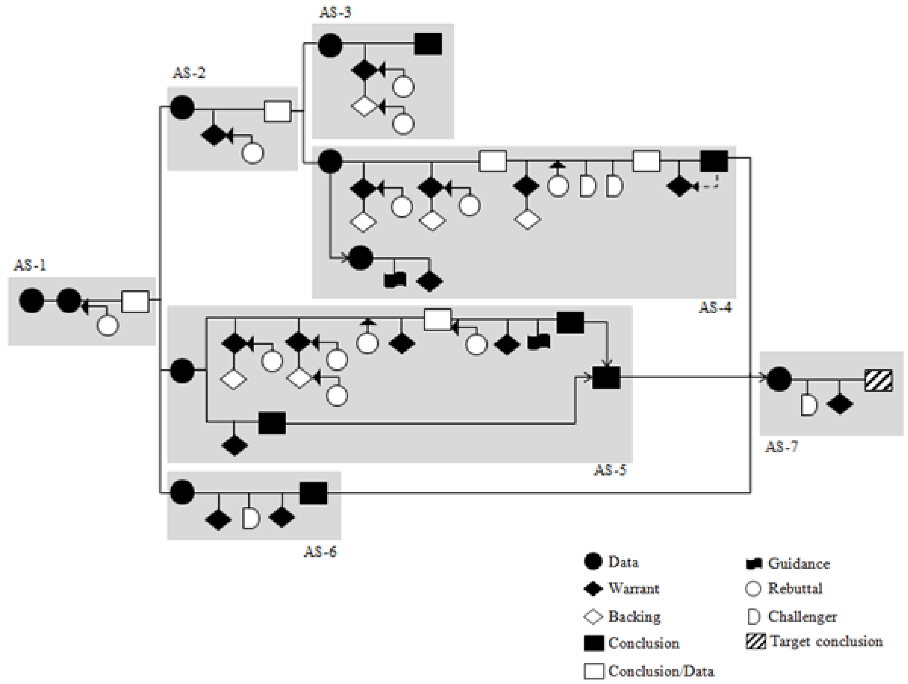


Figure 4. 6. The funneling-structure example from CSG in Activity 2

As stated in the data analysis section, the funneling-structure takes its name from the fact that parallel argumentation streams funnel towards one argumentation stream involving conclusion or target conclusion in the last part of the argumentation schema. The beginning part of the schema was accepted as related to the roots and the global argumentation structure may involve one root or more than one root. In other words, the parallel argumentation streams may be originated from one root or more roots so that it determines the sub-category of the funneling-structure such as the single-rooted funneling-structure and the multiple-rooted funneling-structure. According to Figure 4.6, AS-1 refers to the root of the global argumentation structure. Then, argumentation streams originated from AS-1 which are AS-2, AS-3, AS-4, AS-5, and AS-6 were located as parallel. Finally, these parallel argumentation streams funnel towards one argumentation stream which is AS-7 involving target conclusion of the argumentation. Thus, the global argumentation structure in Figure 4.6 was categorized as the single-rooted funneling-structure since the parallel argumentation streams are originated from one root at the beginning part, the root was followed by parallel argumentation streams in the intermediate part, and it covers the feature of funneling into one argumentation stream involving target conclusion.

In order to offer an in-depth description of the global argumentation structure in Figure 4.6, the contents of the argumentation streams were documented as follows. In AS-1, CSG evaluated the givens in the activity and discussed what was asked. Particularly, they envisioned the altitudes for the given three triangles ΔABC, ΔDEF, and ΔKLM and also hesitated whether the orthocenter is the point of concurrency of the altitudes or the perpendicular bisectors, and finally agreed that, if it exists, it is the point of concurrency of the altitudes. In this regard, conclusion/data in AS-1 summarizes this discussion process and provides data for the following parallel argumentation streams.

Since CSG was acquainted with the construction of the circumcenter of a triangle from the prior cognitive unity based activity, they attempted to construct the perpendicular bisectors of the sides of a triangle so as to construct altitudes of the given acute triangle in AS-2. Thus, CSG decided to check whether the construction of the perpendicular bisectors of the sides of the triangle works or not in line with the purpose of the activity and this idea was represented as conclusion/data in AS-2. Then, the argumentation process symbolized in AS-3 started (See Figure 4.26). That is to say, CSG wondered if the perpendicular bisector of $\overline{EF}$ in ΔDEF passes through the vertex D. They thought that if it passes through the point D, it might also turn out to be the construction of the altitude of $\overline{EF}$. Then, they tried this idea for the given three triangles. By means of the rebuttals presented in AS-3, CSG concluded that the mentioned idea might work for some triangles, but it is not always the case depending on the type of triangle. In AS-4, CSG maintained the idea of using the perpendicular bisectors while aiming to construct the altitudes of given triangles so that AS-4 was originated from the conclusion/data of AS-2 as a parallel argumentation stream to AS-3. At the beginning of AS-4, one participant of CSG offered about drawing the parallel lines to the perpendicular bisectors from the corresponding vertices in order to find the altitudes. They discussed about it, and then decided to determine the smallest distance from a vertex to the perpendicular bisector by drawing isosceles triangles and to transfer this distance to determine the second point by which the parallel line will be drawn. By using this approach, which was explained in the findings related to the third research question in detail, they constructed the altitudes of each side of all triangles throughout the first argumentation step in AS-4. As the conclusion of AS-4, they pointed out the orthocenters of each triangle (See Figures 4.26, 4.27, and 4.28). Moreover, the second argumentation step in AS-4 represents an alternative approach for the construction of the parallel lines. It involves the idea of transferring the angle to draw the parallel line, but they could not reach a solid result regarding this idea and then gave up. Thus, it can be stated that this alternative idea did not avail for the construction of the altitudes in Activity 2. Moreover, the backward reasoning existed in AS-4. Although CSG provided justification for the conclusion in AS-4 which was showed via the warrant component before the conclusion, they declared that the warrant was partially expressed and discussed over the warrant for further justification. Since this movement of the group was quite distinguishable in the argumentation process, it was presented as a backward reasoning with the dotted line in the last part of AS-4.

In the meantime, as represented in AS-5, one participant of CSG initiated another approach for construction based upon the statement "the angle subtended by the diameter of a circle is 90°". Therefore, CSG accepted the sides of the triangle as the diameters and drew circles. Then, they marked the intersection points of circles and sides or extended sides of triangles and signed such a point as the foot of the altitude, and this process enabled them to construct the altitudes. The former argumentation step in AS-5 involves the intertwined application process of the mentioned idea for the construction of the altitudes of the given acute and obtuse triangles (See Figures 4.29 and 4.30) while the latter argumentation step in AS-5 covers the construction of the altitudes of the given right triangle by using the same idea (See Figure 4.31). The last conclusion of AS-5 which was deduced from the examination of the processes expressed in both of the argumentation steps presented the validity of this method and pointed out the orthocenters of each given triangle. Moreover, it can be stated there are comparatively many rebuttals in AS-5 due to the fact that they experienced some disagreements regarding this approach, especially in the construction of the altitudes of the given obtuse triangle.

Through AS-6, CSG tried to construct the altitudes by applying a geometric construction approach which is the one frequently written in the sources. Specifically, it can be stated that the mentioned approach is the basic method used to construct the line perpendicular to a given line from a point, not on the given line. By following this method, they drew the altitudes of each side of all given triangles and pointed out the presence of the orthocenters for all triangles which was represented in the conclusion of AS-6 (See Figures 4.32, 4.33, and 4.34). Finally, a discussion about the presence of orthocenter for all types of triangles appeared in AS-7. By taking into consideration of three valid construction approaches applied to the given three types of triangles and by combining the conclusions reached in AS-4, AS-5, and AS-6, CSG stated target conclusion that the altitudes of a triangle are concurrent and this point is the orthocenter.

To conclude, the global argumentation structure in Figure 4.6 which was entitled as the funneling-structure covers five approaches for construction through AS-3, AS-4, AS-5, and AS-6. One of these approaches could not reach a conclusion which was symbolized in the second argumentation step in AS-4, one of them did not serve as a valid approach for the construction which was signified at AS-3, and the remaining three approaches were conveyed to valid constructions which resulted in the conclusions in AS-4, AS-5, and AS-6. In line with these three approaches offered for construction, CSG reached some pieces of evidence for the conjecture regarding the presence of the orthocenter. By following them, CSG declared the statement that the altitudes of any type of triangle are concurrent which also refers to the presence of the orthocenter for all types of triangles. As stated before, the mentioned construction approaches and the ideas leading to the conjecture in this activity will be considered later in detail while explaining the findings of the related research questions.

4.2.2.1.3. The branching-structure

Similar to the funneling-structure, the branching-structure emerged two times in the study, namely, from the discussion of CSG in Activity 4 and the discussion of GG in Activity 1. As the example for the branching-structure, the global argumentation structure prepared from the collective argumentation of GG in Activity 1 was presented in Figure 4.7 given below. In Activity 1, prospective middle school mathematics teachers were given ΔABC which is an acute triangle and asked to construct the circle passing through the vertices of the given triangle by using compass-straightedge or GeoGebra (See Appendix B). Moreover, the second global argumentation structure categorized as the branching-structure can be seen in Appendix E.

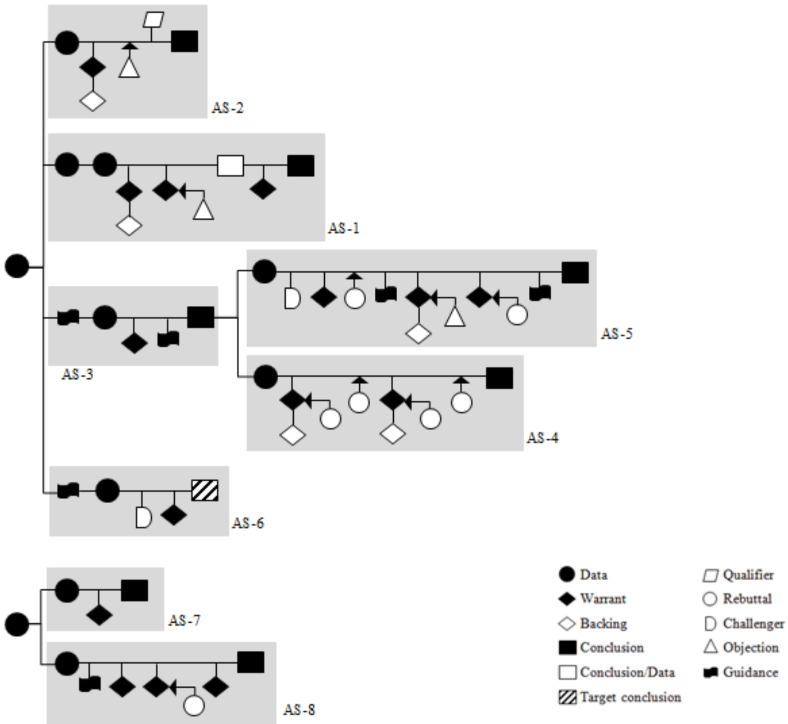


Figure 4. 7. The branching-structure example from GG in Activity 1

As seen in Figure 4.7, the overall argumentation structure of GG involves two argumentation blocks. In spite of the fact that it has a piecewise structure, the reason of not classifying it under the hybrid structures is that each argumentation block could be classified as the same type of mono structure, namely, as the branching-structure. Thus, since it does not have a mixed nature, it was decided to classify such global argumentation structures under the corresponding mono structure instead of the hybrid structure heading.

In the first argumentation block, there is one root which contains only one data instead of an argumentation stream. Then, four parallel argumentation streams which are AS-1, AS-2, AS3, and AS-6 arose from this root, and also two further argumentation streams which are AS-4 and AS-5 branched off from AS-3. Similarly, the second argumentation block includes one data component as a single root and two parallel argumentation streams which are AS-7 and AS-8 originated from this root. As displayed in Figure 4.7, all argumentation streams in the structure ended with a conclusion, but one of them was the target conclusion. By considering these points, it can be stated that this structure is appropriate to categorize under the branching-structure, particularly the multiple-rooted branching-structure.

After the clarification about the entitling the global argumentation structure of GG in Activity 1, each argumentation stream situated in it was described in detail. In the first argumentation block, GG studied with the GeoGebra file in which the tools 'circle through 3 points' and 'circumcircular arc' were restricted. In the data component presented as the root, GG thought about all given data in this activity and aimed to draw the circumcircle of ΔABC. Then, four parallel argumentation streams originated from this root. In AS-2, GG tried to use the theorem "the measure of an inscribed angle of a circle is one-half the measure of its intercepted arc". However, they decided to give up this method since they noticed the absence of a tool to draw an arc by writing a particular measure in GeoGebra. Meanwhile, in AS-1, GG focused on the idea of finding the center of the circle. Their first idea for finding the center is to construct the medians and to find their point of intersection so that the centroid might also be the center of the circle passing through the vertices of ΔABC. However, they resulted in that the intended circle cannot be drawn by accepting the centroid as the center (See Figures 4.45 and 4.46).

After this, GG thought to continue with the construction of the angle bisectors and find the point of intersection of them as symbolized in AS-3. At the conclusion of AS-3, they stated that the intersection point of the angle bisectors gives them the incenter (See Figure 4.47). Then, GG continued to discuss about two issues they noticed during the construction of the angle bisectors which were represented in AS-4 and AS-5. In AS-4, GG focused on finding the radius of the circle. With this purpose, they tried to use Pythagoras theorem and the tool ‘distance and length’ of GeoGebra, but later they noticed the line segment they assumed as the radius was not the radius of the circle. The issue they missed in AS-4 is that the intersection point of the angle bisectors is not the center of the circle. Although they expressed this statement in AS-3, they continued as if it is the center in AS-4 (See Figure 4.48). The second idea arose from the construction of the angle bisectors was discussed throughout AS-5. In more detail, in AS-5, GG formed a larger triangle by drawing lines parallel to the sides of ΔABC and passing through each vertex of ΔABC. Then, they assumed that the incircle of the larger triangle might be the circumcircle of ΔABC and expected the vertices A, B, and C to be the tangency points of the incircle of the larger triangle. However, they concluded that this idea did not work in AS-5 (See Figure 4.49). Moreover, as seen from Figure 4.7, the rebuttal and the objection components which involve rejections were frequently seen in AS-4 and AS-5 compared to other argumentation streams since the ideas were not so clear for some participants so that they queried the process. The last argumentation stream of the first argumentation block is AS-6. After working on the angle bisectors and the medians, GG tried to find the perpendicular bisectors of the sides of ΔABC. They observed that the perpendicular bisectors of the sides were also concurrent and then concluded that this point is the circumcenter. This was represented as a target conclusion in AS-6 (See Figure 4.50).

In the second argumentation block of the global argumentation structure given in Figure 4.7, the GeoGebra file used by GG involves some extra restricted tools. Since prospective middle school mathematics teachers reached the relevance of the perpendicular bisectors of the sides and the circumcircle in the first part of the argumentation, they were expected to try about different construction ideas in the same activity. That is, the already restricted two tools were kept and three more tools which are ‘midpoint or center’, ‘perpendicular line’, and ‘perpendicular bisector’ were also removed from the toolbar. The second argumentation block involves one root which is one data component and two branching and parallel argumentation streams which are AS-7 and AS-8. In AS-7, GG focused on finding the midpoints of the sides of ΔABC so that they measured the lengths of the sides of ΔABC and divided these lengths by 2 to find the midpoints. In the meantime, GG thought about constructing the perpendicular bisectors of the sides of ΔABC as if they were using compass-straightedge. As the final step of this method, they drew the circumcircle of ΔABC. This approach was displayed through AS-8 (See Figure 4.51).

To sum up, the global argumentation structure categorized as the multiple-rooted branching-structure involves eight approaches suggested for the construction of the circumcircle of ΔABC and four conjectures involving the one stated in the target conclusion. Moreover, both the approaches offered for construction and the ideas related to conjectures were described in further sections by referencing the tools used and the geometric figures formed, but this section was mainly about the content of the argumentation streams. At this point, since all mono structures that emerged in this study were explained with examples, the hybrid structures will be described in the next section.

4.2.2.2. Hybrid structures

When the global argumentation structure involves more than one argumentation block and each of them can be classified with different types of mono structures, it was categorized as a hybrid structure. As presented before in Table 4.2, three types of hybrid structures which are the reservoir-funneling-structure, the line-branching-structure, and the line-reservoir-branching-structure among eleven possible ones were observed during the classification of the global argumentation structures in the present study. Each of these hybrid structures emerged once in the activities. In more detail, the global argumentation structure of GG in Activity 3 was appropriate to categorize as the reservoir-funneling-structure since it involves two argumentation blocks named with a different type of mono structures. As the second type of the hybrid structure, the line-branching-structure was constructed from the discussion of CSG in Activity 1. Finally, the global argumentation structure of GG in Activity 4 involves three argumentation blocks so that this argumentation structure was entitled as the line-reservoir-branching-structure. Like the mono structures, the findings regarding three types of hybrid structures were explained in-depth in the following subsections.

4.2.2.2.1. The reservoir-funneling-structure

The reservoir-funneling-structure was seen once in this study in the global argumentation structure of GG in Activity 3, and it was presented in Figure 4.8. Although Activity 3 was explained before while exemplifying the reservoir-structure, it was aimed to remind Activity 3 at this point for the ease of reference. Activity 3 covers the construction of three points which are the orthocenter, the circumcenter, and the orthocenter for a given acute triangle and asks for the examination of the connections/relationships among these three points (See Appendix B).

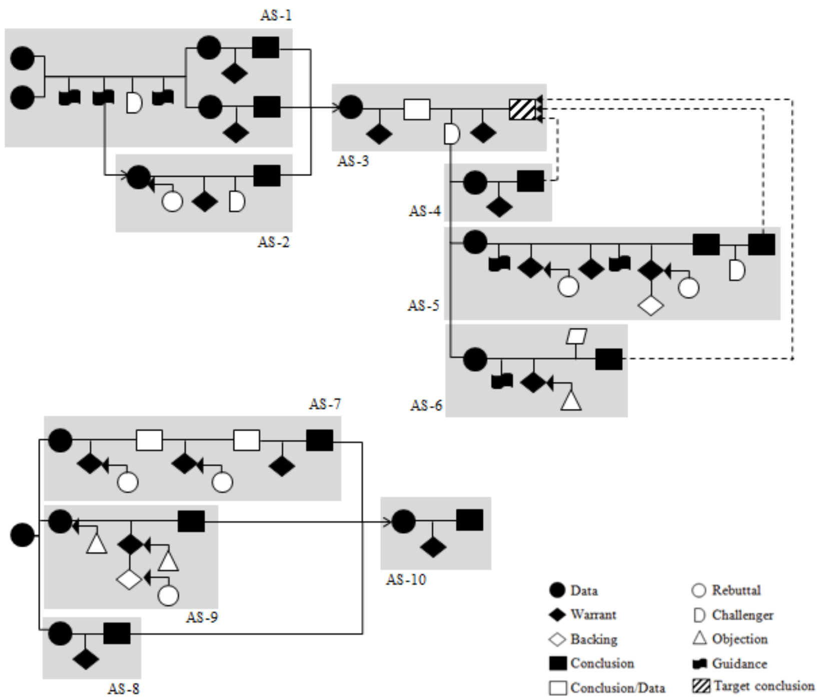


Figure 4. 8. The reservoir-funneling-structure example from GG in Activity 3

Based on the descriptions of the reservoir-structure and the funneling-structure arranged for this study, the first argumentation block in Figure 4.8 was classified as the reservoir-structure while the second argumentation block was classified as the funneling-structure. The argumentation block coded as the reservoir-structure involves two parts. The first part of the reservoir-structure involves AS-1 and AS-2 while the second part of it involves AS-4, AS-5, and AS-6. The presence of an intermediate target conclusion in AS-3 signifies the transition from the first part to the second part of the argumentation. Moreover, there existed the backward reasoning three times. The conclusions of AS-4, AS-5, and AS-6 moved backward to the target conclusion in AS-3 to support it. By taking into consideration the exclusion of characteristics of the reservoir-structure regarding the absence of refutations, it can be stated that all of these features are sufficient to label this argumentation block as the reservoir-structure. Moreover, the second argumentation block was categorized under the funneling-structure. At the beginning part, it involves one root and then three parallel argumentation streams, which were AS-7, AS-8, and AS-9, originated from this root. As the discriminating feature of this type of structure, the parallel argumentation streams funnel through one argumentation stream which involves the conclusion as noted in AS-10. Based on these characteristics, the second argumentation block was accepted as appropriate to be classified as the funneling-structure, particularly the single-rooted funneling-structure. In conclusion, since the global argumentation structure of GG in Activity 3 is a combination of the reservoir-structure and the funneling-structure, it was coded under the hybrid structures and entitled as the reservoir-funneling-structure.

In order to elaborate on this global argumentation structure, the argumentation streams were also discussed one by one. In the first argumentation block, GG was provided a GeoGebra file in which all tools were available. Since the participants of GG were familiar with the constructions of the orthocenter and the circumcenter from the previous cognitive unity based activities, they started with the constructions of them in Activity 3 which was schematized in AS-1. As seen in AS-1, there are two argumentation steps after a guidance component. In the first argumentation step of AS-1, they concluded that the intersection of the altitudes of ΔABC gives them the orthocenter (See Figure 4.65). In the second one, they explicitly stated that the point of concurrency of the perpendicular bisectors of the sides of ΔABC is the circumcenter of that triangle (See Figure 4.66). The remaining point asked to construct in this activity which is the centroid was not so clear for them at the beginning of AS-2. They hesitated whether they can find the centroid via finding the intersection point of the angle bisectors or the medians of ΔABC. However, they continued with the option regarding the medians and stated that the intersection of the medians of ΔABC is the centroid of it which was represented in the conclusion of AS-2 (See Figure 4.67). By correlating the discussions represented in AS-1 and AS-2 and examining the properties of the points in the GeoGebra file, GG declared that the constructed three points are collinear which refers to target conclusion in AS-3 (See Figure 4.68). At this point, the second part of the global argumentation structure started. Due to the challenger displayed in AS-3, which questioned the collinearity of three points in different types of triangles, GG attempted to check it quickly by dragging the elements of ΔABC. Although they were quite sure about their conjecture after dragging, they also intended to check it by constructing some types of triangles separately instead of forming them by dragging. The mentioned checking process was presented via AS-4, AS-5, and AS-6.

In AS-4, GG checked the conjecture about the collinearity of the aforementioned three points for an obtuse triangle. In a similar vein, the content of AS-5 is the examination of the conjecture for an equilateral triangle. They concluded that the orthocenter, the circumcenter, and the centroid refer to the same point in an equilateral triangle and then explicitly stated about the collinearity of these points again so this was presented as a backward reasoning with the dotted line from AS-5 to AS-3. Finally, GG checked the conjecture for a right triangle throughout AS-6. Up to this point, the reservoir-structure was explained in detail. From now on, the argumentation block coded as the funneling-structure was discussed as follows.

In the second argumentation block, GG was given a GeoGebra file in which some tools were removed from the toolbar. The restricted tools are ‘midpoint or center’, ‘perpendicular line’, and ‘perpendicular bisector’. By starting from this data, they derived out three construction ideas, each of which were represented as an argumentation stream, namely, AS-7, AS-8, and AS-9. In AS-7, GG found out another idea for the construction of the perpendicular bisectors of the sides of ΔABC (See Figure 4.70). The remaining two points which are the centroid and the orthocenter were constructed in AS-8 and AS-9 respectively by means of the alternative approaches (See Figures 4.71 and 4.72, respectively). Finally, GG combined the discussion presented in AS-7, AS-8, and AS-9 and then checked whether their construction approaches used in the restricted GeoGebra file were valid or not. In other words, they checked about what they drew can be accepted as a construction by dragging and evaluated the collinearity of three points. Since they stated that the collinearity of points was not violated during the dragging process, it was represented in the conclusion of AS-10 that the geometric figure they formed in the restricted GeoGebra is a construction (See Figure 4.73).

Consequently, it was observed that the reservoir-funneling-structure of GG in Activity 3 involves six approaches offered for construction, three of which are in the first argumentation block and the remaining ones are in the second argumentation block. In addition, GG proposed one conjecture which was represented as the target conclusion, and they also offered other ideas which supported the conjecture. As stated before, the findings related to the mentioned issues were presented in detail in the following sections. It was also noticed that the global argumentation structure of CSG in Activity 3 and the first argumentation block of GG in Activity 3 have a common point that both of them were coded under the same type of structure which is the reservoir-structure.

4.2.2.2.2. The line-branching-structure

As presented in Table 4.2, the second type of hybrid structure is the line-branching-structure and it was formed from the argumentation process of CSG in Activity 1. Although the content of Activity 1 was stated while presenting the findings related to the branching-structure under the heading of mono structures, it was wanted to restate it at this point for the ease of inference. Activity 1 asked prospective middle school mathematics teachers to construct a circle passing through the vertices of the given ΔABC which corresponds to the construction of the circumcircle of an acute triangle (See Appendix B). The only global argumentation structure labeled under the line-branching-structure was indicated in Figure 4.9 given below.

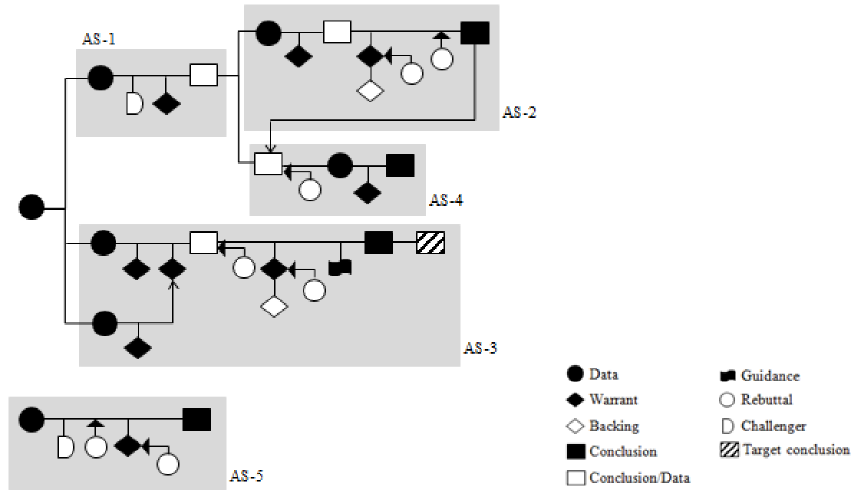


Figure 4. 9. The line-branching-structure example from CSG in Activity 1

As seen from Figure 4.9, the global argumentation structure of CSG in Activity 1 is piecewise. There are an argumentation block and an argumentation stream as separate from the block. Specifically, the first argumentation block was coded as the branching-structure and the separate argumentation stream was coded as the line-structure. Due to its hybrid nature, the overall argumentation structure was entitled as the line-branching-structure. Since the argumentation block involves one root which refers to one data component and four parallel argumentations streams (AS-1, AS-2, AS-3, and AS-4) branched from that root, each of which arrives at a conclusion at the end, it was categorized as the branching-structure, specifically the single-rooted branching-structure. As the second argumentation block, the separate argumentation stream which was represented as AS-5 is suitable for entitling as the line-structure. By considering the cases that the argumentation in AS-5 flows linearly, finishes with a conclusion without linking and spreading to other streams, and it seems like a line, AS-5 was coded under the line-structure. Based on the adaptations related to the line-structure in this study, the characteristics that the transitions are conducted with the component claim/data, which was presented in the studies of Erkek (2017) and Erkek and Işıksal-Bostan (2019), was not accepted directly. This case might be seen in the line-structure, but it was not accepted as a necessary condition for being classified under this type of structure. According to the characteristics of the line-structure and the branching-structure described for this study, the global argumentation structure displayed in Figure 4.9 was classified under the hybrid structures and named as the line-branching-structure.

After the explanation about the classification process of the line-branching-structure, the enumerated argumentation streams were described to provide more insight into the argumentation process of the given example. In the argumentation block coded as the branching-structure, CSG focused on finding the center of the circle in the root so that the data component at the beginning part covers the given data in the activity and their idea about finding the center of the circle that they were asked to construct. Then, four parallel argumentation streams which are AS-1, AS-2, AS-3, and AS-4 were branched off from this root. In AS-1, they aimed to find the centroid of ΔABC since they expected that the centroid might give them the center of the circle passing through the vertices of ΔABC. At this point, they hesitated whether they should construct the angle bisectors or the medians for arriving at the centroid. As presented in the conclusion/data component of AS-1, CSG thought about the potential of having the same point as the intersection points of both the angle bisectors and the medians and decided to try both of them so as to determine the centroid. Hence, two argumentation cases, which were represented as AS-2 and AS-4, derived from the conclusion/data component in AS-1. Along with AS-2, CSG tried to find the centroid by constructing the angle bisectors which is actually a dead end for this purpose. As the conclusion of AS-2, they noticed that the intersection point of angle bisectors gives the incenter (See Figure 4.20). The mentioned conjecture was not one of the expected conjectures related to the circumcircle since it was directly related to the incircle. They also looked for a way to transfer the incircle to the circumcircle by extending it. However, they observed that the incircle could not be conveyed to the circumcircle, and its extending scope did not match with the circumcircle.

As stemmed from the conclusion/data in AS-1 and also affected from the conclusion in AS-2, CSG continued to discuss about the idea that the concurrency points of the angle bisectors and the medians might be the same point and decided to find the intersection of the medians in AS-4. Although they passed to another approach presented in AS-3 after mentioning about the construction of the medians in AS-4, they continued to construct the medians after the application of the approach represented in AS-3. Therefore, the argumentation stream about the medians was enumerated as AS-4. When they found the point of concurrency of the medians, they concluded that they found the centroid of ΔABC, and it was not related to the construction of the circle passing through the vertices of ΔABC. This was the statement signified as conclusion in AS-4 (See Figure 4.21).

Meanwhile, in AS-3, it was seen that two ideas were combined and continued as one argumentation step. As the first idea, one participant of CSG proposed to accept the sides of ΔABC as the chords, to find the midpoints of these chords, and then to draw the lines perpendicular to the chords and passing through the midpoints of them so that the intersection of these perpendicular lines would give them the center of the intended circle. As the second idea, another participant mentioned that she was searching for an approach related to the sides and the perpendicular bisectors of the sides. Then, they noticed that they were focusing on the same issue, but their starting points were different. Although there were some disagreements and unclear points throughout the construction trials in AS-3, they could construct the circle passing through the vertices of ΔABC by taking the distance between the potential center and any vertex as the radius and this was represented via the conclusion in AS-3. Afterwards, they also declared that the intersection of perpendicular bisectors of sides of ΔABC gives the circumcenter. This conjecture was symbolized as target conclusion at the end of AS-3 (See Figure 4.22). At this point, all argumentation streams involved in the branching-structure were explained.

As stated, the second argumentation block which involves one argumentation stream only was classified as the line-structure and it was represented via AS-5. In the first argumentation block, their starting point was to find the center of the circle aimed to construct. In AS-5, CSG focused on finding another construction approach for the intended circle without emphasizing to find the center at the first step. Then, they decided to draw three circles with equal radius by accepting each vertex of ΔABC as center and thought that the intersection points of these circles might be on the circumcircle but observed that this idea did not work. In this process, they drew the lines between the intersection points of each two circles. Then, they noticed that the intersection point of these lines also gave them the circumcenter since all circles used have the equal radius. Thus, they declared this as a new approach for the construction of the intended circle in the conclusion of AS-5 (See Figure 4.23).

To sum up, four approaches were offered and tried in the construction process of the circle passing through the vertices of ΔABC. At the end of these construction trials, three conjectures were stated, one of which was about the circumcircle so that it was accepted as related to the activity and represented as target conclusion. The mentioned approaches for construction and the conjectures were also explained thoroughly in the following sections under the heading of the associated research questions.

4.2.2.2.3. The line-reservoir-branching-structure

The last type of hybrid structure in this study is the line-reservoir-branching-structure. While the previous hybrid structures are involving two different types of mono structures, this structure involves three of them. This type of hybrid structure was observed only in the global argumentation structure of GG in Activity 4. In Activity 4, prospective middle school mathematics teachers were asked to mark random points X, Y, and Z on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of the given ΔABC, respectively. Then, they were asked to construct three circles; the first circle is passing through the points A, X, and Z, the second circle is passing through the points B, Y, and Z, and the third circle is passing through the points C, Z, and Y by using compass-straightedge or GeoGebra. Finally, they were asked to examine the relationships or connections among these circles (See Appendix B). Before describing it in detail, the global argumentation structure of GG in Activity 4 was given in Figure 4.10.

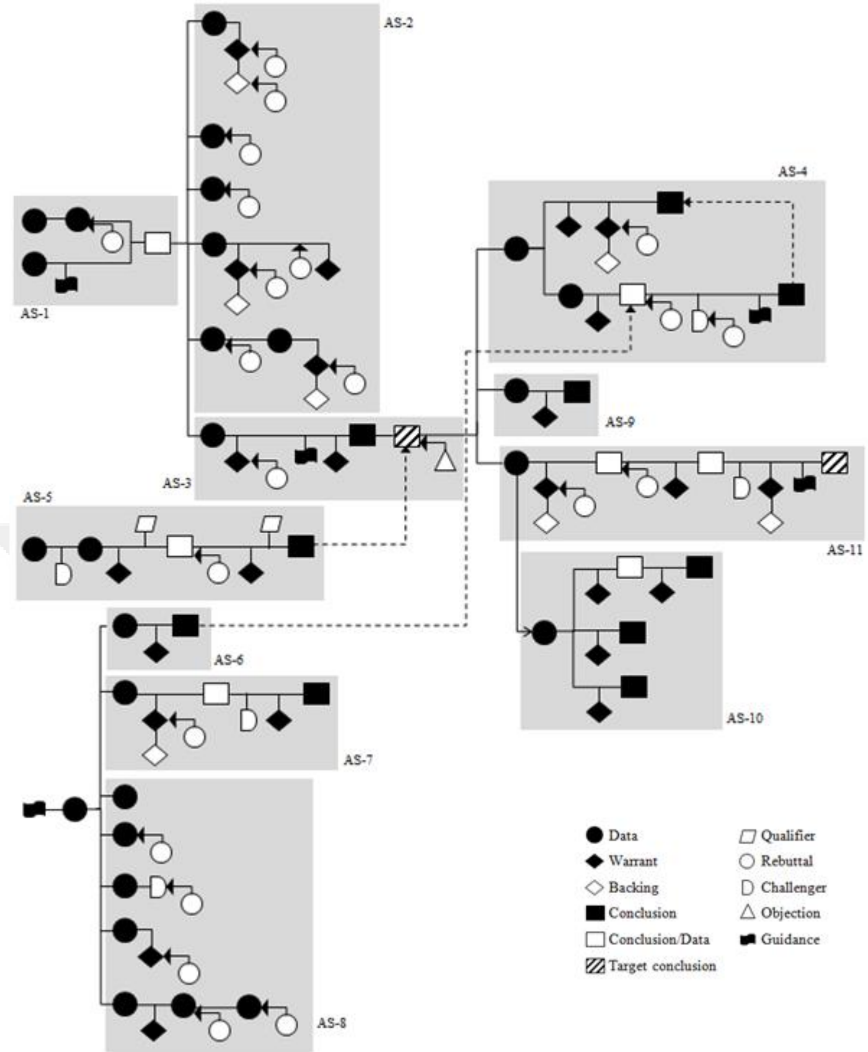


Figure 4. 10. The line-reservoir-branching-structure example from GG in Activity 4

Compared to the previously stated global argumentation structures, it can be stated that the one given in Figure 4.10 is the most comprehensive and complex structure in this study. It was accepted that there are three pieces to focus on. The first argumentation block is comprised of AS-1, AS-2, AS-3, AS-4, AS-9, AS-10, and AS-11, and it was classified under the reservoir-structure. In more detail, the reservoir-structure coded global argumentation block involves two self-contained parts. AS-1 and AS-2 constituted the first part of the reservoir-structure which reached to the intermediate target conclusion in AS-3. After target conclusion which holds all ideas coming from the first part of the argumentation and leads them to move to the second part, the second part of the argumentation which involves AS-4, AS-9, AS-10, and AS-11 started. In addition, the reasoning backward was observed in AS-4.

The second argumentation block involves one argumentation stream only which was represented as AS-5 and coded as the line-structure. Since the argumentation in AS-5 flowed like a line through the conclusion without spreading to other argumentation streams, it was classified under the line-structure.

The last argumentation block which involves AS-6, AS-7, and AS-8 was entitled as the branching-structure, specifically the single-rooted branching-structure. The components guidance and data at the beginning were accepted as the root, and then three argumentation streams which are AS-6, AS-7, and AS-8 branched off from this root. Besides, the mentioned parallel argumentation streams end with multiple conclusions. By considering these points, the last argumentation block was classified as the branching-structure. To sum up, as a combination of these three types of mono structures, the global argumentation structure of GG in Activity 4 was named as the line-reservoir-branching-structure under the heading of the hybrid structures.

It can be seen that there are two dotted lines which represent the backward reasoning between separate argumentations blocks. Specifically, there is a dotted line from AS-5 of the line-structure to AS-3 of the reservoir-structure, and there is another dotted line from AS-6 of the branching-structure to AS-4 of the reservoir-structure. The mentioned dotted lines were not considered as an obstacle for declaring these three argumentation blocks as separate. In other words, although there are the dotted lines representing the backward reasoning between argumentation blocks coded under different types of mono structures, the argumentation blocks were accepted as separate.

After explaining how the global argumentation structure in Figure 4.10 named as the line-reservoir-branching-structure, a more detailed explanation about the enumerated argumentation streams was documented below. First of all, for the participants using GeoGebra, one GeoGebra file in which ‘circle through three points’ and ‘circumcircular arc’ were removed from the toolbar was given in Activity 4. That is, only one GeoGebra file was used by GG during the whole argumentation process presented in Figure 4.10. In AS-1, GG discussed about what the givens are, what is asked in the activity, and which tools are restricted in the GeoGebra file, determined the sides where the points X, Y, and Z were aimed to locate, visualized the intended three circles, and mentioned about the connection of these circles. Moreover, they stated that in the case that the points X, Y, and Z are random, it might be impossible to construct three circles in AS-1, but they managed to comprehend the activity after a discussion process. At the end of the AS-1, they concluded that they should construct three circles in the first phase and then search for the connections/relationships among these circles. This statement was represented via the conclusion/data component in AS-1 which became the starting point of the following argumentation streams, which are AS-2 and AS-3.

AS-2 covers the flow of the ideas offered for the construction of three circles. The ideas presented in AS-2 are the ones that were stated consecutively and refuted by means of a rebuttal or given up at the end of studying about it via GeoGebra quickly or even without working on it. Such methods which did not reach an absolute conclusion were presented within the same argumentation stream, namely, AS-2. As seen in Figure 4.10, AS-2 covers five argumentation steps, each of which is an idea for the construction asked in Activity 4 and will be explained in the findings of the third research question later. The last idea stemmed from the same root given in AS-1 was presented separately in AS-3 since it covers a conclusion at the end. In more detail, in AS-3, GG noticed that the construction of a circle passing through the points, A, X, and Z corresponds to the construction of the circumcircle of ΔAXZ , which is one of the constructions they were familiar with from the previous cognitive unity based activities. In this manner, GG formed three triangles which are ΔAXZ , ΔBYX , and ΔCZY so as to find their circumcircles. Firstly, they looked for a tool which can be used directly to find the circumcenter of triangles. Since they could not find such a tool, they attempted to remember how they constructed the circumcircle in Activity 1 (See Figure 4.82). After the conclusion in AS-3, one of the participants of GG asked what the point of concurrency of these three circles refers to. This statement showed the first instance that GG noticed that three circles in this activity are concurrent at a point so that this was represented as the target conclusion in AS-3. As stated before, the intermediate target conclusion is the evidence for the starting of the second part of the reservoir-structure.

In the second part, GG discussed about the characteristics of the point of concurrency of three circles they noticed recently. In more detail, in AS-4, they checked whether this concurrency point is also the intersection of the angle bisectors of ΔABC, but found out that it is not. This was represented in the first argumentation step of AS-4. In the second argumentation step of AS-4, they attempted to show that the lines passing through the intersection of the circles which they drew while constructing the angle bisectors previously are perpendicular to the sides of ΔABC. However, they saw that these lines were not perpendicular when they measured the angles. By means of the guidance stated after rebuttals and challengers in the second argumentation step of AS-4, they concluded that these lines might not be perpendicular when the points X, Y, and Z were differentiated. Then, they linked this conclusion to the conclusion in the first step of AS-4 with the aim of providing justification for the fact that the intersection of three circles is not the concurrency point of the angle bisectors of the triangle which was represented with the dotted line with the one-way arrow in AS-4.

As stated, GG started to examine the characteristics of the point of concurrency of three circles in AS-4. Then, they made a pause for this discussion for a while and passed to some other issues. That is to say, they started to check the validity of their conjecture in AS-5 and then tried alternative approaches for the construction of three circles in AS-6, AS-7, and AS-8. Afterwards, they continued with searching about the features of the concurrency point of three circles in AS-9, AS-10, and AS-11. To see the flow of the argumentation process, the chronological order of the argumentation streams was pursued while explaining them in detail. Thus, the context of AS-5 was clarified as follows.

In AS-5, GG checked whether the conjecture they produced about the concurrency point of three circles is valid for differently located X, Y, and Z points and the different types of triangles. They checked different cases by dragging, examined the nature of circles and their intersection point under dragging, and then concluded that three circles are concurrent at a point. Thus, this conclusion was presented as a support for the target conclusion in AS-3 and this reasoning backward was symbolized with a dotted line from AS-5 to AS-3. Besides, in the conclusion of AS-5, they expressed that the geometric figure they drew can be accepted a construction.

Afterwards, along with the branching-structure, they started to look for alternative construction approaches for three circles. After guidance and data in the root of the branching-structure which cover the idea of finding alternative construction approaches, three argumentation streams which are AS-6, AS-7, and AS-8 branched. In more detail, in AS-6, GG wondered whether they could derive out an approach for the construction of three circles by constructing the circumcircle of the given ΔABC at first. Although they could not deduce a construction approach from this attempt, they noticed an issue regarding the second argumentation step of AS-4. In AS-4, they worked about whether the lines passing through the intersection of the circles they drew are perpendicular to the sides of ΔABC. They concluded that the circles should be equal to supply the mentioned perpendicularity, and this was schematized in the conclusion of AS-6. Therefore, this relevance was indicated by a dotted line from the conclusion of AS-6 to the corresponding component of AS-4.

In AS-7, GG tried to look from an inverse perspective with the intent of finding an approach for construction. In more detail, they thought about finding the point of concurrency of the angle bisectors of ΔABC, drawing the perpendicular lines from this point to the sides of ΔABC, and then naming the intersection points of the perpendicular lines and the sides as the points X, Y, and Z. However, one participant of GG immediately proposed a rebuttal by evoking about the randomness of these points. Thus, as the conclusion of AS-7, they expressed that it seems not possible to find a working method from this idea.

Similar to AS-2, AS-8 also involves a list of ideas for the construction of three circles which will be explained later in detail. Moreover, as can be seen from Figure 4.10, these ideas were not able to reach an absolute conclusion since they were generally given up at the beginning. After searching about alternative construction approaches for three circles intended in the activity for a while along the argumentation streams AS-6, AS-7, and AS-8, they gave up the attempts regarding the construction approaches. Thus, the branching-structure was concluded in the scope of the global argumentation structure in Figure 4.10. Then, they went back to search for the characteristics of the concurrency point of three circles which located in the second part of the reservoir-structure.

As mentioned, the second part of the reservoir-structure involves the discussion of GG about the characteristics of the concurrency point of three circles. In AS-9, based on the instant geometric figure on GeoGebra screen, GG stated an incorrect conclusion. They declared that they could show that tangents of a circle which meet at the same point are equal in length by this construction. However, this idea is not working for all circles. After that, GG passed another issue which led them another target conclusion which was represented in AS-11. In AS-11, they drew a new triangle by using the centers of three circles as the vertices of the triangle. Then, they concluded that this new triangle is similar to ΔABC, and this was displayed as target conclusion at the end of AS-11 (See Figure 4.82). As it can be seen in Figure 4.10, AS-10 is rooted from the data of AS-11. In AS-10, they looked for a relation between the intersection point of three circles and the new smaller triangle. They tried whether the point of intersection of three circles is the intersection of the perpendicular bisectors, the medians, or the angle bisectors of the smaller triangle. However, they could not find a direct relationship which was represented through the conclusions in AS-10.

To sum up, the first part of the reservoir-structure covers the discussion about possible construction approaches. After producing the first target conclusion which states that three circles are concurrent, the second part of the reservoir-structure which involves the discussion about the characteristics of this point started. Secondly, in the line-structure, the issue of whether there is such an intersection point for differently located X, Y, and Z points and for different types of triangles was discussed. Finally, the branching-structure covers the attempts to construct three circles by means of different approaches. As seen, there is not an ordered system among the argumentation streams involved in these three types of mono structures. Moreover, the global argumentation structure of GG in Activity 4 covers thirteen ideas regarding the construction, but only one of them was proper for the construction. In addition to these ideas, it was seen that they declared two statements, which were coded as target conclusions.

Up to this point, the findings related to the content and the types of global argumentation structures emerged in this study were presented. Now, the components situated in these global argumentation structures will be touched upon.

4.2.3. Components of the Global Argumentation Structures

As mentioned before, the global argumentation structures emerged in the study contain eleven components which are data, warrant, conclusion, backing, rebuttal, qualifier, conclusion/data, target conclusion, guidance, challenger, and objection. These components, which were explained theoretically in the preceding chapter, were instantiated by referencing to the dialogues from the argumentations of groups in this section. Since there is not an argumentation stream involving all components, different argumentation streams were decided to use to exemplify. Thus, three argumentation streams from eight global argumentation structures formed in this study were selected to provide examples for each component. The argumentation streams selected to exemplify the components were presented in the following table.

Table 4. 3

Selected argumentation streams to exemplify components

Argumentation streams	**Components exemplified**
AS-6 in the global argumentation structure of CSG in Activity 4 (See Figures 4.11 and 4.12)	data, warrant, rebuttal, conclusion/data, challenger, qualifier, and target conclusion
AS-1 in the global argumentation structure of GG in Activity 1 (See Figures 4.13 and 4.14)	backing, objection, and conclusion
AS-3 in the global argumentation structure of GG in Activity 4 (See Figures 4.15 and 4.16)	guidance

Table 4.3 summarized which argumentation stream was selected to display examples for which components. AS-6 in the global argumentation structure of CSG in Activity 4 (the branching-structure) covers seven components which are data, warrant, rebuttal, conclusion/data, challenger, qualifier, and target conclusion and they appeared in this order in the stream. Then, the remaining four components were illustrated by means of two more argumentation streams. The components backing, objection, and conclusion were instantiated via AS-1 in the global argumentation structure of GG in Activity 1 (the branching-structure). Since the mentioned two argumentation streams do not involve the guidance component, it was exemplified from AS-3 in the global argumentation structure of GG in Activity 4 (the line-reservoir-branching-structure). Hereby, it was also aimed to instantiate the flow of the argumentation streams that emerged in the study.

First of all, to depict the majority of components, the most comprehensive argumentation stream among the mentioned ones which is AS-6 in the global argumentation structure of CSG during Activity 4 was examined. Before going further about the content of AS-6, it might be better to show the schematic representation of AS-6 in the global argumentation structure. Since only AS-6 was focused at this point, the schematic representation of it was cropped out of the whole structure and presented as follows. The overall argumentation structure where AS-6 belongs can be seen in Appendix E.

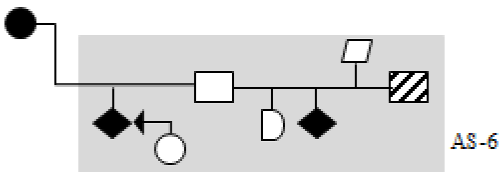


Figure 4. 11. The schematic representation of AS-6 in the global argumentation structure of CSG in Activity 4

The dialogues given as examples in this section were presented in such a way that the explanations regarding participants' unclear expressions and also some extra descriptions to make the dialogues clearer were inserted in the parentheses as italics.

Moreover, triple dots placed between the lines mean that some other conversation took place at that moment, but they were not related to the focused argumentation stream. These conversations were not included in the given excerpts, but they were indicated with the presence of triple dots. The first capital letters on the left side of the dialogues refer to the abbreviations of the pseudonyms of the participants in the groups. The other capital letters which were given in the parentheses after the abbreviations of pseudonyms refer to the components of argumentation that the following lines were coded. The abbreviations of these components were kept the same as it was mentioned in the methodology chapter. For example, (D) refers to the data component, (W) refers to the warrant component, and (R) refers to the rebuttal component. In this respect, the conversation of CSG represented in AS-6 in Activity 4 was as noted below.

F (D)	These are three circles that we drew. I draw a triangle from the centers I obtained from them *(three circles).*
...	
B (C/D)	Does the one emerged at the outside look like a right triangle, doesn't it? *(She noticed that there is a similarity between the right triangle at the beginning and the triangle they drew by accepting centers as the vertices)*
F (W1)	Yes, it is like something of the same triangle.
B (W1)	Like miniature.
F (W1)	Like its symmetric, the version of its direction was changed. We can also find its circumcircle.
...	
F (R)	Hmm, it does not happen like this *(She was stating that being symmetric is not a property of triangles).* Its direction was changed only.
B (R)	I think, it was getting smaller and the direction changed.
...	
F (C/D)	The thing that we draw inside *(the new triangle)* is similar to this *(the first triangle, that is* ΔABC *),* isn't it?
...	
I (CH)	Well, what happens when the points X, Y, and Z change?
B (TC)	They *always* looked like similar.
B (Q)	*Always*
B (W2)	Here, it is like a right triangle, this is too (*She was showing different types of triangles and differently placed X, Y, and Z points)*
B (TC)	Actually this one is *like* similar to this.
B (Q)	*Like*
…	
B (Q)	I think,
B (TC)	Like similar
...	
I	How do you describe your connection now?

B (TC) Here we will say that the point X, Y, and Z on the triangle ABC, well. The centers of the circles passing through the points AXZ, BYX, and CZY are similar to the triangle ABC.

F (TC) The triangle formed from the centers is similar to the triangle ABC.

In the conversation given above, it can be seen that participants of CSG were in a period that they were searching for a possible relationship among three circles. The endeavor of Filiz (F) in terms of examining the characteristics of the geometric figures they performed resulted in the new data in the argumentation structure. Filiz declared that she could draw another triangle by accepting the centers of three circles as the vertices of the triangle. This statement was coded as data and represented with a black-filled circle in argumentation stream as can be observed in Figure 4.11. Although there are two argumentation streams originated from this data which are AS-5 and AS-6, the focus at this point is on AS-6. Thus, the data was displayed outside of AS-6 in the schematic representation in Figure 4.11 due to the existence of AS-5 located above AS-6. This data inspired Bahar (B), and she noticed that the new triangle also looks like a right triangle as well as the triangle they started to work on at first. They continued to work on this issue and brought some justifications to the surface. They expressed some warrants like as follows; the new triangle is a kind of miniature of the first one, the new one is like symmetric of the first one, and the new one is a version of the first one in which its status or direction was changed. These sentences were labeled as warrants and represented with black-filled kite since they functioned as bridges between data and conclusion in AS-6. In the meantime, they expressed issues against some parts of their warrants. In more detail, they attempted to defeat the notion of being symmetric they said earlier. Such expressions were coded as rebuttal which was stated against warrant. As displayed in Figure 4.11, the schematic representation of rebuttal is a white-filled circle accompanied by an arrow. Based on warrant and rebuttal, CSG concluded that the new triangle and the first triangle are similar. This statement was coded as a conclusion/data since the related argumentation continued after this sentence and the idea of similarity performed as data of the following part of the argumentation stream. Moreover, the component data/conclusion was represented schematically with a white-filled rectangle as given in Figure 4.11.

Afterwards, a challenger was put forward by the instructor; “what happens when the points X, Y, and Z change?”. Since this sentence caused the group to hesitate and search on this issue, it was coded as a challenger and represented with a white-filled semicircle in the argumentation stream. Thus, CSG started to query about what they have found recently. Bahar showed the cases for differently placed X, Y, and Z points through different types of triangles. These actions were also coded as the second warrant since she listed the occasions related to the conclusion they produced and the challenger. After these justifications, CSG reached to target conclusion with the presence of a qualifier. They declared the words “like” and “I think” which can be coded as qualifiers while expressing their final conjecture which also corresponds to target conclusion in AS-6. The qualifier was represented with a white-filled parallelogram, and target conclusion was represented with a stripe-filled rectangle. As target conclusion, CSG came up with the conjecture that there is a similarity between the initial triangle and the triangle they formed from the centers of three circles (See Figure 4.43 and Appendix F).

The correspondence of the given dialogue for each component in the argumentation stream was summed up in the following figure. That is to say, Figure 4.12 involves the sentences of CSG during Activity 4 in conjunction with the components in AS-6.

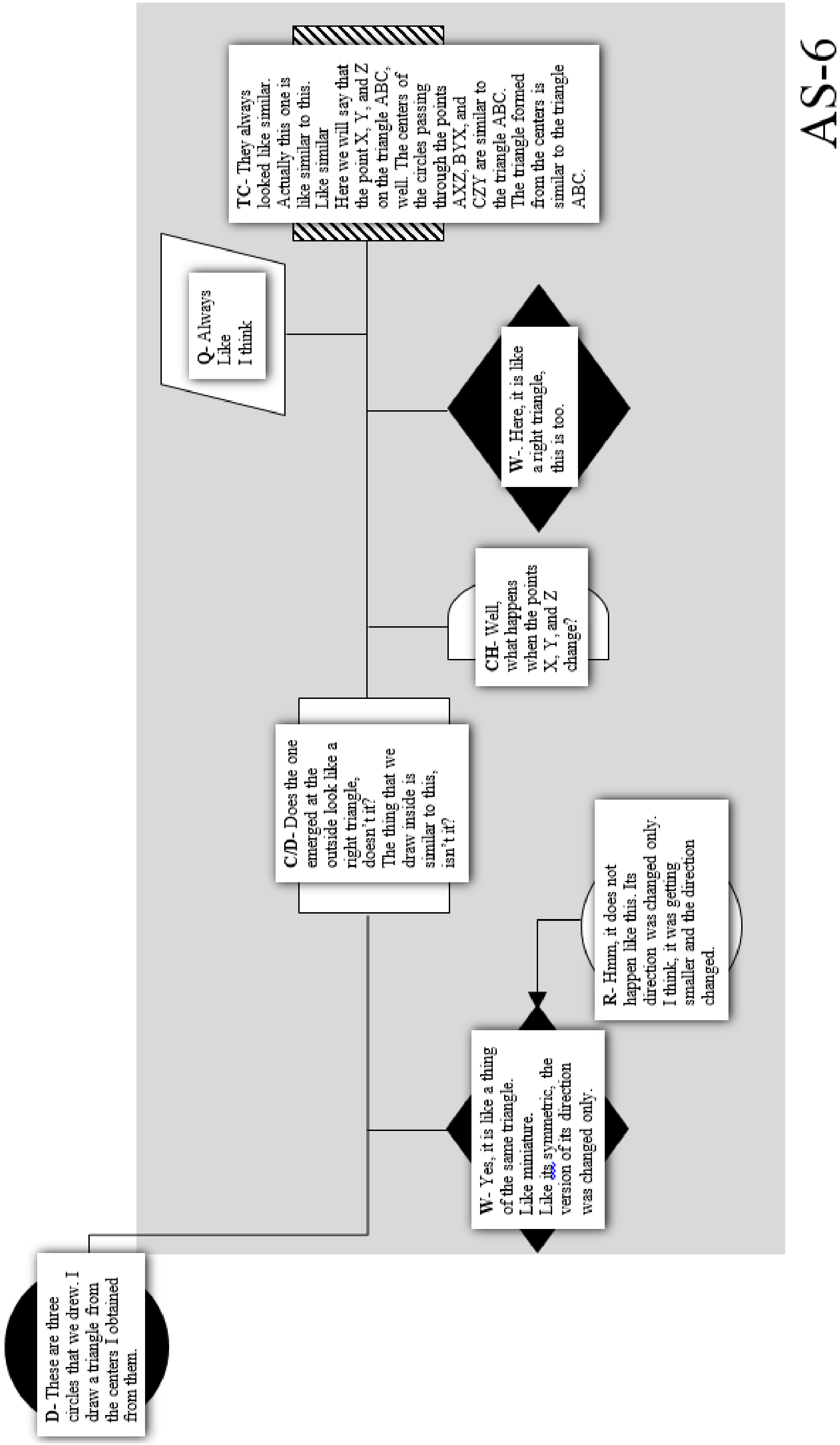
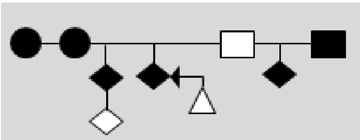


Figure 4. 12. Statements of CSG during Activity 4 in conjunction with the components of AS-6

The second argumentation stream selected to exemplify the components backing, objection, and conclusion is AS-1 which was located in the global argumentation structure of GG in Activity 1. Similar to the argumentation stream in the previous example, the schematic representation of the mentioned AS-1 was given in Figure 4.13 before explaining the details of the conversation. Moreover, AS-1 was connected to some other argumentation streams in the structure (See Appendix E), but it was cut out of the others since the focus is on some particular components of AS-1 at this point.



AS-1

Figure 4. 13. The schematic representation of AS-1 in the global argumentation structure of GG in Activity 1

Although the main purpose is to give instances for the mentioned three components, dialogues regarding some extra components in AS-1 were intentionally included in the following excerpt. More specifically, since backing was provided to support the first warrant in AS-1, the first warrant was presented in the dialogue. In a similar vein, since objection was stated against the second warrant, it was also included. In order not to descend into particulars about the remaining components, the conversation regarding conclusion was directly presented, and the intermediate two components which are a conclusion/data and another warrant which can be seen in Figure 4.13 were omitted. Besides, as presented in Figure 4.13, the conversation related to the first data was not given below, but the statements related to the second data were presented so as to make the subsequent warrants interpretable.

G (D) What if we use the centroid?

B (D) There is a feature about the ratio 2k, k.

G (D) Exactly, it was about the centroid.

B This equals to a too. Then, these three equal to a *(She mentioned the equality of the line segments she drew as radii through the vertices of $\triangle ABC$)*

Z (W1) Now, if we draw the lines passing through the midpoints of these *(the sides of the triangle)*, does the intersection of these three *(three medians)* give the centroid?
G Tell me again.
Z (W1) We found a point by intersecting this and this *(She was moving her hand like drawing two medians. Therefore, she referred to the intersection of two medians as saying point)*.
We combined this with this *(She was pointing the vertex A and the midpoint of $\overline{BC}$. That is, she referred to one median).*
Similarly, I think, the midpoint of this with this gives us the center *(She was pointing the vertex B and the midpoint of $\overline{AC}$).*
B I do not know it.
Z (W1) Because this is the median, as we said about the ratio a and 2a.
G (B) Hmm, we already know that the point of concurrency of the medians is the centroid.
Z (W2) Now, to find the midpoint of this *(the side of the triangle)*, here is the tool for drawing the line passing through the midpoint.
G Okay.
Z (W2) We will find the midpoint from this.
B (W2) That is, is it the perpendicular bisector, oops the midpoint?
Z (W2) We need to find the midpoint. It could be either of them.
...
Z (W2) Similarly, if we accept it as an example, then we combine the vertices. For example, we name them.
...
B (W2) Now, you will combine these two points *(the vertex A and the midpoint of $\overline{BC}$)*
Z (W2) It will be the median. For example, draw a line *(a line passing from the vertex A and the midpoint of $\overline{BC}$)*
B (O) I wonder if you draw a line segment *(instead of drawing line).*
...
Z (C) But, it did not pass *(The circle did not pass through all vertices)*
B (C) But, one minute. Why did not it pass from these? *(the vertices of the triangle)*. But I selected the thing. Since these are not equal. We accepted them as equal and labeled as the radius *(She referred to the fact that the distances between the centroid and the vertices are not equal)*
G (C) Exactly, $|AG|$ and $|BG|$ are not equal.

According to the conversation given above, the participants of GG were trying to find an approach for the construction of the centroid (See Figures 4.45 and 4.46). Since Activity 1 asked them to construct the circle passing through the vertices of $\triangle ABC$, they aimed to find the center of this circle as the first step. However, they considered that the center of the circle could be reached via finding the centroid. Thus, all their statements regarding finding the centroid and the ratio 2k:k which can be seen in the sentences at the beginning of the dialogue were coded as the data. Then, Zuhal (Z) asserted that the point of concurrency of the lines passing from the midpoints of the sides and the vertices of ΔABC might give them the centroid. After explaining her idea for a while, she stated the word median explicitly. These statements of Zuhal were coded as the first warrant in AS-1. After the word median, Güler (G) agreed with this idea and supported the warrant of Zuhal. Güler declared that they already know that the point of concurrency of the medians is the centroid. This sentence was coded as a backing and represented with a white-filled kite as provided in Figure 4.13. Afterwards, Zuhal and Berna (B) focused on the construction of the median by using the tools of GeoGebra. As the first step, they found the midpoint of the sides and then drew lines between the vertices and the midpoints of the sides. In the meantime, Berna warned others about drawing line segments instead of the lines while constructing the medians. This statement was coded as an objection component since she interfered in the construction process without explaining the reasoning of her expression. Moreover, others in GG hesitated for a while during the construction due to this objection but then continued to construct lines. Figure 4.13 shows that the objection component was represented with a white-filled triangle. After this objection, the discussion of the group continued in such a way that there were expressions coded as a conclusion/data and the third warrant. Finally, the issue in this argumentation stream ended up with a conclusion. As seen in the last part of the dialogue, GG observed that the circle they drew by accepting the centroid as the center and the distance between the centroid and the vertex A as the radius did not pass through other vertices B and C in ΔABC. Then, they noticed that they accepted the radius of circle incorrectly since the distances between the centroid and the vertices are not equal. The sentences regarding this result were coded as conclusion in AS-1 and symbolized with a black-filled rectangle schematically, as given in Figure 4.13. Finally, the detailed version of AS-1 of the global argumentation structure of GG in Activity 1 in conjunction with the sentences given in the dialogue was presented in Figure 4.14 as follows.

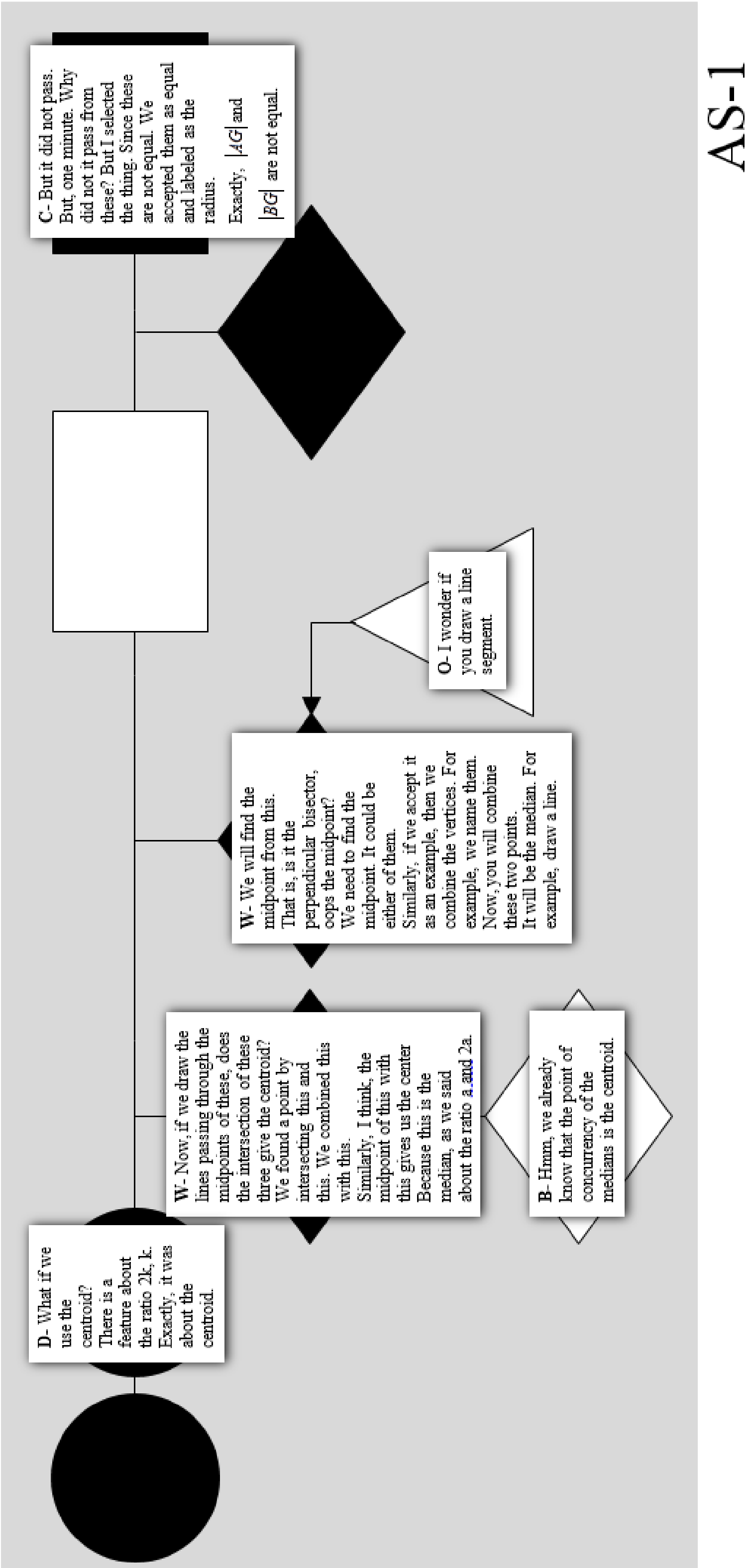


Figure 4. 14. Statements of GG during Activity 1 in conjunction with the components of AS-1

The third argumentation stream presented in order to give an example for guidance is AS-3 from the global argumentation structure of GG formed through Activity 4. The schematic representation of only AS-3 was presented in Figure 4.15. That is, AS-3 was cut out of the global argumentation structure of GG in Activity 4, and any connection of AS-3 was not located in Figure 4.15. The global argumentation structure in which AS-3 was involved can be seen from Appendix E.

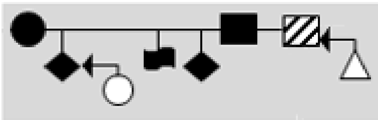


Figure 4. 15. The schematic representation of AS-3 in the global argumentation structure of GG in Activity 4

As illustrated in Figure 4.15, there are three more components which are data, warrant, and rebuttal until guidance in AS-3. The conversation of GG related to the previous components of guidance was also presented below. However, the components after guidance were not taken into consideration at this point.

G (D)	There is a triangle here *(She was pointing ΔAXZ).* If we draw the circumcircle of that triangle.
Z	Hmm
B	Let's look at this *(the tools of GeoGebra).* Is there a tool for this?
G (D)	There is ΔAXZ , let's try to construct the circumcircle of this triangle. *(She noticed that they could form a triangle by using the points A, X, and Z and the circumcircle of ΔAXZ is one of three circles aimed to construct in Activity 4)*
Z (W)	I think, there is a method that we can find the center of the triangle.
G	Where?
Z	In this side, this side *(She was pointing the left side of the toolbar of GeoGebra)*
G (W)	Is this one? The midpoint or center
Z	Try with this.
G	What will I do? Will I click here? *(She referred to clicking the tool)*
Z (W)	It *(the tool)* says two points or a line.
B (W)	I suppose, we will select this and this *(two vertices of the triangle)*
Z (W)	If we select the triangle, it would find the center. I think, we will select the triangle from the vertices.
B (R)	But it goes away after two times, the tool does not let me select the third vertex. *(She showed others that the tool the midpoint or center did not work)*

I (G) I think, it *(the tool)* is accepting the process differently. You may think about the construction of the circumcircle that we did in the previous weeks.

When the excerpt was read, it can be seen that GG was trying an approach for the construction of the circles asked in Activity 4. As seen, Güler (G) noticed that they could draw a triangle from the points A, X, and Z. She also stated that they could construct one of the intended circles by constructing the circumcircle of this triangle. These sentences constituted the data component of AS-3. Based on this data, they started to examine the toolbar of GeoGebra to find a tool to construct the circumcenter. They attempted to use the tool 'midpoint or center' but they could not manage how to use it. The entire process about finding and using the tool was coded as warrant. In the meantime, a rebuttal against warrant was presented by Berna (B) since she could not select all vertices of the triangle by using the mentioned tool. Therefore, they ended up with that the tool did not serve their purpose. Since the instructor observed that they were having difficulty in both using the tool and finding the circumcenter, she was involved in their discussion. The instructor mentioned that the tool might be functioning in a different manner since the center of the triangle that the mentioned tool can supply is the centroid and not the circumcenter. Then, the instructor led them to think about the approach they used while constructing the circumcircle of a given triangle in one of the previous activities. This involvement of the instructor was coded as guidance and it was represented with a black-filled wavy figure schematically, as presented in Figure 4.15. Based on this guidance, GG tried to remember the approach they used and presented some other warrants for finding the circumcenter of ΔAXZ .

An extensive version of the schematic representation of AS-3 by indicating the related statements of the participants of GG on it was given in Figure 4.16 as follows.

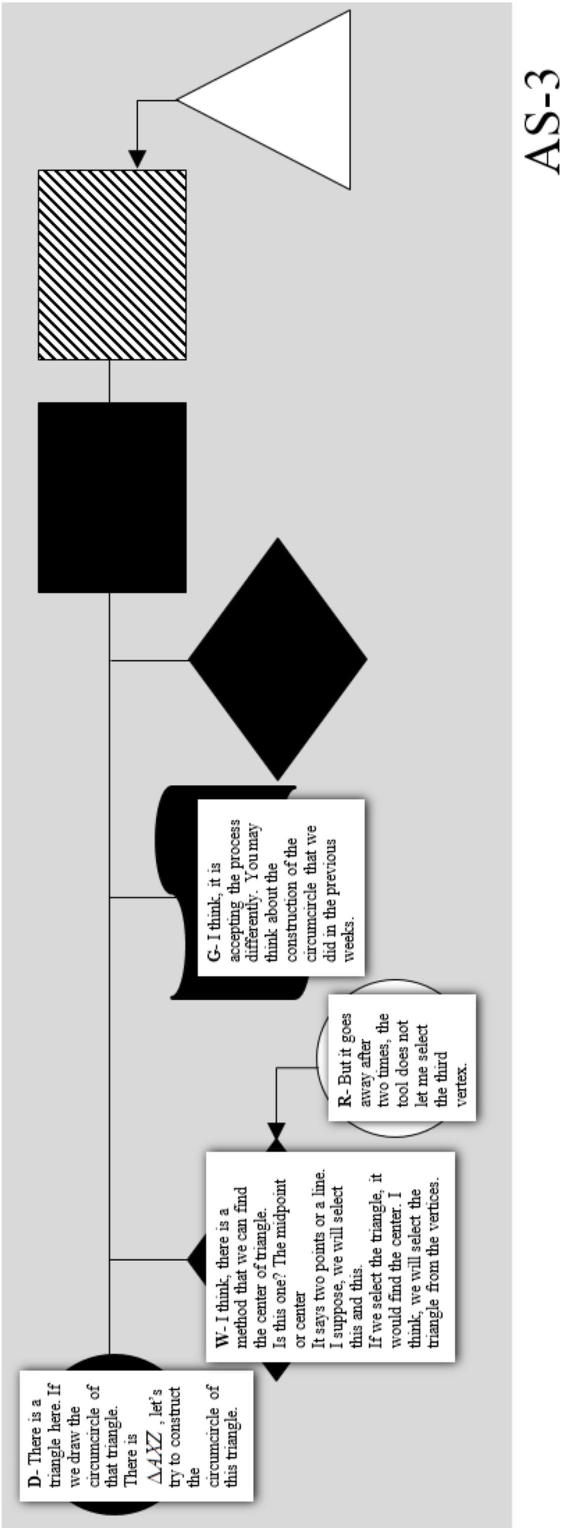


Figure 4. 16. Statements of GG during Activity 4 in conjunction with the components of AS-3

In the next section, the functions of rebuttals situated in the global argumentation structures were investigated thoroughly.

4.2.4. Functions of Rebuttals

To be able to answer the last sub-question of the second research question which aims to investigate the functions of rebuttals situated in the global argumentation structures of prospective middle school mathematics teachers in the conjecture production process in the cognitive unity based activities, all rebuttals were examined thoroughly. Since the addressing process of the second research question covers the preparation of the global argumentation structures as the first step, the section related to rebuttals was handled as the last issue. In the schematic representation of argumentation, rebuttals were represented with white-filled circles accompanied by the arrows. As originated from the study Verheij (2005), the arrows were used to point out the cases which rebuttals were stated against clearly.

During the analysis, all rebuttal components in eight global argumentation structures arranged from four cognitive unity based activities were marked and classified based on the cases that they were declared against. As stated before, rebuttal was basically defined in this study as the conditions under which conclusion would not hold or warrants would not be valid. By considering these underlined features of rebuttal, it can be inferred that there is a defeat or a refutation purpose in the nature of rebuttal. By considering this stance of rebuttals, the functions of them were examined. For example, if a rebuttal was expressed against a warrant, the function of such a rebuttal was described as to refute warrant. In this manner, it can be said that the functions of rebuttals were investigated on the basis of the adaptation of the study of Verheij (2005). In more detail, the functions of rebuttals that appeared in this study were summed up in Table 4.4 given below. Moreover, an outline about from which group and activity the examples for the functions of rebuttals were taken was also presented in the following table.

Table 4. 4

The functions of rebuttals in the global argumentation structures

Functions of rebuttals		Total number of rebuttals	Examples
F1	To refute warrant (W)	32 rebuttals (CSG 12- GG 20)	Activity 2, CSG, AS-3 Activity 4, GG, AS-3
F2	To refute the connection between data and conclusion (D→C)	10 rebuttals (CSG 4- GG 6)	Activity 1, GG, AS-5
F3	To refute conclusion/data (C/D)	9 rebuttals (CSG 4- GG 5)	Activity 1, CSG, AS-4
F4	To refute data (D)	8 rebuttals (CSG 2- GG 6)	Activity 3, GG, AS-2
F5	To refute backing (B)	5 rebuttals (CSG 2- GG 3)	Activity 2, GG, AS-2
F6	To refute conclusion (C)	5 rebuttals (CSG 4- GG 1)	Activity 3, CSG, AS-1
F7	To refute challenger (CH)	2 rebuttals (CSG none- GG 2)	Activity 4, GG, AS-8
F8	To refute target conclusion (TC)	1 rebuttal (CSG 1- GG none)	Activity 4, CSG, AS-4

As can be seen in Table 4.4, the functions of rebuttals were listed as the most frequent one at the top and descending to the least frequent one. Therefore, based on the number of rebuttals emerged, the first function in Table 4.4 became to refute warrant (W) with the occurrence of 32 times. It was noticed that there is a quite decline in the frequency after the first function. Specifically, the second function of rebuttals was to refute the connection between data and conclusion (D→C), and it was seen 10 times. The third function which was to refute conclusion/data (C/D) appeared 9 times in this study. Then, the fourth function of rebuttals which was to refute data (D) was presented in Table 4.4 with the emergence of 8 times. Since the subsequent two functions of rebuttals which are to refute backing (B) and to refute conclusion (C) have the equal frequency, they were presented in Table 4.4 with alphabetical order. Thus, the fifth function of rebuttals was given as the refutation of backing (B), and the rebuttals covering this function were seen 5 times. To refute conclusion (C) was displayed as the sixth function in Table 4.4 since such rebuttals also appeared 5 times in the global argumentation structures. Until that point, the mentioned six functions of rebuttals were seen in the global argumentation structures of both CSG and GG. However, the seventh function which was the refutation of challenger (CH) was seen twice in the global argumentation structure of GG only, and the last function which was the refutation of target conclusion (TC) was observed once in the global argumentation structure of CSG. To present an in-depth description of the nature of the stated eight functions of rebuttals, examples were given for each function. As indicated in the rightmost column of Table 4.4, the examples regarding the functions of rebuttals were intentionally selected from different activities and groups as far as possible. The examples of rebuttals presented in Table 4.4 were explained in detail as follows.

Within four global argumentation structures of CSG, rebuttal component against warrant was observed 12 times. On the other hand, four global argumentation structures of GG involve 20 rebuttal components which aimed to refute warrant. Thus, in total, 32 rebuttals with the function of refutation of the warrant were seen in eight global argumentation structures emerged in the study. Moreover, such rebuttals were seen in the argumentation processes of both groups during all four cognitive unity based activities. The following dialogue taken from the argumentation of CSG while dealing with the construction section of Activity 2 serves as an example for the mentioned function of rebuttal.

F (D) I think, it works for this one. Can we benefit from this? *(She was pointing on* $\overline{EF}$ *of* ΔDEF*)*
Draw the third one. *(She referred to drawing the third perpendicular bisector of* ΔDEF *which is the one of* $\overline{EF}$*)*

B What will I draw?

F (D) Draw from $\overline{EF}$ for D.

…

B (W) Now, I will draw a line when I combine these two points.
(She was offering to draw the perpendicular bisector of $\overline{EF}$ *by combining two points. One of these points is the intersection of the previously constructed two perpendicular bisectors and other point is the midpoint of* $\overline{EF}$*)*
I will extend this perpendicular line to this point *(the vertex D).*
(She was stating that she would extend this line through the vertex D)

F (W) Yes, it *(the extended line)* will be perpendicular *(to* $\overline{EF}$*).*

…

B (R) Look, you say that they become collinear when you combine these points. However, they are not collinear in my drawing.
(She noticed that three points used at this case, which are the vertex D, the intersection point of the perpendicular bisectors, and the midpoint of $\overline{EF}$*, are not collinear)*

Regarding the excerpt given above, two participants of CSG who are Filiz (F) and Bahar (B) were active. This dialogue was represented as a part of AS-3 in the global argumentation structure of CSG in Activity 2 which was described in a more detailed way in the previous section. In short, CSG worked on the construction of the altitudes of the given three triangles and also the orthocenters of them in the case of existence. In AS-2, which is the previous argumentation steam of AS-3, CSG drew the perpendicular bisectors of two sides of ΔDEF. Based on the visual of ΔDEF at that instant, they focused on the third perpendicular bisector which is the perpendicular bisector of $\overline{EF}$. As presented in the dialogue above, Filiz offered to draw and examine whether the third one may work for their purpose in this activity, which is to construct the altitude of $\overline{EF}$. This idea was coded as data of AS-3. Then, Bahar constructed the perpendicular bisector of $\overline{EF}$ and stated that she would extend this line through the vertex D to obtain confirmation from other participants of CSG. Filiz agreed that the mentioned line would be perpendicular to $\overline{EF}$. The construction process of the third perpendicular bisector and the comments based on this were categorized as warrant. However, Bahar drew the mentioned line and saw that three points, which are the intersection point of the perpendicular bisectors, the midpoint of $\overline{EF}$, and the vertex D, are not collinear. She informed others in the group about the fact that three points are not collinear in her drawing. This statement of Bahar was categorized as a rebuttal against warrant which was previously given by Filiz and Bahar. That is, warrant stated for all drawings in this construction attempt was refuted by a rebuttal given by Bahar. To see the whole issue disused in this example, Figure 4.26, which was presented in the following section with respect to the findings of approaches offered for geometric construction, can be examined. Moreover, in the following examples of rebuttals, the related figures if they existed in the following sections were referred in the parentheses to make the examples clearer.

Due to the highest frequency of the function of refuting warrant in the study, another example was also presented. The second example for the refutation of warrant is from AS-3 in the global argumentation structure of GG in Activity 4.

G (D) There is a triangle here *(She was pointing ΔAXZ)*. If we draw the circumcircle of that triangle.
Z Hmm
B Let's look at this *(the tools of GeoGebra)*. Is there a tool for this?
G (D) There is ΔAXZ, let's try to construct the circumcircle of this triangle. *(She noticed that they could form a triangle by using the points A, X, and Z and the circumcircle of ΔAXZ is one of three circles aimed to construct in Activity 4)*
Z (W) I think, there is a method that we can find the center of the triangle.
G Where?
Z In this side, this side *(She was pointing the left side of the toolbar of GeoGebra)*
G (W) Is this one? The midpoint or center
Z Try with this.
G What will I do? Will I click here? *(She referred to clicking the tool)*
Z (W) It *(the tool)* says two points or a line.
B (W) I suppose, we will select this and this *(two vertices of the triangle)*
Z (W) If we select the triangle, it would find the center. I think, we will select the triangle from the vertices.
B (R) But it goes away after two times, the tool does not let me select the third vertex. *(She showed others that the tool the midpoint or center did not work)*

According to the dialogue given above, while searching for an approach to construct three circles, GG noticed that the construction of three circles asked in Activity 4 refers to the construction of the circumcircles of the formed three triangles. This idea was offered by Güler (G) and categorized as data in AS-3. Then, they started to search for a tool to be able to find the center of the triangle. They attempted to use the tool 'midpoint or center' and tried some ways to make this tool serve their purpose. The statements regarding the use of this tool were categorized as warrant since they acted as a bridge between data and conclusion. However, it was noticed that they did not pay attention to the characteristics of the center of the triangle that they can find by means of the tool 'midpoint or center'. They probably assumed that this would be the circumcenter although they did not say it out loud. In fact, the point that the mentioned tool can give them is the centroid of triangle. However, they could not even find this point by using a tool. As seen from the dialogue, Berna (B) stated that they could select only two vertices of the triangle and the tool did not give chance to select the third vertex. They assumed that this tool did not work for their purpose and passed to another trial for the construction of the circumcircles of formed three triangles. Thus, the statement of Berna was coded as a rebuttal since it refuted the first warrant of GG in this argumentation stream.

The second frequent function of rebuttals in this study was to refute the connection between data and conclusion which was symbolized with D→C. Such rebuttals were seen 10 times, four of which were seen in the global argumentation structures of CSG in Activity 1 and Activity 2 evenly. The rest of them were deduced from the discussions of GG during Activity 1, Activity 2, and Activity 4. It was the only type of function in which the rebuttal was not stated against a particular component such as data, conclusion, and backing. Besides, these rebuttals functioned to refute the flow of the argument from data to conclusion. The following conversation presents how GG came up with a rebuttal stated against the connection between data and conclusion in Activity 1.

B (D) By extending this triangle, parallel. By finding the distance between them

Z It will be tangent to the sides *(of the larger triangle)*. But, how will we calculate this?

B The distances between them

…

Z (D) Now, A will be the point of tangency, B too, C too. But, in which way will it be a point of tangency? *(They were talking about drawing a larger triangle so that the vertices of ΔABC will be the tangent points of the incircle of the larger triangle)*

…

Z (CH) I thought something, I will say something. Now, if we assume that there is a circle here, if I put a point of tangency here, will this be parallel to it? *(Firstly, she assumed that they drew the circumcircle of ΔABC. Then, she stated that they will draw the tangent to the circumcircle from the vertices of triangle. Finally, she was asking whether these lines will be parallel to the opposite sides of triangle)*

Z (W) I wondered about it. If you say that it would be, we will draw a parallel line. The tangent line passing from this *(the point A)* is parallel to that *($\overline{BC}$)*, the tangent line passing from this *(the point B)* is parallel to this *()* too. What I mean is that if we draw three lines, then the things *(the lines)* passing from the center is perpendicular *(to the sides of the triangle)*. If we find the perpendicular line to this *(the tangent line passing through A*) and then the perpendicular line of this too *(the tangent line passing through B),* then it will give us the center. After this, from that center and a point *(She referred to use of tool 'circle with center through point')*. For example, this may be a point *(the vertex A).*

…

I	I want to ask you too, what did you do?
Z	We thought about things, we could not go further from the angle bisectors.
I	Angle bisectors, yes.
Z (W)	We could not be sure, but we drew the tangent line of this. It is parallel to it *(the side of the triangle)*
I	Okay
G (W)	We drew the tangent line of this *(tangent line to the circumcircle of ΔABC at any vertex)*, we will draw a parallel line to this *(the side of the triangle)*. That is, we said that what if we draw the tangent line so that it will be parallel to this *(the side of the triangle)* since we saw that there is a circle there.
I (W)	Is it the incircle of that triangle *(the larger triangle)*?
Z	Exactly
…	
G (R)	But this circle is not touching here, there is a gap between them. *(They noticed that they could not draw the incircle properly)*
Z (R)	We drew it randomly then, we could not arrange.

The recently presented dialogue is from AS-5 in the global argumentation structure of GG in Activity 1 and involves all participants of GG and the instructor of the course. In Activity 1, the purpose is to construct the circle passing through the vertices of the given ΔABC. Before AS-5, the construction process related to the angle bisectors of ΔABC was discussed in AS-3. There are two argumentation streams branched from AS-3 and AS-5 is one of these branches. Therefore, GG discussed about an issue that they noticed while working about the angle bisectors in AS-5. More specifically, throughout AS-5, GG attended to draw a larger triangle than the given ΔABC and checked whether the incircle of the larger triangle corresponds to the circumcircle of ΔABC which is the desired construction in the activity (See Figure 4.49). The rebuttal which functions to refute the connection from data to conclusion took place comparatively at the beginning part of AS-5.

As seen from the dialogue above, Berna (B) and Zuhal (Z) were talking about drawing a larger triangle so that the vertices of ΔABC will be the tangency points of the incircle of the larger triangle and then they started to search for a way to draw the larger triangle. Meanwhile, Zuhal presented a challenger by asking others whether the tangent lines would be parallel to the sides of ΔABC. For example, they were challenged whether the tangent line passing through the point A is parallel to $\overline{BC}$. Based on this challenger, they continued with the option that the tangent line and the side are parallel. With this aim, they came up with some ideas which were coded as warrant in the dialogue. Zuhal offered to draw the perpendicular lines to the sides of the larger triangle passing from the points A, B, and C. Then, the intersection of the perpendicular lines would give them the center of the circle aimed to construct. Thus, since the center is known, they would use the tool ‘circle with center through point’ to draw the circle. While Zuhal was explaining their idea to the instructor (I), the issue became more clear about the fact that they started by assuming the idea that incircle of the larger triangle would be the circumcircle of the given ΔABC. While working on this idea, they noticed that they could not even draw the incircle of ΔABC properly which was an issue presented in AS-3 since Güler (G) warned others about the gap between the incircle and the side of the triangle. Thus, GG accepted that they could not draw the incircle correctly, and then they stopped working on this idea for a while and focused on the construction of the incircle. As seen, the noticing of Güler regarding the gap between the incircle and the side of the triangle in GeoGebra file turned out to be the refutation of the combination of the related data and warrant. That is to say, the rebuttal refuted all idea stated in AS-5 so far and the connection from data through conclusion. After this rebuttal, they focused on the proper construction of the incircle.

As seen in Table 4.4, the third common function of rebuttals in this study is to refute conclusion/data. Rebuttals with such a function were observed 9 times. Four of them came from CSG during the first three cognitive unity based activities, and the remaining five instances of this function were from the discussion of GG in the last activity. To provide an example for the refutation of conclusion/data, the related conversation from the argumentation process of CSG in Activity 1 was given below.

F (C/D) Do the angle bisector and this thing *(the median)* have the same manner Bahar? *(She mentioned the presence of the similarity between the angle bisectors and the medians)*

…

B (D) The first issue that I think was the angle bisector.

F (D) Let’s try it too. Now, I will try the angle bisector.

…

F (C) No way, its center is different *(the circumcenter is different).* Then, let’s say that we have such a thing, but we have found incircle.

F (C/D) Then, the concurrency point of the medians is the same too *(She stated that the concurrency point of the medians is the same with the concurrency point of the angle bisectors)*

F (R) These are the angle bisector and the median. However, it *(the median)* divides the sides into two equal pieces, doesn’t it? The angle bisector does

> not bisect the sides into two equal pieces. It is not the same as the previous one *(the construction of angle bisectors)*, I looked. I will write this too.

This conversation involves parts from various argumentation streams of the global argumentation structure of CSG in Activity 1. Since the aim is to present an instance for rebuttal with the function of refuting the conclusion/data from this study, to mention about the statements from the previous argumentation streams which led to the conclusion/data component was considered as necessary. Without dilating upon all process, conclusion/data component of AS-1 and both data and conclusion components in AS-2 were presented since AS-4 which covers the example aimed to give for this function of rebuttals were originated from AS-1 and affected directly from the conclusion of AS-2. The mentioned relevance can be seen in the global argumentation structure of CSG in Activity 1 which was expanded on in the findings of the line-branching-structure in the prior section (See Figure 4.9 and Appendix E).

The first sentence in the quotation above which was articulated by Filiz (F) was the last component of AS-1. It was coded as conclusion/data since it constitutes the origin of the next two argumentation streams, namely, AS-2 and AS-4. In this component, she mentioned a similarity between the angle bisectors and the medians in the given ΔABC. Then, they decided to continue with the construction of the angle bisectors since it was the first notion that they considered while searching for an approach to construct the circle passing through the vertices of ΔABC. This idea was coded as data, and the following attempt of construction was symbolized throughout AS-2. As seen in the dialogue above, the conclusion of AS-2 showed that the center of the circle is not the point of concurrency of the angle bisectors of ΔABC and CSG found out the construction of the incircle by means of the angle bisectors. By combining AS-1 and AS-2, CSG reached to the conclusion/data in AS-4. More specifically, Filiz prompted that the point of concurrency of the angle bisectors is the same with the point of concurrency of the medians in the given triangle. This statement was not only conclusion of AS-1 and AS-2 but also data of the following stream which is AS-4. Meanwhile, Filiz realized that medians divide the sides equally, but the angle bisectors do not have to divide the sides into equal pieces. Thus, the points of concurrency of them cannot be the same point (See Figures 4.20 and 4.21). This awareness about the angle bisectors and the medians refuted the previous conclusion/data which was represented at the beginning of AS-4. After that, CSG turned their direction to the construction of the medians by aiming to construct the circle passing through the vertices of the given triangle.

As the fourth function of rebuttals, Table 4.4 displays the refutation of data since it appeared 8 times in the global argumentation structures of groups. While the global argumentation structures of CSG involved it two times from Activity 2 and Activity 3, the global argumentation structures of GG contained the remaining six exemplars, one of which was from Activity 3 and five of which were from Activity 4. The following dialogue from the argumentation process of GG during Activity 3 can be given as an example of this function of rebuttal.

B (D)	We will find the centroid. Regarding the centroid, there is the ratio 2k:k.
G (D)	Try the angle bisector. It is *(the centroid)* the point of concurrency of the angle bisectors, isn't it?
Z (R)	It *(the centroid)* is the point of concurrency of the medians.
G	Hmm, is it *(the centroid)* the point of concurrency of the medians?

In the quotation above, which was represented as a part of AS-2 of the global argumentation structure of GG in Activity 3, all participants of GG were involved. They were trying to find a method for the construction of the centroid of the given ΔABC since they were asked to construct three points (the orthocenter, the circumcenter, and the centroid) in Activity 3. As seen in the dialogue, Berna (B) and Güler (G) were suggesting the data of AS-2 by mentioning about the ratio 2k:k related to the centroid and stating that the point of concurrency of the angle bisectors can give them the centroid. However, Zuhal (Z) warned others at this point by recalling the fact that the point of concurrency of the medians is the centroid of a triangle. Then, GG continued to suggest warrant to construct the medians and find the intersection point of the medians. As seen, the sentence of Zuhal was coded as a rebuttal stated against data.

The fifth function of rebuttals presented in Table 4.4 is to refute backing with the appearance of 5 times. While two rebuttals with this function were observed in the argumentation process of CSG in Activity 2, the remaining of them were seen one by one during the argumentation of GG in the last three activities. An example to the refutation of backing was selected from AS-2 in the argumentation of GG during Activity 2, and their conversation was given below.

G (W)	Then, this is the diameter *(She was pointing* $\overline{EF}$ *of* ΔDEF *which is an acute triangle)*
B (W)	We accepted this as the center *(They accepted the midpoint of one side of triangle which is* $\overline{EF}$ *as the center)*
…	
G (W)	Here is the diameter, this is the diameter (*She was pointing the sides of* ΔDEF *as diameters)*
…	
G (R)	Okay, I wonder whether this circle passes through these two points. I think, it does not. Does it pass? *(She was working on* ΔDEF *and stated that one circle constructed by accepting any side as the diameter does not give the altitudes of two adjacent sides)*
B (B)	Since the angle subtended by the diameter of a circle is 90°, these two become 90° *(She was pointing the angle between the altitudes and the sides)*
...	
B (B)	Here is 90. Since this is the orthocenter, this is also subtended by diameter. This 90 is also subtended by the diameter. It *(the circle)* has to pass these two points.
G (R)	But here is the question; does it pass through two points? Or do we have to draw another circle? In fact, we guarantee them by drawing two circles, don't we?
…	
G (R)	No matter what, that is, they were not intersecting or they were intersecting, there is no harm in drawing two circles.

As stated before, GG was given three GeoGebra files in which the tool 'perpendicular line' was restricted in Activity 2. The participants were asked to construct the altitudes and the orthocenters of the given three types of triangles. This dialogue is a part of the period that they were working on the given acute triangle ΔDEF and focusing on using the theorem "inscribed angles subtended by a diameter are right angles". Güler (G) offered to accept $\overline{EF}$ as the diameter and draw a circle. Therefore, Berna (B) pointed the midpoint of $\overline{EF}$ as the center of the circle. These statements were coded as a warrant in AS-2. During that time, Güler hesitated and expressed that the circle they drew by accepting any side as the diameter does not give the altitudes of two adjacent sides. This was coded as a rebuttal to the warrant they proposed recently since she was stating about a case in their warrant which may not be working in terms of their purpose. Berna insisted on their ideas and provided support to the warrant. Specifically, she presented the possible locations of two altitudes and the orthocenter, underlined the theorem “the angle subtended by the diameter of a circle is 90°”, and showed the angles between the altitudes and the sides. These sentences are categorized as a backing since their aim was to support the queried warrant. However, Güler was also stable in her idea. In other words, Güler kept questioning whether one circle can give two altitudes and offered to draw more circles to guarantee the construction. Based on these expressions, it can be stated that the sentences of Berna coded as backing could not convince Güler so that she continued to give suggestions which aim to refute the backing. Thus, the last sentences of Güler were coded as another rebuttal, but this one was stated against the backing. What GG reached by applying this idea can be seen in Figure 4.54.

According to Table 4.4, the sixth function of rebuttal is to refute conclusion and it occurred 5 times in the global argumentation structures. Only one of them was from the argumentation of GG in Activity 2 and four of them appeared in the global argumentation structures of CSG in Activity 3 and Activity 4 evenly. Although this function emerged 5 times in the global argumentation structures like the fifth function, they were presented in Table 4.4 in alphabetical order. The instance of this function of rebuttals was taken from the argumentation of CSG in Activity 3 as follows.

G (C/D)	Is this *(the orthocenter)* also the circumcircle? *(She was referring to the center of the circumcircle)*
B	Do you mean the orthocenter?
G (W)	We took two chords.
B (W)	By accepting two things *(two sides of triangle)* as chords, I combine the midpoints of them.
B (Q)	Most likely, as if
B (C)	the centroid and the centers of these things *(the orthocenter and the circumcenter)* will give the same point.
F (R)	It cannot be like this.
B	It will be.
F (R)	They might be but the centroid does not be *(She stated that the orthocenter and the circumcenter might be the same point but the centroid could not be the same point)* I could find the centroid somewhere here *(She was pointing a different place inside the triangle based on her drawings)*

The excerpt given above was displayed schematically in AS-1 of the global argumentation structure of CSG in Activity 3. As stated before, groups were asked to construct three points which are the orthocenter, the circumcenter, and the centroid in

Activity 3. This dialogue covers all participant of CSG and it is about the construction of all three points. Instead of directly presenting the statements coded as conclusion and rebuttal, the former conversations were given in the quotation above to make the flow of argumentation in the stream easier to understand. Regarding the work sharing of CSG in the activity, Gizem (G) was working on the construction of the orthocenter, Bahar was trying to construct the circumcenter, and Filiz (F) was focusing on the construction of the centroid. As seen at the beginning sentence of the dialogue, Gizem (G) proposed a wrong warrant and this led them a mathematically incorrect conclusion. In more detail, based on her drawings, Gizem inferred that the orthocenter and the circumcenter refer to the same point. Since this inference constituted the data of the rest of the argumentation stream, it was coded as conclusion/data. Then, both Gizem and Bahar explained their construction steps briefly as their warrants. Both of them accepted two sides of ΔABC as chords, found the perpendicular bisectors of these sides, and then pointed out the intersection point of these perpendicular bisectors. At this point, Bahar asserted that what she found is the circumcenter which is the correct inference. On the other hand, Gizem followed the same procedure and named the intersection point as the orthocenter which is not mathematically correct since the orthocenter is the point of concurrency of the altitudes of a triangle. Base on this incorrect inference, Bahar asserted that the mentioned three points are most likely the same point. This assertion was coded as a conclusion and the expression most likely as a qualifier. However, Filiz who is the one working on the construction of the centroid offered a rebuttal to this conclusion. By showing through her drawings, Filiz declared that the centroid could not be the point others found since she was about to find a different point inside triangle as the centroid (See Figures 4.37 and 4.38).

According to Table 4.4, the seventh function of rebuttals is to refute challenger and it was seen only twice through AS-4 and AS-8 of the global argumentation structure of GG in Activity 4. As an example, the one in AS-8 was presented via referencing to the related dialogue as follows.

I (G) Well, is there any other method that you find for the construction?
For example, can you construct circles by using a different method? Is there anything that you think about?

Then, maybe, this new method can lead you to something else about the relationship among three circles.

…

Z (D) In fact, we draw a circle passing through four points since three *(three circles)* are concurrent at a point. But, what will I do with this?

Z (CH) I will say something. Does a circle pass through any four points? *(She was asking whether a circle could be drawn from any four points)*

G (R) No

B (R) It passes through three, but it cannot pass through four *(She stated that a circle could be drawn from any three points, but it cannot be drawn from any four points)*

This extract shows the dialogue between the participants of GG while they were searching for a different idea for the construction of three circles. Moreover, it was represented schematically as an argumentation step in AS-8. As seen in the first part, the instructor (I) asked GG to search for any other method for the construction of three circles. This one was coded as guidance since the flow of the argument was affected. Then, they started to look for some other possible methods and ideas for construction. After talking about a few ideas for construction and giving ups, Zuhal (Z) declared that they were drawing a circle passing from four points. The fourth point is the point of concurrency of three circles. Since they used this as a starting point to find an idea for construction, it was coded as the data of the corresponding argumentation step in AS-8. Based on this idea, Zuhal put a challenger forward. She asked others whether a circle can be drawn from any four points. After thinking for a while, Güler (G) and Berna (B) came up with a rebuttal against this challenger. They stated that a circle could be drawn from any three points, but it cannot be drawn from any random four points by showing their drawings from paper, not from GeoGebra. Then, GG gave up this idea and continued with a new idea and this new idea was represented in another argumentation step of AS-8.

As stated in Table 4.4, the last function of rebuttals which is also the least frequent one in this study is to refute target conclusion. Such a rebuttal was seen once in the global argumentation structure formed from the discussion of CSG in Activity 4. The discussion of CSG regarding the mentioned rebuttal was given below.

B Is there a point that these three *(three circles)* are concurrent?

…

F (TC) Yes, there is *(She was stating that three circles are concurrent at a point)*

B (R) I could not draw and find the concurrency point of three of them.

F (Q) I suppose, mine has it *(She stated that her drawing involves the concurrency point of three circles)*
B (R) I found in this one. In my other drawing for example in this one, three of them are like this *(She was showing that the circles are not concurrent at a point in her worksheets)*
G (TC) Ee here is the concurrency point *(She was showing the concurrency point of three circles in another drawing)*
B (R) There is not such a point in which all of them are concurrent.
…
B (R) The ones that I draw are not concurrent.

The last function of rebuttals is to refute target conclusion, it was seen only once in AS-4 in the global argumentation structure of CSG in Activity 4. According to the quotation above, Bahar (B) was asking others whether they have found a particular point that three circles are concurrent. By presenting warrants from their drawings, Filiz (F) declared the presence of the point of concurrency of three circles. Similarly, Gizem (G) showed the point of concurrency of three circles in her drawings. These statements were coded as target conclusion in the global argumentation structure. Moreover, Filiz used the term “I suppose” while talking about the presence of the mentioned concurrency point. This phrase was coded as a qualifier. While Filiz and Gizem were showing the concurrency points from their worksheets, Bahar declared that there was not such a point in her drawings. These expressions of Bahar were coded as a rebuttal against target conclusion since they were suggesting a counter situation for target conclusion. In fact, what Bahar asserted was not mathematically correct and three circles constructed in Activity 4 are concurrent at a point and this point is called as Miquel point. After working for a while, Bahar noticed that she drew incorrectly and agreed with others. The geometric figure that they presented as a group at the end of Activity 4 can be seen in Figure 4.43.

To summarize, within the scope of the second research question and its sub-questions, this section explained the content and types of the global argumentation structures, the components situated in these structures, and the functions of the rebuttal component. In the next section, the findings regarding the approaches prospective middle school mathematics teachers offered for the geometric constructions asked in the cognitive unity based activities will be presented in detail.

4.3. Approaches Offered for Geometric Constructions

In line with the third research question and the sub-questions, the main topic covered in this section is the approaches that prospective middle school mathematics teachers offered to perform the geometric constructions in the cognitive unity based activities. In more detail, this issue was taken into consideration from three aspects; the details of approaches accompanied by the explanations and drawings in the worksheets and GeoGebra files, the final comments of prospective middle school mathematics teachers about the validity of their approaches, and to what extent they performed geometric constructions correctly.

After the formation and investigation of the global argumentation structures of groups within the scope of addressing the second research question, it was noticed that some components in the global argumentation structures could be used to represent the approaches groups offered for construction. In order to see the locations of the approaches in the global argumentation structures and interpret these approaches with respect to the flow of the argumentation, more comprehensive comparisons between the approaches and the global argumentation structures were conducted. Then, the correspondence of each approach in the global argumentation structure was pointed out regardless of the validity of the approach at first. In more detail, it was seen that the final ideas of groups regarding the approach discussed through the argumentation stream were generally coded with any one of the conclusion-based components which are conclusion, conclusion/data, and target conclusion. Whilst the correspondences of the majority of the approaches in the global argumentation structures were the conclusion components, the final decisions of groups about fewer approaches were plotted by virtue of the conclusion/data or the target conclusion components. To sum up, it can be stated that when groups proposed an idea for construction, they first tried this approach and then stated a conclusion regarding the applicability of it. Thus, such approaches were pointed out in conclusion or conclusion/data or target conclusion in the global argumentation structures. On the other hand, when groups offered an idea for construction in which a clear conclusion was not reached, the correspondences of such approaches in the global argumentation structures were shown by means of the data component. In this manner, what each approach corresponds in the global argumentation structures was analyzed, and then conclusion, conclusion/data, target conclusion, and data components referring to an approach were marked by means of red or blue indicators. That is, the approaches stated by groups as invalid in terms of construction asked in the activity were marked with blue indicators whereas the valid ones were marked with red indicators in the global argumentation structures.

Afterwards, all approaches pointed out in the global argumentation structures were explained concisely in the tables. The valid and invalid approaches were tabulated separately. Initially, the tables involving approaches that groups stated as invalid were explained. Then, the tables involving the ones stated by groups as valid for the aimed construction were expanded on.

As expected, the application of all approaches or ideas did not come out as a geometric construction. Accordingly, the groups declared about which approaches of them worked and which approaches did not during the argumentation process. By comparing with the explanations of groups, the approaches were also examined thoroughly. During the analysis, it was seen that the criteria for accepting a geometric figure as a construction differ depending on the tools used. More specifically, while prospective middle school mathematics teachers were using compass-straightedge, the geometric figures they presented were evaluated based upon a list involving six criteria given in Table 4.5.

Table 4. 5

Criteria for accepting a geometric figure as a construction while using compass-straightedge

Criteria	
C1	Geometric figure presented by CSG is proper to the construction asked in the activity
C2	Geometric figure was constructed by using compass-straightedge only
C3	Compass was used properly/correctly
C4	Straightedge was used properly/correctly
C5	Inferences in construction process were mathematically correct
C6	Explanations in construction process were mathematically correct

On the other hand, while prospective middle school mathematics teachers were using GeoGebra in the construction process, the geometric figures they presented were evaluated based on a diagram given in Figure 4.17. As can be seen, by following the questions throughout the given three phases, the geometric figures groups formed were examined thoroughly. This stepwise evaluation process presents whether the figure is a geometric construction or not. If it can be accepted as a construction, it also presents the types of constructions which were identified as construction type A (CTA), construction type B (CTB), and construction type C (CTC) in this study.

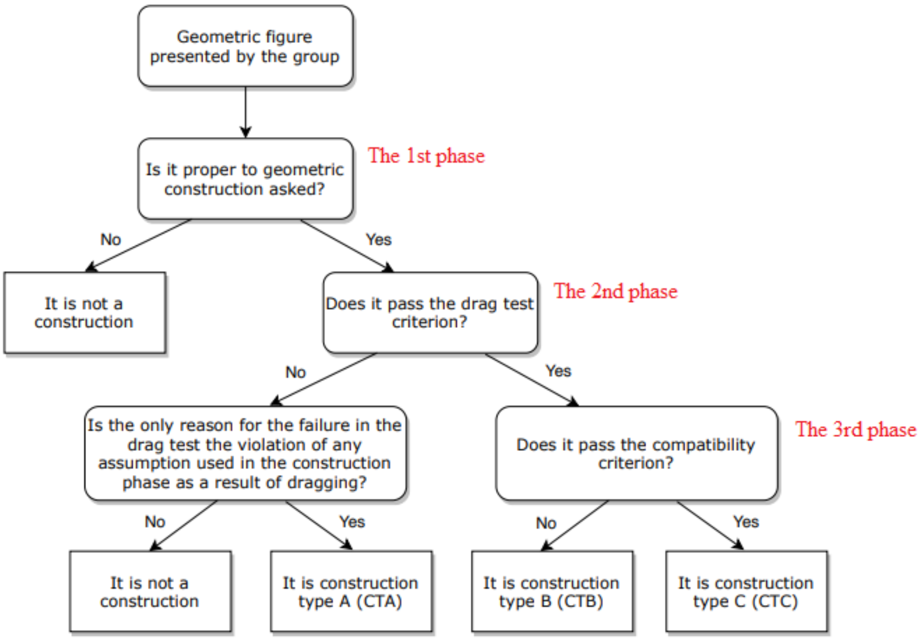


Figure 4. 17. The diagram for accepting a geometric figure as a construction while using GeoGebra

By considering the differentiation related to the criteria to be accepted as a geometric construction when different tools are used, the findings arranged for addressing the third research question were presented under two main sub-sections. In this manner, it can be said that the former sub-section presented the findings deduced from compass-straightedge group (CSG) and the latter sub-section explained the findings obtained from GeoGebra group (GG).

4.3.1. Approaches CSG Offered for Geometric Constructions

The first set of findings related to the third research question is about the approaches prospective middle school mathematics teachers offered for geometric constructions while using compass-straightedge. Since there are four cognitive unity based activities in this study, this topic can best be treated under four sub-headings. Under each sub-heading, the correspondences of approaches offered for construction in the global argumentation structures of CSG were presented as the first step. That is to say, Figures 4.19, 4.25, 4.36, and 4.42 cover the global argumentation structures of CSG for each activity on which the approaches stated for construction were marked. Afterwards, each approach labeled in the global argumentation structures was explained briefly in the tables. That is, Tables 4.6, 4.8, and 4.10 present the summary of the approaches CSG stated as invalid and similarly Tables 4.7, 4.9, 4.11, and 4.12 cover the summary of approaches CSG labeled as valid. In the meantime, all approaches mentioned in the tables were explained in detail by indicating the actions of CSG and tracing over the geometric figures they formed. Finally, the decisions of CSG regarding the validity of the approaches in terms of providing constructions were paid attention and also the validity of the approaches were clarified based on the criteria list given in Table 4.5.

4.3.1.1. Approaches CSG Offered for Geometric Construction in Activity 1

In Activity 1, one worksheet per participant of CSG was given and extra worksheets were presented at the table of CSG to use whenever necessary. In the distributed worksheets, the geometric construction that CSG was expected to perform was written and an acute triangle (ΔABC) was drawn. More specifically, Activity 1 asked CSG the construction of a circle passing through the vertices of ΔABC, which was presented in Figure 4.18, by using compass-straightedge. As tools, one compass and one straightedge were also given to each participant and the substitute pairs of these tools were also present in the table of CSG.

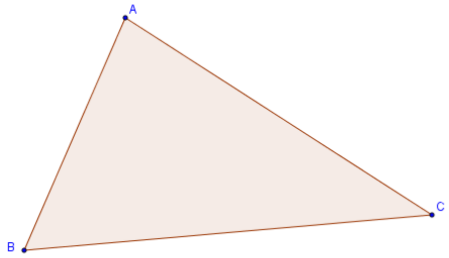


Figure 4. 18. ΔABC given in the worksheets of Activity 1

According to the analysis of the data, it was observed that CSG proposed four approaches to be able to construct the geometric figure asked in Activity 1. The correspondences of four approaches in the global argumentation structure of CSG in Activity 1 were illustrated in Figure 4.19.

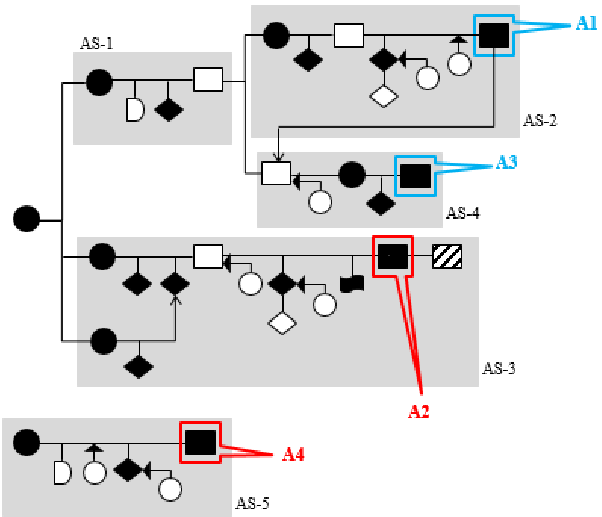


Figure 4. 19. The locations of approaches offered for the construction in the global argumentation structure of CSG in Activity 1

As seen in Figure 4.19, four conclusion components covered in different argumentation streams were marked with the indicators as A1, A2, A3, and A4 in order to point out the corresponding approaches. The enumeration of these approaches was conducted based on the occurrence order. Since CSG worked on the ideas of construction until reaching a conclusion and did not give up in the process, the final ideas about all approaches were schematized via the conclusion component.

Specifically, the approaches that CSG stated as not valid for geometric construction in Activity 1, which are A1 and A3, were marked with blue indicators. On the other hand, the approaches that CSG declared as valid for geometric construction in Activity 1, which are A2 and A4, were marked with red indicators. Based on this difference, first of all, Table 4.6 which involves the summary of the approaches CSG stated as invalid was given below. After the examination of the invalid approaches presented in Table 4.6 in detail, another table (See Table 4.7) which involves the summary of the approaches CSG stated as valid was presented in this section.

Table 4. 6

Approaches CSG stated as invalid for geometric construction in Activity 1

<table>
<tr><th>Approach for construction</th><th>Validity of approach</th></tr>
<tr><td rowspan="2">A1. CSG aimed to find the point of concurrency of the angle bisectors of ΔABC since they assumed that this point might give them the center of the circle passing through the vertices of ΔABC. After the determination of the point of concurrency of the angle bisectors of ΔABC, they noticed that this point is the incenter, not the center of the intended circle.
(written in the worksheet)</td><td>According to CSG
- Invalid approach for Activity 1
- Construction of the incircle of ΔABC</td></tr>
<tr><td>Based on the criteria list (See Table 4.5)
- Invalid approach for Activity 1
- Not construction of the incircle of ΔABC</td></tr>
<tr><td rowspan="2">A3. CSG tried to find the point of concurrency of the medians of ΔABC. However, they concluded that the point of intersection of the medians gives the centroid of ΔABC, not the center of the intended circle.
(written in the worksheet)</td><td>According to CSG
- Invalid approach for Activity 1
- Construction of the medians of ΔABC</td></tr>
<tr><td>Based on the criteria list
- Invalid approach for Activity 1
- Construction of the medians of ΔABC</td></tr>
</table>

As stated, Table 4.6 was prepared in a way that it includes the approaches CSG mentioned as invalid for the geometric construction asked in Activity 1. By descending to particulars, in addition to the summary of such approaches which was presented in the first column, Table 4.6 involves the decisions and comments of CSG about the validity of the related approaches and the evaluation of the approaches based on the criteria list given in Table 4.5 in the second column. According to Table 4.6, A1 and A3 among the four approaches are not the applicable ones for the construction of the circle in this activity. While A1 was about the angle bisectors of ΔABC, A3 was related to the medians of ΔABC. Moreover, CSG wrote down about not only the valid approaches of this activity but also invalid ones in the worksheets which can be reached from Appendix F. Thus, each approach could be explained via the geometric figure formed by CSG. In line with the third research question, it would be benefit to present a more detailed description of the construction approaches offered in Activity 1 at this point. Thus, the reasoning behind the evaluation of the validity of any approach may be clearer and more meaningful.

The first approach tried by CSG to construct the intended circle in Activity 1 was marked with a blue indicator in AS-2 of the global argumentation structure of CSG (See Figure 4.19). Besides, the geometric figure CSG formed by applying A1 was given in Figure 4.20.

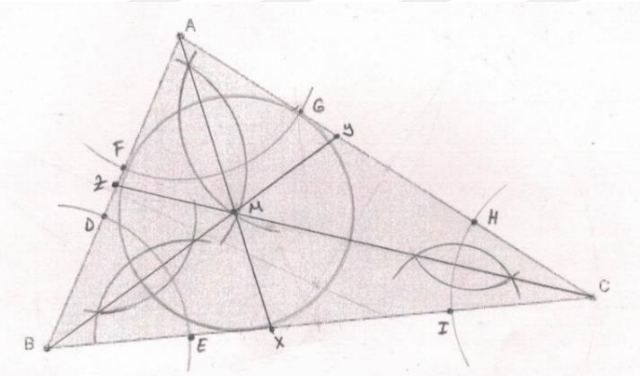


Figure 4. 20. Geometric figure CSG formed through A1 in Activity 1

As the first act in Activity 1, CSG aimed to find the center of ΔABC since they thought that this would also give the center of the circle passing through the vertices of ΔABC. Then, they assumed that they should find the centroid of ΔABC as the center of circle. However, they were not sure about whether the point of concurrency of the angle bisectors or the medians is the centroid. Firstly, they decided to find the intersection of the angle bisectors of ΔABC which was named as A1. For this purpose, CSG started by constructing the bisector of $\angle ABC$. They set the compass for a random radius, placed the stationary point of the compass at the vertex B, drew an arc crossing both $\overline{AB}$ and $\overline{BC}$, and named the intersection points of this arc and the sides as D and E, respectively. Thus, the points D and E are equidistant from the vertex B. Then, they drew another arc with the center D as inside of the triangle by means of a random radius and also drew one more arc with the center E as inside of the triangle by keeping the radius the same. Hereby, they determined the intersection points of the lastly drawn two arcs. Finally, they drew a line passing through the intersection points of these arcs by expecting that this line would also pass through the vertex B. As seen from Figure 4.20, this line became the bisector of $\angle ABC$.

By following the same steps, CSG constructed the bisectors of other interior angles of ΔABC which are $\angle CAB$ and $\angle BCA$. Thus, $\overline{BY}$ is the bisector of $\angle ABC$, $\overline{AX}$ is the bisector of $\angle CAB$, and $\overline{CZ}$ is the bisector of $\angle BCA$. Then, they determined the point of concurrency of the angle bisectors and named this point as M. To check whether the point M is the center of the circle passing through the vertices of ΔABC, they tried to draw the circle, but they observed that this approach did not give them the center of the circle they aimed to construct. That is, according to CSG, A1 was an invalid approach for Activity 1. Based on the criteria list (See Table 4.5), A1 was also an invalid approach for construction since it could not fulfill the first criterion. That is, the geometric figure formed via A1 is not proper to what was asked to construct in Activity 1 since it did not present the intended circle even visually.

Meanwhile, they noticed that the point of intersection of the angle bisectors gives the incenter. Thus, they declared that the geometric figure they formed by applying this approach could be accepted as a construction for the incircle of ΔABC although A1 was not a correct approach for Activity 1. However, the geometric figure cannot be accepted as a construction even for the case that the construction of incircle of ΔABC was asked. The reason of this evaluation is that CSG did not find the radius of the incircle properly. As it can be seen in Figure 4.20, they determined the points of tangency of the incircle randomly or by just looking at it. To accept the figure as a construction of the incircle, they should draw at least one perpendicular line from the point M to any side of ΔABC to be able to determine the radius and one point of tangency of the incircle properly. To conclude, although CSG stated the figure formed by A1 can be accepted as a construction in terms of the incircle of ΔABC, it cannot be accepted as a correct one based on the criteria listed in Table 4.5. The mentioned inference related to the radius of the circle was not correct, and the explanation regarding the steps followed in the determination of the radius of the incircle was inadequate.

After A1, CSG worked on A2 but it will be explained later in this section since A2 was labeled as a valid approach for construction by them. Although they were sure that they performed the construction in Activity 1 correctly by means of A2, they wanted to search for alternative construction approaches. Thus, they came up with A3 for the construction of the circle passing through the vertices of ΔABC. As the next invalid approach offered, A3 was explained at this point. This approach was signified with a blue indicator in AS-4 of the global argumentation structure of CSG given in Figure 4.19 and its scope was described briefly in Table 4.6. Moreover, the geometric figure they submitted at the end of this approach was displayed in Figure 4.21 as given below.

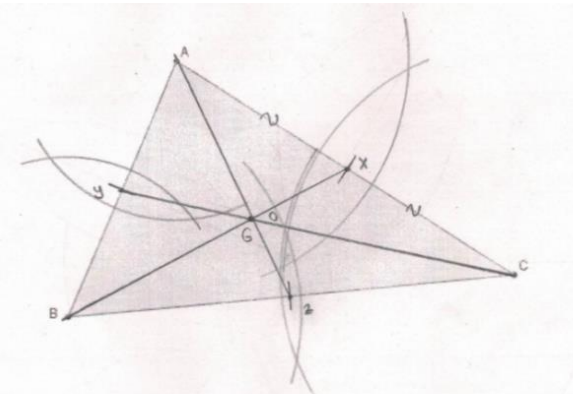


Figure 4. 21. Geometric figure CSG formed through A3 in Activity 1

Since CSG was confused with the terms angle bisector and median, it was observed that they sometimes used them interchangeably in the argumentation process. As a result of this, they discussed whether the point of intersection of the angle bisectors and the medians would give the same result. Therefore, they also wanted to try to construct the medians of ΔABC. They actually remade the steps they performed while constructing the perpendicular bisectors of sides of ΔABC in A2 to find the midpoints of the sides. More specifically, as seen in Figure 4.21, they set the compass more than the half of $|CA|$ and drew two arcs crossing over $\overline{CA}$ by placing the compass needle on the vertices A and C. By arranging the straightedge across the intersections of two arcs and drawing a short line, they determined the midpoint of $\overline{CA}$ which was named as X. By carrying out the same procedure for other sides of ΔABC, the constructions of all midpoints were completed. Moreover, an issue written in the field notes about the argumentation of CSG in Activity 1 was paid attention in the analysis process. It was seen that CSG did not draw the lines passing from the intersections of arcs explicitly and they just marked the midpoints of the sides by using the straightedge to avoid the confusion created by the presence of many drawings over ΔABC. Based on the criteria list in Table 4.5, this was not accepted as an incorrect usage of the straightedge since they seemed to use it simply to draw the lines with little lengths in these cases.

Later, CSG drew the line segments between the midpoints X, Y, and Z and the opposite vertices B, C, and A, respectively. Then, they named the point of concurrency of the medians as G and concluded that it gives the centroid of ΔABC, not the center of the aimed circle. Similar to A1, CSG declared that A3 could be regarded as a valid construction approach if the construction of the centroid of a triangle was asked. Since it was not the case in Activity 1, A3 cannot be accepted as a valid approach for this activity. Since the figure formed via A3 failed in the first criterion given in Table 4.5, A3 was classified as an invalid approach for the geometric construction asked in Activity 1.

To sum up, according to both CSG and the criteria list, A1 and A3 are not valid approaches for the geometric construction in Activity 1. Besides, CSG offered two more approaches which were declared as valid, namely, A2 and A4 (See Figure 4.19). Similar to Table 4.6, another table was prepared for the summary of these approaches as noted below.

Table 4. 7

Approaches CSG stated as valid for geometric construction in Activity 1

Approach for construction	**Validity of approach**
A2. CSG noticed to use the sides of ΔABC as the chords of the circle. In more detail, they used the theorem "perpendicular bisector of a chord passes through the center of the circle" to find the center of the circle. Therefore, they drew the perpendicular bisectors of each side of ΔABC. They concluded that the point of concurrency of the perpendicular bisectors of the sides presented the center so that they could draw the circle passing through the vertices of ΔABC. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 1 - Construction *Based on the criteria list (See Table 4.5)* - Valid approach for Activity 1 - Construction
A4. CSG drew three circles with equal radius by accepting each vertex of ΔABC as center. They thought that the points of intersection of these circles might be on the circumcircle, but noticed that this idea did not work during construction. Then, they drew lines between the intersection points of each pair of adjacent circles while trying to find a different approach. They concluded that the point of intersection of these lines gave the center of the circumcircle. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 1 - Construction *Based on the criteria list* - Valid approach for Activity 1 - Construction

Table 4.7 covers the explanation of approaches in the first column and the evaluation of the validity of the related approaches in the second column. Both A2 and A4 were explained by CSG in the worksheets as well as the geometric figures (See Appendix F). As it will be explained in detail below, A2 and A4 can be considered as the same approach except that the starting points of them are different. Since CSG considered them separately, they were also presented as separate approaches in Table 4.7.

As the second approach CSG stated for construction, A2 was summarized in Table 4.7 and pointed out with a red indicator in AS-3 of the global argumentation structure of CSG (See Figure 4.19). Moreover, the geometric figure CSG submitted as relating to this approach was presented in Figure 4.22 given below.

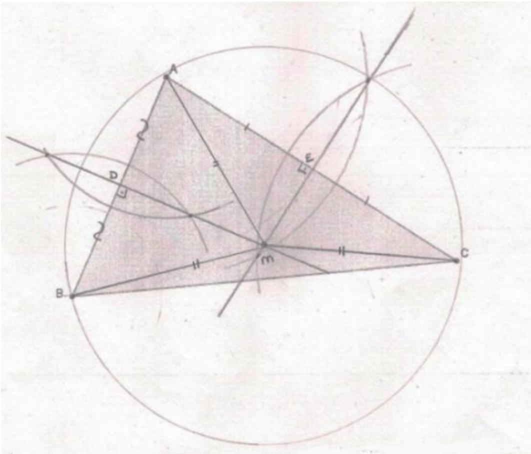


Figure 4. 22. Geometric figure CSG formed through A2 in Activity 1

While examining the process in A2 in depth, it was seen that CSG used the theorem “perpendicular bisector of a chord passes through the center of the circle”. Thus, to be able to find the center of the circle, they accepted the sides of the given triangle as the chords. Therefore, they firstly aimed to construct the perpendicular bisectors of the sides of ΔABC. They started via the construction of the perpendicular bisector of $\overline{AB}$. To that end, they set the compass to a length more than half of $|AB|$ so as to make the arcs intersect. Then, they drew an arc by placing the stationary point of the compass at the vertex A, kept the compass the same, and drew another arc by placing the stationary point of the compass at the vertex B. Afterwards, they determined the intersections of the arcs and drew a line passing through these two points. This line became the perpendicular bisector of $\overline{AB}$. By conducting the same steps, CSG constructed the perpendicular bisector of $\overline{CA}$. Since they noticed that they could determine the point of intersection of the perpendicular bisectors of the sides via constructing two of them, they decided not to draw the perpendicular bisector of $\overline{BC}$. They named the point of intersection of two perpendicular bisectors of ΔABC as the point M, as illustrated in Figure 4.22. Finally, they drew the circle asked in the activity by accepting the center as the point M and the radius as the distance between the point M and the vertex A. They would choose any vertex of ΔABC since the distances between the point M and all vertices are equal. CSG accepted A2 as a valid approach and stated that what they drew could be accepted as a geometric construction. After this, they also stated that the circle they formed is the circumcircle of ΔABC so that the point of concurrency of the perpendicular bisectors is the circumcenter. This expression was represented with the target conclusion component in AS-3 (See Figure 4.19). When the criteria list was checked for the geometric figure formed by applying A2, it was seen that all six criteria were fulfilled by the geometric figure. Thus, it can be accepted as a geometric construction for Activity 1.

As the final approach in Activity 1, A4 was pointed out with a red indicator in AS-5 of the global argumentation structure of CSG in Activity 1 (See Figure 4.19). To make this approach clearer, it was explained as follows by citing the geometric figure they submitted at the end of the activity which was given in Figure 4.23.

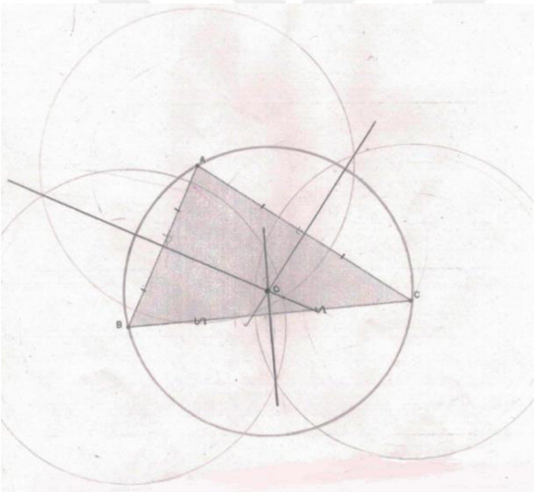


Figure 4. 23. Geometric figure CSG formed through A4 in Activity 1

In A4, CSG attended to alter their starting point related to the construction of the intended circle. In the first three approaches, they focused on finding the center of the circle. In accordance with the global argumentation structure in Figure 4.19 in which A4 was presented in a different argumentation block, the starting point of A4 was not aiming to find the center of the circle. In this manner, CSG wanted to draw three circles with equal radius by accepting each vertex of ΔABC as center. They thought that the intersections of these circles might be on the circumcircle. To that end, they set the compass as more than half of $|BC|$ and checked it whether it is set as more than half of the lengths of other sides too. Then, they placed the fixed point of the compass at the vertices A, B, and C and drew three equal circles. As can be seen in Figure 4.23, it was not possible for the intersections of three circles to be on the circumcircle. Although they did not start with a focus on finding the center, they drew the lines between the intersection points of each pair of adjacent circles while trying to find a different approach. Then, they tried to draw a circle by accepting the point of concurrency of these lines as the center. While drawing the circle, they observed that A4 also resulted in the intended circle in Activity 1. Therefore, CSG regarded A4 as a valid approach and the figure formed at the end as a geometric construction. Similarly, it was concluded that A4 is a valid approach and the figure is a construction based on the criteria list in Table 4.5.

In addition, the field notes concerning CSG in this activity showed that CSG did not notice that A2 and A4 are comparatively similar approaches during the application stage of Activity 1. In other words, CSG was not aware of the fact that they constructed the perpendicular bisectors of the sides of ΔABC by implementing A4. The differences between A2 and A4 are listed as follows. The starting point of A2 is to find the center of the circle while the starting notion of A4 is not about focusing on the center, but about finding an alternative approach. Secondly, they used two equal circles while finding the perpendicular bisector of one side and then used a different pair of circles while working on another side in A2, but all three circles used to draw the perpendicular bisectors of the sides are equal in A4.

4.3.1.2. Approaches CSG Offered for Geometric Construction in Activity 2

In Activity 2, three types of triangles are given to the participants of CSG as one triangle per page. The triangles in the worksheets were presented in the following figure. As seen, ΔDEF is an acute triangle, ΔABC is an obtuse triangle, and ΔKLM is a right triangle. Then, CSG was asked to construct the altitudes and also the orthocenters, if they exist, for the given triangles (See Appendix B). For supplying back up, both extra worksheets and packs of compass-straightedge were available at the table of CSG during Activity 2.

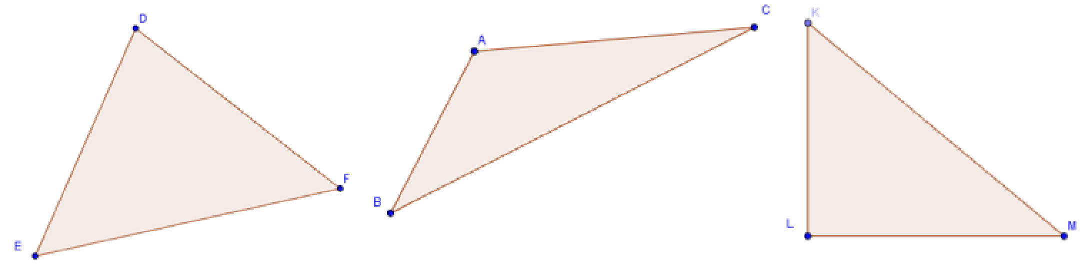


Figure 4. 24. ΔDEF , ΔABC, and ΔKLM given in the worksheets of Activity 2

In the data analysis, it was noted that CSG offered five approaches for the construction asked in Activity 2. The locations of five approaches in the global argumentation structure of CSG in Activity 2 were displayed in Figure 4.25.

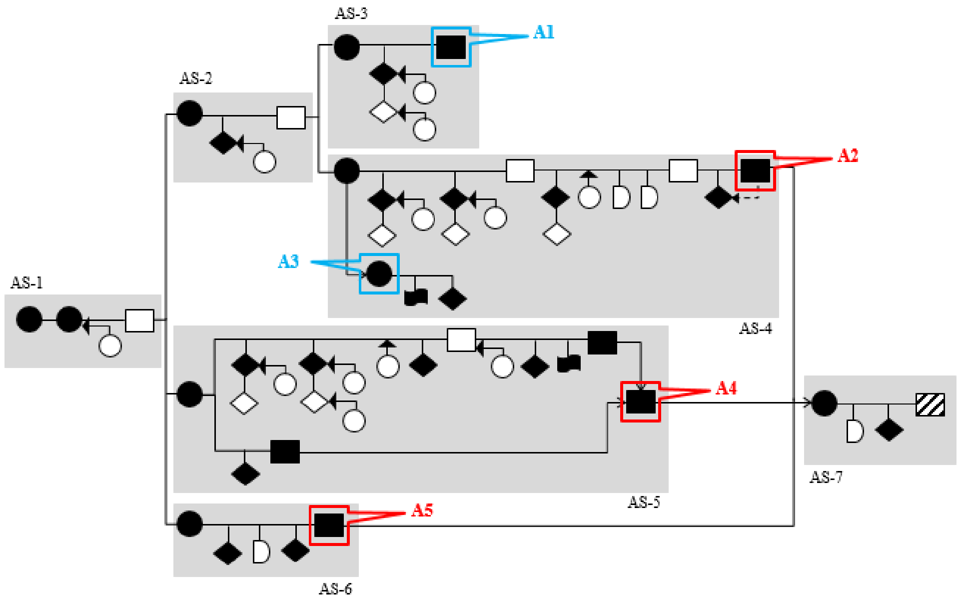


Figure 4. 25. The locations of approaches offered for the construction in the global argumentation structure of CSG in Activity 2

The approaches stated as not working in terms of the construction in Activity 2 by CSG, which are A1 and A3, were signed with blue indicators. The remaining three approaches, which are A2, A4, and A5, were signed with red indicators since CSG labeled them as valid approaches. While four of them were marked via the conclusion components, only A3 was pointed out via the data component since the issue regarding the validity of this approach was not stated in a conclusion component. Moreover, all approaches were numbered based on their appearance order. The contents of these invalid approaches were explained in Table 4.8 and the ones stated as valid by CSG were explained in Table 4.9.

Table 4. 8

Approaches CSG stated as invalid for geometric construction in Activity 2

Approach for construction	**Validity of approach**
A1. CSG extended the perpendicular bisector of a side of a triangle to check whether it passes through the opposite vertex of the triangle. In ΔDEF , the perpendicular bisector of $\overline{EF}$ passes through quite near to the vertex D so that they discussed such an idea. However, CSG observed that it is not a correct idea for all triangles. *(not written in the worksheet)*	*According to CSG* - Invalid approach for Activity 2
	Based on the criteria list (See Table 4.5) - Invalid approach for Activity 2
A3. CSG considered drawing a line passing through a vertex and also parallel to the perpendicular bisector of the corresponding side may present the altitude of that side. Hereby, they tried to remember the basic construction approach to draw parallel lines. However, they stated that they could not remember exactly. Therefore, they gave up this approach. *(not written in the worksheet)*	*According to CSG* - Invalid approach for Activity 2
	Based on the criteria list - Invalid approach for Activity 2 - Not finished

As indicated in Table 4.8, A1 is about the construction of the perpendicular bisectors of the sides in the given triangles and A3 is about the construction of the lines parallel to the perpendicular bisectors of the sides. Although CSG tried to apply A1 in ΔDEF , they could not move forward in A3 since they could not schematize the entire process needed for construction. Since CSG did not write down the explanation for both A1 and A3 during the application of the activity, any figure which is completely related to them could not be displayed. The argumentation processes during A1 and A3 were explained as follows as much as possible in detail to help to envision.

At the beginning of the activity, CSG discussed what the orthocenter is and how they can find it in the case of its existence. They decided that they should construct the altitudes of given three triangles at the first phase and then examine the orthocenter. Therefore, they started to the construction trials with the given acute triangle which is ΔDEF . The figure they presented at the end of the application of A2 also covers the application of A1. That is to say, the geometric figure presented in Figure 4.26 can be examined in terms of both A1 and A2 since CSG started to the construction of A1 in this figure, deleted some parts of it when A1 did not work, and then continued with the construction by means of A2.

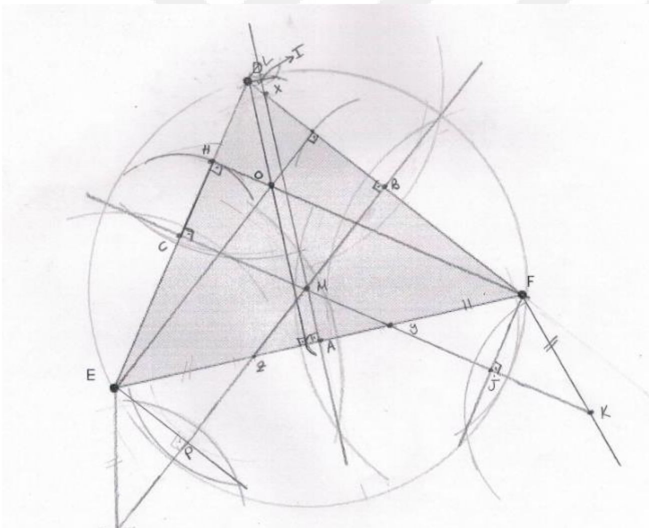


Figure 4. 26. Geometric figure CSG formed through both A1 and A2 in Activity 2

Due to the effect of the prior activity (Activity 1), CSG had a tendency to start what they had already known. Thus, CSG thought that drawing the perpendicular bisectors of the sides of ΔDEF might help them in the construction of the altitudes. This idea appeared at the beginning of AS-2 and then they constructed perpendicular bisectors of two sides of ΔDEF throughout AS-2. In more detail, CSG set the compass more than the half of $|DE|$, put the stationary point of the compass at the vertices D and E, and drew two arcs as passing over $\overline{DE}$. The line passing from the intersections of these arcs gave them the perpendicular bisector of $\overline{DE}$ and also the midpoint of $\overline{DE}$, which was named as C. Then, CSG pursued the same steps for $\overline{FD}$ so that they had the perpendicular bisector and the midpoint of $\overline{FD}$, which was named as B. However, instead of what they expected, they observed that the perpendicular bisectors of $\overline{DE}$ and $\overline{FD}$ did not pass through the vertices F and E, respectively. While they were quitting this idea, one of them offered to draw the third perpendicular bisector of ΔDEF. By focusing on the visual of ΔDEF, they anticipated that the extension of the perpendicular bisector of $\overline{EF}$ might pass through the point D. To that end, after the construction of the perpendicular bisector of $\overline{EF}$, they checked whether the vertex D, the point M, which is the point of concurrency of the perpendicular bisectors of the sides of ΔDEF, and the point A, which is the midpoint of $\overline{EF}$ are collinear or not. Since the perpendicular bisector of $\overline{EF}$ passes through quite near to the vertex D, they came up with such an idea. However, CSG resulted in that the mentioned points D, M, and A are not collinear so that it is not possible to state that the extension of the perpendicular bisector of $\overline{EF}$ passes through the vertex D. Therefore, they concluded that this approach is not a valid one and the perpendicular bisectors of the sides of a triangle do not give the altitudes directly when different types of triangles were considered.

Afterwards, CSG deleted from the worksheet the lastly drawn line to check the collinearity but kept the construction of all perpendicular bisectors of the sides of ΔDEF which was used later by associating with A2 (See Figure 4.26). Since CSG considered that A1 did not work in terms of construction, they did not try A1 for other given triangles (ΔABC and ΔKLM). However, as seen from the movements of CSG in the video recordings, they examined and envisioned the altitudes and the perpendicular bisectors of the sides of both ΔABC and ΔKLM while deciding about the validity of A1. In the same vein, when A1 was checked based on the criteria list given in Table 4.5, it was noted that A1 is not a valid approach for the construction of the altitudes and the orthocenter in Activity 2 since it failed in the first criterion. In other words, there is not a geometric figure presented at the end of A1 as proper to the asked construction in the activity.

Based on A1, CSG offered to draw the lines parallel to the perpendicular bisectors of the sides of ΔDEF in a way that these lines would also pass through the vertices so that the altitudes of ΔDEF would be constructed. Then, they proposed A2 which was explained later in detail in the scope of the valid approaches in Activity 2. While working on A2, one participant of CSG attended to draw the mentioned parallel lines by using another approach. This alternative approach was presented as A3 in the second step of AS-4 in the global argumentation structure of CSG (See Figure 4.25). She endeavored to remember the construction they worked on throughout the teaching sessions at the beginning of the course. By virtue of the question of her, all participants of CSG started to think about it. Since the instructor observed that they got stuck on this idea, she decided to help by giving some clues. The instructor tried to prompt their ideas by stating a clue about using the angles. Then, one of them remembered that they used the similarity and transferred the angle to justify the parallelism. However, CSG failed to conduct what she referred as a warrant for A3. Since they could not resolve all details of the construction related to the idea in A3, they did not continue to work on it and did not state a conclusion regarding it. Thus, CSG declared A3 as an invalid approach for the construction of the altitudes and the orthocenters of the given triangles. Accordingly, since A3 could not be finished and there is not a product related to A3, any of the criteria used to check the validity of an approach was not fulfilled by A3.

After the invalid approaches, it is the turn of three valid approaches suggested in Activity 2. Therefore, the summary and the evaluations of A2, A4, and A5 were described in Table 4.9 below.

Table 4. 9

Approaches CSG stated as valid for geometric construction in Activity 2

Approach for construction	**Validity of approach**
A2. CSG decided to draw parallel lines to perpendicular bisectors so that these parallel lines will pass through the vertices and become the altitudes. Thus, they determined the smallest distance from the vertex to the corresponding perpendicular bisector. Then, they used this distance to determine the second point since the distance between parallel lines are equal. The line drawn from the second point and the vertex became the altitude of one side of triangle. They found the altitudes of each side of the given three triangles with this approach and found the orthocenters. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 2 - Construction *Based on the criteria list (See Table 4.5)* - Valid approach for Activity 2 - Construction
A4. CSG used the theorem that "any diameter of a circle subtends a right angle to any point on the circle". Thus, they accepted the sides of the given triangles as the diameters and drew circles. Then, they marked the intersection points of circles and the sides of the triangle and thought such a point as the foot of the altitude. They found the altitudes of each side of the given three triangles with this approach and found the orthocenters. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 2 - Construction *Based on the criteria list* - Valid approach for Activity 2 - Construction
A5. CSG used the basic geometric construction approach for constructing the line which is perpendicular to a given line from a point not on the given line. They accepted the given line as one side of the triangle and the point not on the given line as the vertex. They found the altitudes of each side of the given three triangles with this approach and found the orthocenters. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 2 - Construction *Based on the criteria list* - Valid approach for Activity 2 - Construction

According to the review of the approaches in Table 4.9, three among five approaches which are A2, A4, and A5 were categorized as valid approaches for construction by CSG. All of these approaches were applied to each one of the given triangles so that each approach was explained by means of three geometric figures formed by CSG. Moreover, the descriptions of CSG for these approaches written in the worksheets were presented in Appendix F. As it can be seen from the summary in Table 4.9, the mutual aim of A2, A4, and A5 is to construct the altitudes of triangles and then the orthocenters of them, but the contexts of these approaches are quite different. In more detail, A2 is about the construction of the parallel line, A4 is about using the theorem “any diameter of a circle subtends a right angle to any point on the circle”, and A5 is related to the basic approach used in the construction of a perpendicular line. Each of these approaches was explained by descending into details as noted below.

As the first valid approach, CSG worked on A2 which was applied to all triangles throughout AS-4 in the global argumentation structure of CSG (See Figure 4.25). In A2, CSG was trying to construct the lines parallel to the perpendicular bisectors of the sides of the given triangles so as to be able to construct the altitudes. The geometric figure CSG put forward by applying A2 for ΔDEF was presented in Figure 4.26 while describing A1. Due to this overlapping issue in A1 and A2, the geometric figure formed with ΔDEF was presented before A2, but the geometric figures formed via A2 while working on ΔABC and ΔKLM were also given after the explanation of the application process of A2 in ΔDEF.

In A2, one of the participants of CSG suggested finding the smallest distance from the vertex to the corresponding perpendicular bisector. For example, as presented in Figure 4.26, she wanted to find the smallest distance from the vertex F to the perpendicular bisector of $\overline{DE}$. Then, all participants of CSG focused on this idea. To this end, they used one of the basic constructions they worked on during the teaching sessions of the course. Specifically, by getting inspired from the construction of a line perpendicular to a given line from a point not on the given line, CSG determined the point Y which is the intersection of the perpendicular bisector of $\overline{DE}$ and $\overline{EF}$. Then, they set the compass to $|FY|$, put the stationary point of the compass at the vertex F, and marked an arc so as to determine the second intersection point of the arc and the perpendicular bisector of $\overline{DE}$ which is the point K. Thus, they constructed an isosceles triangle, which is ΔFYK. They thought that they could draw the perpendicular bisector of $\overline{YK}$ which would also pass through the vertex F. Again, by placing the fixed point of the compass on the points Y and K and drawing arcs of equal radius, they drew the perpendicular bisector of $\overline{YK}$. While doing so, CSG found out the smallest distance between the vertex F and the perpendicular bisector of $\overline{DE}$ which refers to the distance between the points J and F. As the next step, they set the compass to $|JF|$ and drew an arc by placing the compass at the point C. With this move, they aimed to find the second point that could be used to draw the line referring to the altitude of $\overline{DE}$. CSG found that the mentioned point is H in Figure 4.26. By drawing a line passing through the point H and the vertex F, they asserted that the altitude of $\overline{DE}$ was drawn.

Subsequent to the construction of the altitude of $\overline{DE}$, CSG implemented A2 for $\overline{FD}$. CSG started to extend the length of the perpendicular bisector of $\overline{FD}$ so as to determine the smallest distance between the vertex E and the perpendicular bisector of $\overline{FD}$. As can be seen in Figure 4.26, they formed the isosceles triangle ΔENZ. By finding the perpendicular bisector of $\overline{NZ}$, they determined the point P and $|EP|$. By setting compass to $|EP|$ and placing the compass at the point B, they drew an arc. By this way, CSG also constructed the second altitude which is the one for $\overline{FD}$. However, they had difficulty in pursuing A2 for $\overline{EF}$ since the distance between the possible altitude and the perpendicular bisector of $\overline{EF}$ is comparatively small. Specifically, they formed the isosceles triangle ΔDXL and then determined $|DI|$. By placing the compass needle at the point A which the midpoint of $\overline{EF}$, they found the second point so as to draw the altitude of $\overline{EF}$. After they managed the construction of three altitudes of ΔDEF, they also specified the point of concurrency of the altitudes with the letter O, and declared that this point is the orthocenter of ΔDEF. This idea was presented by CSG as a valid approach for the construction of the altitudes and the orthocenter.

By following this approach, CSG drew the altitudes of each side of other two triangles and found the orthocenters. In a similar vein, Figure 4.27 indicates what CSG formed by working on ΔKLM via implementing A2.

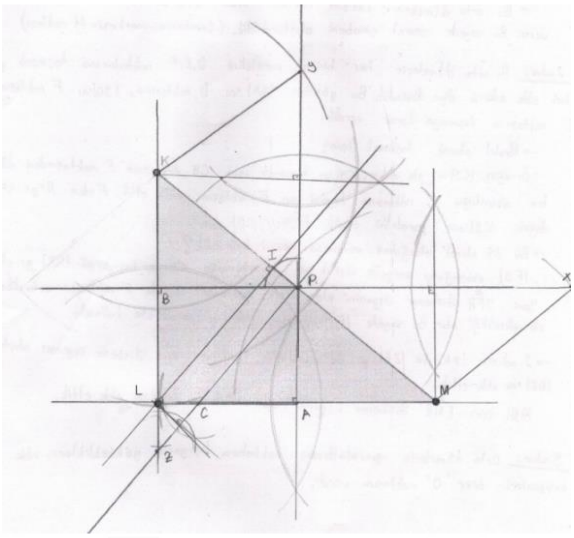


Figure 4. 27. Geometric figure (for ΔKLM) CSG formed through A2 in Activity 2

The first act of CSG while applying A2 to ΔKLM was to determine the perpendicular bisectors of each side. To construct the perpendicular bisector of $\overline{LM}$, they arranged the compass more than half of the length of this side, drew two arcs over $\overline{LM}$ by placing the compass needle at the vertices L and M, and drew a line passing through the intersections of the arcs by using straightedge. Thus, they constructed the perpendicular bisector of $\overline{LM}$ and determined the midpoint of $\overline{LM}$ as the point A. Likewise, CSG constructed the perpendicular bisectors of $\overline{KL}$ and $\overline{MK}$ and also marked the midpoints of them as the points B and P, respectively. Moreover, they notified that P is the point of concurrency of the perpendicular bisectors of the sides of ΔKLM so that it is the circumcenter as well. The next phase of A2 was to draw the lines passing from the vertices and parallel to the perpendicular bisectors of the sides. Hence, they formed three isosceles triangles which are ΔMXP , ΔLZC , and ΔKPY one by one. Then, they constructed the perpendicular bisectors of the bases of these isosceles triangles. The purpose in these actions was to determine the smallest distance between the vertices and the corresponding perpendicular bisectors of the sides. After determination of these distances, they passed to find other points needed to construct altitudes. In more detail, the line passing through the vertices K and L is the altitude of $\overline{LM}$ and similarly the line passing through the vertices M and L is the altitude of

$\overline{KL}$ since ΔKLM is a right triangle. Different from the previous two sides, the line passing through the vertex L and the point I is the altitude of $\overline{MK}$. In conclusion, they put forward that the orthocenter of ΔKLM is the point L since all altitudes are concurrent at this point.

Lastly, ΔABC was focused to construct the altitudes and the orthocenter by applying A2 and the geometric figure formed in the worksheet covering ΔABC was presented in Figure 4.28.

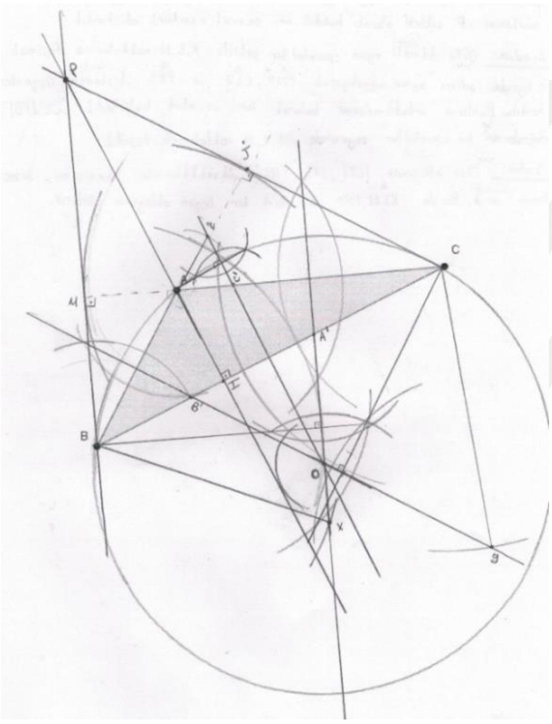


Figure 4. 28. Geometric figure (for ΔABC) CSG formed through A2 in Activity 2

CSG applied A2 for the given obtuse triangle and found the orthocenter of ΔABC as the point P. To be able to evaluate the validity of A2, the construction process of the altitudes prior to the determination of the orthocenter was elaborated as follows. First of all, CSG aimed to find the perpendicular bisectors of each side of ΔABC as usual in A2. In this manner, CSG followed a series of construction steps for each side of ΔABC. For example, the process for $\overline{BC}$ was like this; they set the arms of compass to a proper distance, placed the stationary point of the compass at the vertices B and C, drew two arcs in a way that the arcs were also passing over $\overline{BC}$, and drew a line from the intersections of arcs. This line was the perpendicular bisector of $\overline{BC}$. These steps were also conducted for $\overline{AB}$ and $\overline{CA}$. At this point, CSG had the perpendicular bisectors of the sides and drawing the parallel lines was the next move in A2. To this end, they started with $\overline{BC}$. To find the smallest distance between the vertex A and the perpendicular bisector of $\overline{BC}$, they formed an isosceles triangle $\Delta AC'Z$. By constructing the perpendicular bisector of $\overline{C'Z}$ in $\Delta AC'Z$, they determined the aimed distance. Then, they marked the point I as transferring the distance. They drew a line passing from the vertex A and the point I; hence, they constructed the altitude of $\overline{BC}$.

The constructions of the altitudes of other sides of ΔABC were more complicated since ΔABC is an obtuse triangle. In more detail, CSG constructed the perpendicular bisectors of $\overline{AB}$ and $\overline{CA}$ by using the routine way in A2. To find the smallest distance between the vertex C and the perpendicular bisector of $\overline{AB}$, they formed $\Delta CB'Y$ which is also an isosceles triangle. Then, they found the mentioned distance by constructing the perpendicular bisector of $\overline{B'Y}$. When they attempted to draw an arc by using this distance from the midpoint of $\overline{AB}$, they noticed that the extension of $\overline{AB}$ was needed to determine other point used in the construction of the altitude of $\overline{AB}$ in addition to the vertex C. By using straightedge, they drew a line passing through the vertices A and B and extended it. Then, they could find the intersection of the arc and the extended version of $\overline{AB}$ and named this point as J. The line drawn from the point J and the vertex C presented the altitude of $\overline{AB}$. By following the same process of $\overline{AB}$, CSG constructed the altitude of $\overline{AC}$, as can be seen in Figure 4.28.

All in all, CSG checked the application of A2 for the given three triangles one by one. Then, they concluded that A2 is a valid approach for the construction in Activity 2. By following the criteria listed in Table 4.5, it can be stated that the geometric figures formed via A2, which can be seen in Figures 4.26, 4.27, and 4.28, fulfill all criteria. In more detail, these figures are proper to the geometric construction asked in Activity 2, they were formed by using compass-straightedge only in a proper way, and the inferences stated during the construction process were mathematically correct and proper in terms of the Euclidean restrictions. Thus, these figures were classified as constructions and A2 was regarded as a valid approach.

Although CSG was sure about the fact that they found a valid approach for the construction in Activity 2, they continued to search for some alternative approaches. This is the general stance of groups in the activities since the activities were applied in the scope of an elective course. It can be stated that prospective middle school mathematics teachers kept their effort in high pace during the activities.

Unlike A1, A2, and A3, CSG moved away from the ideas related to the construction of the perpendicular bisectors of the sides of triangles and the construction of the lines parallel to them while enhancing a new approach, namely, A4. In more detail, A4 was marked in AS-5 of the global argumentation structure of CSG (See Figure 4.25). In the first argumentation step of AS-4, CSG endeavored on both ΔDEF and ΔABC due to the interwoven process followed by CSG. The second argumentation step of AS-4 is about performing the aimed construction in ΔKLM via A4.

The scope of A4 covers the utilization of the theorem “any diameter of a circle subtends a right angle to any point on the circle”. Based upon this, CSG attended to accept the sides of a triangle as the diameters and draw circles. Then, CSG expected that the intersections of the circles and the sides would give them the altitudes. When CSG agreed about the possibility that A4 might work, they decided to construct it in all given three triangles. In this respect, CSG started the construction with ΔDEF. First of all, what they constructed by applying A4 in the given worksheet involving ΔDEF was illustrated in Figure 4.29 given below.

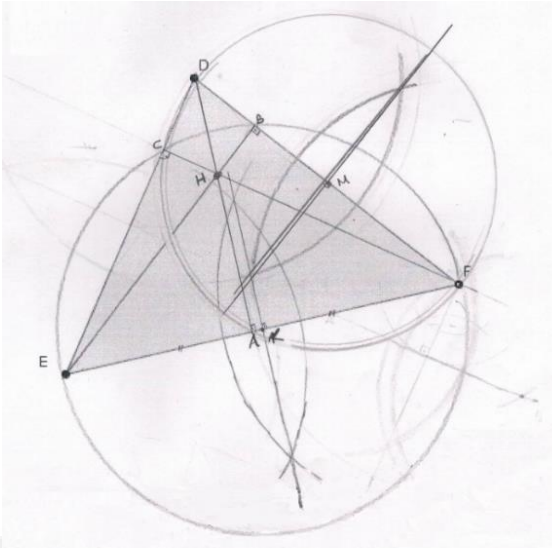


Figure 4. 29. Geometric figure (for ΔDEF) CSG formed through A4 in Activity 2

In the application process of A4, CSG started with ΔDEF like A2. Given that they accepted $\overline{EF}$ as the diameter, they found the midpoint of $\overline{EF}$ so as to find the center of the aimed circle. In more detail, CSG drew two arcs by placing the stationary point of the compass at the vertices E and F and then drew a line passing through the intersections of arcs. Thus, they determined the midpoint of $\overline{EF}$ as the point K. By using compass, they drew a circle with center K and the radius $|EK|$, and this circle intersected both $\overline{DE}$ and $\overline{FD}$. Then, CSG named these points as C and B, respectively. To find the angles which were subtended by the diameter $\overline{EF}$, CSG drew a line segment between the point C and the vertex F and also another line segment between the point E and the vertex B. By pursuing the previously given theorem, they concluded that $\angle FCE$ and $\angle FBE$ are 90º. All this process showed that $\overline{EB}$ is the altitude of $\overline{FD}$ and $\overline{CF}$ is the altitude of $\overline{DE}$.

On the other hand, the altitude of $\overline{EF}$ could not be reached with this circle. Therefore, CSG discussed drawing another circle by accepting any of the sides as diameter again. They selected $\overline{FD}$ as the diameter and followed the same steps for $\overline{FD}$. Accordingly, they drew two arcs passing over $\overline{FD}$ via placing the fixed point of the compass at the vertices D and F, marked the intersections of two arcs, drew a line passing through them, and labeled the intersection of this line and $\overline{FD}$ as the midpoint M. Later, a circle was drawn with the center M and the radius $|DM|$. As the next move, CSG determined the intersection of this circle with $\overline{EF}$ which is the point A so as to find the foot of the altitude of $\overline{EF}$. After drawing a line segment between the point A and the vertex D, they accomplished to finish the construction of three altitudes of ΔDEF and noticed that there is a point of concurrency of these altitudes which was named as H. Hence, the orthocenter of ΔDEF was pointed out as H (See Figure 4.29).

After finishing to work on ΔDEF, CSG continued to work on the obtuse triangle ΔABC by using A4. The geometric figure they submitted was shown in Figure 4.30 as given below.

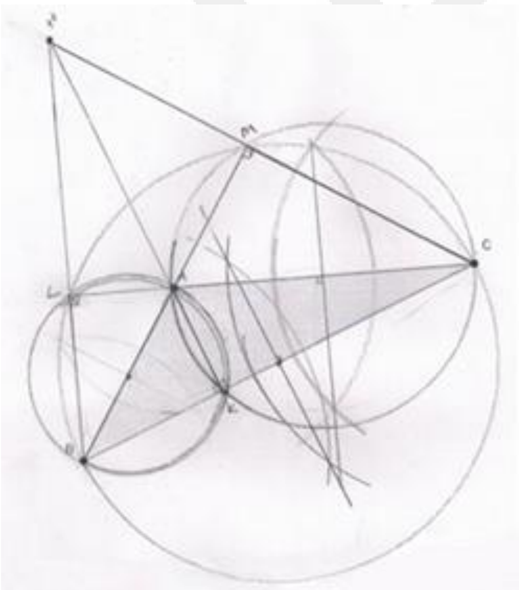


Figure 4. 30. Geometric figure (for ΔABC) CSG formed through A4 in Activity 2

The application of A4 in ΔABC was not as clear as ΔDEF for some participants of CSG since it is an obtuse triangle and the altitudes of two sides were drawn outside of ΔABC. They started with the construction of the circle with the diameter $\overline{CA}$. As seen in Figure 4.30, a line was drawn from the intersections of the arcs marked from the vertices A and C. The intersection of this line and $\overline{CA}$ presented the midpoint of $\overline{CA}$. By accepting the midpoint of $\overline{CA}$ as the center, a circle passing also from the vertex A was drawn. Then, it was time to find the angle subtended by $\overline{CA}$. To that end, CSG marked the intersection of the circle and $\overline{BC}$ as the point K and $\angle AKC$ become 90º. All of these steps formed the altitude of $\overline{BC}$ which is the line passing from the vertex A and the point K. After that, CSG found the midpoint of $\overline{AB}$ by following the usual midpoint construction in A4, so they could draw the circle with the diameter $\overline{AB}$. At this point, CSG noticed that the second circle was also passing through the point K on $\overline{BC}$. That is, another altitude of ΔABC was not designated by way of the second circle. Since the remaining two altitudes could not be established with the mentioned two circles, some of them were frustrated and mentioned about the possibility that A4 might not work for ΔABC.

As the last option with respect to A4, CSG continued with drawing a circle by accepting the last side as the diameter, namely $\overline{BC}$. By finding the midpoint of $\overline{BC}$ and using this midpoint as the center, they drew the aimed circle. However, they noticed that any of these circles were not intersecting with $\overline{AB}$ and $\overline{CA}$. They dealt with this situation by extending the mentioned sides via using the straightedge. While the intersection of the extension of $\overline{CA}$ and circles gave the point L, the intersection of the extension of $\overline{AB}$ and the circles gave the point M. Then, they drew the line between the vertex B and the point L to indicate the altitude of $\overline{CA}$. Similarly, the line passing through the vertex C and the point M constituted the altitude of $\overline{AB}$. To ensure the perpendicularity of these lines to the sides, they mentioned that $\angle CLB$ and $\angle CMB$ are the angles subtended by the diameter $\overline{BC}$. As the final remark, CSG declared that the orthocenter of ΔABC is the point H in Figure 4.30 since the altitudes are concurrent at that point. The implementation of A4 in ΔABC was a struggling circumstance for CSG, but they could find a way out for this trial too.

Since CSG reinforced their anticipation that A4 is a valid approach for the construction of altitudes for any type of triangles on the basis of finding the intended elements in ΔDEF and ΔABC, they moved to apply A4 in the right triangle ΔKLM

with a high expectancy in favor of A4. The figure they drew on the worksheet covering ΔKLM was given in Figure 4.31.

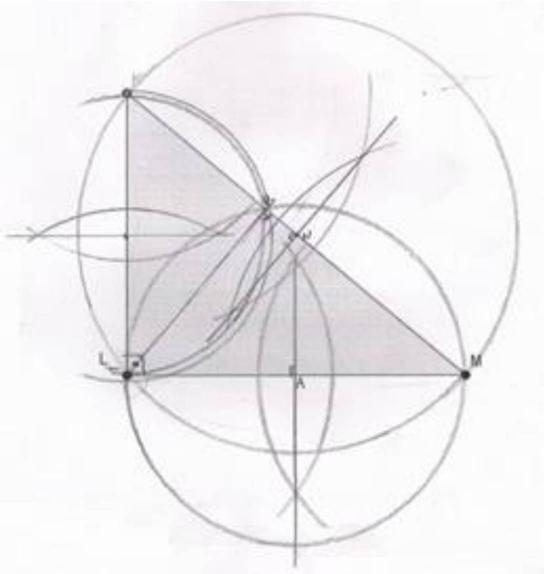


Figure 4. 31. Geometric figure (for ΔKLM) CSG formed through A4 in Activity 2

While working with ΔKLM , the first move of CSG was to draw the circle with the diameter $\overline{LM}$. By means of the drawn arcs and the line like the previous midpoint constructions in A4, they stated the point A as the midpoint of $\overline{LM}$ and drew the circle with center A and the radius $|AL|$. They also named the intersection of this circle and $\overline{MK}$ as the point B. When they connected the points L and B with a line segment, $\overline{LB}$ became the altitude of $\overline{MK}$. By drawing a circle with the diameter $\overline{KL}$ via following the same steps, they reconstructed the altitude of $\overline{MK}$ again. The last side which is $\overline{MK}$ was accepted as the diameter for this time and a circle was drawn with the same steps. Since they observed that this circle was passing through the vertices K, L, and M, they concluded that $\overline{KL}$ is the altitude of $\overline{LM}$ and vice versa. Finally, CSG stated that L is the orthocenter as the point of concurrency of three altitudes of ΔKLM .

According to CSG, all of these trials of A4 with the given triangles pointed out the fact that A4 is a valid approach for the construction asked in Activity 4. Given that the geometric figures CSG formed as illustrated in Figures 4.29, 4.30, and 4.31 are matching all six criteria in Table 4.5, A4 was categorized as a valid approach and the mentioned figures were labeled as constructions.

As the last valid approach, A5 was identified from the argumentation of CSG during Activity 2. It was marked with a red indicator in AS-6 of the global argumentation structure of CSG (See Figure 4.25). In a similar way, the geometric figures drawn in the worksheets of CSG by applying A5 were presented as follows. Specifically, Figure 4.32 displays what they constructed in ΔDEF by means of A5.

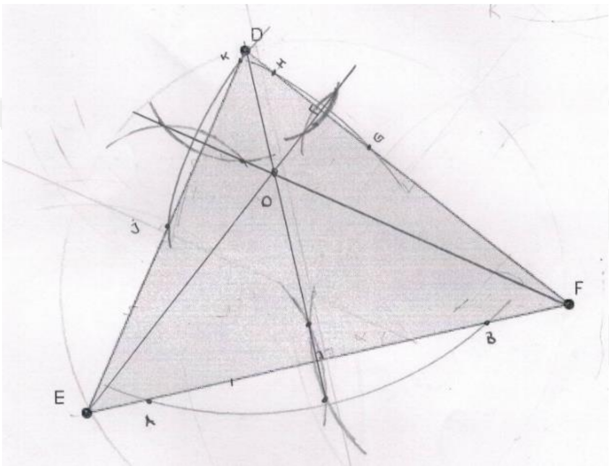


Figure 4. 32. Geometric figure (for ΔDEF) CSG formed through A5 in Activity 2

Similar to A2 and A4, CSG started to work with ΔDEF while implementing A5. At the beginning of the course, the basic geometric constructions were investigated by both CSG and GG and the common approaches were introduced and discussed. By the guidance of the instructor, CSG focused on one of the basic constructions that they worked before. They tried to remember how they performed the construction of the line which is perpendicular to a given line from a point, not on the given line. First of all, CSG accepted $\overline{EF}$ as the given line and the vertex D as the point not on that line. Then, they aimed to construct the line perpendicular to $\overline{EF}$ and passing from the vertex D and this would constitute the altitude of $\overline{EF}$. To that end, they placed the stationary point of the compass on the vertex D, drew an arc crossing $\overline{EF}$ at two points, and named these points as A and B. Then, the aimed construction turned out to be the construction of the perpendicular bisector of $\overline{AB}$. By setting compass more than half of $|AB|$ and placing the fixed point of the compass at the points A and B, they drew two arcs over $\overline{AB}$. The line drawn from the intersections of these arcs gave them the perpendicular bisector of $\overline{AB}$. This line also passed through the vertex D. They accomplished to draw a line passing from the vertex D and perpendicular to $\overline{EF}$ which refers to the altitude of $\overline{EF}$. By using the same steps while constructing the altitudes of $\overline{DE}$ and $\overline{FD}$, they determined the point of concurrency of them as the point O and labeled this point as the orthocenter of ΔDEF.

As a further step in the application of A5, CSG attended to work on ΔABC. What CSG constructed at the end of the implementation of A5 was illustrated in Figure 4.33.

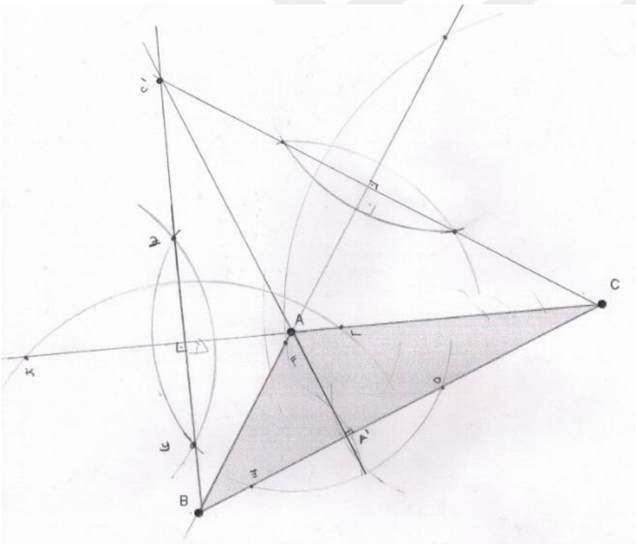


Figure 4. 33. Geometric figure (for ΔABC) CSG formed through A5 in Activity 2

CSG started with an aim of constructing the altitude of $\overline{CA}$ so that they adjusted the idea of A5 in a way that the given line was the extension of $\overline{CA}$, the point not on that line was the vertex B and the aim was to construct a line perpendicular to $\overline{CA}$ passing from the point B. Hence, the altitude of $\overline{CA}$ would be formed. However, CSG needed to extend $\overline{CA}$ to determine two points on that line equidistant to the point

B by virtue of an arc. As seen in Figure 4.33, they drew the arc by placing the stationary point of the compass at the vertex B and termed the mentioned points on the extended line as K and L. At this point, CSG aimed to find the perpendicular bisector of $\overline{KL}$. To that end, they set the compass more than half of $|KL|$ and drew two intersecting arcs by placing the compass needle on the points K and L. The intersections of these arcs were pointed out with the letters Y and Z. They expected that the line passing through Y and Z would also pass through the vertex B. At the end, this line became the altitude of $\overline{CA}$ due to the perpendicularity of $\overleftrightarrow{YZ}$ to $\overleftrightarrow{AC}$. Likewise, CSG extended $\overline{AB}$ and followed the previously given steps to find the line passing through the vertex C and perpendicular to the $\overleftrightarrow{AB}$ which refers to the altitude of $\overline{AB}$. Lastly, they attended to apply A5 to construct the altitude of $\overline{BC}$. Since there was no need to extend $\overline{BC}$ to find the equidistant points from the vertex A, they directly found these points and named them as E and O as the first move. The perpendicular bisector of $\overline{EO}$ presented them the altitude of $\overline{BC}$. All in all, they noticed that three altitudes are concurrent at a point outside of ΔABC which is the orthocenter of ΔABC. They named the orthocenter as the point C′, as displayed in Figure 4.33.

As the last triangle worked via A5, CSG focused on ΔKLM. The geometric figure they formed in the worksheet covering ΔKLM was given in Figure 4.34.

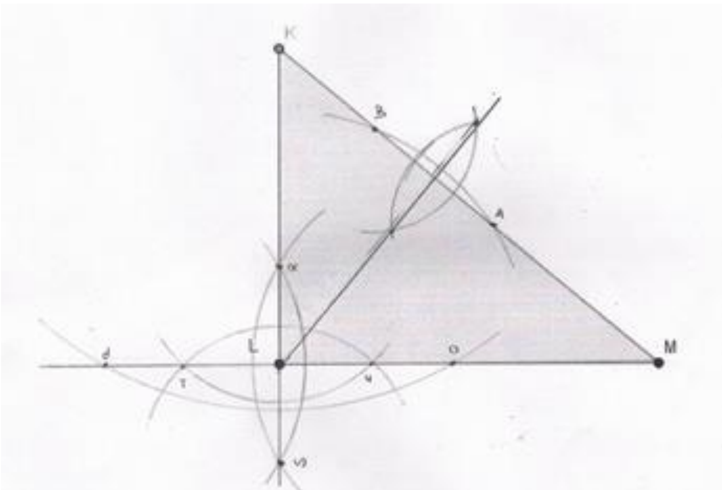


Figure 4. 34. Geometric figure (for ΔKLM) CSG formed through A5 in Activity 2

Regarding ΔKLM, CSG started with the altitude of $\overline{MK}$. That is to say, from the perspective of A5, $\overline{MK}$ refers to the given line, the vertex L refers to the point not on the given line, and the mission is to construct a line perpendicular to $\overline{MK}$ from the vertex L. To determine two points on $\overline{MK}$ which are equidistant to the vertex L, CSG drew an arc with the center L and a proper radius. The mentioned points were labeled as A and B. Then, they drew two arcs by placing the compass at the points A and B. They drew a line passing through the intersections of the arcs which also became the altitude of $\overline{MK}$ since it also passes through the vertex L. In the same manner, based on A5, CSG constructed the line passing through the vertex M and perpendicular to $\overline{KL}$ which corresponds to the altitude of $\overline{KL}$. Moreover, they constructed the line passing through the vertex K and perpendicular to $\overline{LM}$ which refers to the altitude of $\overline{LM}$. To sum up, CSG declared the vertex L as the orthocenter of ΔKLM.

At the end of dealing with all three triangles in the scope of A5, CSG evaluated A5 as a valid approach for the construction asked in Activity 2. Moreover, the figures presented via Figures 4.32, 4.33, and 4.34 were treated as constructions. By examining the given figures and their construction process in detail in the light of six criteria in Table 4.5, the mentioned figures were accepted as constructions and A5 was declared as a valid approach for Activity 2.

4.3.1.3. Approaches CSG Offered for Geometric Construction in Activity 3

In Activity 3, prospective middle school mathematics teachers were given one type of worksheet on which ΔABC and the intended geometric construction were presented. Moreover, one pair of compass-straightedge per participant was distributed. Extra worksheets and substitute tools were presented in the table of CSG. In more detail, CSG was asked to construct the orthocenter, the circumcenter, and the centroid of the given ΔABC and then examine the relationships/connections among these points (See Appendix B). Figure 4.35 illustrates ΔABC situated in the worksheet.

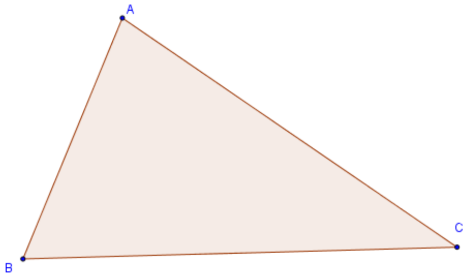


Figure 4. 35. ΔABC given in the worksheets of Activity 3

According to the analysis of the data gathered from Activity 3, it was seen that five approaches were proposed with the aim of construction by CSG. The locations of these approaches in the global argumentation structure of CSG were presented in Figure 4.36.

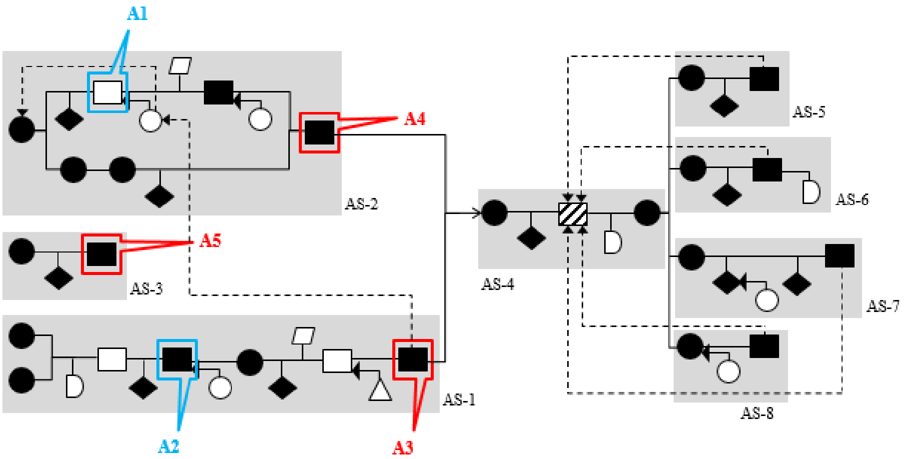


Figure 4. 36. The locations of approaches offered for the construction in the global argumentation structure of CSG in Activity 3

A1 and A2 were accepted as incorrect ideas for construction by CSG while the remaining three were accepted as the valid approaches. Therefore, A1 and A2 were marked with blue indicators whereas A3, A4, and A5 were plotted with red indicators. Since there are three points asked to construct in Activity 3, each of these valid approaches was dedicated to the construction of one of these points. In other words, there are not three different approaches offered for each one of three points. In short, A1 and A4 were related to the construction of the orthocenter, A2 and A3 were about the construction of the centroid, and lastly A5 was regarding the construction of the circumcenter. Moreover, only A1 was pointed out by means of the conclusion/data component, but the remaining four approaches were signified via the conclusion component. As seen, the evaluation of CSG for each idea could be represented via a conclusion-based component since all of them were tried and reached a conclusion regardless of the validity of them. As the first phase, the invalid approaches were summarized in Table 4.10 as follows.

Table 4. 10

Approaches CSG stated as invalid for geometric construction in Activity 3

Approach for construction	**Validity of approach**
A1. CSG thought that they could construct the orthocenter by drawing the perpendicular bisectors of the sides of ΔABC. Then, they noticed that it was not the orthocenter. *(written in the worksheet)*	*According to CSG* - Invalid approach for Activity 3 - Not construction of the orthocenter of ΔABC
	Based on the criteria list (See Table 4.5) - Invalid approach for Activity 3 - Not construction of the orthocenter of ΔABC
A2. CSG assumed that the point of concurrency of the angle bisectors of ΔABC would give the centroid. After finding the mentioned point, they noticed that it was not the centroid. *(written in the worksheet)*	*According to CSG* - Invalid approach for Activity 3 - Not construction of the centroid of ΔABC
	Based on the criteria list - Invalid approach for Activity 3 - Not construction of the orthocenter of ΔABC

Table 4.10 provides the summary of approaches that were categorized as invalid by CSG in the first column and the evaluations regarding the validity of the approaches in the second column. It can be inferred that A1 and A2 are the preliminary approaches for the constructions of the orthocenter and the centroid, respectively. Moreover, CSG firstly worked on the construction of the intended points on different worksheets, discussed the validity of them, and finally wrote the whole process into one worksheet by referencing the geometric figures formed on other worksheets. The

worksheet involving the explanation of CSG was presented in Appendix F and the four geometric figures were displayed in the followings whenever the related case was explained.

At the first phase of Activity 3, CSG decided to focus on the construction of different points individually. In other words, at first, they followed a work-sharing stance in Activity 3. After a while, they started to construct three points in a new worksheet collectively via probing their individual attempts. A1 and A2 are the approaches offered in the mentioned basically individual phase of the activity. After starting an active argumentation process, the participants of CSG noticed and warned others about the misleading and incorrect points they have encountered.

In the first argumentation step of AS-2 in the global argumentation structure of CSG (See Figure 4.36), CSG developed an incorrect justification for the construction of the orthocenter which was noted as A1. One of the participants of CSG associated the orthocenter with a previous approach in which they used the sides of the triangle as chords. What she drew with this intention in A1 was presented in Figure 4.37 given below.

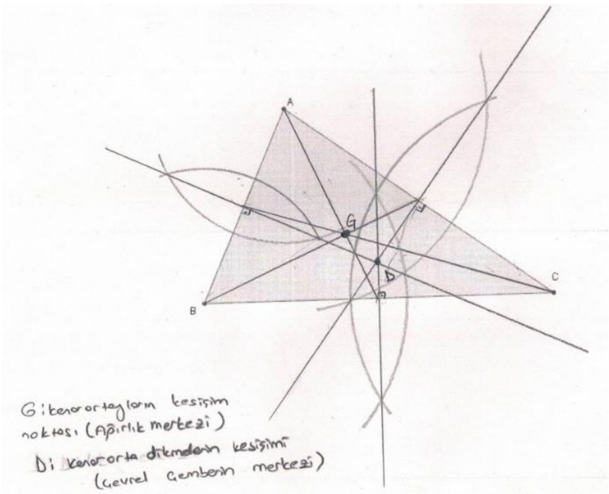


Figure 4. 37. Geometric figure CSG formed through A1 in Activity 3

As can be seen from Figure 4.37, the point D is the subject of A1. Although it also involves the point G which was described as the point of concurrency of the medians on the left side explanation, it was drawn later during the argumentation process. Since the participants continued to work on the same worksheet afterwards, the figure involves the point G, but it is not in the scope of A1. The participant who was assigned by CSG as working on the construction of the orthocenter set out an incorrect plan. More specifically, she set the compass more than half of $|BC|$ and then marked two arcs by placing the stationary point of the compass at the vertices B and C. Thus, she determined the intersection points of these arcs and drew a line passing through these points. Since she used the statement that the perpendicular bisector of a chord passes through the center of the circle, she accepted that the point of intersection of such two lines would give the orthocenter. She carried out the same procedure for other two sides of ΔABC. She missed the point that what they drew is the perpendicular bisectors of the sides of the triangle, not the altitudes of the triangle. Then, CSG realized the mistake related to A1 and offered another approach for the construction of the orthocenter which was marked as A4. Since it was a valid approach, it will be explained in detail in a further phase of this section.

Meanwhile, another participant of CSG was endeavoring the construction of the centroid of ΔABC. Since the centroid was not a familiar construction for CSG, the first approach offered with this purpose which was labeled as A2 could not fulfill the expectations. A2 was displayed in AS-1 of the global argumentation structure of CSG (See Figure 4.36). In this manner, the geometric figure related to A2 was shown in Figure 4.38.

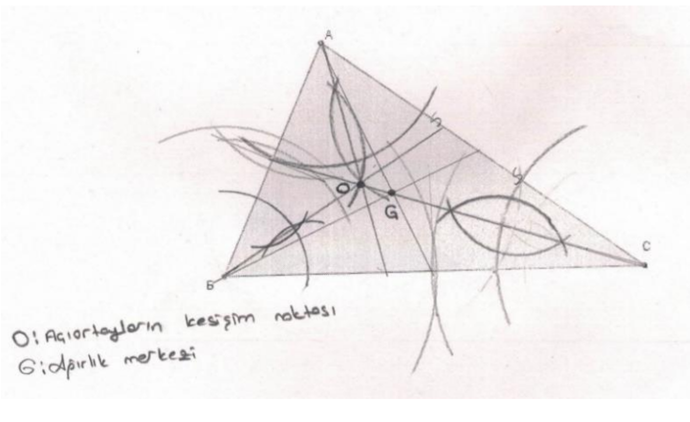


Figure 4. 38. Geometric figure CSG formed through A2 and A3 in Activity 3

From Figure 4.38, the product of both A2 and A3 can be examined. In other words, the point O which was formed as the point of concurrency of the angle bisectors of ΔABC was the product of A2 since CSG asserted that the centroid could be found via the angle bisectors in A2. To that end, one of the participants of CSG focused on the construction of the angle bisectors of ΔABC. She set the compass randomly, placed the compass needle to the vertex B, and marked an arc passing the adjacent sides which are $\overline{AB}$ and $\overline{BC}$. By this way, she determined the points of intersection of the arc and the stated sides. Although she did not name these points, she used them later on. She placed the compass on these points and drew two arcs inside of the triangle with the same compass setting. As it can be seen from the geometric figure given above, the line passing through the intersections of the lastly drawn two arcs also passed from the vertex B. Thus, she constructed the bisectors of $\angle ABC$. By following the explained procedure, she constructed the bisectors of other two interior angles of ΔABC. Afterwards, they concluded that the point of concurrency of the angle bisectors, which was named as O, is the centroid of ΔABC. However, another participant probed this conclusion and showed that the point O could not satisfy the property of the centroid. As background knowledge, they know that the centroid divides the related elements of the triangle, which are the angle bisectors in this case, in a ratio of 2:1. Based on this, CSG started to have second thoughts regarding the point O as well as A2. One of the participants mentioned that the length of the angle bisector between the point O and the vertex C was more than 2 times of the length of the angle bisector from the point O through $\overline{AB}$. Then, another participant stated that the possibility of the fact that the point of the concurrency of the medians of ΔABC would give the centroid. Since this is the subject of A3, it will be explained after the table of valid approaches (See Table 4.11).

In conclusion, CSG stated that A1 and A2 are invalid approaches for the construction of the stated points in Activity 3. As usual, the geometric figures formed by A1 and A2 were also evaluated according to the criteria list given in Table 4.5. However, each of them failed in the first criteria. The geometric figure presented via A1 asserted that it presents the orthocenter, but the point reached via A1 was the circumcenter. In this respect, the geometric figure in A1 was not a proper one in terms of the construction of the orthocenter. In a similar vein, it was asserted that the geometric figure formed via A2 gives the centroid. However, the constructed point was the incenter of ΔABC not the centroid of it. Therefore, A1 and A2 were labeled as invalid approaches based on their aims to construct.

Until that point, the approaches stated as invalid by CSG were explained. Thus, to present the valid ones, the following Table 4.11 was prepared.

Table 4. 11

Approaches CSG stated as valid for geometric construction in Activity 3

Approach for construction	**Validity of approach**
A3. CSG constructed the medians of ΔABC. Then, they concluded that the point of concurrency of the medians of ΔABC gives the centroid. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 3 - Construction of the centroid of ΔABC
	Based on the criteria list (See Table 4.5) - Valid approach for Activity 3 - Construction of the centroid of ΔABC
A4. CSG thought that the point of concurrency of the altitudes of ΔABC gives the orthocenter. Thus, they used the basic approach to construct the altitudes which is the construction of the line perpendicular to a given line from a point, not on the given line. Then, they concluded that the point of concurrency of the altitudes of ΔABC presented the orthocenter. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 3 - Construction of the orthocenter of ΔABC
	Based on the criteria list - Valid approach for Activity 3 - Construction of the orthocenter of ΔABC
A5. CSG noticed to use the sides of ΔABC as the chords of the circle. In more detail, they used the theorem “perpendicular bisector of a chord passes through the center of the circle” to find the circumcenter. Therefore, they drew the perpendicular bisectors of each side of ΔABC. They concluded that the point of concurrency of them presented the circumcircle. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC
	Based on the criteria list - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC

As a continuation of the concept of A2, A3 can be taken into consideration at this point. As stated before, one geometric figure was submitted as the product of the applications of A2 and A3 (See Figure 4.38). In addition to A2, A3 was also marked in AS-1 of the global argumentation structure of CSG (See Figure 4.36).

In accordance with what mentioned above in A2, CSG relied on the refutation based on the ratio 2:1 of the centroid. By taking into account the suggestion related to the medians, they attended to try to construct the medians of ΔABC so as to detect the centroid. CSG started to find the midpoints of $\overline{AB}$. For this purpose, one of the participants set the compass more than half of the length of this side so as to make arcs intersect. Then, she put the stationary point of the compass on the vertices A and B and drew two arcs crossing $\overline{AB}$. She drew a line from the intersections of these arcs. Then, the intersection of the lastly drawn line and $\overline{AB}$ presented the midpoint of $\overline{AB}$. As a final remark, she drew a line passing through the midpoint of $\overline{AB}$ and the vertex C which means the median of $\overline{AB}$. By chasing the same steps, she constructed the median of $\overline{BC}$. From the intersection of these two medians, she marked the point G in Figure 4.38. Therefore, they constructed the median of $\overline{AC}$ by drawing a line passing from the vertex B and the point G without pursuing the steps in the construction of the previous medians. At the beginning of the mentioned process, they doubted once whether the angle bisectors in A2 and the medians in A3 might give the same point of ΔABC, but postponed the interpretation of this idea to the end of the application of A3. Therefore, at the end of A3, they noticed that the mentioned points are different in ΔABC. All in all, CSG concluded that A3 is a valid approach in terms of the construction of the centroid of a triangle.

When it is returned to A1, it can be stated that A1 affected the flow of the argumentation in Activity 3. The participants of CSG who are constructing the orthocenter and the circumcenter were working on different worksheets simultaneously at the beginning part of the activity. As mentioned, in A1, she constructed the perpendicular bisectors of ΔABC and asserted that she has found the orthocenter. Thus, these two participants observed that they were finding the same point in ΔABC. Due to the mistake in A1, they came up with the supposition that the orthocenter and the circumcenter of ΔABC are the same point so that the centroid might also be at the same point. The construction of the centroid in A3 conflicted with this idea. By considering the result of A3, they decided to compare their worksheets and discussed how each point was constructed. This comparison showed them the mistake in A1, so they stepped back related to the mentioned supposition. Then, they started to think about the construction of the orthocenter again which was represented in the second argumentation step of AS-2 (See Figure 4.36). What they presented as the secondary idea for the construction of the orthocenter was labeled as A4.

As a result of the mentioned drawback, CSG concluded that the point of concurrency of the altitudes of ΔABC is the orthocenter. Therefore, they tried to remember how they had constructed the altitudes of the given triangles in Activity 2. Among three valid approaches in Activity 2, they remembered A5 which is actually the basic approach for the construction of a line perpendicular to a given line from a point not on that line (See Figures 4.32, 4.33, and 4.34). According to the occurrence order of approaches in Activity 3, the mentioned approach for the construction of the altitudes was named as A4. Based on this idea, CSG started to work in a new worksheet collaboratively. The geometric figure they presented related to A4 was given below in Figure 4.39.

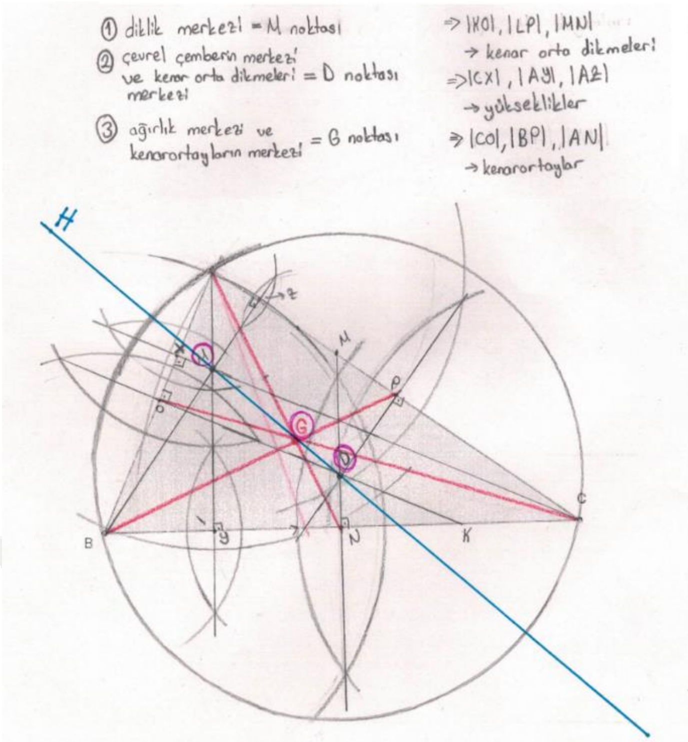


Figure 4. 39. Geometric figure CSG formed through A3, A4, and A5 in Activity 3

As seen, Figure 4.39 covers the constructions of all three points. In this geometric figure, CSG started with the application of A4 to construct the altitudes of ΔABC and then continued to the constructions of the other two points in the same worksheet. The subject of A4 is related to the construction of the point H which is also the one explained at this point.

CSG started with the construction of the altitude of $\overline{AB}$. By adjusting to A4, it can be stated that the given line refers to $\overline{AB}$ and the point not from on that line refers to the vertex C. The task about drawing a line perpendicular to the given line passing from a point not that line refers to the construction of the altitude of $\overline{AB}$. They set the compass to a proper distance that the arc drawn by placing the compass at the vertex C could intersect $\overline{AB}$ at two points. After determining these points on $\overline{AB}$, they drew two arcs passing over $\overline{AB}$. These points were used to place the fixed point of the compass. As can be seen from Figure 4.39, the line passing from the intersections from lastly drawn arcs presented the foot of the altitude of $\overline{AB}$ which was pointed out with the letter X. As the final step, they drew a line passing from the vertex C and the point X and this line constituted the altitude of $\overline{AB}$. By following the same procedure, they determined the point Y as the foot of the altitude of $\overline{BC}$ and constructed the altitude of it. Similarly, they determined the point Z as the foot of the altitude of $\overline{CA}$ and constructed the altitude of the last side. Finally, the point of concurrency of three altitudes was marked as the point M and this point was stated as the orthocenter of ΔABC. As expected, according to CSG, A4 was a valid approach in terms of the construction of the orthocenter of ΔABC. Since it was seen that the geometric figure reached via application of A4 fulfills all criteria listed in Table 4.5, it was decided that A4 is a valid approach for the construction of the orthocenter of ΔABC.

The final valid approach presented in Table 4.11 is A5 which is related to the construction of the circumcircle. In more detail, A5 was marked by means of the conclusion of AS-3 in the global argumentation structure of CSG in Activity 3 (See Figure 4.36). Until that point, it was seen that CSG focused on the construction of the orthocenter and the centroid rather than the circumcenter. After the conflictions emerged related to the orthocenter and the circumcenter, they wanted to check how the circumcircle of ΔABC was constructed. The geometric figure they formed via A5 was illustrated as in Figure 4.40.

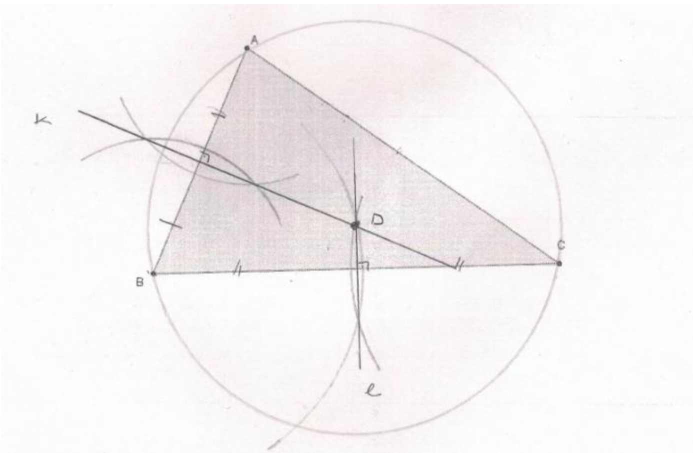


Figure 4. 40. Geometric figure CSG formed through A5 in Activity 3

Similar to A4, CSG got inspired from the valid approach in Activity 1. In other words, accepting the sides of ΔABC as the chords of the circle and drawing the perpendicular bisectors of them was an approach tried during Activity 1. In this activity, this idea was coded as A5. Regarding A5, one participant of CSG worked on it, but the final decision related to the validity of it was stated collaboratively. As seen from Figure 4.40, she dealt with the construction of the perpendicular bisectors of two sides of ΔABC which are $\overline{AB}$ and $\overline{BC}$ in the given order. In more detail, she set the compass more than half of $|AB|$, put the fixed point of the compass at the vertex A, and drew an arc as crossing $\overline{AB}$. By keeping the setting of the compass, she placed it at the vertex B and drew another arc. The line drawn from the intersection points of these arcs became the perpendicular bisector of $\overline{AB}$ and the line was symbolized as *k*. In a similar vein, she constructed the perpendicular bisector of $\overline{BC}$ and named the line as *l*. Then, the point of concurrency of the lines *k* and *l* was pointed out as the circumcenter of ΔABC and termed as the point D. As stated before, according to CSG, A5 is a valid approach for the construction of the circumcenter of ΔABC. Since the geometric figure presented above matches the criteria related to being a construction in Table 4.5, A5 was labeled as a valid approach.

Until this point, it can be observed that the constructions of three points were performed in different worksheets. Specifically, Figure 4.38 partially covers the construction of the centroid via A3, and Figure 4.39 includes the construction of the orthocenter via A4 at the first step, and Figure 4.40 involves the construction of the circumcenter via A5. Then, CSG decided to combine these construction approaches offered for three points in one worksheet, so they could think about the relationships/connections among these points effectively. In spite of starting to work all of them in a blank worksheet, they preferred to add the construction of two points into a worksheet which has already involved the construction of one of the intended points on it. In this regard, they focused on Figure 4.39 which has already covered the construction of the orthocenter and also started the construction of other points on it collaboratively. That is, CSG attended to transfer A3 and A5 to the worksheet on which

A4 was already applied. Because of this case, Figure 4.39 aligned with the explanation A4 covers the constructions of all points.

4.3.1.4. Approaches CSG Offered for Geometric Construction in Activity 4

At the beginning of Activity 4, the worksheets on which ΔABC was drawn and the pairs of compass-straightedge were distributed to the participants of CSG. As usual, the back-up worksheets and pack of tools were present at the table of CSG. In more detail, CSG was asked to mark random points X, Y, and Z on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively and then construct three circles; the first circle passes through the vertex A and the points X and Z, the second circle passes through the vertex B and the points Y and Z, and the third circle passes through the vertex C and the points Y and Z. Finally, they were asked to examine the relationships/connections among these three circles (See Appendix B). Moreover, ΔABC presented in the worksheets were shown in Figure 4.41.

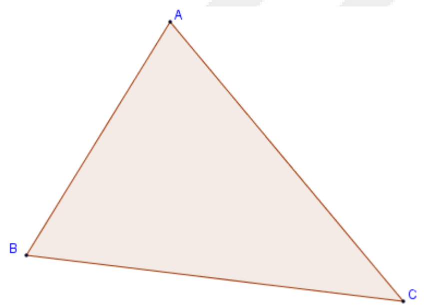


Figure 4. 41. ΔABC given in the worksheets of Activity 4

Different from the previous activities, CSG came up with only one approach in Activity 4. Rather than thinking about the alternative approaches for the construction of three circles, they searched for the relationships among these circles. As a matter of fact, they were expected to try more ideas for the construction in Activity 4 even if not reaching a valid result. The location of this approach in the global argumentation structure of CSG was displayed below.

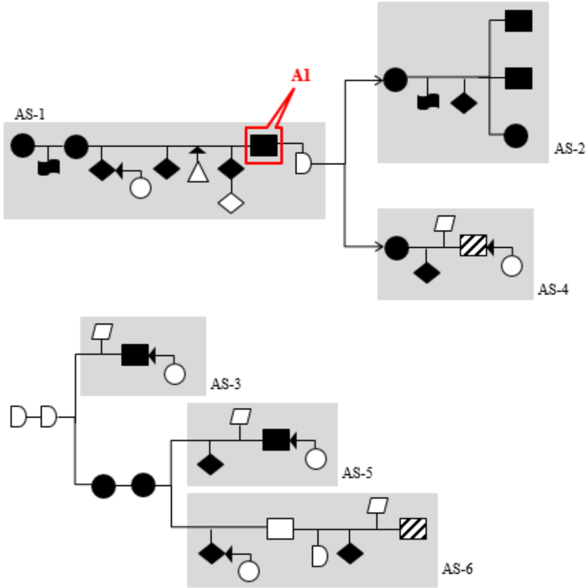


Figure 4. 42. The location of the approach offered for the construction in the global argumentation structure of CSG in Activity 4

What stands out in Figure 4.42 is that CSG could offer one approach for construction and it was marked with a red indicator. Moreover, A1 was marked by means of the conclusion component of AS-1 in the global argumentation structure of CSG. As usual, the summary of A1 and the related evaluations were given below in Table 4.12.

Table 4. 12

Approach CSG stated as valid for geometric construction in Activity 4

Approach for construction	**Validity of approach**
A1. CSG noticed that the construction of three circles corresponds to the construction of the circumcircles of triangles formed by the given three points. Therefore, they formed three triangles which are ΔAXZ , ΔBYZ , and ΔCZY . Then, they constructed the circumcircles of these triangles. *(written in the worksheet)*	*According to CSG* - Valid approach for Activity 4 - Construction
	Based on the criteria list (See Table 4.5) - Valid approach for Activity 4 - Construction

It can be considered that A1 is the repetition of an approach in Activity 1 three times consecutively. Actually, A1 was configured based on an already employed idea in Activity 1. As explained before, CSG offered A2 and A4 which are quite similar approaches for the construction of the circle passing from the vertices of a given triangle in Activity 1. Although CSG tried the construction asked in Activity 4 for differently placed X, Y, and Z points and different types of triangles, they always used A1 while constructing three circles. The explanation of CSG regarding A1 and all geometric figures formed via A1 were presented in Appendix F. The first figure CSG formed via A1 was indicated in Figure 4.43 given below.

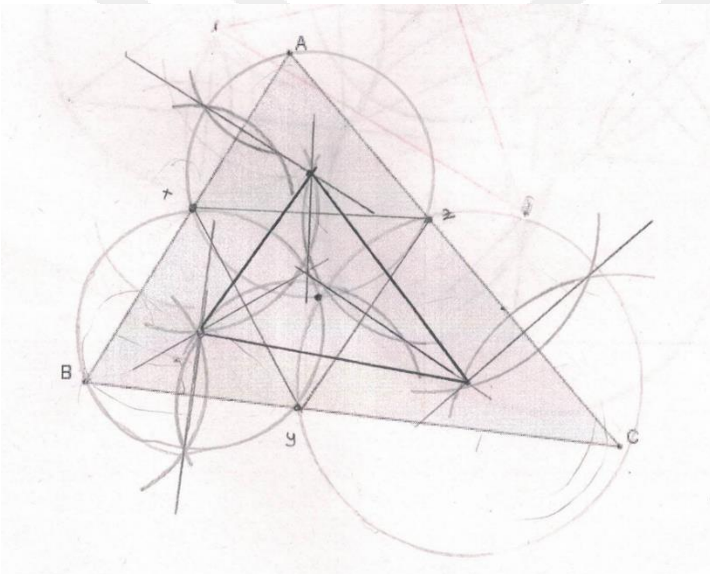


Figure 4. 43. Geometric figure CSG formed through A1 in Activity 4

At the beginning of Activity 4, some participants in CSG had difficulty in comprehending what was given and asked in Activity 4 for a while. Since the points X, Y, and Z were not placed on the sides in the distributed worksheets and participants were asked to place them. This was a questionable case for some of them at first. Then, the one who understood the activity at short notice delineated the possible X, Y, and Z points and also three circles to help others to visualize the geometric figure they intended to construct. The sketch facilitated the process of understanding the activity. They summarized the activity that they should construct three circles by considering the given information in the activity and then they should search for possible relationships among these circles.

As the first idea for construction, one of them noticed that these circles might be considered as the circumcircles of three triangles. Based on this issue, they also thought that the points X, Y, and Z should be placed in such a way that the circumcircles could be drawn. Therefore, they offered to mark the mentioned points near to the midpoints of the sides. However, a rebuttal was stated against this idea since the randomness property could not be provided in such a case. Afterwards, they decided to work on differently located X, Y, and Z points so that extra situation could be detected. For example, one of them placed the points near to the midpoints of the sides whereas another one placed them so close to the vertices.

As the first step in A1, CSG arranged the triangles which are ΔAXZ , ΔBYZ , and ΔCZY by drawing three line segments. They also considered the fact that the point of intersection of the perpendicular bisectors of the sides of the mentioned triangles presents the circumcenter of each triangle. They started with the construction of the circumcircle of ΔAXZ , that is the construction of the circle passing from the vertex A, the points X, and Z. To construct the perpendicular bisector of $\overline{AX}$, they set the compass more than half of the length of $\overline{AX}$, put the compass needle at the vertex A and the point X, and drew two arcs crossing $\overline{AX}$. The line drawn from the intersections of these arcs is the perpendicular bisector of $\overline{AX}$. By following the same steps, they constructed the perpendicular bisectors of $\overline{XZ}$. They accepted the point of concurrency of these lines as the circumcenter and then drew the circumcircle of ΔAXZ by using compass.

As the second triangle, CSG continued with the construction of the circumcircle of ΔBYX . Similar to ΔAXZ , they focused on the construction of the perpendicular bisectors of two sides of ΔBYX which are $\overline{BY}$ and $\overline{YX}$, determined the circumcenter, and constructed the circumcircle of ΔBYX . To construct the last circle in Activity 4, they determined the points of intersection of the perpendicular bisectors of $\overline{CZ}$ and $\overline{ZY}$ and then drew the circumcircle of ΔCZY . Thus, the geometric figure presented in Figure 4.43 was formed by constructing three intended circles. As seen from Table 4.12, CSG accepted A1 as a valid approach for the construction of the intended three circles in Activity 4. According to the criteria list given in Table 4.5, A1 is also a valid approach since the geometric figure formed via A1 (See Figure 4.43) fulfills all criteria.

Until that point, all approaches which CSG offered for the geometric constructions in the cognitive unity based activities were examined and reported. In the following section, the approaches mentioned by GG with the aim of construction throughout the same four cognitive unity based activities will be presented.

4.3.2. Approaches GG Offered for Geometric Constructions

The findings related to the approaches GeoGebra group (GG) stated for construction in the cognitive unity based activities were presented in this section. Approaches offered per cognitive unity based activity were explained under a separate sub-heading. As similar to the report of the findings of the approaches deduced from the geometric construction process of CSG in the previous section, how the activity was presented to groups and the nature of GeoGebra files accompanied to the activity were described as the first step. Thereafter, the correspondences of each approach in the global argumentation structures of GG were indicated by using figures (Figures 4.44, 4.53, 4.64, and 4.79). While the approaches accepted as invalid by GG were pointed out by means of blue indicators, the ones declared as valid by GG were marked with red indicators in the global argumentation structures for all activities. After that, some tables were prepared in order to summarize the approaches and present the evaluation of the validity of these approaches. The invalid approaches were firstly presented via tables (Tables 4.13, 4.15, and 4.18) and then the tables summarizing the valid ones were presented (Tables 4.14, 4.16, 4.17, and 4.19). After these tables, approaches were explained in detail by referencing to the geometric figures GG formed during activities and their usages of GeoGebra in order to explain how the third research question of the study was addressed in detail. Finally, the approaches by which GG declared that they performed a construction correctly were examined thoroughly based on the diagram presented in Figure 4.17 in order to explain the underpinnings of the evaluation process of the validity of the approaches.

4.3.2.1. Approaches GG Offered for Geometric Construction in Activity 1

In Activity 1, prospective middle school mathematics teachers who were allowed to use GeoGebra during the geometric construction were given two GeoGebra files as well as the worksheets. As stated in the worksheets, they were asked to construct a circle passing through the vertices of ΔABC. In both GeoGebra files and the worksheets, ΔABC was presented as ready to work on and what ΔABC looks like was displayed before in Figure 4.18. Moreover, GG was informed about using the GeoGebra files in the given order. In the first GeoGebra file, some tools which directly provide the construction of the intended geometric figure in the activity by way of one or a few clicks were removed from the toolbar. These tools were determined as 'circle through three points' and 'circumcircular arc'. When it was observed that GG reached to the fact that the intersection of the perpendicular bisectors of the sides of ΔABC is the circumcenter and constructed the circumcircle properly during the application of the activity, the participants were asked to conduct the same activity with the second GeoGebra file. In the second GeoGebra file, some extra tools were also removed from the toolbar. That is, the already restricted two tools were kept in the same way, three more tools which are 'midpoint or center', 'perpendicular line', and 'perpendicular bisector' were also removed from the toolbar. Due to the absence of these extra tools, it was expected that the construction would be more challenging for GG and also they would come up with different approaches for the construction of the intended circle in Activity 1.

After the analysis, it was noted that GG discussed eight approaches for the construction of the circle asked in Activity 1 which refers to the presence of more approaches compared to CSG in the same activity. To show where eight approaches located in the global argumentation structure of GG, Figure 4.44 was given as follows.

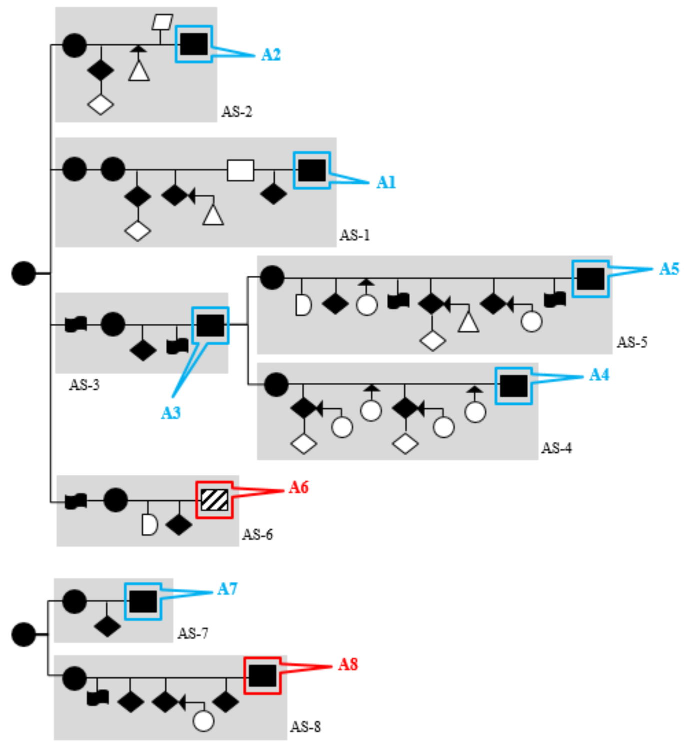


Figure 4. 44. The locations of approaches offered for construction in the global argumentation structure of GG in Activity 1

As it can be seen from Figure 4.44, six approaches which were concluded as invalid for the aimed construction by GG were marked by means of blue indicators and two approaches which were explained as valid by GG were marked with red indicators. Seven approaches were represented via the conclusion component and one of them was represented via the target conclusion component. There is not an approach showed via the data component since GG carried out all approaches until expressing a conclusion. Therefore, it can be stated that each argumentation stream ended up with a conclusion or a target conclusion which indicates an approach. Moreover, the order of approaches was arranged based on the occurrence. The content of the approaches labeled as invalid by GG and the evaluations of these approaches were indicated in Table 4.13.

Table 4. 13

Approaches GG stated as invalid for geometric construction in Activity 1

Approach for construction	**Validity of approach**
A1. GG thought that the centroid of ΔABC gives the center of the circle. Then, they drew the medians of ΔABC and determined the centroid of ΔABC. Since they noticed that the circle did not pass through all vertices, they resulted in that the centroid of the triangle is not the center of the intended circle. *(written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1 *Based on the diagram (See Figure 4.17)* - Invalid approach for Activity 1
A2. GG accepted that the interior angles of ΔABC are the inscribed angles of the circle. They wanted to use the following theorem that "the measure of an inscribed angle of a circle is one-half the measure of its intercepted arc". Thus, they aimed to multiply the measure of an interior angle by 2 and find the measure of intercepted arc. Since they did not find a tool serving their purpose, they gave up this idea. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1 *Based on the diagram* - Invalid approach for Activity 1 - Not finished
A3. GG thought that the point of intersection of the angle bisectors of ΔABC gives them the center of the circle. However, they observed that the mentioned point is the incenter of ΔABC, not the center of the intended circle. *(written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1 *Based on the diagram* - Invalid approach for Activity 1
A4. GG tried to find the radius of the circle by using Pythagoras theorem. Then, they offered to use the tool 'distance or length' to measure it directly. While trying this, they evoked that the determined distance is not the radius of the circle and did not continue on this idea. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1 *Based on the diagram* - Invalid approach for Activity 1 - Not finished
A5. GG drew a larger triangle by drawing lines parallel to the sides of ΔABC and also passing through each vertex of ΔABC. They wondered that the incircle of the larger triangle might be the circumcircle of ΔABC, and the vertices A, B and C would be the points of tangency of the incircle of the larger triangle. However, they observed that the previous assumption was not correct. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1 *Based on the diagram* - Invalid approach for Activity 1

Table 4. 13 (continued)

A7. GG started to work with the second GeoGebra file in which some extra tools were removed from the toolbar. To find the midpoints of the sides of ΔABC, they thought that they could measure the length of the sides of ΔABC and divide by 2 to find the midpoints. Although they stated that they might find the midpoints by using measurement, it was seen that they did not finish this method. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 1
	Based on the diagram - Invalid approach for Activity 1 - Not finished

What stands out in Table 4.13 is that GG offered six approaches inapplicable in terms of conducting a construction for Activity 1. Since any one of these approaches was saved as a GeoGebra file by GG, to explain them in detail, the simulated versions of the geometric figures that GG formed by applying these approaches were preferred to present due to the low-quality of visuals from the video recordings. Besides, GG did not write down most of these approaches in the worksheets, except A1 and A3. Their explanations in the worksheets regarding A1 and A3 were presented in Appendix F. More detailed descriptions of the approaches in Table 4.13 were presented as follows.

As the first idea in Activity 1, GG aimed to proceed by finding the radius and the center of the circle. In this respect, they assumed that the centroid of ΔABC might be the center of the circle they wanted to construct. This idea was marked as A1 in the conclusion component of AS-1 in the global argumentation structure of GG (See Figure 4.44). Since GG could not construct the asked figure in the activity via A1, they preferred not to save this approach as a GeoGebra file, but they wrote about the process of A1 in the worksheet which was presented in Appendix F. In addition, since there is not a saved GeoGebra file related to the application of A1, the screen capture from the video recordings, as presented in Figure 4.45, was aimed to use in this section at first. However, by considering the low-quality of such visuals, it was decided to present the simulated geometric figure formed by using A1, as displayed in Figure 4.46, in order to explain A1 thoroughly. This process was utilized while presenting the findings of the further cases in which related GeoGebra files were not saved by GG and presenting the geometric figure would be better for the clarification of the approach.

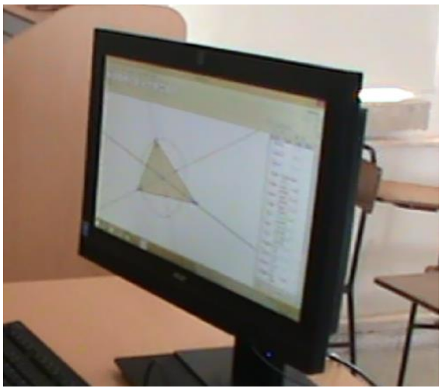


Figure 4. 45. The screenshot of GeoGebra related to A1 from the video recordings

By combining all data sources which are the worksheets of GG, the video recordings, and the audio recordings of GG in Activity 1, a simulated geometric figure of A1 was prepared in a way that given in Figure 4.46. By comparing Figure 4.45 and Figure 4.46, the equity of these figures can be observed.

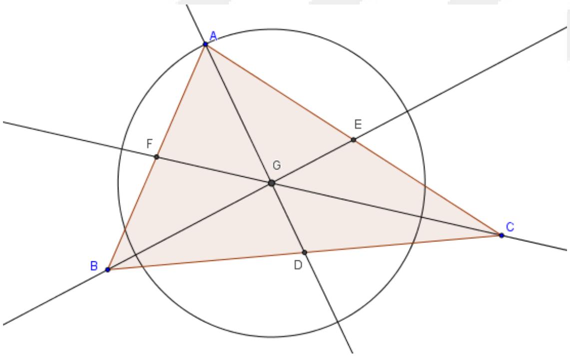


Figure 4. 46. The simulated geometric figure for A1 in the Activity 1

To find the centroid, GG endeavored to draw the medians of ΔABC. Then, they found the midpoints of the sides of ΔABC by using the tool 'midpoint or center' and named the midpoint of $\overline{BC}$ as D, the one of $\overline{CA}$ as E, and the one of $\overline{AB}$ as F. The next move of GG was to draw three lines, each of which was passing through one vertex and the midpoint of the opposite side. Thus, GG constructed three medians of ΔABC. To find the point of concurrency of them, they used the tool 'intersect' and also termed this point with the letter G. Since they expected the centroid to be the center of the circle, they decided to use the tool 'circle with center through point' to construct the circle. While using this tool, a center and another point, which specifies the radius, are needed. Therefore, they continued with the vertex A as accepting it the second point needed. Nevertheless, the circle that they drew did not pass through other vertices B and C which can be seen in Figure 4.46. They concluded that the centroid is not the center of the circle passing through the vertices of ΔABC. Moreover, GG did not consider A1 as a correct approach for the construction since they could not present the circle asked in the activity. According to the diagram given in Figure 4.17, the geometric figure formed via A1 failed in the first phase. That is, it was not a proper geometric figure in terms of what was asked in Activity 1 so that it was not considered as a construction and also A1 was labeled as an invalid approach.

After A1, A2 was pointed out in the conclusion of AS-2 in the global argumentation structure of GG (See Figure 4.44). Regarding A2, GG considered using the theorem "the measure of an inscribed angle of a circle is one-half the measure of its intercepted arc" so as to be able to construct the circle passing through the vertices of ΔABC. To that end, they accepted each interior angle of ΔABC as an inscribed angle of the intended circle. Then, GG measured the interior angles of ΔABC by using the tool 'angle'. Since the measures of these angles are half of the measures of their intercepted arcs, they aimed to multiply the measures of angles by 2. While one of the participants was multiplying the measures by 2, others were looking for a tool in GeoGebra to draw an arc by writing a particular measure. However, they could not find such a tool which could serve to their purpose in A2. Since they reached an impasse in terms of A2, they gave up this idea. Given that GG could not accomplish to present even a geometric figure via A2, they stated that it was not a working approach for the construction. Since there was not an absolute product as a result of A2, there was no need to use the diagram given in Figure 4.17. Moreover, GG preferred neither to write about A2 in their worksheets nor to save as a GeoGebra file. Therefore, all data about this approach were reached from the video recordings of the activity. Since they used GeoGebra while measuring the angles only, it was considered that any simulated figure for A2 was not required for the clarification of this approach.

Since the centroid did not work in A1, GG thought that the center of the circle might be the point of concurrency of the angle bisectors of ΔABC as the third approach. In Figure 4.44, A3 was shown via the conclusion of AS-3 of the global argumentation structure of GG. Due to the fact that they could not reach a valid geometric figure in A3, GG did not save their attempt in A3 as a GeoGebra file, but they wrote about it in the worksheet which can be seen in Appendix F. Similar to A1, the simulated geometric figure for A3 was presented as follows.

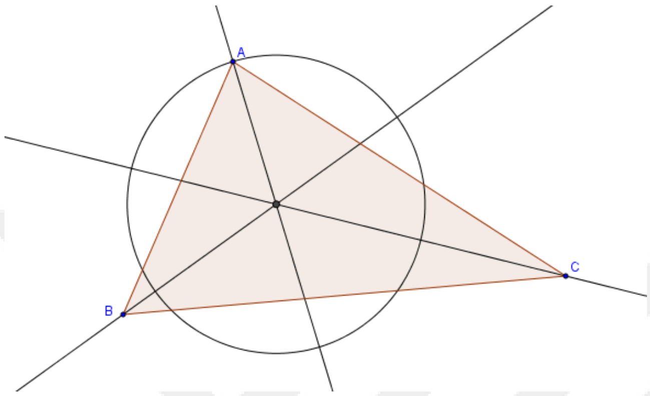


Figure 4. 47. The simulated geometric figure for A3 in the Activity 1

In accordance with the aim of A3, GG started to form the angle bisectors of ΔABC by using the tool 'angle bisector'. It was seen that GG had an inclination to check the tooltip appeared when the mouse cursor was placed over the tool icon before starting the configuration in GeoGebra. In this respect, GG selected the tool 'angle bisector' by clicking on it and then moved the cursor on the tool to look at the tooltip of it which is 'select three points or two lines'. Based on this instruction, GG constructed each angle bisector of ΔABC by clicking the related three points. Then, they determined the intersection of them by using the tool 'intersect'. Similar to A1, GG accepted the recently found intersection as the center and used the tool 'circle with center through point'. They used the vertex A again as the point that the tool asks to determine the radius. However, they observed that the point of concurrency of the angle bisectors was not also the center of the circle since the circle was not passing through all vertices of ΔABC as given in Figure 4.47.

In the meantime, one of the participants of GG recalled that this point might give the incenter. Although they noticed the relationship between the angle bisectors and the incircle, they did not pay attention to the proper construction of the incircle. In conclusion, A3 was not accepted a valid approach for the construction in Activity 1 by GG. Similarly, according to the diagram prepared to evaluate the approaches of construction (See Figure 4.17), it was noted that the figure formed via A3 does not fulfill the first criteria since it was not a proper one in terms of the construction of the intended circle in Activity 1.

After GG worked related to the angle bisectors of ΔABC as A3, they derived two argumentation streams which are AS-4 and AS-5 originated from AS-3. As indicated in Figure 4.44, the fourth approach was handled throughout AS-4 and the fifth approach was discussed in AS-5. Since A4 and A5 originated from the idea in A3, it can be considered that A4 and A5 are like the spinoffs of A3. In more detail, in A4, GG behaved as if they were sure about the case that the point of intersection of the angle bisectors is the center of the intended circle and the problem is to find the measure of the radius. This stance was totally incorrect since the center of the circle passing from the vertices of ΔABC is not the point of intersection of the angle bisectors. Although they noticed and talked about it recently in the previous argumentation stream, they led themselves such a discussion but gave up A4 in a comparatively quick manner. Since GG neither saved this idea as a GeoGebra file nor submitted the worksheet they used while working on it, the simulated geometric figure for A4 which was formed by inspecting the video recordings was presented in Figure 4.48 to be able to present the details of A4.

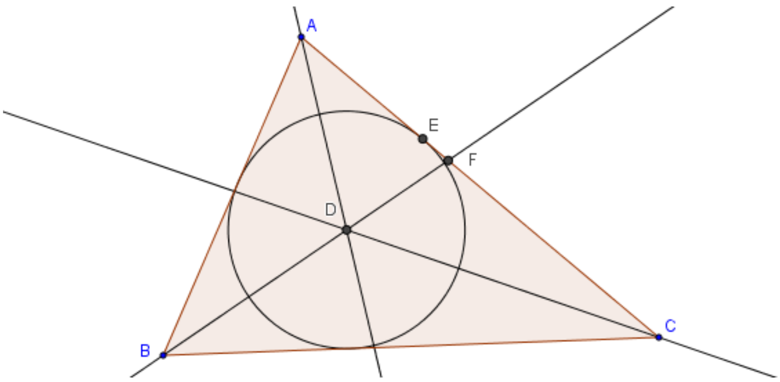


Figure 4. 48. The simulated geometric figure for A4 in the Activity 1

As the first step of A4, they tried to find the radius of the circle by using Pythagoras theorem. As illustrated in the simulated figure of A4 in Figure 4.48, GG thought that $|DC|$ is the radius of circumcircle and planned to use $|DF|^2 + |FC|^2 = |DC|^2$ to find $|DC|$. Then, they noticed that F is not the point of tangency of incircle. Therefore, they planned to find the tangency point E by drawing a perpendicular line from D to $\overline{CA}$, to find $|DC|$ by using Pythagoras theorem, and then to use the tool 'circle with center and radius' while drawing circumcircle. However, they could not apply it in GeoGebra, they just sketched this idea on the worksheet. In the meantime, before going further on this idea in practice, one of them offered to use the tool 'distance or length' to measure $|DC|$ directly. When they attempted to use this tool, they noticed that $|DC|$ is not the radius of the circumcircle and did not continue on this idea. As expected, GG did not label A4 as a valid approach for the construction asked in Activity 1. Since there was not even a proper figure presented at the end of A4, the diagram in Figure 4.17 was not applicable.

Another approach originated from A3 is A5. Since GG did not write A5 in the worksheets and also did not save a GeoGebra file related to this approach, another simulated figure was prepared and presented in Figure 4.49 for the ease of interpretation of A5.

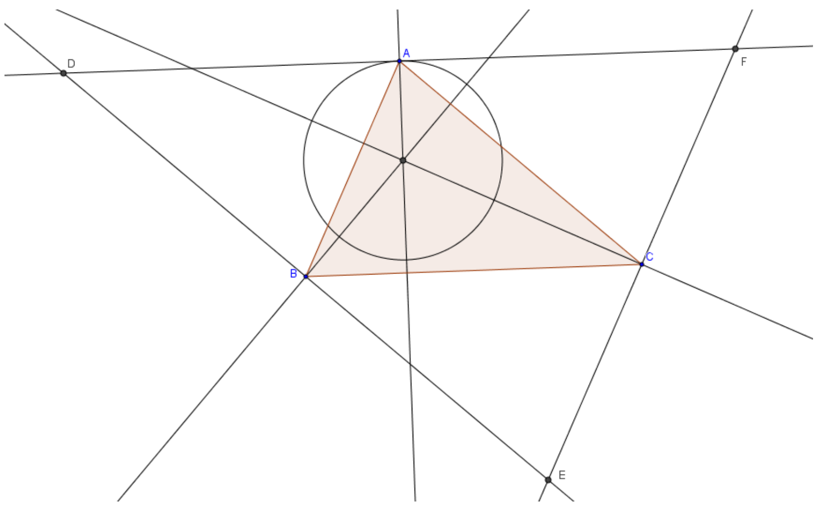


Figure 4. 49. The simulated geometric figure for A5 in the Activity 1

In A5, GG wanted to draw a larger triangle (See ΔDEF in Figure 4.49) by drawing lines passing through each vertex of ΔABC as parallel to the opposite sides. They wondered that the incircle of the larger triangle might be the circumcircle of ΔABC and the vertices A, B, and C might be the tangency points of the incircle of the larger triangle. To that end, GG mentioned drawing the perpendicular lines to each side of the larger triangle passing through the vertices A, B, and C in order to find the center of the incircle of the larger triangle. The reasoning behind that idea is to use the statement "the tangent is perpendicular to the radius at the point of tangency". In other words, a line passing from the center referred to the radius in the statement so that it was deduced that the tangent and a line passing from the center and the point of tangency are perpendicular to each other. Based on this warrant, they wanted to draw the lines perpendicular to the sides of the larger triangle and passing from the points A, B, and C. However, the point of concurrency of the recently stated perpendicular lines cannot always be the incenter of the larger triangle since it is not the point of concurrency of the angle bisectors of the larger triangle. In addition, while drawing the perpendicular line, GG had some difficulties because they tried to use the tool 'angle with given size'. They figured out that they could not draw a robust figure under dragging with this tool. By the guidance of the instructor, they attended to use the tool 'perpendicular line' since it was not a restricted tool in the first GeoGebra file. Hereby, they constructed the perpendicular lines and determined the intersection as presented in Figure 4.49. With an expectation of finding the incircle of the larger triangle as well as the circumcircle of ΔABC, they drew a circle by accepting the previously found intersection point as center and the distance between this point and the vertex A as the radius. However, they observed that the intersection of these perpendicular lines did not give the incenter of the larger triangle. As a result, GG did not label this approach as a valid one for construction. Similarly, the figure formed by A5 was evaluated based on the diagram (See Figure 4.17). However, it failed in the first phase since it was not a proper geometric figure when compared the construction asked in Activity 1. In other words, the figure GG formed by applying A5 did not offer even a circle passing through the vertices of ΔABC.

Following A5, GG offered A6 and concluded that it was a valid approach for the construction asked in Activity 1. Therefore, it will be explained later in this section in detail. Now, it is the turn of the last invalid approach situated in Table 4.13 which is A7. After A6, GG started to work on the secondly given GeoGebra file. While working with this new GeoGebra file, GG offered two ideas for construction, one of which was marked as A7 in the conclusion of AS-7 and the other one was signed as A8 via the conclusion of AS-8 (See Figure 4.44).

In A7, GG tried to find an approach to detect the midpoints since the tools 'midpoint or center' and 'perpendicular bisector' were removed from the toolbar. As the first issue, they thought that they could measure the length of the sides of ΔABC and divide them by 2 so that they anticipated finding the midpoint by using measurement. Although they stated that they might find the midpoint by using measurement, it was seen that they did not finish this idea. They measured the lengths of three sides by using the tool 'distance or length' and then divided these numbers by 2 via the calculator to have the exact numbers since the tool gave them decimal numbers as the lengths of the sides. However, at this point, they stopped and explained to the instructor what they considered in A7. Due to their hesitation, they stated that A7 might work, but they could find a more precise approach and moved on another idea. Besides, GG did not write about A7 on the worksheets and did not submit a GeoGebra file related to it. Since GG did not ensure the validity of A7 in terms of the intended construction, A7 was also listed among the invalid approaches. Moreover, the diagram in Figure 4.17 was not applicable because of the absence of the geometric figure clearly presented by implementing A7.

To sum up, six approaches presented in Table 4.13 were considered as invalid in terms of construction aimed in Activity 1. As stated, GG categorized two approaches as valid in Activity 1 which are A6 and A8. Moreover, GG declared that these approaches resulted in the geometric figures which can be termed as construction. These approaches were summarized in another table given below.

Table 4. 14

Approaches GG stated as valid for geometric construction in Activity 1

Approach for construction	**Validity of approach**
A6. GG drew the perpendicular bisectors of the sides of ΔABC by considering that the intersection of them might give the center of the intended circle. Finally, they concluded that the point of intersection of the perpendicular bisectors of the sides of ΔABC is the circumcenter. *(written in the worksheet, GeoGebra file 1)*	*According to GG* - Valid approach for Activity 1 - Construction
	Based on the diagram (See Figure 4.17) - Valid approach for Activity 1 - Construction (CTB)
A8. GG attempted to construct the perpendicular bisectors as if they were using compass-straightedge. Thus, they drew the perpendicular bisectors by using some tools such as 'circle with center and radius' and 'intersect'. Finally, they used the tool 'circle with center through point' to draw the circumcircle of ΔABC. *(written in the worksheet, GeoGebra file 2)*	*According to GG* - Valid approach for Activity 1 - Construction
	Based on the diagram - Valid approach for Activity 1 - Construction (CTA)

Table 4.14 gives the summary of A6 and A8. While A6 was offered for the first GeoGebra file, A8 was offered for the second GeoGebra file. Moreover, GG noted these approaches in the worksheets which can be seen in Appendix F and also saved as a GeoGebra file per approach. The GeoGebra file saved for A6 was named as 1.ggb whereas the GeoGebra files used in A8 was saved as 2.ggb. The approaches presented in Table 4.14 were expanded on as follows by associating with the reports of GG in the documents and the geometric figures they formed via GeoGebra.

After summarizing their trials to determine the center of the circle by means of finding the points of concurrency of both the medians in A1 and the angle bisectors in A3 which was followed by the extensions A4 and A5, GG maintained the idea of finding the center and noticed that they did not work on the lines drawn as perpendicular to the sides of ΔABC. The next step in the argumentation was about deciding whether they would draw the altitudes or the perpendicular bisectors of the sides which are both perpendicular to the sides in fact. Then, based on their previous knowledge, they decided to continue with the construction of the perpendicular bisectors of the sides of ΔABC and thought that the intersection of them might give the center of the circle. This idea was coded as A6 and marked at the target conclusion component in AS-6 of the global argumentation structure of GG (See Figure 4.44). Before explaining how they utilized A6 in detail, the screenshot of the first GeoGebra file GG saved (1.ggb) was presented in Figure 4.50 as follows.

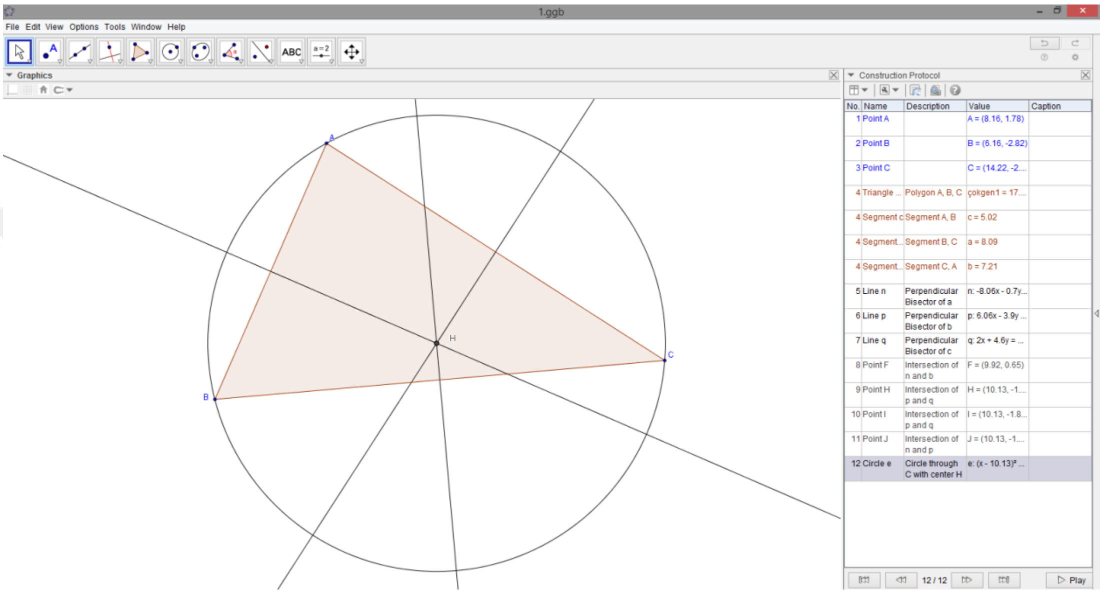


Figure 4. 50. GeoGebra file GG submitted for A6 in Activity 1 (1.ggb)

While applying A6, GG constructed the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively by using the tool ‘perpendicular bisector’. Then, they used the tool ‘intersect’ to find the point of concurrency of them and named it with the letter H. Then, they checked whether this intersection is the center or not by using the tool ‘circle with center through point’. As the name suggests, this tool needs another point in addition to the center H so that they selected the vertex C as the second point. Finally, they concluded that the intersection of the perpendicular bisectors of the sides of ΔABC gave them the center of the circle and this circle is also the circumcircle of ΔABC. The evaluation process based on the diagram in Figure 4.17 will be presented in detail after the explanation of the last approach.

As stated before, since it was encountered that A6 was an effective approach in terms of the construction during the application of the activity by the researcher, they were informed about using the second GeoGebra file in order to construct the same geometric figure when they were ready to work on it. Thus, it can be stated that the last approach offered by GG for the construction of the intended circle in Activity 1 was derived as a consequence of the restriction of the tools in GeoGebra. It was pointed out as A8 in the conclusion of AS-8 of the global argumentation structure of GG (See Figure 4.44). The screenshot of the GeoGebra file submitted at the end of A8 (2.ggb) was displayed in Figure 4.51 given below.

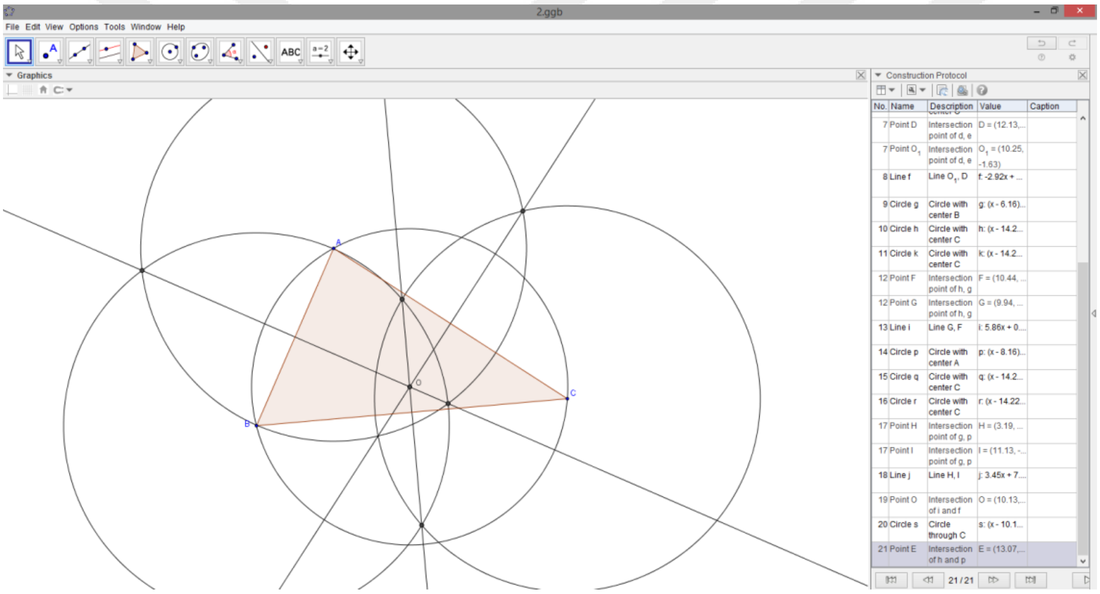


Figure 4. 51. GeoGebra file GG submitted for A8 in Activity 1 (2.ggb)

At the beginning of A8, by the help of the guidance provided by the instructor, GG endeavored to remember how they constructed the perpendicular bisector with compass-straightedge in the teaching sessions at the beginning of the course. What they remembered was that they were drawing two circles or arcs, detecting the intersections, and combining these points with a line. Thus, they started to draw circles and then noticed that they made a mistake about the center of the circles. They took centers as random points for the first attempt and then decided that the centers should be the vertices of ΔABC. While drawing the circles, they decided to use the tool 'circle with center and radius' by arranging the centers as the vertices and typing the radius as 5. Thus, all circles drawn by aiming to draw the perpendicular bisectors of the sides became equal. Then, GG used the tool 'intersect' to determine the intersections of the circles. After that, they drew the perpendicular bisectors by drawing lines from the points of intersection of adjacent circles. By using the tool 'intersect' again, the point of intersection of three perpendicular bisectors of the sides was displayed and named with the letter O. Finally, they constructed the circumcircle of ΔABC by accepting the point O as center and other required point as C via the tool 'circle with center through point'. CSG concluded that A8 was also a valid approach since they could come up with the geometric figure illustrated in Figure 4.51.

To sum up, GG submitted two GeoGebra files which were performed via A6 and A8 as involving constructions. The approaches which GG labeled as giving a construction were evaluated based on the diagram given in Figure 4.17. Although the results of the mentioned evaluation regarding the validity of A6 and A8 was presented in Table 4.14, the underpinnings of the evaluations in conjunction with the diagram were explained and the types of the constructions that geometric figures belong to were determined as follows.

To begin with, both figures in GeoGebra files were categorized as proper to the construction in Activity 1 since they actually present a circle passing through the vertices of ΔABC. Since the answer to the question in the first phase in the diagram is yes for both of them, the diagram led to the second phase which asks whether the geometric figure presented by GG passes the drag test criterion or not. Starting from this phase, A6 and A8 were explained separately.

When the geometric figure presented in the first GeoGebra file by the application of A6 was checked whether it passes the drag test criterion or not, it was seen that it stayed robust under dragging. This was an expected situation since the geometric figure was constructed by using the proper tools of GeoGebra. Thus, it was stated that it passed the drag test criterion. As given in the third phase of the diagram, this answer led to check whether the geometric figure presented passes the compatibility criterion or not. By controlling through the construction protocol, video recordings, and the worksheets, it was concluded that GG did not behave as if they were using compass-straightedge and did not follow the Euclidean restrictions in A6. Thus, it was concluded that the figure could not pass the compatibility criterion. Based on this final answer, it was seen that the geometric figure reached construction type B (CTB) in the diagram.

According to the examination of the geometric figure in the second GeoGebra file which was formed as a result of the application of A8, it was figured out that it could not stay robust under dragging. Thus, it did not fulfill the entailments of the drag test criterion. For example, when the point C was dragged to the right, the circumcircle presented in Figure 4.51 disappeared. Since the answer is no at the second phase, this answer led to the question whether the dragging caused to violate the assumptions GG used during the construction process and it is the only reason for the failure in the drag test. To that end, the hidden elements used during the construction in the second GeoGebra were shown and their movements under dragging were examined thoroughly. To examine the background of the disappearance of the circumcircle in the example, the same dragging which is dragging of the vertex C to the right was conducted. At the time of the disappearance of the circumcircle, it was seen that the circle drawn with the center C was not intersecting with other two circles anymore. Thus, the perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ could not be constructed because the related circles were not intersecting. The mentioned case was presented in Figure 4.52 as noted below.

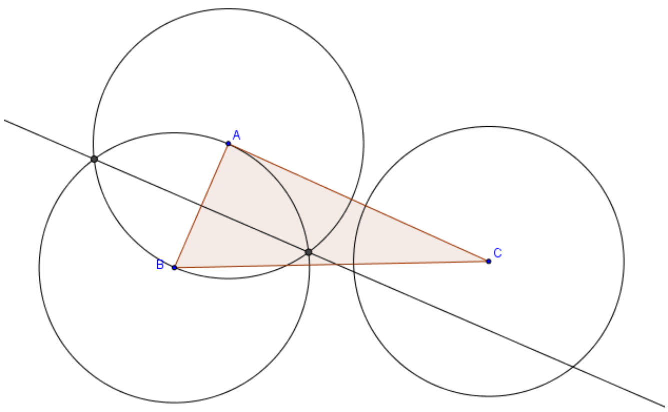


Figure 4. 52. The example for the assumption violation in the geometric figure formed via A8 in Activity 2

To be sure about such violation of the assumptions is the only reason for the failure in the drag test, the further dragging attempts were carried out. However, any case different than the assumption violation was not found in the process. Therefore, it was concluded that the presence of the violation of the assumptions used in the construction process under dragging is the only reason for the failure in the drag test. This final answer let the figure in the second GeoGebra to be categorized in construction type A (CTA).

4.3.2.2. Approaches GG Offered for Geometric Construction in Activity 2

Three triangles which were given in the separate worksheets were presented in the previous section while explaining the approaches proposed by CSG for construction (See Figure 4.24). In short, GG was given both three worksheets like CSG and also three GeoGebra files, each of which involves one type of triangle. As stated before, among three triangles, ΔDEF is an acute triangle, ΔABC is an obtuse triangle, and ΔKLM is a right triangle. GG was asked to construct the altitudes of these triangles and the orthocenters of them if they exist by using the given GeoGebra files. The tool 'perpendicular bisector' was restricted in all GeoGebra files since it was considered as a tool which directly provides the construction of the intended geometric figure in the activity.

From the argumentation of GG while dealing with Activity 2, it was detected that there are three approaches stated for the construction of the altitudes of the given triangles. Some of these approaches involve dimensions which are A1a, A1b, A2a, A2b, and A2c due to the implementation of the same approach to the given different types of triangles. To depict where the approaches emerged in the argumentation process of GG, Figure 4.53 was given below.

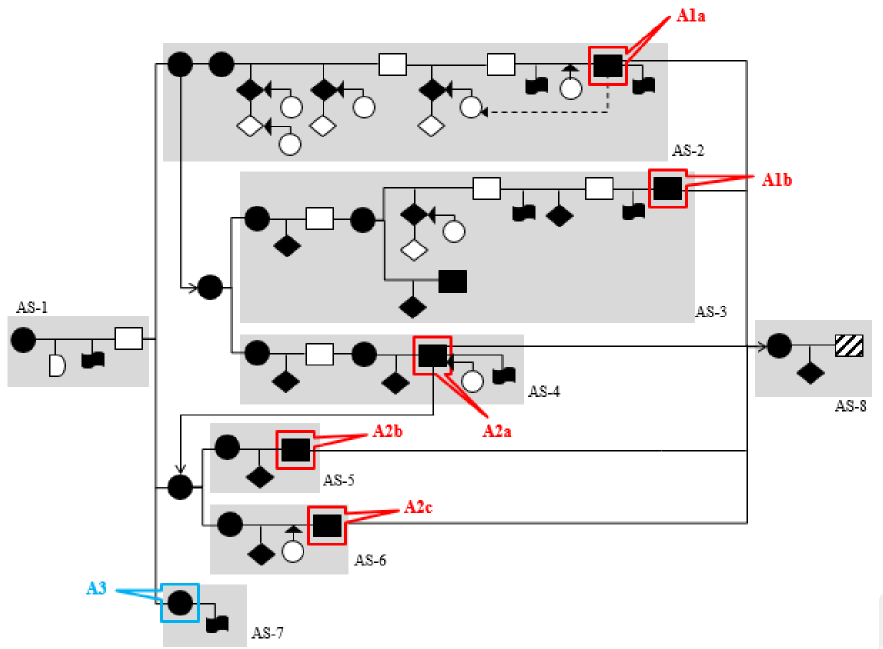


Figure 4. 53. The locations of approaches offered for the construction in the global argumentation structure of GG in Activity 2

Within Figure 4.53 is contained three approaches, two of which were accepted as valid by GG and signed with red indicators and one of which was declared as an invalid approach and marked with a blue indicator. Specifically, A1a, A1b, A2a, A2b, and A2c which are the ones stated as valid by GG were pointed out by the virtue of the conclusion components since they ended up with the products, namely, the geometric figures in GeoGebra files. On the other hand, A3 which was stated as invalid by GG was signed by means of the data component since this approach could not be carried out completely and it stayed as just an idea. As the first phase, A3 was summarized in Table 4.15 and then the remaining approaches were explained in Table 4.16.

Table 4. 15

Approach GG stated as invalid for geometric construction in Activity 2

Approach for construction	**Validity of approach**
A3. GG thought about how they would construct if they are using compass-straightedge. Thus, they mentioned about the construction of a line perpendicular to a given line through an external point. However, this approach could not go further. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 2
	Based on the diagram (See Figure 4.17) - Invalid approach for Activity 2 - Not finished

As indicated in Figure 4.53, A3 is the third approach GG offered and took place in AS-7 of the global argumentation structure of GG in Activity 2. The table above gives information about A3 and also involves the remarks about the validity of it. GG was inclined to consider how they could perform the construction if they were using compass-straightedge instead of GeoGebra as an alternative method. Since all groups in the course participated into the teaching sessions regarding the basic constructions by using both compass-straightedge and GeoGebra, it can be stated that each group was accustomed to the use of all mentioned tools in the study. In this manner, GG tried to remember how they constructed a line perpendicular to a given line through an external point by using compass-straightedge at the beginning of the course. Their purpose in searching for this issue was to adapt this approach to the available tools in GeoGebra. When GG got stuck at the data component of A3, the instructor gave the clues by mentioning about drawing arcs and the similarity between the intended construction in A3 and the construction of the perpendicular bisector or the midpoint of a line segment they performed in Activity 1. Nevertheless, GG could not proceed in A3 and gave up after discussing about it quickly. The reason of the case that GG did not insist on finding the entire process for applying A3 might be that they thought that they already found two valid approaches which are A1 and A2 and checked them for the given different types of triangles. Since they did not go further in A3, there is neither a worksheet nor a GeoGebra file about it which can be presented at this point.

In this manner, A3 was concluded as an invalid approach since there was not a GeoGebra file related to A3 to be used to examine via the diagram in Figure 4.17.

Having explained what A3 means, it is time to continue with the approaches GG accepted as valid for the construction in Activity 2. Table 4.16 showed the mentioned two approaches and their components as follows.

Table 4. 16

Approaches GG stated as valid for geometric construction in Activity 2

Approach for construction	**Validity of approach**
A1a. GG drew a circle by accepting $\overline{EF}$ of ΔDEF as the diameter. Since they know that inscribed angle subtended by a diameter is a right angle, they determined the intersection points of the circle and other sides of ΔDEF and considered these intersection points as the feet of the altitudes. Then, they determined the intersection of two altitudes and also assumed that the third altitude should also pass from the intersection. Based on this, they drew a line passing through the intersection of two altitudes and the last vertex for reaching to the third altitude. *(written in the worksheet, GeoGebra file 1a)*	*According to GG* - Valid approach for Activity 2 - Construction *Based on the diagram (See Figure 4.17)* - Invalid approach for Activity 2 - Not a construction
A1b. GG adapted the idea in A1a for ΔABC. In more detail, they accepted $\overline{BC}$ and $\overline{AB}$ as the diameters and drew two circles. By determining the intersections of these circles with the sides, they found the feet of the altitudes. They also changed the idea of drawing the third altitude in A1a. *(written in the worksheet, GeoGebra file 1b)*	*According to GG* - Valid approach for Activity 2 - Construction *Based on the diagram* - Invalid approach for Activity 2 - Not a construction
A2a. GG offered a new approach while working on ΔKLM. They reflected the vertex L across $\overline{MK}$ and named the reflected point as L′. Then, they presented the altitude of $\overline{MK}$ by drawing a line passing through these points. They did not need to draw the altitude of other sides since they stated them as apparent in a right triangle. *(written in the worksheet, GeoGebra file 2a)*	*According to GG* - Valid approach for Activity 2 - Construction *Based on the diagram* - Valid approach for Activity 2 - Construction (CTB)

Table 4. 16 (continued)

A2b. GG adapted the idea in A2a to ΔDEF. After reflecting the vertices, they drew the line segments between the vertices and the reflected versions of them. They declared that these line segments are the altitudes of ΔDEF. *(written in the worksheet, GeoGebra file 2b)*	*According to GG* - Valid approach for Activity 2 - Construction *Based on the diagram* - Invalid approach for Activity 2 - Not a construction
A2c. GG adapted the idea in A2a to ΔABC. After reflecting the vertices, they preferred to draw the lines between the vertices and the reflected versions of them. They declared that these lines are the altitudes of ΔABC. *(written in the worksheet, GeoGebra file 2c)*	*According to GG* - Valid approach for Activity 2 - Construction *Based on the diagram* - Valid approach for Activity 2 - Construction (CTB)

What is interesting about the approaches is in Table 4.16 is that there are two constituents of A1 and three constituents of A2. That is to say, A1a refers to the use of A1 in ΔDEF and A1b covers the application of A1 in ΔABC. In a similar vein, regarding the application of A2, ΔKLM was worked in A2a, ΔDEF was focused in A2b, and lastly ΔABC was the subject of A2c. Another point important to underline related to Table 4.16 is that it was prepared based on the evaluation of GG during the argumentation process of Activity 2. Thus, some disagreements in the validity of these approaches were observed. Moreover, GG saved five GeoGebra files in Activity, each of which was related to one trial given in Table 4.16. The descriptions of GG related to the mentioned five cases were placed in Appendix F. What follows are the in-depth descriptions of the mentioned approaches.

When GG started to work on Activity 2, their first action was to discuss where the altitudes of the given triangles could be and the relevance of the altitudes and the orthocenter in a triangle. Then, they started to work on the construction of the altitudes of ΔDEF. As the first approach, similar to CSG in Activity 2, GG also deduced a construction approach from the statement "the inscribed angles subtended by a diameter are right angles". A1a was tried by GG through AS-2 of the global argumentation structure of GG in Activity 2 (See Figure 4.53). The following figure presents the screenshot of the GeoGebra file GG saved at the end of A1a (1a.ggb).

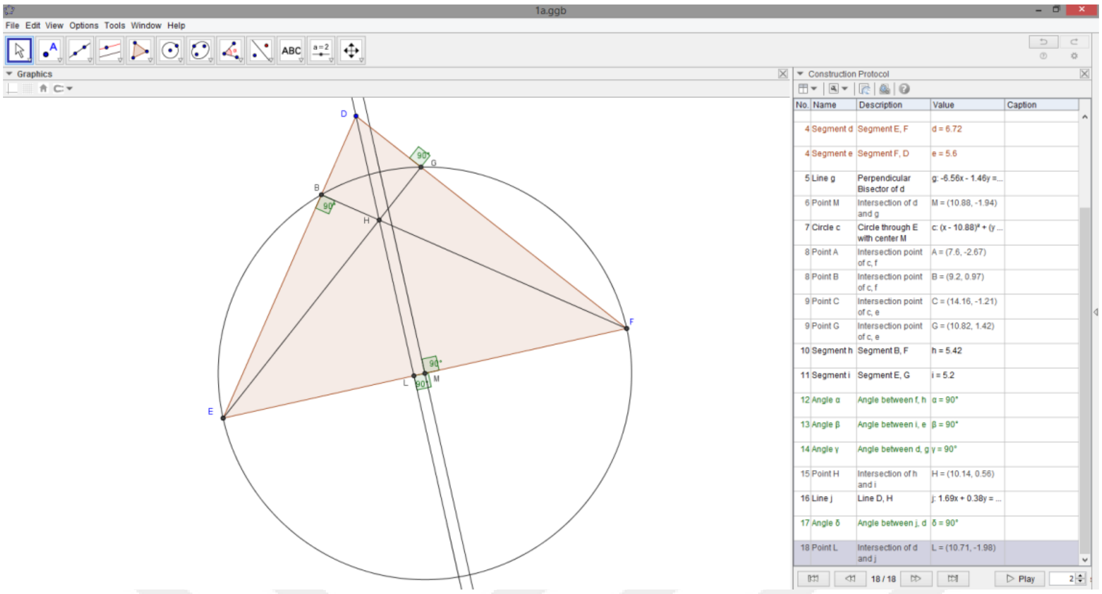


Figure 4. 54. GeoGebra file GG submitted for A1a in Activity 2 (1a.ggb)

GG decided to accept $\overline{EF}$ as the diameter of the circle so they needed a center and a radius. By using the tool ‘perpendicular bisector' instead of the tool ‘midpoint or center’, they determined the center as the point M on $\overline{EF}$. Then, GG drew a circle through E with the center M by using the tool ‘circle with center through point’. After that, they determined the intersections of the circle and other sides of ΔDEF which are $\overline{DE}$ and $\overline{FD}$ by using the tool ‘intersect’. They also named the intersection of circle and $\overline{DE}$ as B and also the intersection of circle and $\overline{FD}$ as G. The next move of GG was to draw the line segments between the points E and G to present the altitude of $\overline{FD}$. Likewise, they drew another line segment between the points B and F so as to construct the altitude of $\overline{DE}$. To check whether these line segments are perpendicular to the mentioned sides, they measured the angles via the tool ‘angle’ and confirmed the perpendicularity of them. At that point, GG noticed that the orthocenter might be the intersection of two altitudes that they recently drew. By using the tool ‘intersect’ again, they determined the aforementioned intersection and named it as H. Based on the presumption that H is the orthocenter, they accepted directly that the altitude of $\overline{EF}$ should also pass through the point H. To that end, GG drew the line passing from the vertex D and the point H and signified this line as the altitude of $\overline{EF}$. They checked whether this line is perpendicular to $\overline{EF}$ by measuring the angle via the tool ‘angle’ again. However, one of the participants of GG hesitated about why they accepted H as the orthocenter and the construction of the last altitude could be done differently. To support this unsure issue, the instructor asked them whether they checked the figure by dragging or not. Despite the fact that the geometric figure did not stay robust under dragging, GG could not grasp this situation and accepted as a construction without putting intense thought on it. The evaluation of A1a based on the diagram given in Figure 4.17 will be presented after the descriptions of all approaches.

Since GG was convinced about the validity of A1a, they continued to apply this approach for another given triangle which is $\triangle ABC$. Thus, this trial was signed as A1b through AS-3 of the global argumentation structure of GG (See Figure 4.53). The screenshot from the GeoGebra file which GG saved by means of the application of A1b (1b.ggb) was displayed in Figure 4.55.

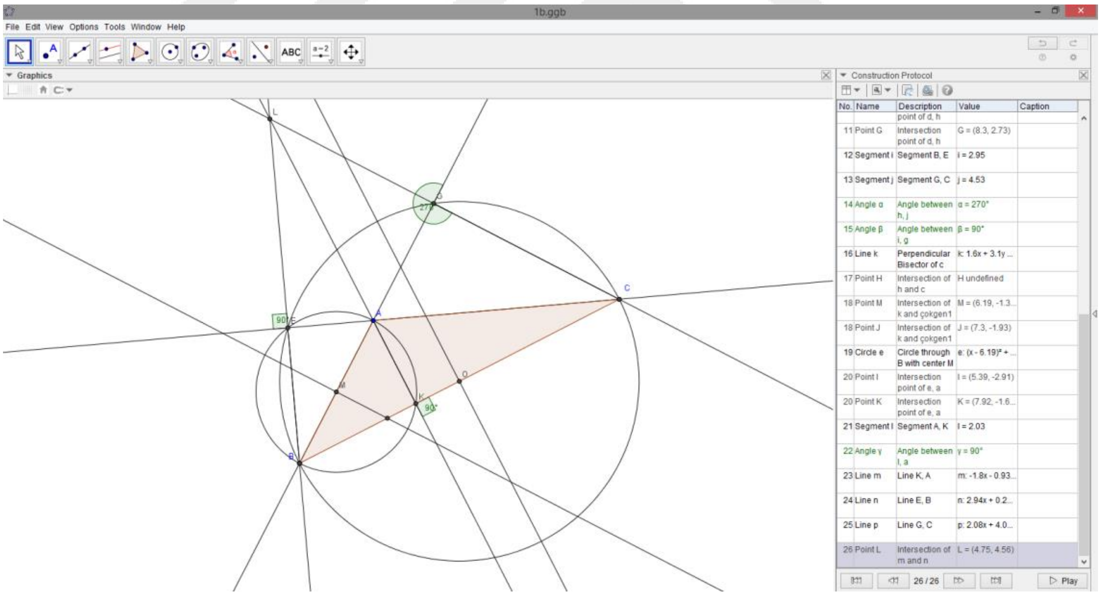


Figure 4. 55. GeoGebra file GG submitted for A1b in Activity 2 (1b.ggb)

In the approach coded A1b, GG worked on $\triangle ABC$ which is an obtuse triangle, as can be seen in Figure 4.55. Firstly, GG intended to draw a circle with the diameter $\overline{BC}$ so as to construct the altitudes of both $\overline{AB}$ and $\overline{CA}$. For this purpose, GG determined the midpoint of $\overline{BC}$ by virtue of the tool ‘perpendicular bisector’ and named with the letter O. Then, they drew the circle with the center O and the radius $|OB|$ by using the tool ‘circle with center through point’. At this point, they noticed that this circle did not intersect with the aimed sides of ΔABC. Since they saw that it was impossible to draw the altitudes in such a case, they decided to extend $\overline{AB}$ and $\overline{CA}$ to have the intersections which serve as the feet of altitudes in A1b. GG drew a line passing through the vertices A and C, determined the intersection of this line and the recently drawn circle, and named as E. Similarly, $\overline{AB}$ was extended by drawing a line passing through the vertices A and B and also the intersection of this line and the same circle was pointed out as the point G. To get the altitudes of $\overline{AB}$ and $\overline{CA}$, GG drew two line segments between the points C and G and also the points B and E, respectively. To ensure this idea, they measured the angles between the line segments drawn and the mentioned sides by using the tool ‘angle’.

At this moment, GG brought up the idea that there is no chance that three altitudes would be concurrent for this triangle since two of the altitudes have already appeared outside of the triangle. In this manner, it was inferred that they considered the case that there might be a triangle which does not have the orthocenter. Actually, each triangle has the orthocenter and this is the conjecture aimed to be reached by GG in Activity 2. After this expression, they continued with the construction of the third altitude which is the one for $\overline{BC}$. The first idea they mentioned to draw the altitude of $\overline{BC}$ but gave up promptly was finding the reflection of point E across the extended line of $\overline{AB}$ since the measure of $\angle BEA$ is 90º. They tried this idea by using the tool ‘reflect about line’ and concluded that the reflection did not give the foot of the altitude of $\overline{BC}$. That is to say, this idea was of no use in terms of the aimed construction. However, GG did not mention that attempt in their worksheet related to A1b. Therefore, GG continued with the idea of drawing a circle by accepting a side as the diameter like they did recently for finding the altitudes of the other two sides of ΔABC .

GG decided to draw a circle with the diameter of $\overline{AB}$ by anticipating that it would give the altitude of $\overline{BC}$. By performing the same procedure, GG determined the midpoint of $\overline{AB}$ and named as M. Then, they constructed the circle passing through B with the center M and then labeled the point of intersection of this circle and $\overline{BC}$ with the letter K. By drawing a line segment between the points A and K, they pointed out the altitude of $\overline{BC}$. Lastly, they also checked whether the measure of $\angle AKC$ is 90º as they expected. Since they completed the construction of the altitudes of ΔABC, they restarted to inquire about the existence of the orthocenter of ΔABC on the contrary of their earlier conclusion about the absence of the orthocenter for ΔABC. To inspect the possible intersection of the altitudes of ΔABC, they offered to draw the lines instead of the line segments for constructing the altitudes of all sides. Thus, they observed that the altitudes of ΔABC were also concurrent at a point which was named as L in the GeoGebra file 1b. Finally, GG expressed that the geometric figure they presented by applying A1b is a construction which makes A1b a valid approach. As stated before, the evaluation of A1b based on the diagram will be explained later in detail.

By focusing on the same idea that the inscribed angles subtended by a dimeter are right angles, GG worked on ΔDEF which was coded as A1a at first and then dealt with ΔABC which was coded as A1b. As expected, they passed to work on ΔKLM which is a right triangle to utilize the same idea. However, they observed that the altitudes of two sides in this triangle were already present since it is a right triangle. Therefore, GG decided to change their route and try an alternative approach for construction in Activity 2. In this respect, it can be stated that GG showed up with a new approach A2 which is related to the reflection of the points without trying A1 for ΔKLM. How CSG used A2 in ΔKLM was represented as A2a and it was marked in the conclusion component of AS-4 of the global argumentation structure of GG (See Figure 4.53). What GG saved as a GeoGebra file after the application of A2a (2a.ggb) was presented in Figure 4.56.

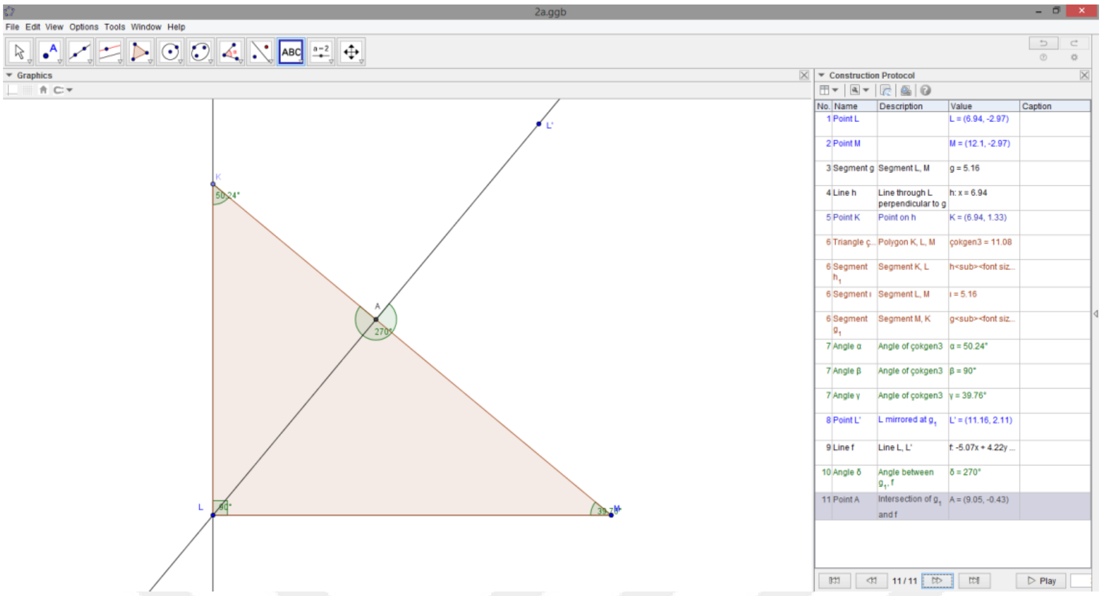


Figure 4. 56. GeoGebra file GG submitted for A2a in Activity 2 (2a.ggb)

As illustrated in the construction protocol in Figure 4.56, GG firstly checked out the measures of the interior angles of ΔKLM. The new approach they came up with was about finding the reflections of the vertices across the opposite sides and drawing lines passing the vertex and the reflection of it. They proposed that this line would be perpendicular to the side used as the reflection axis. As mentioned, the application of this idea in ΔKLM corresponds to A2a. Thus, in A2a, GG started with determining the reflection of the vertex L across $\overline{MK}$. By means of the tool 'reflect about line', they obtained the point L′. After that, they drew a line passing through L and L′ and also named the intersection of this line and $\overline{MK}$ as the point A. Since GG considered that A is the foot of the altitude of $\overline{MK}$, they wanted to measure the perpendicularity by using the tool 'angle' and concluded that the lastly drawn line was the altitude of $\overline{MK}$. Since ΔKLM is a right triangle, they declared that the altitude of $\overline{LM}$ corresponds to $\overline{KL}$ and vice versa. Finally, the orthocenter of ΔKLM was expressed as the point L since all altitudes of ΔKLM are concurrent at L. After checking the validity of A2a by means of dragging, GG categorized the figure they formed as a construction and A2a as a valid approach for Activity 2.

In accordance with what mentioned above, GG tried this new approach related to the reflection in another triangle. A2b was related to ΔDEF and marked in AS-5 of the global argumentation structure of GG in Activity 2 (See Figure 4.53). The geometric figure GG formed via GeoGebra at the end of A2b (2b.ggb) was presented below.

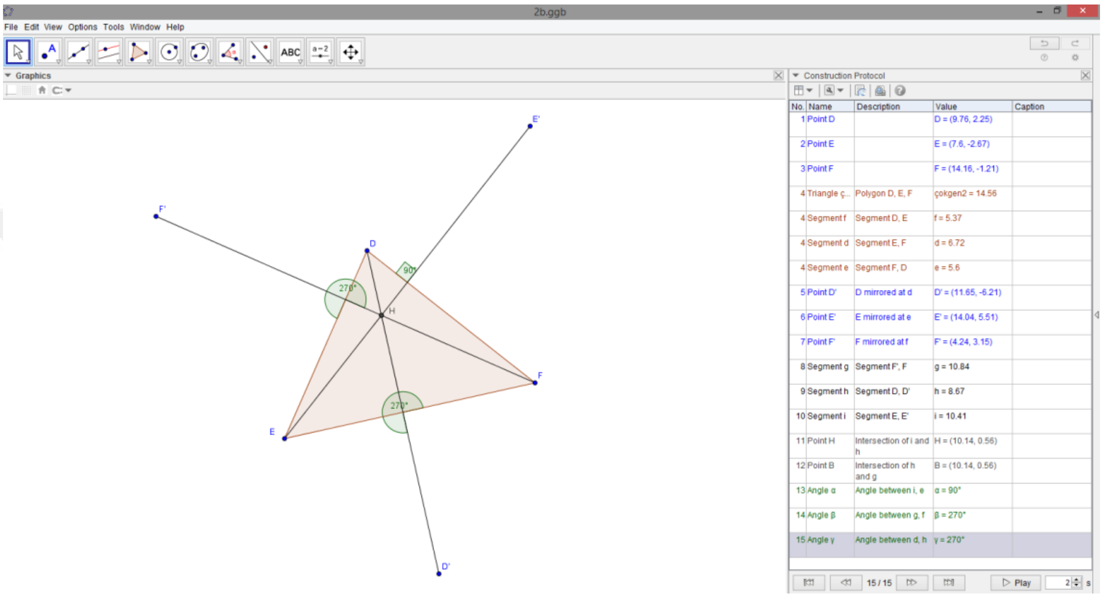


Figure 4. 57. GeoGebra file GG submitted for A2b in Activity 2 (2b.ggb)

According to Figure 4.57, GG reflected the vertex D across $\overline{EF}$ and found the point D′. Likewise, they found the point E′ by reflecting the vertex E across $\overline{FD}$ and the point F′ by reflecting the vertex F across $\overline{DE}$ by using the tool ‘reflect about line’. Then, they drew the line segments between the vertices and the reflected versions of them and accepted them as the altitudes of the sides. That is, according to GG, $\overline{DD'}$ is the altitude of $\overline{EF}$, $\overline{EE'}$ is the altitude of $\overline{FD}$, and $\overline{FF'}$ is the altitude of $\overline{DE}$, as given in Figure 4.57. Finally, they determined the point of intersection of these line segments, named with the letter H, and declared explicitly that the point H is the orthocenter of ΔDEF. GG evaluated A2b as a valid approach for the intended construction in Activity 2 after dragging.

As the final trial, GG adapted the same approach for ΔABC which was entitled as A2c in AS-6 of the global argumentation structure of GG (See Figure 4.53). Since A2c is the third attempt of the same idea, GG did not give a detailed explanation of it in the worksheet, but it was also given in Appendix F. The following Figure 4.58 displays what GG submitted at the end of performing A2c (2c.ggb).

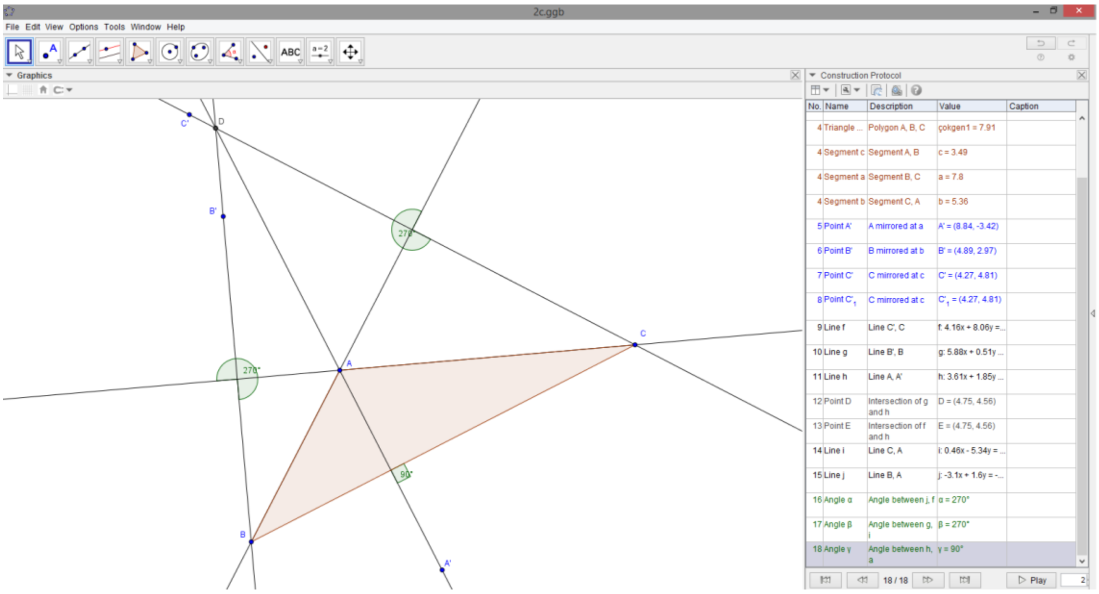


Figure 4. 58. GeoGebra file GG submitted for A2c in Activity 2 (2c.ggb)

As the first move in the GeoGebra file covering ΔABC, GG reflected the vertices A, B, and C across $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively via the tool 'reflect about line' and determined the points A′, B′, and C. Despite the fact that they drew the line segments in ΔDEF , they directly drew the lines between the points A and A′, B and B′, and C and C′ so as to construct the altitudes of each side of ΔABC. They observed that three lines drawn are concurrent at a point which was named as D as presented in Figure 4.58 and GG mentioned this point as the orthocenter. After that, GG wanted to draw the extensions of $\overline{AB}$ and $\overline{CA}$ to be able to measure the perpendicularity of the lines drawn as the altitudes. After checking the measure of the angles between the sides and the lines drawn as aiming the altitudes and also using the dragging function of

GeoGebra, they concluded that A2c was a valid approach for the intended construction of Activity 2.

Until that point, the approaches in Table 4.16 were explained in detail and the evaluations of GG related to these approaches were mentioned. Now, the evaluation based on the diagram given in Figure 4.17 about the validity of A1a, A1b, A2a, A2b, and A2c were examined as follows.

The first phase in this diagram asks whether the geometric figure GG presented is a proper one in terms of the asked construction in the activity. Since GeoGebra files of all approaches presented the altitudes and the orthocenters of the given triangles visually, it can be stated that what was offered by GG is proper in terms of what was asked as construction. Thus, the answer is yes for all approaches for the first phase of the diagram. This case led to the question of whether the geometric figures pass the drag test criterion. Since there was not a clear pattern or answer among five geometric figures in saved GeoGebra files in this phase, each of them was focused on separately from now on.

There is a problem at the end of the application of A1a and it was apparent for the researcher during the activity. Moreover, in the analysis of this approach, field notes, video recordings, worksheet, and the GeoGebra file submitted were taken into consideration as a whole. Firstly, the geometric figure GG formed by means of A1a failed to pass the drag test criterion since both the altitudes and the orthocenter disappeared at a point while dragging. As an example, a screenshot from such a case was given below. As seen in Figure 4.59, the altitudes disappeared when D was dragged through $\overline{EF}$.

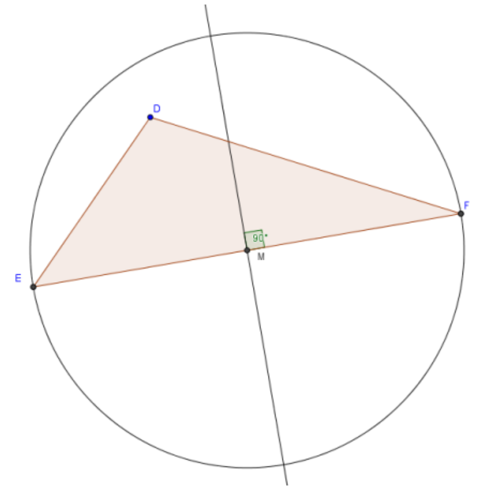


Figure 4. 59. The example for the drag test failure of geometric figure formed via A1a in Activity 2

This answer forwarded the figure in the GeoGebra file 1a to the assumption check question in the third phase. Since it was observed that there are some other problems in this figure such as the construction of the third altitude and drawing line segments for the first two altitudes instead of drawing lines in addition to the assumption violation, this geometric figure was classified as not a construction. Thus, it was coded that A1a is not a valid approach for the construction of Activity 2 although GG regarded A1a as a valid approach for construction.

When it was checked whether the geometric figure in the GeoGebra file submitted as 1b by applying A1b passes the drag test criterion, it was seen that this one also failed to pass it. An example for the case that one of the altitudes disappeared by dragging the vertex C downwards was given below. As seen in Figure 4.60, the altitude of $\overline{BC}$ was not present anymore due to the dragging so that the orthocenter also disappeared.

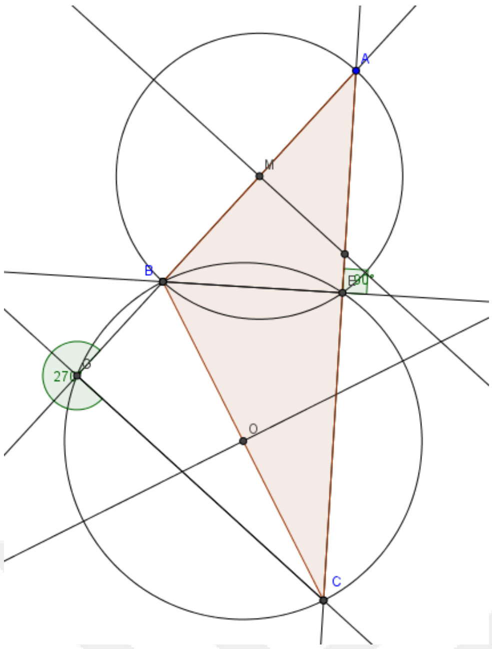
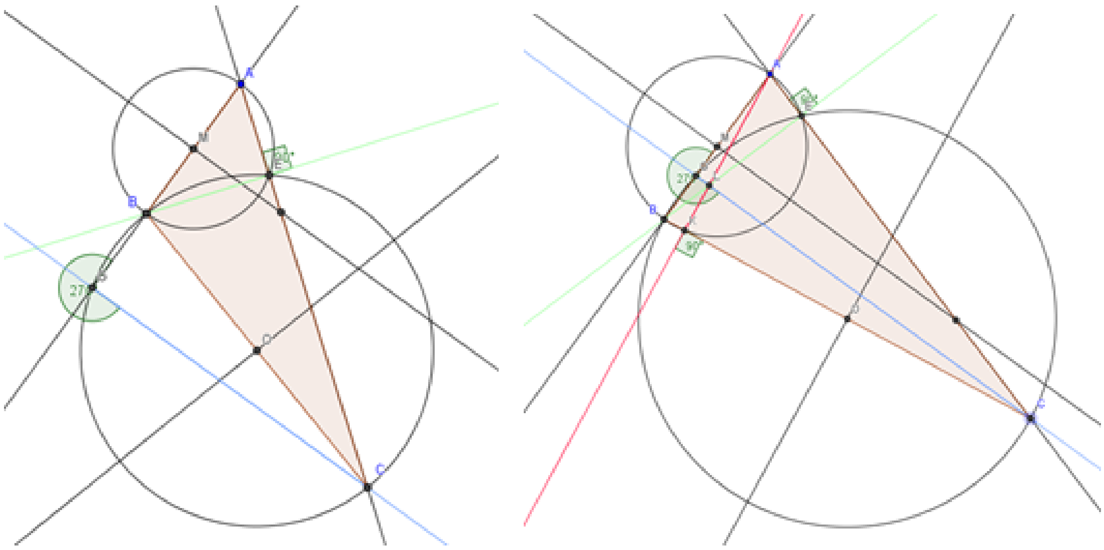


Figure 4. 60. The example for the drag test failure of geometric figure formed via A1b in Activity 2

Since the geometric figure formed via A1b failed to pass the drag test criterion, the evaluation process moved to the third phase. The figure in the GeoGebra file was checked whether any assumption GG used in A1b was violated as a result of dragging or not. To ease this process, the altitudes of sides were colored in a way that the altitude of $\overline{AB}$ is blue, the altitude of $\overline{BC}$ is red, and the altitude of $\overline{CA}$ is green. Moreover, to be able to examine the assumptions under dragging thoroughly, how GG drew the altitudes was focused on and listed as follows. The altitude of $\overline{BC}$ was drawn by means of the vertex A and the point K, which is the point of intersection of the circle with the diameter $\overline{AB}$ and $\overline{BC}$. The altitude of $\overline{CA}$ was drawn on the base of the vertex B and the point E, which the point of intersection of the circle with the diameter $\overline{BC}$ and the extension line of $\overline{CA}$. Lastly, the altitude of $\overline{AB}$ was drawn via the vertex C and the point G, which is the point of intersection of the circle with the diameter $\overline{BC}$ and the extension line of $\overline{AB}$. These were the assumptions GG used while drawing the altitudes of ΔABC.

In the third phase of the diagram, the mentioned issues checked with by performing not extreme dragging at first. It was found out that some assumptions were violated as a result of dragging in the figure submitted in GeoGebra file 1b. An example for this case was presented in the figure below.



(a) (b)

Figure 4. 61. The example for the assumption violation in the geometric figure formed via A1b in Activity 2

As seen in the first part of Figure 4.61, the red line which is the altitude of $\overline{BC}$ was no more present by virtue of dragging. It was assumed that the altitude of $\overline{BC}$ was drawn by means of the vertex A and the point K, but there was not also the point K. Thus, how the point K was formed was focused and it was seen that K is the point of intersection of the circle with the diameter $\overline{AB}$ and $\overline{BC}$. As seen in (a) of Figure 4.61, the circle with the diameter $\overline{AB}$ and $\overline{BC}$ were not intersecting anymore. With the help of the dragging which aims to have the intersection of the mentioned ones so as to have the point K, the red line appeared again which can be seen in (b) of Figure 4.61.

At this point, it seems that the answer for the third phase of the diagram for A1b is yes, but this is one of the challenging cases and the answer should be no. To decide without searching for all possible and also extreme cases by means of dragging might be misleading as in this idea. As mentioned, extreme dragging was also conducted in GeoGebra file 1b. For example, a case when the vertex A was moved below $\overline{BC}$ was presented in Figure 4.62 as given below.

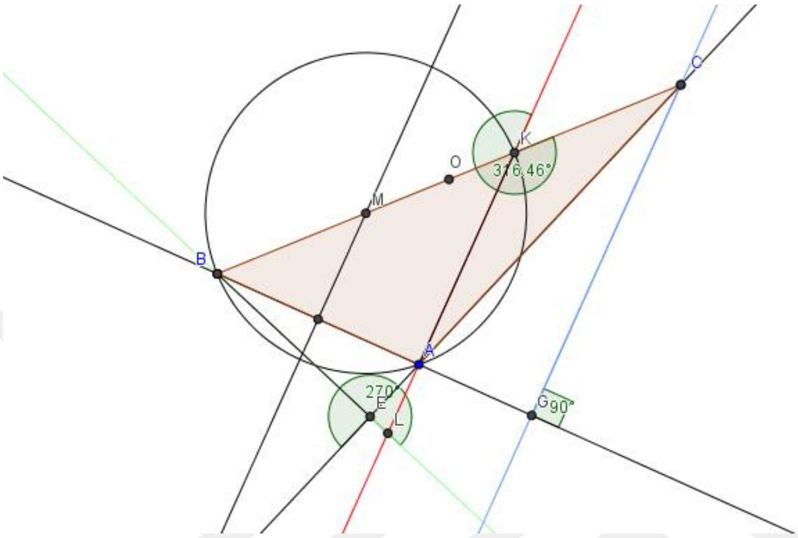


Figure 4. 62. The example for the drag test failure of a geometric figure formed via A1b in Activity 2 without assumption violation

As seen in Figure 4.62, the red line is not the altitude of $\overline{BC}$ and also the blue line which is constructed as the altitude of $\overline{AB}$ is not passing through the previously found orthocenter which is the point L. It can be inferred from this example that the violation of the assumptions used for the construction as the result of dragging is not the only reason for the fact that the geometric figure GG formed via A1b failed in the drag test criterion. By the effect of this last case, the geometric figure saved in the GeoGebra file was coded as not a construction based on the diagram in Figure 4.17.

Since the evaluation of all cases regarding A1 was finished, how the geometric figures formed through three attempts related to A2 will be taken into consideration hereafter. As mentioned before, A2 was firstly used in the right triangle and this process was referred as A2a. Since it is proper to the construction asked in Activity 2, it passed the first phase in the diagram. Afterwards, the geometric figure in GeoGebra file 2a was dragged and checked whether it could stay robust. The given ΔKLM was constructed before presenting to groups in a way that the dragging has no effect on its characteristics of being a right triangle. Thus, GG tested A2a for different right triangles and it was seen that the geometric figure kept its properties under dragging. This result directed to the third phase which seeks for whether the geometric figure presented passes the compatibility criterion or not. Since GG did not drive A2a based on the Euclidean restrictions, it was inferred that the geometric figure formed via A2a did not pass the compatibility criterion. This result indicated that the figure in GeoGebra file 2a is a construction, specifically it fits into construction type B (CTB).

By employing the same technique which is A2, GG worked on ΔDEF . In the mentioned process, the geometric figure formed via A2b was saved in the GeoGebra file 2b. As mentioned, the geometric figure passed the first phase in the diagram since it fits visually what was asked as construction in Activity 2. As usual, in the next phase of the diagram, it should be checked whether the geometric figure passes the drag test criterion. Nevertheless, it could not pass the drag test criterion. An example in which the vertex F was dragged towards $\overline{DE}$ was presented below to clear the underlying reason for the failure.

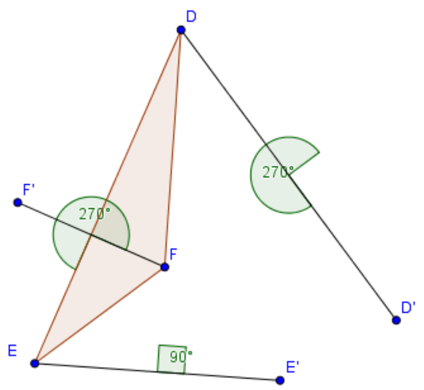


Figure 4. 63. The example for the drag test failure of geometric figure formed via A2b in Activity 2

Since Activity 2 asks for the construction of the altitudes and the orthocenters in the case of the existence of the given triangles, each geometric figure should involve both altitudes and the orthocenter to be classified as a construction. However, Figure

4.63 does not display the orthocenter although they stated the existence of orthocenter of ΔDEF and marked this point as H (See Figure 4.57). Since it failed in the drag test criterion, the third phase offers to check whether the only reason of this fail is the violation of the assumptions used as a base in the construction process due to the dragging. Since it was seen that there is not such a case related to this figure, it was accepted as not a construction so that A2b became an invalid approach for the construction in Activity 1. However, the issue that avoids the geometric figure in GeoGebra file 2b to be a construction is the fact that GG drew the line segments while connecting the vertices and their reflections. If they preferred to draw the lines instead of the line segments, the figure would not fail in the drag test. As the next step, it would not pass the compatibility criterion and fall into construction type B (CTB) based on the diagram. Although the infrastructure of A2b was solid, the application of it had some incorrect points.

The lastly mentioned hypothetical case about classifying the geometric figure formed via A2b as CTB was seen in the application of A2 in ΔABC which was expressed as A2c. In more detail, the geometric figure in the GeoGebra file 2c passed the drag test criterion since they drew the lines passing from the vertices and the reflections of them (See Figure 4.58). Under dragging, both the altitudes and the orthocenter of ΔABC stayed robust. Thus, the third phase came with the demand for checking the compatibility criterion. Since A2 is related to the reflection in nature and it is not an applicable case with respect to the Euclidean restrictions, it was stated that the geometric figure formed via A2c could not pass the compatibility criterion so that it was categorized as construction type B (CTB).

4.3.2.3. Approaches GG Offered for Geometric Construction in Activity 3

Activity 3 supplied not only two GeoGebra files but also the worksheets on which an acute triangle (ΔABC) and the details regarding the expected construction were written to GG. The given triangle was presented in Figure 4.35 earlier in the findings of CSG in Activity 3. The first GeoGebra file was assigned in a way that it keeps the default toolbar but the second GeoGebra file was prepared by removing three tools which are ‘midpoint or center’, perpendicular line’, and ‘perpendicular bisector’ from the toolbar. GG was asked to work with the first GeoGebra file until the notice was given to use the second one. Moreover, they were asked not to open the second one since the restricted tools might be used as the clue and affect their reasoning regarding the construction. To sum up, GG was asked to construct the orthocenter, the circumcenter, and the centroid of ΔABC and then search for the relationships among three points.

It was documented that GG offered six approaches with the aim of construction of the mentioned three points while working on two GeoGebra files. The locations of these approaches in the argumentation process of GG in Activity 3 were illustrated in Figure 4.64 given below.

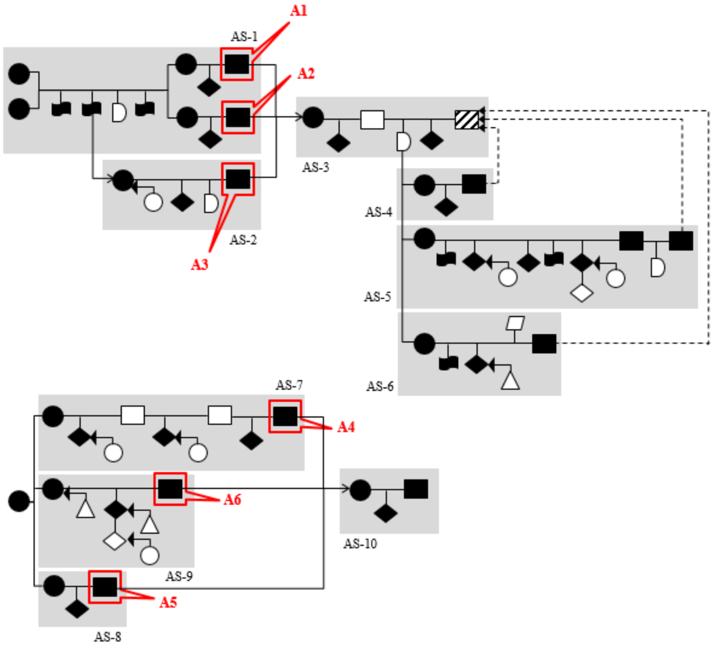


Figure 4. 64. The locations of approaches offered for the construction in the global argumentation structure of GG in Activity 3

As seen, all approaches were marked by means of the conclusion component. The first three approaches, which were offered while working on the first GeoGebra file, were presented in the first argumentation block whereas the last three approaches, which were stated while working on the restricted GeoGebra file, were located in the second argumentation block. Since GG did not state any approach as invalid in the process, there is not any blue indicator in Figure 4.64. All approaches were accepted as valid by GG so that only one table was formed to summarize them.

Table 4. 17

Approaches GG stated as valid for geometric construction in Activity 3

Approach for construction	**Validity of approach**
A1. GG used the tools ‘perpendicular line’, ‘intersection’, and ‘angle’ while constructing the altitudes of ΔABC. Then, they concluded that the point of concurrency of the altitudes presented the orthocenter of ΔABC. *(written in the worksheet, GeoGebra file 1, 2)*	*According to GG* - Valid approach for Activity 3 - Construction of the orthocenter of ΔABC
	Based on the diagram (See Figure 4.17) - Valid approach for Activity 3 - Construction of the orthocenter of ΔABC (CTB)
A2. GG used the tools ‘perpendicular bisector’ and ‘intersection’, in order to construct the perpendicular bisectors of the sides of ΔABC. Then, they stated that the point of concurrency of the perpendicular bisectors of the sides presented the circumcenter of ΔABC. *(written in the worksheet, GeoGebra file 1, 2)*	*According to GG* - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC
	Based on the diagram - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC (CTB)
A3. GG determined the midpoints of each side of ΔABC. Since the perpendicular bisectors of the sides of ΔABC were already drawn via A2, they drew the line segments between the vertices and the midpoints to construct the medians of ΔABC. Then, they concluded that the point of concurrency of the medians presented the orthocenter of ΔABC. *(written in the worksheet, GeoGebra file 1, 2)*	*According to GG* - Valid approach for Activity 3 - Construction of the centroid of ΔABC
	Based on the diagram - Valid approach for Activity 3 - Construction of the centroid of ΔABC (CTB)
A4. GG attempted to construct the perpendicular bisectors of the sides as if they were using compass-straightedge. Thus, they drew the perpendicular bisectors by using some tools such as ‘circle with center through point’ and ‘intersect’. Finally, they restated that the point of concurrency of the perpendicular bisectors of the sides presented the circumcenter of ΔABC. *(written in the worksheet, GeoGebra file 3)*	*According to GG* - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC
	Based on the diagram - Valid approach for Activity 3 - Construction of the circumcenter of ΔABC (CTA)

Table 4. 17 (continued)

A5. GG followed the application of A4 to construct the medians of ΔABC. Since the midpoints of the sides were already constructed in A4, they drew the line segments between the vertices and the midpoints to construct the medians of ΔABC. Then, they restated that the point of concurrency of the medians presented the centroid of ΔABC. *(written in the worksheet, GeoGebra file 3)*	*According to GG* - Valid approach for Activity 3 - Construction of the centroid of ΔABC *Based on the diagram* - Valid approach for Activity 3 - Construction of the centroid of ΔABC (CTA)
A6. GG thought about using the idea that inscribed angle subtended by a diameter is a right angle. Then, they drew three circles by accepting the sides of ΔABC as the diameters and determined the feet of the altitudes from the intersections of these circles and triangle. Then, they restated that the point of concurrency of the altitudes as the orthocenter of ΔABC. *(written in the worksheet, GeoGebra file 3)*	*According to GG* - Valid approach for Activity 3 - Construction of the orthocenter of ΔABC *Based on the diagram* - Invalid approach for Activity 3 - Not construction of the orthocenter of ΔABC

According to Table 4.17, it can be summarized that A1 and A6 were related to the orthocenter, A2 and A4 were about the circumcenter, and lastly A3 and A5 were stated related to the centroid. Moreover, the combination of A1, A2, and A3 was offered to carry out the whole construction process asked in Activity 3. Similarly, the combination of A4, A5, and A6 can be seen another pack of approaches performed for the entire construction in Activity 3. To address the third research question in detail, each of these approaches and the evaluations regarding the validity of them, which were summarized in Table 4.17 above, were paid attention separately as follows.

At the beginning of Activity 3, GG discussed the meaning of these points and the intersection of which elements of the triangle could present them. Although they studied about the circumcenter and the orthocenter in the previous cognitive unity based activities, they could not be sure about the construction process of them for a while. However, they recaptured what they have conducted at the end of the discussion. Since GG was familiar with these two points, they preferred to construct them firstly. In this manner, A1 which was represented in AS-1 of the global argumentation structure of GG was about the construction of the orthocenter (See Figure 4.64). Since GG worked on one GeoGebra file for A1, A2, and A3, the screenshots from GeoGebra file per approach were presented to be able to examine the process in detail. Accordingly, the first screenshot which is related to A1 was given below in Figure 4.65.

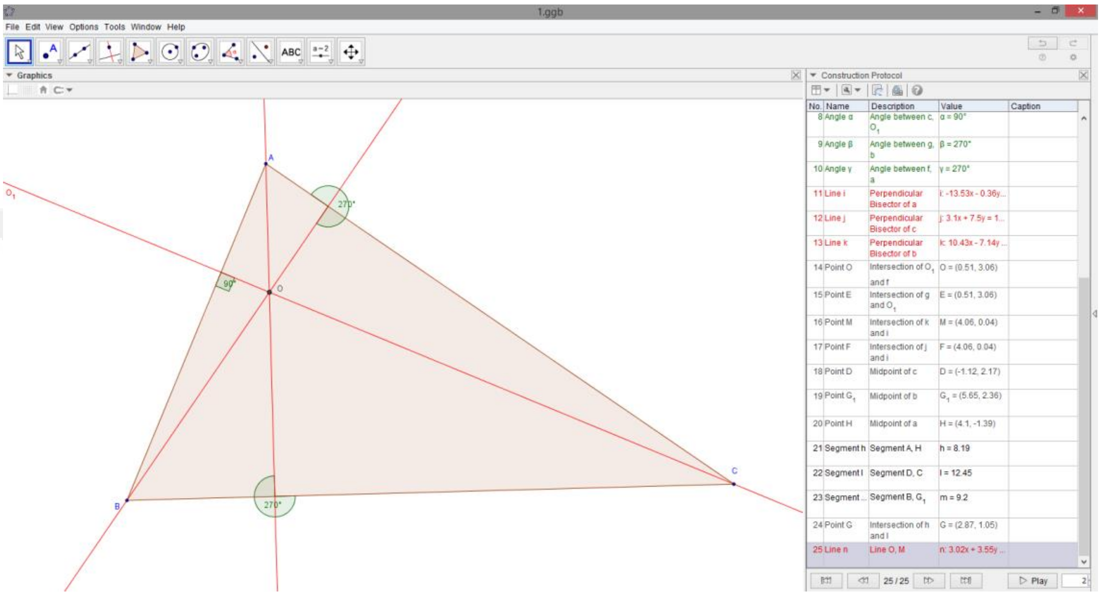


Figure 4. 65. GeoGebra file GG submitted in terms of A1 in Activity 3 (1.ggb)

Since none of the tools was restricted, GG used the most direct and feasible tools while constructing three points. As displayed in Figure 4.65, in A1, GG aimed to draw the altitudes of the sides of ΔABC. Thus, they used the tool ‘perpendicular line’ to draw three lines, each of which are passing through one vertex and perpendicular to the opposite side. While using the mentioned tool, they checked the tooltip which states ‘select point and perpendicular line’. By clicking the vertex and the opposite line, they constructed all altitudes of ΔABC. Then, they checked the perpendicularity of all lines formed with the aim of altitudes to the corresponding sides by measuring the angles with the tool ‘angle’. By using the tool ‘intersect’, they determined the point of concurrency of three altitudes of ΔABC and named as O. Thus, according to GG, A1 was a valid approach in terms of the construction of the orthocenter of ΔABC.

As another approach given in AS-1 of the global argumentation structure of GG (See Figure 4.64), A2 is about the construction of the circumcenter of ΔABC. The screenshot as directly in the scope of A2 only was also indicated in Figure 4.66.

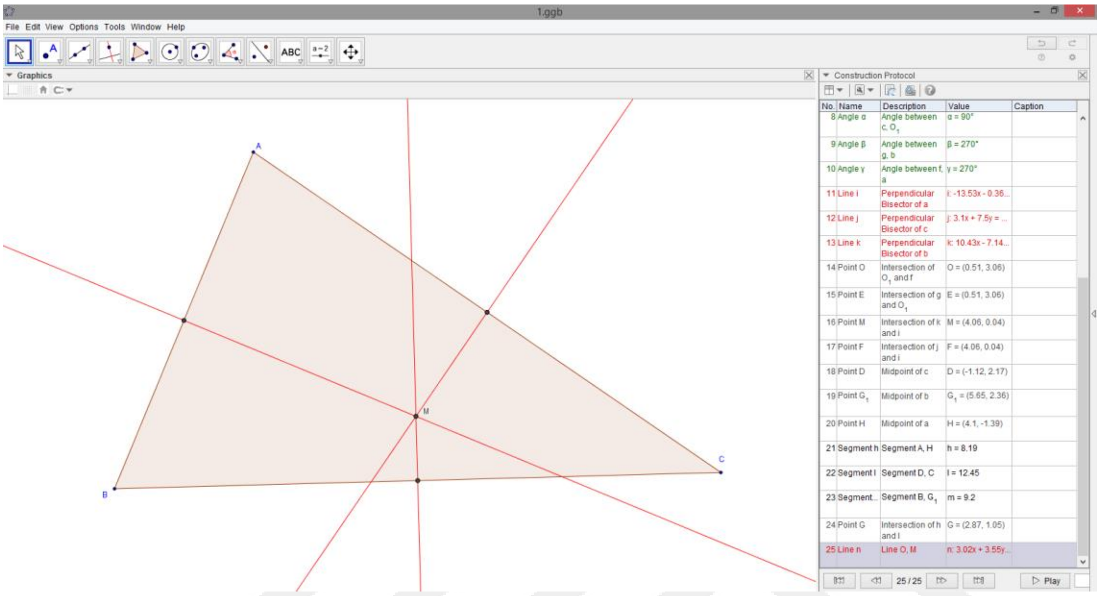


Figure 4. 66. GeoGebra file GG submitted in terms of A2 in Activity 3 (1.ggb)

In A2, GG used the tool 'perpendicular bisector' to construct the perpendicular bisectors of all sides of ΔABC. In addition, they specified the midpoints of each side via the tool 'intersect' which facilitated the application of A3. Again, by using the tool 'intersect', they found the point of concurrency of the perpendicular bisectors of the sides of ΔABC and labeled the point as M. According to GG, the lastly determined point M is the circumcenter of ΔABC so that A2 can be stated as a valid approach in terms of the construction of the circumcenter of ΔABC.

As the last case in the first GeoGebra file, GG aimed to construct the medians of ΔABC which was marked as A3 in AS-2 of the global argumentation structure of GG (See Figure 4.64). Similar to A1 and A2, the screenshot related to the application of A3 only was presented in Figure 4.67.

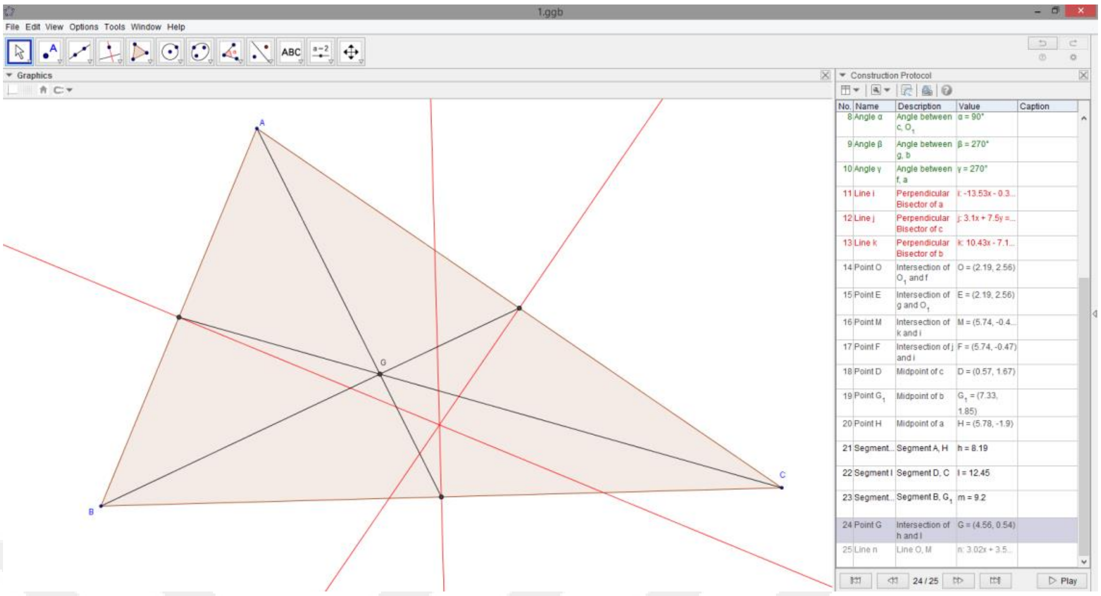


Figure 4. 67. GeoGebra file GG submitted in terms of A3 in Activity 3 (1.ggb)

Since GG already drew the perpendicular bisectors of the sides of $\triangle ABC$ via A2, the midpoints of the sides were present. From a broad perspective, A3 also involves the construction of the midpoints of the sides at the beginning. For the first GeoGebra file, the midpoints of the sides could be found by either the tool 'perpendicular bisector' or the tool 'midpoint or center'. Then, they drew the line segments from each midpoint to the opposite vertices and named the point of concurrency of them as the point G. Since they used the line segments in drawing the medians and the point of intersection of the medians of a triangle is always inside of that triangle, the dragging did not affect the geometric figure in a negative manner. Otherwise, when the line segments are used in the construction of the orthocenter and the circumcenter, the dragging might cause the intersection point to disappear.

After applying these approaches in one GeoGebra file, GG saved and submitted it as 1.ggb. Since the default toolbar was available, it was a quite quick construction for them. As a sum up, the screenshot from the first GeoGebra file GG submitted, which presents the application of A1, A2, and A3, was displayed in Figure 4.68 given below.

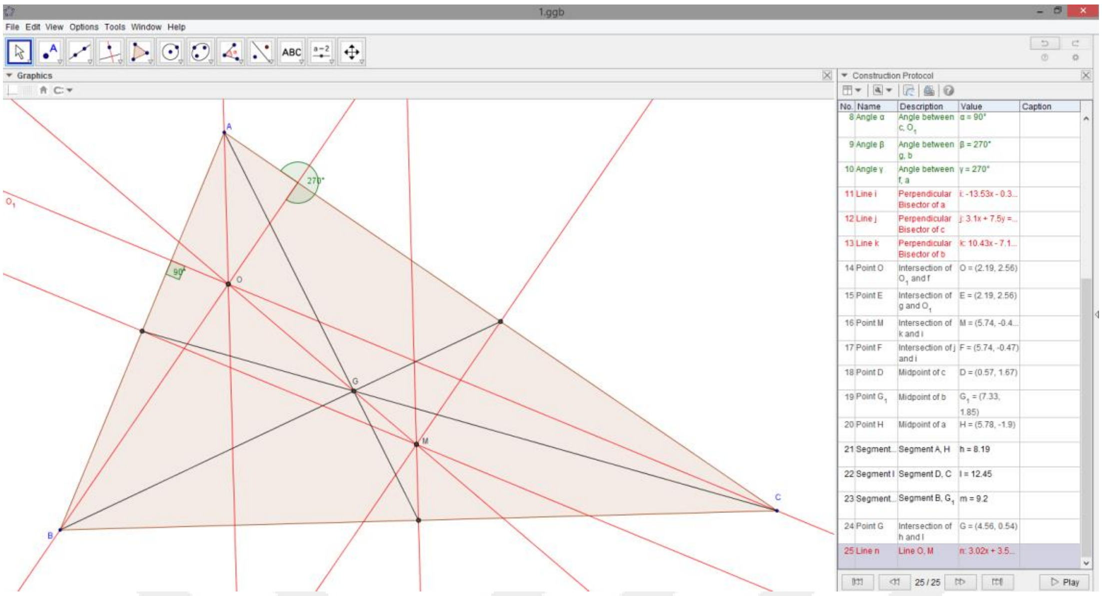


Figure 4. 68. GeoGebra file GG submitted as a combination of A1, A2, and A3 in Activity 3 (1.ggb)

As the next step in Activity 3, GG looked for the relationship between these points and directly stated the possibility of the fact that the mentioned three points are collinear. In addition to the construction of three points in the given ΔABC, GG wanted to construct the same points for different types of triangles to check the conjecture they produced regarding the collinearity. Hereby, they dragged the point A to the left so as to make ΔABC an obtuse triangle and checked the presence of the points as well as the collinearity of them. Afterwards, they opened a new GeoGebra file and constructed an equilateral triangle by using the tool 'regular polygon'. They noticed that the asked three points occurred at the same point in an equilateral triangle. Lastly, they wanted to check it for a right triangle so that they drew a triangle by setting one interior angle as 90°, and constructed three intended points. As a result of all these trials, GG became sure about the conjecture they produced which will be explained in the following section in detail. Moreover, they saved the new GeoGebra file (2.ggb) and submitted it at the end of the activity. Since it involves the construction via applying A1, A2, and A3 in different types of triangles, the screenshot of this file was also presented in Figure 4.69 as given below.

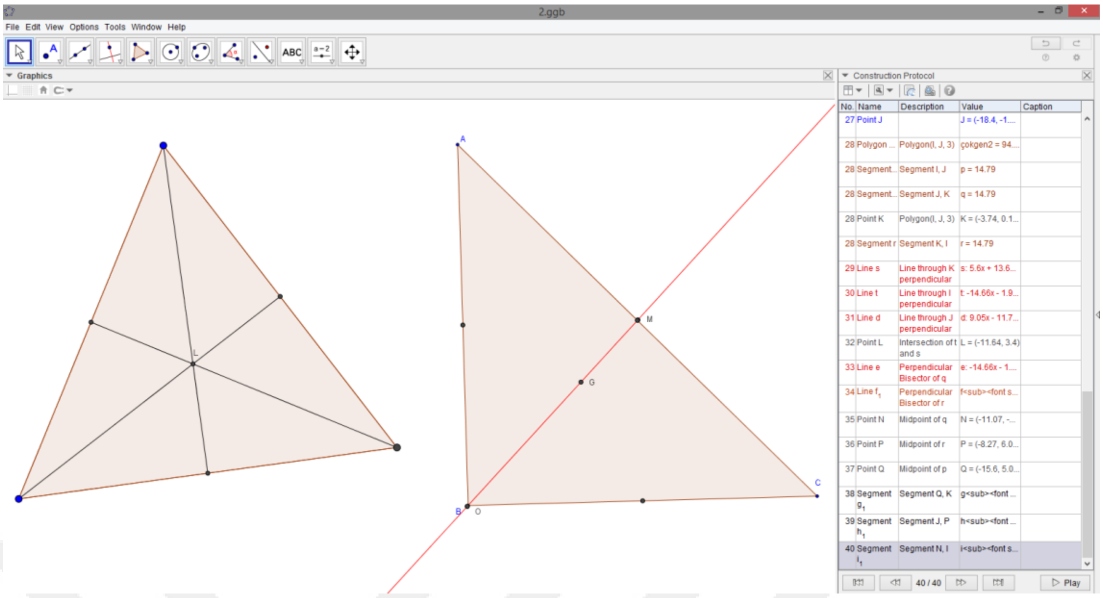


Figure 4. 69. The second GeoGebra file GG submitted as a combination of A1, A2, and A3 in Activity 3 (2.ggb)

After this process, GG was informed about using the restricted GeoGebra file for the construction of the same three points. Likewise the structure in A1, A2, and A3, the approaches GG tried for the construction of each point were taken into consideration separately. Specifically, A4 was about the alternative construction of the circumcenter, A5 was about the construction of the centroid, and finally A6 was about the construction of the orthocenter of the given triangle. They saved the GeoGebra file after the implementation of A4, A5, and A6 so that this file became the last file GG submitted in this activity (3.ggb). The screenshot from this GeoGebra file was given in Figure 4.73 so as to present the total picture in the secondly given GeoGebra file. Before that, each approach was explained by associating with the screenshots separately. In this respect, what GG performed in the restricted GeoGebra file via applying A4 was presented in Figure 4.70.

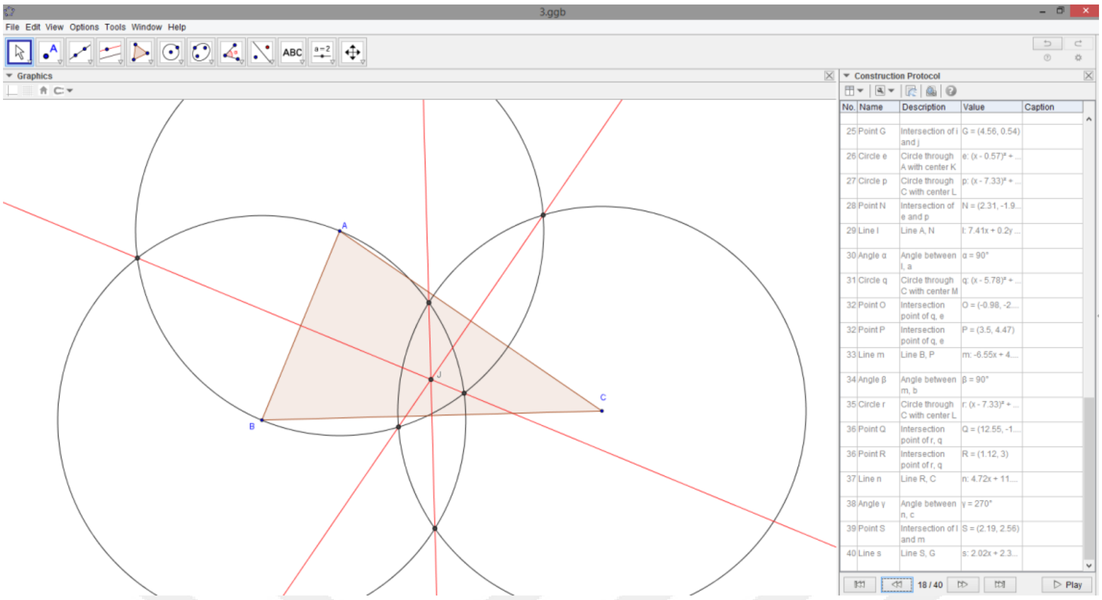


Figure 4. 70. GeoGebra file GG submitted in terms of A4 in Activity 3 (3.ggb)

In A4, GG focused on the construction of the circumcenter and it was marked throughout the conclusion of AS-7 of the global argumentation structure of GG (See Figure 4.64). Since the tool ‘perpendicular line’ was removed from the toolbar, they needed to find another approach. They put forward the idea of drawing the circles that the centers were at the vertices. At first, they ignored the fact that they need equal pairs of circles for constructing the perpendicular bisector of any side. However, they noticed that issue by means of a rebuttal in the process. Thus, they drew a circle with the center A and the radius $\overline{AB}$ by using the tool ‘circle with center through point’. By considering the equality of the circles, they attempted to transfer that circle for other vertices. After drawing three circles, they determined the points of intersection of adjacent circles by using the tool ‘intersect’. This led them to draw three lines passing through the intersections of each pair of the adjacent circles. As seen, A4 was similar to an approach GG used while constructing the circumcenter of the triangle in Activity 1. In conclusion, GG found the point of intersection of the perpendicular bisectors of the sides and named as the point J. Therefore, GG accepted A4 as a valid approach in terms of the construction of the circumcenter of ΔABC.

After the circumcenter, GG started to the construction of the centroid. Similar to the first GeoGebra file, one of them noticed that they have already localized the midpoints of the sides of ΔABC by means of the construction of the perpendicular bisectors of the sides with A4. This approach was coded as A5 at the conclusion of AS-8 of the global argumentation structure of GG (See Figure 4.64). What the GeoGebra file covers at the end of the application of A5 was presented as follows.

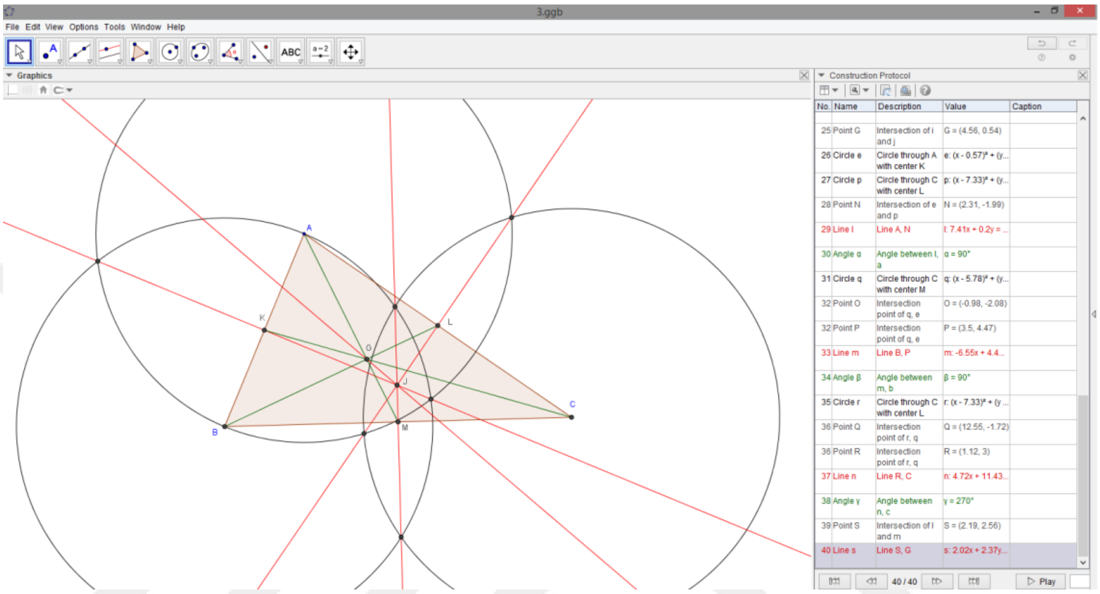


Figure 4. 71. GeoGebra file GG submitted in terms of A5 in Activity 3 (3.ggb)

An interesting issue regarding this approach is that A5 actually covers the process of A4. By comparing Figure 4.70 and Figure 4.71, it can be seen that A4 is a subset of A5. In this respect, when it was considered as a separate approach, A5 also covers drawing three equal circles with the centers as the vertices, determining the intersections of the circles, drawing the lines from the intersections of the circles to determine the midpoints of the sides, and lastly drawing the lines or the line segments from each vertex to the midpoint of the opposite side. Since the GeoGebra file used in this activity has already involved the midpoints of the sides via A4, there were two steps left behind of A4 for completing A5. Thus, GG used the tool ‘intersect’ to specify the midpoints of $\overline{AB}$, $\overline{CA}$, and $\overline{BC}$ and then entitled them as K, L, and M, respectively. Then, they drew three line segments from each vertex to the midpoint of the opposite sides of ΔABC, determined the point of concurrency of the medians, and labeled as the point G. Moreover, GG stated that A5 a valid approach for the construction of the centroid of ΔABC.

The last idea was offered for the construction of the altitudes and the orthocenter of ΔABC which was labeled as A6 and represented via the conclusion of AS-9 of the global argumentation structure of GG (See Figure 4.64). The geometric figure formed by means of A6 was presented below.

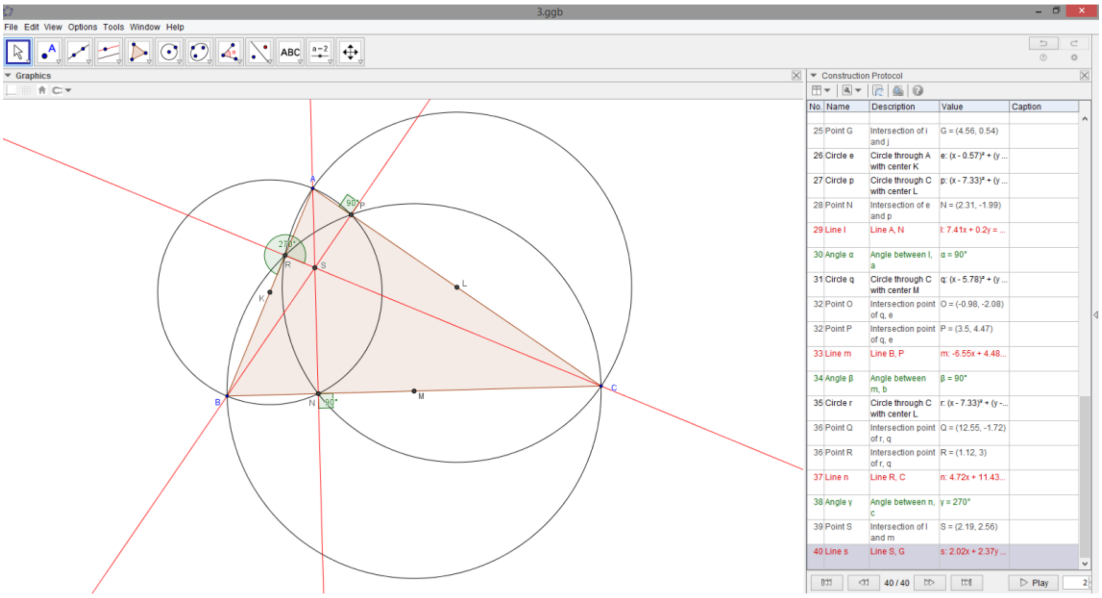


Figure 4. 72. GeoGebra file GG submitted in terms of A6 in Activity 3 (3.ggb)

The aim of A6 is to use the statement that inscribed angle subtended by a diameter is a right angle. To that end, GG aimed to draw two circles with the diameter $\overline{AB}$ and $\overline{CA}$ by using the tool 'circle with center through point' at first. They noticed that the midpoints of the sides of ΔABC were required as the centers of the circles to be able to use the mentioned tool. Therefore, they kept what they have done via A4 in which they determined the midpoints of the sides while drawing the perpendicular bisectors of the sides. Therefore, a similar case to A5 emerged. That is to say, A4 is also a subset of A6. From a broad perspective, A6 covers finding the midpoints of the sides, drawing three circles by accepting the midpoints as centers and the sides as diameters, finding the intersection points of these circles and the sides, accepting them as the feet of altitudes of the sides, and drawing the lines passing through the feet of the altitudes and the corresponding vertices.

In this regard, GG moved on the construction of the altitudes by accepting the presence of the midpoints, namely, K is the midpoint of $\overline{AB}$ and L is the midpoint of $\overline{CA}$. Thus, they drew two circles by accepting K and L as the center and also $\overline{AB}$ and $\overline{CA}$ as the diameters by using the tool ‘circle with center through point’. Then, they determined the intersections of these circles via the tool ‘intersect’. As expected, one of these intersection points was the vertex A and the other one was labeled as the point N on $\overline{BC}$. They drew a line passing through the vertex A and the point N. This practice presented the altitude of $\overline{BC}$ only, they still needed to construct the altitudes of two sides of ΔABC. With this purpose, they accepted the point M which is the midpoint of $\overline{BC}$ as center and then drew the third circle with the diameter $\overline{BC}$. Then, they determined the intersections of the third circle with $\overline{AB}$ and $\overline{CA}$ and also named these points as R and P, respectively. By accepting the points R and P as the feet of the altitudes of $\overline{AB}$ and $\overline{CA}$, they drew two lines, one of which passes from the vertex B and the point P and other one passes from the vertex C and the point R. Then, they determined the point of concurrency of the altitudes as the orthocenter of ΔABC and termed as the point S.

As stated, A4, A5, and A6 were applied in the restricted GeoGebra file. Indeed, the combination of them was needed to perform the construction asked in Activity 3. Thus, the final form of the geometric figure arranged by applying three of them was presented below as a screenshot from the GeoGebra file saved as 3.ggb.

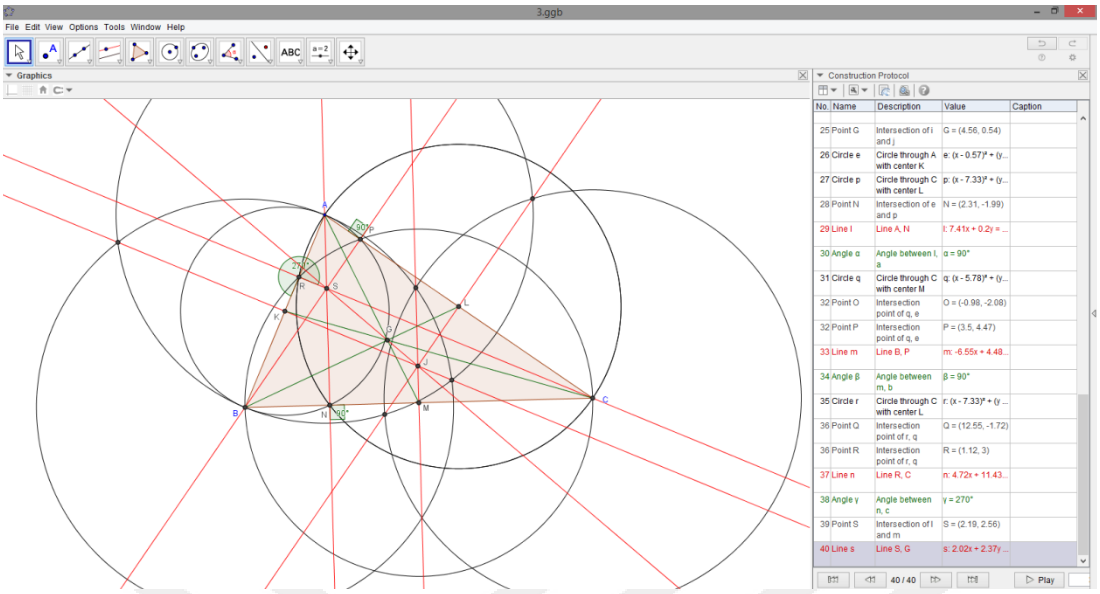


Figure 4. 73. The third GeoGebra file GG submitted as a combination of A4, A5, and A6 in Activity 3 (3.ggb)

As mentioned, GG accepted all approaches as valid for the construction of the particular points asked in Activity 3. The evaluation of the mentioned approaches based on the diagram in Figure 4.17 was explained thereafter. It was aimed to evaluate A1, A2, and A3, which were applied in one GeoGebra file saved as 1.ggb, at the same time. Since there was not a restricted tool in that GeoGebra file, the possibility of having a case which might cause a problem in terms of labeling the figure as a geometric construction was expected as low. In the first phase of the diagram, the geometric figure that GG formed was found out as proper in terms of the construction asked in Activity 3. Thus, this answer directed to the drag test criterion in the second phase. In any occasion under dragging, it kept its properties. In other words, it was seen that the geometric figure in 1.ggb always covered the orthocenter, the circumcenter, and the centroid in a collinear manner. Thus, it passed the drag test criterion. Based on this result, the third phase asked whether it passes the compatibility criterion or not. Since GG did not focus on the Euclidean restrictions during A1, A2, and A3, it was concluded that it could not pass the compatibility criterion. As the final step, the geometric figure formed as a combination of A1, A2, and A3 fell into construction type B (CTB).

On the contrary, the applications of A4, A5, and A6 were evaluated separately since they were used in the restricted GeoGebra file. There might be extra situations which might interfere in the evaluation process so that they should be taken into consideration one by one. First of all, the elements formed via A5 and A6 were hidden in the GeoGebra file so that the elements displayed in the construction of the circumcenter via A4 could be examined clearly. Since the application of A4 ended up a proper visual in terms of the geometric construction asked in the activity, the first phase of the diagram directed to the drag test criterion in the second phase. When the vertex C was dragged through the right side, the perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ disappeared as illustrated in Figure 4.74. Due to this situation, the circumcenter was not present anymore in the geometric figure.

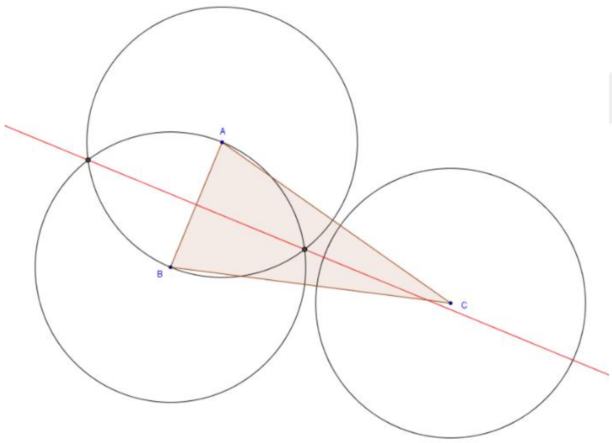


Figure 4. 74. The example for the drag test failure of geometric figure formed via A4 in Activity 3

Since the figure did not pass the drag test criterion, the question in the third phase was whether the only reason of the failure in the drag test is the violation of an assumption that GG used as the base in the construction. To answer this question, the dragging process was continued, but it was concluded that it is the only reason and any other case was not found. As can be seen in Figure 4.74, GG started with finding the intersection of two equal circles drawn by accepting centers as the vertices to be able to draw the perpendicular bisector of a side. However, they violated this assumption by dragging. That is, the idea in A4 keeps working unless the circles leave intersecting. In conclusion, A4 was stated as valid and the way that circumcenter was formed fell into construction type A (CTA).

In a similar vein, the elements only related to A5 were left apparent in the GeoGebra file and the remaining elements were hidden. Since A5 offered visually a proper figure to the geometric construction asked, the answer was ‘yes’ in the first phase of the diagram. Then, it led to control whether the figure formed via A5 passes the drag test or not. Since the beginning parts of A4 and A5 are the same, a similar situation occurred while dragging the figure formed via A5. For example, what happened to the figure when the vertex C was dragged through the right side was presented in Figure 4.75.

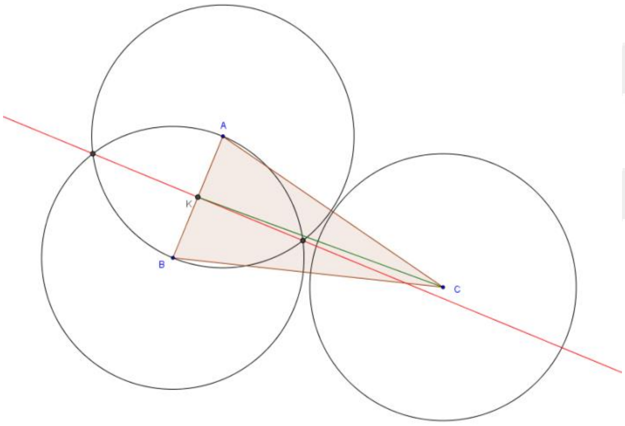


Figure 4. 75. The example for the drag test failure of geometric figure formed via A5 in Activity 3

According to the case in Figure 4.75, the dragging of the vertex C blocked the intersection of the circle with center C with the other two circles. Since these circles were used to determine the midpoints of the sides, the midpoint L of $\overline{CA}$ and the midpoint M of $\overline{BC}$ were not present. Therefore, the medians of $\overline{BC}$ and $\overline{CA}$ and also the point of concurrency of the medians disappeared. Due to this unstable situation, it could not meet the entailments of the drag test. Afterwards, the third phase directed the question of whether the violation of assumptions is the only reason for the failure in the drag test. The further examination of the figure under dragging presented that it was the only reason like A4. In summary, A5 was declared as a valid approach for the construction of the centroid of a triangle. In this respect, the way that the centroid formed via A5 was classified as construction type A (CTA).

In the evaluation of the final approach, a similar procedure was followed and the elements unrelated to A6 were hidden in the GeoGebra file. Since the figure seen in the screen of the GeoGebra file presents the orthocenter, it was accepted as a proper one to the asked construction. Thus, the figure was checked in terms of the dragging in the second phase. Since it was observed that to follow the elements in the geometric figure was difficult under dragging, the colorization was utilized as presented in Figure 4.76. More specifically, the perpendicular bisectors of the sides of ΔABC were kept as red, the circles drawn to find the perpendicular bisectors of the sides were kept as black, the circles drawn by accepting the sides as the diameters were colored as green, the altitudes of the sides were colored as blue, and the point of the concurrency of the altitudes which the point S was colored as red.

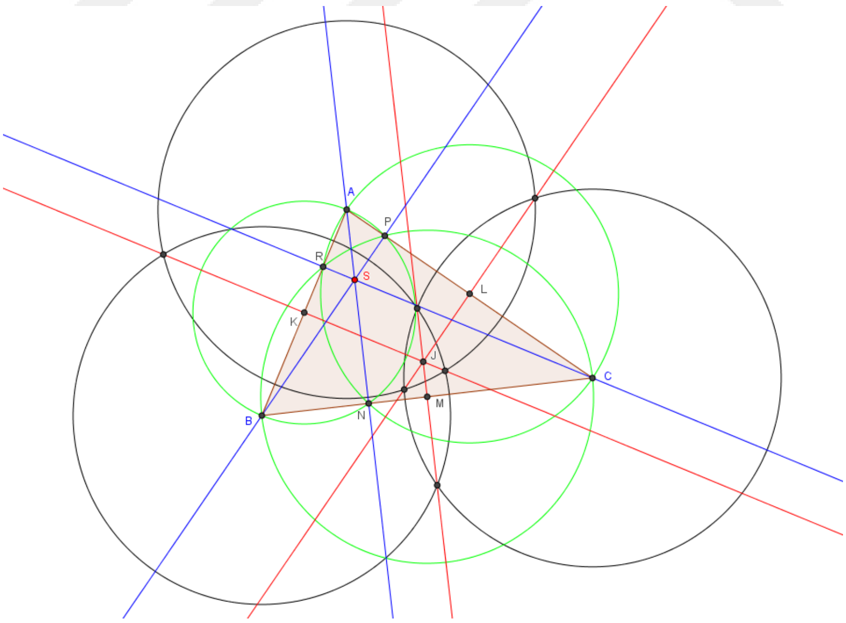


Figure 4. 76. The geometric figure formed via A6 before dragging to check the drag test creation

After arranging the figure in the GeoGebra file 3.ggb as presented in Figure 4.76, the dragging was applied. By following the pattern in the previous examples, the case that the vertex C was dragged through the right was displayed in Figure 4.77.

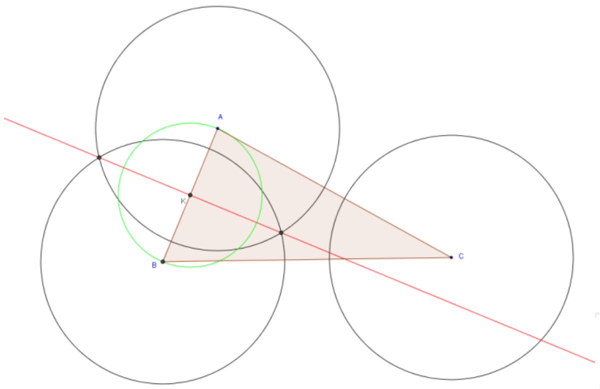


Figure 4. 77. The example for the drag test failure of geometric figure formed via A6 in Activity 3

As seen, the dragging of the vertex C caused the disappearance of the midpoints L and M due to the fact that the circle with the center C is not intersecting with the other two circles anymore. This case caused the disappearance of two of the green circles since the midpoints L and M are also the centers of the circles with the diameters $\overline{CA}$ and $\overline{BC}$. Due to the disappearance of the lastly mentioned two circles, all altitudes of the sides, which can be seen as blue lines in Figure 4.76, disappeared. Because of the sequence of such disappearances, it was stated that the geometric figure did not stay robust under dragging. Thus, it was concluded that the geometric figure could not pass the drag test criterion. Afterwards, the third phase asks whether the violation of the assumptions used in the construction process is the only reason for the failure in the drag test. To that end, GG continued with some other dragging trials to check whether there is a case other than the assumption violation caused the failure in the drag test. Nevertheless, such cases were noticed during the dragging. As an example, the dragging of the vertex C to the left side of $\overline{AB}$ was explained as follows. Moreover, the screenshot of the mentioned case was illustrated in Figure 4.78.

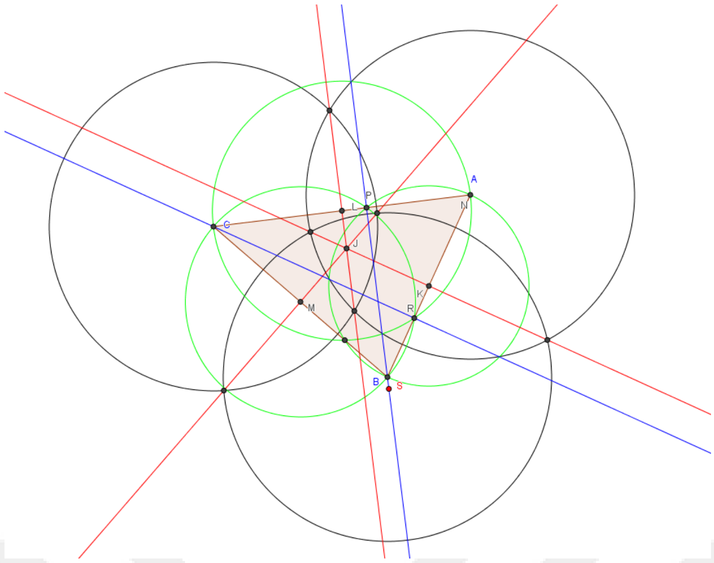


Figure 4. 78. The example for the drag test failure of a geometric figure formed via A6 in Activity 3 without assumption violation

As seen from Figure 4.78, the altitude of $\overline{BC}$, which is the third blue line, was not present anymore. When how this altitude was formed was examined (See Figure 4.72), it was seen that it was formed from the intersections of two circles with the diameters $\overline{AB}$ and $\overline{CA}$. Although these circles are still present in the figure presented above, the points of intersection of them which are the vertex A and the points N overlapped so that there is not the altitude of $\overline{BC}$ in Figure 4.78. Besides, the point S was found out as the point of concurrency of the altitudes formed via applying A6 in Figure 4.72. However, it can be observed that the point S lost this property under dragging as presented in Figure 4.78. In this respect, it was concluded that the assumption violation is not the only reason for the failure of the figure formed via A6 in the drag test. Therefore, it was labeled as not a construction based on the diagram in Figure 4.17.

4.3.2.4. Approaches GG Offered for Geometric Construction in Activity 4

In the last sub-heading in this section, the approaches that GG proposed for the geometric construction in Activity 4 were explained. During the administration of Activity 4, GG was given one GeoGebra file which was customized in a way that the tools ‘circle through three points’ and ‘circumcircular arc’ were extracted from the toolbar. As usual, the reason for the restriction of the mentioned tools is to avoid the tools which constitute the intended figure by means of one or a few clicks without paying attention to the reasoning much. In addition to the GeoGebra file, the worksheets involving ΔABC, which was displayed before in Figure 4.41, were also distributed to the participants of GG.

It was documented that GG offered thirteen ideas with the aim of finding a way for the construction. Although the majority of them have no avail in terms of conducting a construction except for A6, it was seen that the highest number of ideas for construction was offered by GG in this activity compared to all other activities of both CSG and GG. The locations of these approaches throughout the global argumentation structure of GG were indicated in Figure 4.79.

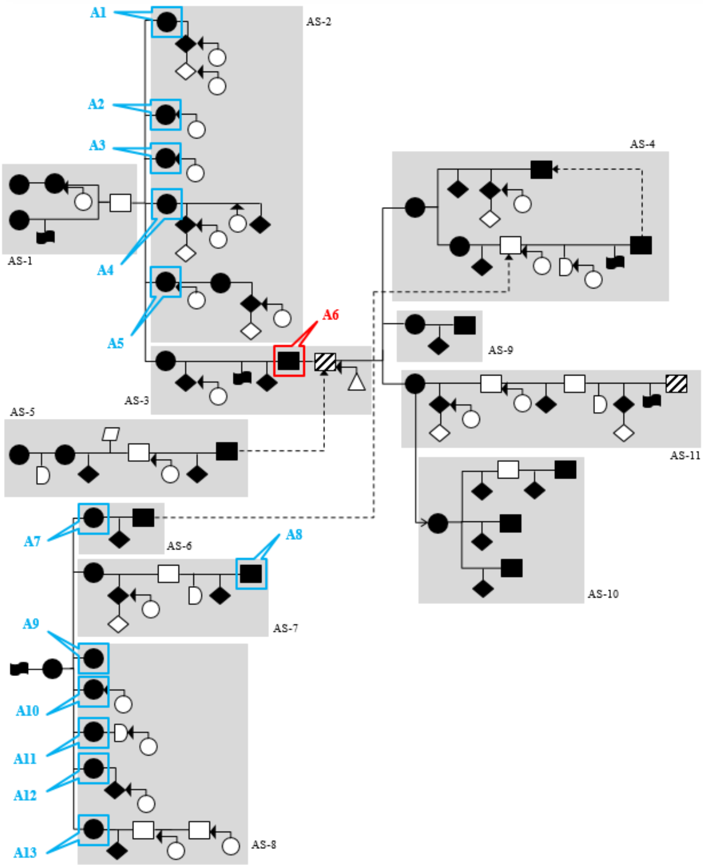


Figure 4. 79. The locations of approaches offered for the construction in the global argumentation structure of GG in Activity 4

As indicated in Figure 4.79, the approaches GG considered as invalid for construction were plotted with blue indicators whereas one approach which was stated as valid by GG was marked with a red indicator. The ideas of the invalid approaches were offered through two periods; the first period covers the A1, A2, A3, A4, and A5 while the second period involves the remaining seven approaches. Although GG considered that they had a valid one via A6, they continued to search for the alternative construction approach which brought up the sequence of approaches in the second period. Moreover, most of the invalid ones were marked via the data component since they were given up quickly and only A8 was marked through the conclusion component. On the other hand, the valid one which is A6 was marked via the conclusion component since it reached a product at the end and the final idea about the validity of the approach was apparent in conclusion. Initially, the invalid ones were probed based on the summary in Table 4.18 and then the valid approach was taken into consideration in a subsequent table, namely, Table 4.19.

Table 4. 18

Approaches GG stated as invalid for geometric construction in Activity 4

Approach for construction	Validity of approach
A1. GG focused on the toolbox related to drawing of a circle. However, they could not find a tool so that all points will be on the circle. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram (See Figure 4.17) - Invalid approach for Activity 4 - Not finished
A2. GG offered to use the tool 'circle through three points'. However, this idea was not applicable since this tool was removed from the toolbar. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished
A3. GG thought about using the tools related to arcs to be able to construct the circles passing through the determined three points. However, they could not find a working specific idea for construction so that they gave up. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished

Table 4. 18 (continued)

A4. GG attempted to find an approach for construction based on a figure that they are familiar. They drew circles by accepting the vertices as centers and drew the lines passing from the intersections of these circles. However, they could not come up with a reasonable approach for the intended construction so that they quitted. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished
A5. GG tried to find an approach by drawing circles like A4. For this time, they wanted to try two circles, each of which has the center as a vertex but also passing from the randomly located points on the adjacent sides separately. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished
A7. GG offered to construct the circumcenter and the circumcircle of ΔABC and then search for a possible idea that might be deduced from this construction. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished
A8. GG focused on an inverse point of view. They offered to determine the locations of X, Y, and Z by constructing the point of concurrency of the angle bisectors and also drawing the lines from this point as perpendicular to the sides of ΔABC. Based on these points, they offered to search for a starting point for the construction. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished
A9. GG tried to use some other tools which are 'circular sector' and 'circumcircular sector' to be able to construct the intended three circles. However, they could not construct the circles although there is a working case via using the tool 'circumcircular sector'. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4
	Based on the diagram - Invalid approach for Activity 4 - Not finished

Table 4. 18 (continued)

A10. GG formed a large triangle by drawing parallel lines to the sides of ΔABC as passing from each vertex. Then, they aimed to accept the vertices of ΔABC like the points X, Y, and Z for the larger triangle. Similar to the previous attempts, they sought for a starting point for the construction based on this idea. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4 *Based on the diagram* - Invalid approach for Activity 4 - Not finished
A11. GG offered to accept the fact that they were drawing a circle passing through four points as the starting point to find an alternative construction approach. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4 *Based on the diagram* - Invalid approach for Activity 4 - Not finished
A12. GG offered to use the arc related tools for the construction since they wanted to use the theorem 'the sum of the opposite angles in any quadrilateral inscribed in a circle is 180°'. However, they could not come up with a working idea and gave up again. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4 *Based on the diagram* - Invalid approach for Activity 4 - Not finished
A13. GG followed the idea in A11 with a different perspective. They considered finding the fourth point on the sides by using the angle related tools. However, this idea did not end with a working approach for construction. *(not written in the worksheet, no GeoGebra file)*	*According to GG* - Invalid approach for Activity 4 *Based on the diagram* - Invalid approach for Activity 4 - Not finished

As indicated above, Table 4.18 gives an outline about twelve approaches that GG did not categorize as valid for the intended construction of Activity 4. It also involves evaluations regarding the validity of these approaches. None of these approaches were noted down in the worksheets by GG and any of these attempts was saved as a GeoGebra file. Due to the lack of a saved GeoGebra file related to these approaches, some simulated figures were formed when required while explaining them. These approaches were examined more closely in the followings of this section.

When GG started to work on Activity 4, they discussed the givens and placed the points X, Y, and Z on the corresponding sides of ΔABC in the GeoGebra file to conceive what was asked in the activity in-depth. At the end of a preliminary reasoning process, they decided to search for an approach for construction at first and then work on the relationships among three circles. Thus, they posited ideas with the aim of finding a usable approach for construction.

The first five approaches were located in AS-2 of the global argumentation structure of GG, each of which was an argumentation step of AS-2 (See Figure 4.79). As described before, a range of ideas which was offered consecutively, given up or refuted quickly, and served the same purpose was presented via one argumentation stream. As the first idea, A1 was marked in the first step of AS-2 of the global argumentation structure of GG. In A1, GG considered how they could use the tools resembling the compass. In this manner, they examined the toolbox related to drawing of a circle which involves the tools such as ‘circle with center through point’, ‘circle with center and radius’, and ‘compass’. By reading the tooltips of these tools, they expressed the need for a center and a radius or another point or a line segment to have the radius. Meanwhile, they tried to find a starting point to use these tools in conjunction with the construction in Activity 4. For example, one of them offered to accept $\overline{AX}$ as the radius and the center as the point A, but this idea was refuted since the point A cannot be on the circle in this case and they need a circle passing from both of these points as well as the point Z. The presence of two rebuttals which functioned as the refutation of the whole process can be seen in the first step of AS-2. Finally, they gave up this idea since they did not know about the centers and the radii of the intended circles.

The second idea stated by one of the participants of GG as a possible approach for construction in AS-2 was the use of the tool ‘circle through three points’. This idea was confuted by a rebuttal which reminded that this tool was restricted in Activity 4, as can be seen in the second step of AS-2, and then they did not continue to work about this idea.

As the third approach, the idea offered was related to drawing arcs to be able to construct the circles passing through the determined three points. However, one of them made group give up this idea by stating that she tried about the arcs and could not come up with a working case for construction. The rest of the group was convinced instantly and did not insist on searching for any possible case related to arcs. It was seen from the video recordings that the participant who tried the arc related tools attended to use the tools ‘semicircle through two points’ and ‘circular arc’. Due to the nature of these tools, it was not possible to adjust an approach for the aimed construction.

In A4, GG did not attend to use the GeoGebra file and they simply draw the idea on the worksheet. Moreover, it can be stated that they put forward A4 without having a clear idea, but their aim was to find a starting point by virtue of conducting the familiar construction and to associate it somehow to the intended construction in Activity 4. More specifically, one of the participants of GG offered to draw circles by accepting the vertices of ΔABC as centers and to draw the lines passing through the intersection points of each pair of circles. Then, she declared that these lines would be perpendicular to the sides of ΔABC. While explaining this idea to others, she delineated two circles by accepting the vertices A and C as the centers and also a line passing through the intersection of the mentioned circles on the worksheet. By focusing on this particular drawing, they attempted to derive an approach for the construction of three circles, but their efforts did not land up a clear approach for construction. Since the rest of GG could not grasp what was offered throughout the process, two rebuttals can be seen in the fourth argumentation step of AS-2 (See Figure 4.79).

Unlike A4, GG used the GeoGebra file while explaining the idea of A5. Moreover, GG maintained the idea of drawing two circles in A5 which is the last argumentation step in AS-2. The simulated geometric figure formed based on A5 by pursuing the video recordings was given as in Figure 4.80.

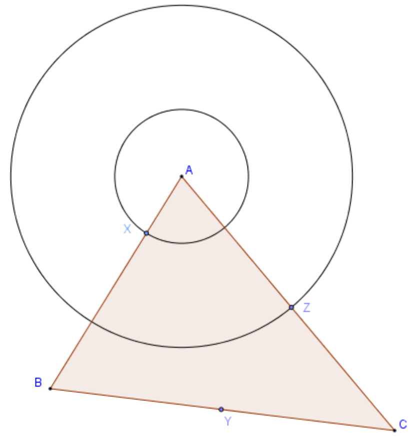


Figure 4. 80. The simulated geometric figure for A5 in the Activity 4

The centers of both circles were the vertex A and they were also passing through the points X and Z on the adjacent sides of the vertex A separately as the first step. Moreover, GG formed these circles by using the tool ‘circle with center through point’. Then, they searched for whether they could continue from that point. However, they refuted this idea by stating that there should be one circle passing through all of these three points instead of two separate circles. Based on this refutation, they thought about determining the points X, Y, and Z on the sides in a more feasible way such as near to the midpoint of the sides. At this point, they proposed another rebuttal again by stating the points X, Y, and Z should be determined randomly. Then, they left this idea and did not think about it further. Thus, GG concluded that five ideas offered to find a construction ideas, which were located in AS-2, could not work as proper approaches for the construction of three circles.

After that, the second period related to the construction, which involves A7, A8, A9, A10, A11, A12, and A13, was also explained by descending into particulars as follows. As seen, it involves seven ideas offered for construction, each of which was seen as invalid throughout the construction process by GG, located in three argumentation streams which are AS-6, AS-7, and AS-8.

As the first idea in the second period of the construction attempts, A7 was marked in the data of AS-6 of the global argumentation structure of GG (See Figure 4.79). Although AS-6 involves the conclusion component, they moved away from the construction focus after data so that it would be better to represent A7 with data rather than conclusion. In more detail, one of the participants of GG offered to draw the circumcircle of ΔABC via finding the circumcenter at first and then she offered to search for a possible idea that might be deduced from this construction. However, they noticed something not directly related to the construction and focused on it after the data. Therefore, GG did not apply A7 in the GeoGebra file and also did not continue to work on this issue. Thus, it was involved in the invalid approaches.

The conclusion of AS-7 of the global argumentation structure of GG (See Figure 4.79) was marked as A8. While searching for the alternative approaches for construction, GG had an idea related to the locations of the points X, Y, and Z on the sides and suggested to search for an approach for construction later based on this issue. In more detail, one of the participants of GG offered to construct the angle bisectors of ΔABC at first, determine the point of concurrency of the angle bisectors, and then draw the perpendicular lines from this point to the sides. Thus, the intersections of these lines and the sides would be accepted as the points X, Y, and Z. However, another participant did not agree with this starting idea since she declared that the randomness of the points could not be provided in such an application. Despite this rebuttal, they continued to think about whether any approach for the construction of three circles might be deduced from this idea. However, they could not put forward such an idea. Moreover, they concluded that this attempt refers to the beginning of the construction, presents a way to determine the points X, Y, and Z only, and does not provide an approach in terms of the intended construction. As expected, they did not write about A8 in the worksheets and also did not try it even in the GeoGebra file.

In the first argumentation step of AS-8, A9 was marked via the data component (See Figure 4.79). In A9, one participant tried to use the tools ‘circular sector’ and ‘circumcircular sector’ but could not construct the circles. In fact, she could draw the circumcircle of a triangle by combining three circumcircular sectors. However, she expressed that she was using the mentioned tools and gave up quickly this idea as it can be seen in the step that there was not even an opposition said explicitly in the group.

In the second argumentation step of AS-8, which was marked as A10, GG worked on the GeoGebra files and used some tools. Since they did not save a GeoGebra file related to A10, the simulated figure was presented below.

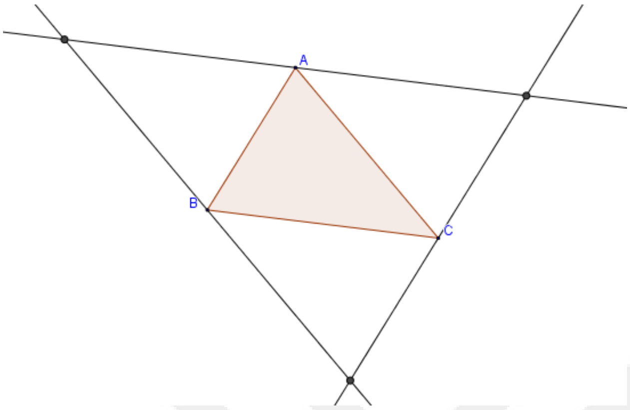


Figure 4. 81. The simulated geometric figure for A10 in the Activity 4

As the first step of A10, GG drew the lines parallel to the sides of ΔABC through the vertices of ΔABC by using the tool 'parallel line' so that they formed a larger triangle. Then, they thought about accepting the points A, B, and C as randomly determined points on the sides of the larger triangle like the points X, Y, and Z so as to search for a possible approach for the construction of three circles. However, since they could not come up with an approach which is primarily related to the construction, they stopped working on it.

In the third argumentation step of AS-8, A11 was marked by means of the data component. GG declared that they were actually drawing circles passing through four points since there was the fourth point which is the concurrency point of three circles. They offered to accept the fact that they were drawing a circle passing through four points as the starting point to find an alternative construction approach. However, this idea was followed by a challenger and rebuttal so that they could not go further on this idea.

In the fourth step of AS-8 in which A12 was marked, GG suggested to use the arcs and to examine the arc related tools for the construction. Since they noticed the existence of a cyclic quadrilateral while working on the third step of AS-8, they mentioned about using the theorem related to cyclic quadrilaterals that ‘the sum of the opposite angles in any quadrilateral inscribed in a circle is 180°’. Since they were not completely sure about this theorem and also could not associate the theorem to a construction approach of the intended circle, they gave up again after a rebuttal.

The idea in the last argumentation step of AS-8 was marked as A13 and it is also related to the third step of AS-8. In more detail, one of them insisted on the construction of the circle passing through four points but from a different point of view. That is, they considered finding the fourth point by using the angle related tools. To that end, they measured $\angle CAB$ by using the tool ‘angle’. Then, they determined a random point X on $\overline{AB}$, placed a random point D inside the triangle, and accepted the point D as the point of concurrency of three circles. At this point, their aim was to find a point Z on $\overline{CA}$ by using the tool ‘angle with given size’ in a way that the mentioned four points would be used to draw a circle. However, in the end, they concluded that they could not draw a circle passing from any four points so this idea would not lead them any convenient approach for construction.

As seen from the mentioned twelve attempts for the construction asked in Activity 4, GG generally tried to find a different starting point to be able to produce an alternative construction approach. During these attempts, they delineated the idea on the worksheet or used some tools of GeoGebra. However, in all of them, GG gave up and accepted that they could not state an applicable approach for the aimed construction. The mentioned withdrawals, especially in AS-8, can be seen through the presence of the rebuttal components (See Figure 4.79). Moreover, these approaches were also evaluated based on the diagram given in Figure 4.17. Since any of them was able to present a geometric figure proper to the intended construction, all of them failed in the first phase of the diagram. Thus, these approaches were also coded as invalid due to their unfinished status.

After the explanation of the invalid approaches stated by GG, it is the turn of one valid approach suggested in Activity 4. Therefore, the content of A6 was presented briefly in Table 4.19 given below.

Table 4. 19

Approach GG stated as valid for geometric construction in Activity 4

Approach for construction	**Validity of approach**
A6. GG noticed that the construction of the asked three circles corresponds to the construction of the circumcircles of triangles formed by the given three points. Thus, GG formed three triangles which are ΔAXZ , ΔBYX , and ΔCZY . Then, they formed the circumcircles of these triangles by means of drawing the perpendicular bisectors of the sides. *(written in the worksheet, GeoGebra file 1)*	*According to GG* - Valid approach for Activity 4 - Construction
	Based on the diagram (See Figure 4.17) - Valid approach for Activity 4 - Construction (CTB)

Table 4.19 covers the summary of A6 in the first column and the evaluations of the validity of A6 in the second column. The only approach GG explained in the worksheet was A6 and it can be examined from Appendix F. Besides, GG saved the geometric figure formed via A6 as a GeoGebra file by naming as 1.ggb. What GG formed by applying A6 was displayed in Figure 4.82.

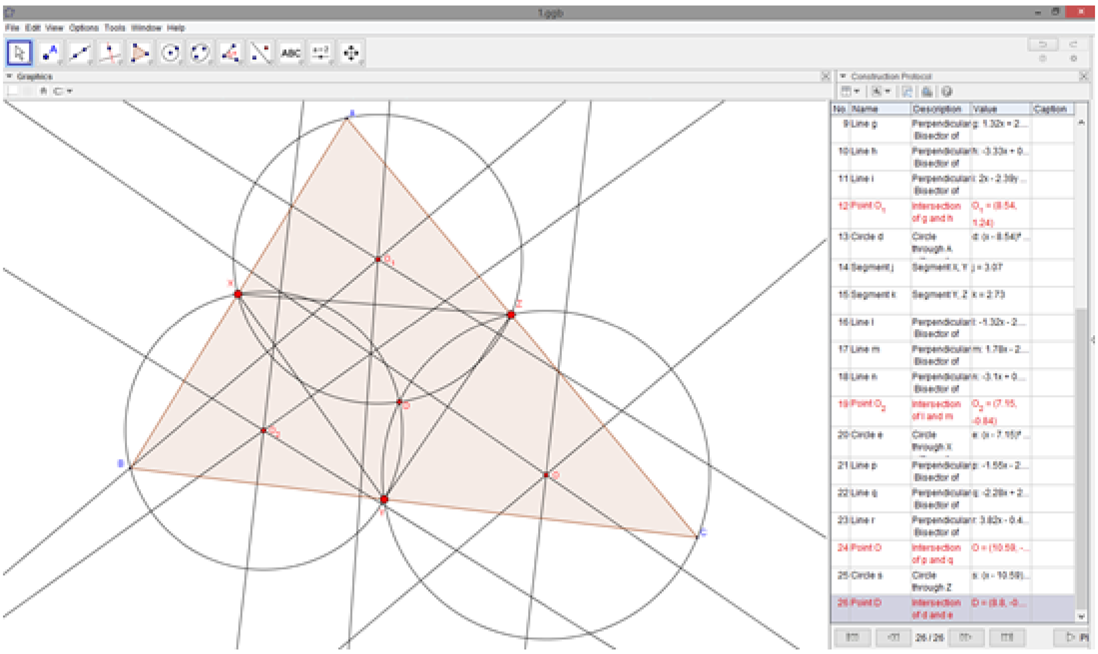


Figure 4. 82. GeoGebra file GG submitted in terms of A6 in Activity 4 (1.ggb)

As can be seen from Figure 4.79, A6 was contained in the conclusion component of AS-3 of the global argumentation structure of GG. They noticed that the construction of three circles corresponds to the construction of the circumcircles of three triangles. To that end, the first move of GG in A6 was to draw the line segment from the point X to the point Z so as to signify ΔAXZ . Similarly, two more line segments were drawn, one of which from the point X to the point Y to determine ΔBYX and other one was drawn from the point Y to point Z to present ΔCZY . Then, they tried to remember how they constructed the circumcircle from the previous cognitive unity based activities. Since three circles which constitute the basis of A6 were present in the GeoGebra file, they started to the construction of the circumcircles of each triangle one by one.

First of all, GG started to work with ΔAXZ . In more detail, GG constructed the perpendicular bisectors of each side of ΔAXZ by using the tool ‘perpendicular bisector’ and then determined the point of concurrency of them, which was named as O1, via the tool ‘intersect’. Afterwards, they pursued the same procedure for ΔBYX and ΔCZY to construct the circumcircles of them. As the final step, GG used the tool ‘intersect’ gain to determine the point of concurrency of three circles that they noticed while dragging.

As mentioned, it was also checked whether the approach that GG stated as valid actually constitutes a valid case for the activity based on the diagram in Figure 4.17. In the first phase of the diagram, the figure GG presented was examined and stated as a proper one for the construction asked in the activity. This answer directed to the drag test criterion in the second phase of the diagram. It was also observed that the geometric figure kept its properties under dragging. Since the figure passed the drag test criterion, it led to the check of the compatibility criterion in the third phase. Since GG did not pay attention to the Euclidean restrictions in A6 and used the most appropriate tools of GeoGebra during the implementation of A6, it cannot be stated that the figure formed via A6 passed the drag test criterion. The stepwise evaluation resulted in that the construction of three circles via A6 fell into construction type B (CTB) in the diagram.

4.3.3. Summary of Approaches Offered for Geometric Constructions

The approaches CSG offered throughout the cognitive unity based activities were summarized in Table 4.20. Actually, this table can be seen as a summary of the seven tables given within this section, which are Tables 4.6-4.12.

Table 4. 20

Summary of the approaches CSG offered for geometric constructions

	Invalid approaches	Valid approaches
Activity 1	A1, A3	A2, A4
Activity 2	A1, A3	A2, A4, A5
Activity 3	A1, A2	A3, A4, A5
Activity 4	-	A1

The approaches were evaluated whether a geometric figure which can be labeled as a construction could be presented by applying them. Since the evaluations of CSG related to the validity of the approaches are consistent with the results obtained after the evaluation based on the criteria list given in Table 4.5, the approaches were presented as a whole in Table 4.20. As seen, CSG was successful in the determination of the validity of the approaches. Moreover, CSG could present at least one valid approach for the geometric constructions asked in the activities.

The approaches GG offered throughout the cognitive unity based activities were summarized in Table 4.21. Similarly, this table can be seen as the summary of a series of tables given within this section, which are Tables 4.13-4.19.

Table 4. 21

Summary of the approaches GG offered for geometric constructions

	Approaches GG stated as invalid	Approaches GG stated as valid	Based on the diagram given in Figure 4.17
Activity 1	A1, A2, A3, A4, A5, A7	A6, A8	A6- CTB A8- CTA
Activity 2	A3	A1a, A1b, A2a, A2b, A2c	A1a, A1b, A2b- Not a construction A2a, A2c- CTB

Table 4. 21 (continued)

Activity 3	-	A1, A2, A3, A4, A5, A6	A1, A2, A3- CTB A4, A5- CTA A6- Not a construction
Activity 4	A1, A2, A3, A4, A5, A7, A8, A9, A10, A11, A12, A13	A6	A6- CTB

Since the evaluations of GG and the results obtained after the evaluation based on the diagram given in Figure 4.17 are not consistent, an extra column was added in Table 4.21. In more detail, the approaches GG stated as invalid were found out as invalid according to the evaluation conducted based on the diagram. However, there are some discrepancies for the ones GG stated as valid. As seen, in the rightmost column of Table 4.21, the results obtained from the evaluation conducted based on the diagram were presented. Thus, the approaches which are decided as invalid although GG declared as valid can be seen. Moreover, the types of constructions reached at the end of the diagram were indicated in the mentioned column of Table 4.21. As the majority of geometric figures arranged by means of the application of the valid approaches was categorized as CTB, a few of them were categorized as CTA, and there was not a geometric figure can be coded as CTC in this study.

Until this point, the approaches offered to perform geometric constructions asked in the cognitive unity based activities were explained in detail. In the next section, the findings related to the arguments of CSG and GG presented while aiming to prove the conjectures produced in the argumentation process will be presented.

4.4. Proof of the Conjectures Produced in the Argumentation Process

The findings reported in this section are based on the data obtained from both the argumentation process of prospective middle school mathematics teachers while working on geometric constructions and the proof process of the conjectures produced in the mentioned argumentation process. More specifically, in the direction of the fourth research question and the sub-questions, it can be stated that this section involves three foci which are the content of the conjectures that groups produced during the argumentation process, the determination of the ones to ask for proof, and the examination of the validity of the arguments that groups proposed to prove the conjectures as a result of collective argumentation.

In more detail, firstly, the locations of the conjectures which groups produced during the argumentation process while performing geometric constructions were indicated by means of the global argumentation structures, which were elicited from the findings of the second research question. All conjectures that groups produced were presented in tables. Then, the conjectures represented with the target conclusion component were pointed out with a different color in these tables. As mentioned earlier, a list of possible conjectures directly related to each activity, which can be coded as target conclusion in the argumentation structure, was prepared in advance of the application of each activity. As based on the cognitive unity concept, it was also planned to ask groups to prove one of the conjectures which they produced. This particular conjecture was selected among the ones represented with target conclusion. That is to say, four conjectures in total were clarified at this step and each of them was asked to prove in one of the activities. Finally, the worksheets of each group for all four activities which were presented as proofs were evaluated in terms of their validity. At this step, the validation process of the arguments of groups was conducted by adapting the validation stages and strategies mentioned in some studies in the literature (e.g., Alcock & Weber, 2005; Bleiler, Thompson, & Krajčevski, 2014; Ko & Knuth, 2013; Selden & Selden, 2003; Stylianides, 2015; Weber, 2008; Weber & Alcock, 2005). Moreover, the proofs of four conjectures were presented in Appendix G to exemplify what kind of arguments were expected from the participants and in which cases their argument could be labeled as valid proof.

To report the findings of the fourth research question, five sub-sections were deployed. The first four sub-sections were designed in a way that each of them unfolded the mentioned foci for one of the cognitive unity based activities. In parallel to that, the whole processes of both CSG and GG in Activity 1 were taken into consideration in the first sub-section. In the followings, each activity will be approached with the same perspective via separate sub-sections. Then, in the last sub-section, to what extent the groups came up with valid proofs will be summarized.

4.4.1. Proof of the Conjecture in Activity 1

As mentioned, via the worksheet A of Activity 1, it was asked to construct the circle passing from the vertices of the given acute triangle by using compass-straightedge or GeoGebra. Actually, they were not directly asked either to find a connection or to state a conjecture. However, it was expected that the argumentation process in the geometric construction section of the activity would lead them to produce conjectures. As can be seen from the subsequent figures (See Figures 4.83 and 4.84) and tables (See Tables 4.22 and 4.23), CSG produced three conjectures whereas GG produced four conjectures throughout the construction process in Activity 1. First of all, the locations of the conjectures in the global argumentation structures of CSG were given in Figure 4.83. Then, accordingly, which conjectures CSG came up with will be presented.

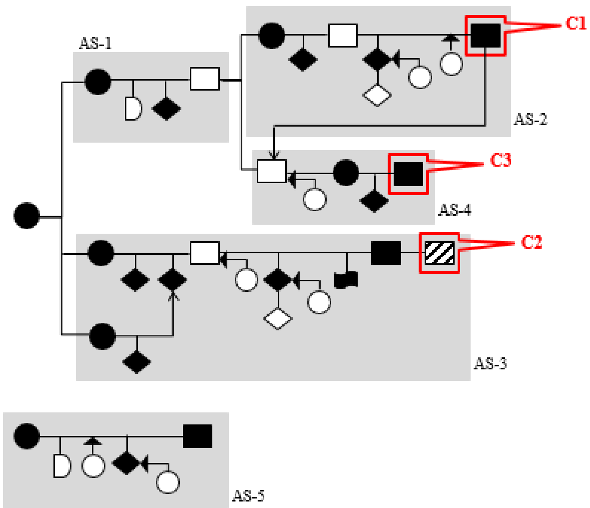


Figure 4. 83. The locations of the conjectures produced by CSG in Activity 1

According to Figure 4.83, C1 and C3 are not among the particularly aimed conjectures for this activity and they were marked by virtue of the conclusion component. On the other hand, C2 was represented with a target conclusion since it was a conjecture directly related to the whole concept of the activity. The marked conjectures were stated in Table 4.22, as given below.

Table 4. 22

The conjectures that CSG produced in Activity 1

Conjectures	
C1	The angle bisectors of ΔABC are concurrent and this point is the incenter of ΔABC.
C2	The perpendicular bisectors of the sides of ΔABC are concurrent and this intersection point is equidistant to the vertices of ΔABC. Thus, the point of concurrency of the perpendicular bisectors of the sides is the circumcenter of ΔABC.
C3	The medians of ΔABC are concurrent and this point is the centroid of ΔABC. Since ΔABC is a scalene triangle, the centroid and the incenter are not the same points. For an equilateral triangle, these two points coincide.

As expected, all conjectures are not directly related to the context of Activity 1. In more detail, C1 covers the fact that the point of concurrency of the angle bisectors of a triangle is the incenter of that triangle whereas C3 declares the case that the point of concurrency of the medians of a triangle is the centroid of that triangle. It can be stated that C1 and C3 were produced as a result of the attempts to find an approach for the construction of the circle passing through the vertices of the given triangle. Since the approaches leading to C1 and C3 were not valid in terms of the desired construction, the conjectures derived from these approaches were not among the expected ones and their proofs were not asked. Between these conjectures, CSG came up with the idea that they should try the perpendicular bisectors of the sides of the given triangle so as to construct the aimed circle. Thus, C2 propounds the fact that the point of concurrency of the perpendicular bisectors of the sides of a triangle is the circumcenter of that triangle. That is to say, via C2, CSG reached a conjecture possible to ask in terms of proving. Therefore, after the construction phase of the activity, CSG was handed the worksheets which ask to prove the conjecture denoted as C2, which was signified with a different color in the table above.

As mentioned, in Activity 1, GG declared one more conjecture compared to CSG. As usual, where these conjectures located in the global argumentation structure of GG was presented to be able to unveil the appearance order of these conjectures in the flow of the argumentation.

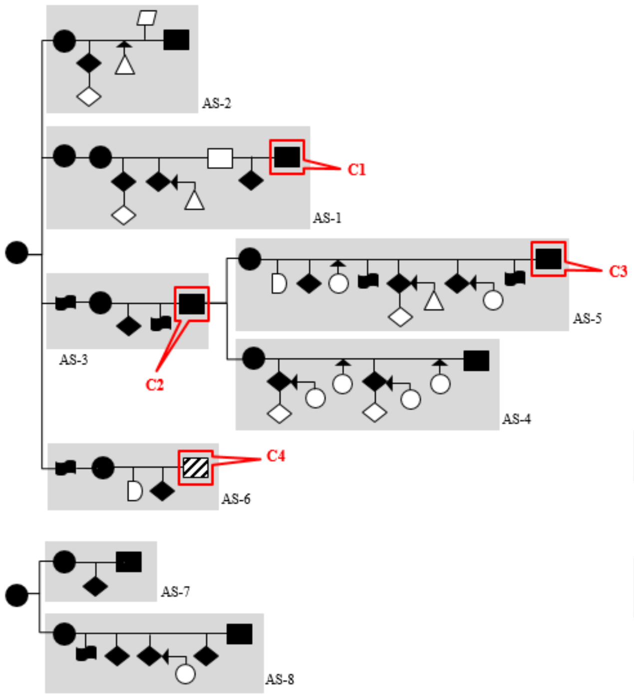


Figure 4. 84. The locations of the conjectures produced by GG in Activity 1

The presence of one more conjecture in the argumentation of GG did not result in more than one conjecture which can be asked for proving. In other words, only one conjecture, which is C4, was represented via target conclusion. The first three conjectures were not directly related to the scope of the activity so that they were signified with the conclusion component. The details of four conjectures produced by GG can be found in Table 4.23 given below.

Table 4. 23

The conjectures that GG produced in Activity 1

Conjectures	
C1	The centroid of ΔABC, which the point of concurrency of the medians, is not the center of the circle passing from the vertices of ΔABC. The centroid is not equidistant to the vertices of ΔABC.

Table 4. 23 (continued)

C2	The angle bisectors of ΔABC are concurrent and this point is the incenter of ΔABC.
C3	A larger triangle was constructed by drawing lines as passing through the vertices of ΔABC and parallel to the sides of ΔABC. The incircle of the larger triangle is not the circumcircle of ΔABC. This statement is not true for every triangle.
C4	The perpendicular bisectors of the sides of ΔABC are concurrent and this intersection point is equidistant to the vertices of ΔABC. Thus, the point of concurrency of the perpendicular bisectors of the sides is the circumcenter of ΔABC.

It was noticed that C1, C2, and C4 stated by GG are nearly the same with C3, C1, and C2 produced by CSG, respectively. Thereby, C3 produced by GG is the extra conjecture compared to the ones of CSG. As seen, the conjecture production sequences of the groups are different. In more detail, GG underlined the fact that the centroid of ΔABC can be constructed by finding the point of concurrency of the medians via C1. Moreover, C1 covers the fact that the centroid is not the center of the circle passing from the vertices of ΔABC since the centroid is not equidistant to the vertices of ΔABC. As the second conjecture, which was marked as C2, GG asserted that the point of concurrency of the angle bisectors of ΔABC gives the incenter, not the center of the intended circle. Followed by C1 and C2, GG put forward another conjecture which was denoted as C3. It can be stated that C3 was reached as a consequence of an invalid trial for construction. What was coded as C3 is that the incircle of a larger triangle, which was formed by drawing the lines from the vertices of the given triangle as parallel to the sides of that triangle, does not always give the circumcircle of the given triangle. After the mentioned not working trials, GG thought about trying the construction of the perpendicular bisectors of the sides of ΔABC to be able to construct the aimed circle. This approach ended up with the last conjecture, which was named as C4. In more detail, C4 refers to the fact that the point of intersection of the perpendicular bisectors of the sides of ΔABC is the circumcenter since it is equidistant to the vertices of ΔABC.

Since Activity 1 is related to the construction of the circumcircle of ΔABC by means of the perpendicular bisectors of the sides, the conjectures pertaining to these issues were represented with the target conclusion component in the global argumentation structures. The number of possible and acceptable conjectures in terms of this focus was not much in Activity 1. Two conjectures which were expected to be produced and planned to be asked for proof in case of existence are stated as follows; "the perpendicular bisectors of the sides of a triangle are concurrent" and "the point of concurrency of the perpendicular bisectors of the sides of a triangle is the circumcenter". As colored in Table 4.22, C2 produced by CSG meets the mentioned possible conjectures for Activity 1. Similarly, as colored in Table 4.23, C4 stated by GG also corresponds to the mentioned possible conjectures for Activity 1. Therefore, for CSG and GG, the statement given in the worksheet B of Activity 1 was that "the perpendicular bisectors of the sides of a triangle are concurrent and this point is the circumcenter of the triangle". That is, they were asked to prove the conjecture they produced recently.

Moreover, the proof of the statement given above, which was prepared based on the proofs presented in some textbooks (e.g., Alexander & Koeberlein, 2011, p.332; Gutenmacher & Vasilyev, 2004, p.35; Leonard et al., 2014, p.42-43; Serra, 2003, p.180), was displayed in Appendix G. As the last stage of this sub-section, the evaluation of the validity of groups' arguments in Activity 1 will take the turn at this point.

As the first step in the evaluation process of the validity of the arguments of the groups, the overall structure of the argument was checked by following the guidelines, which were explained in the data analysis heading of the methodology chapter. Then, the line-by-line checking was conducted as the second step of the evaluation process. By tracking the results of the mentioned evaluation process, the final decisions regarding the validity of the arguments in Activity 1 were conducted and presented as in Table 4.24. In addition, this table displays the extraneous errors in the arguments which were decided as the issues which do not directly affect the validity of the arguments.

Table 4. 24

Validity of the arguments of CSG and GG in Activity 1

Group	Validity of the argument	Extraneous errors
CSG	Invalid argument with warrant error	- Notation errors - Term and expression errors - Unclear inferences and steps - Lack of presentation of some assumptions
GG	Valid proof	- Notation errors - Term and expression errors - Unclear inferences and steps - Lack of presentation of some assumptions

The argument of CSG cannot be categorized as a valid one due to the major errors in warrants offered in some points throughout the argument. The argument of GG was decided to be labeled as valid proof in spite of the listed extraneous errors. The analysis process of these arguments will be explained from this point on.

To begin with, each line of the arguments of both CSG and GG in Activity 1 were numbered regardless of the content and meaning of the lines. Hence, the arguments were set for the evaluation process and these arguments were displayed in Appendix H. Since the arguments were written in Turkish by the groups, it was decided that to display them in the appendix. However, the translated versions of the sentences in the arguments were involved while explaining the line-by-line analysis of them. As usual, preceded by the argument of GG, the one proposed by CSG will be taken into consideration as follows.

According to the result of the examination of the overall structure of the argument of CSG in Activity 1, it was found that the argument is a completed one. The aim is to show that the statement is true by utilizing direct proof as the method. In a general sense, it can be stated that the argument was written clearly since it involves the related drawings and it is easy to follow. After this general examination, the line-by-line analysis of the argument was carried out. The argument involves a figure at the beginning which is an acute triangle. To explain the underlying reason for this evaluation, the figure in the argument of CSG was presented as in Figure 4.85

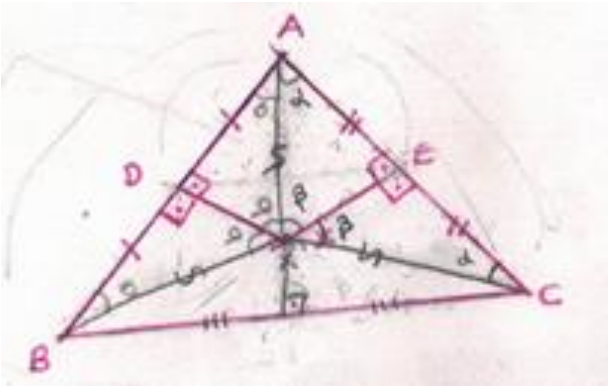


Figure 4. 85. ΔABC in the argument of CSG in Activity 1

Having started to read the first line of the argument, the lack of some assumptions was noticed. For example, an introductory line such as "let ΔABC be a triangle" was not written. The first two lines presented some other assumptions used at the beginning of the argument and drawn in ΔABC. CSG wrote that "Assume that DX and EX which are the perpendicular bisectors of the sides intersect at the point X. We drew the line segments $|AX|$, $|BX|$, and $|CX|$". As seen, they did not use the correct notation while stating the perpendicular bisectors of the sides and the line segments drawn. Moreover, they did not mention that $\overline{DX}$ and $\overline{EX}$ belong to which sides of ΔABC exactly. The mentioned cases can be regarded as examples for the items notation error and lack of presentation of some assumptions listed under the extraneous errors column in Table 4.24.

In lines 3, 4, 5, and 6, CSG focused on two triangles, which are ΔAXE and ΔCXE, and attempted to set out side-angle-side congruence (S-A-S) for these triangles. Nevertheless, their reasoning to justify this congruence was not correct, which caused the argument to be labeled as an invalid argument with warrant error. CSG stated that "the angles $\angle XCE$ and $\angle XAE$ are the same since they are opposite to the same side" and also "the angles $\angle CXE$ and $\angle AXE$ are the same since they are opposite to the congruent sides". Lastly, they stated that $\angle CEX$ and $\angle AEX$ were accepted as 90º from the assumption that $\overline{XE}$ is the perpendicular bisector of $\overline{CA}$. By combining these ideas, CSG wrote down that S-A-S congruence was reached for ΔAXE and ΔCXE so that "$\Delta AXE = \Delta CXE$" was stated. However, how they arranged this congruence was incorrect although the mentioned triangles are actually congruent. They behaved like $\angle CXE$ and $\angle AXE$ are the elements of the same triangle. Although these angles are opposite to $\overline{XE}$, there existed two triangles at stake. In a similar manner, the equality of the measures of $\angle CXE$ and $\angle AXE$ was deduced from an incorrect case which is being opposite to the congruent sides. This idea does not guarantee the equality of the mentioned angles due to the case that these angles are from different triangles. Based on three angles related to the issue, they assumed that they have side-angle-side congruence which is also an incorrect conclusion. To have such a congruence, they would state that $\overline{XE}$ is the common side of ΔAXE and ΔCXE, $|AE| = |EC|$, and $\angle CEX = \angle AEX$ = 90º due to the assumed perpendicular bisector of $\overline{CA}$, so that $\Delta AXE \cong \Delta CXE$ can be stated by means of side-single-side congruence. After this congruence, in line 6, CSG stated that "$|CX|$ and $|AX|$ are equal since they are opposite sides to the equal angles". That is to say, the warrant of these conclusion contains the critical errors. Besides, during these steps, some notation errors such the congruence notation were seen and some terms such as same, congruence, and equal were not used properly so that this issue was an example for the term and expression errors in Table 4.24.

Along with lines 7 and 8, CSG mentioned applying the same process for ΔAXD and ΔBXD. However, they skipped the process and directly concluded that the last mentioned triangles are congruent from S-A-S congruence. In line 9, they stated that they found out $|BX| = |AX| = |CX|$ without stating the main issue of the last congruence which is $|AX| = |BX|$. These three lines would be stated more clearly and step by step. These lines can be seen as examples for the item unclear inferences and steps from the extraneous errors in Table 4.24.

From the equality of the lengths of the line segments given in line 9, CSG came up with the case that ΔBXC is an isosceles triangle in line 10. Since the altitude of the base of an isosceles triangle bisects the base, they stated that the perpendicular line passing through the point A divides $|BC|$ into two equal parts, as written in lines 10 and 11. This was a reasonable inference. Then, in line 12, CSG stated that they could see that the perpendicular bisectors of the sides are concurrent. The concurrency point was the point X in Figure 4.85. Thus, the proof of the first statement was finished.

In this respect, the second statement is that the point of concurrency of the perpendicular bisectors of a triangle is the circumcenter of that triangle. The last three lines of the argument of CSG are about the proof of this statement. In more detail, since $|BX| = |AX| = |CX|$ was found in line 9, they named each part as the radius of the circle and also concluded that the point X became the circumcenter of $\triangle ABC$. All in all, due to the errors in the warrants stated while arranging the congruence, the argument of CSG was categorized as an invalid argument with warrant error.

As can be seen from Table 4.24, the argument of GG in Activity 1 was classified as valid proof although some extraneous errors existed. As the first step of the evaluation, the overall structure of the argument of GG was examined. It was noted that it was a completed argument which aimed to prove that the statement in the worksheet is true since any counterexample was not present in the argument. They used direct proof although they did not express this explicitly. Moreover, it can be stated that the appearance of the argument was neat and it is easy to follow. After this general overview of the argument, the line-by-line analysis was conducted and the necessary points were stated below.

It was seen that the starting points of both groups are the same. In other words, they assumed that the perpendicular bisectors of two sides intersect at a point and then they aimed to show the third one also passes from the same point. However, their ways of showing this case were different. While CSG followed the incorrect reasoning in this manner, GG employed a correct, but not clearly described the idea. While writing proof, GG decided to use $\triangle ABC$ printed on the worksheet A the activity. What they drew in association with the proof of the statement in Activity 1 was presented in Figure 4.86 to make easier to describe the evaluation of the argument. As stated, the whole argument of GG in which the numbered lines were presented explicitly can be found in Appendix H.

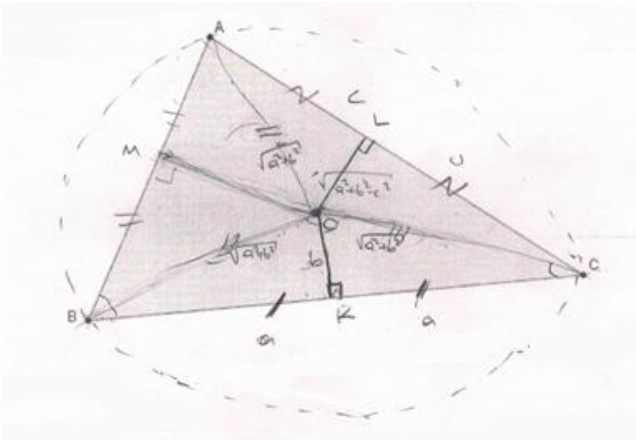


Figure 4. 86. ΔABC in the argument of GG in Activity 1

The line-by-line analysis of the argument of GG presented that the first five lines were reserved for the description of the assumptions and drawings in the argument. In lines 1 and 2, $\overline{OK}$ and $\overline{OL}$ were assumed to be the perpendicular bisectors of two sides and intersect at the point O, as presented in Figure 4.86. In lines 3, 4, and 5, it was written that the lines between three pairs of points were drawn. However, the line segments which are $\overline{AO}$, $\overline{BO}$, and $\overline{CO}$ were drawn instead of the lines. This is an example of the terms and expression errors. Moreover, this part involves the sentence that "the perpendicular bisector OK divided BC into two equal pieces". As seen in the last sentence, the notation error occurred. Moreover, it was not explicitly stated that $\overline{OK}$ and $\overline{OL}$ are accepted to be the perpendicular bisectors of which sides. This case can be considered as an example of the lack of presentation of some assumptions.

In the subsequent three lines, CSG focused on ΔOKC and ΔOBK. Although there were some notation errors in these lines, they were preferred to convey in an accurate way at this point since the explanation would be more meaningful then. How the actual notation was in the argument of GG can be seen through Appendix H. It was declared that they found out $|BO|$ and $|CO|$ as equal by using Pythagorean Theorem in the aforementioned triangles. Thus, it was stated that ΔBOC is an isosceles triangle. Although this process was not written in detail, the geometric figure presents some evidence for the use of Pythagorean Theorem. As seen, $|BO|$ and $|CO|$ were written as $\sqrt{a^2+b^2}$ since $|BK|$ and $|KC|$ were represented with the letter *a* and $|OK|$ was represented with the letter *b*.

In lines 9 and 10, it was stated that the same steps were conducted for ΔAOC and concluded that "AO=BO=CO". However, they should use the following notation $|AO|=|BO|=|CO|$ instead. Again, these steps were not presented thoroughly in the argument. Similarly, the geometric figure involves the evidence pertaining to these unspecified steps. By starting from $|CO|=\sqrt{a^2+b^2}$ and symbolizing $|AL|$ and $|LC|$ with the letter *c*, it was found that $|OL|=\sqrt{a^2+b^2-c^2}$ and $|AO|=\sqrt{a^2+b^2}$ based on Pythagorean Theorem. It would be better if the process was written in a more detailed way. Thus, this case can be given as an example for the item unclear inferences and steps listed among the extraneous errors in Table 4.24.

The remaining part of the argument of GG was nearly the same with the argument of CSG. In line 11, ΔAOB was introduced as an isosceles triangle. In lines 11 and 12, they stated the same reasoning for the concurrency of three perpendicular bisectors of ΔABC at the point O. The final three lines presented the proof for the second statement which can be considered as a valid one for showing that the mentioned point is the circumcenter of ΔABC. To sum up, since the main errors do not exist in the argument, it was evaluated as a valid proof for the conjecture in Activity 1.

4.4.2. Proof of the Conjecture in Activity 2

As mentioned, in the construction section of Activity 2, groups were given three types of triangles and asked to construct the altitudes and the orthocenters of them in the case of existence. The meaning of the orthocenter was not directly given in the activity so that the hesitation regarding the existence of the orthocenter for each triangle was aimed to set up. By doing so, it was targeted that the groups would discuss what the orthocenter mean, work on three different types of triangles while constructing the altitudes, and have the chance to see that the altitudes of all triangles are concurrent and this point is named as the orthocenter. Although the groups were not directly asked to find a relation or connection among the elements of the activity, the context of the activity led them to investigate and state a generalization.

When the conjectures produced by CSG and GG were contrasted, it was seen that there existed four conjectures in the argumentation of CSG and six conjectures in the argumentation of GG. Similar to Activity 1, the similarities between the conjectures of the groups were detected in Activity 2. To begin with, where the conjectures of CSG emerged was illustrated in Figure 4.87 by means of the global argumentation structure of CSG.

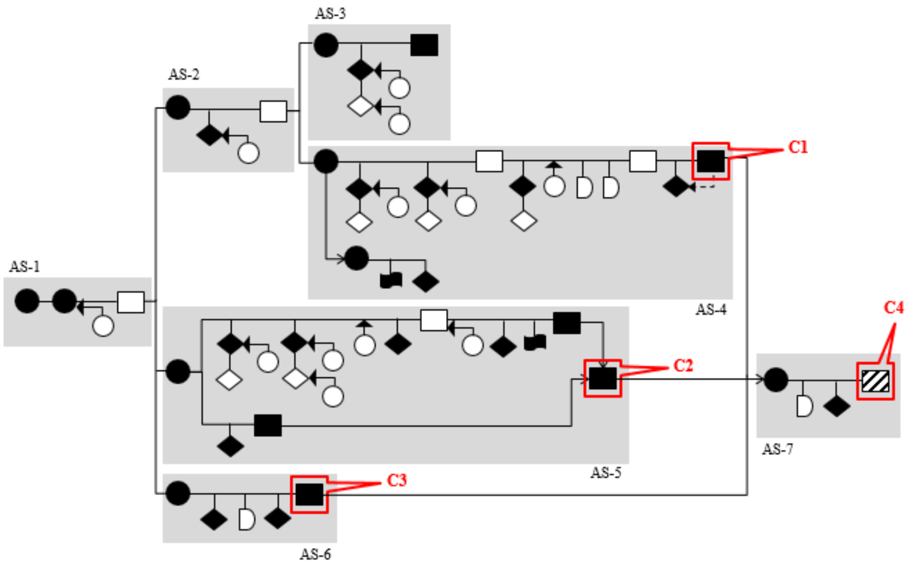


Figure 4. 87. The locations of the conjectures produced by CSG in Activity 2

As seen from Figure 4.87, any of the first three conjectures, which were pointed out as C1, C2, and C3, was not the one asked for the proof since they were symbolized with the conclusion component. On the other hand, it can be stated that C4 was a generalization of the previous three conjectures so that it was the one asked for proving in Activity 2. What the conjectures signified in Figure 4.87 mean was noted in Table 4.25 given below.

Table 4. 25

The conjectures that CSG produced in Activity 2

	Conjectures
C1	A2 was applied in ΔDEF , ΔABC, and ΔKLM . The point of concurrency of the altitudes presented the orthocenters of the given triangles.
C2	A4 was applied in ΔDEF , ΔABC, and ΔKLM . The point of concurrency of the altitudes presented the orthocenters of the given triangles.
C3	A5 was applied in ΔDEF , ΔABC, and ΔKLM . The point of concurrency of the altitudes presented the orthocenters of the given triangles.
C4	By comparing the applications of different construction approaches in the different type of triangles, which were presented as C1, C2, and C3, it was stated that the altitudes of any triangle are concurrent and this point is called as the orthocenter. In other words, every triangle has an orthocenter.

Along with C1, C2, and C3, CSG found out the same result which is the point of concurrency of the altitudes of a triangle is the orthocenter by applying different valid construction approaches. Since these conjectures were produced as a result of different argumentation process and different drawings, they were tabulated separately in Table 4.25. The approaches A2, A4, and A5, which were expanded on in the previous section, were used as promoters while reaching the final conjecture. Moreover, three approaches were used in all of the given triangles. By reuniting C1, C2, and C3, CSG supported the idea in C4. More specifically, via C4, CSG summed up that the altitudes of a triangle are concurrent regardless of the type of the triangle and this point is called as the orthocenter of the triangle.

In the argumentation process of GG in Activity 2, two more conjectures were detected. The locations of six conjectures of GG were presented in the following figure.

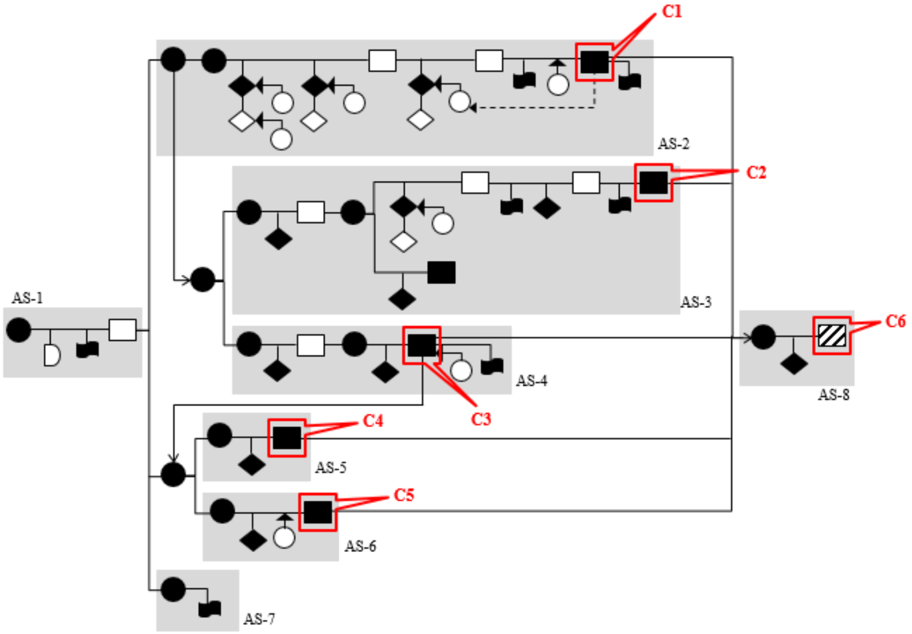


Figure 4. 88. The locations of the conjectures produced by GG in Activity 2

In addition to the equivalence of the types of global argumentation structures of both CSG and GG in Activity 2, which were entitled as the branching-structure, the conjectures of each group were set forth in the same manner. As displayed in Figure 4.88, the first five conjectures were produced by virtue of some construction approaches and marked in the conclusions of the parallel argumentation streams. The last conjecture, which was denoted as C6, took place like the generalization statement of the other conjectures. The details of these conjectures were given in Table 4.26.

Table 4. 26

The conjectures that GG produced in Activity 2

Conjectures	
C1	A1a was applied in ΔDEF and the point of concurrency of the altitudes presented the orthocenter of ΔDEF. The orthocenter is interior to ΔDEF.
C2	A1b was applied in ΔABC and the point of concurrency of the altitudes presented the orthocenter of ΔABC. The orthocenter is exterior to ΔABC.
C3	A2a was applied in ΔKLM and the point of concurrency of the altitudes presented the orthocenter of ΔKLM. The orthocenter of ΔKLM is the vertex L.

Table 4. 26 (continued)

C4	A2b was applied in ΔDEF and the fact that the point of concurrency of the altitudes presented the orthocenter of ΔDEF was ensured.
C5	A2c was applied in ΔABC and the fact that the point of concurrency of the altitudes presented the orthocenter of ΔABC was ensured.
C6	By comparing the application of different construction approaches in the different type of triangles, which were presented as C1, C2, C3, C4, and C5, it was stated that the altitudes of any triangle are concurrent and this point is called as the orthocenter.

The approach stated in the first five conjectures, which are A1a, A1b, A2a, A2b, and A2c, were explained in-depth in the previous section. In short, C1 and C4 refer to the fact that the altitudes of an acute triangle are concurrent by virtue of A1a and A2b respectively, C2 and C5 refer to the fact that the altitudes of an obtuse triangle are concurrent by virtue of A1b and A2c respectively, and C3 refers to the fact that the altitudes of a right triangle are concurrent by virtue of A2a. In addition, GG uttered the locations of the orthocenters in different types of triangles. For example, the orthocenter of an acute triangle is inside of that triangle while the orthocenter of an obtuse triangle is outside of that triangle. As paying regard to the applications of two main approaches to the given three different types of triangles as a whole, GG asserted another conjecture which was denoted as C6 and it refers to the fact that the altitudes of any triangle are concurrent and this point is named as the orthocenter of the triangle.

The conjectures which might be asked to the groups to prove were represented with target conclusion in the global argumentation structures of the groups (See Figures 4.87 and 4.88) and also pointed out in the tables with a different color (See Tables 4.25 and 4.26). The expected conjectures before the application of Activity 2 were quite similar to the ones reached by groups. Some of the expected ones such as “the orthocenter of an acute triangle is inside of the triangle”, “if the orthocenter of a triangle is outside of that triangle, then it is an obtuse triangle”, and “the altitudes of a right triangle intersect at the right-angled vertex” were considered as the auxiliary ones to reach more general conjectures. On the other hand, more inclusive and general conjectures such as “the altitudes of a triangle are concurrent” and “each triangle has an orthocenter" were also among the expected ones. All in all, by taking into consideration the conjectures of groups and the flow of their argumentation process, it was decided to ask them to prove a general conjecture. Thus, the conjecture asked to prove in the worksheets of Activity 2 was stated as follows; "The altitudes of a triangle are concurrent". Since not only C4 of CSG but also C6 of GG cover this conjecture, the proof of the same statement was asked to each group.

Three alternative proofs for this conjecture, which were seen in the textbooks and other sources (e.g., Aarts, 2008, p.30; Alexander & Koeberlein, 2011, p.333-334; Bottema, 2008, p.13-14; Altshiller-Court, 1952, p.94; Gutenmacher & Vasilyev, 2004, p.36-37; Hajja & Martini, 2013, p.5-11; Leonard et al., 2014, p.44), were presented in Appendix G. In the following part, the arguments that groups submitted at the end of the proof process in Activity 2 will be examined whether they can be stated as valid proofs or not.

As indicated previously, the former step of the evaluation of the validity of the arguments is the examination of the overall structure of the arguments while the latter step is to conduct the line-by-line checking. The application of these steps provided Table 4.27 which covers the results of the inspection of the validity of the arguments of both groups as well as the extraneous errors appeared.

Table 4. 27

Validity of the arguments of CSG and GG in Activity 2

Group	Validity of the argument	Extraneous errors
CSG	Invalid argument with structural error	- Notation errors - Term and expression errors - Unclear inferences and steps - Complex flow of the argument
GG	Valid proof	- Notation errors - Unclear inferences and steps - Lack of presentation of some assumptions - Complex flow of the argument

Whilst the argument of CSG was classified as an invalid argument with structural error, the one GG proposed was classified as a valid proof. As stated previously in the findings of Activity 1, the arguments of the groups were presented in Appendix H while the geometric figures they contain were integrated into the explanation of the evaluation process. Firstly, the argument of CSG for the conjecture in Activity 2 was examined as follows.

As the first step, the overall structure of the argument of CSG was reviewed. There were two pages submitted as proof for this activity. The first page was written within the context of an acute triangle while the second page covers the same proof idea in terms of an obtuse triangle. Although the last sentence of the first page signs the end of the argument, the second page just involves some mathematical equations and some inferences without explanation involving a complete sentence. Thus, the second page did not seem like a completed one. It was noted that the aim was to show that the conjecture is true by using the direct proof. However, the writing style was complicated. In other words, which lines are the next in the flow of the writing is not so clear. Therefore, by tracking from the video recordings particularly, the lines were numbered to make it ready for the line-by-line-analysis (See Appendix H). In the line-by-line analysis of the first page, the geometric figure drawn at the upper left side of the page, which was displayed as (a) in Figure 4.89, was examined at first.

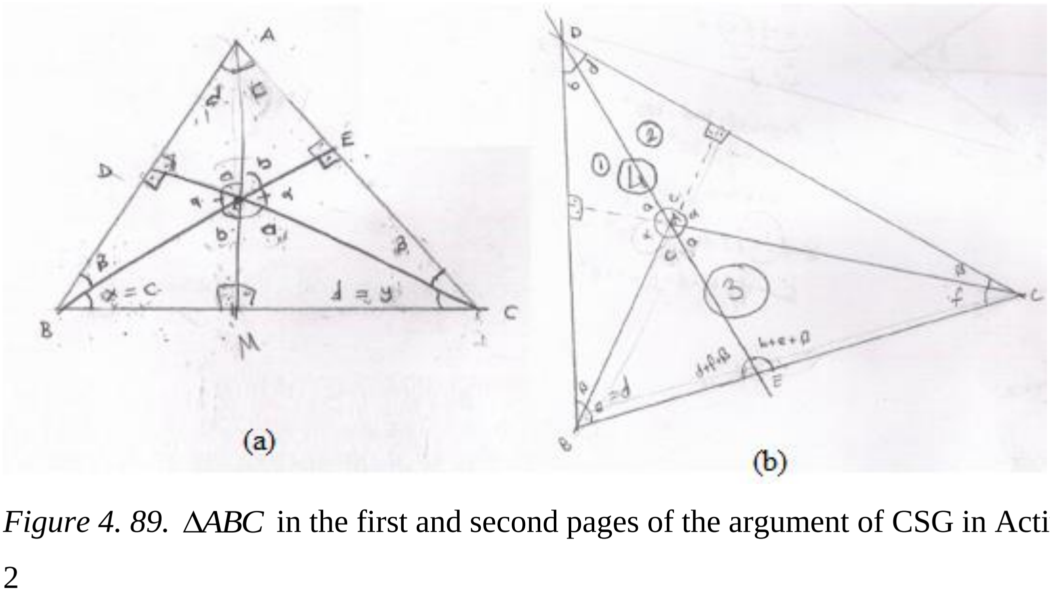


Figure 4. 89. ΔABC in the first and second pages of the argument of CSG in Activity 2

The first four lines of the argument contain the assumptions by referring to the geometric figure. CSG assumed that the altitudes of $\overline{AC}$ and $\overline{AB}$ intersect at the point K. Afterwards, they named $\angle DKB$ and $\angle EKC$ as α by offering that they are alternate interior angles. However, the term used here is not correct for the mentioned angles. This can be seen as an example for the term and expression errors listed in the extraneous error. Moreover, CSG represented $\angle DBE$ and $\angle ECD$ as β without stating the reasoning behind this which can be considered as an example for the unclear inferences and steps among the extraneous errors. Then, they presented the assumption that the line passing from the vertex A and the point K intersects $\overline{BC}$ at the point D. In line 5, CSG introduced the aim of the argument in the given setting which is to present that $\angle BMA$ is 90º to prove that three altitudes are concurrent. After that, CSG kept determining the names of some angles along with line 6 and 7. They arrayed that $\angle MKB$ and $\angle EKA$ are congruent and represented with the letter *b* and also $\angle AKD$ and $\angle CKM$ are congruent and represented with the letter *a* by asserting that each pair of angles are also alternate interior angles. This reasoning is incorrect again since the mentioned angles are not alternate interior angles. Instead, they should express that the given pairs of angles are vertical angles. Moreover, CSG represented $\angle A$ of ΔDKA with the letter *d* and $\angle A$ of ΔEAK with the letter *c*. Thus, in line 8, they reach the following equalities $a+d=b+c=\alpha+\beta=90°$.

From this point on, CSG changed the writing order on the page and moved to write on the upper right side of the page. At this point, the problematic part, which caused it to be categorized as an invalid argument with structural error, has started. In lines 9 and 10, they assumed what they aimed to show in line 5 and continued to make inferences based on this. More specifically, $c+\beta+y=90°$ was written for ΔAMC and $d+\beta+x=90°$ was written for $\angle AMB$. In line 11, they eliminated β by subtracting these equations and came up with the equation $c+y=d+x$ which was numbered as 1 by CSG. After that, along with line 12 and 13, CSG focused on the quadrilateral ADKE and $\angle AMB$. Based on this, they asserted that $c+d=x+y$. This equation was also numbered as 2 by CSG as in line 14. By operating the numbered two equations through lines 15, 16, and 17, they concluded that $c = x$ and $y = d$ in line 18. After these lines, they continued to write through the end of the page.

In the next five lines, CSG integrated the lastly found equations $c = x$ and $y = d$ into $\angle AMB$ and ΔABC. By conducting some operations, they stated that $\angle AMB$ and $\angle AMC$ are right angles. However, they did not notice that they used this as assumption through lines 9 and 10. In the end, with a precautionary stance, CSG wanted to try the same idea with an obtuse triangle, they enumerated this trial as the second page and added information related this through the last two lines of the first page.

The second page of the argument involves an obtuse triangle, as presented in (b) of Figure 4.89. Although it involves 16 lines aligned with this figure, what they have conducted was not clear since there is not a complete sentence on the page. According to the analysis of the video recordings, it was seen that CSG could not be sure the operations in that page, erased it so many times, and tried again. However, they left it as unfinished so that they did not write it in detail. To sum up, since they continued to work on the argument by assuming a case which was needed to show at the end for a valid proof, it was categorized as an invalid proof with a structural error by following the coding of the study of Ko and Knuth (2013).

As different from the approach of CSG in writing the proof, GG tried various cases in the proof process and finally assumed that one of them as valid by the help of the guidance. This argument is actually a basic proof for the concurrence of the altitudes of a triangle offered by Gauss (Bottema, 2008). In more detail, Gauss formed a larger triangle by drawing lines parallel to the sides of the given triangle. By following a series of inferences, this idea was converted to a case that the altitudes of the given triangle turned out to be the perpendicular bisectors of the sides of the larger triangle. This proof was also presented among the proofs related to Activity 2 in Appendix G. Besides, how GG wrote the mentioned idea can be seen in Appendix H.

According to the examination of the overall structure of the argument written by GG to prove the conjecture in Activity 2, it was observed that the argument is finished by using the direct proof. Thus, the aim of it is to show that the conjecture is true. Although there existed a complex flow in the argumentation, this case did not severely interfere in tracking the steps of it. Before depicting the line-by-line analysis of the argument of GG, the geometric figure formed was cut off from the whole argumentation and presented in Figure 4.90 to make the explanation clearer.

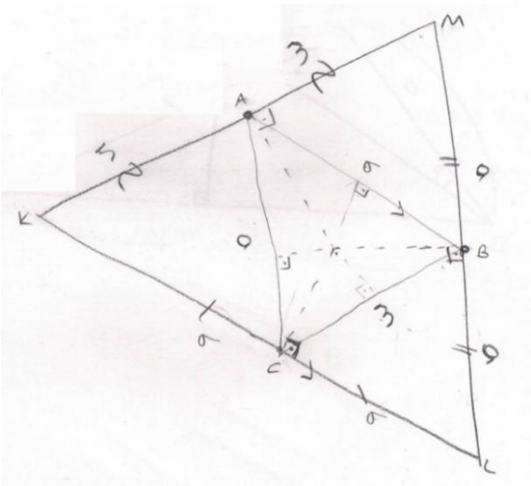


Figure 4. 90. ΔABC and ΔKLM in the argument of GG in Activity 2

When started to conduct the line-by-line analysis, it was noticed that the onset of the argument constitutes a brief summary of the whole idea. The first three lines of the argument were translated as follows. Line 1 covers that "we first formed the large triangle", line 2 states that "the midpoints of the large triangle became the vertices of the small triangle", and line 3 presents that "the perpendicular bisectors of the sides of the large triangle became the altitudes of the small triangle". As seen, the whole idea was stated directly without giving any reasoning for the process resulted in this idea. Although there are still some missing and unclear points, the following lines cover some details about the process.

Lines 4, 5, and 6 listed the parallel sides of two triangles presented in Figure 4.90 which are ΔABC and ΔKLM. That is, $|BC| \parallel |KM|$, $|AB| \parallel |KL|$, and $|AC| \parallel |LM|$ were expressed in each line. Lines 7 and 8 cover the explanation that "the perpendicular line drawn from the midpoint of KM becomes perpendicular to BC because of the parallelism. That is, it is the altitude of BC". Although these sentences involve some notation deficiencies, they explicate the reasoning for some parts of the issues summarized in the first three lines. However, there are still ambiguous points regarding some deductions. For example, how the given parallel sides linked to the fact that the points A, B, and C are the midpoints of the sides of ΔKLM. Since these lines do not involve incorrect expressions which ruin the validity of the argument in an obvious manner, such deficiencies were considered among the extraneous errors.

Then, in line 9, it was stated that this situation is applicable to the other two sides. However, some extra information regarding the process might be given at this point again. GG evoked the conjecture from the previous activity to apply for the present argument in line 10. That is to say, GG stated that the perpendicular bisectors of the sides of a triangle are concurrent. Since it was the conjecture asked to prove in Activity 1, it was regarded that they can use this theorem during the proof of another statement so that no extra proving was seemed to be needed at this point. In lines 11 and 12, “the altitudes of the small triangle correspond to these perpendicular bisectors” was proposed. Since this sentence is a repetition of line 3, it would be better if it was not written at the beginning by considering the flow of the argument. This case was depicted as the complex flow of the argument among the extraneous errors. Then, the final expression that the altitudes of the triangle are also concurrent was presented in lines 13, 14, and 15. As expected, the geometric figure formed in association with the text contributed to the comprehension process of the argument of GG. In conclusion, although it involves the extraneous errors given in Table 4.27, the argument was concluded a valid proof for the given conjecture in Activity 2.

4.4.3. Proof of the Conjecture in Activity 3

In Activity 3, the groups were asked to construct the orthocenter, the circumcenter, and the centroid of a given acute triangle. Then, as different from the previous two activities, they were asked to search for a relationship or a connection among these points. Accordingly, the conjectures produced in this activity have two facets; some conjectures appeared while aiming the construction of the mentioned points and some of them were produced while searching for a possible connection among these points. In this respect, more conjectures were found out from the argumentation process of both CSG and GG. While CSG was producing ten conjectures despite the fact that some of them were related and pointing out the similar cases, GG came up with seven conjectures in a similar manner. As the first step, how the conjectures of CSG spread out in the global argumentation structure of CSG was displayed in Figure 4.91.

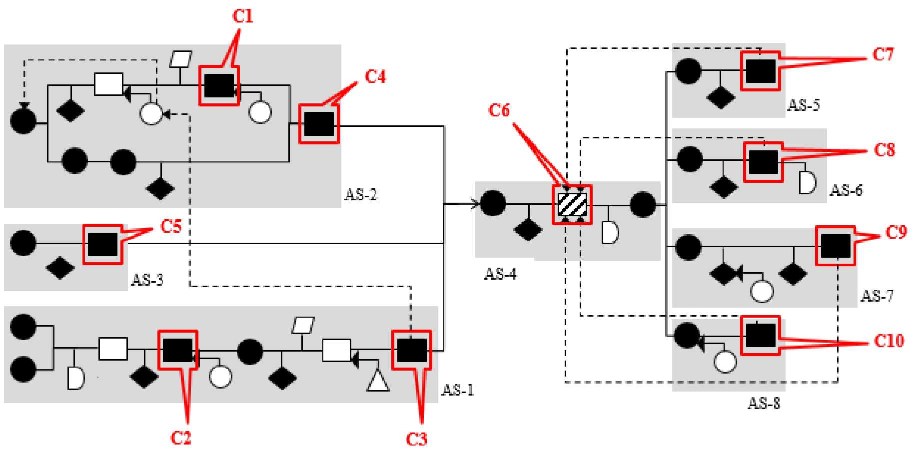


Figure 4. 91. The locations of the conjectures produced by CSG in Activity 3

As it can be deduced from the global argumentation structure of CSG in Figure 4.91 and from the comparison of the related findings presented in the previous sections, C2, C3, C4, and C5 which appeared in the left side of the structure were produced while searching approaches for construction. On the other hand, C1 and the conjectures starting from C6 till C10 were asserted as related to the connections among these points with respect to the different types of triangles other than the given acute triangle. Moreover, only C6 was represented with the target conclusion component since it was the only conjecture matching the expected ones before the application of the activity. The remaining nine conjectures were pointed out by means of the conclusion component. The details of these conjectures were presented as in Table 4.28.

Table 4. 28

The conjectures that CSG produced in Activity 3

Conjectures	
C1	The orthocenter, the circumcenter, and the centroid of ΔABC are concurrent.
C2	The point of concurrency of the angle bisectors is the centroid of ΔABC.
C3	The point of concurrency of the medians is the centroid of ΔABC.
C4	The point of concurrency of the altitudes is the orthocenter of ΔABC. This conjecture refuted C1 since the point of concurrency of the perpendicular bisectors of the sides is not the orthocenter of ΔABC. Therefore, the orthocenter and the circumcenter are not concurrent.
C5	The point of concurrency of the perpendicular bisectors of the sides of ΔABC is the circumcenter.
C6	The orthocenter, the circumcenter, and the centroid of ΔABC are collinear.
C7	In an equilateral triangle, the orthocenter, the circumcenter, and the centroid coincide.
C8	In an obtuse triangle, the orthocenter, the circumcenter, and the centroid are collinear.
C9	In a right triangle, the orthocenter, the circumcenter, and the centroid are collinear.
C10	In an isosceles triangle, the orthocenter, the circumcenter, and the centroid are collinear.

It can be stated that C1 and C2 are false statements. Via C1, it was emphasized that the orthocenter, the circumcenter, and the centroid of ΔABC are concurrent. It is valid only if ΔABC is an equilateral triangle. However, ΔABC which is the triangle they worked on in Activity 2 is not an equilateral triangle. While searching an approach to construct the centroid of ΔABC, CSG concluded that the point of concurrency of the angle bisectors could give them the centroid at first. Nevertheless, it was not a correct case for all types of triangles, except for the equilateral triangle due to the concurrency of three points. Afterwards, C3 and C4, which might be expressed as the refutations of the previous two conjectures, were posited. C3 corrected the case about the centroid by stating that the point of concurrency of the medians is the centroid of ΔABC. Thus, C3 refuted C2 since the point of concurrency of the angle bisectors is not the centroid of ΔABC. Moreover, C3 refuted C1 since the centroid of ΔABC appeared at a different point than assumed to be. Another conjecture which supported the refutation of C1 is C4 which declared that the point of concurrency of the altitudes is the orthocenter of ΔABC. That is to say, C4 also refuted C1 by proposing the fact that the orthocenter and the circumcenter are not concurrent since the point of concurrency of the perpendicular bisectors of the sides is not the orthocenter of ΔABC . Regarding the construction of the final point which is the circumcenter of ΔABC, C5 was produced. C5 refers to the fact that the point of concurrency of the perpendicular bisectors of the sides of ΔABC is the circumcenter of that triangle.

After being sure about the construction of three intended points, CSG started to look for a connection among three points. In this manner, they produced the first conjecture that the orthocenter, the circumcenter, and the centroid of ΔABC are collinear and it was marked as C6. That is, Then, CSG was challenged whether this conjecture is valid for different types of triangles or they wrapped up the case with overgeneralization. Therefore, to check the veracity of this conjecture, CSG started to scrutinize it for different types of triangles. Then, accordingly, a quadripartite sequence of conjectures was formed. Firstly, by working on an equilateral triangle, C7 which refers to the fact that the mentioned three points coincide in an equilateral triangle was asserted. After that, C8, which means that these three points are collinear for an obtuse triangle, was put forward. As another conjecture which repeated the collinearity of three points in a right triangle was stated and marked as C9. Finally, CSG asserted C10 which refers to the fact that the mentioned three points are collinear when the given is an isosceles triangle.

When the conjectures produced by GG were examined, it was seen that how the conjectures were formed has similarities with the process of CSG. That is, the production process of the conjectures of GG has a dual nature. The first type involves the conjectures emerged while searching approaches for the construction of the asked elements in the activity and the second type covers the conjectures stated while searching possible connections among three points. Moreover, it was observed that the majority of the conjectures of both groups are the same. The locations of seven conjectures of GG in the flow of the argumentation were given as in Figure 4.92.

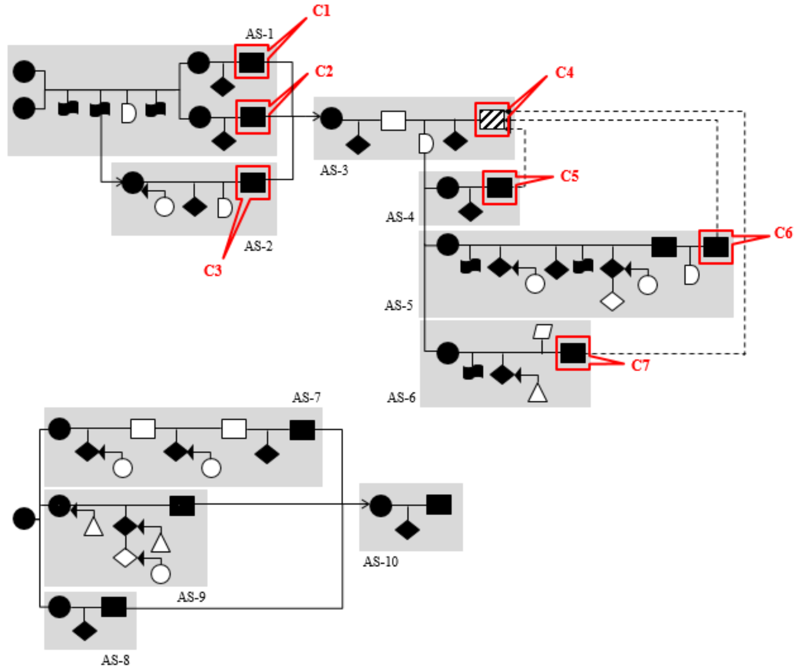


Figure 4. 92. The locations of the conjectures produced by GG in Activity 3

While C1, C2, and C3 belong to the mentioned first type and the remaining four conjectures belong to the second type. Another point noticed is that only C4 was represented with the target conclusion component since it is one of the expected conjectures in advance of the application of the activity. The overlaps between the conjectures of CSG and GG will be explained after Table 4.29 which specifies the details of the conjectures of GG.

Table 4. 29

The conjectures that GG produced in Activity 3

Conjectures	
C1	The point of concurrency of the altitudes is the orthocenter of ΔABC.
C2	The point of concurrency of the perpendicular bisectors of the sides is the circumcenter of ΔABC.
C3	The point of concurrency of the medians is the centroid of ΔABC.
C4	The orthocenter, the circumcenter, and the centroid of ΔABC are collinear.

Table 4. 29 (continued)

C5	In an obtuse triangle, the orthocenter, the circumcenter, and the centroid are collinear.
C6	In an equilateral triangle, the orthocenter, the circumcenter, and the centroid coincide.
C7	In a right triangle, the orthocenter, the circumcenter, and the centroid are collinear.

As can be seen from Table 4.29, the conjectures of GG constitutes a subset of the conjectures of CSG. CSG expressed three more conjecture compared to GG in Activity 3. More specifically, GG did not come up with the false conjectures that CSG stated and the mentioned false conjectures were C1 and C2 of CSG. Moreover, GG did not check the collinearity of three points for an isosceles triangle which corresponds to C10 produced by CSG. To sum up, it was seen that C1, C2, and C10 stated by CSG were absent in the list of the conjectures of GG. However, the remaining seven conjectures are the same but their appearance order was different.

By means of C1, GG stated the orthocenter of ΔABC as the point of intersection of the medians. In C2, GG asserted that the point of concurrency of the perpendicular bisectors of the sides is the circumcenter of ΔABC. For the third point, GG declared that the point of concurrency of the medians is the centroid of ΔABC as in C3. When compared, it can be seen that C1, C2, and C3 of GG are the same with C4, C5, and C3 of CSG, respectively. After the construction section, GG noticed the collinearity of three points which was denoted as C4. Then, they checked the same notion for different types of triangles. C5 covers the collinearity of three points in an obtuse triangle, C6 stated that the mentioned three points coincide in an equilateral triangle, and C7 declared the collinearity of three points in the case of a right triangle. That is to say, C4, C5, C6, and C7 of GG are the same with C6, C8, C7, and C9 of CSG, respectively.

As mentioned, there are overlapping or at least intersecting conjectures produced in the activities so far mentioned. Inevitably, in Activity 3, both groups were asked to prove the same conjecture. Both CSG and GG produced one conjecture directly related to the context of the activity, as colored in the related tables. Moreover, C6 of CSG and C4 of GG were represented with target conclusion. This conjecture was decided to ask the groups to prove in a formal manner. Another conjecture which was expected and would be represented with target conclusion in the case of existence was “the distance from the centroid to the orthocenter is twice of the distance from the centroid to the circumcenter”. However, any of the groups did not posit such a property.

The conjecture asked to prove in Activity 3 was stated as; “The circumcenter, the orthocenter, and the centroid of a triangle are collinear”. Moreover, the straight line containing the circumcenter, the orthocenter, and the centroid of a triangle is called the Euler line of the triangle (Altshiller-Court, 1952, p.102). The proof of this conjecture was presented in Appendix G. As usual, it was arranged based on the review of the related proofs involved in some textbooks (e.g., Altshiller-Court, 1952, p.101-102; Coxeter, 1967, p.18-19; Leonard et al., 2014, p.190; Venema, 2013, p.28). Now, it is the turn of the findings related to the validation check of the arguments of the groups in Activity 3.

The arguments submitted at the end of this activity by each group can be seen in Appendix H. The version of the arguments in Appendix H is the ones which were set for the line-by-line analysis. That is, they involve the numbers in the left margin to specify the lines referred to in the following detailed explanations of the evaluation process. In order to document the results of the analysis process regarding Activity 3, Table 4.30 was prepared.

Table 4. 30

Validity of the arguments of CSG and GG in Activity 3

Group	Validity of the argument	Extraneous errors
CSG	Invalid argument with structural error	- Notation errors - Term and expression errors - Unclear inferences and steps - Lack of presentation of some assumptions
GG	Invalid argument with structural error	- Notation errors - Term and expression errors - Unclear inferences and steps

None of the groups could write a valid proof for the conjecture of Activity 3. The argument of CSG has the main critical and structural error. In other words, what CSG aimed to show at the end of the argument cannot fulfill the main conditions needed to be in the proof of the given conjecture. On the other hand, although it seemed that the argument of GG was built on more solid base at the beginning, another major error in the structure of the argument was noticed towards the end of it. The presentation of the extraneous errors would be more effective for an argument which was labeled as valid proof since a better version of the argument could be carried out by correcting them. For the invalid ones like in this activity, the presentation of the extraneous errors would be of benefit to uncover the line-by-line analysis thoroughly. In this respect, the extraneous errors were also presented in Table 4.30.

First of all, the overall structure of the argument of CSG was inspected. Since the last sentence indicated that the aim of the argument was reached, it was assumed to be a finished argument. Besides, it was observed that the aim was to show that the conjecture is true instead of refuting it and it is easy to follow. However, a discrepancy between the argument and the content of the conjecture was noticed at this point. To be able to detect and investigate it deeply, the line-by-line analysis was applied. Hereby, the geometric figure given at the beginning of the argument was presented below.

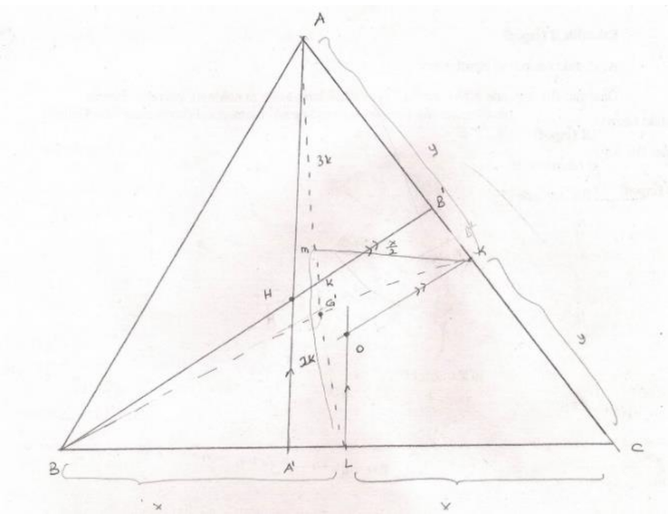


Figure 4. 93. ΔABC in the argument of CSG in Activity 3

The first three lines give the information about the assumption and the aimed notion of the argument; “Assume a point G′ which is collinear with H (the orthocenter) and O (the circumcenter). At this point, if I show that the ratio 2k k as $\frac{G'A}{G'L} = 2$, then the proof ends”. These sentences are quite improper in terms of the structure of the argument. That is to say, they involve the conclusion tried to be reached actually as the assumption by accepting three points are collinear. In addition to this major error, the extraneous errors were also detected such as the lack of presentation of some assumptions at the beginning and notation errors. Anyway, the rest of the argument was continued to examine for having an accurate and in-depth understanding of the whole argument.

In lines 4 and 5, it was written that $\overline{AL}$ and $\overline{BK}$ were drawn in a way that they intersect at the point G′. Moreover, another line segment which is $\overline{KM}$ was formed in a way that $|KM| \parallel |BC|$ was provided. Lastly, based on the previous sentence, it was given that if $\overline{CL} = x$, then $|KM| = \frac{x}{2}$. When compared these sentences with the geometric figure in Figure 4.93, some problematic cases were seen in terms of the given in the first three lines. As mentioned, their aim was to show the ratio 2:1 to point out that G′ is actually the centroid of ΔABC. However, by drawing $\overline{AL}$ and $\overline{BK}$ like in the figure, they assumed that these line segments are the medians since the points K and L were signified as the midpoints of two sides. Thus, the intersection of $\overline{AL}$ and $\overline{BK}$ is automatically the centroid. This case leads to inquiring about what they assumed and what they aimed to present due to the mentioned conflictions.

When followed the line-by-line analysis, line 6 presented the equality of $|AM|$ and $|ML|$ by giving $\frac{AM}{ML} = \frac{AK}{KC}$. Then, in lines 7, 8, and 9, CSG focused on $\Delta KG'M$ and $\Delta BG'L$. Based on the butterfly-like shape constituted by these triangles, they stated that $\frac{KM}{BL} = \frac{G'M}{G'L}$ and $\frac{x}{2x} = \frac{G'M}{G'L}$. Lastly, they declared that

$\frac{G'A}{G'L} = \frac{G'M + MA}{G'L} = 2$ by following the cases that $|G'M| = k$, $|MA| = 2k$, and $|G'L| = 2k$ and this presented the end of the proof for them. In addition, regarding the extraneous errors, it can be stated that the recently explained lines involve some notation errors, informal expressions, and skipped steps.

To sum up, it was not possible to accept this argument as a valid one due to the major errors involved in it and the case that it does not actually offer a proof for the asked conjecture. In this respect, based on the study of Ko and Knuth (2013), such arguments were categorized as an invalid argument with structural error.

Another argument which fell into the same category in the analysis was presented by GG in the same activity. Firstly, the overall structure of the argument of GG was investigated. According to the last sentence, it was seen that it was a complete argument which aims to prove that the conjecture is true by means of direct proof. Moreover, the way it was written is clear from the perspective of the reader. After this general overview, the line-by-line analysis was explained below. As usual, the geometric figure that GG placed at the beginning of the argument was given below.

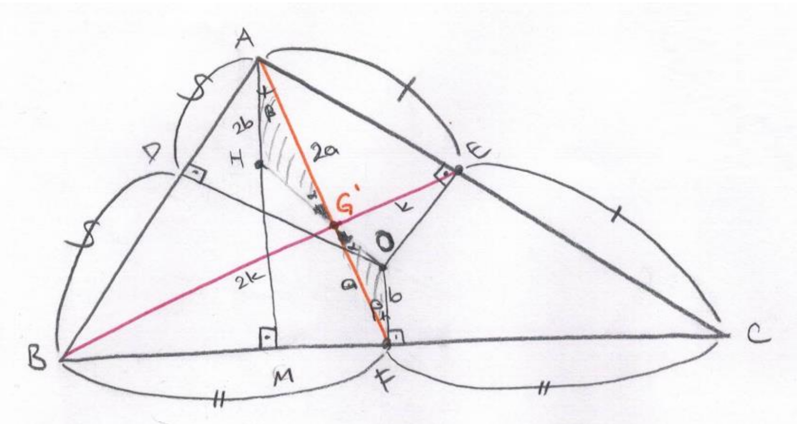


Figure 4. 94. ΔABC in the argument of GG in Activity 3

According to the overall structure analysis, the presence of two phases in the argument was noticed. This aspect was taken into consideration during the line-by-line analysis. The first phase aims to present that the point G′ is actually the centroid of the given triangle, which was questioned in terms of the necessity of presenting it in the argument. The second phase aims to show that the points H, G′, and O are collinear, which was questioned in terms of how it was structured in the followings.

Along with lines 1, 2, and 3, GG explained the assumptions that the midpoints of the sides of ΔABC were determined as the points D, E, and F, the point O was presented as the circumcenter, and the point H was placed as the orthocenter. Then, GG used the extra drawings in the figure such as drawing $\overline{AF}$ and $\overline{BE}$ and also marking the intersection of these line segments as the point G′, as given in lines 4 and 5. After this introduction section, GG declared the use of the Menelaus Theorem in line 6 which followed by two lines involving the application the Menelaus Theorem as displayed below.

$$\frac{|BF|}{|BC|}\cdot\frac{|EC|}{|EA|}\cdot\frac{|AG'|}{|G'F|}=1\Rightarrow\frac{1}{2}\cdot 1\cdot\frac{|AG'|}{|G'F|}=1\Rightarrow\frac{|AG'|}{|G'F|}=\frac{2a}{a}$$

$$\frac{|AE|}{|AC|}\cdot\frac{|CF|}{|FB|}\cdot\frac{|BG'|}{|G'E|}=1\Rightarrow\frac{1}{2}\cdot 1\cdot\frac{|BG'|}{|G'E|}=1\Rightarrow\frac{|BG'|}{|G'E|}=\frac{2k}{k}$$

Subsequently, GG deduced from these equations that G′ is the centroid in line 9. The lines mentioned so far constitute the first phase of the argument. Here, the arguable point is the same with the argument of CSG. That is to say, $\overline{AF}$ and $\overline{BE}$ that GG drew are already known as the medians of ΔABC and this brings up the case that their intersection which was named as G′ already presents the centroid. In this respect, the intent of using the Menelaus Theorem was inquired.

With line 10, the second phase of the argument, which aims to present that H, G′, and O are collinear, started. Thereby, GG used the butterfly-like status of $\Delta AHG'$ and $\Delta OFG'$ to reach this conclusion, as seen in line 11. The first issue considered in this path given in line 12 was correct. GG stated the presence of $\angle OFG' = \angle G'AM$ since $|AM| \,||\, |OF|$. The case was ruined in line 13 which states that "$\angle OG'F$ and $\angle AG'H$ are congruent because of the opposite angles". By declaring the opposite angles, which was also stated as vertical angles, GG accepted the collinearity of the points H, G′, and O again. At this point, it can be noted that this consequence presented a structural error. To avoid misjudgment regarding the validity of the argument of GG, the examination of the remaining lines were maintained.

Although it was not written explicitly, the geometric figure showed that $\angle OG'F$ and $\angle AG'H$ were represented with α, and $\angle OFG'$ and $\angle G'AM$ were represented with β. In lines 14 and 15, they declared that the ratio of the lengths of the sides opposite to the angle α equals to the ratio of the lengths of the sides opposite to the angle β because of the triangle similarity. In these lines, they did not mention the underlying reason for this conclusion and triangle similarity was set. After this expression, in line 16, GG concluded that the points H, G′, and O become collinear by this way. As seen, in addition to the major errors, it also involves extraneous errors which are notation errors, term and expression errors, and unclear inferences and steps. To conclude, given that the argument of GG comprises major structural errors, it was classified as the invalid argument with structural error.

4.4.4. Proof of the Conjecture in Activity 4

In Activity 4, groups were asked to determine one point randomly from each side of the given ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point Z is on $\overline{CA}$. Then, they were asked to construct three circles, each of which is passing through one vertex and two points marked on the adjacent sides. For example, one of the circles is passing through the vertex A, the points X, and Z. As the final step of Activity 4, they were asked to search for any relationship or connection among these circles. In that respect, Activity 4 is similar to the structure of Activity 3 which means that groups were directly asked to search for relations as wells as the construction. When the conjectures produced during the argumentation process were examined by means of the sequent two figures (See Figures 4.95 and 4.96) and two tables (See Tables 4.31 and 4.32), it was seen that GG depicted more conjectures compared to CSG. To begin with, the points where CSG stated a conjecture was displayed below via the global argumentation structure of CSG.

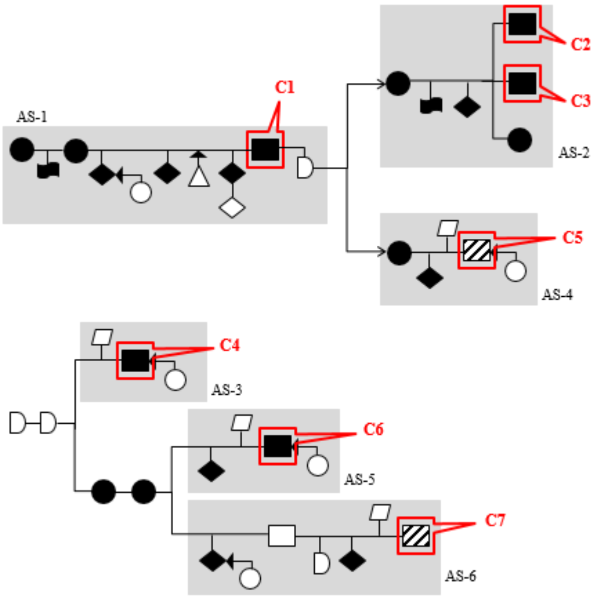


Figure 4. 95. The locations of the conjectures produced by CSG in Activity 4

Unlike the previously mentioned three activities, which involves one conjecture represented with the target conclusion component, there existed two conjectures which were considered as possible to ask for the proof. As seen from Figure 4.95, both C5 and C7 were represented with target conclusion. The others were pointed out with the conclusion component which means that they do not have the potential to be asked for proof in this activity. The details of these conjectures were noted as in Table 4.31.

Table 4. 31

The conjectures that CSG produced in Activity 4

Conjectures	
C1	To construct the circumcircle of a triangle, the circumcenter can be found by means of forming the point of concurrency of the perpendicular bisectors of the sides of that triangle.
C2	Three circles have more than one intersection point
C3	Three circles are concurrent at a point for this triangle at this setting. This point was checked whether it is the circumcenter of ΔABC and observed that it is not. In addition, it was checked whether this point is the incenter of ΔABC and observed that it is not.

Table 4. 31 (continued)

C4	The centers of three circles are not always inside of the circumcircle of $\triangle ABC$. This idea did not work for different types of triangles and random points X, Y, and Z.
C5	Three circles are always concurrent. It is a valid conjecture for different types of triangles and randomly placed X, Y, and Z points.
C6	A new triangle was formed by accepting the centers of three circles as the vertices. The circumcircle of $\triangle ABC$ is not always inside of this new triangle. This idea did not work for different types of triangles and random points X, Y, and Z.
C7	The triangle drawn by accepting the centers of the circles as the vertices is similar to $\triangle ABC$. The type of the given triangle and the formed triangle are the same. For example, if $\triangle ABC$ is an obtuse triangle, then the constructed triangle is also an obtuse triangle.

As it was explained in the findings of the third research question, CSG was quite quick in finding an approach to construct three circles which meet the criteria given in the activity. The consequence of this approach posed the first conjecture in the argumentation which was denoted as C1. By means of C1, it was reunderlined the fact that the circumcircle of a triangle can be constructed via accepting the point of concurrency of the perpendicular bisectors of the sides of that triangle as the circumcenter. Moreover, CSG did not focus on finding another approach for construction much. Instead, they went for searching the relationship among these circles, which brought the sequent six conjectures up as presented above. Expectedly, CSG noticed that the intersections of these circles at first glance which was coded as C2. They declared and pointed out the presence of more than one intersection point of three circles with the hesitated intonation since they were not considering such a case among the asked connections. Therefore, they kept looking for other connections related to three circles they constructed recently. Based on a drawing in the worksheet on the table, one of them stated that three circles are concurrent at a point for this triangle when three points X, Y, and Z were placed like presented in the worksheet. Thus, they surmised that the concurrency of three circles at one point is peculiar to the mentioned case only for a while. Meanwhile, they maintained to produce some ideas for this setting by aiming to find a conjecture. Specifically, they checked whether this point might be either circumcenter or incenter of ΔABC, but it was concluded that none of the mentioned cases held true. This case was listed as C3 in Table 4.31. Afterwards, CSG shifted the focus away from the intersections and came up with another conjecture which was marked as C4. Along with C4, CSG mentioned the possibility of the fact that the centers of all circles are always interior to the circumcircle of ΔABC. As soon as this idea was verbalized by one of the participants of CSG, another one presented a case to refute the mentioned idea. Thus, they concluded that this idea did not work for different types of triangles and random points X, Y, and Z, as presented the explanation of C4.

While constructing the different types of triangles and differently located points on the sides, they noticed that the concurrency of three circles was still valid. Then, they decided to go back to the idea mentioned as the base of C3. As a consequence of the construction trials with careful use of the compass and straightedge, the majority of CSG was sure that three circles are always concurrent not matter what the type of the given triangle is and where the points X, Y, and Z were placed. However, one participant had some hesitations at the time of the declaration of this idea. Then, she also agreed with this conjecture. This conclusion was represented with C5 and it was one of the expected ones prior to the application of the activity and represented with the target conclusion component (See Figure 4.95).

Since CSG kept searching for any other connections among three circles, they could put forward another conjecture which can be asked for the proof which was denoted as C7. Preceded by C7, another conjecture coded as C6, was also stated. Moreover, C6 can be considered as a step leading to C7 since it covers the idea related to the formation of another triangle by accepting the centers of the circles as the vertices. However, C6 states the case that the circumcircle of ΔABC is not always inside of this new triangle. Then, they noticed that the newly constructed triangle is similar to ΔABC which was coded as C7. By checking for various type of triangles and points, they became sure that C7 is a valid conjecture. All in all, C5 and C7 were marked as the conjectures which can be asked for proof.

On the other hand, GG put forward ten conjectures during the argumentation process of Activity 1. As indicated previously, in Activity 4, GG focused on not only the construction ideas of three circles but also the possible connections among these circles. The locations of these conjectures were illustrated in Figure 4.96 via the global argumentation structure of GG obtained from the findings of the second research question.

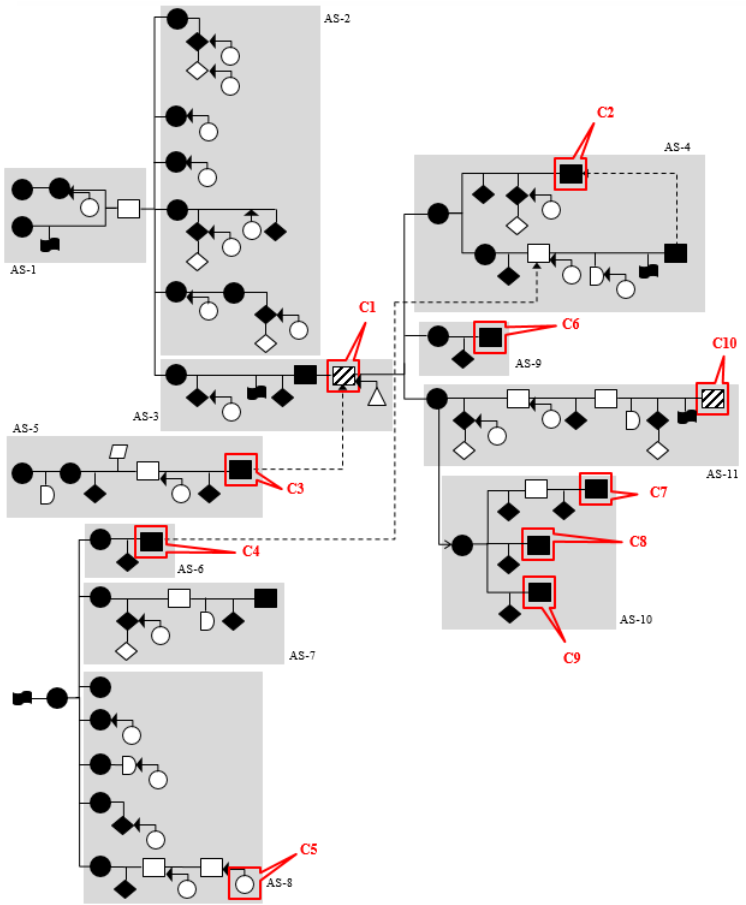


Figure 4. 96. The locations of the conjectures produced by GG in Activity 4

Similar to CSG, two conjectures which were coded as C1 and C10 were represented with the target conclusion component since it was decided that they had the potential to be asked for the proof in advance of the application of the activity. As usual, that type of conjectures was colored in the following table to draw the attention particularly on them. As a new case stated for this section, one of the remaining conjectures, which is named as C5, was represented with the rebuttal since the information was used for refutation throughout the argumentation. The others were marked by means of the conclusion component. The contents of these conjectures were listed as in Table 4.32.

Table 4. 32

The conjectures that GG produced in Activity 4

Conjectures	
C1	Three circles are concurrent.
C2	The point of concurrency of the circles is not the intersection of the angle bisectors of ΔABC.
C3	Three circles are always concurrent. This point is on the hypotenuse for a right triangle, it is inside of the triangle if it is an acute triangle, and it is outside of the triangle if it is an obtuse triangle.
C4	If two equal circles with the centers at two vertices of a triangle were drawn, the line passing through the intersections of these circles is perpendicular to the corresponding side of the triangle. This idea is directly related to the perpendicular bisector construction.
C5	Each quadrilateral cannot be circumscribed so that it can be stated that a circle cannot be drawn from any four points.
C6	Two tangents drawn from a point outside of the circle are the same length.
C7	Another triangle was constructed by accepting the centers of three circles as vertices. The point of concurrency of three circles is not always the orthocenter of the new triangle.
C8	The point of concurrency of three circles is not always the centroid of the new triangle.
C9	The point of concurrency of three circles is not always the incenter of the new triangle.
C10	The new triangle and ΔABC are similar.

The first issue seen clearly from the table given above is that the origin of the conjectures is twofold; some of them appeared as a result of the construction attempts of three circles such as C1, C3, C4, and C5 while some of them were stated during searching for the connections among them such as C2, C6, C7, C8, C9, and C10. The second issue noticed from the table is that the start and the closure of the conjectures have the property to be asked to prove in the final section of the activity. That is to say, C1 and C10 of GG were quite the same with C5 and C7 produced by CSG. While C1 covers that three circles are concurrent, C10 declares that another triangle which was formed by accepting the centers of the circles as the vertices are similar to ΔABC. There some similarities between the other conjectures of CSG and GG which will be seen through the explanation of each one as follows.

After C1, since GG noticed that three circles are concurrent at a point, they checked whether this point is the intersection of the angle bisectors of ΔABC and observed that it is not. This conjecture was presented as C2. During searching for possible connections, they checked the concurrency of three circles stated as C1 and concluded that it is always valid for different types of triangles and randomly placed X, Y, and Z points. Moreover, this point is on the hypotenuse for a right triangle, it is inside of the triangle if it is an acute triangle, and it is outside of the triangle if it is an obtuse triangle. All the given extra information was coded as C3. While trying some ideas for construction, they declared that the lines passing through the intersections of each pair of two congruent circles are perpendicular to the sides of ΔABC and the congruence is the key element in this conjecture. The whole idea was labeled as C4. Another conjecture which appeared in the scope of a rebuttal component and coded as C5 stated that it is not always possible to construct a circle passing from any four points.

Along with C6, GG mentioned that the statement “two tangents drawn from a point outside of the circle are the same length” can be reached via the figure seen in the screen at that moment. In more detail, they assumed that the tangents of a circle correspond to the radii of another triangle and the point which tangents are drawn corresponds to the center of that circle. Since the radii of a circle are equal in lengths, they inferred the equality of tangents from a point outside of the circle. Although this statement is correct, it is not possible to deduce such a statement from the drawings in the activity. Then, one of the participants of GG offered to draw another triangle by accepting the centers of the circles as the vertices, which was also presented by CSG. These ideas led a triadic sequence of conjectures which are C7, C8, and C9. In more detail, GG looked for a relation between the intersection point of three circles and this triangle. They concluded that the intersection point of three circles is not the intersection of the altitudes, the medians, and the angle bisectors of the new triangle, respectively. Finally, they noticed the similarity between these triangles which was represented as C10.

All in all, it can be stated that both groups came up with a great number of conjectures due to the dual nature of the activity. Before the application of the activity, the possible conjectures that groups might produce were listed by considering the difficulty level of presenting the proof of them and three aspects of the conjectures were noticed. The first aspect covers the ones that most likely to be produced by groups which are “three circles are concurrent” and “the new triangle formed by referring the centers of the circles as the vertices of it is similar to the given triangle”. The second aspect covers the statement which might be considered as auxiliary due to either their restricted nature or the content of their proofs. For example, “the point of concurrency of three circles is on the hypotenuse for a right triangle”, “the point of concurrency of three circles is inside of the triangle if it is an acute triangle”, and “the point of concurrency of three circles is outside of the triangle if it is an obtuse triangle”. The last aspect involves the more advanced conjectures and such conjectures are the least likely ones that groups would produce. For example, “if the points placed on the sides or sidelines of the triangle are collinear, then the circumcircle of the triangle passes through the Miquel point” which is related to the Simson line.

As seen from the tables (See Tables 4.31 and 4.32), both CSG and GG reached two conjectures from the set of more possible ones. In this respect, C5 and C7 produced by CSG and C1 and C10 produced by GG were represented with the target conclusion component in the global argumentation structures and marked by coloring in the aforementioned tables. Among these two possible conjecture, the most general one was decided to ask for proof for both groups so that all groups could work on the proof of the same statement at the end. The conjecture decided to ask for the proof in Activity 4 was given as follows; “Suppose that the point X, Y, and Z are placed at random on the sides of ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point

Z is on $\overline{CA}$. Then, in every case, the circles AXZ, BXY, and CYZ are concurrent". Similar to the previous activities, the proof of this conjecture was prepared based on the proofs presented in some textbooks (e.g, Aarts, 2008, p.158-159; Honsberger, 1995, p.79-81; Leonard et al., 2014, p.171-177; Venema, 2013, p.101). In addition, two related extensions of the proof were presented in Appendix G.

In the light of the fourth research question, the last issue of this sub-section is the analysis of the validity of the arguments of the groups offered by aiming to present a proof for the conjecture given above. Drawing on the results of the examination of the overall structure and the line-by-line analysis of the worksheets submitted by groups at the end of Activity 4, Table 4.33 was prepared to depict the validity of the arguments.

Table 4. 33

Validity of the arguments of CSG and GG in Activity 4

Group	Validity of the argument	Extraneous errors
CSG	Valid proof	- Notation errors - Unclear inferences and steps
GG	Valid proof	- Notation errors - Unclear inferences and steps

To begin with, the overall structure of the argument of CSG was examined. It was observed that it is a completed and clearly presented argument. Since their aim to show that the conjecture is true, they used direct proof instead of stating counterexamples. Then, the line-by-line checking was conducted and the outcomes of this process were outlined at this point.

Since the conjecture presented the notations of the triangle, the points on the sides and the circles formed explicitly, CSG was not expected to state extra assumption like "let ΔABC be a triangle" at the beginning of the argument. Thus, it was not written as a deficiency under the heading of the extraneous errors. The whole argument of CSG can be found in Appendix H but the geometric figure in it was also presented below to make the explanation of the findings clearer.

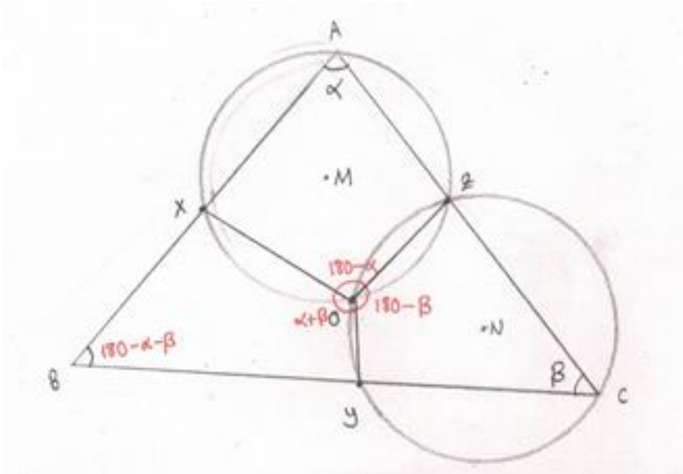


Figure 4. 97. ΔABC in the argument of CSG in Activity 4

In the first three lines, CSG clearly explained what was accepted and what was aimed to show in the followings. That is, CSG stated that "Place the points X, Y, and Z on the sides of ABC triangle. The circles passing through the points, A, X, Z and C, Y, Z, which have the centers M and N, intersect at the point O. Does the circle passing through the points B, X, Y pass through the point O?". Then, in line 4, they described that $\angle BAC = \alpha$ and $\angle ZCY = \beta$ and also labeled AXOZ and ZOYC as the cyclic quadrilaterals. In line 5, they started to arrange the relations of the mentioned elements of ΔABC as $\angle XOZ = 180 - \alpha$ and $\angle ZOY = 180 - \beta$. Since they did not state the reasoning behind the given equations directly and stated that it is because of the properties of the cyclic quadrilaterals, it could be written in detail. By the help of the figure, it can be understood that the statement that the opposite angles of a cyclic quadrilateral are supplementary was used at that line.

After that, in line 6, CSG focused on the angles around the point O. It was written that "To make it 360º, we have $\angle XOY = 360 - \left[(180-\alpha)+(180-\beta)\right] = \alpha + \beta$" which is correct. In line 7, their focus deflected to ΔABC and they displayed this angle operation "$\angle B + \alpha + \beta = 180° \Rightarrow \angle B = 180° - \alpha - \beta$". Along with line 8, they worked on the quadrilateral BXOY. That is, they added the measures of the opposite angles as "$\angle ABY + \angle XOY = 180° - \alpha - \beta + \alpha + \beta = 180°$". Along with the last three lines, CSG linked this conclusion to the aimed idea in the argument. In more detail, they stated that the quadrilateral BXOY becomes a cyclic quadrilateral. Since the cyclic quadrilateral has only one circle which circumscribes it, it can be stated that this circle passes through the point O. Thus, the concurrency of three circles was presented in the argument. As seen, although there are some unclear inferences and steps and a few notation errors, the argument of CSG was stated as a valid proof.

When the overall structure of the argument of GG was examined, it was seen that it was completed and its flow was easy to follow. Since they used direct proof, it can be stated that their aim was to show that the given conjecture is true. Given that it was passed to the presentation of the line-by-line analysis, the geometric figure GG aligned with the argument was illustrated as in Figure 4.98.

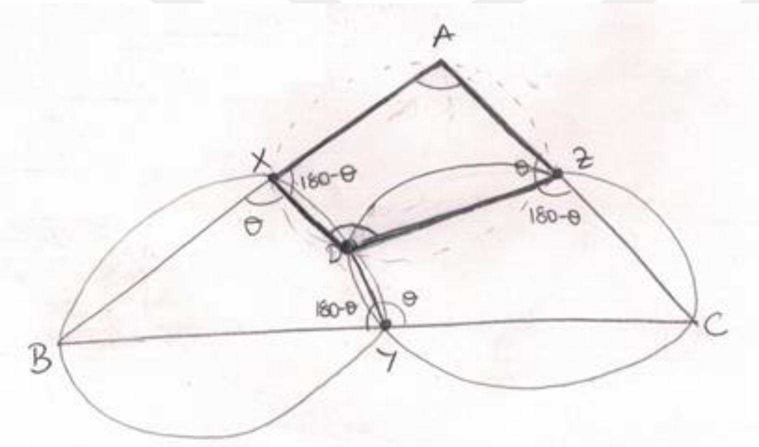


Figure 4. 98. $\triangle ABC$ in the argument of GG in Activity 4

The first five lines in the argument cover what they assumed and what they drew. GG assumed that the circles BYX and CZY intersect at the points Y and D and drew the line segments between the points D and X and also the points D and Z. In addition, they named $\angle BXD$ as θ. After this, GG started to suggest some mathematical deductions. That is, in line 5, they also concluded that the measure of the intercepted arc of the angle θ becomes 2θ without giving the underlying reason for this conclusion. It would be better if the statement "the measure of an inscribed angle of a circle is one-half the measure of its intercepted arc" was presented at this point. In line 6, the remaining arc of this circle was described as "$BXD = 360 - 2\theta$" and then the angle subtended by BXD, which is $\angle BYD$, becomes "180-θ".

This sequence of ideas brought the argument to other circle assumed which is the one passing form the points C, Y, and Z. In line 7, based on the last operation, the measurement of $\angle DYC$ was presented as "$\angle DYC = 180 - (180 - \theta) = \theta$". As the intercepted arc of $\angle DYC$, it was stated that "$DZC = 2\theta$". Along with lines 8, 9, and 10, GG kept working on the angles and arcs. In more detail, since $DZC = 2\theta$, the remaining arc of this circle becomes "$DYC = 360 - 2\theta$". Then, the angle subtended by DYC, which is $\angle DZC$, becomes "$180 - \theta$". Afterwards, GG presented that "$\angle DZA = 180 - (180 - \theta) = \theta$". According to the assumption, $\angle BXD$ was represented with θ which brings the case that "$\angle DXA = 180 - \theta$". Line 10 ended with this conclusion. Along lines 11, 12, 13, and 14, GG stated that "since the addition of $\angle DXA$ and $\angle DZA$ is 180º, AXDZ becomes the cyclic quadrilateral". In the final two lines, GG concluded that the circumscribed circle of AXDZ makes three circles concurrent at the point D. Based on this analysis, the argument of GG in Activity 4 was categorized as a valid proof although there existed some notation errors and unclear inferences and steps as listed among the extraneous errors in Table 4.33.

Having explained the conjecture production and proof processes, a summary will be presented below to approach the results of this section from a broader perspective.

4.4.5. Summary of the Evaluation of the Validity of the Arguments

In parallel to both previous four sub-sections, each of which was oriented to the examination of the proving processes of the conjectures produced per activity, and the fourth research question, to what extent the groups could conduct valid proofs for the conjectures was taken into consideration at this point. In this manner, previously given four tables were combined in one table as noted below.

Table 4. 34

The summary of the validity of the arguments

Activity	The arguments of CSG	The arguments of GG
Activity 1	Invalid argument with warrant error	Valid proof
Activity 2	Invalid argument with structural error	Valid proof
Activity 3	Invalid argument with structural error	Invalid argument with structural error
Activity 4	Valid proof	Valid proof

Drawing on the findings of this section, three categories were organized pertaining to the validity of the arguments that the groups offered. The first category is the valid proof which covers the arguments having a rationale flow and logically correct deductions in spite of the presence of the extraneous errors. Since all arguments evaluated in this study have a few extraneous errors, it was decided to not count such errors as a factor affecting the validity of the argument. However, to have a clear insight regarding the arguments, this type of errors were listed in the related tables. As seen from Table 4.34, half of the arguments presented by groups were categorized as valid proofs. Moreover, it was seen that both CSG and GG were successful in presenting a valid proof for the conjecture in Activity 4. While CSG could not come up with any valid proof in other conjectures, GG was able to present valid proof for the conjectures in Activity 1 and Activity 2.

The second category that emerged in the study is the invalid argument with structural error. The arguments in this category have the major structural errors leading the whole argument to be labeled as invalid. According to Table 4.34, three among eight arguments were classified under this category. In more detail, the mentioned three arguments either assumed the ideas or conclusions needed to be reached at the end throughout another section of the argument or offered an argument irrelevant to the conjecture. In addition, it was seen that the arguments of both groups in Activity 3 were coded as an invalid argument with structural error. When the conjectures in each activity were compared, it was seen that the third conjecture might be the most complicated one since it asks for the proof of the collinearity of three points. Therefore, it was expected that they have difficulty in stating valid proof for the conjecture in third activity although they produced it as a result of an argumentation process.

The last category which emerged once in the study was the invalid argument with warrant error. As Table 4.34 presents, the argument CSG offered for the proof of the conjecture in Activity 1 was the only one categorized under this category. When compared to the second one, this category involves the argument which has a structure closer to the valid proof. In other words, the flow of the argument is logically correct but the warrant used at a step to deduce the next step is not correct. Based on the study of Alcock and Weber (2005), which stated that when the warrant is false, the argument is accepted as invalid even though the data and conclusion are correct, such arguments were decided to be coded as invalid. To specify the category, it was named as the invalid argument with warrant error.

CHAPTER 5

CONCLUSIONS AND DISCUSSION

The purpose of the current study is fourfold. The first purpose is to examine how prospective middle school mathematics teachers' argumentation process while producing conjectures relates to the process of proving. The second purpose is to examine the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities, the components inherent in these argumentation structures, and the functions of the rebuttal component. The third purpose is to investigate the approaches prospective middle school mathematics teachers offered to perform geometric constructions asked in the cognitive unity based activities, how they evaluated the validity of their approaches, and to what extent they could perform geometric constructions correctly while using compass-straightedge and GeoGebra. The last purpose is to scrutinize the conjectures prospective middle school mathematics teachers put forward during argumentation and whether they could present valid proofs for the recently produced conjectures.

The previous chapter included a comprehensive presentation of the findings related to the purposes of the study. The present chapter entails three sections, each of which draws conclusions and discusses the findings of the study. In the first section, the findings deduced from the first and the fourth purposes are discussed collectively to present the whole picture regarding the concepts of cognitive unity and proof within the context of geometry. Subsequently, in the second section, the discussion of the results within the scope of the second purpose of the study is presented. In the third section, some critical points in the findings deduced from the third purpose of the study are discussed. These sections on conclusions are followed by the implications and limitations of the study, and recommendations are made for further research studies.

5.1. The Concept of Cognitive Unity and Proof within the Context of Geometry

As stated previously, the present study utilized the cognitive unity perspective of Pedemonte (2002b, 2007a, 2007b), which is associated both with argumentation within the process of producing conjectures and with proof within the process of presenting the validation of the conjectures. With the purpose of seeking the response to the first research question of the present study, how prospective middle school mathematics teachers relate the argumentation process, in which conjectures are produced, to the process of providing proof of a recently produced conjecture was investigated. To this end, the discussion processes of the groups in both phases of cognitive unity, which are conjecturing and proving, were taken into consideration. According to the themes that emerged in the findings, it can be stated that there are both positive and negative aspects of being involved in the argumentation process before providing proof in the context of geometry (See Table 4.1), which will be discussed in this section. Moreover, since it was seen that the codes and the themes reached in the data analysis related to the first research question did not differ in terms of the groups, the related findings and discussion were not handled separately. Since how the groups' argumentation process while producing conjectures relates to the proving process of conjectures is not an issue directly dependent on the tools used during geometric constructions, similar codes might be seen in both groups during the mentioned analysis. It might be better to consider that how argumentation relates to proof is an issue dependent on the mathematical domain, which is geometry in the present study.

As the first issue to mention herein, some positive and negative affective occasions related to being involved in an argumentation process before proof were noticed. In the cognitive unity based activities, it is the flow of the argumentation of groups which specifies the conjectures. Accordingly, the mentioned argumentation process determines what would be proved in the following proof section of the activities. This might be the reason underlying the participants' confidence in providing proof to the statements. That is, they might think that they were able to find the statements asked to prove by exploring and performing constructions; hence, they were also capable of conducting the proofs of these statements. On the other hand, in cases where the conjecture production process was considerably difficult for them and they got bored during the process, they might have been reluctant to work on the proving phase even though they had found the statements. As aforementioned, the participants worked as a group. If a productive, supportive, and motivative atmosphere during the collective argumentation process can emerge among the participants, this might increase their self-efficacy in the proof section of the activities. However, it has been generally reported that proof is a difficult concept for students at all levels (Ellis et al., 2012; Mejia-Ramos & Inglis, 2009; Moore, 1994; NCTM, 2000; Reid & Knipping, 2010; Reiss et al., 2008; Stylianides & Stylianides, 2017). In this respect, the participants' affective stances during the argumentation process might be considered critical since proof is already a difficult and tedious task in its own right.

Geometric constructions in the cognitive unity based activities were not easy for the participants due to their lack of knowledge in basic concepts of geometry. As the participants could not accurately present the constructions aimed at or reach the intended connections, this may have made them feel frustrated and discouraged, which might have been one factor affecting the whole proof process negatively. Indeed, incorrect approaches and results in argumentation are important since they affect the flow and structure of argumentation (Slob, 2006). Results of analyses clearly indicated that incorrect results are also important in terms of their potential to lead the participants through the valid ones. Thus, depending on the difficulty level of the conjecturing phase of the activities, the participants' positions related to the phase of providing proof might vary both negatively and positively.

In the present study, it was concluded that the participants had the opportunity to arrange the knowledge base related to the concepts of the activities by being involved in the argumentation in which conjectures were produced. Developing students' reasoning and thinking skills and supporting them in their development of a predisposition towards producing conjectures and proposing related plausible arguments are critical due to the fact that these issues constitute the basis for further experiences (NCTM, 2000). Therefore, not only proving but also conjecturing is a theme that constantly emerges in the school mathematics (Yevdokimov, 2006). Similarly, conjecturing can be considered as a point of entry into both an activity and a reasoning process in the mathematics domain (Lannin et al., 2011). Ellis et al. (2012) stated that students' argumentation process can enable them to notice various patterns and relationships and to produce interesting and related conjectures, which were not particularly aimed for in the activity. In this respect, the term "conjecture walls" was used by Ellis et al. (2012) to refer to the set of conjectures that students discovered, but not proved right after. They also mentioned that these conjectures could provide a source for their future activities and furnish their inquisitiveness in the related concepts.

While performing the intended constructions and searching for the relations in the first section of the cognitive unity based activities, the participants evoked the previous knowledge of both themselves and that of the others in the group. As stated by Boero (2017), the concept of cognitive unity was offered to present a smooth approach to proving. Since the necessary knowledge of the participants is prompted in argumentation before proving, it can be stated that it is a step that is considerably effective in terms of preparing students to the phase of providing proof. Similarly, Hewitt (2005) stated that the studies in the related literature present evidence for the fact that being involved in an argumentation enhances the learning of students. In the absence of a conjecturing process, students might need to make more effort to provide valid arguments (Antonini & Mariotti, 2008). The more students become familiar with the concepts in mathematics, the more possible it is for them to present valid proofs (Barrier et al., 2009). Moreover, to participate in an activity which involves the production of conjectures might offer the opportunity to activate their hidden cognitive processes.

When the difficulties prospective middle school mathematics teachers experienced during the activities were considered, the importance of being competent in geometric concepts became prominent. For example, they got confused while using terms such as the perpendicular bisector of the sides and the altitude in a triangle. If they had a sound understanding of such concepts, the activities would be easier and more meaningful for them. At least, the difficulties which prevented them from proceeding in the activities would be fewer when compared to the actual ones. According to Jones et al. (2012), the definitions of concepts in geometry have an important role in the proof process of students. In the present study, the participants had the chance to underpin and check the descriptions of the related geometric concepts via constructions. This situation might be considered as a development for the participants even though they could not end up with a valid argument at the end of the activities.

Garuti et al. (1996), whose study introduced the concept of the cognitive unity, proposed the presence of a possible cognitive continuum between the production process and the proving process of a statement. Within the context of geometry, visualization in the argumentation process might be considered as a critical issue in terms of the mentioned continuum. Sinclair et al. (2012a) stated that "geometric images provide the content in relation to which properties can be noticed, definitions can be made, and invariances can be discerned" (p.8). In this respect, in the argumentation process, the participants of the study might notice many cases visually, work on extra and even unrelated cases of the concepts before proof, and see the cases which have the potential to help them while proving. For example, as mentioned in the previous chapter, both groups noticed the cyclic quadrilaterals while searching for the connections in Activity 4. Then, they used the cyclic quadrilaterals while proving the statement. Moreover, it was highlighted that the development of students' geometric reasoning and awareness is directly related to construction and visualization (Köse et al., 2012; Sinclair et al., 2012a). To use figures while trying to solve a geometry problem is an effective method (Duval, 1995b). Thus, in the present study, while working on the geometric constructions, the participants might have developed their geometric reasoning with the help of visual elements before proving process. Another point related to the visual aspect is the auxiliary lines, which Fan, Qi, Liu, Wang, and Lin (2017) mentioned. They underlined that "in geometric proof, adding auxiliary lines is often helpful and in many cases necessary" (p.230). In the present study, the participants might have had the opportunity to see the potential auxiliary drawings which may have been used while proving.

The examination of the veracity of the conjectures is one of the main steps of the conjecture production process. It can be stated that the prospective middle school mathematics teachers in the present study were sure about the fact that the statement given to them for proof, which were also produced by them recently, were true. Thus, they did not focus on the idea of finding counterexamples. A consistent finding was reported in a study of Boero et al. (1995). They observed that middle grade students who participated in a conjecture production process started to work on the proving stage one step ahead since they already had the idea related to the validity of the statement they aimed to prove. As Fiallo and Gutiérrez (2017) stated, cognitive unity could have an effect on the types of proof such as deductive or empirical. Since the participants had taken the Discrete Mathematics course in their first year in the program, they were aware of the difference between proving that a statement is true and refuting a statement. Thus, they did not even mention the presence of a counterexample in any activity. It was seen that the participants' initial attempt was to use direct proof during the proving process.

As a negative aspect, the participants were rarely seen to experience difficulty in differentiating the concepts of conjecturing and proving. This result coincides with the finding reported in a previous study of Pedemonte (2008) that students might be experiencing difficulty in proof due to the fact that they cannot transform the argumentation structure during conjecturing into a deductive structure. Moreover, Boero et al. (1999) stated that students could be pursuing the same mental events during both conjecture generation and the proving phases by employing different functions. Pedemonte (2002b) mentioned the similarities of students' arguments in the conjecturing and proving phases. In a similar vein, the participants of the present study might have tried to transfer their actions during conjecture production to the proving phase as well. Moreover, this result might have occasioned by the fact that they continued to think in a way that they were still endeavoring on the geometric constructions in the proof section of the activity.

Although Leung and Lopez-Real (2002) predicated that "Euclidean geometry has been regarded as a formal system and proofs in it are deductive in nature" (p.145), drawing figures without writing proofs in a logical way might cause students to think that they have already proved the statement with the geometric figure. It is indisputable that drawings constitute a core component of both learning and teaching geometry. On the other hand, the possible obstructions inherent in them in terms of mathematical proof should not be ignored. The case where engaging or at least seeing a diagram like the ones in the book of Nelsen (1993), which includes examples for the issue of “proofs without words”, may enable students to query the necessity of conducting a formal proof in the presence of such a diagram.

Although it was consistently stated that mathematics teachers should have the necessary knowledge related to proof in geometry, studies revealed that prospective mathematics teachers had difficulty in even basic concepts related to proof, such as noticing the difference between an empirical argument and a deductive argument (Buchbinder & Cook, 2015; Stylianides & Ball, 2008). In this study, it was observed that the participants experienced confusion related to the meaning of formal proof in a few occasions. However, since they worked on the activities as a group, they informed each other about what should be presented as proof in such cases. The participants were aware of the difference between empirical and deductive arguments although they could not present valid arguments for some of the statements that they were required to prove in the activities.

As the last issue of this section, the focus on the discussion of the findings related to cognitive unity shifted to the discussion of the findings deduced from the fourth purpose. As aforementioned, the last purpose of the study involves the investigation of the validity of the arguments that the groups presented as proof at the end of the cognitive unity based activities. Thus, the related conclusions are reviewed and the points noticeable in this process are dwelt upon as follows.

When argumentation is particularly devoted to the conjecture production phase, proof can be regarded as the final product of the whole process (Pedemonte, 2007a). The findings reported that the majority of the conjectures of CSG and GG in Activities 1 and 2 were similar, half of their conjectures in Activity 3 were similar, and a few conjectures of CSG and GG in Activity 4 were similar. As can be realized, the outcomes of the groups’ conjecture production process have varying degrees of similarities. The fact that the groups had produced different conjectures could be attributed to the tools used while producing conjectures and to the nature of the activity. For example, GG might pass to another idea more quickly since GeoGebra provides the opportunity to check the validity of the produced statement by means of dragging. Moreover, some activities involve both performing the geometric constructions and searching for the connections such as Activity 3 and Activity 4. The diversity in the nature of the activities might lead groups to different conjectures. However, since all the groups were able to produce the expected conjectures per activity, the statements given to both groups to prove in the second worksheets were the same.

Based on the review of the related literature, it can be stated that when students are involved in an argumentation process to produce a conjecture, the proof of the recently stated conjecture becomes more accessible to them (e.g. Antonini & Mariotti, 2008; Boero, 2017; Boero et al., 1996; Boero et al., 2010; Fiallo & Gutiérrez, 2017; Garuti et al., 1996; Garuti et al., 1998; Mariotti et al., 1997; Pedemonte, 2007a, 2007b; Pedemonte & Balacheff, 2016). According to Sinclair et al. (2012b), in a task which directly poses the proof of a given statement, the possibility of producing an argumentation leading to proof is less compared to a task which involves the process of producing the conjecture as well. However, in this study, all the arguments presented by groups at the end of the cognitive unity based activities could not be classified as valid proof. Although prospective middle school mathematics teachers were aware of the entailments of a proof, they still had difficulty in conducting valid proofs at a point when the case or statement they aimed to prove through conducting logical inferences were comparatively difficult.

How the proof is taken into consideration in collegiate level mathematics and mathematics education courses is also one of the investigated themes (Fukawa-Connelly, 2014) since it influences the overall understanding of proof. In the Discrete Mathematics course, there is a chapter devoted to mathematical logic and proving. In addition, there is another undergraduate level course, Geometry, in which prospective middle school mathematics teachers deal with the proof of statements in geometry. The participants have comparatively higher grades in these courses. However, it can be inferred that these courses were not enough to develop the content knowledge and the proof ability of the participants since they submitted some invalid arguments as proof. Moreover, Ellis et al. (2012) underlined that mathematics teachers might need to ask students to think in a different way to prove the statement and give some ideas regarding a different statement which can be proved in a similar manner in cases where they actually needed help. In this regard, another reason underlying this result might be the fact that the groups sometimes ignored the clues and guidance of the instructor while searching for a proof. For example, in the proof of Activity 2, CSG preferred to work on finding congruence by means of angles although it was a dead end for them, and the instructor gave some clues about the method to utilize and other possible ideas on proof. Besides, since the argumentation while producing conjectures by means of geometric constructions does not have a deductive nature, the participants might have difficulty in transferring their ways of thinking to a deductive nature during the proof section of the activities. In a similar vein, Pedemonte (2007a) warned about this issue. If the domain worked on is geometry, then structural continuity might not furnish the proving process and, thus, might be regarded as an obstacle for it (Pedemonte, 2007a).

According to the analysis, the arguments of the groups were coded under three categories, namely invalid argument with structural error, invalid argument with warrant error, and valid proof. As the name suggests, the one coded as invalid argument with warrant error covers warrant errors. It emerged only once in the argument of CSG in Activity 1. More specifically, CSG could not set out side-angle-side congruence correctly in two situations while trying to prove the statement in Activity 1. However, the triangles they worked on at these steps were actually congruent. If they had moved through these steps properly, their argument would have been categorized as valid proof. Since proof in mathematics involves “a series of assertions” (Weber & Alcock, 2005, p.37), incorrect steps of CSG led to negative results. The reason underlying this situation might be the lack of geometry knowledge of the participants of CSG. Due to their insufficient prior knowledge, they might have difficulty in arranging the proper concepts to suggest the congruence of triangles. Among these three categories, the invalid argument with structural error could be considered as the most problematic case in terms of proving. For the ones coded as the invalid argument with warrant error, the argument has the potential to be coded as valid in cases where the warrant was proposed correctly. However, the invalid arguments with structural error pose more vital problems in terms of being a valid argument. Improving the argument is not simple as to correct the warrant, the whole structure needs to be revised. Moreover, since the groups did not use the tools while working on proving, it was seen that there was no clear evidence regarding the relevance of tools to proving. In this respect, proof related findings were taken into consideration without particularly focusing on the difference between the groups.

Fiallo and Gutiérrez (2017) also claimed that students' ability to provide proof is not constant; it changes depending on the difficulty level of the problems. In this respect, the findings deduced from Activity 3 and Activity 4 can be considered. More specifically, the arguments of both groups were classified as invalid argument with structural error in Activity 3 while both groups presented a valid proof in Activity 4. This situation might have originated from the difficulty levels of the statements required to be proved. It can be stated that the proof of the statement in Activity 3 was not of a common type for the participants. Thus, they might not have been able to think of a starting point to prove the collinearity of three points. This was actually an expected situation since proving the collinearity of three points is not an easy task if they have not gained experience in such a proving process before. They might not be able to frame how the collinearity of three points can be proved. Both students at various levels and mathematics teachers have difficulty at the beginning phase of a proof since arranging the genesis of a proof is a challenging issue (Sinclair et al., 2012b). This issue was also observed in the findings of the current study. It was seen that prospective middle school mathematics teachers allocated a great deal of time to search for a way to start the proof of the conjectures or statements they produced. In addition, it was also seen that the mentioned time was the highest in the stage of providing proof for the statement given in Activity 3. Since it was expected that the participants would experience difficulty in providing proof for the statement in this activity, the guided questions and some clues were prepared by the researcher before the implementation of the activity. As an example for the mentioned clues, it can be stated that the participants were offered to search for a congruence by following the parallelism in the geometric figure they drew. As seen, such clues depend on the discussion of the groups regarding the activity. The guidance and mentoring in some occasions such as the development of students' capability in proving are of critical importance.

On the other hand, the only activity in which both CSG and GG were able to present proof for the statement was Activity 4. Since it was the last activity, the participants might have gotten used to working on proof. Moreover, they might have developed their understanding of proof for the concurrency of some elements by the help of the previous activities. The idea regarding the development of the ability to prove was also seen in the study of Fiallo and Gutiérrez (2017). More specifically, Fiallo and Gutiérrez (2017) reported that the participants were giving empirical arguments at the beginning of the classroom-based intervention but as weeks passed by, they started to offer deductive proofs.

The findings pointed out that all arguments, regardless of whether they were classified as valid proof or not, involved some extraneous errors. These extraneous errors were categorized under five themes, which are listed as notation errors, term and expression errors, unclear inferences and steps, lack of presentation of some assumptions, and complex flow of the argument. It was decided that such errors do not cause the argument to be labeled as invalid. This idea is consistent with the study of Selden and Selden (2003). The cases which neither affect the correctness of the next expression nor affect the whole structure of the arguments in terms of its validity were named as the extraneous errors (Selden & Selden, 2003). As one of the prevailing issues regarding the arguments of both groups, they involved unclear inferences and steps and lack of presentation of some assumptions. The occurrence of these two errors could be attributed to the enthymemes that the participants observed until that point. According to Fallis (2003), the arguer might present less than the reasoning process involved by skipping some steps. Walton and Reed (2005) described enthymemes as “arguments with missing (unstated) premises or conclusions” (p.339). The proof in the textbooks might have a substantive effect on the fact that participants have the tendency to skip some points while writing proof.

All in all, it is not feasible to assert that each argumentation process ends up with a proof since each of them has a different nature and the rules needed to be followed (Barrier et al., 2009). Unlike the proving process, the baseline of the argumentation does not have to span a theoretical nature since some other frames and contents intellectually formed are also the parts of argumentation (Pedemonte &

Balacheff, 2016). The argumentation process in which prospective middle school mathematics teachers produced conjectures was very effective in terms of facilitating the proving process but did not suffice for providing valid proofs for all the statements asked in the activities. Although the participants could not present arguments which could be concluded as being valid in the activities, this does not indicate the absence of cognitive unity. Moreover, since the participants maintained the use of drawings, notations, theorems, and definitions in their argumentations while trying to prove, it can be deduced that there is also referential cognitive unity based on the framework of Pedemonte (2005, as cited in Fiallo & Gutiérrez, 2017).

In the section that follows, the findings related to argumentation during the conjecture production process are discussed by giving references to the related literature.

5.2. Global Argumentation Structures

As stated previously, one of the purposes of the current study was to document the global argumentation structures of prospective middle school mathematics teachers while they were endeavoring on producing conjectures by means of the cognitive unity based activities. With this core goal in mind regarding the types of global argumentation structures, the components situated in these structures and the functions of the rebuttal component were also investigated so as to capture different aspects of the argumentation process. In this section, the findings drawn from the analysis related to the global argumentation structures are discussed.

The argumentation of prospective middle school mathematics teachers while producing conjectures was examined both holistically and analytically. That is, the holistic approach was employed to focus on the types of global argumentation structures. Then, the analytic approach was employed to focus on the components situated in the global argumentation structures and the functions of the rebuttal component in particular. It can be stated this perspective has a resemblance with the frameworks used in the studies of Knipping (2008), Knipping and Reid (2013, 2015, 2019), and Erkek (2017), which investigated not only the global argumentation but also the local argumentation. For example, Knipping (2008) emphasized the importance of comparing argumentations not only at the local level but also at the global level to be able to determine different types of argumentation structures. In a similar vein, Pedemonte (2007a) stated that using Toulmin's model provides the opportunity to examine both the elements within the argument and the structure of a particular step within the overall process. When the examination of a complex and lengthy argumentation process is aimed at, the two options of looking at it as a whole or focusing on the details based on some actions in the argumentation might be utilized (Walton, 2006). In more detail, Walton (2006) examined the global and local aspects of argumentation with respect to logic. It was stated that the term argument at the local level was used to refer to the one-step argument. To examine argumentation from a local aspect is critical when the aim is to investigate particular instances and how the focused notions affect the following argumentation steps as well as the whole argumentation. On the other hand, the argument at the global level involves a series of connected local arguments to represent a long discussion or dialogue. Moreover, to examine argumentation from a global stand is also for the benefit of cases where the argumentation is a comprehensive one and the context of argumentation is known so that the flow and direction of it become more interpretable (Walton, 2006).

In the literature, some other terms, such as diagram, scheme, and model were observed to be used to have a similar meaning with the structure of argumentation. More specifically, according to Walton, Reed, and Macagno (2008), a diagram can be constructed to represent the connections of a set of propositions in argumentation, and the inferences among them can be signified by means of a set of arrows. Moreover, they underlined the benefits of using diagrams as a technique to visualize the overall argumentation and reasoning. These kinds of diagrams might be used to analyze each step of reasoning, to notice the relationships among the critical points in the argumentation, to make inferential and implicitly stated cases more visible, to present the principals of reasoning and where the sub-arguments emerged, and to catch the missing statements which are critical in terms of supporting the conclusion. In addition, preparing such diagrams might be beneficial for teachers. By integrating the use of diagrams into teaching, teachers might pursue how argumentation in the classrooms emerge and detect the necessary revisions for better teaching (Walton et al., 2008). In a similar vein, Slob (2006) supported the significance of using diagrams as a strategic tool to illustrate and also to concretize the nature of argumentation and expressed the importance in terms of the analysis of argumentation process as follows: "Not only because diagramming is a valuable tool to gain insight in the structure of argument, but also because vice versa: diagrams show important features of argument analysis. And they may show possible weaknesses perhaps more clearly than words alone can do" (p.175-176). Actually, it might be considered that what they called a diagram is similar to one argumentation step formed by using the model of Toulmin or the combination of argumentation steps like the ones presented by Knipping (2003, 2004, 2008), Reid and Knipping (2010), Knipping and Reid (2013, 2015, 2019), Erkek (2017), and Erkek and Işıksal-Bostan (2019). The model of Toulmin affected the trend of graphical representation used today while analyzing argumentation (Hitchcock & Verheij, 2005).

First of all, findings related to the global aspect of prospective middle school mathematics teachers' argumentation will be focused on at this point. It was seen that the argumentation model of Toulmin is an effective and prevailing tool to examine both informal and formal argumentation (Knipping, 2003, 2004, 2008; Krummheuer, 1995; Pedemonte, 2007a; Reid & Knipping, 2010), but how it is applied is also critical. Toulmin's model of argumentation was recommended to be used as a tool to portray how learning takes place in the classroom (Yackel, 2001). On the other hand, it was also stated that argumentation in a classroom cannot be presented merely based on the argumentation model of Toulmin involving six components due to the complex nature of the process. Toulmin's model can be used to unfold distinct argumentation streams seen throughout the classroom discussion. To this end, this model can be used as a basis for arranging a system for the overall argumentation (Knipping, 2003, 2004). As Knipping (2008) underlined, Toulmin's model can be used to form one step of the argumentation which is referred to as either argumentation step or local argument. Thus, attention was drawn to the requirement of presenting a theoretical framework to explain and analyze the overall processes and structures of complex argumentations (Knipping, 2003, 2008). In other words, since argumentation streams generally intertwine in a complex way rather than being a linear chain, there was a need to develop a framework to be able to describe the global argumentation structure of a long discussion process (Knipping, 2003).

The global argumentation structures that emerged in the study were reported under two main headings, which are the mono structures and the hybrid structures. While the mono structures cover four types of global argumentation structures, namely the reservoir-structure, the line-structure, the funneling-structure, and the branching-structure, the hybrid structures entail any combination of the mono structures. Particularly, by means of eight global argumentation structures that emerged in the study, six types were reported. It was found out that one of them was labeled as the reservoir-structure, two of them were coded as the funneling-structure, and two of them were entitled as the branching-structure. Thus, five of them were framed under the mono structures. The remaining three global argumentation structures were coded under the hybrid structures. Among eleven possible combinations of the mono structures, each of them was coded as a different kind of hybrid structure, namely the reservoir-funneling-structure, the line-branching-structure, and the line-reservoir-branching-structure (See Table 4.2).

While examining the different types of global argumentation structures that emerged, two factors might be considered, namely the groups and the activities. In terms of the groups, it was reported that three of four global argumentations of CSG have the mono structures and one global argumentation has the hybrid structure, while two global argumentations of GG have the mono structures and the remaining two global argumentations have the hybrid structures. In terms of the cognitive unity based activities, regardless of the groups, it can be stated that the mono structures were observed at least once in every activity. Since the global argumentation structures of both CSG and GG in Activity 2 were categorized under the same type of mono structure, which is the funneling-structure, the hybrid structures were seen all activities except for Activity 2. Besides, some commonalities were also noticed. For example, the global argumentation structure of CSG in Activity 3 and the first argumentation block of GG in Activity 3 have a common point. Both of them were coded under the same type of structure, which is the reservoir-structure. Due to the fact that the second argumentation block of GG in Activity 3 was entitled as the funneling-structure, both of the groups could not be categorized under the same type of global argumentation structure. Similarly, both CSG and GG had two argumentation blocks in Activity 1, and the first argumentation block of each group was coded as the branching-structure. Consequently, although there are some similarities among the global argumentation structures of the groups, there is not a final rule or result which can be prescribed for the types of global argumentation structures. Thus, it could be stated that the type of global argumentation structure has a clear and direct dependency on neither the group type nor the content of the activity. The underlying reason for this situation might be that the key factor of an argumentation is the arguer rather than the property of the groups, such as using GeoGebra. Since the global argumentation structure is a kind of representation of the actions of arguers in the process, the characteristics of arguers might be considered as the origin of the emerging structure. Besides, many variables which organize the stance of an arguer in the argumentation might be considered. For example, in the case where the arguers in a group already knew the issue asked in the activity, their argumentation structure would be simpler. In contrast, when the arguers do not have the necessary knowledge of the issue in the activity, they would need to conduct a deeper exploration process to be able to reach the intended statement. Then, the global argumentation structure would be a more complicated one. In the case where only one arguer cannot follow the discussion of others but keeps presenting oppositions, this situation might cause the presence of a structure involving lots of rebuttals and objections. To sum up, the properties of the arguers might be considered as powerful determinants related to the global argumentation structures rather than the common characteristics of a group such as the use of GeoGebra and the activities focused on.

As stated previously, the six types of global argumentation structures in the accessible literature, which are the source-structure, the spiral-structure, the reservoir-structure, the gathering-structure (Knipping, 2003, 2004, 2008; Knipping & Reid, 2013, 2015, 2019; Reid & Knipping, 2010), the line-structure, and the independent arguments-structure (Erkek, 2017; Erkek & Işıksal-Bostan, 2019) were observed. Furthermore, some of them were adapted for the analysis of the current study. More specifically, the reservoir-structure and the line-structure were used with some modifications. By means of these revisions, the aim was to describe types of global argumentation structures in a more general sense and by visual means rather than focusing on some specific characteristics such as backward reasoning. The remaining four types of structures were not appropriate to utilize based on the analysis of the data obtained. This coincides with the studies of Erkek (2017) and Erkek and Işıksal-Bostan (2019). Although the mentioned four types of global argumentation structures offered by Reid and Knipping (2010) were taken into consideration in the analysis by Erkek (2017) and Erkek and Işıksal-Bostan (2019), it was pointed out that their findings yielded no structure that could be coded under the gathering-structure, whereas the remaining three types were utilized. In addition to these three types, they offered two new types, which are the line-structure and the independent arguments-structure. In a similar vein, this study presented two new types of global argumentation structures, namely the funneling-structure and the branching-structure.

Evidently, there are some inconsistencies in studies in terms of the application of the existing types of global argumentation structures to the new ones in the studies. Thus, it was needed to offer new types of structures in the mentioned studies. The current study is among these in that new types of structures were offered at the end of the analysis. Since new components, such as challenger and objection were considered as the components of argumentation in the present study, this addition led to diversity in the schematic representations of the argumentation process, and this might be one of the reasons for not fitting into the different types of global argumentation structures in the literature. Another reason for the presentation of various types of argumentation structures might be associated with cultural issues. Knipping (2008) conducted a study related to the global argumentation structures in German and French classrooms, and Reid and Knipping (2013, 2015) investigated the global argumentation structures in German and Canadian classrooms. Erkek (2017) and Erkek and Işıksal-Bostan (2019) investigated the issue with undergraduate students in Turkey. The variation in the types of structures in the mentioned studies might have originated from the fact that the participants were from different countries. Since the participants in different countries have different educational experiences, their actions in an argumentation process might be shaped based on different educational backgrounds. In this respect, it might be expected that this study would have more common results with the ones conducted by Erkek (2017) and Erkek and Işıksal-Bostan (2019) due to the fact they were conducted in the same country, and that the participants were prospective middle school mathematics teachers. However, the similarities between the global argumentation structures that emerged in the mentioned studies and the present study were not as much as the expected. Specifically, there are three common types of structures, which are the source-structure, the reservoir-structure, and the line-structure. Since the current study needed to make modifications in the properties of these types, it can be stated that the features of these types are not directly the same although the names of the structures used in these studies are the same. As Erkek (2017) mentioned, the types of structures might vary in different universities as well as in different cultures. Since the participants of this study and the participants of the studies Erkek (2017) and Erkek and Işıksal-Bostan (2019) were from different universities and different year levels of the program, the mentioned differences in types of the global argumentation structures might arise.

All in all, the study aimed to present type of structures which have some basic characteristics without descending into particulars regarding the features of the types of structures possess. For example, according to Reid and Knipping (2010), there are no refutations in the reservoir-structure. When a global argumentation structure involves all main features of the reservoir-structure but also has some refutations, it brings out the controversy in terms of entitling the type of it. For instance, the presence of argumentation streams not connected to the main structure was given as a characteristic of the spiral-structure by Reid and Knipping (2010). If a global argumentation structure without any distinct stream but in consistency with all the other characteristics of the spiral-structure appears, it is considered another point of conflict for identification of the type of structure. In this respect, the present study set out to outline the basic and more visual-oriented characteristics for types of global argumentation structures by aiming to increase the applicability of this classification in other occasions.

It was noted that the line-structure was not encountered on its own as a mono structure type but involved in the hybrid structures. The underlying reason for this might be that the overall argumentation process in the cognitive unity based activities had a comprehensive nature since each one of them lasted at least one and a half hour with the participation of three prospective middle school mathematics teachers. Besides, the conjecture producing section of the cognitive unity based activities was organized by virtue of geometric construction, which is regarded as demanding work (Djorić & Janičić, 2004; Sarhangi, 2007; Stupel & Ben-Chaim, 2013). Moreover, the groups continued to work on their attempts to perform geometric constructions although they thought that they had found one approach resulting in a valid geometric construction. The cognitive unity based activities were administered within the scope of an elective course, and the participants were the ones who comparatively had the highest scores in the program. These two cases might also contribute to the presence of more complex argumentation structures. Thus, there was not a compact and short argumentation process in the study, so there was not a global argumentation coded as the line-structure only. Moreover, to present some argumentation streams in a linear manner does not provide an interpretable source regarding the whole argumentation structure in that such streams indeed emerged in a more complicated and interwoven way (Knipping, 2003, 2008).

When the series of studies conducted either by Knipping or by Knipping and Reid together (e.g., Knipping, 2003, 2004, 2008; Knipping & Reid, 2013, 2015, 2019; Reid & Knipping, 2010) were combined, it was seen that they put forward four types of global argumentation structures in total. More specifically, Knipping (2003, 2004, 2008) explained the source-structure and the reservoir-structure, Reid and Knipping (2010) purported all of the four argumentation structures, Knipping and Reid (2013, 2015) focused on the source-structure and the spiral-structure, and lastly Knipping and Reid (2019) mentioned three structures except for the gathering-structure. Evidently, the only common type of structure in all the mentioned studies is the source-structure. However, any of the global argumentation structures that emerged in this study was not proper to be coded under the source-structure. Since the fundamental characteristics of the source-structure was described as "arguments and ideas [that] arise from a variety of origins, like water welling up from many springs" (Knipping, 2008, p.437). Due to the limited information given in the worksheets of the cognitive unity based activities, the participants were not able to offer many ideas to perform the geometric construction at the beginning. Besides, the participants always worked as a group of three prospective teachers in the activities so that the beginning parts of the activities cover the process in which they tried to understand what is given and what is asked in the activities collectively. Therefore, the prelude sections of the majority of global argumentation structures were not appropriate to cover the argumentation streams arising from a variety of sources. In contrast, in such cases, the global argumentation structure was appropriate to entitle as the reservoir-structure since the argumentation continued with a second part like what happens in the reservoir-structure.

Different from what global argumentation structure related studies proposed regarding this point, Walton (2006) presented some basic types of arguments while working on diagramming, which are single argument, linked argument, convergent argument, divergent argument, and serial argument. Since these arguments were not divided into many components like those presented in Toulmin's model, it can be stated that they have a more basic nature involving only premises and conclusions. In more detail, a single argument involves one premise, which can be used as a support for the conclusion; a linked argument covers more than one premise functioning together to support the conclusion; a convergent argument involves premises, each of which supports the conclusion in its own way; in a serial argument, the conclusion of an argument is the premise of the next argument; and a divergent argument involves different conclusions drawn from the same premise. Evidently, such a classification is in contradiction with the argumentation structures that emerged in the present study. Since the structures of the argumentation process in which the participants not only engaged in geometric construction but also searched for the possible relationships among the geometric concepts in the activities were generated in this study, their categorization was handled by proposing the hybrid structures. Since there are argumentation structures with a comparatively complex nature accompanied by lots of different components, the types of argumentation structures proposed by Walton (2006) were not applicable for the data obtained in this study. However, they could be used for smaller dialogues.

Another point to note is related to the GeoGebra files given to GG throughout the four cognitive unity based activities and the nature of global argumentation structures. As stated previously, in some activities, GG was provided with more than one GeoGebra file, each of which covered different toolbars because of the restriction. In the case where more than one GeoGebra file were submitted to GG, the expectation was to have independent argumentation blocks in the global argumentation structure, which might be a factor leading to its being coded under the hybrid structures. While the global argumentation structures of GG in Activities 1 and 2 were coded under the mono structures, the remaining two were coded under the hybrid structures. To remind, the GeoGebra files given in the activities were explained herein by contrasting the type of the global argumentation structure that appeared. In Activity 1, two GeoGebra files involving the same triangle were given, the first one covered two restricted tools, which are ‘circle through three points’ and ‘circumcircular arc’, while the second one involved three more restricted tools, namely which are ‘midpoint or center’, ‘perpendicular line’, and ‘perpendicular bisector’. As expected, the global argumentation structure of GG in this activity involved more than one argumentation block, each of which covers the study on the files separately. However, since each block could be properly coded as the branching-structure, the overall structure was categorized under the mono structures. In Activity 2, three GeoGebra files, each of which involved one type of triangle, were given to GG, and the tool ‘perpendicular bisector’ was removed from all the GeoGebra files. Since all the GeoGebra files served the same purpose and had the same restrictions, the participants established relations and drew inferences among the triangles in these files. Thus, there was only one argumentation block in the global argumentation structure of GG in Activity 2, which was coded as the funneling-structure under the mono structures. In Activity 3, two GeoGebra files covering the same triangle were given. The first one kept the default toolbar, but the second one did not include three tools, namely ‘midpoint or center’, ‘perpendicular line’, and ‘perpendicular bisector’. This time, the global argumentation structure of GG, namely the reservoir-funneling structure, was coded under the hybrid structures. What was expected was seen in this structure only. The use of different GeoGebra files resulted in two argumentation blocks, which were coded under a different mono structure, which led to a hybrid structure. In Activity 4, one GeoGebra file in which 'circle through three points' and 'circumcircular arc' were removed from the toolbar was given to GG. Especially, the global argumentation structure of GG in Activity 4 is the most comprehensive one since it involves the combination of three types of mono structures, namely the line-reservoir-branching-structure. It can be stated that this global argumentation structure refuted the mentioned expectation of the study since it involved one GeoGebra file but also the most complex structure. Consequently, it can be stated that the involvement of more than one GeoGebra files might have an effect on the presence of piecewise nature in the structures but the types of global argumentation structures of GG were not completely dependent on the content of GeoGebra files.

After all, there was no type of global argumentation structure which could be asserted to be superior to the others. Similarly, the presence of the hybrid structures does not make the arguments stronger when compared to the mono structures. In a general manner, it can be stated that the current study aimed to make a contribution to the literature by offering a classification of global argumentation structures applicable to all strands of mathematics and also to other disciplines. Moreover, according to the accessible literature, the previous studies conducted on the global argumentation structures did not particularly focus on the conjecture production process, which came to the fore in this study due to the description of the argumentation in terms of the construct of cognitive unity. The studies of Reid and Knipping related to the global argumentation structures were concerned with the proving processes in classroom discussions. On the other hand, the studies conducted by Erkek and Işıksal-Bostan focused on the global argumentation structures while solving geometry tasks. What this study focused on as argumentation is the conjecture production process of prospective middle school mathematics teachers within the context of geometric construction. As it is seen, the ways by which all these studies approached the argumentation processes and the contents that the participants were dealing were quite different.

5.2.1. Components Situated in Global Argumentation Structures

After arranging the global argumentation structures in this study, the components of the argumentation were also focused on. Since the components in Toulmin's model were not sufficient to code the argumentation in the study, five additional components were also utilized. This situation led to the presence of a sub-question in the second research question of the study, which aimed to investigate the components situated in the global argumentation structures. At this point, the conclusions and the discussion related to this issue are presented.

Different from the analysis in formal logic, which focuses solely on the dichotomy involving premises and conclusions throughout the examination of arguments, the argumentation model of Toulmin offers six components for the analysis of arguments (Verheij, 2009). The mentioned six components in the model are data, conclusion, warrant, backing, rebuttal, and qualifier (Toulmin, 2003). However, Toulmin's model of argumentation is found to be limited in terms of analyzing the complex structure of arguments in practical discourse. To put it differently, the possibility of the fact that this model might be too simple and static to reflect the fine details of reasoning since it presents an oversimplified idea involving one claim and one data. It is not the case when the argumentation has a comprehensive structure. Thus, researchers continued conducting studies to unfold new models and theories (Knipping, 2008; Reid & Knipping, 2010; Tans, 2006). Since arguments are considerably complicated processes, Toulmin's model of argumentation may require some adaptations (Conner et al., 2014a). What Knipping (2008) and Reid and Knipping (2010) offered is an alternative model to schematize the global argumentation structures especially in cases where a complex and multilayered argumentation process was needed to be handled. Therefore, prima facie, the six components proposed in Toulmin's model which are also involved in the framework of Knipping (2008) seemed to be sufficient in order to analyze and arrange the structures of argumentation in the data analysis of this study. Since Knipping (2008) was observed to have engaged in a complex process of investigating the proving process of a classroom, it was believed that the same framework could be utilized in the present study to examine the argumentation of the groups while they were producing conjectures. However, it was noticed that there was a need for some extra components to present a detailed analysis of the argumentation. Thus, five more components were added to the argumentation model, which are conclusion/data, target conclusion, guidance, challenger, and objection. Thus, the number of the components of argumentation reached eleven. The need for the extra components might have originated from the fact that the discussion of the groups in this study was covering the long periods and there were some statements which could not be categorized under the existing components. To consolidate the analysis with respect to the content of argumentation, all instances during the argumentation were taken into consideration by aligning them with the intonations in the video recordings. It may have been this detailed analysis perspective that led to the presence of the extra components. Moreover, it could have stemmed from the fact that the functions of some existing components in Toulmin's model were simplified and divided into different components in this study. For example, the statement referred to as objection might be addressed under rebuttal in other studies. Similarly, the statements coded as the guidance component in this study could be considered as a part of one of the six known components such as data and warrant depending on the function in the discussion.

One of the newly used components in the argumentation is challenger. This component might partially originated from the nature of the concept of questioning proposed by Walton (2006). When an arguer questions a statement, the aim does not have to be to show that the statement at stake is false or true. That is, questioning can take a neutral stance or just refer to a phrase of doubt. Walton (2006) described it as "questioning a proposition represents a weaker kind of commitment than asserting it" (p.26). Originating from the instinct of questioning at a particular degree, the participants might have put forward some issues which were challenging for the rest of the group. The challenging issue was not asserted as true or false, but actually required an identification regarding the validity of the projected issue. To sum up, the unclear stance of the questioning act in terms of being valid or not might have turned into the statements which created a challenging environment in the collective argumentation. Since it was observed that none of the existing components of Toulmin's model completely addressed the statements leading to a challenging issue, causing to have question marks, and directing the arguers to new attempts regarding the issue, the study implied the need for a component which was referred to as challenger.

While students are dealing with a problem involving proof, mathematics teachers might be coaching them by deflecting their attention to the needed issues in the problem and offering the theorems possible to use in the proving process (Jones et al., 2012). In this respect, how the instructor during the course in which the activities were applied would be included in the process while groups were working on the activities was determined prior to the main study. By considering the analysis of the pilot study and the role of teachers in the argumentation process underlined in the literature, it was decided that the instructor would assume to the roles of a guide, coach, and facilitator. Thus, the clues which were anticipated to be affecting the argumentation process and the achievement of the groups in different degrees were determined prior to the implementation of the activities. Moreover, Knipping (2003) stated that not only were students expected and encouraged to present one justification to explain the statement but also the teacher had a critical role at this point such as asking questions. As seen, the behaviors of the teacher in the classroom have an impact on how the argumentation in the classroom is framed. It was stated that “arguments are produced by several students together, guided by the teacher” (Knipping, 2008, p.432). However, Toulmin’s model does not address a particular component to represent the stance of such actions of the teacher. To this end, the guidance component was employed during the analysis of the present study. The statements of the instructor throughout the collective argumentation of the groups, which facilitated the argumentation process, were labeled as guidance. However, it does not mean that all the statements of the instructor were coded as guidance. Since it is inevitable that argumentations are co-generated by students and the teacher (Knipping, 2008), there were also instructor statements which were coded as other components. That is, the ones which possessed the features of guidance were coded attentively. Moreover, the instructor presented the guidance depending on the difficulty levels of the activities for the participants.

The addition of guidance to the layout of argumentation was also observed in the study of Lin (2018), in which this component undertook three main functions, which are to complete conjecture, revise conjectures, and evoke argumentation. Moreover, it can be stated that the guidance role of the instructor has a critical position in argumentation. Since guidance was provided to the participants when they got stuck while working on their activities, it might be concluded that guidance facilitated argumentation. Since the participants paid attention to what the instructor provided in the activities, their global argumentation structures might have been more comprehensive due to the guidance effect. Besides, the instructor guided the groups not only in the argumentation process but also in the proving process. Since finding a starting point to prove is a difficult issue, the guided questions presented by the instructor might have been helpful for the groups in terms of conducting proofs. Moreover, it was seen that the participants had an inclination to ask the instructor whether their ideas and approaches were correct or not throughout the activities. For example, they asked whether the approach they suggested for the construction was correct or not, whether the geometric figure they formed could be accepted as construction or not, the idea related to proof would end with a proof or not, and the argument they presented as proof was valid or not. Such a tendency to gain the approval of the instructor might be related to their previous learning experiences. At this point, the instructor was careful not to make a judgement related to the accuracy of the ideas. Thus, the instructor encouraged the students to hold a discussion within their group and arrive at decisions collaboratively. Presenting such guidance in the activities may have enabled the participants to intensively look into the issues and the presence of comprehensive argumentation structures.

In addition to the challenger and guidance components mentioned above, some extra components mentioned in the subsequent studies of Reid and Knipping were also employed with some modifications since the study followed the three-stage process of Knipping (2008) as a foundation while arranging the global argumentation structures of the groups. The mentioned components were data/conclusion, which refers to the transition from one part to another in a discourse, and target conclusion, which stands for the final and main conclusions throughout the argumentation (Knipping, 2003, 2004, 2008; Reid & Knipping, 2013, 2015). As mentioned, the term target conclusion was kept, but its function was slightly changed. However, the term data/conclusion was reversed as conclusion/data because of the order of the functions of the combined components in the argument. Actually, the conclusion/data component has correspondence in the study of Walton (2006). It was asserted by Walton (2006) that the conclusion of an argument can function as a premise of the next argument. Thus, the connections of such arguments were referred to as the chain of argumentation. In this respect, "the chains of argumentation" stated by Walton (2006) might be regarded as a phrase used instead of the global argumentation structures proposed by Knipping (2008).

Due to the differences between asserting that a statement is false and criticizing its validity (Walton, 2006), it was decided that all negative utterances in the argumentation of groups cannot be coded under the same component. When the components in the argumentation model of Toulmin were examined, it was seen that rebuttal undertakes the mentioned negative stance since it "provides conditions of exception for the argument" (Verheij, 2005, p.348). However, during the analysis of the study, another component was required to be able to provide a deep understanding of the argumentation. More specifically, two types of oppositions were noticed in the argumentation. The first one entails a stronger opposition by representing the reasoning aiming to refute the statement at stake, and the second one has a weaker opposition since it only involves a negative utterance without presenting any reason to support. Thus, one more component referred to as objection came into play in the analysis of the data of the present study. In instances where the objection was uttered by a participant in a high interrogative and doubtful manner without stating even the reason, the rest of the group was led to have doubts and sometimes to give up the issue argued against. That is, the objection might have had the power to make others give up the issue argued against without presenting any solid counterargument. This finding is in accordance with what Walton (2006) proposed, which is two possible ways of attacking an argument. The first way is to present a counterargument to a statement by stating the underlying reasons and the second way is to utter a doubt regarding the statement without presenting the reason so that it cannot actually be rebutted due to the lack of a solid counterargument. Such a distinction noticed during the analysis of the present study led to the fine tuning of the scope of the rebuttal component. More specifically, arguments in which the participants expressed the reason for presenting the counterargument or the background for refutation were coded as rebuttal. This is similar to the first way of attacking an argument suggested by Walton (2006). On the other hand, arguments where the participants did not present the reason why they tended to object to the statement but simply interfered with the argumentation using negative statements were coded as objection. This is similar to the second way of attacking an argument, as stated by Walton (2006). As can be understood, the objection component was the last new component included in the argumentation model in this study.

Due to the close nature of the rebuttal and objection components, some questions may arise, such as whether objection covers rebuttal, whether it is vice versa, or they are mutually exclusive. From the perspective of the present study, rebuttal and objection are not mutually exclusive since both of them were expressed against a statement. Moreover, it can be stated that one is not a subset of the other. In addition, some other terms, which are counterexamples and refutations, might be considered in the comparison process of rebuttal and objection. It can be inferred that the relations among these terms depend on the perspective. Since the presence of a counterexample in the argument poses an invalid circumstance, such statements might be considered as the rebuttal (Pedemonte & Buchbinder, 2011). When rebuttal and refutation are compared, it can be stated that refutation has a more powerful nature since a rebuttal is presented against a statement, but it does not have to refute it. However, the statement referred to as refutation defeats the opposed argument since it possesses the necessary content to overpower the intended statement. After all, it is stated that "a refutation is something like a strong rebuttal or a rebuttal that has active force in successfully attacking the argument it is aimed at" (Walton et al., 2008, p.220). According to Knipping and Reid (2019), what they regarded as rebuttal serves, as stated in Toulmin's model, to specify "circumstances where the conclusion does not hold" (p.5), which is different from the rebuttal perspective of Aberdein (2006). The mentioned difference may be occasioned by the fact that Aberdein (2006) integrated the refutation into the term rebuttal, but Knipping and Reid (2019) differentiated these terms by stating the fact that rebuttal is local since it affects a particular step of argumentation, but refutation affects a broader aspect and causes the disproval of some sections of the arguments. As another perspective, Walton (2009) described refutation as a “successful rebuttal” (p.5) which brings out a subset relation among them. In the present study, whether the rebuttal components were successful in terms of defeating the aimed statement was not particularly focused on. That is, the statement coded as rebuttal could have been based on incorrect reasoning.

As the name suggests, collective argumentation is generated by several participants. Due to this social aspect, it is expected that argumentation cannot always be progressed in a regular and positive manner. That is, having some disagreements in argumentation, which might result in some modifications, revisions, and changes, is a usual situation (Krummheuer, 1995). Since conjecturing might end up with both true and false statements, validation and refutation are also important in the argumentation process (Lannin et al., 2011). When the aim is to examine the process of the overall argumentation rather than the result of it, the refuted cases are also important to portray the context of argumentation (Knipping & Reid, 2019). In a similar vein, Hitchcock and Verheij (2006) emphasized this issue as a point to which attention had to be drawn as follows: “reasoning and argument involve not only support for points of view, but also attack against them” (p.3). This sentence can be regarded as an illustration of the rebuttal component. Despite the wide appreciation of the significance of rebuttal in the determination of the flow of the argumentation, it was seen that it is a relatively less examined aspect in argumentation. In this respect, by employing the framework of Verheij (2005), the functions of rebuttals situated in the eight global argumentation structures emerging in the study were also examined.

The findings reported that there existed eight functions of rebuttal, each of which was stated against a piece of argumentation by aiming to refute it. These functions are listed as follows; to refute warrant, to refute the connection between data and conclusion, to refute conclusion/data, to refute data, to refute backing, to refute conclusion, to refute challenger, and to refute target conclusion. Among these functions, to refute warrant was the most prevalent function of rebuttals that emerged in the present study. This finding matched with the definitions of the warrant and rebuttal components proposed in various studies. For example, Conner et al. (2007b) defined warrant as “link between data and claim” (p.184) and defined rebuttal based on the warrant; “circumstances under which the warrant is not valid” (p.184). Similarly, Toulmin (2003) explained rebuttal as “indicating circumstances in which the general authority of the warrant would have to be set aside” (p.94). Since the statements coded as warrant involves justifications and reasons used to proceed in the argument (Conner et al., 2014b, Hollebrands et al., 2010), any opposition regarding the justification might resonate as a rebuttal to the global argumentation structure. On the other hand, the schematic representation of the argumentation model of Toulmin (2003, p.97) did not place rebuttal directly connected to warrant, and this representation seemed to propose that rebuttal is more relevant to qualifier (See Figure 2.3).

As mentioned, the study of Verheij (2005) constituted the basis of the analysis related to the rebuttal component in this study although it was not used completely in the same manner. Verheij (2005) offered a layout starting with the data at the bottom and continuing to claim upwards, which does not match with what Toulmin (1958, 2003) developed as the layout of an argument. Since the schematic representation proposed by Knipping (2008) was used while forming the global argumentation structures in the current study, the layout of Verheij (2005) regarding rebuttals could not be completely used. In order to show the cases against which the rebuttals were stated, the connection of the rebuttal to the argumentation streams in the schematic representation was pointed out by the use of an arrow. In addition, the functions of rebuttals and their schematic representations were arranged throughout the analysis process of the study. While Toulmin (2003) did not directly focus on any type of classification regarding rebuttal, Verheij (2005) purported five types of rebuttal; in other words, five cases could be argued against in argumentation (See Figure 2.9). The first four of the rebuttal types Verheij (2005) stated were consistent with the findings. That is, there are rebuttals stated against data, claim or conclusion, warrant, and the connection between data and claim, respectively. However, the fifth type argumentation in Verheij’s (2005) model was combined with the third one. In other words, the rebuttals regarding the warrant, which refer to the third and the fifth types of rebuttal in Verheij's model (2005) were taken into account as one function of rebuttal during the analysis. The remaining four functions of rebuttal that emerged, namely the refutation of backing, conclusion/data, challenger, and target conclusion, were not among the types of rebuttals explained by Verheij (2005). When one of the participants was not convinced with the warrants put forward by others, she presented a rebuttal against the warrant. In a case where the backing presented to support the previously given warrant was also not convincing enough, another rebuttal could emerge. The second rebuttal in such a case was considered to be given against the backing component. The function regarding the refutation of backing was also mentioned by Lin (2018). It was underlined that rebuttal might present a contradiction to various components of the argument listed as data, warrant, backing, and qualifier (Lin, 2018). Contrary to what Lin (2018) stated, findings showed that there was no occasion where rebuttal was stated against a qualifier. Moreover, since conclusion/data, challenger, and target conclusion are the components offered in the present study, other research mentioning them and aligned with the rebuttal component was not encountered in the accessible literature.

Moreover, as Walton et al. (2008) stated, there might be three ways to refute or attack an argument, but what they regard as an argument has a basic structure with premises and conclusions based on them. The first one is asserting that the premises are not true; the second one is asserting that the conclusion is not deduced from the premises; the last one is stating that the conclusion is not true. This categorization is quite simple when compared to the components of Toulmin's model. As seen, it can be inferred that the more concepts or instances were taken into consideration in the argumentation, the more possibilities there were of ways to refute the argumentation. This result might have stemmed from the fact that the participants were occasionally motivated by the instructor to discuss all their ideas within the group, and it was explained that any of their ideas might be a clue or a source leading to valid results for other participants in the group. Besides, the participants had already known each other since they had taken courses together for two years in the program. Since the first cognitive unity based activity was applied in the eighth week of the elective course, it can be stated that they got used to working together. Thus, it can be inferred that the participants felt comfortable to say out loud any point they noticed during the argumentation. Therefore, the number of oppositions in the argumentation sections in the cognitive unity based activities was considerably high. This situation also led to the presence of rebuttal against any component of the argumentation.

When the frequency of rebuttals in the groups with respect to their functions was examined, it was seen that the global argumentation structures of GG had more rebuttals than the ones of CSG in terms of the six functions of rebuttals, except for the functions involving the refutation of the conclusion and the refutation of the target conclusion (See Table 4.4). Specifically, there were five rebuttals stated against the conclusion, one of which was uttered by GG and four of which were uttered by CSG. The only instance where the statement was stated against target conclusion was seen in the global argumentation structure of CSG in Activity 4. One of the participants could not be sure of the fact that three circles constructed coincided at a point due to her drawing error while using compass-straightedge. Since she could not reach the intersection mentioned in target conclusion in her worksheet, the force of the target conclusion weakened from her perspective, and she resisted for a while. However, they agreed on the mentioned intersection in the parts of the argumentation that followed. Evidently, GG did not have the tendency to offer a rebuttal against a conclusion-related component. Since the participants were able to construct and check the issues by using the dragging feature of GeoGebra, they were already sure about the validity of the statement coded as conclusion. That is, the dynamic nature of GeoGebra could provide support in terms of agreeing on the final statements of the issue, which were either conclusions or target conclusions.

Another point to note herein is that the social norms, which are arranged through the interactions in the classroom (Yackel, 2001), and the sociomathematical norms, which are identified as the norms particular to mathematics (Yackel & Cobb, 1996), have some overlapping points with the functions of the components of argument situated in the present study. For example, as a social norm, students were anticipated to justify their ideas and reasoning in the classroom, which could be considered as a feature underlying the warrant component. Students were also expected to probe questions in circumstances there were disagreements in, which could be considered to be related to the functions of the components of rebuttal and objection in the argumentation depending on the presence of the reasoning proposed for the statement. To set up challenges to enrich the issue discussed in the classroom was also mentioned among the norms, which is quite relevant to the challenger component. As such social norms signify the characteristics of the interactions taking place in classrooms (Yackel, 2001), it can be analogized that the components of argumentation characterize how argumentation is enhanced by small groups or the whole classroom. In a similar vein, to promote the discussion in an inquiry-based mathematics classroom is one of the roles which were cast to teachers (Yackel & Cobb, 1996). In this respect, the mentioned role of the teachers can be considered to be associated with the guidance component of the argumentation that emerged in the analysis.

Having discussed the findings specific to the argumentation process in detail, it is time to expand upon the discussion of the findings drawn from the concept of geometric construction.

5.3. Geometric Constructions by using Compass-Straightedge and GeoGebra

In the previous chapter, the findings related to the geometric construction were depicted in detail so as to portray the mindset of groups regarding the geometric constructions with respect to the use of different tools, namely compass-straightedge and GeoGebra. Moreover, all approaches that both CSG and GG offered to carry out the geometric constructions were explained thoroughly to shed more light on what the participants were capable of regarding the issue at stake. Clearly, the focus regarding the geometric constructions was not on the approaches that the participants offered per se, but on the validity of these approaches depending on the tools used. At this point, the findings related to geometric constructions which are related to the third research question will be discussed with a general perspective.

The sets of geometric figures can be constructed and also how they are constructed while using compass-straightedge or a dynamic geometry program may differ. When different tools are used during the geometric construction, the rules of the game become necessarily different. For example, trisecting a given angle can be performed with a dynamic geometry program such as GeoGebra, while it is not possible to construct with compass-straightedge (Baragar, 2002; Shen, 2018; Stupel & Ben-Chaim, 2013; Stylianides & Stylianides, 2005). Due to the possible differences, the findings of CSG and GG were handled separately although in some cases they offered similar approaches for the intended geometric constructions. In the restricted versions of GeoGebra, GG had the tendency to think about how they would construct the intended geometric figure if they were using compass-straightedge. For example, in Activity 1, GG was given two GeoGebra files, the first one of which had two restricted tools, which were 'circle through three points' and 'circumcircular arc', whereas the second one included three more restricted tools, namely 'midpoint or center', 'perpendicular line', and 'perpendicular bisector'. Moreover, Activity 1 asked the participants to construct a circle passing through the vertices of the given triangle. While GG was working on the first GeoGebra file, they presented one valid construction approach (A6) by drawing upon the tools peculiar to GeoGebra such as 'perpendicular bisector'. Then, it turned out to be that some of the tools previously used by GG were also restricted in the second GeoGebra file. At that point, GG started to think about what they would do by using compass-straightedge and came up with another valid approach (A8). In cases such as this one, the approaches offered by CSG and GG have a more similar nature.

At this point, it would be better to mention why it was decided to customize the toolbar in some GeoGebra files. In fact, the property to customize the toolbar is one of the reasons for the selection of GeoGebra as a tool to use during the geometric construction among all the available options of DGS. For example, in Activity 4, GG was given a triangle and asked to point random points on each side of the triangle. Then, GG was asked to construct three circles, each of which had to pass through one vertex and the randomly placed two points on the adjacent sides of that vertex. Since GeoGebra includes the tool 'circle through three points', GG could directly construct the required geometric figure with a few clicks. However, an argumentation process was aimed in the activities, and it was also aimed to arrange an environment that leads the participants to geometric thinking by presenting a challenging environment. Thus, the mentioned tool was removed from the GeoGebra file given to GG in Activity 4 by using the property that GeoGebra provides to customize the toolbar. By means of this property of GeoGebra, the users can specify the tools in the interface they want for the construction of a particular figure (Hohenwarter & Preiner, 2007). This could provide mathematics teachers with numerous opportunities to configure the task according to the needs of their students. Moreover, teachers can arrange the degree of freedom, limitation, and guidance, which can be given to students by means of customizing the interface of GeoGebra. In fact, while using interactive applets of GeoGebra, the tools presented could be used as clues regarding the task by students (Hohenwarter, Hohenwarter, Kreis, & Lavicza, 2008). In addition, the restrictions given in some GeoGebra files in the activities serve to the problem solving strategy of what-if-not, proposed by Brown and Walter (2005). In this strategy, the givens, conditions or constraints in a problem are changed. Since some tools were removed from the default version of the toolbar of GeoGebra, prospective middle school mathematics teachers had the opportunity to encounter a new version of the geometric construction in the activity, attend further investigation, participate in a more challenging environment, and come up with different conjectures and statements throughout the process (Sinclair et al., 2012b).

As the name suggests, GeoGebra presents a combination of geometry, algebra, and calculus. Moreover, GeoGebra is an open-source software, which means that it is free of charge and can be reached without any restrictions (Hohenwarter et al., 2008), so it is used by millions of people around the world (Botana et al., 2015). Although GeoGebra presents a combination of computer algebra systems (CAS), such as Derive and Maple, and dynamic geometry software (DGS), such as Cabri and Sketchpad (Hohenwarter et al., 2008; Hohenwarter & Jones, 2007), the present study mainly made use of the graphics view of GeoGebra. Since the algebra view and the graphics view are dynamically linked, it was seen that GG sometimes used the algebra view to show some previously hidden objects during the construction attempt. Besides, it was observed that GG sometimes checked the construction protocol to go over what they had done and to determine a point to move on with a different approach. Moreover, it was seen that GG got used to considering the tooltip by moving the cursor over the related tool in the toolbar as they proceeded in the activities throughout the Teaching of Geometry Concepts course. However, no evidence was noticed in the case where they did not use the tooltip idea by moving the cursor on the object situated in the graphics view. As can be seen, some examples regarding the participants' use of GeoGebra have mentioned above. The way the participants used GeoGebra might have had an impact on their argumentation in the cognitive unity based activities, and this situation might be related to the functional fixedness. In some occasions, the solution process might have lasted long or delayed due to the functional fixedness (Adamson, 1952). The term functioned fixedness, which was introduced by Duncker (1945), refers that how an object was used in the previous occasions might have affected the new situation in which a different function of the object was intended and prevented one from noticing other uses of the object (Adamson, 1952). The functional fixedness was described as follows: "use of an object or idea in one context prevents the problem solver from recognizing a potential use of that object or idea in a subsequent problem situation" (Greeno, Magone, & Chaiklin, 1979, p.458). For example, GG generally tried to use the first tool given in the toolbar to draw a circle among all other options. That is, it was observed that they had the inclination to use the tool 'circle with center through a point' instead of some other tools such as 'circle: center & radius' and 'compass', which also serve the same purpose. In this respect, the functional fixedness might have prevented the participants from finding some other approaches for geometric constructions since they had the tendency to use the same tools if applicable throughout the activities.

According to the findings of the study, both CSG and GG could offer at least one approach leading them to valid geometric constructions in the cognitive unity based activities. Thus, it can be stated that both groups were successful in geometric constructions at varying degrees. In more detail, in Activity 1, which asked the participants to construct the circle passing through all vertices of the given triangle, CSG offered one valid approach out of four attempts, whereas GG presented two valid approaches out of eight attempts. In Activity 2, the participants were asked to construct the altitudes of the given three different types of triangles and construct their orthocenters, if they existed. It was seen that CSG presented three approaches leading to valid constructions out of five trials and GG unfolded two working approaches out of three trials. In Activity 3, which demanded for the construction of the orthocenter, the circumcenter, and the centroid of a given triangle, CSG offered three valid approaches out of five trials, each of which was related to one of the mentioned points. GG worked on two approaches for each one of the intended points, but five attempts among the six offered were valid in terms of leading to a geometric construction. Finally, in Activity 4, the participants were given a triangle, asked to point random points on each side of the triangle, and construct three circles, each of which had to pass through one vertex and the randomly placed two points on the adjacent sides of that vertex. CSG offered only one valid approach and then continued to search for possible connections among these circles during the argumentation, while GG tried thirteen approaches to be able to construct these circles with only one of them being valid. Based on the high number of approaches tried by GG in Activity 4, it can be inferred that the participants' hesitated, unsure, and trial-and-error based construction attempts did not dwindle away as they continued to work on geometric constructions throughout the weeks of the course.

Moreover, since the participants were not familiar with the concept of geometric construction in any of the undergraduate courses they took until that semester and since teaching sessions regarding geometric construction did not directly present the steps of even basic geometric constructions, they did not have inflexible rules or facts regarding construction. To put it differently, the participants had an innovative and developing stance regarding the geometric construction and did not have knowledge of construction based on memorization, so they presented a repertoire of different approaches for the geometric constructions they were required to do. However, it was observed that the approaches presented in the previous activities or by the other group sometimes affected the following activities. It should not be considered as an undesirable case since the development of the argumentation of the participants leading to proving is an important issue herein. For example, regarding A2 in Activity 1, it can be stated that CSG noticed a connection between Activity 1 and the first orientation activity in the Teaching of Geometry Concepts course. In that activity, prospective middle school mathematics teachers worked on the construction of the center of a given circle. According to the documents of CSG in the mentioned activity, the video recordings, and the field notes, it was seen that CSG used a different version of A2 while attempting to find the center of the circle. Therefore, it can be stated that CSG evoked their previous construction attempts and made modifications to be able to construct the required figure in Activity 1. Moreover, as mentioned earlier, there were two groups in each section, but the data obtained from the triads were presented in the study. Since the groups in each section conducted a classroom argumentation and explained their approaches for the required geometric construction at the end of the lesson each week, it should be considered that the approaches offered by the quadruplets could have had an impact on the approaches of the triads in the following activities. Whether the groups affected each other could be examined as an extension of the study.

As expected, GG came up with more diverse approaches when compared to CSG in the activities, except for Activity 2. This finding partially confirms the fact that using DGS helps both students and the teacher to quickly construct and observe the changes in geometric figures and get a grip on the concepts worked on and the relations among them since it provides them with the chance to endeavor on a series of cases or examples of the concepts at stake (Jones, 2002). The reason behind this might be the opportunities that GeoGebra brought to the participants. For example, it might be related basically to the dragging feature of GeoGebra, which enables students to examine not only the given versions but also the prototypes of the geometric figures. Leung and Lopez-Real (2002) underscored the importance of dragging as follows: “drag-mode in dynamic geometry seems to be a kernel that is potent with rich didactic possibilities” (p.146). It was stated that dragging is not only a tool to control whether the construction is correct or not but also a sign for following theoretical aspects of the related geometric concept (Mariotti et al., 1997). DGS offers an environment that is suitable for the application of open problems involving conjecturing since the dragging property of such software provides a dynamic nature (Baccaglini-Frank & Mariotti, 2010). In addition, using DGS in the classrooms provides students with an environment in which they can examine the applications of the theorems in geometry by means of dragging and then reach some generalizations, conjectures or properties (de Villiers, 1998, Stupel et al., 2018).

The utilization of various dragging modalities provides both students and teachers with the opportunity to examine the properties of geometric figures and present generalizations regarding geometry (Leung & Lopez-Real, 2002). The dragging modalities that Baccaglini-Frank and Mariotti (2010) dwelled upon in their study are wandering/random dragging, maintaining dragging, dragging with trace activated, and dragging test. In more detail, wandering dragging, which was stated as the harmony of wandering dragging and guided dragging, offered by Arzarello et al. (2002), refers to a random dragging act when searching for different configurations of the figure. Maintaining dragging, which was emphasized as partially different from dummy locus dragging, described by Arzarello et al. (2002), means dragging by keeping the particular properties of the figure. Dragging with trace activated, as the name suggests, involves the activation of the trace option of the program for one or more points of the figure while dragging. Finally, the dragging test refers to checking whether the figure keeps its aimed properties under dragging. The dragging test is usable in terms of deciding whether or not to attempt a new construction, and whether to select a different point to control the figure and the produced conjecture (Baccaglini-Frank & Mariotti, 2010). In a similar vein, Arzarello et al. (2002) listed the dragging test among the seven dragging modalities, namely wandering dragging, bound dragging, guided dragging, dummy locus dragging, line dragging, and linked dragging. The most recurring use of the drag test in the literature is to test whether the figure is constructed correctly or not, in other words, whether the figure stays robust under dragging (Baccaglini-Frank & Mariotti, 2010; Olivero, 2002; Laborde, 2005). On the other hand, the findings drew attention to the fact that care should be paid to avoid the overreliance on the dragging feature of GeoGebra while deciding whether the formed figure can be regarded as a geometric construction.

For DGS, the most cited criterion is that the solution of a geometric construction problem is labeled as valid if it can pass the drag test (Jones, 2000; Mariotti, 2001; Stylianides & Stylianides, 2005). Stylianides and Stylianides (2005) pointed out that stating the drag test as both necessary and adequate criterion for the validation of a solution of the construction problem causes to the emergence of an inconsistency. In more detail, when some measurement tools offered in DGS were used in the construction, the geometric figure formed would pass the drag test. However, such a construction seems to be inconsistent with the classical geometry where the only tools used are compass and straightedge. This inconsistent situation led to the question of whether the drag test is a sufficient criterion or not (Stylianides & Stylianides, 2005). In this respect, Stylianides and Stylianides (2005) focused on the analogue of the validity criterion in the compass-straightedge environment for dynamic geometry environments. Therefore, a new criterion for the validity determination in dynamic geometry environments, which is stated as the compatibility criterion, was introduced by Stylianides and Stylianides (2005). To sum up, Stylianides and Stylianides (2005) mentioned two criteria for the validation of construction problem solutions in dynamic geometry software. These criteria were named as the drag test criterion and the compatibility criterion. While Stylianides and Stylianides (2005) discussed the drag test criterion based on the remarks of Jones (2000) and Mariotti (2001), they introduced the compatibility criterion.

Among the applicable criteria in the literature, the ones signified by Stylianides and Stylianides (2005) were employed as the starting point in the present study, but then some modifications were applied and a diagram for accepting a geometric figure as a construction while using GeoGebra was presented (See Figure 4.17). As seen in the first step of this diagram, the geometric figure presented by GG was checked to see whether it is proper to what is asked to construct in the activity. If GG presented an irrelevant figure in terms of the intended geometric construction, it was directly classified as not a construction without proceeding further to try out the mentioned criteria. If the answer was ‘yes’ in the first phase, then the concern of the second stage was to ask whether the geometric figure passed the drag test or not. Stylianides and Stylianides (2005) described the drag test criterion as follows: the solution of the given construction problem is considered as valid if it keeps its properties while dragging. The answer to this phase constituted the prelude of the conflicting case noticed from the findings of the present study, which is explained with an example below.

As described in detail in the previous chapter, GG asserted that the approach coded as A6 could be regarded as a valid geometric construction in Activity 1. By means of A6, GG used the tool ‘perpendicular bisector’ and constructed the circle passing through all the vertices of the given triangle (See Figure 4.50). Then, it was concluded that this figure passed the drag test criterion. As the criteria presented by Stylianides and Stylianides (2005) were aimed to be utilized in the study, the next move was to check the compatibility criterion. In the compatibility criterion, the solution of the construction problem is accepted to be valid if its steps are appropriate to the construction restrictions. The term 'construction restriction' refers to using the properties and tools of dynamic geometry software as if compass and straightedge were used in the construction (Stylianides & Stylianides, 2005). For example, while constructing the circumcircle of a triangle, using 'circle through three points' tool of GeoGebra violates the restrictions and makes the geometric figure invalid for the compatibility criterion. Since being robust under dragging is a necessary but not sufficient condition for the compatibility criterion, it can be stated that the compatibility criterion covers the drag test criterion. It was persistently emphasized that "one criterion is not in general more correct than or superior to the other" (Stylianides & Stylianides, 2005, p.38). Moreover, according to Stylianides and Stylianides (2005), if the geometric figure cannot pass the drag test, then it is accepted as invalid. Moreover, if the geometric figure can pass the drag test, then it should be checked whether there is a violation of the construction restrictions in a dynamic geometry environment. However, it was seen that the figure formed via A6 did not pass the compatibility criterion. What GG offered by using A6 was still counted as a geometric construction. There was no problem at that point in terms of using the mentioned two criteria directly in the analysis of the data obtained in the study.

As stated earlier, GG asserted another approach, A8, to be valid in Activity 1, which was conducted with the second GeoGebra file involving more restricted tools (See Figure 4.51). Checking whether the geometric figure presented by GG by means of A8 passed the drag test unfolded the conflicting issue. More specifically, since the tool 'perpendicular bisector' was restricted, GG acted in A8 as if they were using compass-straightedge. While dragging, the circle passing through the vertices of the given triangle disappeared when one of the points was dragged through a far point. When the approach was examined in detail, it was seen that it was a correct approach in terms of using compass-straightedge. In other words, CSG followed the same approach and their geometric figure was proper to be coded as a geometric construction. In this respect, the subject geometric figure of GG would be passing the compatibility criterion since it is proper to ‘construction restriction’ in this criterion. However, Stylianides and Stylianides (2005) situated passing the drag test as a necessary condition for the compatibility criterion. This gap which the participants fell into opened a new window to the issue. That is, a need for an intermediate step or category emerged at this point of the analysis. After the detailed examination, it was seen that the mentioned disappearance of the circle of GG in Activity 1 and the failure in the drag test criterion stemmed from the violation of the assumption used in the construction phase as a result of dragging. Therefore, another question regarding this case was added to the diagram as the third phase (See Figure 4.17). There were two questions in the third phase of the diagram. According to the answers in this phase, four possibilities emerged. One of these possibilities was leading the geometric figure to be labeled as not a construction while the remaining three cases determined the type of geometric construction after the application of the mentioned sequence of criteria.

After the analysis of all the geometric figures that GG presented by asserting them as constructions in the four cognitive unity based activities, it was seen that some of them were not actually constructions even though they thought so. Moreover, there were geometric constructions which were categorized as construction type A (CTA) and construction type B (CTB). However, there was not a geometric construction which was categorized as construction type C (CTC). Therefore, CTC remained as a hypothetical case for this study. In more detail, there was not a geometric figure which was appropriate for the geometric figure asked and passed both the drag test criterion and the compatibility criterion. Although this study did not put forward such a case, one of the examples related to the construction of the angle bisector given by Stylianides and Stylianides (2005) can be considered as proper to CTC. Since the mentioned construction, which was named as Construction A in that study, passed both the drag test criterion and the compatibility criterion, it might be coded as CTC based on the diagram used in the analysis.

As mentioned, some approaches declared by GG to be leading to geometric construction were not actually proper to be labeled as construction. However, such a case was not observed in CSG. The approaches CSG mentioned as not working in terms of reaching the intended geometric construction were actually invalid. The ones they asserted to be ending up with a valid geometric construction were indeed appropriate to label as a geometric construction. Thus, it can be concluded that CSG was more successful in attaining the validity of their approaches to geometric construction. This conclusion might be occasioned by the fact that CSG had to present a solid background for their actions so as to end up with a correct approach. They should know the underlying logic of even basic constructions while using compass-straightedge so that they would be capable of pursuing more complicated and intertwined cases and offer valid approaches. For example, they should even be careful about setting the compass to a proper length. Thus, they should find out the necessary conditions by using their previous knowledge, exploring the case, and discussing it with others. Since they needed to think about the case in-depth during the construction process and query about each move conducted with compass-straightedge, they could make the approaches ground on more solid bases. Moreover, likewise the dynamic geometry environments, Stylianides and Stylianides (2005) also described the criterion for determining the validity of a solution for a construction problem in paper-pencil environment as the use of compass and straightedge only in the solution process. However, this study was aimed to list a series of conditions for the use of compass-straightedge and presented the criteria list involving six items (See Table 4.5).

All in all, geometric construction can be considered a substantial concept for both geometry and geometry education (Djorić & Janičić, 2004; Kostovskii, 1961; Napitupulu, 2001; Usiskin, 1987) and it is a main component of training of mathematicians throughout history (Sarhangi, 2007). Students' ability to construct geometric figures is directly related to the development of their geometric reasoning (Köse et al., 2012), and their awareness of geometry can be developed by means of participating in visualization and construction related tasks (Sinclair et al., 2012a). Geometric construction involves the identification and description of some properties and relations peculiar to a specific geometric figure. For example, to construct an isosceles triangle, the issue to which attention needs to be paid is the congruency of two sides of the triangle since it is a relation or characteristic of the mentioned triangle

(Sinclair et al., 2012a). Despite the benefits of geometric construction tasks for students in geometry teaching, it was also mentioned that there might be a pitfall in practice in the case where teachers directly give the construction steps to the students (Kuzle, 2013). In cases where students are not asked to simply follow the given instructions in a geometric construction, they can develop their problem solving by means of the reasoning process they are involved in (Pandisico, 2002). On the other hand, when students are asked to pursue the given steps in a geometric construction, they do not need to think about the underpinnings of these steps. Thus, it becomes an ineffective task in terms of leading students to develop their problem solving skills, geometry knowledge, and mathematical thinking (Schoenfeld, 1988; Kuzle, 2013).

5.4. Implications of the Study

Based on the findings and conclusions of the study and the related discussion, the present study can be claimed to have the potential to contribute to the literature and to provide insight and implications to prospective mathematics teachers, in-service mathematics teachers, teacher educators, stakeholders and policy makers in terms of teacher education programs, curriculum developers, and researchers who are working on the concepts of cognitive unity, argumentation, proof, and geometric construction.

Based on the findings of the study, the content of the courses in teacher education programs should be determined carefully. In this manner, this study could contribute to the development of the content of geometry related courses in mathematics teacher education programs. In more detail, the content of geometry related courses could be modified according to the characteristics of prospective middle school mathematics teachers' geometric reasoning process while using different tools, the nature of their arguments, and the effects of arguments on proving. It was concluded that the participants had deficiencies in even basic geometry concepts. Thus, prospective middle school mathematics teacher should gain the necessary content knowledge in geometry concepts before graduation. More precisely, the participants of the study had taken the course Geometry, which provided the substantial support to their content knowledge regarding geometry concepts. It was offered as a must course in Elementary Mathematics Teacher Education program.

Since geometry was a challenging issue for both teachers and learners (Arıcı & Aslan-Tutak, 2015; Fischbein & Nachlieli, 1998; Hollebrands et al., 2010; Laborde et al., 2006; Sinclair et al., 2012a, 2012b; Stone, 1971), it was observed that there were points in which prospective middle school mathematics teachers had difficulty in the cognitive unity based activities depending on their deficiencies in geometry. On the other hand, according to the revision of teacher education programs conducted by the Council of Higher Education (CoHE, 2018), Geometry is neither a must course nor one of the suggested elective courses in the program. The deficiencies in the geometry knowledge of prospective middle school mathematics teachers in this study who took this course might present the clue to foresee the possible shortcomings in geometry knowledge of prospective middle school mathematics teachers who are subject to the revised program. In this respect, mathematics teacher education programs might take precautions related to this issue by offering an elective course directly related to Geometry, like the previous version of the program.

As Mariotti (2006) proposed, cognitive unity can be utilized as a basis to examine complicated relations between argumentation and proof. This study approached cognitive unity within the context of geometry. Cognitive unity could be utilized in a way that it might be helpful in terms of meeting students' needs related to argumentation and proof. Looking at the cognitive unity from an educational perspective might widen the applicability of it in practice. While teaching proof-related concepts in undergraduate courses and helping prospective teachers to integrate the core mathematics with mathematics teaching, cognitive unity might be considered as an effective tool. In this respect, it could be integrated into some courses in mathematics teacher education programs in appropriate occasions. Moreover, to overcome the undergraduate students' difficulties in proof due to the quick transition to formal proof in a variety of domains in mathematical courses, it was suggested that a transition to proof course at the beginning of the undergraduate program could be offered to prepare a bridge through formal proof and to help them to use mathematical language effectively (Moore, 1994). In terms of the present study, the concept of cognitive unity could be considered as a technique to facilitate students' learning in such kind of transition to proof courses. Moreover, when the issue is not advanced mathematics, but middle school students, the cognitive unity activities which end up with informal proofs or justifications could be used to improve students' reasoning.

Since geometric construction is chosen as the set up for the argumentation phase of the cognitive unity in which the conjecture production is aimed at, the possibility of using different tools brings about another aspect for the study. Moreover, the literature review showed the scarcity of the studies related to geometric construction involving the use of compass-straightedge, so this study might contribute to the related body of knowledge. Besides, since the present study involves the use of GeoGebra while conducting constructions, it could present findings about to what extent prospective teachers can use GeoGebra in constructions. When applied properly, the geometric construction has an inevitable potential to develop students' reasoning in geometry. In this respect, to present frameworks peculiar to the tools which can be used to determine the validity of the geometric figures formed is needed. To propose the criteria to check the validity of a geometric figure carries the utmost importance in terms of the scope of the concept of geometric construction for further studies.

Regarding argumentation, this study might serve to the related literature on three aspects, which are global argumentation structures, components of them, and the functions of rebuttal. It was also stated that the research pertaining to argumentation centered on the structure or the content of the arguments (Inglis et al., 2007; Pedemonte, 2007b). By means of the second research question, this study investigated both the structure and the content of argumentation. This study might be considered as a support to the use of Toulmin's model in spite of the criticisms that exist in the literature (e.g., Conner et al., 2014b; Mariotti et al., 2018; Metaxas et al., 2016; Pedemonte & Balacheff, 2016). Since Toulmin's model was used along with some modifications in a variety of studies (e.g., Conner et al., 2014a; Knipping, 2008; Reid & Knipping, 2010; Verheij, 2005), the adapted version of the model, which was reconstructed with the inclusion of new components as well as keeping all six component situated in the default version of Toulmin's model, might be used while investigating and analyzing the complex argumentation process. The frame followed to arrange the global argumentation structures might be used in other domains of mathematics, different from geometry. In addition, since Toulmin's model of argumentation was declared to be independent of other disciplines, the adapted version in this study could be applied to other disciplines. All in all, the outputs of the study in terms of the categorization of the global argumentation structure brought along a more general case to the literature. In this respect, the types of structures were employed in a way by which basic and comparatively non-overlapping occasions were considered.

5.5. Assumptions and Limitations of the Study

In the present study, it was taken the road with some assumptions. Since the participants are junior prospective middle school mathematics teachers, they did not take a course directly involving the geometric construction and GeoGebra at the time of the data collection. Therefore, teaching sessions, which lasted four weeks, were prepared for the beginning section of the Teaching of Geometry Concepts course. These teaching sessions aimed to make the participants ready and more qualified in terms of geometric construction, use of compass-straightedge and GeoGebra, and argumentation, reasoning, and proof in mathematics education. Besides, during this process, the participants were asked work on GeoGebra and get prepared for the applications of the next week which would be conducted with GeoGebra. To this end, extra handouts regarding GeoGebra were sent to the participants by e-mail. Upon finishing the mentioned teaching sessions, the participants started to work on the activities. Moreover, to make them learn to use the mentioned tools and get experience with geometric constructions and proof, two orientation activities were administered after teaching sessions and before the cognitive unity based activities. Thus, after all these preparations, it was assumed that the participants got experienced about the main issues in the study before engaging with the cognitive unity based activities and also they were aware of the expectancies regarding the cognitive unity based activities.

The participants were assumed to be presenting high effort during all weeks of the course especially while working on the cognitive unity based activities. Since they knew that the course covers the data collection process of the study conducted by the instructor, it was observed that they tried to keep their involvement in the study at the high level as much as possible. That is to say, it can be stated that the participants' engagements during the course were not carried out in a superficial manner. It was especially inferred from some occasions noticed in the video recordings. For example, the participants did not stop working on the activity even if they thought that they had found an approach presenting a valid geometric construction or the statements they came up with regarding the activity were correct. They were ungrudging to discuss a variety of possible ideas regarding the proof of the asked statement. Moreover, during the interviews which were conducted after the preliminary analysis of the data collected during the elective course, they allocated plenty of time and they were eager to participate in the interviews. Therefore, it was assumed that the participants tried to make their best during each step of the course as well as the data collection process of the study.

It can be stated that junior prospective middle school mathematics teachers who took the Teaching of Geometry Concepts course and I, as the instructor-researcher, developed a relationship based on trust during the data collection due to the familiarity coming from the previous courses they took in the program. As mentioned, I knew the undergraduates in this year level in the program quite well since I assisted two Calculus courses they took in their second year. Since the same activities were usually applied to the first section on Mondays and to the second section on Tuesdays, it was quite important for the sake of the study that they would not share anything about the content of the activities. This issue was explained to all prospective teachers in the elective course at the beginning week while they were deciding about whether they would take the course or not. Thus, it was assumed that they did not share the details with the ones in other section during the whole course and any occasion which presented the opposite was not noticed during the application and the analysis process. To ensure this, they were sometimes warned about the sensitivity of this issue during the process. Moreover, it was assumed that my status at the program as a research assistant did not affect the performances of the participants in a negative manner since the data collected from the participants in conjunction with the purposes of the study do not have an affective aspect.

In addition to the assumptions, some issues could be regarded as the limitations of the study were explained as follows. The first issue which can be considered as the limitation is that the data was collected from six junior prospective middle school mathematics teachers in a state university in Ankara. The findings of the study would be different if some other juniors were selected from either the subject university or another university. In a similar vein, if the participants were selected from other year levels in the same program such as freshmen, sophomores, and seniors, which was the case in the pilot study, the findings would be different. Moreover, since the participants were selected by purposeful sampling, they might not be representative to other prospective middle school mathematics teachers. In this respect, the findings can be considered as less generalizable to other occasions. Actually, it can be stated that the generalization was not the aim of the study due to its qualitative nature. Another point regarding the participants was that there were seven junior prospective middle school mathematics teachers in each section of the Teaching of Geometry Concepts course and they formed two groups as involving three and four participants. However, the analysis of the data obtained from the triads in each section was presented in the study. Actually, since the aim was to conduct a deep investigation of the issues at stake in the study, to focus on one group in each section might provide the opportunity to deepen the understanding of the concepts.

Another issue as the limitation of the study is that the data mainly were gathered by means of the four cognitive unity based activities administered within an elective course, namely, Teaching of Geometry Concepts. Since data collection was centered on the four activities, which were prepared as related to triangles and circles, the findings of the study are limited to these activities. Moreover, during the application of these activities, I was the instructor-researcher, as stated before. The juniors in the elective course were expected to be involved in a collective argumentation in which I had the role of guidance. Thus, I had a direct effect on the argumentation processes of the groups. This situation was tried to be controlled by means of being careful about following an equal stance to all groups and configuring the guidance in a systematic manner as much as possible.

As mentioned, the tool used by the participants in the first section of the course during geometric construction was compass-straightedge and the tool used in the second section of the course was GeoGebra. The case that juniors in a section have lack of experience related to the tool used in other section might be considered as a limitation. To handle this issue, in the teaching sessions at the beginning of the course, the same procedure was followed in each section. That is, juniors in each section were introduced the use of both compass-straightedge and GeoGebra in geometric constructions and they endeavored on basic geometric constructions by using all the mentioned tools. Also at the end of the interviews, they were asked whether there is any construction they want to apply or talk about by using other tool. Depending on students' demands, some differences between the uses of tools were mentioned and the application of the other tool in some activities was conducted collectively.

Another limitation of the study is that the computer used by GeoGebra group was recorded by a camera focused on the screen and also an audio recorder was utilized during the application of the cognitive unity based activities and interviews. However, there are some occasions in which students blocked the lens of the camera for a few seconds and some lightening changes during the activities affected the quality of the recordings. Although what was conducted with GeoGebra was quite clear in the analysis when aligned with students' sentences at that moment, their documents in which they wrote down what they conducted, and GeoGebra files saved during the activity, this issue might be considered as a limitation of the study.

As a delimitation of the study, it can be stated that the argumentation process of the conjecturing phase in the cognitive unity based activities was particularly focused during the analysis of the second research question. That is, the global argumentation structures of this phase were arranged so as to provide a detailed picture of it. However, such an in-depth process was not followed for the proof section of the cognitive unity based activities, this might be considered as a delimitation. However, based on the approach of Pedemonte which refers to argumentation as "the process of conjecturing" (Reid & Knipping, 2010, p.163), proof was deduced as the product of the overall process. Another delimitation of the study is that the construct of cognitive unity was offered and examined in the scope of geometry.

As the final section of this chapter, by considering the findings, implications, and limitations of the study, some recommendations for further research were presented below.

5.6. Recommendations for the Further Research

Originated from the overall process of the study, some recommendations for further research studies were mentioned hereafter. Firstly, the construct of cognitive unity was taken into consideration with respect to prospective middle school mathematics teachers solely within the domain of geometry. Therefore, in further studies, it might be worth to investigate cognitive unity with the participation of different levels of students and within the other areas of mathematics. Indeed, drawing on the findings of the studies conducted by Pedemonte (2007a, 2008) which pointed out some differences between geometry and algebra in terms of the cognitive unity, such studies conducted with other mathematical domains have the potential to contribute to the related literature and provide comparisons. Thus, it was highly recommended that cognitive unity should be considered within the scope of the other domains of mathematics. As further research, the concept of cognitive unity as outlined in this study could be as a tool while teaching the reasoning and proof needed areas.

The second research question of the study intends to investigate the global argumentation structures of prospective middle school mathematics teachers while producing conjectures in the cognitive unity based activities. In this manner, as an extension of this study, the global argumentation structures that emerged while prospective middle school mathematics teachers working on proving phase might be investigated. Thus, the global argumentation structures in argumentation and proof with respect to the definition of cognitive unity might be compared. The findings also underlined the need for studies regarding the global argumentation structures in different areas of mathematics. Moreover, while arranging the global argumentation structures, who said which component might be considered. That is to say, whether the data component was given by student, teacher or both of them might be taken into consideration in the analysis. In this study, the only component which is known by whom it was presented is guidance.

The statements coded as guidance were the ones stated by the instructor only. However, the remaining all components were not clear in this perspective. As related to this idea, Conner et al. (2014b) offered a system as mentioned before. They used different frames while presenting the component depending on the person who declared the related statement. Another point mentioned in other studies is that all components of argumentation might not be stated explicitly. The presence of some implicit components might also be signified in the global argumentation structures. For example, Knipping (2008) used a box framed with the dotted line while presenting the components emerged implicitly in the flow of the argumentation stream.

In this study, the concept of geometric construction was undertaken to provide a collective argumentation during the conjecture production phase in the cognitive unity based activities. Since geometric construction is a demanding and challenging task and a critical tool for investigation of geometry (Kostowskii, 1961) and the participants were not so familiar with the basis of geometric construction, it was an efficacious decision. With the core idea of cognitive unity in mind, depending on the mathematical domain at stake, another concept might be employed to facilitate the conjecture production phase and to lead to display a rich collective argumentation environment. Another point to note related to the geometric construction is the tools used which were compass-straightedge and GeoGebra in the current study. Alternatively, some other tools such as ruler and Geometer's Sketchpad might also be considered in further studies. While investigating an issue through which the participants use any technological tools such as GeoGebra, a program might be deployed to save screen recording. When the screen of the computer was recorded an outsider camera, the study might subject to some unclear cases. To avoid having the unclear seconds in the video recordings, screen recording might be used in future studies. Moreover, the collaboration of groups can be arranged in a face-to-face setting around the computer, which is the case in this study, or in a networked learning environment such as Virtual Math Teams (Lipponen, 2002; Öner, 2008; Stahl, 2009). Öner (2016) described Virtual Math Teams as follows; "VMT is an open-source, virtual, collaborative learning setting that affords synchronous text-based interaction (chat) with an embedded multi-user dynamic geometry application, GeoGebra" (p.60). In this respect, further researchers might use Virtual Math Teams (VMT) in this concept.

Geometric constructions embedded in the cognitive unity based activities brought to surface the significance of the issue in terms of the development of geometric reasoning and argumentation. A further study might be conducted to investigate to what extent prospective and in-service mathematics teachers are able to use dynamic geometry programs and compass-straightedge in geometric construction, how effectively they can integrate these tools into teaching, and how the use of these tools affects students' understanding. In addition, in the study of Karakuş (2014), prospective middle school mathematics teachers reported that geometric construction activities supported the conceptual understanding of students. In this respect, further research might be conducted to investigate the relationship between geometric constructions and the conceptual understanding of geometric concepts. Besides, it can be stated that the number of studies about geometric constructions in terms of affective factors is also limited in the accessible literature. In this manner, a further study might be conducted to investigate geometric constructions as related to affective factors such as self-efficacy, attitude, and belief. Such studies might be helpful to understand why teachers do not integrate geometric construction into teaching. The present study offered criteria to control whether a geometric figure can be labeled as a geometric construction in different settings, that is, when compass-straightedge and GeoGebra are used. These frameworks involving a list and a diagram regarding criteria might be implemented in further studies to evaluate whether it can be used as an effective tool.

As an extension of this study, it might be investigated that how prospective middle school mathematics teachers prepare activities or lesson plans on triangle and circle related objectives in middle school mathematics curriculum by aiming to integrate the conjecturing and dynamic exploration like the ones they have been involved in the cognitive unity based activities. Since middle school mathematics curriculum does not cover formal proving, prospective middle school mathematics teachers might focus on producing conjectures, reasoning, and justification. Such a study might also fill the gaps in terms of the practical considerations.

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APPENDICES

APPENDIX A. COURSES IN ELEMENTARY MATHEMATICS TEACHER EDUCATION

The lists of both must courses and elective courses offered in Elementary Mathematics Teacher Education program of the selected university were given below. The abbreviations T and P which are presented on the right of the tables refer to the theory and practice class hours of the courses and also ECTS (European Credit Transfer and Accumulation System) refers to the credits of the courses.

Elementary Mathematics Teacher Education program courses list

The 1st semester	**T**	**P**	**ECTS**
Ataturk's Principles and History of Revolutions I	2	0	2
Basic Information and Commutation Technologies	0	2	2
Introduction to Education	3	0	4
Turkish I Written Expression	2	0	2
General Mathematics	4	2	7
Computer I	2	2	6
Basic English I	2	2	3
Elective			4
			30
The 2nd semester			
Ataturk's Principles and History of Revolutions II	2	0	2
Educational Psychology	3	0	4
Turkish II Oral Expression	2	0	3
Discrete Mathematics	3	0	7
Geometry	3	0	7
Basic English II	2	2	3
Elective			4
			30
The 3rd semester			
Instructional Principles and Methods	3	0	4
Calculus I	4	2	5
Linear Algebra I	3	0	6
Scientific Research Methods	2	0	3

Teaching Elementary School Mathematics	3	0	4
Elective			5
			30
The 4th semester			
Measurement and Assessment	3	0	4
Calculus II	4	2	9
Instructional Technologies and Material Design	2	2	4
Mathematical Modeling in Elementary Education	2	2	4
Middle School Mathematics Curriculum	3	0	4
Elective			5
			30
The 5th semester			
Calculus III	3	0	5
Statistic and Probability I	2	2	4
Introduction to Algebra	3	0	4
Methods of Teaching Mathematics I	2	2	7
Elective			10
			30
The 6th semester			
Information and Communi. Techn. Assisted Math. Instruction	2	2	4
Analytic Geometry	3	0	4
Community Service	1	2	3
Methods of Teaching Mathematics II	2	2	7
Elective			12
			30
The 7th semester			
Classroom Management	2	0	3
Guidance	3	0	4
School Experience	1	4	8
Misconceptions in Elementary Mathematics Education	3	0	4
Elective			11
			30
The 8th semester			
Turkish Educational System and School Management	2	0	3
Teaching Practice	2	6	15
Elective			12
			30
Total			240
Must Courses Total ECTS			177
Elective Courses Total ECTS			63

Elective courses offered in the Elementary Mathematics Teacher Education program

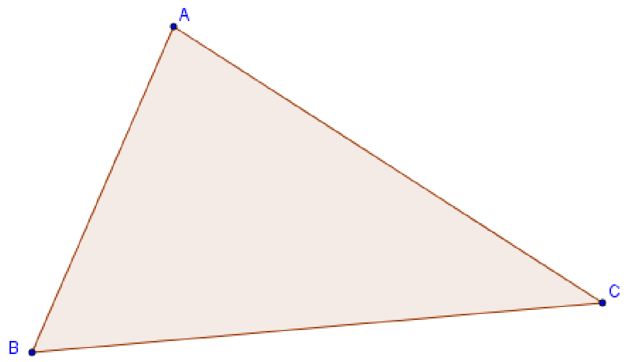
Fall semester	T	P	ECTS
Creative Drama in Education I	3	0	4
School Mathematics in the World	2	0	4
History of Mathematics	2	0	4
History of Science	2	0	2
Effective Learning in Mathematics Education	2	0	4
Computer Assisted Mathematics Instruction	2	0	4
Mental Games I	2	1	4
Elementary Number Theory	3	0	4
Teaching of Geometry Concepts	3	0	4
Creative Drama in Mathematics Education	3	0	4
Visual Art Culture	2	0	4
Fall semester			
Computer II	2	2	4
Creative Drama in Education II	3	0	4
Linear Algebra II	3	0	4
Problem Solving Approaches in Mathematics	2	0	4
Differential Equations	4	0	4
Statistic and Probability II	2	2	4
Communication	2	0	4
Mental Games II	2	1	4
Philosophy of Mathematics	2	0	4
Micro Teaching in Mathematics Education	3	0	4
Test Development in Mathematics	3	0	4
Any semester			
Programming Supported Mathematics Teaching	2	2	4
Measurement and Assessment in Mathematics Education	3	0	4
Origami in Mathematics Education	3	0	4

APPENDIX B. COGNITIVE UNITY BASED ACTIVITIES

ACTIVITY 1

Worksheet A

Construct the circle passing through the vertices of the given ΔABC by using compass-straightedge/GeoGebra.



Worksheet B

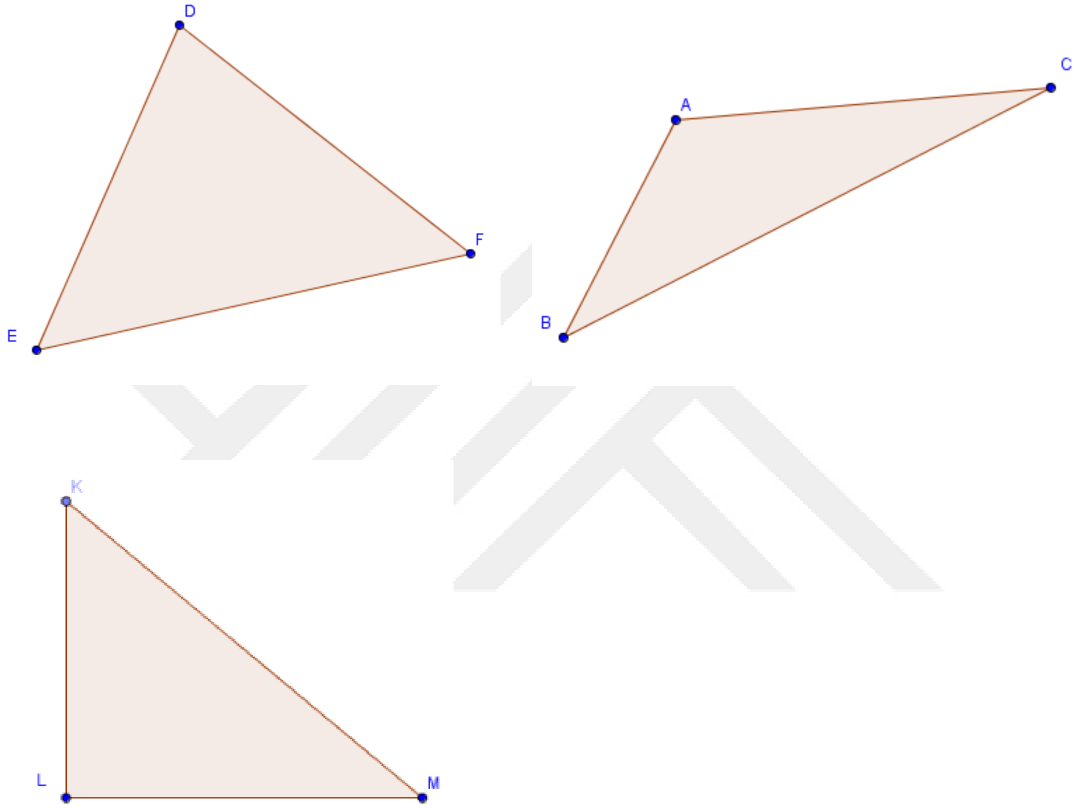
Prove the statement given below.

The perpendicular bisectors of the sides of a triangle are concurrent and this point is the circumcenter of the triangle.

ACTIVITY 2

Worksheet A

Construct the altitudes and the orthocenters (if exist) of the given ΔDEF , ΔABC, and ΔKLM by using compass-straightedge/GeoGebra.



Worksheet B

Prove the statement given below.

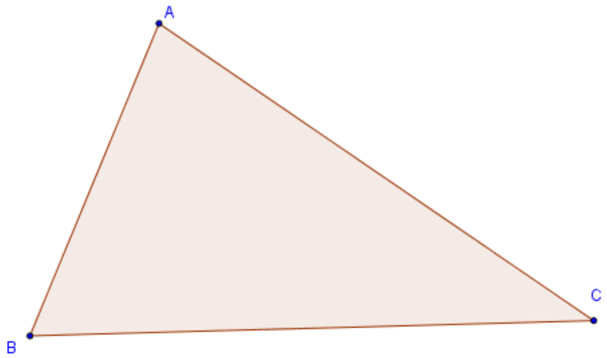
The altitudes of a triangle are concurrent.

ACTIVITY 3

Worksheet A

Construct the orthocenter, the circumcenter, and the centroid of the given ΔABC by using compass-straightedge/GeoGebra.

Examine the connection/relationship among these points.



Worksheet B

Prove the statement given below.

The circumcenter, the orthocenter, and the centroid of a triangle are collinear.

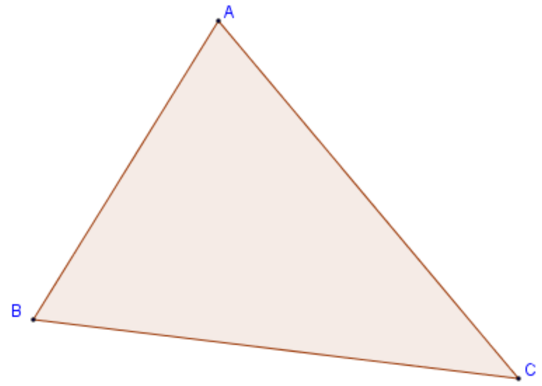
ACTIVITY 4

Worksheet A

Mark random points X, Y, and Z on the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of the given ΔABC respectively.

Construct the first circle passing through the points A, X, and Z, second circle passing through the points B, Y, and Z, and the third circle passing through the points C, Z, and Y by using compass-straightedge/GeoGebra.

Examine the connection/relationship among these circles.



Worksheet B

Prove the statement given below.

Suppose that the point X, Y, and Z are placed at random on the sides of ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point Z is on $\overline{CA}$. Then, in every case, the circles AXZ, BXY, and CYZ are concurrent.

APPENDIX C. INTERVIEW QUESTIONS EXAMPLE

CSG, Activity 1

Conjecture production section

The written works of the group related to the conjecture production section of Activity 1 were given and asked them to examine.

1. You stated that the first and the third approaches for geometric construction (A1 and A3) did not work. Do you agree now?

2. You stated that the second and the fourth approaches for geometric construction (A2 and A4) resulted in a valid construction. Do you agree now?

3. What would you do if you use GeoGebra in this activity?

4. Is there any issue about the first section of Activity 1 that you wanted to mention?

Proof section

The written works of the group related to the proof section of Activity 1 were given and asked them to examine.

1. What is the proof method you used here?

2. Do you think that this argument can be labeled as a valid proof?

3. Is there any point you wanted to change in this argument?

4. Is there any issue about the second section of Activity 1 that you wanted to mention?

Activity 1

1. How did you work in two sections of the activity?

2. Do you think that any of the mentioned sections affected the other or not? How?

GG, Activity 1

Conjecture production section

The written works of the group related to the conjecture production section of Activity 1 were given and asked them to examine. Moreover, GeoGebra files they submitted were presented.

1. You were given two GeoGebra files. While working on the first GeoGebra file, you tried some approaches for geometric construction. You stated that this approach for geometric construction (A6) resulted in a valid construction and submitted the related GeoGebra file saving as 1.ggb. Do you agree now?

2. While working on the second GeoGebra file, you tried some approaches for geometric construction. You stated that this approach for geometric construction (A8) resulted in a valid construction and submitted the related GeoGebra file saving as 2.ggb. Do you agree now?

3. If you drag the point C to the right side in 2.ggb file, the intended circle passing through the vertices of the triangle disappeared at a point. Could you try it in 2.ggb file? What do you think about the validity of this approach?

4. What would you do if you use compass-straightedge in this activity?

5. Is there any issue about the first section of Activity 1 that you wanted to mention?

Proof section

The written works of the group related to the proof section of Activity 1 were given and asked them to examine.

1. What is the proof method you used here?

2. Do you think that this argument can be labeled as a valid proof?

3. Is there any point you wanted to change in this argument?

4. Is there any issue about the second section of Activity 1 that you wanted to mention?

Activity 1

1. How did you work in two sections of the activity?

2. Do you think that any of the mentioned sections affected the other or not? How?

APPENDIX D. ETHICS COMMITTEE APPROVAL

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

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ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

01 EYLÜL 2016

Sayı: 28620816 / 352

Konu: Değerlendirme Sonucu

Gönderilen: Doç. Dr. Mine Işıksal BOSTAN

İlköğretim Bölümü

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın : Doç. Dr. Mine Işıksal BOSTAN

Danışmanlığını yaptığınız doktora öğrencisi Esra DEMİRAY'ın "Ortaokul Matematik Öğretmen Adaylarının Pergel-Çizgeç ve Dinamik Geometri Programının Kullanıldığı İnşa ve İspat Etkinliklerindeki Akıl Yürütme Sürecinin İncelenmesi" başlıklı araştırması İnsan Araştırmaları Kurulu tarafından uygun görülerek gerekli onay **2016-EGT-133** protokol numarası **03.10.2016-30.06.2017** tarihleri arasında geçerli olmak üzere verilmiştir

Bilgilerinize saygılarımızla sunarız.

Prof. Dr. Canan SÜMER

İnsan Araştırmaları Etik Kurulu Başkanı

Prof. Dr. Meliha ALTUNIŞIK
İAEK Üyesi

Prof. Dr. Ayhan SOL
İAEK Üyesi

Prof. Dr. Mehmet UTKU
İAEK Üyesi

Prof. Dr. Ayhan Gürbüz DEMİR
İAEK Üyesi

Yrd .Doç .Dr. Pınar KAYGAN
İAEK Üyesi

Yrd. Doç. Dr. Emre SELÇUK
İAEK Üyesi

ODTÜ 2015

BU BÖLÜM, İLGİLİ BÖLÜMLERİ TEMSİL EDEN İNSAN ARAŞTIRMALARI ETİK ALT KURULU TARAFINDAN DOLDURULACAKTIR.

Protokol No: 2016-EGT-133

İAEK DEĞERLENDİRME SONUCU

Sayın Hakem,

Aşağıda yer alan üç seçenekten birini işaretleyerek değerlendirmenizi tamamlayınız. Lütfen "Revizyon Gereklidir" ve "Ret" değerlendirmeleri için gerekli açıklamaları yapınız.

Değerlendirme Tarihi: 01-09-2016

Ad Soyad:

☑ **Herhangi bir değişikliğe gerek yoktur. Veri toplama/uygulama başlatılabilir.**

☐ **Revizyon gereklidir**

☐ Gönüllü Katılım Formu yoktur.

☐ Gönüllü Katılım Formu eksiktir.

Gerekçenizi ayrıntılı olarak açıklayınız:

☐ Katılım Sonrası Bilgilendirme Formu yoktur.

☐ Katılım Sonrası Bilgilendirme Formu eksiktir.

Gerekçenizi ayrıntılı olarak açıklayınız:

☐ Rahatsızlık kaynağı olabilecek sorular/maddeler ya da prosedürler içerilmektedir.

Gerekçenizi ayrıntılı olarak açıklayınız:

☐ Diğer.

Gerekçenizi ayrıntılı olarak açıklayınız:

☐ **Ret**

Ret gerekçenizi ayrıntılı olarak açıklayınız:

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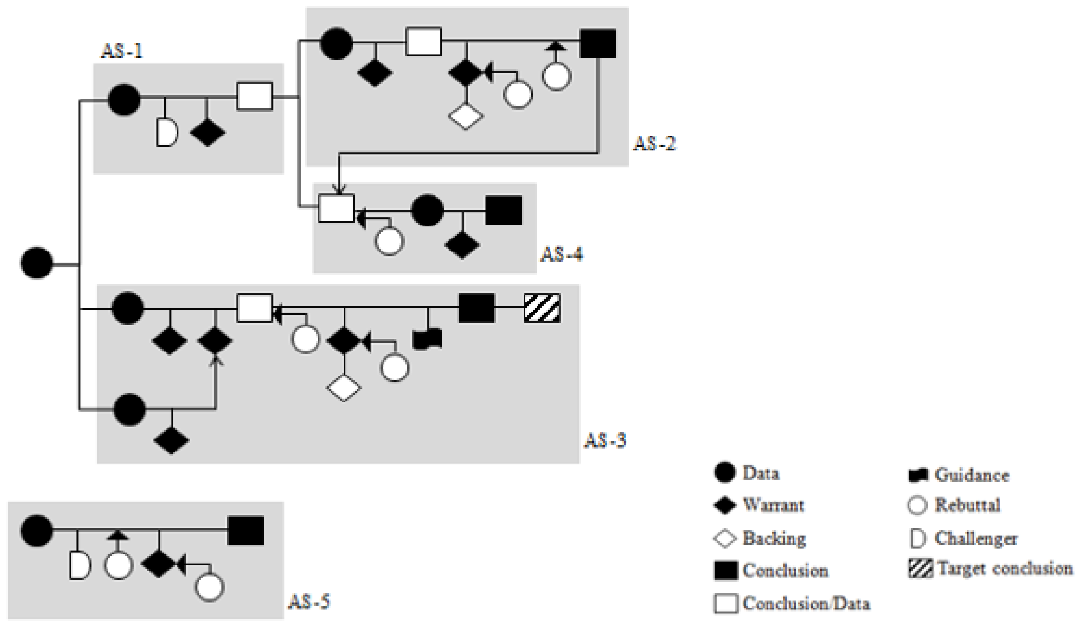
APPENDIX E. GLOBAL ARGUMENTATION STRUCTURES

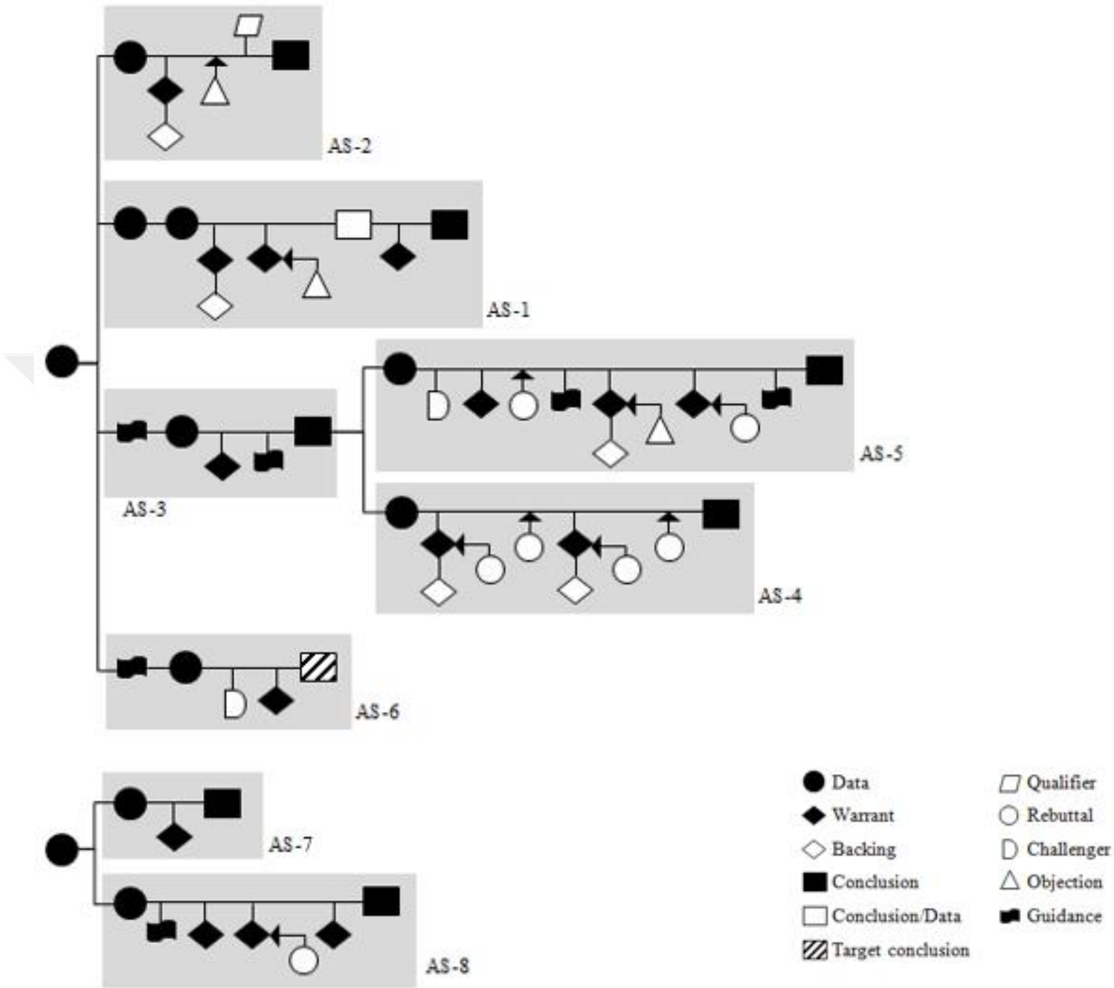
The global argumentation structure of CSG in Activity 1

The line-branching-structure



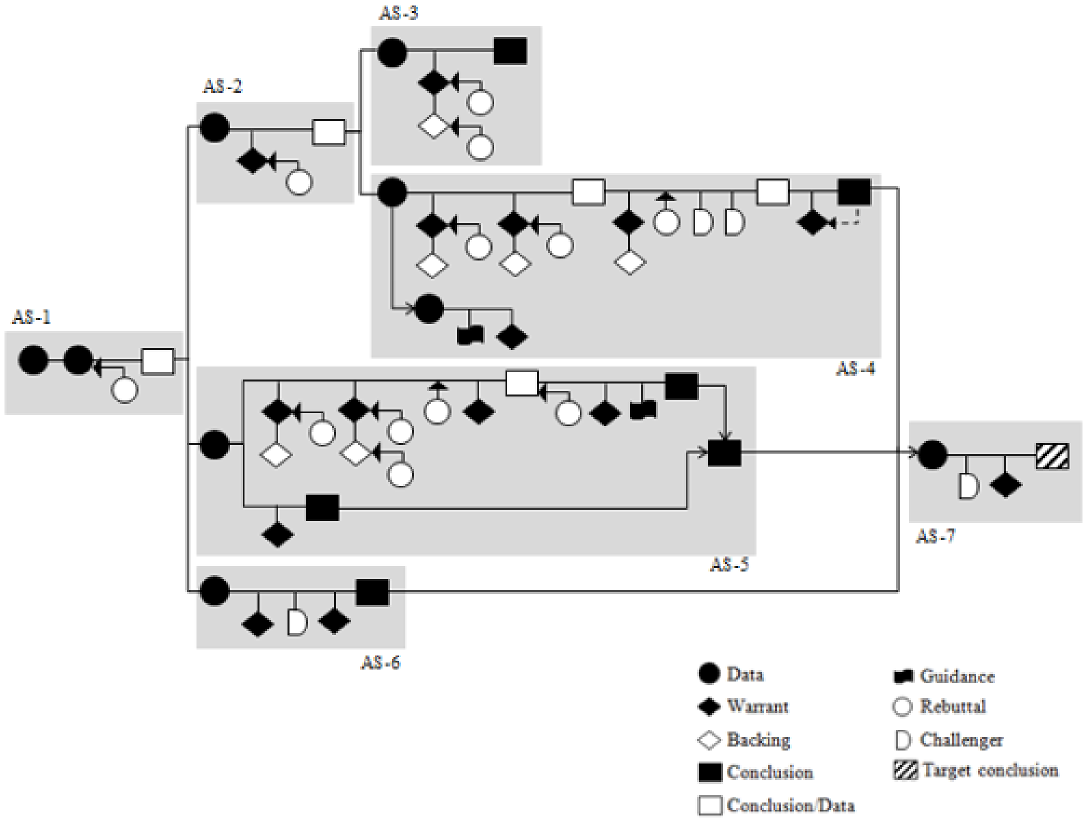
The global argumentation structure of GG in Activity 1

The branching-structure (multiple-rooted branching structure)



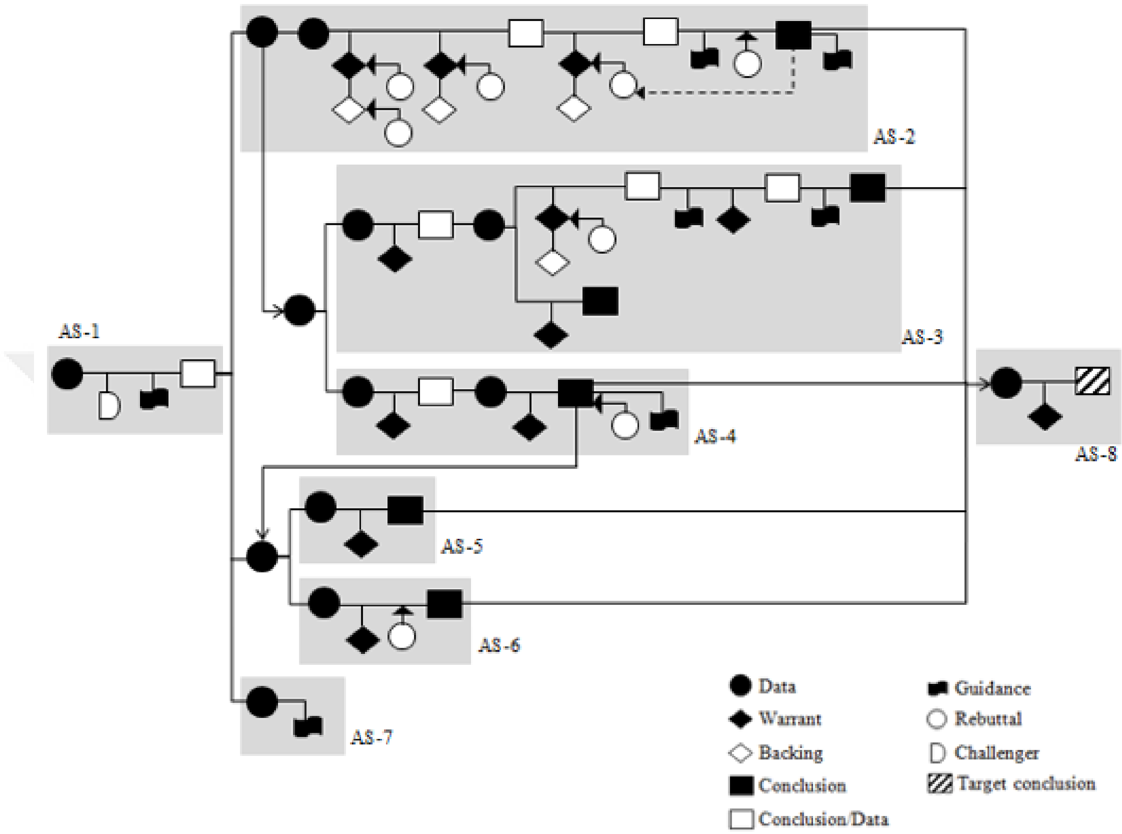
The global argumentation structure of CSG in Activity 2

The funneling-structure (one-rooted funneling-structure)



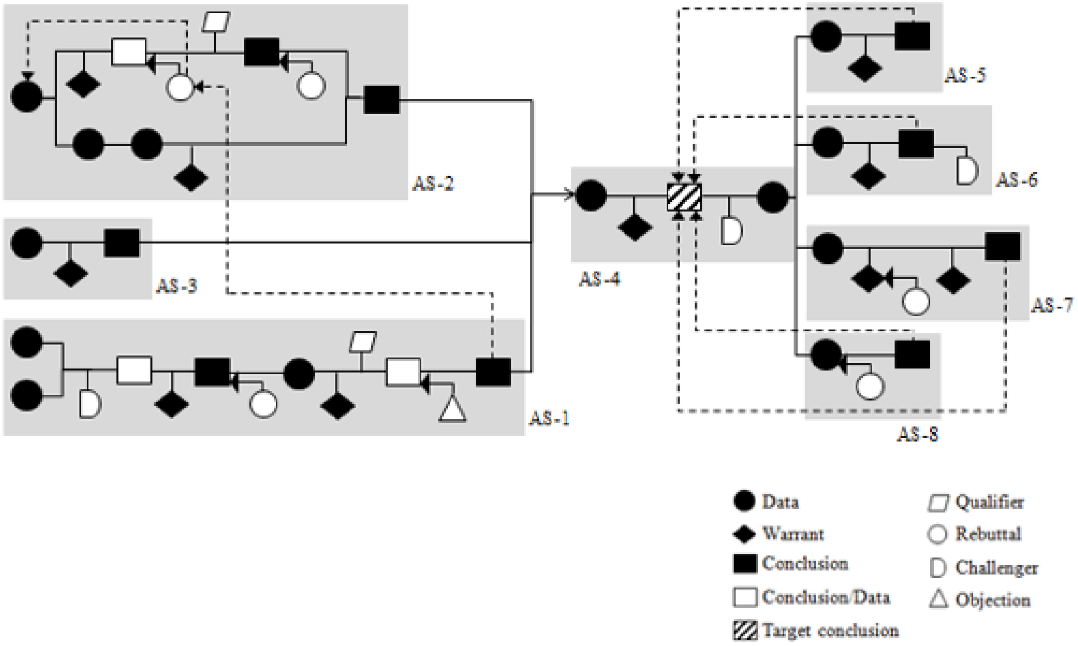
The global argumentation structure of GG in Activity 2

The funneling-structure (one-rooted funneling-structure)



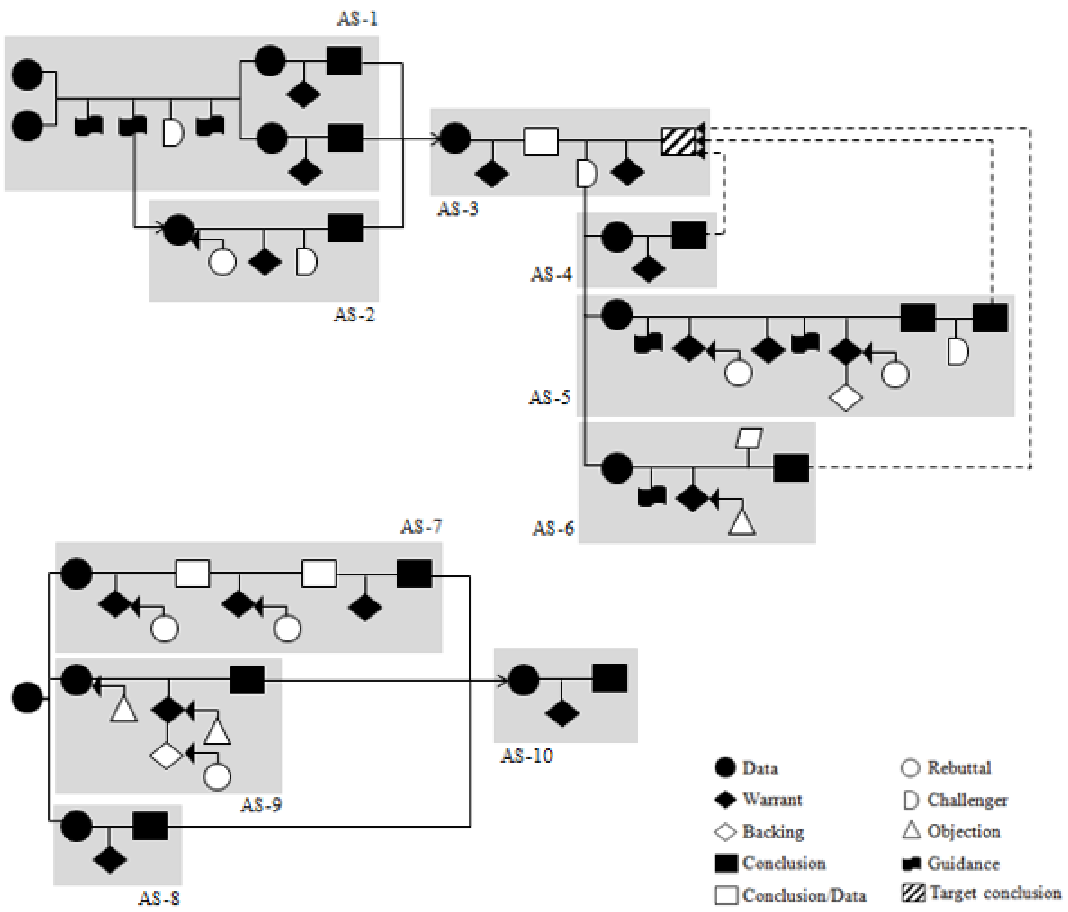
The global argumentation structure of CSG in Activity 3

The reservoir-structure



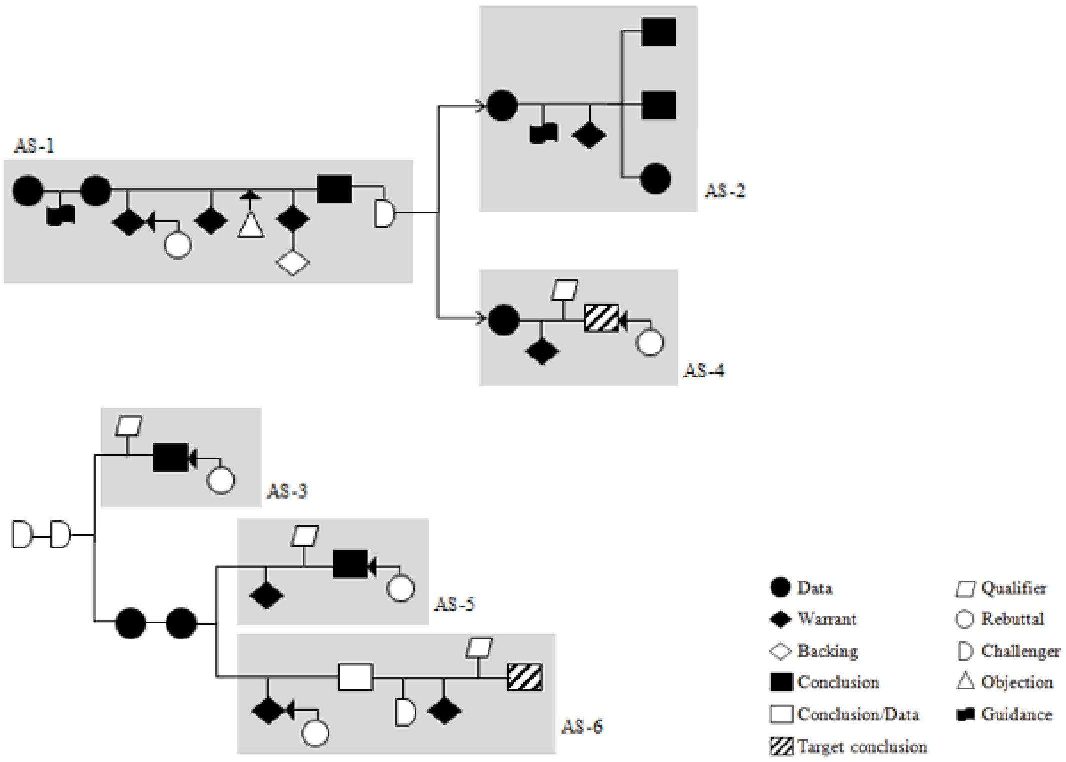
The global argumentation structure of GG in Activity 3

The reservoir-funneling-structure



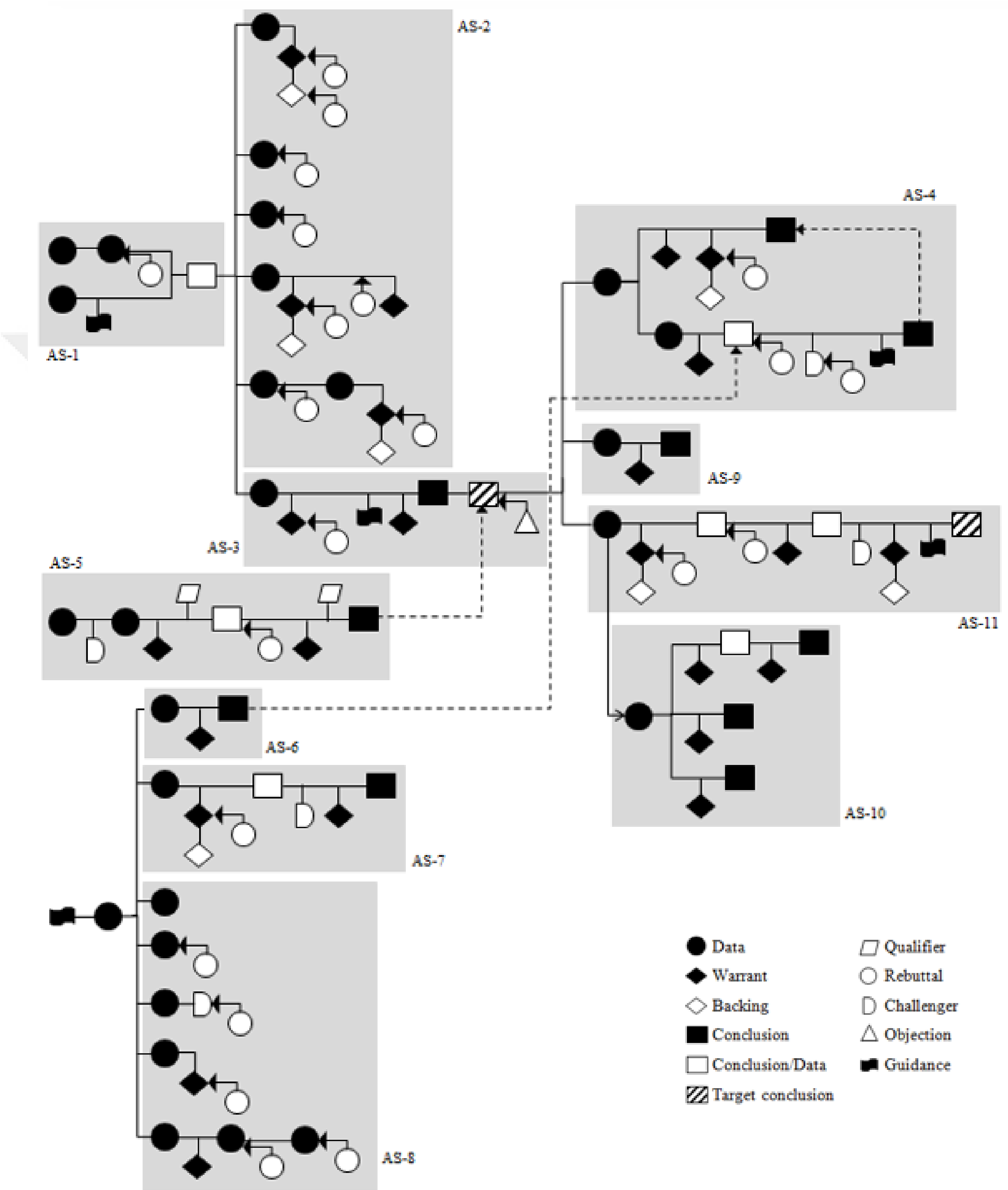
The global argumentation structure of CSG in Activity 4

The branching-structure (multiple-rooted branching-structure)



The global argumentation structure of GG in Activity 4

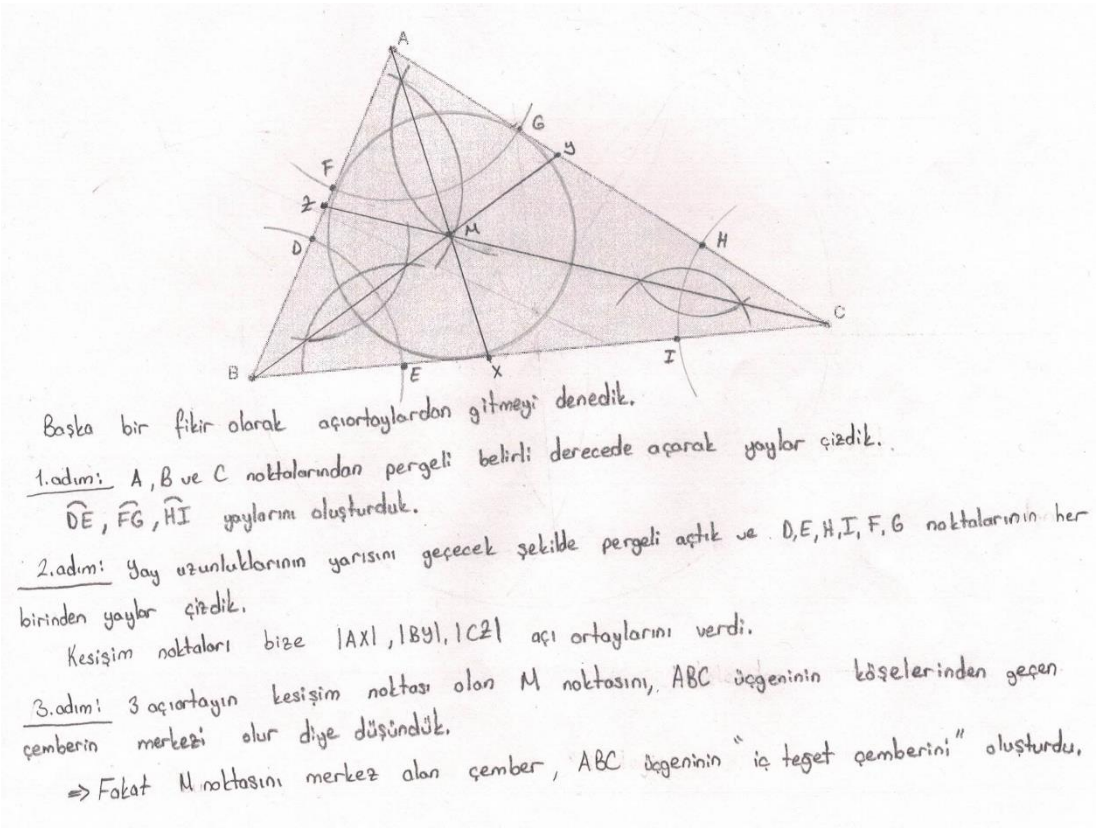
The line-reservoir-branching-structure



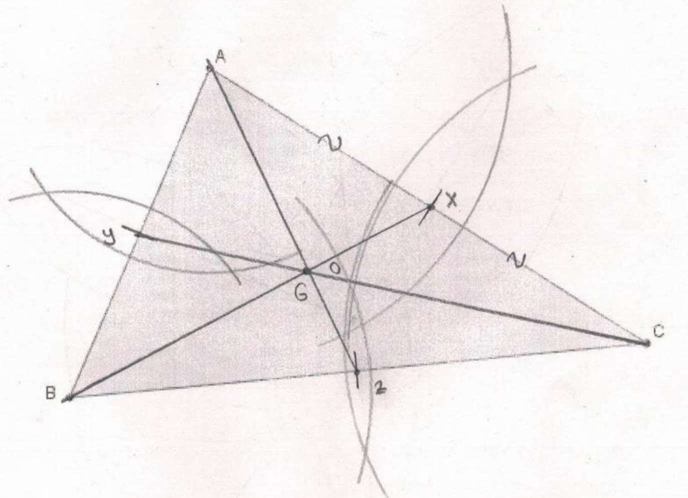
APPENDIX F. APPROACHES OFFERED FOR GEOMETRIC CONSTRUCTION

Activity 1, CSG

The explanation of CSG related to A1 in Activity 1



The explanation of CSG related to A3 in Activity 1



1. adım: |AC| 'nın orta noktasını bulmak için AC uzunluğunun yarısından fazla o.ş. pergeli açtık ve A,C noktalarından iki yay çizdik. Bu yayların kesişimi bize X noktasını verdi.

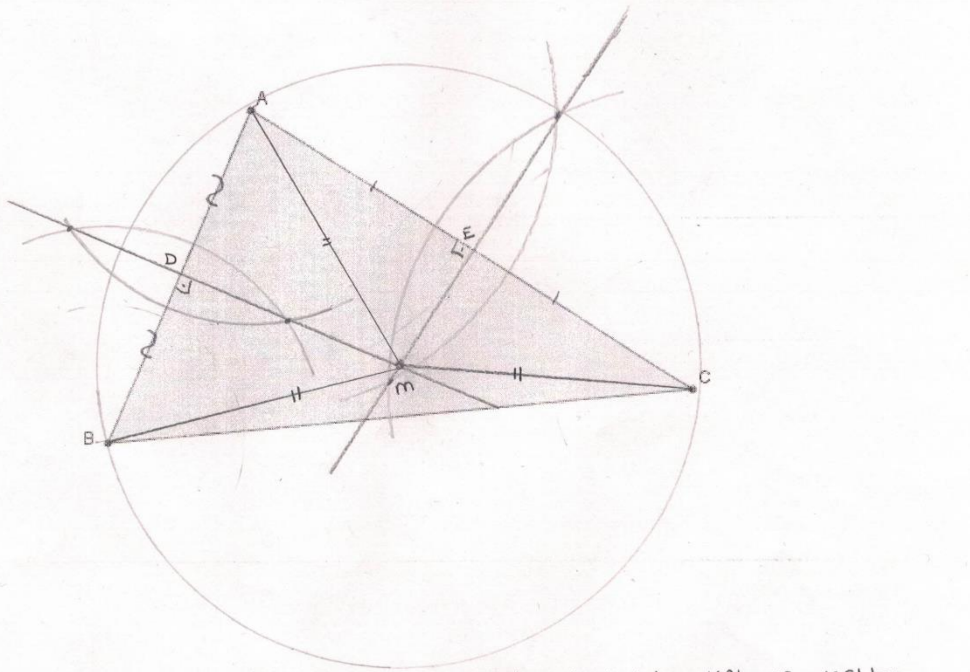
|AX| = |XC|

2. adım: 1. adımı |AB| ve |BC| 'larına da uyguladık ve aynı şekilde Y ve Z noktalarını buld

|AY| = |YB| ve |BZ| = |ZC|

3. adım: |CY|, |AZ| ve |BX| doğru parçaları bize kenarortayları verdi. Bunların kesişim noktası bize çemberin merkezini verir sandık ama bu G noktası bize $\overset{\Delta}{ABC}$ 'nin ağırlık noktasını verdi. Ayrıca $\overset{\Delta}{ABC}$ çeşit kenar üçgen old. için ②. sayfadaki gibi içteğet çember de oluşturamadık.

The explanation of CSG related to A2 in Activity 1

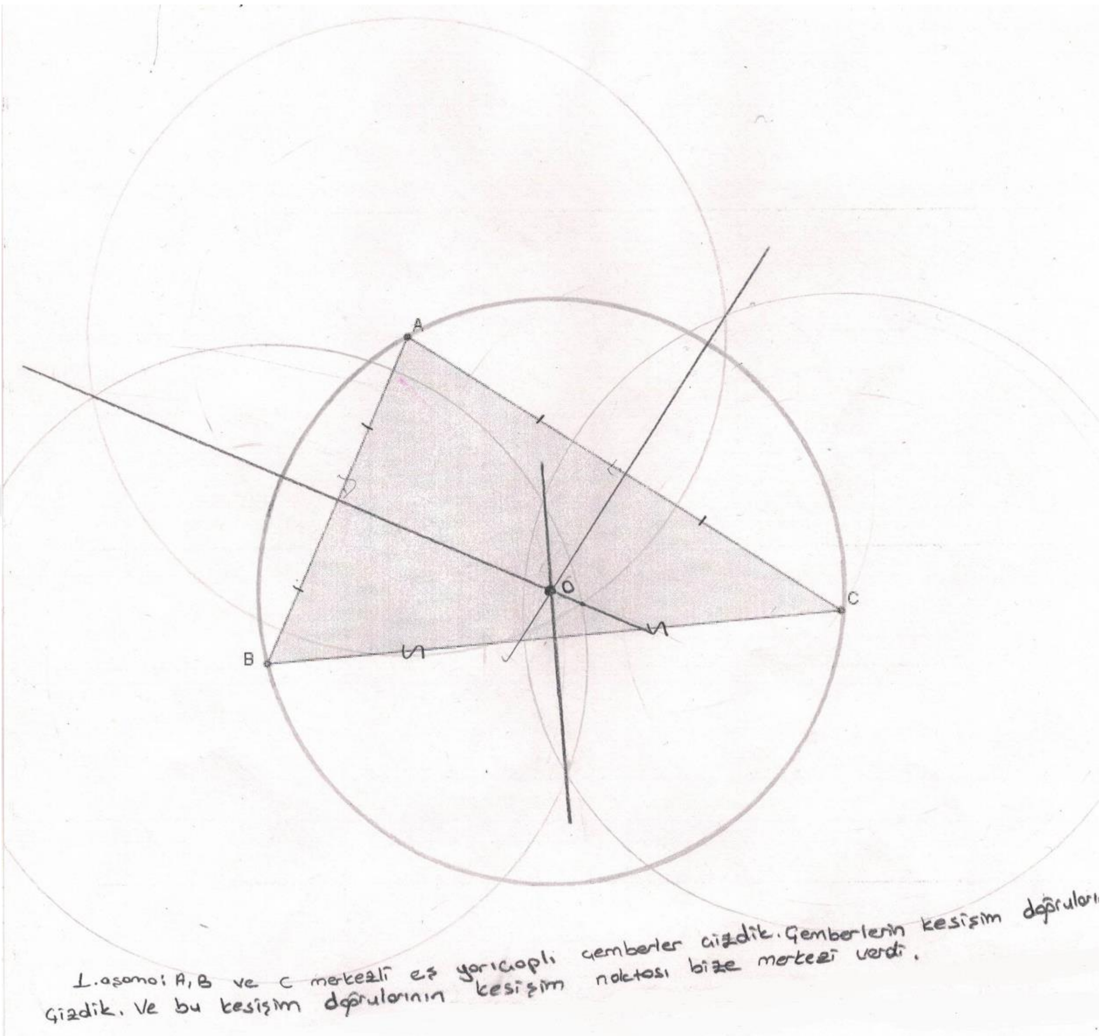


1. Aşama; A, B, C noktalarından geçen çemberi bulmak için |AB| ve |AC|'nu kiriş kabul ettik. Geçen haftalardan kirişlerin kenar orta dikmesinin kesişimlerinin merkezi verdiğini biliyoduk. (şekil 1)

2. Aşama; |AB|'nin kenar orta dikmesini; AB uzunluğunun yarısından fazla a.ç açacak A ve B merkezli yaylar çizerek kesişim noktalarını birleştirerek bulduk. Aynı şekilde |AC|'nin de kenar orta dikmesini bulduk.

3. Aşama; |DM|'nin ve |EM|'in kesiştiği yer çizecek olduğum çemberin merkezi oldu. m nok. köşelere olan uzaklığı birbirine eşit geldi ve yarıçapları oluşturdu.

The explanation of CSG related to A4 in Activity 1



Activity 2, CSG

The explanation of CSG related to A2 in Activity 2 (ΔABC)

1. üçgenin açıklaması:

1.adım: $\overset{\Delta}{EDF}$'nin çevrel çemberini bulmak için |EF|, |FD|, |DE| 'nin orta dikmelerini bulduk.
→ |EF|'nin orta dikmesini, |EF|'nin yarısından daha uzun bir pergel açıklığı alıp, E ve F noktalarından yaylar çizerek ve bu yayların kesişim noktalarını çizerek bulduk.
→ Aynı adımı |FD| ve |DE| doğru parçalarına da uyguladık.
→ Bu orta dikmelerin kesişim noktası bize $\overset{\Delta}{EFD}$'nin çevrel çemberinin merkezini verir. Bu sayede çevrel çemberi oluşturabildik. (Çemberin merkezi = M noktası)

2.adım: Bu orta dikmelerin her birinin paralelini D, E, F noktalarına taşırsak yükseklikleri elde ederiz diye düşündük. Bu yüzden |XA|'nı D noktasına, |YC|'nı F noktasına, |ZB|'nı E noktasına taşımaya karar verdik.
→ Paralel olarak taşımak için;
Örneğin |CY|'nı ele alalım. |CY|'nı taşımak için CY doğrusuna F noktasından dik çizip bu uzunluğu C noktasına koyduk ve H noktasını elde ettik. F'den H'ye çizdiğimiz doğru |CY|'nin paralelini çizdik. |CY| // |FH| olur.
→ Bu dik olarak aldığımız uzunluğu nasıl belirledik?
|FY| uzunluğunu pergelle ölçtük ve bu uzunluğu CY'ye taşıyarak |FK| yı oluşturduk. Yani $\overset{\Delta}{YFK}$ ikizkenar üçgenini elde ettik. İkizkenar üçgende F noktasının orta dikmesi |YK| nin yüksekliği olur. Bu sayede |FJ|'nı elde ettik ve paralellikte kullandık.
→ 2. adımı |XA| ve |ZB|'ye de uyguladık. |XA| için $\overset{\Delta}{XDL}$ ikizkenar üçgenini oluşturduk ve |DI|'nı elde ettik.
|ZB| için $\overset{\Delta}{ENZ}$ ikizkenar üçgenini oluşturduk ve |EP|'nı elde ettik.

3.adım: Orta dikmelerin paralellerini noktalara taşıyarak yükseklikleri elde ettik ve kesişimleri bize "O" noktasını verdi.

The explanation of CSG related to A2 in Activity 2 (ΔKLM)

2. üçgenin açıklaması;

1. adımı 1. üçgen için uyguladığımız 1. adımı bu üçgene de uygulayarak çevrel çemberin merkezini P noktası olarak bulduk ve çevrel çemberi oluşturduk.

2. adım: Orta dikmeli yine paralel bir şekilde K, L, M noktalarına taşımak istedik. 1. üçgendeki adımı aynen uygulayarak $\overset{\Delta}{PMX}$, $\overset{\Delta}{CLZ}$ ve $\overset{\Delta}{PKY}$ ikizkenar üçgenlerini oluşturduk. Bunların ortadikmelerini bularak bir uzunluk belirledik ve |PB|, |PA|, |CP| doğrularını bu uzunluklar sayesinde K, L, M noktalarına taşıdık.

3. adım: Elde ettiğimiz |LI|, |KL|, |ML| yüksekliklerinin kesişimi bize L noktasını verdi. Bu da $\overset{\Delta}{KLM}$'nin dik açılı bir üçgen olduğunu gösterir.

The explanation of CSG related to A2 in Activity 2 (ΔDEF)

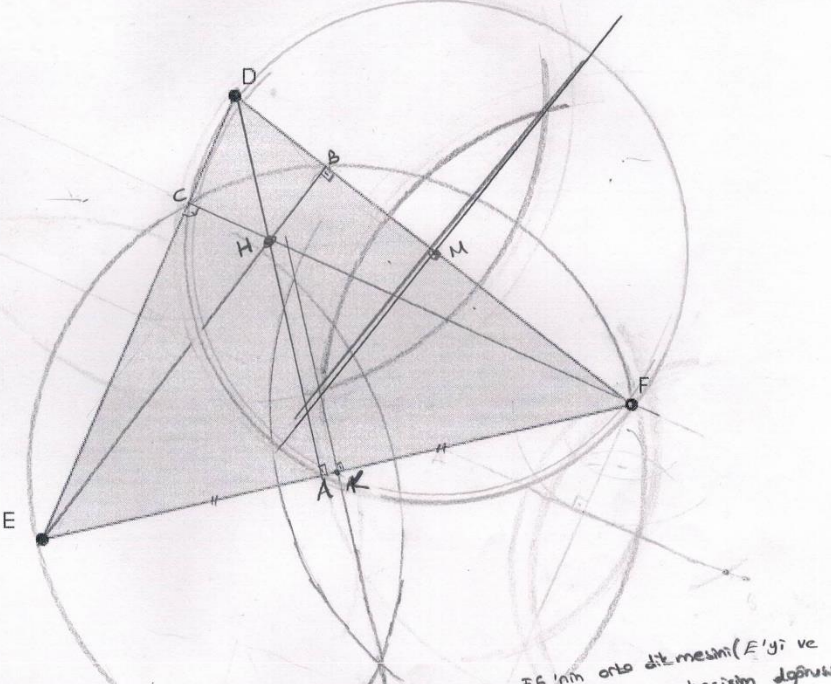
3. üçgenin açıklaması;

1. adım: 1. üçgendeki 1. adımı buraya da uygulayarak çevrel çemberin merkezini O noktası olarak bulduk. $\overset{\Delta}{ABC}$'nin çevrel çemberini oluşturduk.

2. adım: Orta dikmelerin paralelini A, B, C noktalarına taşıyarak yükseklikleri elde ettik. Taşımak için kullandığımız uzunlukları da $\overset{\Delta}{CB'Y}$, $\overset{\Delta}{BA'X}$, $\overset{\Delta}{AC'Z}$ ikizkenar üçgenlerini oluşturarak elde ettik. Orta dikmeleri bu sayede noktalara taşıdık.

3. adım: Elde ettiğimiz |AI|, |CJ|, |BM| yüksekliklerinin kesişimi bize P noktasını verdi. Bu da geniş açılı üçgenlerde yüksekliklerin kesişiminin üçgen dışında olduğunu gösterir.

The explanation of CSG related to A4 in Activity 2 (ΔDEF)



1.aşama; |EF|'yi çap kabuleden çember çizdik; Çemberi çizerken ilk olarak EF'nin orta dikmesini (E'yi ve F'yi merkez kabul edecek şekilde |EF|'nin yarısından fazla olacak yaylar çizdik. Yayların kesişim doğrusu bize orta dikmeyi verdi.) bulduk. EF'nin yarısı merkezi K oldu. K merkezli |KE|=|KF| yarıçaplı çember çizdik. |EB| doğru parçasını çizdik. $E\hat{B}F$ çapı gördüğü için 90°'dir. Ve |EB|, |DF|'ye inilen dikmedir, yüksekliktir.

2.aşama; |FC|'yi birleştirdik. $F\hat{C}E$ çapı gördüğü için 90° dir. Buradan da görülüyor ki F'den |DE|'ye inilen yükseklik |FC|'dir.

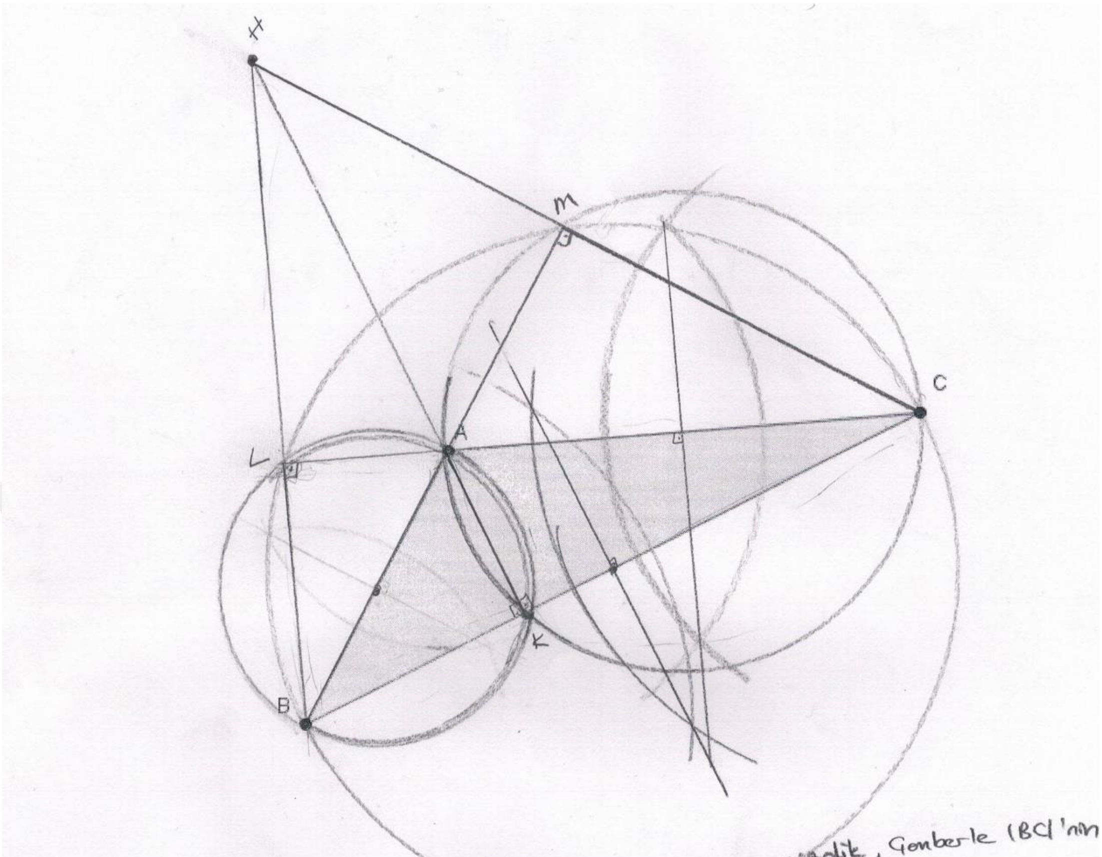
Şimdiye kadar F'den |DE|'ye inilen dikmeyi, E'den |DF|'ye inilen dikmeyi bulduk.

3.aşama; |DF|'yi çap kabul eden çember çizdik; İlk olarak DF'nin orta dikmesini bulduk. |DF| ile orta dikmenin kesişimi M noktasını yani çizeceğim çemberin merkezini verdi. M merkezli |DM|=|MF| yarıçaplı çember çizdik.

4.aşama; |DF| çaplı çemberin |DE| kenarı ile kesişimi C noktasını, EF kenarı ile kesişimi A noktasını verir. C ve F noktalarını birleştirerek |CF|'yi, D ve A noktalarını birleştirerek DA'yı bulduk. Ve $D\hat{C}F$, $D\hat{A}F$ açıları çapı gördüğü için 90° dirler. |CF|'nin yükseklik olduğunu 2.aşamada belirtmiştik. Burada da yine bulmuş olduk. D'den |EF|'ye inilen dikme de |AD| yüksekliğini verdi.

5.aşama; |AD|, |EB|, |CF| yükseklikleri bir noktada kesiştiler. Bu noktayı H ile gösterdik.

The explanation of CSG related to A4 in Activity 2 (ΔABC)



Gemberle |BC|'nin

1.aşama; Yine aynı işlemleri yaparak AC çaplı çember çizdik. ve |AKC çapı gördüğü için 90° dir.

kesişim noktası K'dır. |AK|'yı birleştirdik.

A'dan BC'ye inilen yükseklik AK'dır.

aynı işlemleri uygularsak

2.aşama; |AB| yi çap kabul eden çember çizip, |AK|'nın yükseklik olduğunu bulmuş oluruz.

|AK|⊥|KB| 'yi buluruz. Yani yine

3.aşama; |BC|'yi çap kabul eden çember çizdik yine aynı işlemlerle. |CA|'yı uzatarak C,A,L'den

ve çizdiğimiz çember üçgenimizle kesişmiyor, |BL|⊥|CL| oldu. BLC çapı gördü.

geçen |CL|'yi bulduk. |BL|'yi birleştirdik.

B'den inen yükseklik |BL| oldu.

4.aşama |BA|'yı uzatarak B,A,M'den geçen |BM|'yi çizdik. |CM|'yi birleştirdik.

BMC çapı gördüğü için 90° dir.

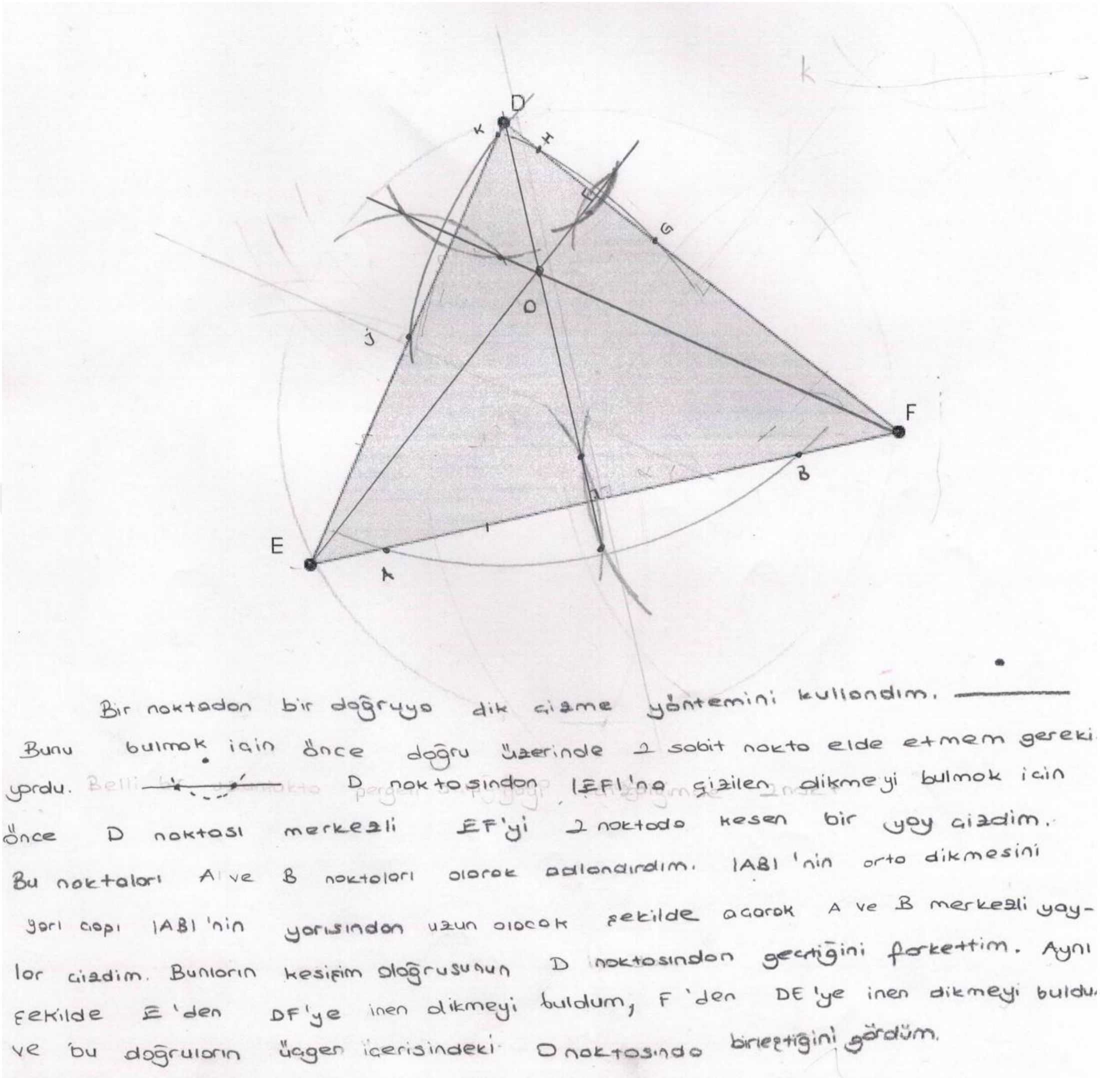
Buradan da C'den inilen dikme |CM|'dir.

5.aşama; |BL|, |CM|, |AK| yüksekliklerini uzatarak kesiştikleri H noktasını bulduk.

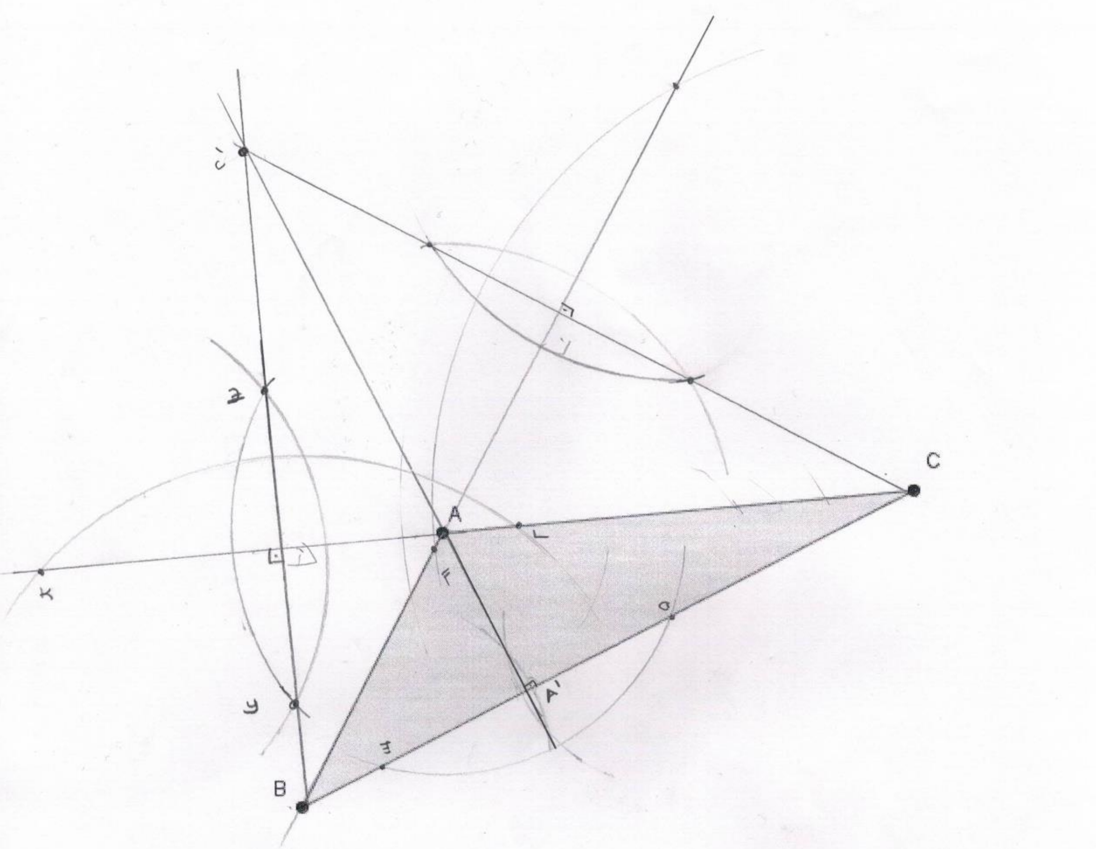
The explanation of CSG related to A4 in Activity 2 (ΔKLM)

1.aşama; |LM|'yi çap kabul eden çember
çizdik, |LM|'nin orta dikmesini bularak çizeceğimiz
çemberin merkezini bulduk(A) ve A merkezli |AL|=|LM| yarıçaplı çember çizdik. Çember ile
|KM|'nun kesişimine B dedik ve |LB|'yi birleştirdik. $L\hat{B}M$ çapı gördüğü için 90° dir.
L'den |KM|'ye inilen dikmeyi |LB| olarak bulduk.
2.aşama; 1.aşamadaki işlemi |KL| kenarı için de uyguladık. |KL| çaplı çember çizdi
Çember B, K, L noktalarından geçiyor. |LB| zaten birleştirilmişti ilk aşamadan. Burada
yine |LB| yüksekliğini bulmuş olduk.
3.aşama; |KM| çaplı çember çizdik; Orta dikmesini bulduk, Orta dikme ile |KM|'nin
kesiştiği nokta merkezi verdi. merkezi bilinen yarıçapı |KN|=|NM| dan olan çember
çizdik, Çember K, L, M noktalarından geçiyor. Buradan |KL| ve |LM| uzunluğunu
bulduk.
4.aşama; |KL|, |LM| ve |LB| nin kesişimi bize yüksekliklerin kesişim
noktasını verdi.
Yani L noktasını bulmuş olduk.

The explanation of CSG related to A5 in Activity 2 (ΔDEF)

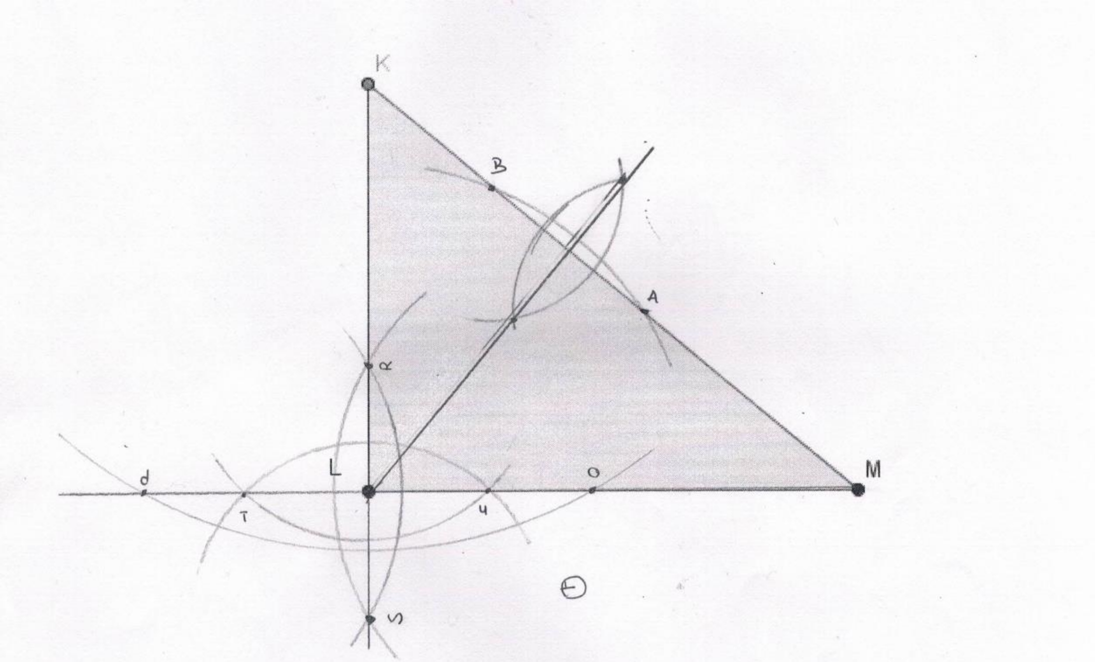


The explanation of CSG related to A5 in Activity 2 (ΔABC)



önceki üçgende kullondığım yöntemi uygulomok üzere B'den AC doğrusuna dik buldum. Bu dikmeyi bulurken |AC| doğru parcası üzerinde 2 nokta elde edemedim. AC doğrusunu uzatarak bu doğru üzerinde K ve L noktaları elde ettim. KL doğru porcasına inen dikmeyi bulmak adına orta dikmeyi buldum ve Y, Z, B noktalarını birleştirdim. Aynı şekilde C'den AB'ye inen dikmeyi bulmok icinde AB doğrusu uzattım. ve CC' doğrusunu elde ettim A'dan BC doğrusuna dik bulmak için BC'den 2 nokta (D ve E) elde ettim. Aynı şekilde DE doğrusunun kenar orta dikmesini buldum. 3 dikmenin uzantısını birleştirdiğimde C' noktasını dikmelerin kesişimi olarak buldum. Dışarda bir nokta olduğunu farkettim.

The explanation of CSG related to A5 in Activity 2 (ΔKLM)



1. Aşama; L'den KM'ye dik çizmek için L merkezli bir yay çizdim ve A, B noktalarını elde ettim. AB noktasının kenar orta dikmesini diğer yerlerde uyguladığım yöntemler ile buldum ve bu dikmenin L noktasından geçtiği gördüm

Aynı şekilde M'de KL'ye dik bulmak için KL doğru parçasını uzattı M merkezli yay çizerek R ve S noktalarını buldum. RS doğru parçasının kenar orta dikmesinin LM doğrusu olduğunu gördüm.

Aynı şekilde K'den LM'ye dik bulmak için LM'yi uzattım K merkezli yay çizerek T ve M nok. buldum. TM doğru parçasının kenar orta dikmesinin KL doğrusu olduğunu gördüm. Bu üçgenin diklik merkezinin L noktası olduğunu

Activity 3, CSG

The explanation of CSG related to A1, A2, A3, A4, and A5 in Activity 3

1. Önce çevrel çember ve diklik merkezini bildiğimizi düşündük. Çevrel
çemberi merkezini AB ve BC'yi kiriş kabul ettik. Bu doğru parçala-
rının orta noktasından geçen k ve ℓ doğrularının kesiştiği nokta bize
çevrel çemberin merkezini verdi (1. kağıt) - D noktası -

Diklik merkezini kenarorta dikmelerinin kesişim noktası olduğunu
düşündük ve (Kağıt 2)deki inşayı yaptık. Bu kağıttaki D noktası bize
kenarorta dikmelerinin merkezini verdi. Ama kenarorta dikmelerin
merkezi diklik merkezi mi? bunu yazarken aramızda fikir birliği
olmadı ve diklik merkezinin farklı bir şey olacağını düşündük. Diklik
merkezi köşelerden karşı kenarlara inilen diklerin (yüksekliklerin) kesişi-
mi olacaktı. Kağıt 4'de köşelerden kenarlara dik inşa ettik ve bun-
ların kesişim noktası m çıktı. Yani diklik merkezini m olarak bul-
duk. (Kağıt 4)

Ağırlık merkezinin ne olduğunu önce bilemedik. Sonra açıortayların
kesişim noktası olabileceğini düşündük ve 3. kağıttaki inşayı yaptık O nok-
tasını ağırlık merkezi gibi kabul ettik ama burada 2k'ya k oranını
bulamadık. Ağırlık merkezi açıortayların kesişimlerinde farklı bir şey olmalıy-
dı. Sonrasında kenarortaylarının kesişimleri olabileceğin düşündük
ve kağıt 3'de kenarortayların kesişim noktasını inşa ettik. G noktası
bize ağırlık merkezi verdi. Ve bu noktanın ağırlık merkezi olduğuna
emindik benzerlikten 2k'ya k oranını görmüş olduk.

En sonda bu 3 noktanın konumlarını kağıtta görebilmek için 4. kağıt
ta hepsini inşa ettik. m noktası diklik merkezi, D çevrel çemberin merkezi
ve G noktası ağırlık merkezi olmak üzere bu 3'ünün doğrusal olduğunu
keşfettik. Dar açılı üçgende durum böyleydi. Fakat bu üçgen eğer
h doğrusu üzerinde bulunmakta (kağıt 4)

eşkenar olsaydı, bu noktalar tek bir nokta olacaktı. Kağıt 5'in arka tarafın-
da şekil 1'de bunu gösterdik (a noktası). Geniş açılı üçgen olsaydı yine nokta-
lar doğrusal olacaktı. Kağıt 5'de şekil 2'de bunu gösterdik. (ℓ doğrusu)
Dik açılı üçgen olsaydı kağıt 5'de şekil 3'de yine doğrusal olacaktır.
İkizkenar üçgen bu üçgenin özelliğine göre değişebilir.

Activity 4, CSG

The explanation of CSG related to A1 in Activity 4

Bağlantı; AXZ, BYX ve CZY noktalarından geçen çemberlerin merkez noktaları ile oluşan üçgen ABC üçgeni ile benzerdir.

1. aşama; AB doğrusu üzerinde rastgele bir X, BC doğrusu üzerinde rastgele bir Y, AC doğrusu üzerinde rastgele bir Z noktası aldık. (6. kağıtta)

2. aşama; A, X, Z noktalarından geçen çemberi bulurken;
AX doğru parçasının orta dikmesini bulduk. (Orta dikmeyi bulurken A ve X merkezli eşit yarıçaplı yaylar çizerek bulduk.)
AZ doğru parçasının da orta dikmesini aynı işlemlerle bulduk.
AX ve AZ doğru parçalarının orta dikmelerinin kesişim noktası A, X, Z noktalarından geçen çemberin merkezini verir.

3. aşama; 2. aşamadaki işlemleri tekrarlayarak BYX noktalarından ve CZY noktalarından geçen çemberleri bulduk.

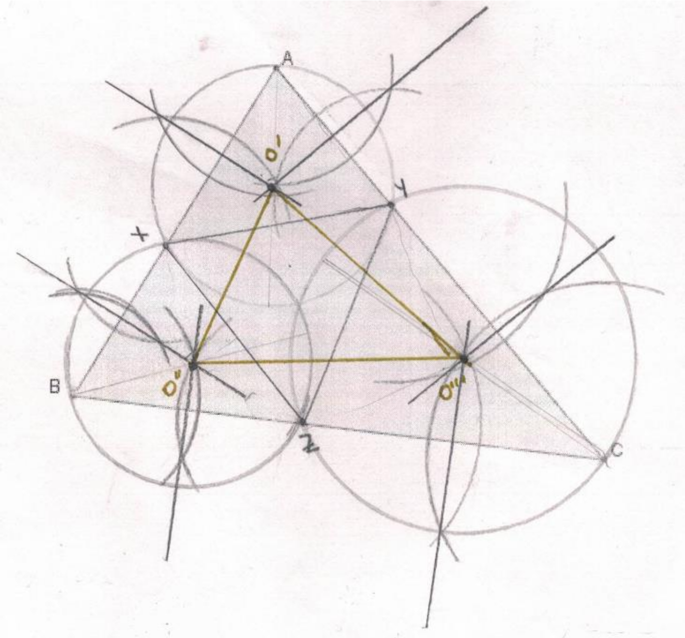
4. aşama; AXZ, BYX ve CZY noktalarından geçen çemberlerin merkez noktalarını birleştirdik.

5. aşama; Elde ettiğimiz üçgen ABC üçgeni ile benzer çıktı.

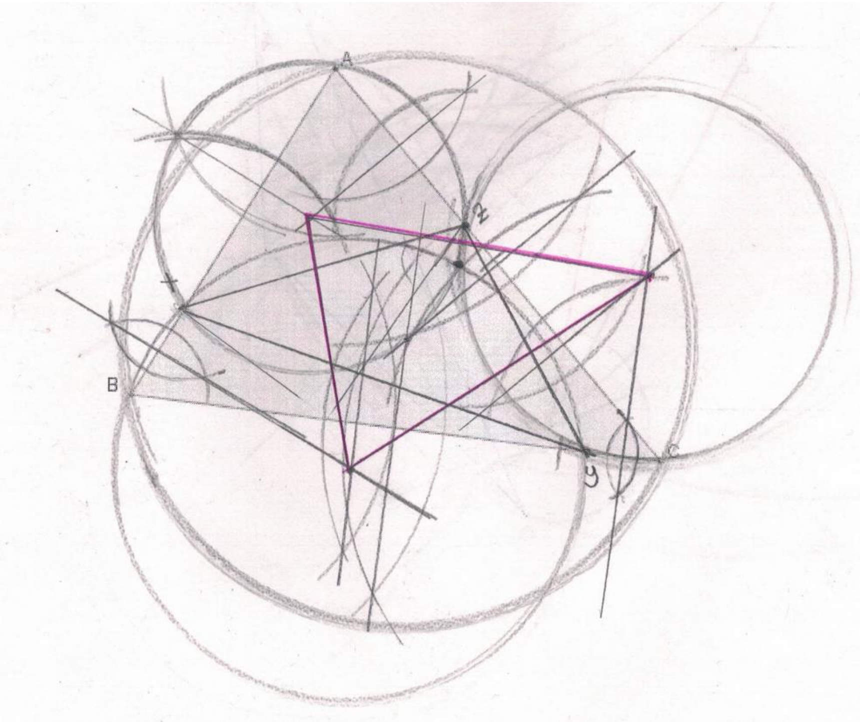
6. aşama; Bunun çalışıp çalışmadığını kontrol etmek için farklı durumlar deriedik. Kağıt 6'da dik üçgeni sabit tutup noktaların yerlerini değiştirdik. Ve yine aynı sonucu elde ettik.

Diğer tüm kağıtta da bulduğumuz bağlantı uyuyor mu, uymuyor mu diye renkli kalemlerle çizip inceledik.

The second geometric figure formed via A1 in Activity 4 by considering differently located X, Y, and Z points

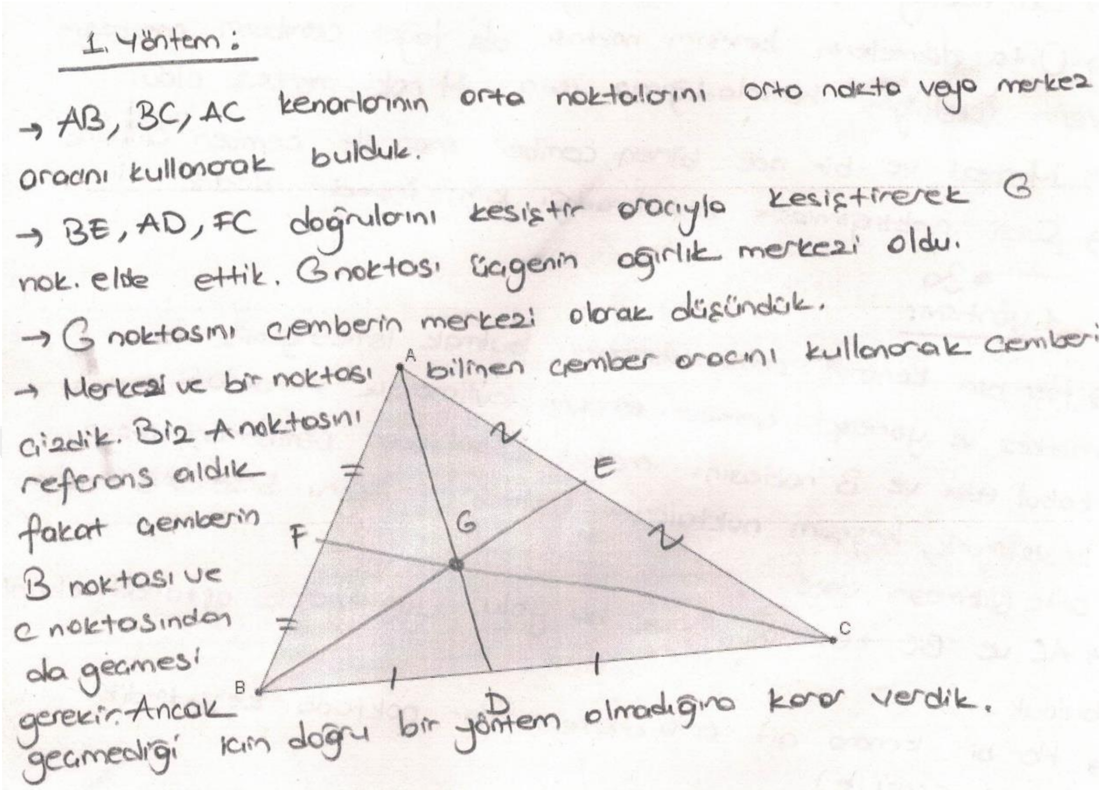


The third geometric figure formed via A1 in Activity 4 by considering differently located X, Y, and Z points

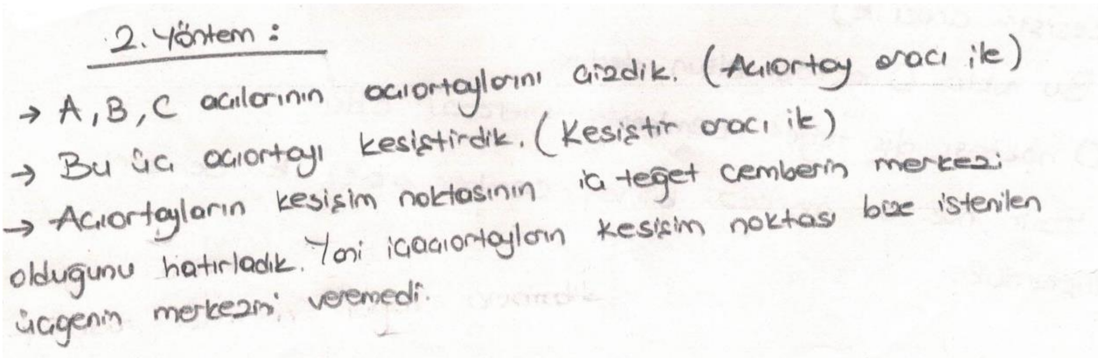


Activity 1, GG

The explanation of GG related to A1 in Activity 1



The explanation of GG related to A3 in Activity 1



The explanation of GG related to A6 in Activity 1

3. Yöntem:

→ AB, BC, AC kenarlarının orta dikmelerini bulduk. (Orta dikme aracı ile

→ Bu orta dikmelerin kesişim nok. bulduk. (Kesiştir aracı ile)

→ Bu noktaya H noktası dedik.

→ Orta dikmelerin kesişim noktası dış teğet çemberin merkezini verir özelliğini hatırladığımız için H nok merkez oldu.

→ Merkez! ve bir nok bilinen çember aracı ile çemberi çizdik

→ Şekli çektiğimizde bozulmadığı için inşadır dedik.

The explanation of GG related to A8 in Activity 1

1. yöntem:

bulmak istediğimiz için

→ Her bir kenarın orta dikmesini

çember aracını kullanarak A noktasını merkez

merkez ve yarıçapla

kabul eden ve B noktasını merkez kabul eden çemberleri çizip, kesiştirdik, kesişim noktalarını birleştiren doğru bize AB kenarının orta dikmesini verdi.

→ AC ve BC kenarları içinde bu yolu uygulayarak orta dikmelerin bulduk.

→ Her bir kenara ait orta dikmeleri bir noktada kesiştirdik. (Kesiştir aracı ile)

→ Bu nokta O noktası olsun dedik.

→ O noktası dış teğet çemberin merkezi oldu.

→ Bir nok. ve merkezi bilinen çember aracı ile çemberi oluşturduk.

Activity 2, GG

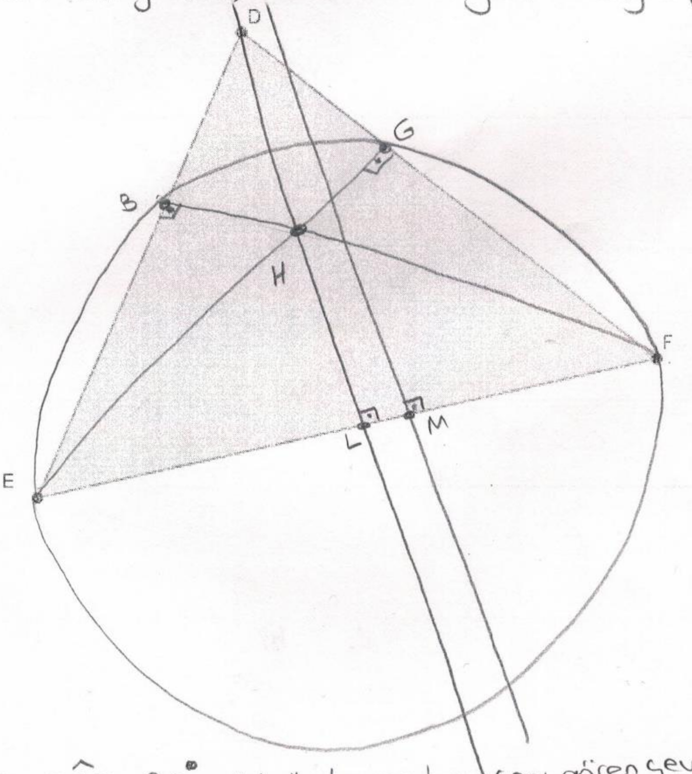
The explanation of GG related to A1a in Activity 2

→ Orta dikme aracını kullanarak EF'nin orta noktasını bulduk (M noktası)

→ EF çap olacak şek. merkez ve bir noktadan geçen çember aracını kullanarak M merkezli bir çember oluşturduk.

→ Çember ve üçgenin kesişim noktalarını B ve G olarak belirle-dik.

→ B ve F'den geçen, G ve E'den geçen doğru parçalarını oluştur-duk.



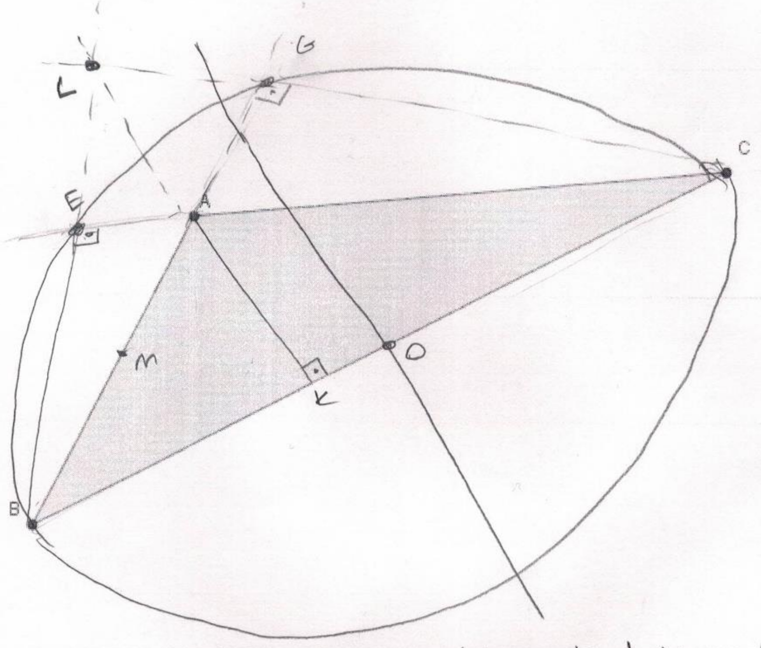
→ $E\hat{B}F = E\hat{G}F = 90^\circ$ çünkü bu açılar çapı gören çevre açıdır.

→ Açı aracını kullanarak bu açıların 90° olduğunu kontrol ettik ve |BF| ve |EG|'yi kesiştirdik. (H noktası)

→ D ve H'den geçen doğru'yu doğru aracını kullanarak çizdik ve bu doğru üçgeni L noktasında kesti. $D\hat{L}F = 90^\circ$ oldu. (Açı aracını kullanara

The explanation of GG related to A1b in Activity 2

→ BC'yi çap kabul eden çember çizdik.
→ BC'nin orta noktasını orta dikme aracıyla bulduk (O noktası)
→ A ve C noktalarından geçen, B ve G noktalarından geçen doğruları çizdik. Bunlar çemberi E ve G noktalarında kesti.
→ $\widehat{BEC} = \widehat{BEG} = 90°$ (Açı aracını kullanarak)
→ BA'nın orta noktasını orta dikme aracını kullanarak belirledik (m noktası) ve merkez ve bir noktadan geçen çember aracını kullanarak M merkezli çember oluşturduk.



→ Bu çemberin BC'yi kestiği noktaya K dedik ve |AK|'yı oluşturduk. Açı aracını kullanarak $\widehat{AKC} = 90°$ olduğunu belirledik. (Çapı gören çevre açı)
→ C ve G'den geçen doğru, B ve E'den geçen ve K ve A'dan geçen doğruları çizdik bunları kesiştirdik (L noktası) Diklik merkezimiz oldu.
→ Taşı aracını kullanarak şekli sürüklediğimizde diklik merkezi bozulmadığı için inşadır.

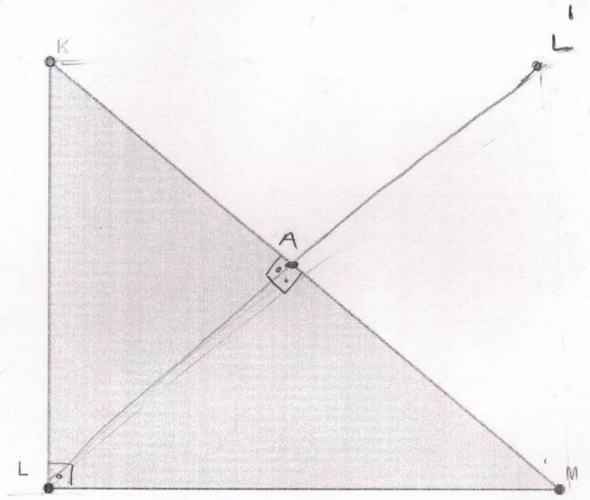
The explanation of GG related to A2a in Activity 2

→ L noktasını doğruda yansıt aracını kullanarak |KM|'ye göre yansımasını bulduk (L' noktası)

→ L ve L' den geçen doğruyu çizdik. Bu doğru üçgeni A noktasında kesti ve $L\hat{A}M = 90°$ oldu. (Açı aracıyla)

→ Verilen üçgen dik olduğu için $K\hat{L}M = 90°$'ydi.

→ K'dan LM'ye çizilen ve M'den KL'ye çizilen doğrular L noktasında dik kesişir. KM'ye çizilen yükseklikte L noktasından geçer o yüzden L diklik merkezi olur.

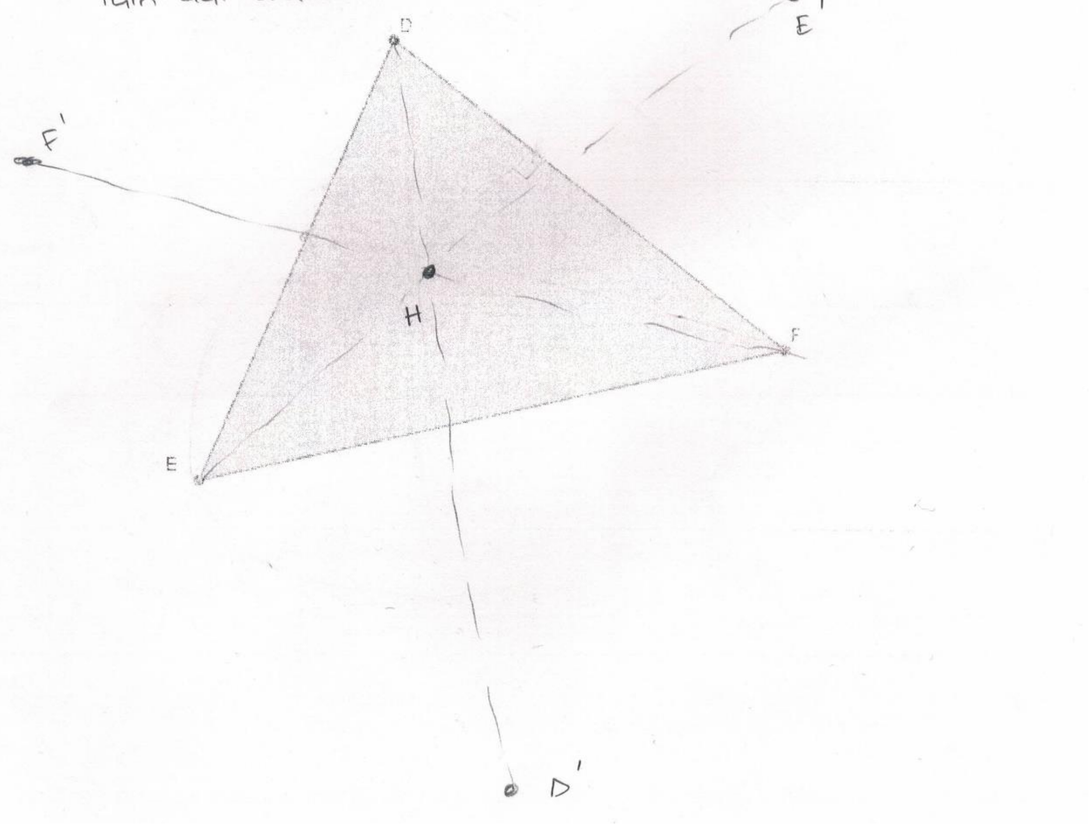


→ Taşı aracıyla şekli sürüklediğimizde diklik merkezi değişmedi şekil inşadır.

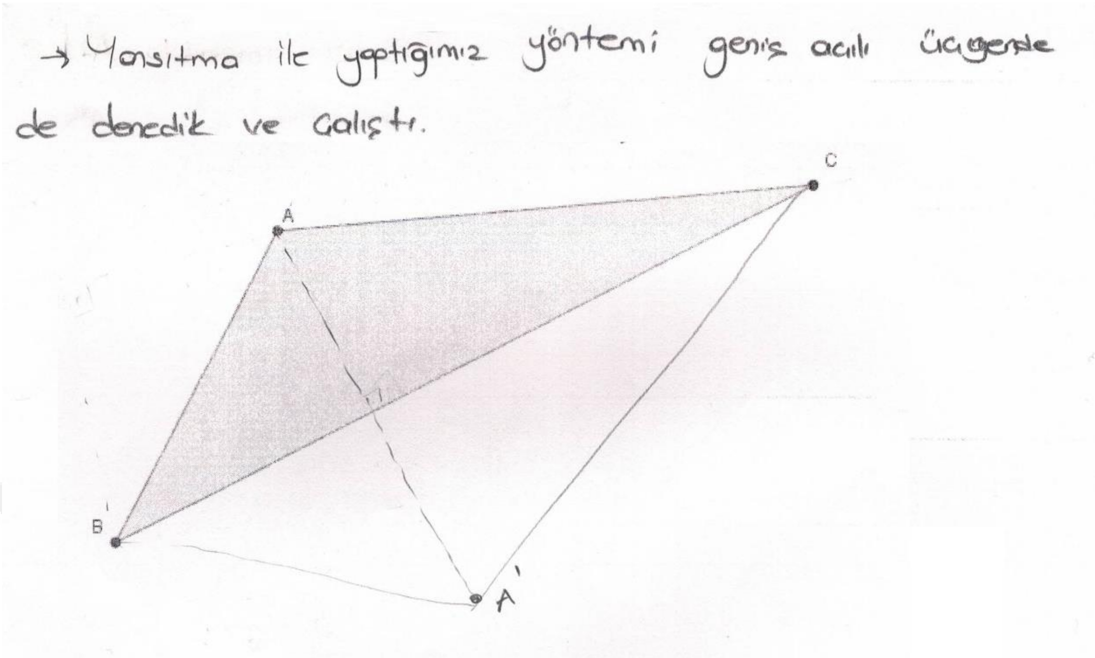
The explanation of GG related to A2b in Activity 2

→ D noktasının EF'ye göre, E noktasının DF'ye, F noktasının DE'ye göre yansımasını aldığımızda (doğruda yansıt aracını kullanarak) E', F' ve D' noktalarını elde ederiz.

→ DD', EE' ve FF' doğrularını kesiştirdiğimizde H noktasını elde ettik. Bu doğruların kenarlara dikliğini kontrol etmek için açı aracını kullandık. H noktası diklik merkezimiz oldu



The explanation of GG related to A2c in Activity 2



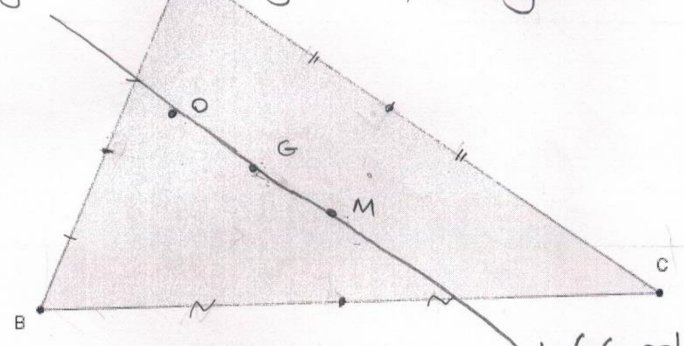
Activity 3, GG

The explanation of GG related to A1, A2, and A3 in Activity 3

→ Her bir kenara dik doğru aracıyla dikler indik. Bu üç doğrunun kesişim noktası (O noktası) diklik merkezini verdi.

→ Orta dikme aracını kullanarak her bir kenarın ortadikmesini bulduk, bunların kesişim noktası çevrel çemberin merkezini verdi. (m noktası)

→ orta nokta veya merkez aracını kullanarak her kenarın orta noktasını bulduk. Bu noktalarla üçgenin kenarlarını doğru parçasıyla birleştirdiğimizde, bu doğru parçalarının kesişim



noktası bize ağırlık merkezini verdi (G noktası)

→ Bu üç nokta arasındaki ilişkiyi bulmak için doğru yardımıyla birleştirdiğimizde doğrusal olduğunu gördük.

→ Taşı aracıyla A noktasını sürüklediğimizde doğrusallık değişmedi.

→ A noktasını taşıdığımızda farklı üçgenlerde de doğrusallık değişmediği için her üçgen için uygulanabilir.

→ A noktasını sürükleyerek geniş açılı üçgen elde ettiğimizde O ve M nok. dışarı çıktığını fakat doğrusallığın değişmediğini gördük.

→ Düzgün çokgen aracı ile eşkenar üçgen çizdik. Yukarıdaki yöntemlerle dik. mer., çevrel çemberin mer., ağ. mer. çizdik.

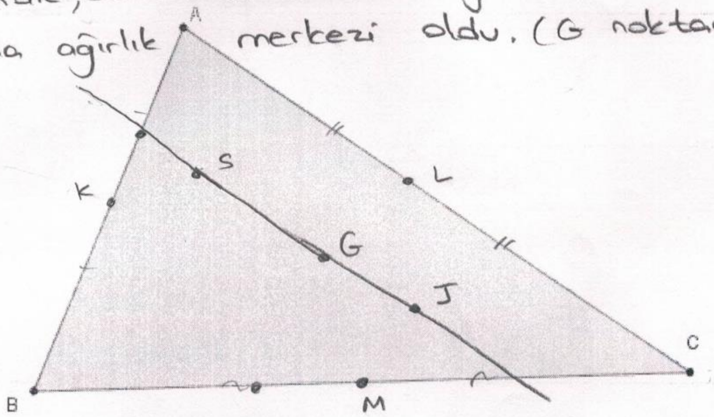
→ Bu üç noktanın aynı yerde oluştuğunu gördük.

→ A noktasını sürükleyerek dik üçgene benzettik. Burada gördük ki yine üç nokta doğrusal oldu.

The explanation of GG related to A4, A5, and A6 in Activity 3

→ Merkezi A ve B olan 2 eş çember çizdik. Bu çemberlerin kesişiminden geçen doğru |AB|'nin orta dikmesi olur. |AC| ve |BC| içinde aynısını uyguladık ve bu üç orta dikmeyi kesiştirdiğimizde çevrel çemberin merkezini verdi. (J noktası)

→ K ile C'yi, A ile M'yi, B ile L'yi birleştirdik, bu üçünü kesiştirdik, bu nokta kenarortayların kesişim noktası aynı zamanda ağırlık merkezi oldu. (G noktası)



→ K merkezli AK yarıçaplı, M merkezli BM yarıçaplı, L merkezli CL yarıçaplı çemberler çizdik. K merkezli çember ile M merkezli çemberin kesişiminden geçen doğru |AC|'ye dik dur, M merkezli çember ile L merkezli çemberi kesiştirirsek |AB|'ye dik olur, L merkezli ile K merkezliyi kesiştirirsek de |BC|'ye dik dur. Bu üç doğrunun kesişimi bize diklik merkezini verir (S noktası)

→ S, G ve J noktaları doğrusal oldu.

→ Şekli sürüklediğimizde değişmediği için inşa oldu.

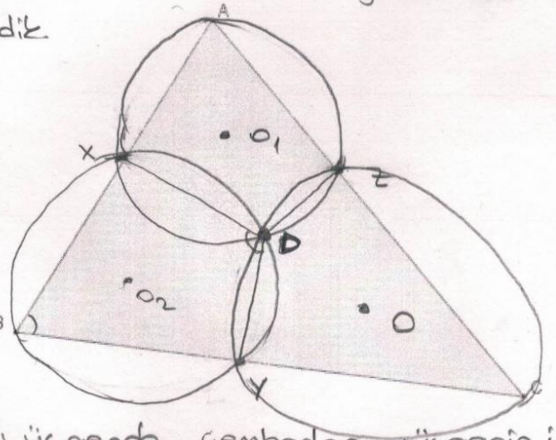
Activity 4, GG

The explanation of GG related to A6 in Activity 4

→ Bir üçgende kenar orta dikmelerin kesişimi bize çevrel çemberin merkezini verir. O yüzden $\overset{\triangle}{AXZ}$'inde kenar orta dikmelerin kesişimini bulduk, bu bize AXZ noktalarından geçen çemberin merkezini verdi.

→ Merkez ve bir noktadan geçen çember aracıyla AXZ noktalarından geçen çemberi oluşturduk.

→ BYX ve CZY noktalarından geçen çemberleri de aynı şekilde çizdik.



→ Dar açılı üçgende çemberler üçgenin içinde bir noktada, dik açılı üçgende hipotenüs üzerinde bir noktada, geniş açılı üçgende ise üçgenin dışında bir noktada kesişti

→ sonuç olarak bu üç çember bütün üçgenlerde 1 noktada kesişir. Taşı aracıyla şekli taşıdığımızda inşa bozulmadı.

→ BXDY, CYDZ, AXDZ kirişler dörtgenleri oluşturduk bu dörtgenlerin birer köşesi de D noktasında kesişti

→ OO_1O_2 noktalarından geçen $A\overline{BC}$'inin içerisinde bir üçgen oluşturduk. Uzunluk veya uzaklık aracıyla AB ve O_1O_2 uzunluğunu ölçtüğümüzde bir oran elde ettik aynı şekilde O_1O ile AC'yi, OO_2 ile BC'yi ölçtüğümüzde 2 katına yakın oran elde ettik.

→ Dik üçgen, geniş açılı üçgen ve dar açılı üçgende yine aynı oranı elde ettik (Taşı aracıyla bu üçgenleri oluşturduk.

APPENDIX G. PROOF OF THE STATEMENTS IN THE COGNITIVE UNITY BASED ACTIVITIES

Proof of the statement in Activity 1

Statement asked to prove in Activity 1

The perpendicular bisectors of the sides of a triangle are concurrent and this point is the circumcenter of the triangle.

The following proof was prepared as an accumulation of the proofs of this statement presented in some textbooks (e.g., Alexander & Koeberlein, 2011, p.332; Gutenmacher & Vasilyev, 2004, p.35; Leonard et al., 2014, p.42-43; Serra, 2003, p.180).

Proof of the statement in Activity 1

Suppose that we have ΔABC which is an acute triangle. Let $\overleftrightarrow{OD}$ and $\overleftrightarrow{OE}$ be the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ respectively and these lines are concurrent at the point O. Moreover, let the midpoints of $\overline{AB}$ and $\overline{BC}$ be the points D and E respectively. Draw the line segments from the point O to the vertices A, B, and C as indicated in Figure 6.1. Now, the purposes with respect to proving the statement are to present that the perpendicular bisector of $\overline{CA}$ also passes through the point O and $|AO| = |BO| = |CO|$.

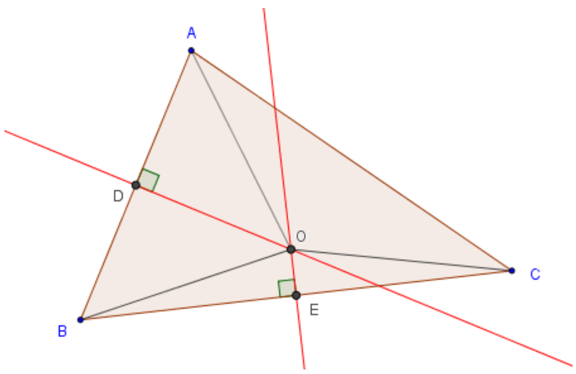


Figure 6.1. ΔABC in the proof of the statement in Activity 1

Since $\overleftrightarrow{OD}$ is the perpendicular bisector of $\overline{AB}$, we have $\angle ODA = \angle ODB$=90° and $|AD| = |DB|$. By considering $\overline{OD}$ as a common side, it can be stated that $\Delta ADO \cong \Delta BDO$ based on Side-Angle-Side congruence (SAS congruence). Thus, we have $|AO| = |BO|$.

In the same manner, since $\overleftrightarrow{OE}$ is the perpendicular bisector of $\overline{BC}$, we have $\angle OEB = \angle OEC$=90° and $|BE| = |EC|$. By considering $\overline{OE}$ as a common side, it can be stated that $\Delta BEO \cong \Delta CEO$ based on Side-Angle-Side congruence (SAS congruence). Thus, we have $|BO| = |CO|$.

By the transitive property, it follows that $|AO| = |CO|$. Based on the theorem that “if a point is equidistant from the endpoints of a line segment, then it lies on the perpendicular bisector of the line segment” (Alexander & Koeberlein, 2011, p.326), it can be inferred that the point O must be on the perpendicular bisector of $\overline{CA}$.

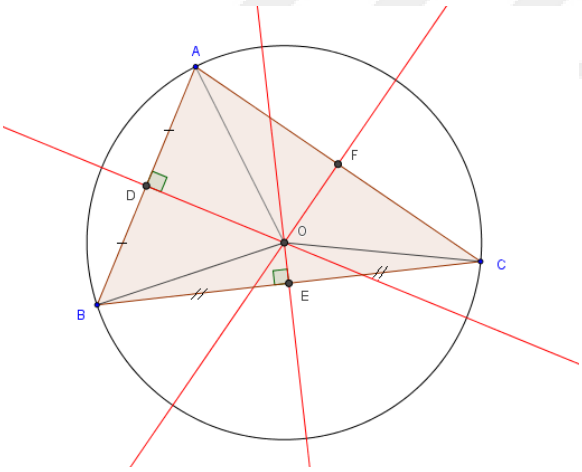


Figure 6.2. The circumcircle of ΔABC in the proof of the statement in Activity 1

Thus, the perpendicular bisectors of the sides of ΔABC are concurrent and the point of concurrency of them is the point O, as illustrated in Figure 6.2. Since it was also found that $|AO| = |BO| = |CO|$, the point O is the circumcenter of ΔABC. ■

Proof of the statement in Activity 2

Statement asked to prove in Activity 2

The altitudes of a triangle are concurrent.

The first proof was presented below by combining the related proofs given in some sources (Aarts, 2008, p.30; Alexander & Koeberlein, 2011, p.333-334; Bottema, 2008; p.13; Gutenmacher & Vasilyev, 2004, p.36; Hajja & Martini, 2013, p.5; Leonard et al., 2014, p.44).

Proof 1 of the statement in Activity 2

Draw three auxiliary lines which are through the vertex A and parallel to $\overline{BC}$, through the vertex B and parallel to $\overline{CA}$, and through the vertex C and parallel to $\overline{AB}$, as illustrated in Figure 6.3. Thus, a larger triangle, which was denoted as ΔDEF, was formed and the given ΔABC was embedded in it.

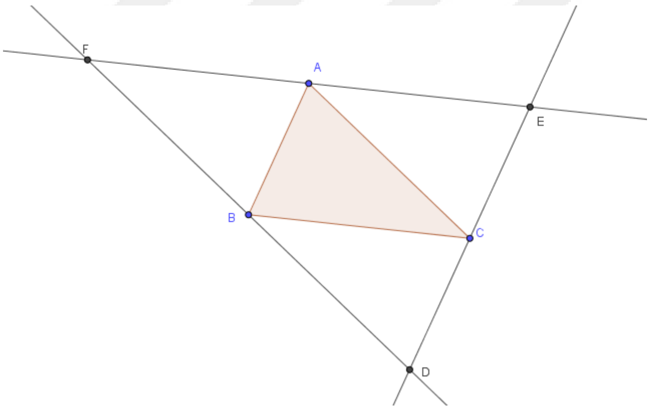


Figure 6.3. ΔABC and ΔDEF in the proof 1 of the statement in Activity 2

Since we know that $\overline{BC} \parallel \overline{AE}$ and $\overline{AB} \parallel \overline{EC}$, it was seen that ABCE is a parallelogram. Since the opposite sides of a parallelogram are congruent, it was stated that $|BC| = |AE|$ and $|AB| = |EC|$. In the same manner, due to the following properties $\overline{BC} \parallel \overline{FA}$ and $\overline{AC} \parallel \overline{FB}$, it was seen that AFBC is a parallelogram. Since the opposite sides of a parallelogram are congruent, it was seen that $|BC| = |FA|$ and $|AC| = |FB|$. By the transitive property, it was stated that $|AE| = |FA|$. That equity presents that the point A is midpoint of $\overline{EF}$.

By following a similar argument for another pair of parallelograms which are AFBC and ABDC, it can be reached that the point B is midpoint of $\overline{FD}$. Moreover, by comparing another pair of parallelograms which are ABDC and ABCE, it can be written that the point C is midpoint of $\overline{DE}$.

Based on the mentioned midpoints of the sides of ΔDEF, the perpendicular bisectors of the sides of ΔDEF were also drawn, as can be seen via the red lines in Figure 6.4.

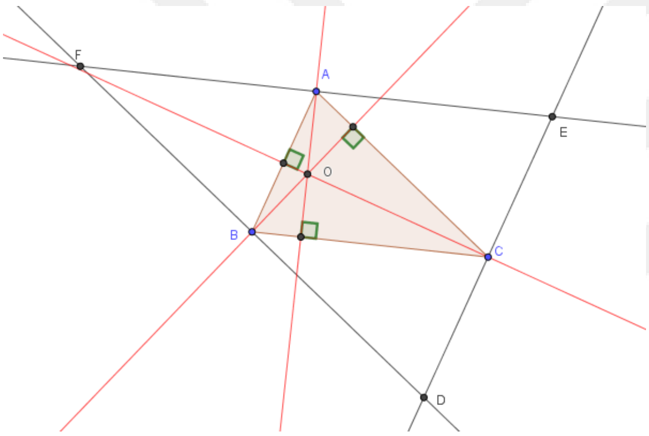


Figure 6.4. The perpendicular bisectors of the sides of ΔDEF in the proof 1 of the statement in Activity 2

Hereby, $\overleftrightarrow{AO}$ is the perpendicular bisector of $\overline{EF}$, $\overleftrightarrow{BO}$ is the perpendicular bisector of $\overline{FD}$, and $\overleftrightarrow{CO}$ is the perpendicular bisector of $\overline{DE}$. Moreover, the perpendicular bisectors of the sides of ΔDEF are concurrent at a point, which was denoted as O. Then, since $\overline{BC} \parallel \overline{EF}$ and $\overleftrightarrow{AO} \perp \overline{EF}$, we have that $\overleftrightarrow{AO} \perp \overline{BC}$. In a similar way, $\overline{CA} \parallel \overline{FD}$ and $\overleftrightarrow{BO} \perp \overline{FD}$, we have that $\overleftrightarrow{BO} \perp \overline{CA}$. Finally, $\overline{AB} \parallel \overline{DE}$ and $\overleftrightarrow{CO} \perp \overline{DE}$, we have that $\overleftrightarrow{CO} \perp \overline{AB}$. Thus, it can be stated that $\overleftrightarrow{AO}$, $\overleftrightarrow{BO}$, and $\overleftrightarrow{CO}$ turned out to be the altitudes of ΔABC. Since the perpendicular bisectors of the sides of ΔDEF are concurrent at the point O, so as the altitudes of ΔABC. ■

Another type of proof for the concurrence of the altitudes of a triangle was seen in the textbooks of both Altshiller-Court (1952, p.94) and Hajja and Martini (2013, p.6-7). Based on them, the second proof of the statement was illustrated and explained below.

Proof 2 of the statement in Activity 2

Assume that ΔABC is an acute triangle, $\overleftrightarrow{BE}$ is the altitude of $\overline{CA}$, $\overleftrightarrow{CF}$ is the altitude of $\overline{AB}$, and these two altitudes intersect at the point H. Moreover, let $\overleftrightarrow{AH}$ meet $\overline{BC}$ at the point D. Now, the aim is to show $\overleftrightarrow{AH} \perp \overline{BC}$ so that it can be stated that the altitude of $\overline{BC}$ also passes through the point H as well as the altitudes of $\overline{AB}$ and $\overline{CA}$.

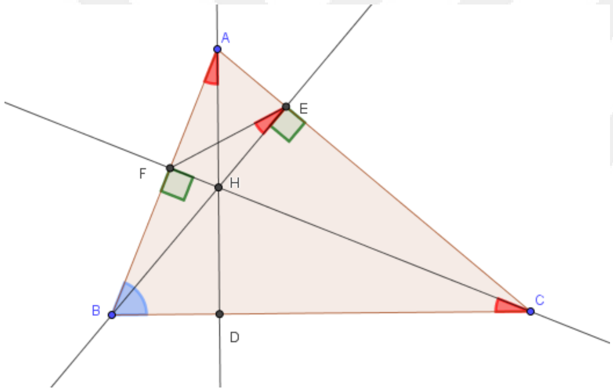


Figure 6.5. ΔABC in the proof 2 of the statement in Activity 2

As presented in Figure 6.5, a circle with the diameter $\overline{BC}$ and passing from the points E and F can be drawn. Thus, a cyclic quadrilateral BCEF was formed. Since the inscribed angles intercept the same arc are congruent, it can be stated that $\angle FCB = \angle FEB$ which was colored as red.

In the similar vein, another cyclic quadrilateral AFHE was noticed since a circle with the diameter $\overline{AH}$ could be drawn. Then, it was reached that $\angle FAH = \angle FEH$, as displayed with red in Figure 6.5. As a combination of all equalities, it can be stated that $\angle BAD = \angle FAH = \angle FEH = \angle FEB = \angle FCB$.

Moreover, we have that $\angle FCB + \angle CBF$ =90º based on the interior angles of ΔBCF . In this respect, it was found that $\angle FCB + \angle CBF = \angle BAD + \angle DBA$ =90º in ΔABD . So the third interior angle of ΔABD which is $\angle ADB$ =90º. That is, the line passing though the points A, H, and D is the altitude of $\overline{BC}$. Thus, the altitudes of ΔABC are concurrent at the point H. ■

The third proof for the statement asked to prove in Activity 2 was directly taken from the textbook of Gutenmacher and Vasilyev (2004, p.36-37).

Proof 3 of the statement in Activity 2

We know that the set

$$\{M : |MA|^2 - |MB|^2 = d\}$$

is a straight line perpendicular to AB. Choose d such that this straight line contains the vertex C. To do this, we must take $d = |CA|^2 - |CB|^2$. Thus, the straight line

$$h_c = \{M : |MA|^2 - |MB|^2 = |CA|^2 - |CB|^2\}$$

contains the altitude of the triangle dropped from the vertex C.
One can consider the straight lines containing two other altitudes of the triangle in a similar way.

$$h_a = \{M : |MB|^2 - |MC|^2 = |AB|^2 - |AC|^2\},$$

$$h_b = \{M : |MC|^2 - |MA|^2 = |BC|^2 - |BA|^2\}.$$

Suppose the first two lines h_c and h_a intersect at the point H. Then when M coincides with this point, both of the following equations hold:

$$|HA|^2 - |HB|^2 = |CA|^2 - |CB|^2,$$

$$|HB|^2 - |HC|^2 = |AB|^2 - |CA|^2.$$

Adding these two equalities, we obtain

$$|HA|^2 - |HB|^2 = |AB|^2 - |CB|^2.$$

Hence, the point H also belongs to the third straight line h_b . ■

Proof of the statement in Activity 3

Statement asked to prove in Activity 3

The circumcenter, the orthocenter, and the centroid of a triangle are collinear.

The proof of this statement was presented based on the review of the related proofs involved in some textbooks (e.g., Altshiller-Court, 1952, p.101-102; Leonard et al., 2014, p.190; Venema, 2013, p.28).

Proof of the statement in Activity 3

Assume that we have an acute triangle which was represented as ΔABC. Let O be the circumcenter of ΔABC, G be the centroid of ΔABC, H be the orthocenter of ΔABC, and E be the midpoint of $\overline{BC}$. Place the points O and G on ΔABC and draw a line passing from the points O and G, as illustrated in Figure 6.6.

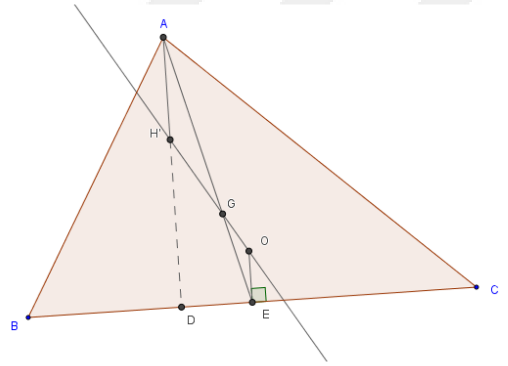


Figure 6.6. ΔABC with respect to the altitude of $\overline{BC}$ in the proof of the statement in Activity 3

Since the centroid divides each medians with a ratio 2:1. That is to say, we have $|AG|:|GE| = 2:1$ based on this ratio.

Since the point O is the point of concurrency of the perpendicular bisectors of the sides of ΔABC and the point E is the midpoint of $\overline{BC}$, it can be stated that $\overline{OE} \perp \overline{BC}$.

By drawing a line passing through the points O and G, determine a point H' in a way that it will have the property $|H'G|:|GO| = 2:1$. Then, form the triangles $\Delta H'GA$ and

ΔOGE. Since we have $\angle H'GA = \angle OGE$ and $|H'G|:|GO| = |AG|:|GE| = 2:1$, we can say that $\Delta H'GA \sim \Delta OGE$ based on Side-Angle-Side (SAS) similarity theorem. Therefore, all corresponding angles of the triangles are congruent. That is, $\angle GAH' = \angle GEA$ and $\angle AH'G = \angle EOG$. The stated congruence of the angles resulted in the fact that $\overline{AH'} \parallel \overline{OE}$. Since $\overline{OE} \perp \overline{BC}$, we have that $\overline{AD} \perp \overline{BC}$. Thus, $\overline{AD}$ is the altitude of $\overline{BC}$. Since H' lies on $\overline{AD}$, it can be stated that H' is a point on the altitude of $\overline{BC}$.

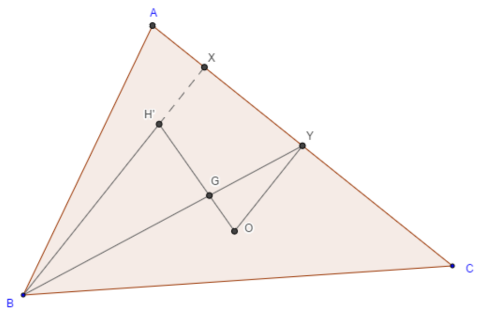


Figure 6.7. ΔABC with respect to the altitude of $\overline{CA}$ in the proof of the statement in Activity 3

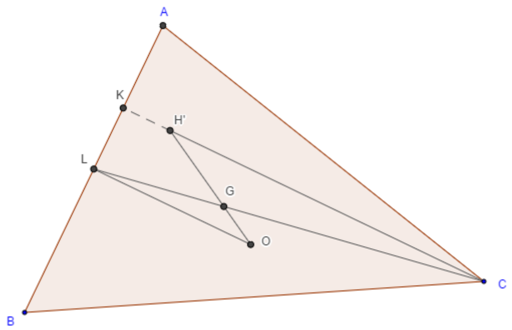


Figure 6.8. ΔABC with respect to the altitude of $\overline{AB}$ in the proof of the statement in Activity 3

As presented in Figure 6.7 and Figure 6.8, when the same steps were followed for the other sides, the altitudes of $\overline{AB}$ and $\overline{CA}$ also pass through H'. This presents that H' is the point of concurrency of the altitudes of ΔABC. ■

Proof of the statement in Activity 4

Statement asked to prove in Activity 4

Suppose that the point X, Y, and Z are placed at random on the sides of ΔABC. That is, the point X is on $\overline{AB}$, the point Y is on $\overline{BC}$, and the point Z is on $\overline{CA}$. Then, in every case, the circles AXZ, BXY, and CYZ are concurrent.

The following proof was prepared as a combination of the proofs of this statement presented in some textbooks (e.g., Aarts, 2008, p.158-159; Honsberger, 1995, p.79-80; Leonard et al., 2014, p.172-173; Venema, 2013, p.101). After this, the proof of the given statement in terms of some extra conditions were also presented, based on the textbook of Honsberger (1995, p.80-81).

Proof 1 of the statement in Activity 4

Based on the givens in the statement, assume that ΔABC is an acute triangle and the points X, Y, and Z were placed on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. The circle passing from the points A, X, and Z and also another circle passing from the points B, X, and Y were drawn, as presented in Figure 6.9. Let these circles meet at the points X and O. Now, to prove the given statement, the aim is to show the case that the circles passing from the points C, Y, and Z also passes from the point O.

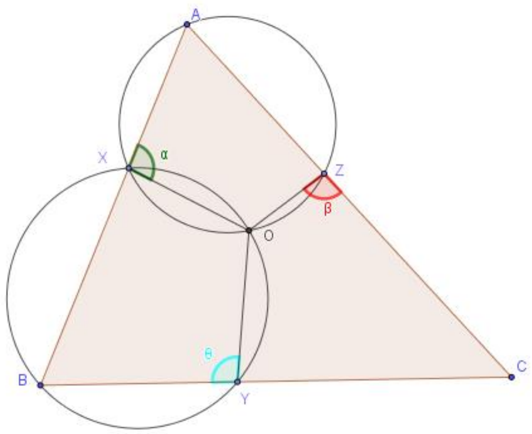


Figure 6.9. ΔABC in the proof 1 of the statement in Activity 4

The cyclic quadrilateral AXOZ presents that $\alpha = \beta$ and another cyclic quadrilateral BYOX gives that $\alpha = \theta$. Since each exterior angle of a cyclic quadrilateral is congruent to the opposite interior angle, we have the given equalities. Based on this, it was reached that $\beta = \theta$ which means that CZOY is also a cyclic quadrilateral. Therefore, it can be stated that three circles given in Figure 6.9 are concurrent at the point O. ■

In the review of the related literature, it was seen that Honsberger (1995, p.80-81) presented two extensions of the proof given above. The first one deals with the proof of the same statement in the case that the given triangle is an obtuse triangle. This extension was presented below as labeling it the second proof. The other case considers the proof when the points X, Y and Z were placed on the sidelines of ΔABC rather than the sides. Similarly, this extension was given below as the third proof of the statement. In this respect, some adjustments in the first proof were also needed to be conducted.

Proof 2 of the statement in Activity 4

Assume that ΔABC is an obtuse triangle and the points X, Y, and Z were placed on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. The circle passing from the points A, X, and Z and also another circle passing from the points B, X, and Y were drawn, as presented in Figure 6.10. Let these circles meet at the points X and O. Now, to prove the given statement, the aim is to show the case that the circles passing from the points C, Y, and Z also passes from the point O.

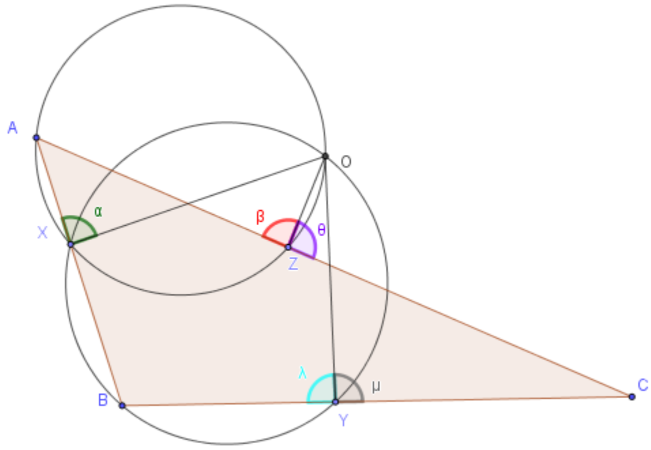


Figure 6.10. ΔABC in the proof 2 of the statement in Activity 4

From the cyclic quadrilateral AXZO, we have that $\alpha = \beta$ since they are subtended by the same arc. From the cyclic quadrilateral BYOX, we have that $\alpha = \lambda$ since $\angle OXB + \lambda = \angle OXB + \alpha = 180°$. Then, we have $\theta = 180° - \beta = 180° - \lambda = \mu$. This equality refers to the fact that COZY is also a cyclic quadrilateral. Thus, the circle passing from the points C, Y, and Z passes through the point O. Therefore, it can be stated that three circles given in Figure 6.10 are concurrent at the point O. ■

The Miquel theorem is also valid in the case that three points were placed at the extensions of the sides (Honsberger, 1995, p.81).

Proof 3 of the statement in Activity 4

Assume that ΔABC is a triangle, the point Y is placed on the extension of $\overline{BC}$, the point Z is placed on the extension of $\overline{CA}$, and the point X is placed on the extension of $\overline{AB}$. The circle passing from the points A, X, and Z and also another circle passing from the points B, X, and Y were drawn, as presented in Figure 6.11. Let these circles meet at the points X and O. Now, to prove the given statement, the aim is to show the case that the circles passing from the points C, Y, and Z also passes from the point O.

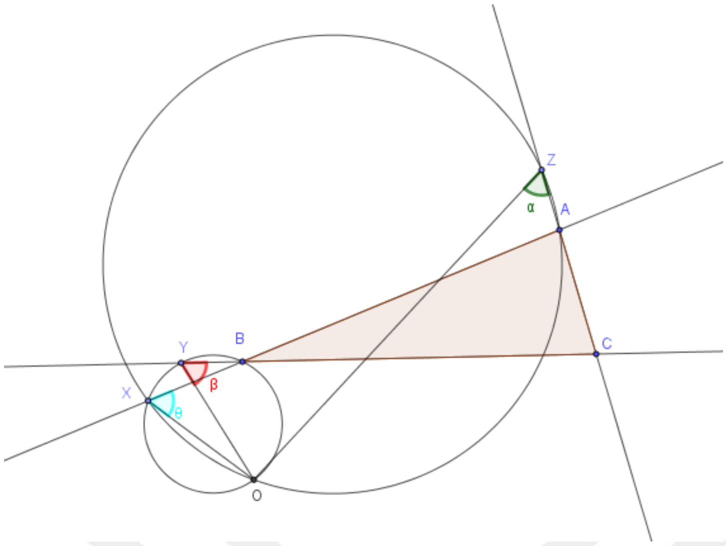


Figure 6.11. ΔABC in the proof 3 of the statement in Activity 4

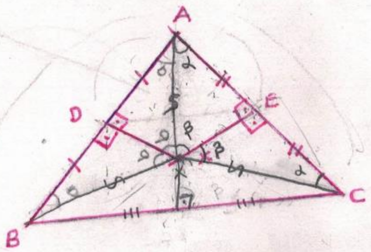
As it can be seen from Figure 6.11, the angles α and θ were subtended by the same arc in the circle passing from the points Z, X, and the vertex A so that we have $\alpha = \theta$.

In the similar vein, the angles β and θ were subtended by the same arc in the circle passing from the points Y, X, and the vertex B so that we have $\beta = \theta$.

Thus, it can be stated $\alpha = \beta$ which means that there existed a cyclic quadrilateral CZYO. Thus, the circle passing from the points C, Y, and Z passes through the point O. Therefore, it can be stated that three circles given in Figure 6.11 are concurrent at the point O. The point O is called as Miquel point. ■

APPENDIX H. ARGUMENTS OF GROUPS PRESENTED AS PROOF

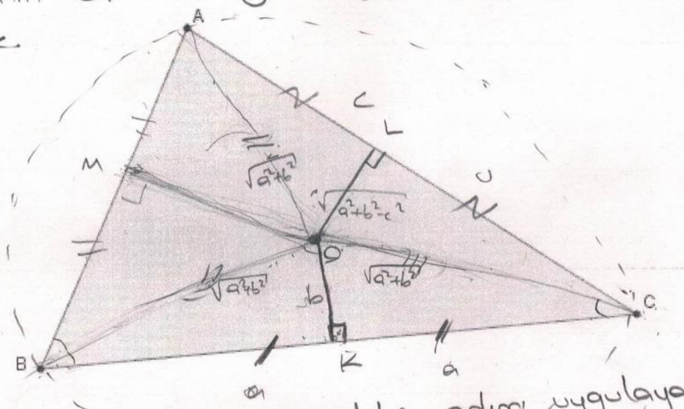
The argument of CSG in Activity 1



1 DX ve EX kenar orta dikmelerinin x noktasında kesiştiğini kabul edelim.
2 |AX|, |BX|, |CX| doğru parçasını çizdik.
3 $\hat{AXE}$ ve $\hat{CXE}$ üçgenlerini inceledik. ∠XCE ve ∠XAE açıları aynı kenarı gördüğü için aynıdır ve a…
4 ∠CXE ve ∠AXE açısı eş kenarları gördüğü için aynıdır.
5 ∠CEX ve ∠AEX açısı kabulden 90° dir ve eşittir. Yani K-A-K eşitliğinden $\hat{AEX} = \hat{CEX}$ olur.
6 Bu sayede |CX| ve |AX| aynı açıya gördüğünden eş olur.
7 Aynı işlemi $\hat{AXD}$ ve $\hat{BXD}$ için de uyguladık. K-A-K eşitliğinden bu iki üçgen de eşit
8 çıkmıştır.
9 Bu sayede |BX| = |AX| = |CX| olmuştur.
10 $\hat{BXC}$'i ikizkenar üçgendir. İkizkenar üçgende inilen dikme kenarı 2 eş parçaya böler.
11 kuralından X'den inilen dikme |BC|'yi 2 eş parçaya ayırır.
12 Buradan da kenar orta dikmelerin tek bir noktada kesiştiğini görüyoruz.
13 |BX| = |AX| = |CX| olduğu için çemberin merkezi x 'dir.
14 → yarıçaplar.
15 Ve bu çember ABC'nin çevrel çemberidir.

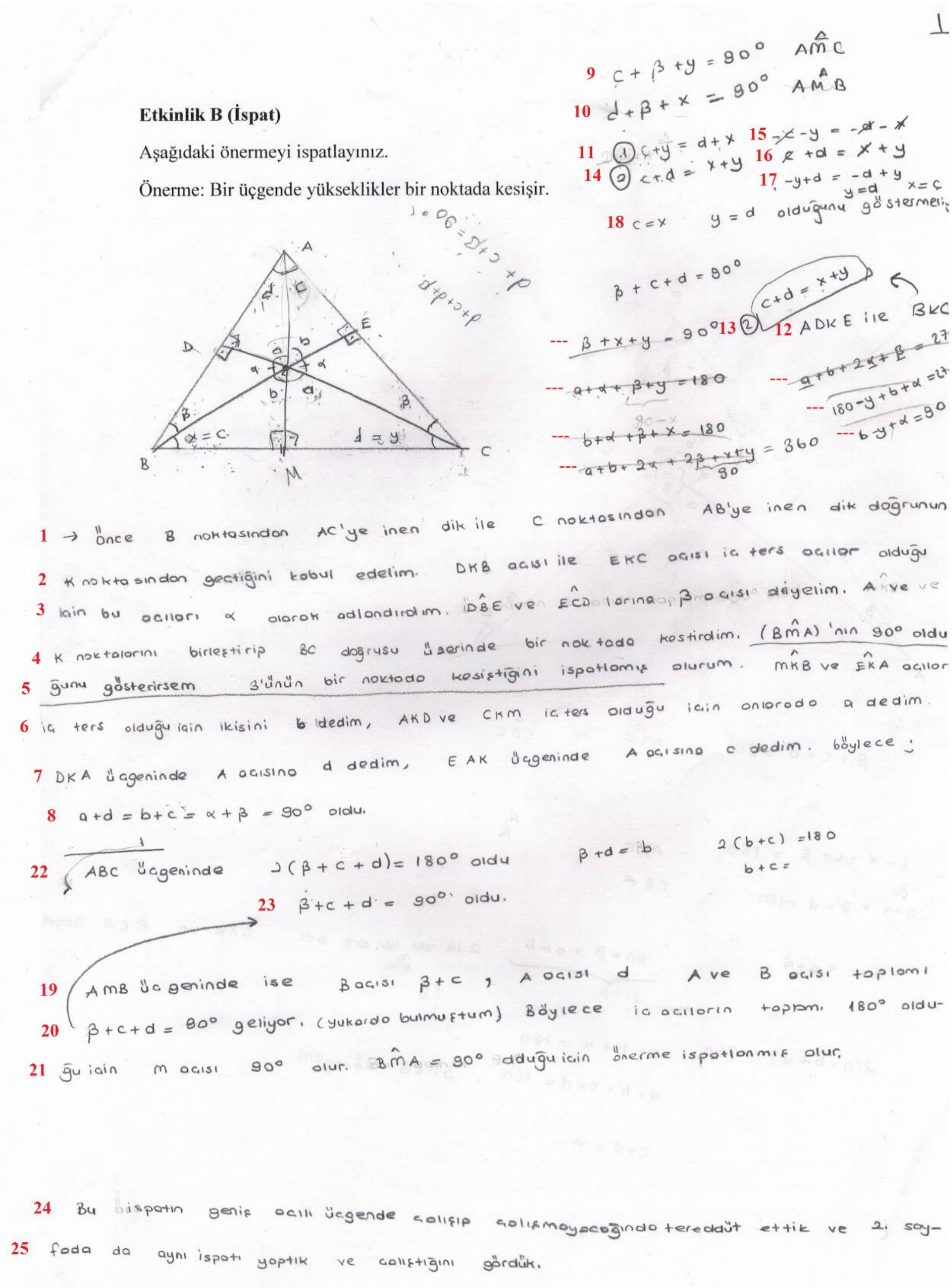
The argument of GG in Activity 1

→ OK ve OL doğru parçalarını orta dikme olarak kabul ettik ve O noktasında kesiştirdir.
→ A ile O, C ile O ve B ile O noktalarını bir doğru yardımıyla birleştirdik. OK orta dikmesi BC'yi iki eşit parçaya ayırdı.
→ $O\hat{K}C$ ve $O\hat{B}K$'inde pisagor teoremi uygulayarak BO ile OC'nin eşit olduğunu gördük. BOC ikizkenar üçgeni elde ettik.

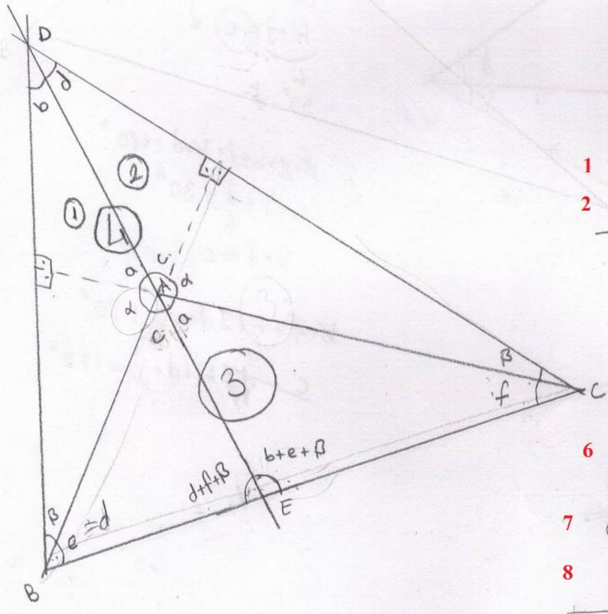


→ AOC üçgeninde de yukarıdaki adımı uygulayarak AO=BO=OC sonucuna vardık.
→ $A\hat{O}B$ ikizkenar üçgen oldu. O noktasından AB'ye indirdiğimiz dikme AB'yi iki eşit parçaya ayırdı. $A\hat{B}C$'inde OM, AB kenarının orta dikmesi oldu.
→ Sonuç olarak orta dikmeler bir noktada kesişti.
→ AO=BO=OC=r olduğu için bu nokta çevrel çemberin merkezi oldu.

The argument of CSG in Activity 2



(2)



1 $\beta + \alpha + a + b = 180°$

2 $\beta + \alpha + c + d = 180°$

3 $a + b = c + d$

4 $a + b = 90°$ (1)

5 $c + d = 90°$ (2)

6 $\beta + e + f = 90°$ (3)

7 $d + b + e + \beta + \beta + f = 180°$

8 $d + b + \beta = 90°$

9 $e + f = d + b$

10 $\beta + b + a + \alpha = 180°$

11 $\beta + \alpha = 90°$

12 $e + f = \alpha$

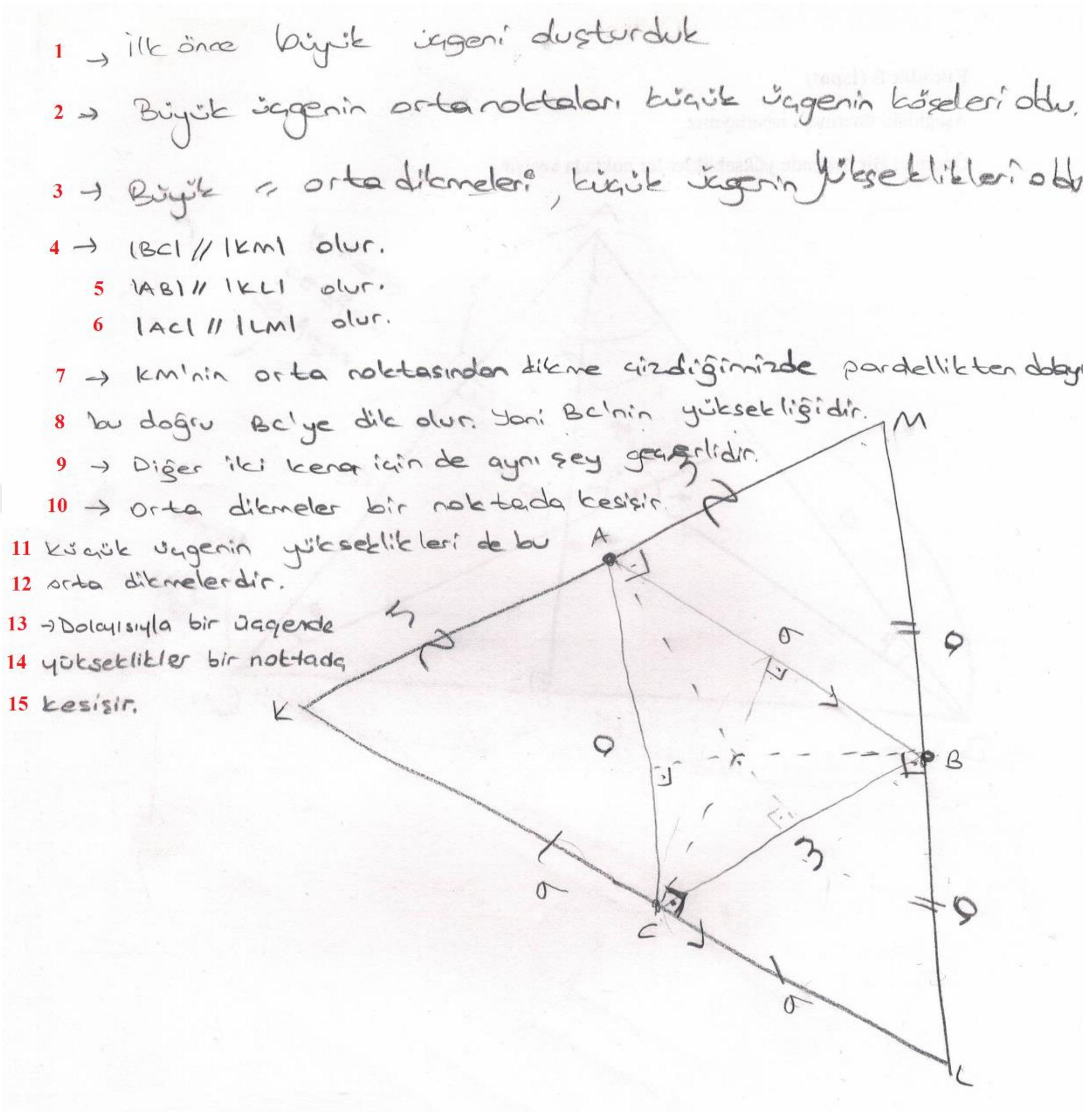
13 $e + d + f + \beta = a + \alpha$

14 $d + \beta = a$

15 $90° - c + \beta = a$

16 $a + c = \beta + 90°$

The argument of GG in Activity 2



1 → ilk önce büyük üçgeni oluşturduk

2 → Büyük üçgenin orta noktaları küçük üçgenin köşeleri oldu.

3 → Büyük " orta dikmeleri, küçük üçgenin yükseklikleri oldu

4 → |BC| // |KM| olur.

5 |AB| // |KL| olur.

6 |AC| // |LM| olur.

7 → KM'nin orta noktasından dikme çizdiğimizde paralellikten dolayı

8 bu doğru BC'ye dik olur. Yani BC'nin yüksekliğidir.

9 → Diğer iki kenar için de aynı şey geçerlidir.

10 → Orta dikmeler bir noktada kesişir.

11 Küçük üçgenin yükseklikleri de bu

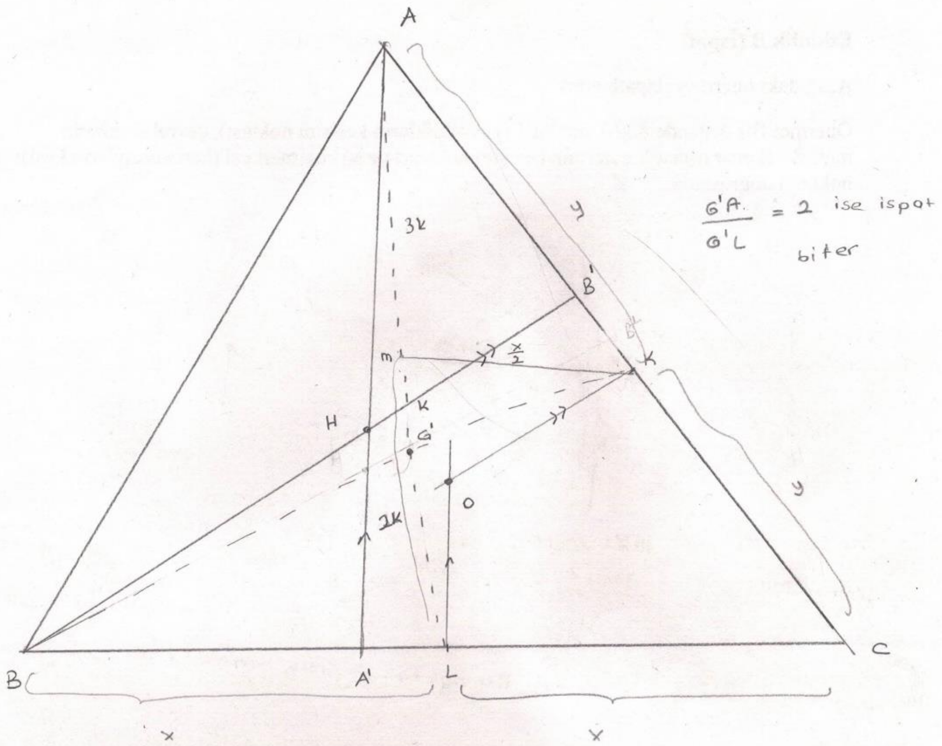
12 orta dikmelerdir.

13 → Dolayısıyla bir üçgende

14 yükseklikler bir noktada

15 kesişir.

The argument of CSG in Activity 3



1 H (diklik merkezini) ve O (çevrel çember merkezi) ile doğrusal G' noktası
2 kabul edelim. Bu noktada 2k'ya k'lık oranı $\frac{G'A}{G'L} = 2$ olduğunu gösterirsem ispat
3 biter.

4 AL ile BK'yı birleştirdim. G''de kesişti |KM| // |BC| buldum. |CL| = x ise

5 |KM| $\frac{x}{2}$

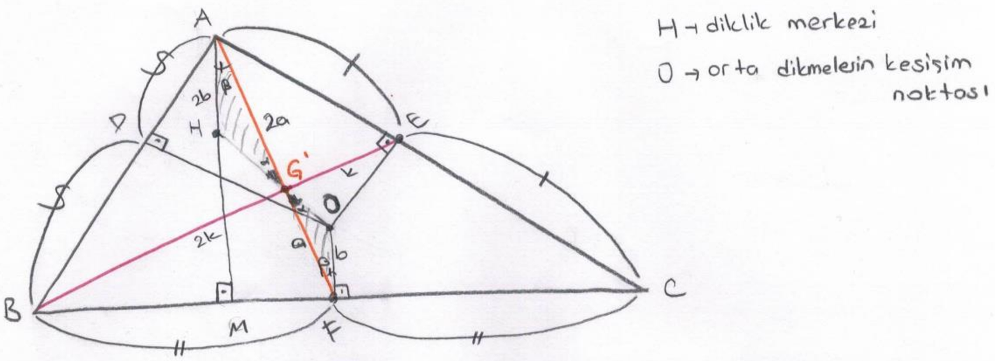
6 $\frac{\overset{\frown}{AM}}{ML} = \frac{\overset{y}{AK}}{\underset{y}{KC}}$. AM = 3k olsun ML'de 3k olmalı

7 KG'M BG'L üçgenlerinde kelebek yaptığımda $\frac{KM}{BL} = \frac{G'M}{G'L}$

8 $\frac{x}{2.x} = \frac{G'M\ k}{G'L\ 2k}$ olduğunu buldum.

9 $\frac{G'A}{G'L} = \frac{\overset{k}{G'M} + \overset{3k}{MA}}{G'L \rightarrow 2k} = 2$ olduğunu göstermiş oluruz.

The argument of GG in Activity 3



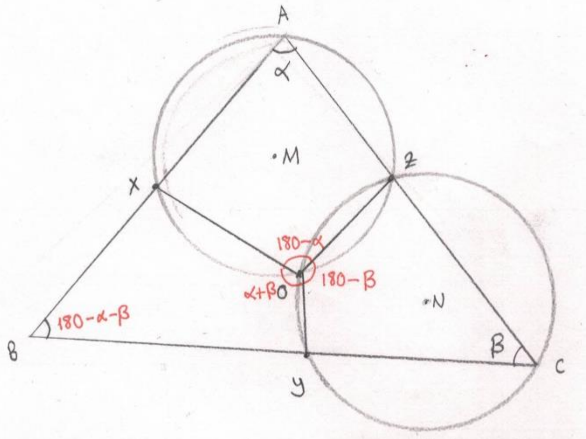
→ ABC üçgeninde kenarların orta noktalarını bulalım (D,E,F noktal
→ çevrel çemberin merkezi olan O noktası, diklik merkezi olan
H noktasını ele alalım
→ A ile F noktalarını, B ile E noktalarını doğru parçasıyla birleş-
tirip kesiştirdiğimizde G' noktasını elde ederiz
→ şekil üzerinde menelaus uyguladığımızda

① $\frac{|BF|}{|BC|} \cdot \frac{|EC|}{|EA|} \cdot \frac{|AG'|}{|G'F|} = 1 \Rightarrow \frac{1}{2} \cdot 1 \cdot \frac{|AG'|}{|G'F|} = 1 \Rightarrow \frac{|AG'|}{|G'F|} = \frac{2a}{a}$

② $\frac{|AE|}{|AC|} \cdot \frac{|CF|}{|FB|} \cdot \frac{|BG'|}{|G'E|} = 1 \Rightarrow \frac{1}{2} \cdot 1 \cdot \frac{|BG'|}{|G'E|} = 1 \Rightarrow \frac{|BG'|}{|G'E|} = \frac{2k}{k}$

→ Buradan G' ağırlık merkezi olur.
→ HG'O 'nun doğrusal olduğunu göstermek için $\overset{\triangle}{AHG'}$ ile $\overset{\triangle}{OFG'}$
inde kelebek uygularız
→ $|AM| \,//\, |OF|$ olduğu için $(\widehat{OFG'}) = (\widehat{G'AM})$ olur
→ $(\widehat{OG'F})$ ile $(\widehat{AG'H})$ ters açılardan birbirine eşit olur.
→ üçgen benzerliğinden α'nın karşısındaki kenarların oranı, β'nın
karşısındaki kenarların oranına eşit olur.
→ Sonuç olarak HG'O noktaları doğrusal olur.

The argument of CSG in Activity 4



Kabul: ABC üçgeni üzerinden X, Y, Z noktaları alınsın. A, X, Z ve C, Y, Z noktalarından geçen M ve N merkezli çemberler, O noktasında kesişsin. (B, X, Y noktalarından geçen çember de O noktasında kesişir mi?)

→ $\angle BAC = \alpha$ ve $\angle ZCY = \beta$ olsun. (AXOZ) ve (ZOYC) kirişler dörtgeni olur. Kirişler dörtgeninin özelliklerinden dolayı $\angle XOZ = 180 - \alpha$ ve $\angle ZOY = 180 - \beta$ olur.

→ 360° olması için $\angle XOY = 360 - [(180-\alpha) + (180-\beta)] = \alpha + \beta$ olur.

→ $\hat{ABC}$ 'nde $\angle B + \alpha + \beta = 180^\circ \Rightarrow \angle B = 180^\circ - \alpha - \beta$ olur.

→ (BXOY) dörtgenine bakarsak $\angle ABY + \angle XOY = 180^\circ - \alpha - \beta + \alpha + \beta = 180^\circ$ olur.

Yani (BXOY) dörtgeni, kirişler dörtgeni olur. Kirişler dörtgeninden de yalnızca bir çember geçer ve o da O noktasından geçer. Yani bu üç çember bir noktada kesişir.//

The argument of GG in Activity 4

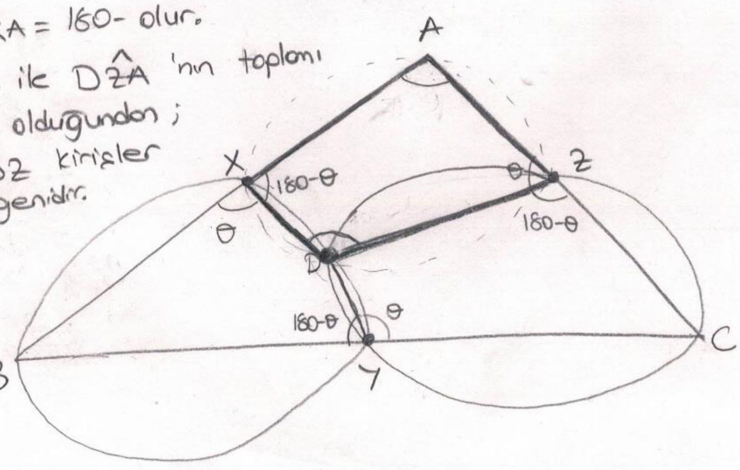
→ BYX ve CZY çemberleri bir noktada kesişsin kabulünden yola çıktık.

→ Bu iki çemberin kesişim noktaları Y ve D oldu.

→ D ile X'i, D ile Z noktalarını birleştirdik.

→ $\widehat{BXD} = \theta$ olsun. θ'nın gördüğü yay 2θ olur. Geriye kalan $\overset{\frown}{BXD} = 360-2\theta$ olur. $\overset{\frown}{BXD}$ yayını gören $\widehat{BYD}$ açısı $180-\theta$ olur. $\widehat{DYC} = 180-(180-\theta) = \theta$ olur. $\overset{\frown}{DZC} = 2\theta$'dır. Geriye kalan $\overset{\frown}{DYC} = 360-2\theta$ olur. $\overset{\frown}{DYC}$ yayının gören $\widehat{DZC}$ açısı $180-\theta$ olur. $\widehat{BXD}=\theta$ olsun demiştik. O halde $\widehat{DZA} = 180-(180-\theta)=\theta$ olur. $\widehat{DXA} = 180-$ olur.

$\widehat{DXA}$ ile $\widehat{DZA}$'nın toplamı 180° olduğundan; AXDZ kirişler dörtgenidir.



AXDZ dörtgeninden çember geçirdiğimizde 3 çemberin de D noktasında kesiştiğini gördük.

APPENDIX I. CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Demiray, Esra
Nationality: Turkish (TC)
Date and Place of Birth: 1988, Bilecik
email: esdemiray@gmail.com

EDUCATION

Degree	Institution	Year of Graduation
PhD	Middle East Technical University Elementary Education	2019
MS	Middle East Technical University Elementary Mathematics and Science Education	2013
BS	Middle East Technical University Elementary Mathematics Teacher Education	2010

WORK EXPERIENCE

Year	Place	Enrollment
2013- Present	Hacettepe University	Research Assistant
2011-2013	Zonguldak Bülent Ecevit University	Research Assistant

PUBLICATIONS

Journal Papers

Demiray, E., & Işıksal-Bostan, M. (2017). An investigation of pre-service middle school mathematics teachers' ability to conduct valid proofs, methods used, and reasons for ınvalid arguments. *International Journal of Science and Mathematics Education, 15*(1), 109–130.

Demiray, E., & Işıksal-Bostan, M. (2017). Pre-service middle school mathematics teachers' evaluations of discussions: The case of proof by contradiction. *Mathematics Education Research Journal*, *29*(1), 1-23.

Conference Presentations

Demiray, E., & Işıksal, M. (2012). The pre-service elementary mathematics teachers' views about proof. *Paper presented at the European Conference on Educational Research (ECER),* Cadiz, Spain.

Demiray, E., & Çapa-Aydın, Y. (2015). Development of geometric construction self-efficacy scale for pre-service middle school mathematics teachers. *Paper presented at the European Conference on Educational Research (ECER),* Budapest, Hungary.

Demiray, E., & Işıksal-Bostan, M. (2016). Ortaokul matematik öğretmen adaylarının aksine örnek verme yöntemini yorumlamaları. *Paper presented at the 3rd International Eurasian Educational Research Congress (EJER),* Muğla, Turkey.

Demiray, E., & Işıksal, M. (2016). Ortaokul matematik öğretmen adaylarının rastgelelik kavramını yorumlamalarının incelenmesi. *12. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi*, Trabzon, Turkey.

Demiray, E., & Işıksal-Bostan, M. (2017). Pre-service middle school mathematics teachers' interpretation of statements regarding proof by contrapositive. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education* (pp. 139-147). Dublin: DCU Institute of Education and ERME.

Demiray, E., & Işıksal-Bostan, M. (2017). The meaning of mathematical proof for prospective middle school mathematics teachers. *Paper presented at the European Conference on Educational Research (ECER),* Copenhagen, Denmark.

Demiray, E., & Saygı, E. (2017). Prospective middle school mathematics teachers' interpretations of graphs related to integral. *Paper presented at the European Conference on Educational Research (ECER),* Copenhagen, Denmark.

Zeybek, N., Demiray, E., & Saygı, E. (2018). Use of mental games in mathematics lesson plans. *Paper presented at International Conference on Mathematics and Mathematics Education (ICMME),* Ordu, Turkey.

Demiray, E., & Işıksal-Bostan, M. (2018). Prospective middle school mathematics teachers' geometric constructions via compass and straightedge. *Paper presented at the European Conference on Educational Research (ECER),* Bolzano, Italy.

Demiray, E., Zeybek, N., & Saygı, E. (2018). Pre-service middle school mathematics teachers' ıntegration of mental games to mathematics lesson plan. *Paper presented at the European Conference on Educational Research (ECER),* Bolzano, Italy.

APPENDIX J. TURKISH SUMMARY / TÜRKÇE ÖZET

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ BİLİŞSEL BÜTÜNLÜK BAĞLAMINDA ARGÜMANTASYON, İSPAT VE GEOMETRİK İNŞA SÜREÇLERİNİN İNCELENMESİ

1. Giriş

Matematiksel ispat üzerine yapılan çalışmalar son yıllarda ivme kazanmıştır (Arzarello, Micheletti, Olivero ve Robutti, 1998; Arzarello ve Sabena, 2011; Stylianides, 2019; Stylianides, Bieda ve Morselli, 2016; Stylianides ve Stylianides, 2017). Ulusal Matematik Öğretmenleri Konseyi'ne (NCTM, 2000) göre, matematiksel ispat "belirli akıl yürütme ve gerekçeleri ifade etmenin formal bir yoludur" (s.56). Hem matematikte hem de matematik eğitiminde ispatın önemi birçok araştırmanın hemfikir olduğu bir konudur (Arzarello ve Sabena, 2011; Edwards, 1997; Ellis, Bieda ve Knuth, 2012; Hanna, 2018; Mariotti, Durand-Guerrier ve Stylianides, 2018; Stylianides, 2019; Stylianides ve Stylianides, 2017; Tsamir, Tirosh, Dreyfus, Barkai ve Tabach, 2009).

İspat kavramı ile birlikte akıl yürütme ve argümantasyon kavramları da son yıllarda matematik eğitimi araştırmalarında ön plana çıkmaktadır (Reiss, Heinze, Renkl ve Groß, 2008). Hanna'ya (2014) göre matematikteki farklı kullanımlarından dolayı "argümantasyon, akıl yürütme ve ispat sınırları iyi tanımlanmamış olan kavramlardır" (s.404). Alanyazın taramasında argümantasyon, akıl yürütme ve ispat kavramlarının genellikle ikili bağlantılar şeklinde ele alındığı görülmektedir (örneğin, Boero, Fenaroli ve Guala, 2018; Boero, Garuti ve Mariotti, 1996; Conner, Singletary, Smith, Wagner ve Francisco, 2014a; Mariotti vd., 2018; Pedemonte, 2007a, 2018a; Reid ve Knipping, 2010; Reiss vd., 2008; Stylianides, 2008; Stylianides, Stylianides ve Shilling-Traina, 2013; Tsamir vd., 2009). Bu çalışma ise temel olarak argümantasyon ve ispat kavramları üzerine yapılandırılmıştır.

Argümantasyon ve ispat ile ilgili farklı fikirlerin varlığı araştırmalarda çeşitli yaklaşımların takip edilmesine neden olmuştur. Bu yaklaşımlar üç kategoride toplanmıştır. İlk olarak, argümantasyon ve ispat arasında net bir fark olmadığını ileri süren çalışmalar görülmektedir (Pedemonte, 2007a). İkinci yaklaşım ise, matematikte argümantasyon ve ispat kavramları arasında farklılık olduğuna dikkat çekmektedir (örneğin, Antonini ve Mariotti, 2008; Boero vd., 1996; Douek, 1998, 2010; Fiallo ve Gutiérrez, 2017; Garuti, Boero, Lemut ve Mariotti, 1996; Mariotti, Bartolini-Bussi, Boero, Ferri ve Garuti, 1997; Pedemonte, 2002b, 2007a, 2007b; Reid ve Knipping, 2010). Argümantasyon ve ispatla ilgili son yaklaşım ise, ikinci yaklaşımdaki perspektifi dikkate almakta, ayrıca konuyu eğitim odaklı bir perspektiften de ele almaktadır (Mariotti, 2006; Pedemonte, 2007a). Daha ayrıntılı olarak, bu yaklaşım varsayım oluşturma sürecini ifade eden argümantasyon ve bu varsayımları ispatlama süreci arasındaki sürekliliği vurgulamaktadır. Diğer bir deyişle, odaklanılan konu varsayım oluşturma aşamasının ispat ile ilişkisidir (Pedemonte, 2007a). Bu bağlamda, Garuti vd. (1996) teoremlerin bilişsel bütünlüğü kavramını ortaya koymuştur. Bu çalışma argümantasyon ve ispata dair bahsedilen yaklaşımlardan sonuncusu, yani bilişsel bütünlük kavramı çerçevesinde yürütülmüştür.

Pedemonte (2007a) bilişsel bütünlük sağlandığında, varsayım oluşturmayı içeren argümantasyon aşamasının ispat aşamasını desteklediğini ifade etmektedir. Bilişsel bütünlük kavramı varsayım oluşturma sürecindeki unsurların ispat sürecindeki olası kullanımı üzerinde durmaktadır (Garuti vd., 1996). Gerekli koşullar sağlandığında ve kritik noktalara dikkat edildiğinde, öğrencilerin argümantasyon süreci aracılığıyla ürettikleri bir önermenin ispatı üzerine çalıştıkları etkinlikler, ispatlama yeteneklerini geliştirmek adına önemli bir potansiyel oluşturmaktadır (Garuti vd., 1996). Böyle bir etkinliğin temel unsurları aşağıdaki gibi örneklendirilebilir; sınıfta didaktik bir ortam sağlayarak açık uçlu bir problem (open problem) üzerinde çalışılabilir, problemin çözülmesinin amaçlandığı bir argümantasyon süreci aracılığıyla varsayım ya da önermelere ulaşılabilir ve etkinlik ispat aşamasına hazırlayan grup ve sınıf tartışmalarını içerebilir (Garuti vd., 1996).

İspat yapma matematikteki belirli bir alana ya da derse uygun bir etkinlik olarak ifade edilmemektedir. Yani, ispatlama uygun bir şekilde ayarlandığında alandan bağımsız olarak matematik öğretiminde etkili bir şekilde kullanılabilir (Ellis vd., 2012). Bu çalışmada, bilişsel bütünlük bağlamında geometri alanında çalışılmasına karar verilmiştir. Bunun nedeni ise, geometrinin "ispatla ilgili kavramları geliştirmek için fırsatlar sunan zengin bir kaynak" oluşu (Jones, 2002, s.125) ve "geometrik bilginin oluşturulmasında akıl yürütmenin gerekliliği"dir (NCTM, 2000, s.7). Ayrıca, geometri öğrencilerin akıl yürütme ve gerekçelendirme becerilerini geliştirmek için uygun bir alan (NCTM, 2000) ve varsayım oluşturma ile araştırma süreçleri için fırsat sağlamaktadır (Gillis, 2005). Bazı müfredatlarda ortaöğretim seviyesindeki geometri dersinin, ispatın formal açıdan ele alındığı ilk aşama olduğu görülmektedir. İspatın öğrencilere nasıl tanıtıldığı cebir, trigonometri ve istatistik gibi diğer matematik konularındaki öğrenmelerini de etkilemektedir (Ellis vd., 2012). Yani, geometri ispatın nasıl öğrenilmesi gerektiği konusunda ilk olarak başvurulan matematik alanıdır (Pedemonte, 2007b). Bunun yanında, didaktik akıl yürütme Öklid geometrisindeki kavramları öğretme ve öğrenme sürecinin önemli bir bölümünü oluşturmaktadır (Leung ve Lopez-Real, 2002). Öğrencilerin geometrinin pratik ve teorik yönleri arasında geçiş yapabildikleri durumlar hazırlamak, matematik eğitimi açısından kritik ve zorlu bir görevdir (Fujita, Jones ve Kunimune, 2010).

Ayrıca, İtalyan araştırmacılar bahsi geçen çalışmalarında (örneğin, Boero vd., 1996; Garuti vd., 1996; Garuti, Boero ve Lemut, 1998; Mariotti, 2001; Mariotti vd., 1997), açık uçlu problemlerin ispat öğrenmede etkili bir yol olduğunu vurgulamaktadır, çünkü varsayım oluşturma sürecindeki argümantasyon ve ispat arasındaki bilişsel bütünlük böyle bir durumda fark edilir hale gelmektedir (Pedemonte, 2011). Benzer şekilde, açık uçlu soruların ispatı destekleme, problemin merak edilmesine neden olma, net olarak görülebilen bir argümantasyon sürecinin oluşmasını sağlama (Baccaglini-Frank ve Mariotti, 2010) ve ispata başlama açısından önemli olan varsayım oluşturma sürecini gerektirme gibi durumlar için önemli bir potansiyele sahip olduğu belirtilmiştir (Baccaglini-Frank, 2010; Pedemonte, 2007b). Bu bağlamda, açık uçlu problemlerin özellikleri göz önüne alınarak, bu çalışmadaki etkinliklerin varsayım oluşturma aşamasında, geometrik inşa konusunun dahil edilmesine karar verilmiştir. Diğer bir deyişle, bu çalışmada katılımcıların üçgen ve çember ile ilgili geometrik inşalar üzerine çalışırken varsayımlar ürettikleri bir argümantasyon sürecine dahil olmaları amaçlanmıştır. Çeşitli araçlar kullanılarak yapılan geometrik inşalar öğrencilerin geometrik bağlantıları anlamalarına ve daha soyut seviyede akıl yürütmenin gerekli olduğu geometrik genellemeler üzerinde düşünmelerine yardımcı olmaktadır (Arıcı, 2012). Kullanılan araç ne olursa olsun, geometrik inşa problemlerini çözme sürecinin öğrenciler için birçok faydası vardır (Djorić ve Janičić, 2004). Örneğin, Fujita vd. (2010), geometrik inşaların öğrencileri önermelerin ispatı üzerinde çalışmaya, bazı varsayımlar üretmeye ve ayrıca argümantasyon, akıl yürütme ve ispatla ilgili becerilerini geliştirmeye teşvik ettiğini belirtmiştir.

Bahsedilen konular dikkate alınarak tasarlanan çalışmanın amaçları ve cevap bulmak istenen araştırma soruları aşağıda detaylıca açıklanmıştır.

1.1. Çalışmanın Amaçları ve Araştırma Soruları

Çalışmanın ilk amacı, ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerde varsayım oluşturma aşamasındaki argümantasyon süreçlerinin, oluşturdukları varsayımları ispatlama süreçleriyle nasıl bir ilişkisi olduğunu incelemektir. Görüldüğü gibi, bu amaç bilişsel bütünlüğün temelindeki bağlamdan bahsetmektedir. Çalışmanın ikinci amacı ise, ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerde varsayım oluşturma sürecindeki global argümantasyon yapılarını araştırmaktır. Bu kapsamda, global argümantasyon yapılarındaki bileşenler ve çürüten bileşenin işlevlerinin de araştırılması amaçlanmıştır. Üçüncü amaç, ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerdeki geometrik inşalar için önerdikleri yaklaşımları incelemektir. Ayrıca, katılımcıların yaklaşımlarının geçerliğini nasıl değerlendirdiklerini ve pergel-çizgeç ve GeoGebra kullanırken ne derece doğru geometrik inşa yapabildiklerini araştırmak da hedeflenmiştir. Çalışmanın son amacı ise, ortaokul matematik öğretmen adaylarının argümantasyon sırasında oluşturdukları varsayımları belirlemek ve bu varsayımlara geçerli ispat yapıp yapamadıklarını araştırmaktır. Bu amaçlar doğrultusunda, aşağıda listelenen araştırma sorularına cevap aranmıştır.

1. Ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerde varsayım oluşturma aşamasındaki argümantasyon süreçlerinin, oluşturdukları varsayımları ispatlama süreçleriyle nasıl bir ilişkisi vardır?
2. Ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerin varsayım oluşturma sürecindeki global argümantasyon yapıları nelerdir?

2.1. Ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerin varsayım oluşturma sürecindeki global argümantasyon yapılarının bileşenleri nelerdir?

2.2. Ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerin varsayım oluşturma sürecindeki global argümantasyon yapılarında bulunan çürüten bileşeninin işlevleri nelerdir?

3. Ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerdeki geometrik inşalar için önerdikleri yaklaşımlar nelerdir?

3.1. Ortaokul matematik öğretmen adayları geometrik inşalar için önerdikleri yaklaşımların geçerliğini nasıl değerlendirmektedir?

3.2. Ortaokul matematik öğretmen adayları pergel-çizgeç kullanırken ne derece doğru geometrik inşa yapabilmektedir?

3.3. Ortaokul matematik öğretmen adayları GeoGebra kullanırken ne derece doğru geometrik inşa yapabilmektedir?

4. Ortaokul matematik öğretmen adaylarının oluşturdukları varsayımları ispatlama amacıyla ortaya koydukları argümanlar nelerdir?

4.1. Ortaokul matematik öğretmen adaylarının argümantasyon sürecinde oluşturdukları varsayımlar nelerdir?

4.2. Ortaokul matematik öğretmen adayları argümantasyon sürecinde oluşturdukları varsayımlara ne derece geçerli ispat yapabilmektedir?

1.2. Çalışmanın Önemi

İspatla ilgili araştırmaların odağının öğrencilerin ispat sırasında karşılaştıkları problemleri rapor etmekten ziyade, bu problemleri gidermek amaçlı sınıfta yapılacak uygulamalar gibi yöntemleri içeren çalışmalara yönlendirilmesi gerektiği sıklıkla önerilmektedir (Stylianides ve Stylianides, 2017; Stylianides ve Stylianides, 2018). Ancak birçok çalışmada öğrencilerin ispat yaparken zorluk yaşadığı konusunda bir fikir birliği olmasına rağmen, bu tür zorlukların üstesinden gelmek için bir müdahale tasarımı sunan araştırmaların azlığı da dikkat çekmektedir (Stylianides, Bieda ve Morselli, 2016; Stylianides, Stylianides ve Weber, 2017; Stylianides ve Stylianides, 2017; Stylianides ve Stylianides, 2018). Bu konuyla ilgili olarak, bilişsel bütünlük öğrencilerin ispatla ilgili kavramları öğrenmeleri, süreçte karşılaşılan olumsuz durumlarla başa çıkmaları ve sonunda formal ispat hedeflenen konularda kavramsal anlamaya ulaşmaları için umut verici bir yöntem olarak düşünülebilir.

Argümantasyon ve ispat kavramlarını ayrı bir şekilde ele almak yerine, her ikisine de odaklanmak ispatlamaya ilişkin daha geniş bir konu aralığını değerlendirme fırsatı sunabilir (Stylianides vd., 2016). İspat ile ilgili olarak argümantasyon sürecinin mantığını, kolaylıklarını ve kısıtlamalarını bulmak, ispat öğretirken kullanılan yöntemleri geliştirmek için yararlı olabilir (Reid ve Knipping, 2010). Dolayısıyla, bu çalışmanın bilişsel bütünlük kavramının hem olumlu hem de olumsuz yanlarını hesaba katması nedeniyle, geometri bağlamında yeni bir bakış açısı getirmesi beklenmektedir.

Bir etkinliğin argümantasyon aşaması varsayım üretmeyi ve bir ürün olarak ispatı içerdiğinde, ispatın başladığı noktanın belirlenmesi öğrenciler açısından kritik öneme sahiptir. Böyle bir etkinlik öğrencilerin argümantasyon ve ispat arasındaki ilişkiyi ve farklılığı yorumlamaları ve aynı zamanda varsayımı teoremden ayırt etmeleri için fayda sağlamaktadır (Pedemonte, 2007a). Bu bağlamda, bu çalışmadaki bilişsel bütünlük temelli etkinlikler gibi iki aşamalı etkinlikleri uygulamak, öğrencilerin ispatlama aşamasının başlangıcını belirlemeleri ve sezgisel bir yaklaşım geliştirmeleri için faydalı olabilir.

Ellis vd.'ne (2012) göre, ispat sürecinin kapsamlı bir yapısı vardır ve bu yapı varsayım oluşturma, genelleme amaçlı araştırma, örnekleri ve karşıt örnekleri göz önünde bulundurma ve durumlar arasında benzerlikler arama gibi ispat yazmaya yönlendiren çeşitli faaliyetleri içermektedir. Bu çalışma ispat amaçlı bir süreçte yer alan bu tür unsurların önemini de dikkate almaktadır. Bu bağlamda, bu çalışmanın ortaokul matematik öğretmen adaylarına bu tür bir öğrenme ortamını farklı yönlerden deneyimleme fırsatı sunduğu sonucuna varılabilir. Ortaokul matematik öğretmen adayları, bilişsel bütünlük temelli etkinlikleri hem öğretmen hem de öğrencilerin bakış açısından inceleyebilirler. Ayrıca, matematik öğretmenleri öğrencilerin bilinçlendirilmesinde ve ispatın gerekliliğine ilişkin fikirlerini geliştirmede kritik bir role sahiptir (Stylianides vd., 2016). Geleceğin matematik öğretmenleri olarak kabul edilen ortaokul matematik öğretmen adaylarının bu etkinliklere aktif katılmaları bilişsel bütünlük kavramının eğitsel açısından özünü yakalamalarına yardımcı olabilir.

Bu çalışmanın diğer bir ana konusu ise matematik eğitiminde argümantasyondur. Bilişsel bütünlük kavramının ilk tanımı ve argümantasyonu “varsayım oluşturma süreci” olarak ifade eden Pedemonte’nin yaklaşımı temel alınarak (Reid & Knipping, 2010, s.163), argümantasyon süreci, çalışmada varsayım oluşturma aşamasına karşılık gelmektedir. Bu aşamada global argümantasyon yapıları ikinci araştırma sorusunu cevaplamak amacıyla oluşturulmuştur. Ayrıca, etkinliklerde katılımcıların kolektif argümantasyon olarak adlandırılan şekilde, yani bir grup olarak çalışması beklenmiştir. Tekin-Dede (2018) kolektif argümantasyon teriminin ne Türkiye'deki matematik müfredatında ne de argümantasyona dair yapılan araştırmalarda doğrudan yer almadığına, ancak matematik eğitimi ile ilgili olarak argümantasyon terimini kullanan çalışmalar olduğuna vurgu yapmaktadır. Bu nedenle, çalışmanın özellikle ulusal bağlamda kolektif argümantasyon açısından alanyazına katkıda bulunması beklenmektedir. Genel anlamda, çalışmanın bulguları matematik eğitiminde argümantasyonun etkinliğini incelemek için kaynak olarak düşünülebilir.

Mevcut araştırmanın argümantasyonla iligili olan bölümü global argümantasyon yapıları üzerine konumlandırılmıştır. Toulmin’in modeli argümantasyonun bir adımının açıklanmasında etkili bir araçtır, bu nedenle kapsamlı bir argümantasyon sürecinde yer alan ayrı argümanları belirtmek için kullanılabilir. Ancak, söz konusu kapsamlı argümantasyonun genel yapısını gösterebilmek için bu modelden daha fazlasına ihtiyaç duyulmaktadır (Knipping, 2008; Reid ve Knipping, 2010). Global argümantasyon yapıları, argümantasyon sürecinin bütününün görüntüsünü sunmak ve bileşenlerin özelliklerinden yola çıkarak bireysel argümanları ortaya çıkarmak adına kullanılabilir (Reid ve Knipping, 2010). Ayrıca, global argümantasyon yapıları, öğrencilerin tartışma sırasında nasıl hareket ettikleri, bileşenlerin argümantasyon akışını nasıl etkilediği ve argümantasyonun nasıl zenginleştirilebileceği konusunda fikir verebilir. Bunun dışında, ilgili alanyazın global argümantasyon yapısının matematik alanında nispeten daha az çalışılmış bir kavram olduğunu ve bu kavram hakkında daha fazla araştırmaya ihtiyaç duyulduğunu göstermektedir. Bu çalışma global argümantasyon yapılarına dair bir sınıflandırma sunması nedeniyle diğer çalışmalar için bir referans noktası sağlayabilir.

Bu çalışma, Toulmin'in argümantasyon modelindeki tüm bileşenleri kullanmaya odaklanmıştır, ancak bulgular belirtilen altı bileşenden daha fazla bileşen ihtiyacını ortaya koymuştur. Bu nedenle, bulgulara dayanarak oluşturulan ekstra bileşenler, Toulmin'in modelini kullanmayı planlayan diğer çalışmalara geri bildirimler sağlayabilir. Bu şekilde, çalışma ile argümantasyonun gerçekleştiği bağlama göre farklı bileşenlerin bulunma ihtimaline de dikkat çekilmektedir. Ayrıca, beklenildiği üzere, bir argümantasyonda katılımcıların her zaman olumlu bir duruşa sahip olmaması nedeniyle bazı itirazların varlığı kaçınılmazdır. Argümantasyonun bileşenleri arasında yer alan çürüten (rebuttal) bu amaca hizmet etmektedir. Ayrıca, ulaşılabilen alanyazında, gerekçe (warrant) ve destek (backing) gibi bileşenlere kıyasla, çürütücünün içeriğine odaklanan çalışmaların azlığı dikkat çekmektedir. Karşıt örnekler bir varsayımı çürütebildiğinden ve aynı zamanda varsayımlarda değişikliklere neden olabileceğinden (Sinclair, Pimm, Skelin ve Zbiek, 2012a), bir çürütme önermek daha üst düzey düşünme becerilerini gerektirir ve öğrencinin tartışması sırasında zor bir görev olarak görülme potansiyeli yüksektir (Lin ve Mintzes, 2010). Bu nedenle, ortaokul matematik öğretmen adaylarının argümantasyon sürecinde nasıl çürütücü buldukları oluşturdukları varsayımlar veya önermeler açısından kritik bir rol oynamaktadır.

Daha önce de belirtildiği gibi, bu çalışmanın matematik alanı bağlamındaki içeriği geometridir. Geometriyi anlama, hem ortaokul hem de ortaöğretim seviyesindeki öğrencilerin gelecekteki öğrenme deneyimleri açısından bir gerekliliktir (Sinclair vd., 2012a, 2012b). Öğrencilerin geometriyi kalıcı bir şekilde anlamalarını sağlamanın ilk adımı, matematik öğretmenlerinin kavramları derinlemesine anlamalarıdır. Başka bir deyişle, matematik öğretmenlerinden geometri kavramlarını öğretmeleri beklendiğinden, aynı zamanda geometri konusunda da yeterli olmaları ve etkili bir şekilde geometri öğretme yöntemlerini bilmeleri gerekmektedir. Bu noktada, belirtilen durumlar nedeniyle odak noktası matematik öğretmenlerinin yeterliliklerinden öğretmen eğitimi programlarına çevrilmektedir. Alan bilgisinin gelişimi açısından, öğretmen eğitimi programlarının kritik bir rolü vardır. Bu bağlamda, mevcut araştırmanın katılımcıları ortaokul matematik öğretmen adayları olduğundan, bu çalışmanın öğretmen adaylarının geometrideki bilgi düzeyleri, doğru çıkarımları ve yanlış yorumlamalarına ek olarak geometri içeren lisans derslerinin içeriğinin nasıl düzenlenmesi gerektiği hakkında bilgi vermesi beklenmektedir. Ayrıca, bu araştırma, ortaokul matematik öğretmen adaylarının öz-değerlendirme yapabilecekleri, geometri ile ilgili eksikliklerini fark edebilecekleri ve öğretmen olmadan önce bu konularla ilgili önlemleri alabilecekleri bir ortam sunmaktadır.

Bu çalışmanın gerekçesiyle ilgili olarak bahsedilecek son konu, bilişsel bütünlük temelli etkinliklere dahil edilmiş geometrik inşa konusudur. Geometrik inşa, geometrinin keşfedilmesi hedeflendiğinde uygun bir araçtır (Kostovskii, 1961). Duval (1998) inşayı bilişsel bir süreç ve ispat için bir yol olarak tanımlamıştır. Ortaokul matematik öğretmen adayları geleceğin matematik öğretmenleri olarak görüldüklerinden geometrik inşaları ne ölçüde yapabildiklerinin araştırılması önem taşımaktadır. Alanyazın taraması pergel-çizgeç kullanımını içeren geometrik inşalar ile ilgili çalışma sayısının sınırlı olduğunu gösterdiğinden, bu çalışmanın ilgili alanyazına katkıda bulunması beklenmektedir. Ayrıca, çalışmada hem pergel-çizgeç hem de GeoGebra kullanıldığından, belirli bir geometrik şeklin inşa olarak kabul edilip edilemeyeceğini belirlemek için ileriki çalışmalarda kullanılabilecek kriterleri sunmak amaçlanmıştır. Geometride ispatlama becerisinin gelişimi açısından geometrik inşanın önemine de dikkat çekmek hedeflenmiştir.

2. Alanyazın Taraması

Alanyazın taramasında, farklı amaçlarla bilişsel bütünlük kavramı üzerine ağırlıklı olarak veya kısmen odaklanan çalışmalar olduğu görülmüştür (örneğin, Antonini ve Mariotti, 2008; Fiallo ve Gutiérrez, 2017; Garuti vd., 1998; Garuti vd., 1996; Mariotti vd., 1997; Pedemonte, 2007a, 2007b, 2008). Bunun yanında, doğrudan bilişsel bütünlük kavramı üzerine yapılanmayan fakat bu kavramı vurgulayan ve atıfta bulunan birçok çalışma olduğu görülmüştür (örneğin, Arzarello ve Sabena, 2011; Baccaglini-Frank, 2010; Conner vd., 2014b; Stylianides vd., 2016; Yan, Mason ve Hanna, 201). Daha önce de belirtildiği gibi, ispatlamaya dair bir yaklaşım önermek adına uygun koşulları belirlemek için yaklaşık yirmi yıl önce bilişsel bütünlük kavramı önerilmiştir (Boero, 2017). Bilişsel bütünlük üzerine yapılan öncü çalışmalarda, öğrencilerin varsayım üretmek için bir argümantasyon sürecine girdikten sonra ulaşılan varsayımların ispatının onlar için daha erişilebilir bir durum haline geldiği ifade edilmiştir (Boero vd., 1996; Boero vd., 2010; Garuti vd., 1996; Garuti vd., 1998). Üstelik, öğrencilerin önerme üretim aşamasındaki kişisel argümanlarının, ispatlama aşamasında aynı şekilde görüldüğü ve akıl yürütme türünü sürdürme, benzer ifadeler kullanma ve benzer adımları takip etme eğiliminde oldukları görülmüştür (Boero vd., 1996; Garuti vd., 1996; Mariotti vd., 1997).

Argümantasyon ise matematik sınıflarında sıkça görülen bir yapı olmasına rağmen matematik eğitiminde tanımlanması zor bir kavramdır (Pedemonte, 2007a). Temel anlamda, Cross (2009) argümantasyonu tartışma sırasında gerçekleştirilen eylemlere odaklanarak "matematiksel fikirlerin paylaşılması, açıklanması ve gerekçelendirilmesi" (s.908) olarak tanımlamıştır. Ayrıca, Conner vd. (2014a), kolektif argümantasyonu birden fazla kişinin çoğu zaman fikir birliği ile bir sonuca vardığı matematiksel tabanlı tartışmalar olarak tanımlamaktadır. İlgili alanyazın taramasına göre, Toulmin'in (1958, 2003) argümantasyon modeli, herhangi bir disipline uygulanacak şekilde tasarlandığından, farklı disiplinlerde çeşitli amaçlar izleyerek argümanları incelemek için kullanılmaktadır (Knipping, 2008; Knipping ve Reid, 2015). Toulmin'in (1958, 2003) temel argümantasyon modeli, her biri argümantasyon boyunca farklı bir role sahip olan veri (D), gerekçe (W) ve iddia (C) olmak üzere üç bileşeni içermektedir (Fukawa-Connelly, 2014). Toulmin (2003) farklı gerekçe türlerinin olması ve bu gerekçelerin sonuçlar üzerinde farklı etkilere sahip olabilmesi nedeniyle argümantasyon modelinde ek açıklamalara ihtiyaç duyulabileceğini belirtmiştir. Farklı argümanların özellikleri göz önüne alındığında, argümantasyon modeli daha karmaşık hale gelmektedir. Bu bağlamda, temel argümantasyon modelindeki bileşenlere ek olarak, niteleyen (Q), çürütücü (R) ve destek (B) bileşenleri modele dahil edilmiştir. Toulmin'in modelini doğrudan kullanmayan ve çalışmalarının amaçları ve bağlamı ışığında bazı değişiklikler yapan çalışmalar da vardır. Örneğin, global argümantasyon yapılarına ilişkin olarak, Reid ve Knipping bir dizi çalışma yürütmüştür (örneğin, Knipping, 2003, 2004, 2008; Knipping ve Reid, 2013, 2015, 2019; Reid ve Knipping, 2010). Toulmin'in argümantasyon modeli söz konusu çalışmaların temelini oluşturmaktadır (Reid ve Knipping, 2010).

Her ne kadar araştırmalarda ispat ve ispatla ilgili kavramların matematik eğitimindeki önemi vurgulansa da sadece farklı seviyelerdeki öğrencilerin değil matematik öğretmenlerinin de ispat içeren görevlerin yerine getirilmesinde güçlük yaşadığı altı çizilen sonuçlardandır (Ellis vd., 2012; Jones, 2002; Moore, 1994; NCTM, 2000; Reid ve Knipping, 2010; Reiss, Klieme ve Heinze, 2001). Dahası, hangi argümanların ispat olarak sayılması gerektiği, hem matematik hem de matematik eğitimi kapsamında bu terimin birçok farklı bakış açısı ve kullanımı nedeniyle, açıkça ortaya konması zor bir konudur (Stylianides, 2019; Weber ve Czocher, 2019). Örneğin, Ko ve Knuth'a (2013) göre, satır-satır analizin amacı, tüm adımların doğru olarak sunulup sunulmadığını ve önceki iddiaların mantıksal olarak takip edilip edilmediğini inceleyerek geçerli ispat sunulup sunulmadığını değerlendirmektir. Benzer şekilde, Alcock ve Weber (2005), argümanların değerlendirilmesi sürecinde satır-satır kontrol edilmesi gerekliliğinden bahsetmiştir. Weber (2008) çalışmasındaki katılımcıların ilk adım olarak argümanların genel yapısını incelediğini ve daha sonra satır-satır inceleme sürecine devam ettiğini ifade etmektedir.

Geometri matematik tarihinin başlangıcından bu yana matematiğin temel kavramlarından biri olarak sayılmıştır (Albrecht, 1952; Bayerthal, 1988; Jones, 2002; Mariotti, 1995; Stupel, Sigler ve Tal, 2018). Ayrıca, geometri okul matematiğinin temel bir alanıdır (Clements, 2003; Clements ve Battista, 1992; NCTM, 2000; Sinclair vd., 2012a, 2012b). Geometri ile ilgili olarak alanyazında bilişsel geometri süreçleri ve geometrik düşüncenin gelişim aşamaları gibi farklı yönlere odaklanan çeşitli teorik çerçeveler görülmektedir. Örneğin, Duval (1998), görselleştirme, inşa ve akıl yürütme olmak üzere geometrinin bilişsel sürecinde üç bileşenin varlığına odaklanmıştır.

Geometrik inşa sunulan verileri takip ederek ve pergel-çizgeç gibi bazı araçları kullanarak istenen geometrik figürün oluşturulduğu bir problem durumu olarak tanımlanmaktadır (Albrecht, 1952). Benzer bir şekilde, inşa sürecinde hangi araç veya araçların kullanıldığına bakılmaksızın, geometrik inşa terimi "inşa problemlerinin geçerli çözümleri" olarak da tanımlanmaktadır (Stylianides ve Stylianides, 2005, s.32). Geometrik inşanın matematik tarihi boyunca popüler kavramlardan biri olmasının yanı sıra (Karakuş, 2014; Sarhangi, 2007; Stupel, Oxman ve Sigler, 2014), günümüzdeki matematik eğitiminde de önemli bir konu olarak görülmektedir (Djorić ve Janičić, 2004; Stupel ve Ben-Chaim, 2013). Pergel ve çizgeç matematiğin tarihi boyunca geometrik inşaya ilişkin yaygın olarak kullanılan araçlar olmasına rağmen (Kuzle, 2013; Pandisico, 2002), inşa sürecinde kullanılabilecek başka araçlar da vardır (Gibb, 1982; Schreck, Mathis ve Narboux, 2012). Geometrik inşalarda Geometer's Sketchpad ve GeoGebra gibi programlar da kullanılmaktadır (Arıcı, 2012). Her ne kadar farklı çeşit dinamik geometri programları olsa da, hepsi Öklid geometrisini modellemek ve inşalarda geometriyi desteklemek için tasarlanmıştır (Hoyles ve Noss, 2003). Bilişsel bütünlük temelli etkinliklerde yer verilen geometrik inşa sürecinde farklı araçların kullanımının argümantasyon ve ispat üzerindeki olası etkileri göz önüne alınarak, bu çalışmada hem pergel-çizgeç hem de GeoGebra kullanılmıştır.

3. Yöntem

3.1 Araştırma Deseni

Bu çalışmanın amacı, ortaokul matematik öğretmeni adaylarının bilişsel bütünlük temelli etkinliklerdeki uygulamalarını özellikle argümantasyon, ispat ve geometrik inşa gibi kavramlar açısından derinlemesine araştırmak olduğu için araştırma deseni olarak durum çalışması belirlenmiştir. Alanyazında durumun büyüklüğü, araştırmacının ilgisi ve bu tasarımı kullanma amacı gibi bazı kriterlere dayanan farklı durum çalışması sınıflamaları bulunmaktadır (Creswell, 2007; Merriam, 2009). Yin'in (2014) sınıflamasına göre bu çalışmada iç içe geçmiş çoklu durum deseni (multiple-case embedded design) kullanılmıştır.

3.2. Çalışmanın Bağlamı ve Katılımcılar

Bu çalışma ortaokul matematik öğretmen adayları ile gerçekleştirilmiştir. Bu durumda, katılımcılar Eğitim Fakültesinin Matematik ve Fen Bilimleri Eğitimi bölümü altında yer alan İlköğretim Matematik Öğretmenliği programına kayıtlıdır. Söz konusu program dört yıllık bir lisans programıdır ve katılımcıların kayıtlı olduğu İlköğretim Matematik Öğretmenliği programındaki zorunlu ve seçmeli derslerin listesi Ek A'da verilmiştir.

Katılımcıları belirlemek için, araştırmacıların derinlemesine bir inceleme yapmak hedefiyle gerekli ve zengin verileri sağlama potansiyeline sahip katılımcıları seçtikleri amaçlı örnekleme kullanılmıştır (Creswell, 2007, 2012; Frankel ve Wallen, 2005; Merriam, 2009; Patton, 2002). Katılımcıları belirleme sürecinde odaklanılan ilk kriter, derinlemesine bir araştırma yapılabilmesi için veri toplama sürecinde katılımcılarla bolca zaman geçirilmesi göz önünde bulundurularak, araştırmacı için erişilebilirlik olmuştur. Bu nedenle, Ankara'da bir devlet üniversitesindeki ortaokul matematik öğretmen adayları seçilmiştir. İkinci kriter kapsamında, çalışma bazında detaylı bilgi toplama potansiyeline sahip olmaları beklendiğinden 4. sınıf ortaokul matematik öğretmen adayları seçilmiştir. Ancak, 4. sınıf öğretmen adayları ile yapılan pilot çalışma bu kriteri değiştirmiştir. 4. sınıf öğretmen adaylarının pilot çalışmanın yapılacağı dönemde sadece iki ders alması nedeniyle üniversitede az vakit geçirmeleri, buna istinaden etkinlikleri uygulama saati belirlemede yaşanan zorluklar, mezun olduklarında girecekleri Kamu Personeli Seçme Sınavı için kursa gitmeleri ve yoğun olarak bu sınava hazırlanmaları gibi durumlar nedeniyle sınıf seviyesinin değiştirilmesi gündeme gelmiştir. Pilot çalışmanın analizinden sonra, danışmanlar ve tez izleme komitesi ile görüşerek 4. sınıfların katılımı ile ilgili kriter değiştirilmiş ve 3. sınıf ortaokul matematik öğretmen adaylarının ana uygulamanın katılımcıları olmasına karar verilmiştir. Ayrıca, ana uygulama için bilişsel bütünlük temelli etkinlikleri seçmeli bir derse dahil etmeye ve haftada bir etkinlik uygulanmasına karar verilmiştir. Programdaki seçmeli dersler arasında yer alan Geometri Öğretimi dersinin çalışmanın amacına en uygun ders olacağı belirlenmiştir.

Geometri Öğretimi dersini alacak 14 3. sınıf ortaokul matematik öğretmeni adayı, ilgili derslerinin notları, ortalamaları ve gönüllü olma durumları dikkate alınarak belirlenmiştir. Geometri Öğretimi dersi pergel-çizgeç ve GeoGebra kullanma ayrımına göre iki şube olarak açılmıştır ve her şubede 3 ve 4 kişilik olmak üzere 2 grup oluşturulmuştur. Çalışmanın başlangıcında tüm gruplardan gelen verilerin incelenmesi amaçlanmasına rağmen, analiz sırasında tüm verilerin analizini sunmanın mümkün olmadığı görülmüştür. Bu nedenle, bu çalışma kapsamında üç öğretmen adayını içeren gruplardan, yani pergel-çizgeç grubundan (PÇG) ve GeoGebra grubundan (GG) edinilen bulguların sunulmasına karar verilmiştir. Sonuç olarak, katılımcılar Ankara'daki bir devlet üniversitesinde öğrenim gören 6 3. sınıf ortaokul matematik öğretmen adayı olarak belirlenmiştir.

3.3. Veri Toplama Süreci

Veri toplama sürecinin ilk adımı etkinliklerin hazırlanması olup, uzmanların etkinlikler ile ilgili görüşlerini aldıktan ve etkinlikleri önerilere ve düzeltmelere göre düzenledikten sonra pilot çalışma yapılmıştır. Pilot çalışma, 2015-2016 eğitim-öğretim yılının bahar döneminde Ankara'daki bir devlet üniversitesinde öğrenim gören ortaokul matematik öğretmen adayları ile yürütülmüştür. Pilot çalışmanın katılımcıları elverişli örneklem yöntemiyle belirlenmiştir. Pilot çalışmanın analizini ve ayrıca çalışmanın amaçlarını dikkate alarak ana uygulamanın veri toplama süreci planlanmıştır. Ana uygulamadaki Geometri Öğretimi dersi bir tanışma haftası, dört öğretim oturumu haftası ve dördü bilişsel bütünlük temelli etkinlikler olan sekiz etkinlik haftasından oluşmaktadır. Ayrıca, ders süresince araştırmacı etkinliklerin uygulanması sırasında ve sonrasında alan notları almıştır. Ana uygulamada toplanan verilerin ön analizinden sonra bilişsel bütünlük temelli etkinlikler üzerine odak grup görüşmeleri yapılmıştır. Böylece, bu çalışmadaki veri toplama süreci tamamlanmıştır.

Bu çalışmada veriler bilişsel bütünlük temelli etkinlikler sırasındaki grupların ses ve video kayıtları, grupların dokümanları ve her etkinliğin sonunda teslim edilen GeoGebra dosyaları, alan notları ve odak grup görüşmeleri yoluyla toplanmıştır. Belirtildiği gibi, çalışmanın verileri Ek B'de sunulan dört bilişsel bütünlük temelli etkinlik aracılığıyla toplanmıştır. Bu etkinlikler üçgen ve çember üzerine hazırlanmıştır ve ikili yapısından dolayı iki çalışma sayfasını kapsamaktadır. Birinci bölümde, katılımcıların geometrik inşalar aracılığıyla varsayımlar oluşturmaları beklenmiştir. İkinci bölümde ise, katılımcıların oluşturdukları varsayımlardan birini ispatlamaları istenmiştir.

3.4. Veri Analizi

Bahsi geçen kaynaklar aracılığıyla veriler toplandıktan sonra, bilişsel bütünlük temelli etkinliklerinden elde edilen tüm video ve ses kayıtları ve bu etkinliklerle ilgili olarak yapılan odak grup görüşmeleri dikkatlice transkript edilmiş ve analiz için hazırlanmıştır.

3.4.1. Birinci Araştırma Sorusunun Veri Analizi

Birinci araştırma sorusunu cevaplamak için nitel analiz yazılımlarından biri olan MAXQDA kodlama sürecinde kullanılmıştır. Elo ve Kyngäs'in (2008) belirttiği gibi, içerik analizi hem nicel hem de nitel verilerle kullanılabilir. Dahası, içerik analizi tümdengelim yaklaşımını veya tümevarım yaklaşımını kapsayabilir (Cho ve Lee, 2014; Elo ve Kyngäs, 2008). Alanyazındaki çalışmalar odaklanılan olgu hakkında yeterli bilgi sağlayamadığı ve toplanan verilerden kodların çıkarılması gerektiğinde, tümevarım yaklaşımından yararlanılabilir (Elo ve Kyngäs, 2008). Çalışmanın birinci araştırma sorusunu araştırmak için yapılacak olan veri analizi bu duruma uygundur. Bu amaçla, "açık kodlama, kategori oluşturma ve soyutlama" olacak şekilde tümevarım içerik analizi aşamaları takip edilmiştir (Elo ve Kyngäs, 2008, s.109). Her bilişsel bütünlük temelli etkinlikte grupların hem varsayım üretirken hem de ispat sürecindeki argümantasyon süreci ayrıntılı bir şekilde incelenmiş ve karşılaştırılmıştır, daha sonra ilgili kodlar ve temalar düzenlenmiştir.

3.4.2. İkinci Araştırma Sorusunun Veri Analizi

İkinci araştırma sorusunun analizinde bazı çalışmalarda belirtilen teorik çerçeveler çeşitli adaptasyonlar yapılarak kullanılmıştır. Daha detaylı olarak, verilerin analizi sırasında Knipping (2008) tarafından önerilen üç-aşamalı süreç takip edilmiştir. Ayrıca, global argümantasyon yapılarını oluştururken, Verheij'in (2005) çalışması çürüten bileşeninin şematik gösterimi ile ilgili olarak kullanılmıştır. Ancak, hem pilot çalışmadan hem de ana uygulamadan elde edilen verilerin analizi sırasında, çalışmanın verileri ile ilgili teorik çerçeveler arasında bazı boşluklar ve tutarsızlıklar olduğu fark edilmiştir. Bu nedenle, bu çalışmada argümanların düzenlenmesiyle ilgili olarak, etkinliklerin bağlamı, doğası ve içerdiği sorular, gruplardaki katılımcı sayısı, eğitmenin rolü ve argümantasyon sırasında yapılan itirazların doğası dikkate alınarak bazı uyarlamalar yapılmıştır.

Bu çalışmada, global argümantasyon yapılarını isimlendirirken ilk adım olarak alanyazındaki çalışmalarda ifade edilen tüm argümantasyon yapı çeşitlerinin özellikleri incelenmiş ve bu çalışmalarda verilen argümantasyon sürecinden örnekler ve alıntılar ayrıntılı bir şekilde incelenmiştir. Alanyazın taramasında kaynak-yapı (source-structure), spiral-yapı (spiral-structure), rezervuar-yapı (reservoir-structure), toplanma-yapı (gathering-structure), çizgi-yapı (line-structure) ve bağımsız argümanlar-yapı (independent arguments-structure) olmak üzere altı çeşit global argümantayon yapısına ulaşılmıştır (Erkek, 2017; Erkek ve Işıksal-Bostan, 2019; Knipping 2003, 2004, 2008; Knipping ve Reid, 2013, 2015, 2019; Reid ve Knipping, 2010). Mevcut çalışmada ortaya çıkan argümantasyan yapıları ve alanyazındaki altı çeşit global argümantasyon yapısı karşılaştırıldığında, mevcut çeşitlerin uygulanabilirliği açısından bazı tartışmalı noktaların olduğu görülmüştür. Bu nedenle, mevcut yapı çeşitleriyle ilgili olarak bazı düzenlemelere ihtiyaç duyulmuştur ve çalışmada ortaya çıkan global argümantasyon yapılarını sınıflandırabilmek için bazı yeni çeşit yapılar tanımlanmıştır. Genel olarak, bu çalışmada oluşturulan global argümantasyon yapıları mono yapılar ve hibrit yapılar olmak üzere iki başlık altında toplanmıştır. Daha ayrıntılı olarak, çalışmada global argümantasyon yapılarının özelliklerinin analizinden sonra, çizgi-yapı (line-structure), rezervuar-yapı (reservoir-structure), kanallaşma-yapı (funneling-structure) ve dallanan-yapı (branching-structure) olmak üzere dört çeşit yapı düzenlenmiştir. Söz konusu dört çeşit mono yapıyı birleştirerek, hibrit yapılar altında sınıflandırılan farklı global argümantasyon yapıları üretilebilir. Örneğin, rezervuar-kanallaşma-yapı (reservoir-funneling-structure), çizgi-dallanan-yapı (line-branching-structure) ve çizgi-rezervuar-dallanan-yapı (line-reservoir-branching-structure) olmak üzere üç hibrit yapı çalışmanın bulgular bölümünde sunulmuştur.

İkinci araştırma sorusunun son alt sorusunu ele alırken temel alınan çalışma Verheij (2005) tarafından yürütülen çalışmadır. Çürütenlerin hedef aldığı bileşenler ya da durumlar, Verheij'in (2005) çalışmasındakine benzer şekilde bir ok kullanılarak belirtilmiştir. Çürütenlerin bulunduğu durumlar listelenmiş ve bu çalışmada çürüten bileşenin sekiz işlevi olduğu belirlenmiştir.

3.4.3. Üçüncü Araştırma Sorusunun Veri Analizi

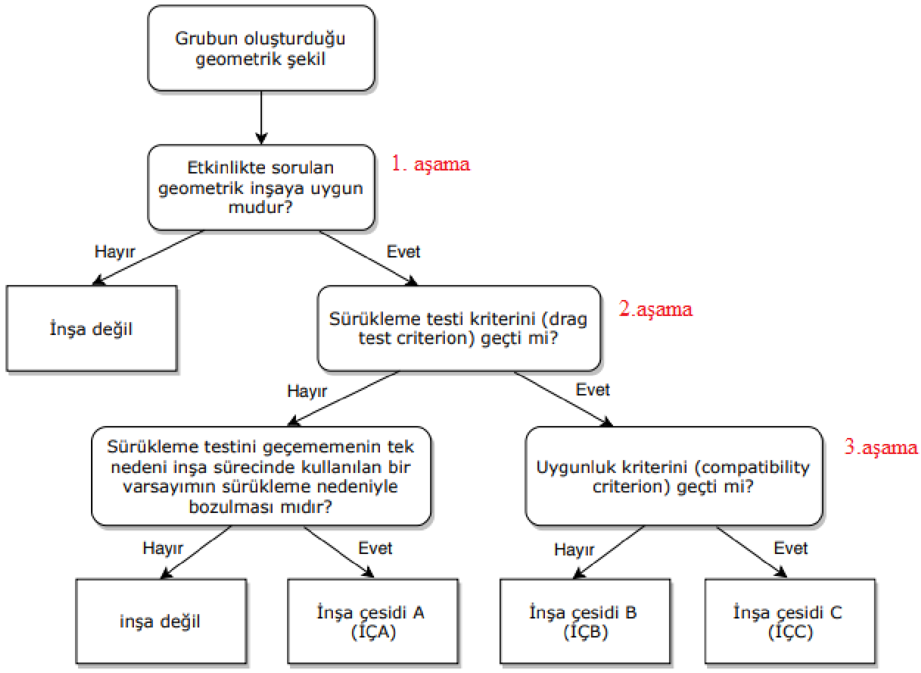
Üçüncü araştırma sorusunun analizinde, gruplar tarafından etkinliklerde sorulan geometrik inşaları gerçekleştirme yolu olarak ifade edilen yaklaşımlar tablolarla listelenmiştir. Bu adımdan sonra, grupların yaklaşımlarının geçerliğini nasıl yorumladıklarına etkinliklerin video kayıtları ve odak grup görüşmeleri aracılığıyla odaklanılmıştır. Pergel-çizgeç ve GeoGebra'yı kullanarak oluşturulan geometrik şekillerin inşa olarak kabul edilmesi için ele alınan kriterlerin farklı olması sebebiyle, gruplardan elde edilen verilerin analiz süreçleri ayrı ayrı açıklanmıştır. PÇG tarafından sunulan geometrik şekiller, Tablo 3.1'de sunulan kriterlerin yerine getirilmesi durumunda inşa olarak kabul edilmiştir. Aşağıdaki kriterler listesi araştırmacı tarafından ilgili alanyazın taraması ve uzmanlar tarafından önerilen düzeltmeler sonucu oluşturulmuştur.

Tablo 3. 1

PÇG için kullanılan geometrik inşa ile ilgili kriter listesi

Kriterler	
K1	PÇG tarafından sunulan geometrik şekil, etkinlikte istenen inşaya uygundur
K2	Geometrik şekil sadece pergel ve çizgeç kullanılarak inşa edildi
K3	Pergel doğru kullanıldı
K4	Çizgeç doğru kullanıldı
K5	İnşa sürecindeki çıkarımlar matematiksel olarak doğrudur
K6	İnşa sürecindeki açıklamalar matematiksel olarak doğrudur

GG tarafından sunulan geometrik şekiller ise aşağıdaki diyagrama göre değerlendirilmiştir.



Şekil 3.1. GG için kullanılan geometrik inşa ile ilgili diyagram

Şekil 3.1'de belirtildiği gibi, diyagram GG tarafından sunulan yaklaşımın geçerli olup olmadığını belirlemek için kullanılan soruları kapsayan üç aşamayı içermektedir. Aynı zamanda, diyagram GG tarafından sunulan geometrik şekillerin bir inşa olarak ifade edilip edilemeyeceğini belirlemek için de kullanılabilir.

3.4.4. Dördüncü Araştırma Sorusunun Veri Analizi

Genel anlamda, bu çalışmada grupların argümanları eğer matematiksel ve mantıksal kuralları doğru uygularlarsa ve istenen sonuca ulaşırlarsa ispat olarak kabul edilmiştir. Daha ayrıntılı olarak açıklamak gerekirse, grupların argümanlarını analiz ederken, iki adım kullanılmıştır. İlk adım olarak, Bleiler vd. (2014), Ko ve Knuth (2013) ve Weber (2008) tarafından yürütülen çalışmalara benzer şekilde, argümanın genel yapısı incelenmiştir. Ardından, ikinci adım olarak satır-satır analiz yapılmıştır. Bilişsel bütünlük temelli etkinlikler sonunda PÇG ve GG'nin argümanları gerekçe hatalı geçersiz argüman, yapısal hatalı geçersiz argüman ve geçerli ispat olmak üzere üç kategoride sınıflandırılmıştır. Ayrıca, bir sonraki ifadenin doğruluğunu etkilemeyen ya da geçerlilik açısından argümanın yapısını etkilemeyen durumlar ikincil hatalar olarak adlandırılmıştır (Selden ve Selden, 2003).

4. Sonuçlar ve Tartışma

Çalışmanın sonuçları bir araştırma sorusunu ele alacak şekilde dört ana bölüm altında açıklanmıştır. Birinci araştırma sorusuna cevap vermeyi amaçlayan ilk bölüm, ortaokul matematik öğretmen adaylarının varsayım oluşturma sürecindeki argümantasyonlarının ispatlama süreci ile nasıl ilişkili olduğuna dair sonuçları içermektedir. İkinci araştırma sorusu ve alt soruları ile ilgili olan ikinci bölüm, bilişsel bütünlük temelli etkinliklerin varsayım üretme sürecinde ortaya çıkan global argümantasyon yapılarını, bu global argümantasyon yapılarının bileşenlerini ve çürüten bileşeninin işlevlerini sunmaktadır. Üçüncü bölümde, bilişsel bütünlük temelli etkinliklerde yer alan geometrik inşa kavramı daha yakından incelenmiştir. Bir başka deyişle, pergel-çizgeç grubu (PÇG) ile GeoGebra grubunun (GG) geometrik inşalar için önerdikleri yaklaşımlar ve geçerli inşalar yapıp yapamadıkları incelenmiştir. Bu sonuçlar, üçüncü araştırma sorusu ve alt soruları açıklamaktadır. Dördüncü araştırma sorusuna cevap vermeyi amaçlayan son bölümde ise, gruplar tarafından üretilen varsayımlar ve bu varsayımlardan bazıları için geçerli ispat yapıp yapamadıkları açıklanmıştır.

4.1. Varsayım Oluşturma Sürecindeki Argümantasyonun İspatlamaya Bağıntısı

Bilişsel bütünlük kavramı bu çalışmanın başlangıç noktası sayıldığından, ilk araştırma sorusu özellikle bilişsel bütünlük boyutunun araştırılmasını amaçlamaktadır. Fiallo ve Gutiérrez (2017) öğrencilerin varsayım aşamasındaki argümantasyonun ispat oluşturmasına yardımcı olduğu zaman bilişsel bütünlük olduğunu ifade etmiştir. Yalnızca argümantasyonun ispata olası yardımına odaklanmadan, ilk araştırma sorusu grupların geometrik inşalar aracılığıyla varsayım oluşturdukları argümantasyon sürecinin varsayımların ispatlanması süreciyle nasıl ilişkili olduğunu incelemeyi amaçlamaktadır. Analiz sonrasında, Tablo 4.1'de gösterildiği üzere, geometri bağlamında ortaokul matematik öğretmen adaylarının varsayım oluşturma sürecindeki argümantasyanları ispatlarla hangi açıdan ilişkilendirildiği belirlenmiştir. Ayrıca, geometrik inşalarda farklı araçların kullanılmasının analiz sırasında düzenlenen kod ve temalarda belirgin bir farklılık göstermediği de görülmüştür. Bu nedenle, bulgular çalışmadaki gruplara ayrı ayrı odaklanılmadan rapor edilmiştir.

Tablo 4.1

Argümantasyonun ispata ilişkilendirildiği yönler

Yönler	**Kodlar**
İspatlamaya başlamadan önce argümantasyon sürecinin olumlu yönleri	Olumlu duyuşsal durumlar Etkinliğin içeriği ile ilgili bilginin düzenlenmesi Görselleştirme İspatı sorulan önermenin doğruluğu
İspatlamaya başlamadan önce argümantasyon sürecinin olumsuz yönleri	Olumsuz duyuşsal durumlar Varsayım oluşturma ve ispatlama arasındaki farka dair karışıklık

Tablo 4.1'de görüldüğü üzere, ortaokul matematik öğretmen adaylarının varsayım oluşturma sürecindeki argümantasyonlarının hem olumlu hem de olumsuz açıdan ispatla ilgili olduğu belirlenmiştir. İspatlamaya başlamadan önce bir argümantasyon sürecine dahil olmanın olumlu yönleri, olumlu duyuşsal durumlar, etkinliğin içeriği ile ilgili bilginin düzenlenmesi, görselleştirme ve ispatı sorulan önermenin doğruluğu olmak üzere dört kod kapsamında incelenmiştir. Diğer yandan, ispatlamaya başlamadan önce bir argümantasyon sürecine katılmanın olumsuz yönleri, olumsuz duyuşsal durumlar ve varsayım oluşturma ve ispatlama arasındaki farka dair karışıklık olmak üzere iki kodla incelenmiştir.

Bilişsel bütünlük temelli etkinliklerde, grupların argümantasyon süreci varsayımları, yani etkinlikte neyin ispatlanacağını belirlemektedir. Bu durumda, katılımcılar ispatlamaları istenen önermeleri keşfederek bulabildiklerinden dolayı bu önermelerin ispatlarını da yapma konusunda yeterli hissediyor olabilirler. Diğer yandan, katılımcıların varsayım oluşturma sürecinde zorlanmaları ve sıkılmaları durumunda, önermeyi bulmuş olsalar bile ispatlamak için çaba harcamak istemeyebilirler. Ayrıca, ispat kendi başına zor ve sıkıcı bir görev olduğu için katılımcıların argümantasyon sürecindeki duyuşsal durumları kritik bir öneme sahiptir. Ayrıca, katılımcıların gerekli bilgilerini ispat öncesinde aktifleştirmelerinin ispata hazırlanma açısından oldukça etkili bir adım olduğu söylenebilir. Benzer şekilde, Hewitt (2005) alanyazındaki çalışmaların bir argümantasyona katılmanın öğrencilerin öğrenimini artırdığına dair kanıtlar sunduğunu belirtmiştir. Bir varsayım süreci olmadığında, öğrencilerin geçerli argümanlar sunmak için daha fazla çaba göstermeleri gerekebilir (Antonini ve Mariotti, 2008).

4.2. Global Argumantasyon Yapıları

Bu bölümde, ikinci araştırma sorusu ve alt soruları kapsamında ulaşılan sonuçlar açıklanmıştır. Grupların bilişsel bütünlük temelli etkinliklerde varsayımlar oluştururken global argümantasyon yapıları incelenmiştir. Ek olarak, bu global argümantasyon yapılarında ortaya çıkan bileşenler ve çürüten bileşeninin işlevlerine de odaklanılmıştır. Bilişsel bütünlük perspektifinden bakıldığında, varsayım üretirken gerçekleştirilen argümantasyon süreci, ispatını yapabilmek adına kritik bir öneme sahiptir (Baccaglini-Frank, 2010; Boero vd., 1996; Fiallo ve Gutiérrez, 2017; Garuti vd., 1996; Pedemonte ve Buchbinder, 2011). Bu çalışmada, varsayımların oluşturulduğu kritik argümantasyon süreci, bilişsel bütünlük temelli etkinliklerin geometrik inşa bölümleri sırasındaki grup tartışmalarına tekabül etmektedir. Bu bağlamda, ortaokul matematik öğretmen adaylarının varsayım oluşturdukları argümantasyon süreci ayrıntılı bir şekilde incelenmiş ve ikinci araştırma sorusuna cevap vermek için global argümantasyon yapıları oluşturulmuştur.

Bu amaç doğrultusunda ilk olarak, her etkinlik için PÇG ve GG'nin global argümantasyon yapıları oluşturulmuştur. Grup başına her etkinlikteki varsayım oluşturma tartışmasından bir global argümantasyon yapısı oluşturulduğundan, PÇG ve GG olmak üzere iki gruba odaklanıldığından ve dört etkinlik yapıldığından, toplanan verilere dayanarak toplamda sekiz global argümantasyon yapısı düzenlenmiştir. Bu yapıların oluşumundan sonra, alanyazındaki mevcut global argümantasyon yapı çeşitlerinin uygunluğu bu çalışmada oluşan yapıları isimlendirmek adına incelenmiştir. Daha önce belirtildiği gibi, kaynak-yapı (source-structure), spiral-yapı (spiral-structure), rezervuar-yapı (reservoir-structure), toplanan-yapı (gathering-structure), çizgi-yapı (line-structure) ve bağımsız argümanlar-yapısı (independent arguments-structure) olmak üzere altı tür global argümantasyon yapısına alanyazında rastlanmıştır. Bu çalışmada, kaynak-yapı, rezervuar-yapı ve çizgi-yapı bazı uyarlamalar yapılarak kullanılmıştır. Daha detaylı olarak, Reid ve Knipping (2010) tarafından belirtilen kaynak-yapının ve rezervuar-yapının ve ayrıca Erkek (2017) tarafından belirtilen çizgi-yapının, bu çalışmada ortaya çıkan global argümantasyon yapılarını kategorize etmek için doğrudan uygun olmadığı belirlenmiştir. Bu nedenle, rezervuar-yapının ve çizgi-yapının bazı özellikleri gözden geçirilmiş ve bu iki yapının bazı temel ve görsel özelliklerinin ağırlıklı olarak üzerinde durulmuştur. Rezervuar-yapı ve çizgi-yapı terimleri doğrudan kullanılmış, kaynak-yapının özelliklerinin uygun olmamasına rağmen, içerdiği bir özelliğin bu çalışmada da kullanılabilir olduğu görülmüştür. Yani, kanallaşma etkisine dair kaynak-yapının görsel bir özelliğine odaklanılmıştır. Bu nedenle, kaynak-yapı terimini kullanmak yerine, kanallaşma-yapı (funneling-structure) olarak yeni bir yapı türü önerilmiştir. Mevcut çalışmada sunulan bir başka yeni global argümantasyon yapısı türü de dallanan-yapı (branching-structure) olarak adlandırılmıştır.

Her etkinlikte PÇG ve GG'nin tartışmalarından çıkan global argümantasyon yapılarının türleri aşağıdaki tabloda sunulmuştur.

Tablo 4.2

Global argümantasyon yapılarını çeşitleri

Global argümantasyon yapılarının çeşitleri		PÇG	GG
Mono yapılar	Rezervuar-yapı	Etkinlik 3	-
	Kanallaşma-yapı	Etkinlik 2	Etkinlik 2
	Dallanan-yapı	Etkinlik 4	Etkinlik 1
Hibrit yapılar	Rezervuar-kanallaşma-yapı	-	Etkinlik 3
	Çizgi-dallanan-yapı	Etkinlik 1	-
	Çizgi-rezervuar-dallanan-yapı	-	Etkinlik 4

Analiz sonucu dört çeşit global argümantasyon yapısı oluşturulmasına rağmen, sadece çizgi-yapı altında kategorize edilebilecek bir global argümantasyon yapısı gözlenmemiştir. Çizgi-yapı hibrit yapıların bir parçası olarak yer almaktadır. Bu bağlamda, Tablo 4.2'de görüldüğü gibi, çalışma kapsamında rezervuar-yapı, kanallaşma-yapı ve dallanan-yapı olmak üzere üç çeşit mono yapı görülmüştür. Dört çeşit mono yapı olması sebebiyle bu yapıların kombinasyonu olan on bir hibrit yapının ortaya çıkabileceği öngörülmektedir. Fakat Tablo 4.2'de görüldüğü gibi, bu çalışmada rezervuar-kanallaşma-yapı, çizgi-dallanan-yapı ve çizgi-rezervuar-dallanan-yapı olmak üzere üç çeşit hibrit yapı ortaya çıkmıştır. Sonuçta, PÇG'nun dört global argümantasyon yapısının üçü mono yapıda iken, GG'nin dört global argümantasyon yapısından ikisinin mono yapıda olduğu görülmüştür. Bu durumda, global argümantasyon yapı çeşidinin ne gruba ne de etkiniğin içeriğine doğrudan bağlılığı olmadığı söylenebilir. Bunun sebebi olarak, argümantasyonun temel faktörünün GeoGebra kullanmak gibi grupların özelliklerinden ziyade gruptaki tartışmacıların bireysel özellikleri gösterilebilir. Ayrıca, çizgi-yapının, mono yapılarda ortaya çıkmamasının sebebi olarak, bilişsel bütünlük temelli etkinliklerdeki argümantasyon sürecinin üç öğretmen adayının katılımıyla en az bir buçuk saat süren, kapsamlı bir yapıya sahip olması düşünülebilir.

Daha önce de belirtildiği gibi, ikinci araştırma sorusunun ilk alt sorusu kapsamında çalışmada ortaya çıkan global argümantasyon yapıları bileşenleri açısından incelenmiştir. Sonuç olarak veri (data), gerekçe (warrant), sonuç (conclusion), destek (backing), çürüten (rebuttal), niteleyen (qualifier), sonuç/veri (conclusion/data), hedef sonuç (target conclusion), rehber (guidance), meydan okuma (challenger) ve itiraz (objection) olan on bir bileşeni içerdiği görülmüştür.

İkinci araştırma sorusunun son alt sorusuna cevap verebilmek için, ortaokul matematik öğretmen adaylarının global argümantasyon yapılarında yer alan çürütenlerin işlevleri detaylıca incelenmiştir. Analiz sırasında, dört bilişsel bütünlük temelli etkinlikten ortaya çıkan sekiz global argümantasyon yapısındaki tüm çürüten bileşenleri işaretlenmiş ve neyi çürütmek için ifade edildiklerine göre sınıflandırılmıştır. Mevcut çalışmada ortaya çıkan çürütenlerin işlevleri, aşağıda verilen Tablo 4.3'te özetlenmiştir.

Tablo 4.3

Global argümantasyon yapılarındaki çürütenlerin işlevleri

Çürütenlerin işlevleri		Çürüten sayıları
İ1	Gerekçeyi çürütmek (W)	32 çürüten (PÇG 12- GG 20)
İ2	Veri ve sonuç arasındaki bağlantıyı çürütmek (D→C)	10 çürüten (PÇG 4- GG 6)
İ3	Sonuç/veriyi çürütmek (C/D)	9 çürüten (PÇG 4- GG 5)
İ4	Veriyi çürütmek (D)	8 çürüten (PÇG 2- GG 6)
İ5	Desteği çürütmek (B)	5 çürüten (PÇG 2- GG 3)
İ6	Sonucu çürütmek (C)	5 çürüten (PÇG 4- GG 1)
İ7	Meydan okumayı çürütmek (CH)	2 çürüten (PÇG yok- GG 2)
İ8	Hedef sonucu çürütmek (TC)	1 çürüten (PÇG 1- GG yok)

Tablo 4.3'te görülebildiği gibi, çürütenlerin işlevleri en sık görülenden en aza doğru listelenmiştir. Bu nedenle, ortaya çıkan çürütücü sayısına bağlı olarak, Tablo 4.3'teki ilk işlev, 32 kez görülmesi nedeniyle gerekçeyi çürütmek olmuştur. Sonuç olarak, argümantasyonda daha fazla kavram ya da durum dikkate alındığında, tartışmayı çürütmek için daha fazla olası durum ortaya çıkmaktadır. Bu sonuç, katılımcıların eğitmen tarafından tüm fikirlerini tartışmak için motive edilmesinden ve gruptakilerin birbirini iyi tanıdıkları için itirazlarını rahatça ifade ediyor olmalarından kaynaklanmış olabilir.

4.3. Geometrik İnşalar için Önerilen Yaklaşımlar

Üçüncü araştırma sorusu ve alt sorular doğrultusunda, bu bölümde ele alınan asıl konu, ortaokul matematik öğretmen adaylarının bilişsel bütünlük temelli etkinliklerde geometrik inşaları gerçekleştirmek için önerdiği yaklaşımlardır. Daha ayrıntılı olarak, bu konu üç açıdan ele alınmıştır. Çalışma sayfalarındaki ve GeoGebra dosyalarındaki açıklamaların ve çizimlerin eşlik ettiği yaklaşımların detayları, ortaokul matematik öğretmen adaylarının yaklaşımlarının geçerliği hakkındaki son yorumları ve geometrik inşaların ne derece doğru yapıldığı analiz edilmiştir. Global argümantasyon yapılarında yaklaşımların yerlerini görmek ve bu yaklaşımları argümantasyonun akışına göre yorumlamak için, yaklaşımlar ve global argümantasyon yapıları arasında daha kapsamlı karşılaştırmalar yapılmıştır. Ardından, her bir yaklaşımın global argümantasyon yapısındaki yeri işaretlenmiştir. Grupların etkinliklerde sorulan inşa açısından geçersiz olarak belirttiği yaklaşımlar global argümantasyon yapılarında mavi göstergelerle, geçerli olarak belirttikleri yaklaşımlar ise kırmızı göstergelerle işaretlenmiştir.

Bilişsel bütünlük temelli etkinliklerde PÇG tarafından önerilen yaklaşımlar Tablo 4.4'te özetlenmiştir.

Tablo 4.4

Geometrik inşalar için PÇG'nin önerdiği yaklaşımlar

	PÇG'nin geçersiz olarak ifade ettiği yaklaşımlar	PÇG'nin geçerli olarak ifade ettiği yaklaşımlar
Etkinlik 1	Y1, Y3	Y2, Y4
Etkinlik 2	Y1, Y3	Y2, Y4, Y5
Etkinlik 3	Y1, Y2	Y3, Y4, Y5
Etkinlik 4	-	Y1

Tablo 4.4'te belirtildiği üzere, PÇG her bilişsel bütünlük temelli etkinlik için en az bir geçerli yaklaşım önermiştir. Bu yaklaşımlar sonucu oluşturulan geometrik şeklin bir inşa olarak kabul edilip edilemeyeceği Tablo 3.1'de verilen geometrik inşa ile ilgili kriter listesini kullanarak değerlendirilmiştir. PÇG'nin yaklaşımların geçerliği ile ilgili değerlendirmelerinin, kriter listesine göre yapılan değerlendirmeden sonra elde edilen sonuçlarla tutarlı olduğu görülmüştür. Yani, PÇG'nin geçersiz olarak nitelendirdiği yaklaşımlar kriter listesine göre yapılan değerlendirme sonunda geçersiz bulunmuştur. Aynı şekilde, PÇG'nin geçerli yaklaşım ve oluşan geometrik şeklin inşa olduğunu ifade ettikleri durumlar kriter listesine göre yapılan değerlendirmede geçerli yaklaşım ve geometrik inşa şeklinde bulunmuştur. Sonuç olarak, yaklaşımların geçerliğinin belirlenmesinde PÇG'nin başarılı olduğu söylenebilmektedir.

Bilişsel bütünlük temelli etkinlikler boyunca GG'nin sunduğu yaklaşımlar Tablo 4.5'te özetlenmiştir.

Tablo 4.5

Geometrik inşalar için GG'nin önerdiği yaklaşımlar

	GG'nin geçersiz olarak ifade ettiği yaklaşımlar	GG'nin geçerli olarak ifade ettiği yaklaşımlar	Şekil 3.1'deki diyagrama göre yaklaşımlar
Etkinlik 1	Y1, Y2, Y3, Y4, Y5, Y7	Y6, Y8	Y6- İÇB Y8- İÇA
Etkinlik 2	Y3	Y1a, Y1b, Y2a, Y2b, Y2c	Y1a, Y1b, Y2b- İnşa değil Y2a, Y2c- İÇB
Etkinlik 3	-	Y1, Y2, Y3, Y4, Y5, Y6	Y1, Y2, Y3- İÇB Y4, Y5- İÇA Y6- İnşa değil
Etkinlik 4	Y1, Y2, Y3, Y4, Y5, Y7, Y8, Y9, Y10, Y11, Y12, Y13	Y6	Y6- İÇB

GG'nin geçersiz olarak belirttiği yaklaşımların diyagrama dayanarak yapılan değerlendirmeye göre geçersiz olduğu tespit edilmiştir. Fakat GG'nin geçerli olduğunu belirttiği yaklaşımlar için bazı tutarsızlıklar görülmüştür. Tablo 4.5'te en sağdaki sütunda, diyagrama dayanarak yapılan değerlendirmeden elde edilen sonuçlar sunulmuştur. Buradan GG'nin geçerli olarak ifade etmesine rağmen geçersiz olduğu belirlenen yaklaşımlar görülebilir. Ayrıca, diyagramın sonunda ulaşılan inşa çeşitleri de Tablo 4.5'in en sağ sütununda belirtilmiştir. Geçerli yaklaşımların uygulanmasıyla düzenlenen geometrik şekillerin çoğunluğu inşa çeşidi B (İÇB) olarak kategorize edilirken, birkaçı inşa çeşidi A (İÇA) olarak kategorize edilmiştir. Bu çalışmada inşa çeşidi C (İÇC) olarak kodlanabilecek geometrik bir şekil bulunmamaktadır. Fakat, Stylianides ve Stylianides (2005) tarafından verilen açıortay inşa örneği ile ilgili örneklerden birinin İÇC'ye uygun olduğu görülmüştür. Söz konusu geometrik şekil hem sürükleme testi kriterini hem de uygunluk kriterini geçtiğinden, bu çalışmada sunulan diyagramda İÇC olarak kodlanabilir. Diğer bir konu ise, kısıtlamalı GeoGebra dosyalarında çalışırken GG'nin pergel-çizgeç kullansalardı amaçlanan geometrik şekli nasıl inşa edeceklerini düşünme eğiliminde oldukları görülmüştür. Ayrıca, PÇG'nin

Etkinlik 2 dışında GG'ye göre daha fazla sayıda yaklaşım önerdiği görülmüştür. Bu durum temelde GeoGebra'nın sürükleme özelliğinin getirdiği farklı durumları hızlıca inceleme fırsatından kaynaklanıyor olabilir.

4.4. Argümantasyon Sürecinde oluşturulan Varsayımların İspatları

Bu bölümde son araştırma sorusu ve alt sorularına dair ulaşılan sonuçlara yer verilmiştir. İlk olarak, grupların oluşturduğu varsayımların yerleri global argümantasyon yapılarında işaretlenmiş ve oluşturulan tüm varsayımlar tablolarda açıklanmıştır. Her bir etkinliğin uygulanmasından önce, argümantasyon yapısında hedef sonuç olarak kodlanabilecek ve etkinliğin içeriğiyle doğrudan ilgili olası varsayımların bir listesi hazırlanmıştır. Bilişsel bütünlük kavramına dayanarak, gruplardan ürettikleri varsayımlardan birini ispat etmeleri istenmiştir. İspatı istenen varsayım, hedef sonuç bileşeni ile temsil edilenler arasından seçilmiştir. Yani, toplamda dört varsayım belirlenmiş ve karşılık gelen etkinliğin son bölümünde ispatlanması istenmiştir. Son olarak, grupların ispat amacıyla oluşturduğu argümanları geçerlikleri açısından değerlendirilmiştir. Grupların etkinliklerde sorulan önermeler için ne derece geçerli ispat yapabildikleri Tablo 4.6 gösterilmektedir.

Tablo 4.6

Argümamların geçerliğinin özeti

Etkinlik	PÇG'nin argümanları	GG'nin argümanları
Etkinlik 1	Gerekçe hatalı geçersiz argüman	Geçerli ispat
Etkinlik 2	Yapısal hatalı geçersiz argüman	Geçerli ispat
Etkinlik 3	Yapısal hatalı geçersiz argüman	Yapısal hatalı geçersiz argüman
Etkinlik 4	Geçerli ispat	Geçerli ispat

Bilişsel bütünlük temelli etkinlikler sonunda PÇG ve GG'nin argümanları; gerekçe hatalı geçersiz argüman, yapısal hatalı geçersiz argüman ve geçerli ispat olmak üzere üç kategoride sınıflandırılmıştır. Ortaokul matematik öğretmen adaylarının etkinliklerdeki tüm önermeler için geçerli ispat yapamamalarının bir nedeni geometri alan bilgilerinin eksikliği olabilir. Ayrıca, Fiallo ve Gutiérrez (2017) öğrencilerin ispat konusundaki gelişiminin sabit olmadığını, problemlerin zorluk seviyesine bağlı olarak değiştiğini göstermiştir. Bu bağlamda, Etkinlik 3 ve 4'te ulaşılan sonuçlara dair çıkarım yapılabilir. Yani, Etkinlik 3'te iki grup da geçerli ispat yapamazken Etkinlik 4'te iki grubun da geçerli ispat yapabilmesi, önermelerin ispatlarının zorluk seviyesinden kaynaklanıyor olabilir. Örneğin, Etkinlik 3'teki üç noktanın doğrusal olduğunun ispatlanması öğretmen adaylarının derslerde ya da kitaplarda sıklıkla karşılaştığı bir ispat içeriği olmadığından ispata başlama aşamasında zorluk yaşamış olabilirler. Diğer yandan, öğretmen adayları son etkinlik olduğu ve bu tarzda ispatlar üzerine önceki etkinliklerde de çalıştıkları için Etkinlik 4'te geçerli ispat sunmuş olabilir.

Bu çalışmada, ortaokul matematik öğretmen adaylarının katılımı ile geometri alanında bilişsel bütünlük kavramı incelenmiştir. İleride yapılacak çalışmalarda farklı düzeylerdeki öğrencilerin katılımıyla ve matematiğin diğer alanlarında bilişsel bütünlük kavramı araştırılması önerilmektedir. Bu çalışmanın kapsamına ek olarak, ispat sürecindeki global argümantasyon yapıları da araştırılabilir. Ayrıca, ileride yapılacak araştırmalarda, ortaokul matematik öğretmen adaylarının matematik müfredatındaki üçgen ve çemberle ilgili kazanımlar için varsayım oluşturma ve aktif araştırma bölümleri içeren etkinlikleri hazırlama ve uygulama süreçlerinin incelenmesi de önerilmektedir.

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