Geometry of Moving Curves

and

Soliton Equations

By

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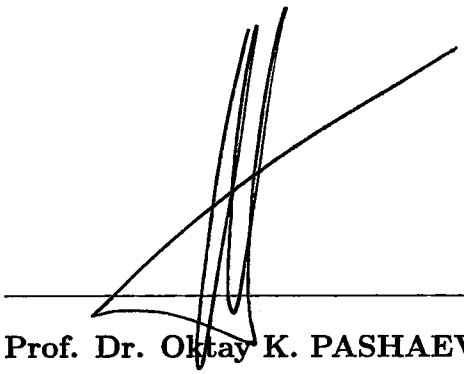

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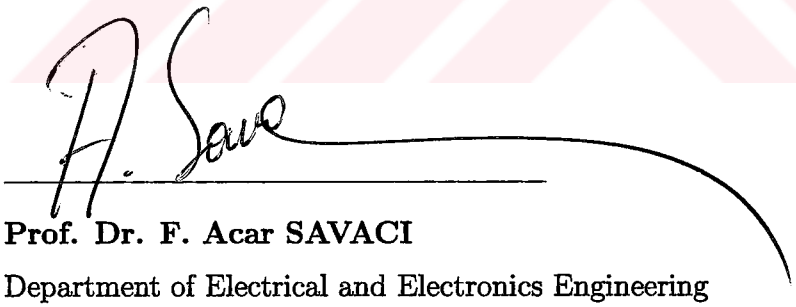

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ABSTRACT

In this thesis we study relations between the motion of curves in classical differential geometry and nonlinear soliton equations. For the planar motion of curves we found hierarchy of MKdV (Modified Korteweg-de Vries) equations generated by corresponding recursion operator. By integration of natural equations of curves, we found soliton curves and their dynamical characteristics. Under negative power recursive reduction we construct Sine-Gordon hierarchy and corresponding soliton curve. For three dimensional motion of curves relation with NLS (Nonlinear Schrödinger) equation and complex MKdV are constructed.

ÖZ

Bu tezde klasik diferansiyel geometrideki eğrilerin hareketi ile doğrusal olmayan soliton denklemleri arasındaki bağlantılar incelendi. Eğrilerin düzlemdeki hareketi için, ilgili rekursiyon operatörü ile elde edilen MKdV (Modified Korteweg-de Vries) denklem hiyerarşisi bulundu. Eğrilerin doğal denklemlerinin integrasyonu ile soliton eğrileri ile soliton eğrileri ve bunların hareketsel özellikleri elde edildi. Rekursiyon operatörünün eksi kuvveti ile Sine-Gordon denklem hiyerarşisi ve ilgili soliton eğrisi oluşturuldu. Üç boyutta eğri hareketi konusunda NLS (Nonlinear Schrödinger) denklemi ve komplexs MKdV arasındaki bağlantılar incelendi.

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Chapter 1

INTRODUCTION

The foundations of differential geometry of curves and surfaces were laid in the early part of the nineteenth century with the works of Monge (1746 - 1818), Gauss (1777 - 1855), Liuoville (1809 - 1882), Frenet (1816 - 1888), Serret (1819 - 1885), Bertrand (1822 - 1900) and Saint - Venant (1796 - 1886) [1]. Monge's major contributions were collected in his "Applications de L'Analyse à la Géometrie (1807)". Gauss' treatise on the geometry of surfaces was the "Distiquistiones Generales Circa Superficies Curvas (1828)". Therein, Gauss set down the system of equations that bears his name and which time has shown to be fundamental to the analysis of surfaces. This Gauss' system establishes the remarkable connection between classical differential geometry, nonlinear partial differential equations and modern soliton theory. Nonlinear partial differential equations have been studied in classical works on differential geometry in 19th century by Darboux, Bäcklund, Riemann and others [1, 2]. But only after discovering soliton equations and methos of solving these equations, in second half of XX century started new systematic approach to study relations between differential geometry and soliton structure [2]. Soliton equations as an integrable systems are Hamiltonian systems with infinite number of degrees of freedom [3, 4], related with symplectic geometrical structure. Manifestation of this infinite dimensional hamiltonian structure is in the existence of infinite hierarchy of soliton equations [5, 6]. This hierarchy is generated by integro-differential recursion operator, playing key role in the soliton theory [6].

The origins of the soliton theory are found in the early part of the nineteenth century. Thus, it was in 1834 that the Scottish engineer John Scott Russell recorded the first sighting, along a channel near Edinburg, of the solitary hump-shaped wave to be rediscovered in 1965 in the context of the celebrated Fermi-Pasta-Ulam problem by Kruskal and Zabusky and termed a *soliton* [7]. Scott

Russell observed that his so-called *great wave of translation* proceeded with a speed proportional to its height [8].

It was in 1895 that Korteweg and de Vries [9], derived the nonlinear wave equation for such a wave, which now bears their name and adopts the canonical form

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.1}$$

This equation has been shown to be a canonical model for a rich diversity of large amplitude wave systems arising in the theory of solids, liquids and gases [10, 11]. In 1965 the KDV equation was rediscovered in the context of Fermi-Pasta-Ulam problem [7]. In a pioneering study by Kruskal and Zabusky [7], the KDV equation was obtained from the lattice model. The existence of solitary waves in this nonlinear model, which possess the remarkable property that they preserve both their amplitude and speed subsequent upon interaction, was revealed via a computational study.

In 1968 Miura [12] obtained the transformation which relates KdV equation and its modified counterpart called Modified Korteweg de Vries (MKdV) equation:

$$\phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0. \tag{1.2}$$

Miura transformation

$$u(x,t) = \phi(x,t)^2 + \phi_x(x,t),$$

connects a solution ϕ of MKdV equation (1.2) with solution u of KdV equation (1.1). So MKdV equation shares all the beatiful solitonic properties with the KdV equation.

It turns out that remarkably, a generic method for the description of soliton interaction has its roots in a type of transformation originally introduced by Bäcklund in the nineteenth century to generate pseudospherical surfaces, that is, surfaces of constant negative Gaussian curvature $\kappa = -1/\rho^2$ [12]. The study of such surfaces goes back at least to Edmond Bour in 1862, who generated the celebrated Sine-Gordon equation

$$\omega_{uv} = \frac{1}{\varphi^2}\sin\omega, \tag{1.3}$$

from the Gauss-Mainardi-Codazzi system for pseudospherical surfaces parametrised in terms of asymptotic coordinates. The Sine-Gordon was subsequently rederived independently by both Bonnet in 1867 and Enneper in 1868 in a similar manner [2]. A purely geometric construction for pseudospherical surfaces was reformulated in mathematical terms as a transformation by Bianchi in 1879. In 1882, Bäcklund published details of his celebrated transformation $\mathbb{B}_\sigma$ which allows the iterative construction of pseudospherical surfaces. In 1892, under the title "Sulla Transformazione di Bäcklund per le Superficie Pseudosferiche " [13], Bianchi demonstrated that the Bäcklund transformation $\mathbb{B}_\sigma$ admits a commutativity property $\mathbb{B}_{\sigma_2}\mathbb{B}_{\sigma_1} = \mathbb{B}_{\sigma_1}\mathbb{B}_{\sigma_2}$ a consequence of which is a nonlinear superposition principle embodied in what is termed a *permutability theorem.*

In 1973, Washlquist and Estabrook demonstrated that the KDV equation, like the Sine-Gordon equation admits invariance under a Bäcklund-type transformation and moreover possesses an associated permutability theorem [12]. In 1974, a Bäcklund transformation for the nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + v|q|^2 q = 0, \tag{1.4}$$

was constructed by Lamb [14] using a classical method developed by Clairin in 1910. The NLS equation has important applications in fibre optics [15]. In 1968, Zakharov derived the NLS equation in a study of deep water gravity waves [16]. Hasimoto [17] in 1971 obtained the same equation in an approximation to the hydrodynamical motion of a thin isolated vortex filament. Implicit was a geometric derivation of the NLS equation wherein it is associated with a motion of an inextensible curve in $\mathbb{R}^3$. This association of an integrable equation with the spatial motion of an inextensible curve arises naturally in the study of the geometry of solitons. In 1991 Goldstein and Petrich [18] have been related integrable evolution equations from MKdV hierarchy to motions of closed curves in a plane. It turns out that being integrable, these motions conserve infinitely many global invariants. The local curve dynamics has similarity with geometric models of interface evolution, proposed to study a crystal growth [19]. In fact, a variety of physical processes can be modeled in terms of the motion of curves, including the dynamics of vortex filaments in fluid dynamics, and more generally, the planar motion of interfaces [20].

There exist now several new developed methods to solve soliton equations: The Inverse Scattering Method [3, 21], Bäclund Transformation [12], Darboux Transformation [22], Hirota Bilinear Method [23, 24] and the others. These meth-

ods allows one to construct exact N soliton solutions and to study their dynamics. This achievement of soliton theory [25]-[29] has impact on classical differential geometry where they have appeared in the first time. In this case, methods of integration of Soliton Equations provide exact tools to study characteristics of special kind of curves and surfaces which called soliton curves and soliton surfaces.

The goal of the present thesis is to study relations between motion of curves in plane R^2 and space R^3 with the soliton equations [30]-[32], [33]-[47], [48, 49].

In Chapter 2 we give main definitions and examples related with the curve theory. In Section 2.1 we introduce basic definitions of the local curve. Serret-Frenet (S-F) equations in natural parametrisation are studied in Section 2.2 and integration of natural equations of a curve in Section 2.3. In Section 2.4 we show that zero torsion curve is the planar curve and then find S-F equations in arbitrary parametrisation in Section 2.5.

In Chapter 3 we study the motion of a curve confined in the plane (the planar curve motion). In Section 3.1 we formulate the main idea related with evolution of a curve with time. Time evolution equations for two dimensional motion are derived in Section 3.2. In Section 3.3 we find that under the natural conditions on the evolution of a curve, the curvature is subject to evolution equation generated by integro-differential operator R. This integro-differential operator is determined as the recursion operator of so called MKdV hierarchy. In Section 3.4 nonlinear evolution hierarchy generated by R is defined.

In Chapter 4 we study MKdV equation and characteristics of the corresponding curves. Section 4.1 is devoted to the MKdV hierarchy. By the Hirota bilinear method we construct exact one soliton solution of MKdV equation in Section 4.2. In Section 4.3 we integrate the natural equations and find MKdV one soliton curve as a loop soliton. We determine some time invariant characteristics of this soliton curve: the area characteristic and the angle characteristic in Sections 4.4 and 4.5 respectively. In Section 4.6 we construct two soliton solution of MKdV, describing collision of two loop solitons (Appendix). In the last part of the chapter, Section 4.7, we discuss complex Miura transformation relating MKdV and KdV equations.

In Chapter 5 the Sine-Gordon equation and corresponding curve are studied. In Section 5.1 we consider negative power recursion operator, generating the Sine-Gordon hierarchy. We construct one soliton solution by Hirota Method in

Section 5.2. Then, integrating equations of motion we construct the soliton curve corresponding to Sine-Gordon equation in Section 5.3.

Chapter 6 is devoted to the motion of curves in three dimensional space. In Section 6.1 equations of time evolution are constructed. In this case, under natural conditions on evolution of curves in Section 6.2, combining curvature and torsion to the complex function we find Nonlinear Scrödinger Equation (N.L.S) in Section 6.3. In Section 6.4 integrating N.L.S for one soliton solution we find motion of a curve as a constant torsion loop soliton. In the last section of this chapter we find Complex MKdV equation as a time evolution equation.

In Conclusions we discuss main results of application soliton theory to differential geometry of curves. In Appendix we analyse MKdV two soliton solution in asymptotic regions, describing collision of two loop soliton curves.

Chapter 2

LOCAL CURVE THEORY

We will begin our study with an investigation of curves in 3-dimensional Euclidean space R^3 [50]. The curve is a geometric set of points in R^3 parametrized by some real parameter α and can be considered as a path traced out by a particle moving in R^3, where parameter α has meaning of time.

2.1 Basic Definitions

Definition 2.1.0.1 *A regular curve in R^3 is a function $\boldsymbol{r}: (a,b) \to R^3$ which is of class C^k for some $k \geq 1$ and for which $\frac{d\boldsymbol{r}}{d\alpha} \neq 0$ for all $\alpha \epsilon (a,b)$.*

Definition 2.1.0.2 *The velocity vector of a regular curve $\boldsymbol{r}(\alpha)$ at $\alpha = \alpha_0$ is the derivative $\frac{d\boldsymbol{r}}{d\alpha}$ evaluated at $\alpha = \alpha_0$. Then the vector valued function $\left(\frac{d\boldsymbol{r}}{d\alpha}\right)$ determines the velocity vector field.*

Definition 2.1.0.3 *The tangent vector field to a regular curve $\boldsymbol{r}(\alpha)$ is the vector valued function*

$$\boldsymbol{t}(\alpha) = \frac{d\boldsymbol{r}/d\alpha}{|d\boldsymbol{r}/d\alpha|}. \tag{2.1}$$

It determines $\boldsymbol{t}(s)$ as the unit vector $|\boldsymbol{t}| = 1$ in the direction of the velocity vector.

Example 1.(Right circular helix)
Let $\mathbf{r}$:$R \to R^3$ be given by parametrization $\mathbf{r}(\alpha) = (a\cos\alpha, a\sin\alpha, h\alpha)$ where

$h > 0$ and $\alpha > 0$ are constants.
Then,

$$\frac{d\mathbf{r}}{d\alpha} = (-a\sin\alpha, a\cos\alpha, h)$$

so that,

$$\frac{d\mathbf{r}}{d\alpha} \neq 0 \quad \forall \alpha \in R.$$

So $\mathbf{r}(\alpha)$ is a regular curve. At $\alpha = \alpha_0$ the tangent vector to the curve is

$$\mathbf{t} = \frac{1}{\sqrt{a^2 + h^2}}(a\cos\alpha_0, a\sin\alpha_0, h).$$

Definition 2.1.0.4 *Let $\boldsymbol{r}$: $(a, b) \to R^3$ be a regular curve and let $\alpha_0 \in (a, b)$. The function*

$$h(\alpha) = \int_{\alpha_0}^{\alpha} \left|\frac{d\boldsymbol{r}}{d\alpha}\right| d\alpha, \tag{2.2}$$

is called the arc length along the curve $\boldsymbol{r}(\alpha)$ and will be denoted as $h(\alpha) = s$.

Example 2.(The circle with radius a)
Let $\mathbf{r}(\alpha) = (a\cos\alpha, a\sin\alpha, 0)$ is a circle of radius $a = constant$. Then

$$\left|\frac{d\mathbf{r}}{d\alpha}\right| = \sqrt{a^2\sin\alpha + a^2\cos\alpha} = a$$

and

$$h(\alpha) = a\alpha, \quad \alpha = s/a.$$

It provides parametrization of the circle by the arc length in the form

$$\mathbf{r}(\alpha(s)) = (a\cos(s/a), a\sin(s/a), 0).$$

2.2 The Serret-Frenet Equations

Definition 2.2.0.5 *A regular curve* $\boldsymbol{r}$*:* $(a,b) \to R^3$ *is called a unit speed curve if* $\left|\frac{d\boldsymbol{r}}{d\alpha}\right| = 1$ *for* $\forall \alpha \in (a,b)$.

Note that for a unit speed curve the arc length is,

$$s = \int_{\alpha_0}^{\alpha} d\alpha = \alpha - \alpha_0.$$

If we set $\alpha_0 = 0$ then, $\alpha = s$. It means that the unit speed curve is the curve which parametrised by the arc length parameter s. The tangent vector for this unit speed curve is just

$$\mathbf{t} = \mathbf{t}(s) = \frac{d\mathbf{r}}{ds}. \tag{2.3}$$

Definition 2.2.0.6 *The curvature of a unit speed curve* $\boldsymbol{r}(s)$ *is defined as*

$$\kappa(s) = \left|\frac{d\boldsymbol{t}}{ds}\right| = \left|\frac{d}{ds}\left(\frac{d\boldsymbol{r}}{ds}\right)\right|. \tag{2.4}$$

Example 3.

For a straight line it is clear that

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds} = \mathbf{a}, \quad \left|\frac{d\mathbf{t}}{ds}\right| = 0,$$

so the curvature $\kappa(s) = 0$.

Example 4.

Given the circle with parametrization $\mathbf{r}(s) = (a\cos(s/a), a\sin(s/a), 0)$, the tangent vector for $\mathbf{r}(s)$ is

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds} = (-\sin(s/a), \cos(s/a), 0).$$

Then

$$\left|\frac{d\mathbf{t}}{ds}\right| = 1/a,$$

so for a circle of radius a, the curvature is a constant $\kappa(s) = 1/a$.

Lemma 1

In Euclidean space R^3, the constant speed vector $\mathbf{t}(s)$ is orthogonal to the vector $\frac{d\mathbf{t}}{ds}$.

Proof

Differentiating $\mathbf{t}^2(s) = const$ according to s, we get:

$$2\mathbf{t}(s)\frac{d\mathbf{t}}{ds} = 0,$$

and as follows,

$$\mathbf{t}(s) \perp \frac{d\mathbf{t}}{ds}.$$

From this lemma it follows that for the unit speed curve, the vector $\frac{d\mathbf{t}}{ds}$ (the acceleration vector) is orthogonal to the tangent vector $\mathbf{t}(s)$.The length of this acceleration vector is the curvature $\kappa(s)$ of the unit speed curve.

Definition 2.2.0.7 *The unit length acceleration vector*

$$\boldsymbol{n} = \frac{d\boldsymbol{t}/ds}{|d\boldsymbol{t}/ds|}, \tag{2.5}$$

is called the normal vector to the curve.

From the above Lemma 1 it follows that the vector $\mathbf{n}(s)$ is orthogonal to the tangent vector $\mathbf{t}(s)$.

Definition 2.2.0.8 *The vector $\boldsymbol{b}(s)$ defined as*

$$\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s), \tag{2.6}$$

is called the binormal vector to the curve.

Since $\boldsymbol{b}^2 = 1$ then, according to Lemma1 we get that $\boldsymbol{b}(s) \perp \frac{d\boldsymbol{b}}{ds}$ and $\frac{d\boldsymbol{b}}{ds}$ belongs to the plane $(\boldsymbol{t}, \boldsymbol{n})$. From Definition 2.2.0.8

$$\frac{d\mathbf{b}}{ds} = \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds},$$

and due to

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n},$$

we get

$$\frac{d\mathbf{b}}{ds} = \kappa(\mathbf{n} \times \mathbf{n}) + \mathbf{t} \times \frac{d\mathbf{n}}{\partial s}.$$

So we have that

$$\frac{d\mathbf{b}}{ds} = \mathbf{t} \times \frac{d\mathbf{n}}{ds}$$

and as follows,

$$\frac{d\mathbf{b}}{ds} \quad \perp \quad \mathbf{t}.$$

Therefore the vector $\frac{d\mathbf{b}}{ds}$ is directed in $\mathbf{n}$. The coefficient $\tau(s)$ of expansion $\frac{d\mathbf{b}}{ds} = \tau\mathbf{n}$ is called torsion of the unit speed curve.

Definition 2.2.0.9 *The torsion of the unit speed curve is defined as*

$$|\tau| = \left|\frac{d\mathbf{b}}{ds}\right|. \tag{2.7}$$

Then $\frac{d\mathbf{b}}{ds} = \tau\mathbf{n}$. According to Lemma 1 the vector $d\mathbf{n}/ds$ is orthogonal to $\mathbf{n}$ and therefore it belongs to the plane $(\mathbf{t}, \mathbf{b})$. Then, from Eq. (2.6)

$$\mathbf{n} = \mathbf{b} \times \mathbf{t},$$

and as follows

$$\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds}.$$

Therefore

$$\frac{d\mathbf{n}}{ds} = \tau(\mathbf{n} \times \mathbf{t}) + \kappa(\mathbf{b} \times \mathbf{n}) = (\tau)(-\mathbf{b}) + (\kappa)(-\mathbf{t}),$$

so that

$$\frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} - \tau\mathbf{b}.$$

From this definitions and explanations it follows that three vectors $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ which represent local basis for the curve at a fixed points, called the **Serret-Frenet basis**, are subject to the following theorem:

Theorem 2.2.0.10 ***(Serret-Frenet Equations)***

Let $\boldsymbol{r}(s)$ be a unit speed curve with the curvature $\kappa(s) \neq 0$ and the torsion $\tau(s)$. Then, the Serret-Frenet basis $\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$ satisfies the system:

$$\frac{d\boldsymbol{t}}{ds} = \kappa(s)\boldsymbol{n}(s), \tag{2.8}$$

$$\frac{d\boldsymbol{n}}{ds} = -\kappa(s)\boldsymbol{t}(s) - \tau(s)\boldsymbol{b}(s), \tag{2.9}$$

$$\frac{d\boldsymbol{b}}{ds} = \tau(s)\boldsymbol{n}(s). \tag{2.10}$$

Example 5.(The unit speed circular helix)
Consider the curve given by the parametrization

$$\mathbf{r}(s) = (a\cos\omega s, a\sin\omega s, h\omega s),$$

where,

$$\omega = (a^2 + h^2)^{-1/2}.$$

The tangent vector is

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds} = \omega(-a\sin\omega s, a\cos\omega s, h),$$

and

$$\frac{d\mathbf{t}}{ds} = -\omega^2 a(\cos\omega s, \sin\omega s, 0),$$

so that

$$\left|\frac{d\mathbf{t}}{ds}\right| = \omega^2 a\sqrt{\cos^2\omega s + \sin^2\omega s} = \omega^2 a.$$

Then by Definition 2.2.0.6 we get for curvature the constant value,

$$\kappa(s) = \omega^2 a,$$

and by Definition 2.2.0.7

$$\mathbf{n} = \frac{d\mathbf{t}/ds}{|d\mathbf{t}/ds|}.$$

It follows that

$$\mathbf{n}(s) = (-\cos\omega s, -\sin\omega s, 0),$$

and by Definition 2.2.0.8 we can find,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}.$$

Therefore,

$$\mathbf{b}(s) = \omega(h\sin\omega s, -h\cos\omega s, a),$$

and

$$\frac{d\mathbf{b}}{ds} = \omega^2 h(\cos\omega s, \sin\omega s, 0) \quad .$$

Finally by Definition 2.2.0.9, we find that the torsion of circular helix is a constant,

$$\tau = |\frac{d\mathbf{b}}{ds}| = \omega^2 h\sqrt{\cos^2\omega s + \sin^2\omega s} = \omega^2 h.$$

From the Serret-Frenet Equations and the theory of the ordinary differential equations follows the next theorem [50, 51].

Theorem 2.2.0.11 ***(Fundamental Theorem of The Curve Theory)***

Given any two functions of C^1 class $f_1(s)$, $f_2(s)$, of which the former is positive for all values of s within a certain domain; there exists a one and only one curve, up to rigid motion in the space, for which the curvature $\kappa = f_1(s)$, the torsion $\tau = f_2(s)$ and s is the arc length parameter ,for all values of s in the given domain.

So according to this theorem the curvature and torsion as functions of natural parameter, determines the curve up to rigid motion (translations, rotations) in three dimensional space.

2.3 Integration of Natural Equations of a Curve

Definition 2.3.0.12 *Equations $\kappa = \kappa(s)$, $\tau = \tau(s)$ determining a curve called natural (intirinsic) equations of a curve [51].*

The problem of finding parametric form of the curve for cartesian coordinates $x = x(s)$, $y = y(s)$, $z = z(s)$ is called integration of natural equations of the curve.

Example 6.
Let us consider a curve with zero curvature, $\kappa(s) = 0$, then for tangent vector

$$\mathbf{t} = \frac{d\mathbf{r}}{ds},$$

from Serret-Frenet equation (2.8), follows that

$$\frac{d\mathbf{t}}{ds} = 0.$$

This equation shows that the vector $\mathbf{t}(s)$ is a constant vector,

$$\mathbf{t}(s) = \mathbf{t}_0 = const.$$

Then, integrating once equation $\frac{d\mathbf{r}}{ds} = \mathbf{t}$ we get:

$$\mathbf{r}(s) = \mathbf{t}_0 s + \mathbf{r}_0.$$

where $\mathbf{r}_0$ is a constant vector.

Three vectors $\mathbf{r}, \mathbf{r}_0, \mathbf{t}_0$ determine three points in R^3:

$$\mathbf{r}(x, y, z), \quad \mathbf{r}_0(x_0, y_0, z_0), \quad \mathbf{t}_0(a, b, c).$$

Then, vector equation

$$\mathbf{r} - \mathbf{r}_0 = \mathbf{t}_0 s,$$

in components implies the linear system

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = s.$$

It proves that the zero curvature curve in R^3 is a straight line.

2.4 The Zero Torsion Curve

Now let us consider a curve with a torsion identically equal to zero

$$\tau(s) = 0.$$

Then, from the last Serret-Frenet equation (2.10) we have

$$\frac{d\mathbf{b}}{ds} = 0,$$

and as follows $\mathbf{b} = \mathbf{b}_0 = constant$. Let us choose the fixed orthonormal coordinate system with axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ such that $\mathbf{e}_3 \equiv \mathbf{b}$. Then, vectors $\mathbf{e}_1, \mathbf{e}_2$ would be orthogonal to the binormal vector $\mathbf{b}$. Since the tangent and normal vectors $\mathbf{t}, \mathbf{n}$ are orthogonal to the same binormal vector $\mathbf{b}$, they belong to the plane determined by fixed vectors $\mathbf{e}_1, \mathbf{e}_2$. If we define by θ an angle between $\mathbf{t}$ and $\mathbf{e}_1$, then we have the following linear representation of our vectors.

$$\mathbf{t} = \quad \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta, \tag{2.11}$$

$$\mathbf{n} = -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta. \tag{2.12}$$

It means that our curve completely belongs to the plane and it is a planar curve. From another side, if we have the planar curve, then vectors $\mathbf{t}$ and $\mathbf{n}$ belongs to the curve's plane so that $\mathbf{b}$ is orthogonal to the plane. Thus $\frac{d\mathbf{b}}{ds} = 0$ and torsion $\tau(s) = 0$. So we have the next theorem.

Theorem 2.4.0.13 *A C^3 curve (other than a straight line) is a plane curve if and only if its torsion vanishes.*

Differentiating (2.11),(2.12) respectively we have

$$\frac{d\mathbf{t}}{ds} = (-\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta)\frac{d\theta}{ds},$$

$$\frac{d\mathbf{n}}{ds} = (-\mathbf{e}_1 \cos\theta - \mathbf{e}_2 \sin\theta)\frac{d\theta}{ds}.$$

Then, comparing with the Serret-Frenet equations (2.8), (2.9)

$$\frac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}(s),$$

$$\frac{d\mathbf{n}}{ds} = -\kappa(s)\mathbf{t}(s),$$

we find that the curvature

$$\kappa(s) = \frac{d\theta}{ds}. \tag{2.13}$$

Let us suppose that the natural equation of a curve is given by the equation

$$\kappa(s) = f(s),$$

where $f(s)$ a is given function. Then, from the equation (2.13) we have

$$\frac{d\theta}{ds} = f(s).$$

Integrating once we get

$$\theta(s) = \phi(s) + \theta_0, \quad \phi(s) = \int_0^s f(s')ds',$$

where $\theta_0 = constant$ is the initial value of the function $\theta(s)$: $\theta(0) = \theta_0$.

If the curve is determined by the equation $\mathbf{r} = \mathbf{r}(s)$, then

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}(s) = \mathbf{e}_1 \cos(\phi(s) + \theta_0) + \mathbf{e}_2 \sin(\phi(s) + \theta_0),$$

or

$$\frac{d\mathbf{r}}{ds} = \mathbf{e}_1(\cos\phi(s)\cos\theta_0 - \sin\phi(s)\sin\theta_0) + \mathbf{e}_2(\sin\phi(s)\cos\theta_0 + \cos\phi(s)\sin\theta_0).$$

Integrating we have

$$\mathbf{r}(s) - \mathbf{r}_0 = \mathbf{e}_1(x(s)\cos\theta_0 - y(s)\sin\theta_0) + \mathbf{e}_2(y(s)\cos\theta_0 + x(s)\sin\theta_0),$$

where

$$x(s) = \int_0^s \cos\phi(s')ds', \quad y(s) = \int_0^s \sin\phi(s')ds',$$

and

$$\theta_0 = constant, \quad \mathbf{r}_0 = (X_0, Y_0) = constant.$$

If we define $X(s), Y(s)$ as follows

$$X(s) = x(s)\cos\theta_0 + y(s)\sin\theta_0 + X_0,$$

$$Y(s) = y(s)\cos\theta_0 - x(s)\sin\theta_0 + Y_0,$$

or in the matrix form

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} + \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix},$$

then

$$\mathbf{r}(s) = X(s)\mathbf{e}_1 + Y(s)\mathbf{e}_2.$$

It is easy to see that new coordinates correspond to a new coordinate system, rotated on the angle θ_0 and with beginning of coordinates at the point $\mathbf{r}_0 = X_0\mathbf{e}_1 + Y_0\mathbf{e}_2$. So, it can be considered as following rigid transformation of the plane with the curve:

(a) Translation on vector $\mathbf{r}_0 = X_0\mathbf{e}_1 + Y_0\mathbf{e}_2$,

(b) Rotation on the angle θ_0.

This is just an illustiration of Theorem 2.2.0.11.

2.5 Serret-Frenet Equations in Arbitrary Parametrization

In section 2.1 we have considered the Serret-Frenet equations in terms of natural parametrization of a curve (unit speed curve). In this section we will determine the Serret-Frenet equations for a curve given by an arbitrary parametrization. Let $\mathbf{r}(\alpha)$ be a regular curve parametrized by α and let $s(\alpha)$ denote the arc length parameter. Then,

$$\mathbf{r}(\alpha) = \mathbf{r}(s(\alpha)), \quad \textit{where} \quad \frac{ds}{d\alpha} = \left|\frac{d\mathbf{r}}{d\alpha}\right| > 0.$$

We wish to determine the Serret-Frenet equations in terms of the variable α. Then the derivatives with respect to α we denote by primes:

$$\mathbf{r}' \equiv \frac{d\mathbf{r}}{d\alpha}, \quad \mathbf{r}'' \equiv \frac{d^2\mathbf{r}}{d\alpha^2}, \quad \mathbf{r}''' \equiv \frac{d^3\mathbf{r}}{d\alpha^3}$$

Theorem 2.5.0.14 *If $\boldsymbol{r}(\alpha)$ is a regular curve in R^3, then The Serret-Frenet basis in this parametrization is given by:*

$$\boldsymbol{t} = \boldsymbol{r}'/|\boldsymbol{r}'| \tag{2.14}$$

$$\boldsymbol{b} = [\boldsymbol{r}' \times \boldsymbol{r}'']/|\boldsymbol{r}' \times \boldsymbol{r}''| \tag{2.15}$$

$$\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t} \tag{2.16}$$

The curvature $\kappa(\alpha)$ and the torsion $\tau(\alpha)$ for the curve are defined as follows:

$$\kappa = |\boldsymbol{r}' \times \boldsymbol{r}''|/|\boldsymbol{r}'|^3, \tag{2.17}$$

$$\tau = [\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''']/|\boldsymbol{r}' \times \boldsymbol{r}''|^2. \tag{2.18}$$

where $[\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''']$ is the triple product of vectors.

Since $\mathbf{r}(\alpha) = \mathbf{r}(\alpha(s))$, we have $\mathbf{r}' = \dot{\mathbf{r}}s' = s'\mathbf{t}$, where we have used the definition (2.3) So we get the relation (2.14).

We know that $\mathbf{r}'' = s''\mathbf{t} + s'\mathbf{t}' = s''\mathbf{t} + \kappa(s')^2\mathbf{n}$ where we have applied the Serret-Frenet Eq. (2.8). Hence $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{b}$, and $\kappa(s')^3 = |\mathbf{r}' \times \mathbf{r}''|$. So, (2.17) is valid.

Since for $\kappa \neq 0$, $\mathbf{b} = \frac{[\mathbf{r}' \times \mathbf{r}'']}{\kappa(s')^3}$ and $\kappa(s')^3 = |\mathbf{r}' \times \mathbf{r}''|$, we find (2.15).

Eq. (2.16) is obvious by orthogonality condition of Serret-Frenet vectors.

To prove formula for torsion (2.18) we have

$$\mathbf{r}''' = s''\mathbf{t} + s''\mathbf{t}' + (\kappa(s')^2)'\mathbf{n} + \kappa(s')^2\mathbf{n}'.$$

Simplifying this expression by Serret-Frenet Eqs. (2.8)-(2.10), we get:

$$\mathbf{r}''' = (s'' - \kappa^2(s')^3)\mathbf{t} + (\kappa s' s'' + (\kappa(s')^2)')\mathbf{n} + \kappa\tau(s')^3\mathbf{b}.$$

Hence $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = \tau|\mathbf{r}' \times \mathbf{r}''|^2$. Thus we get the relation (2.18).

In Theorem (2.2.0.10) we formulated Serret-Frenet equations in natural parametrization s. To define these equations in arbitrary parametrization α, we use the chain rule such that:

$$\frac{d()}{d\alpha} = \frac{d()}{ds}\frac{ds}{d\alpha} = \frac{d()}{ds}v.$$

where $v = ds/d\alpha = |d\mathbf{r}/d\alpha|$.

Applying this formula to vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in Serret-Frenet Eqs. (2.8)-(2.10) we have the next theorem.

Theorem 2.5.0.15 ***(Serret-Frenet Equations in Arbitrary Parametrization)*** *Let $\boldsymbol{r}(\alpha)$ be a regular curve in R^3, and let $v(\alpha) = |\boldsymbol{r}'|$. Then,*

$$\frac{d\boldsymbol{t}}{d\alpha} = \kappa v\boldsymbol{n}, \tag{2.19}$$

$$\frac{d\boldsymbol{n}}{d\alpha} = -\kappa v\boldsymbol{t} + \tau v\boldsymbol{b}, \tag{2.20}$$

$$\frac{d\mathbf{b}}{d\alpha} = -\tau v\mathbf{n}. \tag{2.21}$$

Chapter 3

THE PLANAR CURVE MOTION

3.1 Evolution of a Curve with Time

In this chapter we consider the motion of a curve according to the time variable t [52]. Let the vector function $\mathbf{r}(\alpha, t)$ for any fixed time t denotes the smooth curve in R^3, which is parametrized by $\alpha \epsilon R$. Then, the Riemannian metric of a one dimensional space on the curve is defined as:

$$g(\alpha, t) = \frac{\partial \mathbf{r}}{\partial \alpha} \frac{\partial \mathbf{r}}{\partial \alpha}, \tag{3.1}$$

and the arc length parameter s along the curve is given as follows

$$s(\alpha, t) = \int_0^\alpha \sqrt{g(\alpha', t)} d\alpha' = \int_0^\alpha |\frac{\partial \mathbf{r}}{\partial \alpha'}| d\alpha'. \tag{3.2}$$

Then, we will consider the pair (α, t) or (s, t), as coordinates of a point on the curve at time t. From equations (2.3) and (3.2) we have tangent vector to the curve in the form:

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s} = g^{-1/2} \frac{\partial \mathbf{r}}{\partial \alpha}. \tag{3.3}$$

Let $(\mathbf{t}(s,t), \mathbf{n}(s,t), \mathbf{b}(s,t))$ is Serret-Frenet basis for our curve at the fixed time t. Then, it satisfies the Serret-Frenet equations (2.8), (2.9), (2.10),

$$\frac{\partial \mathbf{t}}{\partial s} = \kappa \mathbf{n}, \quad \frac{\partial \mathbf{b}}{\partial s} = -\tau \mathbf{n}, \quad \frac{\partial \mathbf{n}}{\partial s} = -\kappa \mathbf{t} + \tau \mathbf{b},$$

where $\kappa = \kappa(s,t)$ is the curvature and $\tau = \tau(s,t)$ is the torsion of the curve at the time t. The motion of a point on the curve, that is the time evolution of $\mathbf{r}(\alpha,t)$, can be specified in the form

$$\frac{\partial \mathbf{r}}{\partial t} = \dot{\mathbf{r}} = U\mathbf{n} + V\mathbf{b} + W\mathbf{t}, \tag{3.4}$$

where U, V, W as functions of s and t, correspond to the normal, binormal and tangent projections of the velocity. Below we restrict our attention to a purely local form for these velocities, $U = U(\kappa, \kappa_s, ..., \tau, \tau_s, ...)$, $V = V(\kappa, \kappa_s, ..., \tau, \tau_s, ...)$, $W = W(\kappa, \kappa_s, ..., \tau, \tau_s, ...)$. Evolution in time according to Eq. (3.4) must be compatible with equations of curve (Serret-Frenet equations). We will require compatibility conditions which are given below:

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{r}(\alpha,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{r}(\alpha,t)}{\partial \alpha}, \tag{3.5}$$

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{t}(\alpha,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{t}(\alpha,t)}{\partial \alpha}, \tag{3.6}$$

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{n}(\alpha,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{n}(\alpha,t)}{\partial \alpha}, \tag{3.7}$$

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{b}(\alpha,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{b}(\alpha,t)}{\partial \alpha}, \tag{3.8}$$

To have equation in natural parametrization in the above formulas we can use Eq. (3.3), relating differentiating parameters by following equality:

$$\frac{\partial}{\partial \alpha}() = g^{1/2}\frac{\partial}{\partial s}(). \tag{3.9}$$

where parenthesis we can apply to the vectors, $\mathbf{r}, \mathbf{t}, \mathbf{n}, \mathbf{b}$.

3.2 Two Dimensional Motion

Now we specify the time evolution. As it was shown in Theorem (2.3.1.1), if we restrict our planar curve to be bounded on the plane for any time t then, $\tau(s,t) = 0$ and as follows in Eq. (3.4) $V \equiv 0$. In this case the Serret-Frenet equations reduce to the following system:

$$\frac{\partial \mathbf{t}}{\partial s} = \kappa \mathbf{n}, \tag{3.10}$$

$$\frac{\partial \mathbf{n}}{\partial s} = -\kappa \mathbf{t}, \tag{3.11}$$

$$\frac{\partial \mathbf{b}}{\partial s} = 0. \tag{3.12}$$

For the planar case the motion of a point on the curve can be specified in the form:

$$\frac{\partial \mathbf{r}}{\partial t} = \dot{\mathbf{r}} = U\mathbf{n} + W\mathbf{t}. \tag{3.13}$$

Now we want to find the evolution in time for variables $\mathbf{t}, \mathbf{n}, g, \kappa, s$. By applying (3.9) to $\frac{\partial \mathbf{r}}{\partial t}$ and then using (3.13) we have:

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{r}(\alpha,t)}{\partial t} = g^{1/2}\frac{\partial}{\partial s}\left(\frac{\partial \mathbf{r}}{\partial t}\right) = g^{1/2}\frac{\partial}{\partial s}\left(U\mathbf{n} + W\mathbf{t}\right),$$

Using Leibnitz rule for differentiation and substituting equations (3.10), (3.11),

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{r}(\alpha,t)}{\partial t} = g^{1/2}\left(\left(\frac{dU}{ds} + W\kappa\right)\mathbf{n} + \left(\frac{dW}{ds} - U\kappa\right)\mathbf{t}\right). \tag{3.14}$$

From another side by using Eq. (3.9) we have

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{r}(\alpha,t)}{\partial \alpha} = \frac{\partial}{\partial t}\left(g^{1/2}\frac{\partial \mathbf{r}}{\partial s}\right),$$

or applying Eq. (3.3) we get:

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{r}(\alpha,t)}{\partial \alpha} = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial g}{\partial t}\mathbf{t} + g^{1/2}\frac{\partial \mathbf{t}}{\partial t}. \tag{3.15}$$

By using the compatibility condition (3.5) and Eqs. (3.14), (3.15) projected in direction of $\mathbf{t}$, we find:

$$\frac{\partial g}{\partial t} = 2g\left(\frac{dW}{ds} - U\kappa\right). \tag{3.16}$$

Then, differentiating Eq. (3.3) according t and using (3.13) we have:

$$\frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial s}\left(U\mathbf{n} + W\mathbf{t}\right),$$

or

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial U}{\partial s} + W\kappa\right)\mathbf{n} + \left(\frac{\partial W}{\partial s} - U\kappa\right)\mathbf{t}.$$

Since $\frac{\partial \mathbf{t}}{\partial t}$ and $\mathbf{t}$ are orthogonal vectors, the last term has to be vanishing,

$$\frac{\partial W}{\partial s} - U\kappa = 0, \tag{3.17}$$

and after integration we have for W the next form

$$W = \int^{s} U\kappa ds'. \tag{3.18}$$

Then, the time evolution equation for $\mathbf{t}$ is as follows:

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial U}{\partial s} + W\kappa\right)\mathbf{n}. \tag{3.19}$$

Applying (3.9) to this expression,

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{t}(\alpha,t)}{\partial t} = g^{1/2}\frac{\partial}{\partial s}\frac{\partial \mathbf{t}}{\partial t},$$

and substituting equations (3.11) and (3.19) we get:

$$\frac{\partial}{\partial\alpha}\frac{\partial \mathbf{t}(\alpha,t)}{\partial t} = g^{1/2}\left(\left(\frac{\partial^2 U}{\partial s^2}+\frac{\partial W}{\partial s}\kappa + W\frac{\partial \kappa}{\partial s}\right)\mathbf{n} + \left(\frac{\partial U}{\partial s}\kappa+\kappa^2 W\right)\mathbf{t}\right). \quad (3.20)$$

From another side, according to Eq. (3.9),

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{t}(\alpha,t)}{\partial \alpha} = \frac{\partial}{\partial t}\left(g^{1/2}\frac{\partial \mathbf{t}}{\partial s}\right)$$

By using equations (3.9), (3.10) and (3.16) we get:

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{t}(\alpha,t)}{\partial \alpha} = g^{1/2}\left(\left(\frac{\partial W}{\partial s}\kappa - U\kappa^2 + \frac{\partial \kappa}{\partial t}\right)\mathbf{n} + \kappa\frac{\partial \mathbf{n}}{\partial t}\right). \quad (3.21)$$

Compatibility condition (3.6) with Eqs. (3.20), (3.21) projected on $\mathbf{n}$ direction results in evolution equations:

$$\frac{\partial \kappa}{\partial t} = \left(\frac{\partial^2 U}{\partial s^2} + W\frac{\partial \kappa}{\partial s} + \kappa^2 U\right),$$

and

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(\frac{\partial U}{\partial s} + \kappa W\right)\mathbf{t}.$$

In Eq. (3.2) we defined the arc length function. Now, the time evolution of s by using (3.16) and (3.17) is:

$$\frac{\partial s}{\partial t} = \int_0^\alpha g^{1/2}\left(\frac{dW}{ds} - U\kappa\right)d\alpha' = \int_0^\alpha \frac{dW}{d\alpha'}d\alpha' - \int_0^s U\kappa ds',$$

Then we get:

$$\frac{\partial s}{\partial t} = W - \int_0^s U\kappa ds'.$$

As a result, time evolution equations for variables $\mathbf{t}, \mathbf{n}, g, \kappa, s$ are given below:

$$\frac{\partial s}{\partial t} = W - \int_0^s U\kappa ds'. \quad (3.22)$$

$$\frac{\partial g}{\partial t} = 2g\left(\frac{dW}{ds} - U\kappa\right), \tag{3.23}$$

$$\frac{\partial \kappa}{\partial t} = \left(\frac{\partial^2 U}{\partial s^2} + W\frac{\partial \kappa}{\partial s} + \kappa^2 U\right), \tag{3.24}$$

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial U}{\partial s} + W\kappa\right)\mathbf{n}, \tag{3.25}$$

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(\frac{\partial U}{\partial s} + \kappa W\right)\mathbf{t}, \tag{3.26}$$

3.3 Relation with Nonlinear Evolution Equations

Serret-Frenet equations (3.10), (3.11)and time evolution equations (3.25), (3.26) can be written as a couple of matrix linear systems:

$$\frac{\partial}{\partial s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}, \tag{3.27}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial U}{\partial s} + W\kappa \\ -(\frac{\partial U}{\partial s} + W\kappa) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}. \tag{3.28}$$

or

$$\frac{\partial}{\partial s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = i\sigma_2\kappa\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}, \tag{3.29}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = i\sigma_2(\frac{\partial U}{\partial s} + W\kappa)\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}, \tag{3.30}$$

where σ_2 is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The compatibility condition of these matrix systems is

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} \tag{3.31}$$

Then

$$\frac{\partial \kappa}{\partial t} + \kappa i\sigma_2(\frac{\partial U}{\partial s} + W\kappa) = \left(\frac{\partial^2 U}{\partial s^2} + \frac{\partial W}{\partial s}\kappa + W\frac{\partial \kappa}{\partial s}\right) + \kappa i\sigma_2(\frac{\partial U}{\partial s} + W\kappa).$$

By the above equation and (3.17) we get that:

$$\frac{\partial \kappa}{\partial t} = \left(\frac{\partial^2 U}{\partial s^2} + U\kappa^2 + \frac{\partial \kappa}{\partial s}\int^s \kappa(s')U(s')ds'\right), \tag{3.32}$$

or

$$\frac{\partial \kappa}{\partial t} = \left(\frac{\partial^2}{\partial s^2} + \kappa^2 + \frac{\partial \kappa}{\partial s}\int^s \kappa(s')ds'\right)U.$$

If we fix $U = U(\kappa, \kappa_s, ...)$ then, it provides closed nonlinear evolution equation for the curvature $\kappa(s,t)$. Integro-differential operator in paranthesis:

$$R = \frac{\partial^2}{\partial s^2} + \kappa^2 + \frac{\partial \kappa}{\partial s}\int^s \kappa(s')ds',$$

is called the recursion operator of the mKdV hierarchy [5].

So as we can see our general nonlinear evolution equation appears as the compatibility condition of linear systems. These linear systems are considered as an auxilary linear problem corresponding to the nonlinear evolution equation for the curvature.

3.4 The Nonlinear Evolution Hierarchy

Let us consider the hierarchy of functions [18],

$$U^{(n)} = R^n U^{(0)} = \left(\frac{\partial^2}{\partial s^2} + \kappa^2 + \frac{\partial \kappa}{\partial s} \int^s \kappa(s') ds' \right)^n U^{(0)}$$

$$= R^{n-1} U^{(1)} = ... = R U^{(n-1)},$$

and let us call corresponding time evolution parameters as $t_1, t_2, ..., t_n$. Then evolution of the curvature according to t_n is,

$$\frac{\partial \kappa}{\partial t_n} = R^n U^{(0)} \quad n = 1, 2, ... \tag{3.33}$$

Evolution equations for $\forall n$ evaluated with respect to (3.33) with fixed $U^{(0)}$, we call the evolution hierarchy.

Chapter 4

THE MODIFIED KORTEWEG-DE VRIES EQUATION

4.1 The Modified Korteweg-de Vries Hierarchy

The simplest choice for the hierarchy is,

$$U^{(0)} = 0.$$

Then,

$$U^{(1)} = RU^{(0)} = -\frac{\partial \kappa}{\partial s} \times const.$$

Fixing inessential constant to one we have for the first member of the hierarchy:

$$U^{(1)} = -\frac{\partial \kappa}{\partial s}.$$

It determines the linear wave equation

$$\kappa_{t_1} + \kappa_s = 0.$$

Then, applying recursion operator we will have for the next member of the hierarchy:

$$U^{(2)} = -(\kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s),$$

and

$$\kappa_{t_1} = -(\kappa_{sss} + \kappa^2\kappa_s + \kappa_s \int \kappa\kappa_s ds'),$$

or after simplification,

$$\kappa_{t_1} + \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s = 0.$$

This equation is known as Modified Korteweg-de Vries (MKdV) equation [11, 21]. When we apply the recursion operator R on $U^{(2)}$ we get,

$$\kappa_{t_3} + (\kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s)_{ss} + \kappa^2(\kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s) + \kappa_s \int \kappa(\kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s)ds' = 0,$$

or

$$\kappa_{t_3} + \kappa_{5s} + 3\kappa_s^3 + 9\kappa\kappa_s\kappa_{ss} + \frac{5}{2}\kappa^2\kappa_{sss} + \frac{15}{8}\kappa^4\kappa_s + \frac{3}{2}\kappa_s \int \kappa\kappa_{sss}ds' = 0.$$

Integrating by parts we evaluate the integral,

$$\kappa_{sss}ds = dv \quad \kappa_{ss} = v \quad \kappa = u \quad \kappa_s = du,$$

then

$$\int \kappa\kappa_{sss}ds' = \kappa_{ss}\kappa - \frac{1}{2}\kappa_s^2,$$

so that

$$\kappa_{t_3} + \kappa_{5s} + 10\kappa\kappa_s\kappa_{ss} + \frac{5}{2}\kappa^2\kappa_{sss} + \frac{5}{2}\kappa_s^3 + \frac{15}{8}\kappa^4\kappa_s = 0.$$

This equation represents the next member of MKdV hierarchy. Then we find $U^{(3)}$,

$$U^{(3)} = -(\kappa_{5s} + 10\kappa\kappa_s\kappa_{ss} + \frac{5}{2}\kappa^2\kappa_{sss} + \frac{5}{2}\kappa_s^3 + \frac{15}{8}\kappa^4\kappa_s).$$

When we apply recursion operator R on $U^{(3)}$, we get the next member

$$\kappa_{t_4} + \kappa_{7s} + \frac{91}{2}\kappa_s\kappa_{ss}^2 + \frac{63}{2}\kappa_s^2\kappa_{sss} + 35\kappa\kappa_s\kappa_{ss} + 21\kappa\kappa_s\kappa_{4s} + \frac{7}{2}\kappa^2\kappa_{5s} + \frac{105}{4}\kappa^2\kappa_s^3$$

$$+35\kappa^3\kappa_s\kappa_{ss}+\frac{35}{8}\kappa^4\kappa_{sss}+\frac{35}{16}\kappa^6\kappa_s=0.$$

As we can see all functions $U^{(n)}$ are local functions of $\kappa, \kappa_s, \ldots$. Substituting $U^{(n)}$ to the evolution hierarchy we obtain the so called MKdV hierarchy.

Below we give the first four members of the hierarchy,

$$\kappa_{t_1}+\kappa_s=0. \tag{4.1}$$

$$\kappa_{t_2}+\kappa_{sss}+\frac{3}{2}\kappa^2\kappa_s=0. \tag{4.2}$$

$$\kappa_{t_3}+\kappa_{5s}+10\kappa\kappa_s\kappa_{ss}+\frac{5}{2}\kappa^2\kappa_{sss}+\frac{5}{2}\kappa_s^3+\frac{15}{8}\kappa^4\kappa_s=0. \tag{4.3}$$

$$\kappa_{t_4}+\kappa_{7s}+\frac{91}{2}\kappa_s\kappa_{ss}^2+\frac{63}{2}\kappa_s^2\kappa_{sss}+35\kappa\kappa_s\kappa_{ss}+21\kappa\kappa_s\kappa_{4s}+\frac{7}{2}\kappa^2\kappa_{5s}+\frac{105}{4}\kappa^2\kappa_s^3$$

$$+35\kappa^3\kappa_s\kappa_{ss}+\frac{35}{8}\kappa^4\kappa_{sss}+\frac{35}{16}\kappa^6\kappa_s=0. \tag{4.4}$$

As we can see every member of the hierarchy contains highest derivative power $2n-1$. The MKdV hierarchy is an example of integrable hierarchy of evolution equations [61]. These equations can be integrated by several exact techniques as the inverse scattering method [3, 11, 21], Bäcklund transformation [2, 12] and others [22]. For our purpose, it is convinient to use rather simple direct method called Hirota bilinearization [23, 24, 53, 54].

4.2 Hirota Bilinear Method and MKdV Equation

In 1971 Hirota introduced a new direct method for constructing soliton solutions to integrable nonlinear evolution equations [24]. The idea is to make transformation to new variables, so that in these variables a nonlinear evolution equation become represented in the bilinear form, and multisoliton solutions appear in particularly simple form. Multisoliton solutions can of course be derived by many other methods, by the inverse scattering transform [3], dressing method [3], Bäcklund [55]and Darboux transformations [22], and so on. Particularly, the Inverse Scattering Method (ISM) [21] is very powerful, but at the same time it is

most complicated and needs information about analytic behaviour of scattering data. Comparing with this, the advantage of Hirota's method is its algebraic rather than analytic structure. It allows one to construct soliton solution in a simple algebraic form avoiding analytic difficulties of ISM.

Basic ideas in this method are as follows:

(1) Introduce a dependent variable transformation. The transformation should reduce the nonlinear evolution equation to the quadratic one in the dependent variables. Hirota has developed a novel differential calculus and it is convenient to use it at this stage.

(2) Introduce a formal perturbation expansion into this bilinear equation. In the case of soliton solutions this expansion truncates.

(3) Use mathematical induction to prove that the suggested soliton form is indeed correct for arbitrary number of solitons.

4.2.1 Hirota derivatives and its properties

In this subsection we list some properties of the Hirota derivative operators D_t,D_x, [23] defined by equation

$$D_x^n(f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x} \tag{4.5}$$

or in more general form

$$D_t^n D_x^m(f \cdot g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m f(t,x)g(t',x')|_{t=t',x=x'}. \tag{4.6}$$

From the definition above we can find the general expression for n-th Hirota derivative

$$\begin{aligned} D_x^n(f \cdot g) &= (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x} \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \partial_{x_1}^k \partial_{x_2}^{(n-k)} f(x_1)g(x_2)|_{x_1=x_2=x} \end{aligned} \tag{4.7}$$

or

$$D_x^n(f\cdot g)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) f^{(n-k)}(x)g^{(k)}(x)(-1)^k. \tag{4.8}$$

For the first few derivatives we have explicitly

$$\begin{aligned} D_t(f\cdot g) &= f'g-g'f\ , \\ D_t^2(f\cdot g) &= f''g-2f'g'+g''f\ , \\ D_t^3(f\cdot g) &= f'''g-3f''g'+3g''f'-g'''f\ , \\ D_t^3(f\cdot g) &= f^{(IV)}g-4f'''g'+6f''g''-4f'g'''+fg^{(IV)}\ , \\ &\cdots \end{aligned} \tag{4.9}$$

The following properties are easily seen from the definition

1. $D_x^m(f\cdot 1)=(\frac{\partial}{\partial x})^m f$

2. $D_x^m(f\cdot g)=(-1)^m D_x^m(g.f)$

3. $D_x^m(f\cdot f)=0$ for odd m.

4. $D_x^2(f\cdot f)=2f''f-2f'^2$

5. $D_x^m(f\cdot g)=D_x^{m-1}(f_x\cdot g-f\cdot g_x)$

6. $D_xD_t(f\cdot f)=2D_x(f_t\cdot f)=2D_t(f_x\cdot f)$ for even m.

7. $D_x^m(e^{p_1x}\cdot e^{p_2x})=(p_1-p_2)^m e^{(p_1+p_2)x}$

8. $D_x^m(e^{\Omega_1 t+p_1x}\cdot e^{\Omega_2 t+p_2x})=(p_1-p_2)^m e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

9. $D_t^n(e^{\Omega_1 t+p_1 x}\cdot e^{\Omega_2 t+p_2 x}) = (\Omega_1-\Omega_2)^n e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

10. Let $P(D_t, D_x)$ be a polynomial of D_t and D_x, we have
$P(D_t, D_x)(e^{\Omega_1 t+p_1 x}\cdot e^{\Omega_2 t+p_2 x}) = P(\Omega_1-\Omega_2, p_1-p_2)e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

11. $e^{(\varepsilon D_x)}(f(x)\cdot g(x)) = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}D_x^n(f\cdot g)$

$$= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\sum_{n=0}^{k}\begin{pmatrix} n \\ \\ k \end{pmatrix}(-1)^{k-n}f^{(n)}(x)g^{(k-n)}(x)$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}\varepsilon^n\sum_{m=0}^{\infty}\frac{g^{(m)}(x)}{m!}(-\varepsilon)^m$$

where k = m+n. As a result we get that

$$e^{(\varepsilon D_x)}(f(x)\cdot g(x)) = f(x+\varepsilon)g(x-\varepsilon)$$

12. $D_x(fg\cdot h) = (\frac{\partial f}{\partial x})gh + fD_x(g\cdot h)$

13. $D_x^2(fg\cdot h) = (\frac{\partial^2 f}{\partial x^2})gh + 2(\frac{\partial f}{\partial x})D_x(g\cdot h) + fD_x^2(g\cdot h)$

14. $D_x^m((e^{px}f)\cdot(e^{px}g)) = e^{2px}D_x^m(f\cdot g)$

The following formulas are useful for transforming nonlinear differential equations into bilinear forms.

15. $\frac{\partial}{\partial x}(\frac{g}{f}) = \frac{D_x(g\cdot f)}{f^2}$

16. $\frac{\partial^2}{\partial x^2}(\frac{g}{f}) = \frac{D_x^2(g\cdot f)}{f^2} - \frac{g}{f}\frac{D_x^2(g\cdot f)}{f^2}$

17. $\frac{\partial^3}{\partial x^3}(\frac{g}{f}) = \frac{D_x^3(g\cdot f)}{f^2} - 3[\frac{D_x^2(g\cdot f)}{f^2}\frac{D_x^2(f\cdot f)}{f^2}]$

18. $\frac{\partial^2}{\partial x^2}(logf) = \frac{D_x^2(f\cdot f)}{2f^2}$

19. $\frac{\partial^4}{\partial x^4}(logf) = \frac{D_x^4(f\cdot f)}{2f^2} - 6[\frac{D_x^2(f\cdot f)}{2f^2}]^2$

4.2.2 MKdV One Soliton Solution by Hirota Method

In this section we determine the Hirota bilinear representation for MKdV equation (4.2). The solution of equation we suppose in the form

$$\kappa = \frac{G}{F},$$

where G and F are real functions of s and t. Then first we write (4.2) in terms of Hirota derivatives,

$$\frac{1}{F^2}(D_t(G\cdot F)+D_s^3(G\cdot F))-\frac{3}{F^2}D_s(G\cdot F)D_s^2(F\cdot F)+\frac{3G^2}{2F^2}D_s(G\cdot F)=0.$$

Since we have freedom to choose one of our functions we separate terms of the equation multiplied by $1/F^2$ and $1/F^4$. Thus we obtain the bilinear systems of equations:

$$(D_t+D_s^3)(G\cdot F)=0, \tag{4.10}$$

$$D_s^2(F\cdot F)=\frac{1}{2}G^2. \tag{4.11}$$

Now we write the formal expansions of F and G in powers of some parameter ϵ such that:

$$F=1+\epsilon^2F_2+\epsilon^4F_4+\epsilon^6F_6...,$$

$$G=\epsilon G_1+\epsilon^3G_3+\epsilon^5G_5....$$

When we substitute expansions of F and G to (4.10), (4.11) we get:

$$(D_t+D_s^3)(\epsilon(G_1\cdot 1)+\epsilon^3(G_1\cdot F_2+G_3\cdot 1)+\epsilon^5(F_2\cdot G_3+G_1\cdot F_4)...)=0,$$

$$D_s^2(1\cdot 1+2\epsilon^2F_2\cdot 1+\epsilon^4(F_2\cdot F_2+2(F_4\cdot 1))+\epsilon^6(2F_2\cdot F_4+2(F_6\cdot 1))...)$$

$$=\frac{1}{2}(\epsilon^2G_1^2+\epsilon^4(2G_1G_3)+\epsilon^6(2G_1G_5+G_3^2)...).$$

From this system at ϵ order zero we have identically,

$$D_s^2(1 \cdot 1) = 0.$$

For ϵ order one we find the equation,

$$(D_t + D_s^3)(G_1 \cdot 1) = 0,$$

or

$$(\partial_t + \partial_s^3)G_1 = 0.$$

The simplest nontrivial solution of this equation we search in the form,

$$G_1 = e^{\eta_1}, \quad \eta_1 = k_1 s + \omega t + \eta_1^{(0)}.$$

where dispersion must be fixed as $\omega = -k_1^3$. Then it gives,

$$G_1 = e^{\eta_1}, \quad \eta_1 = k_1 s - k_1^3 t + \eta_1^{(0)}. \tag{4.12}$$

For ϵ order two we have,

$$D_s^2(F_2 \cdot 1) = \frac{1}{4}G_1^2,$$

or

$$\partial_s^2 F_2 = \frac{1}{4}e^{2\eta_1}.$$

Integrating above equation twice and neglecting integration constants we get:

$$F_2 = \frac{1}{16k_1^2}e^{2\eta_1}, \quad \eta_1 = k_1 s - k_1^3 t + \eta_1^{(0)}. \tag{4.13}$$

For ϵ order three we have,

$$(D_t + D_s^3)(G_3 \cdot 1 + G_1 \cdot F_2) = 0.$$

By explicit calculation we have:

$$(D_t + D_s^3)(G_1 \cdot F_2) = e^{3\eta_1}\frac{1}{16k_1^2}((1 - 6 + 12 - 8 + 1)k_1^3) = 0.$$

We can see that contribution of the last term is zero, so that

$$(D_t + D_s^3)(G_1 \cdot F_2) = 0.$$

Then, we conclude that,

$$(D_t + D_s^3)(G_3 \cdot 1) = 0.$$

or

$$(\partial_t + \partial_s^3)G_3 = 0.$$

The simplest choice for the solution of this equation is

$$G_3 = 0.$$

This choice with conditions:

$$F_i = 0, \quad i \geq 4, \quad G_j = 0, \quad j \geq 3,$$

truncates infinite series and provides exact one soliton solution of MKdV equation:

$$\kappa(s,t) = \frac{e^{\eta_1}}{1 + e^{2\eta_1}/16k_1^2},$$

or

$$\kappa(s,t) = \frac{4k_1}{4k_1 e^{-\eta_1} + e^{\eta_1}/4k_1}.$$

This expression is invariant under the change of sign of parameter k_1. This is why it is sufficient to choose $k_1 > 0$. Let $\frac{1}{4k_1} = e^{\phi}$, then

$$\kappa(s,t) = \frac{2k_1}{\cosh(\eta_1 + \phi)}.$$

Adding ϕ term to the constant part of the η_1,

$$\eta_1 + \phi = k_1 s - k_1^3 t + \eta_1^{(0)} + \phi^{(0)},$$

finally we obtain the one soliton solution of MKdV equation (curvature soliton),

$$\kappa(s,t) = \frac{2k_1}{\cosh(\eta_1)}. \tag{4.14}$$

4.3 Recovering MKdV One Soliton Curve

Explicit form of (4.14) for curvature is the traveling wave,

$$\kappa(s,t) = \frac{2a}{\cosh a(s - a^2t - s_0)}, \tag{4.15}$$

(where $a = k_1$), with velocity $v = a^2$ and amplitude $2a$. Now we are going to recover the curve corresponding to the one soliton solution (4.15) by the equation (2.13),

$$\frac{\partial\theta}{\partial s} = \frac{2a}{\cosh a(s - a^2t - s_0)}.$$

Integrating once we get,

$$\theta(s,t) = \int^s \frac{2a}{\cosh a(s' - a^2t - s_0)} ds' + \theta_0.$$

Then, choosing

$$a(s - a^2t - s_0) \equiv z$$

so that

$$ads = dz,$$

the integral turns into the next form:

$$\theta(s,t) = 2\int \frac{1}{\cosh z} dz + \theta_0.$$

or

$$\theta(s,t) = 4\int \frac{e^z}{e^{2z}+1} dz + \theta_0.$$

Changing variables,

$$e^z \equiv u, \quad e^z dz = du,$$

we get the expression,

$$\theta(s,t) = 4\int \frac{1}{1+u^2} du + \theta_0,$$

so that after integration, the angle θ has the form,

$$\theta(s,t) = 4\arctan e^{a(s-a^2t-s_0)} + \theta_0.$$

Now we fix the constant θ_0 from the boundary conditions. Let

$$\hat{\theta} = \theta - \theta_0 \quad \Rightarrow \quad \tan\frac{\hat{\theta}}{4} = e^z.$$

From this expression for asymptotics at infinities we have:

$$z \quad \to -\infty \quad \Rightarrow \frac{\hat{\theta}}{4} = 0, \quad \hat{\theta} = 0.$$

and

$$z \quad \to \infty \quad \Rightarrow \frac{\hat{\theta}}{4} = \frac{\pi}{4}, \quad \hat{\theta} = 2\pi.$$

where

$$0 < \hat{\theta} < 2\pi.$$

Then the solution can be written as:

$$\theta(s,t) = 4\arctan e^{a(s-a^2t-s_0)}. \tag{4.16}$$

As the next step we will find the explicit form of the curve corresponding to our solution. By definitions (2.11) and (3.3),

$$\frac{\partial \mathbf{r}}{\partial s} = \mathbf{e}_1\cos\theta(s) + \mathbf{e}_2\sin\theta(s).$$

Integrating once we get the parametric form of our curve,

$$\mathbf{r}(s,t) = \mathbf{e}_1\int^s \cos\theta(s')ds' + \mathbf{e}_2\int^s \sin\theta(s')ds'. \tag{4.17}$$

To take the above integrals explicitly we will proceed as below: Let

$$4\arctan e^{a(s-a^2t-s_0)} = \alpha.$$

Then by identity

$$\cos\theta = 1 - 2\sin^2 2\alpha,$$

or

$$\begin{aligned}
\cos\theta &= 1 - (8\tan^2\alpha/(1+\tan^2\alpha)^2) \\
\Rightarrow \quad \cos\theta &= 1 - (8e^{2z}/(1+e^{2z})^2) \\
\Rightarrow \quad \cos\theta &= 1 - (8/(e^{-z}+e^{z})^2) \\
\Rightarrow \quad \cos\theta &= 1 - (2/(\cosh^2 z)).
\end{aligned}$$

Then by explicit integration,

$$\int \cos\theta(s')ds' = \frac{1}{a}\int 1 - (2/(\cosh^2 z)dz = \frac{1}{a}(z - 2\tanh z),$$

or

$$\int \cos\theta(s')ds' = (s - a^2t - s_0) - \frac{2}{a}\tanh a(s - a^2t - s_0). \tag{4.18}$$

For the second integral we use identity

$$\sin\theta = \sin 4\alpha = 2\sin 2\alpha\cos 2\alpha,$$

or

$$\begin{aligned}
\sin\theta &= 4\tan\alpha(1-\tan^2\alpha)/(1+\tan^2\alpha)^2 \\
\Rightarrow \quad \sin\theta &= (4e^{z}/1+4e^{2z})(1-e^{2z}/1+e^{2z}) \\
\Rightarrow \quad \sin\theta &= -2\sinh z/\cosh^2 z.
\end{aligned}$$

Then we have explicit form for the integral,

$$\int \sin\theta(s')ds' = \frac{1}{a}\int -2\sinh z/\cosh^2 z dz = \frac{1}{a}(2/\cosh z),$$

or

$$\int \sin\theta(s')ds' = \frac{2}{a} \quad \frac{1}{\cosh a(s - a^2t - s_0)}. \tag{4.19}$$

Substituting to (4.17) we get:

$$\mathbf{r}(z) = \mathbf{e}_1 \frac{1}{a}(z - 2\tanh z) + \mathbf{e}_2 \frac{1}{a}(2/\cosh z). \tag{4.20}$$

where $z = a(s - a^2 t - s_0)$,
or

$$\mathbf{r}(s,t) = \mathbf{e}_1((s - a^2 t - s_0) - \frac{2}{a}\tanh a(s - a^2 t - s_0)) + \mathbf{e}_2 \frac{2}{a}\frac{1}{\cosh a(s - a^2 t - s_0)}. \tag{4.21}$$

The curve corresponding to the one soliton solution of MKdV is called the *soliton curve.* Parametric form of this curve is:

$$x = \frac{1}{a}(z - 2\tanh z), \quad y = \frac{2}{a}\frac{1}{\cosh z}, \quad -\infty < z < \infty. \tag{4.22}$$

Combining these equations as

$$\tanh z = \left(\frac{z - ax}{2}\right)^2, \quad \frac{1}{\cosh^2 z} = \left(\frac{ay}{2}\right)^2,$$

and adding together we have relation

$$\left(x - \frac{z}{a}\right)^2 + y^2 = \left(\frac{2}{a}\right)^2.$$

To exclude parameter z from this equation we use the second Eq. of (4.22)

$$z = \cosh^{-1}\frac{2}{ay}.$$

As a result we get equation of one soliton curve in the form

$$\left(x - \frac{1}{a}\cosh^{-1}\frac{2}{ay}\right)^2 + y^2 = \left(\frac{2}{a}\right)^2. \tag{4.23}$$

This equation shows that the soliton curve is not the algebraic curve but the transcendental one. It has a simple geometrical interpretation. Let

$$x_0(y) \equiv \frac{1}{a}\cosh^{-1}\frac{2}{ay}. \tag{4.24}$$

Then Eq. (4.23) describes the set of points (x, y) equidistant $r_0 \equiv \frac{2}{a}$ from the points $(x_0(y), 0)$: the last set of points is determined from Eq. (4.24) by the curve,

$$y = \frac{r_0}{\cosh\frac{x_0}{r_0}}.$$

Constructing this curve we find that, it represents a loop in the plane. If in Eq. (4.20) we change $z \to -z$ then $x \to -x$, $y \to y$. This means that our curve is symmetrical under axis y. The maximum of the loop is determined by the solution of the equation,

$$\frac{dy}{dx} = \frac{dy/dz}{dx/dz} = -2\frac{\sinh z}{\cosh^2 z - 2},$$

and corresponds to the value $z = 0$. At this point $x(0) = 0$ and the maximum of the curve is,

$$y(0) = \frac{2}{a} = r_0.$$

The curve has other intersection points with axis y, determined by equation

$$x(z) = 0.$$

Except $z = 0$, another couple of roots of this equation is given by symmetrical pairs z_0 and $-z_0$ as solutions of transcendental equation,

$$\tanh z_0 = \frac{z_0}{2}.$$

Approximate value of $z_0 \approx 0.93$. Then, these roots determine intersection point on the soliton curve with coordinates $(0, y(z_0))$. The soliton loop is illustrated in Figure 4.1.

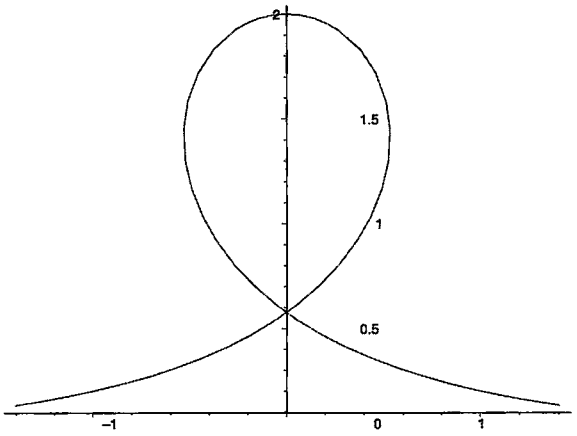

Figure 4.1: Loop soliton

Since coordinates of curve in parametric form are dependent only of size of curve a and parameter z, but not of time t explicitly, the shape and position of curve at any time t is the same. In this case the time evolution is realized by reparametrization of curve's parameter z:

$$z \to \tilde{z} = z - a^2 t.$$

So that the motion of curve is just uniform motion with velocity $v = a^2$ along the shape fixed loop soliton curve.

4.4 The Area Characteristic of MKdV One Soliton Curve

As we have seen in (4.20) the parametric form of the MKdV one soliton curve is,

$$x = \frac{1}{a}(z - 2\tanh z), \quad y = \frac{1}{a}(2/\cosh z),$$

where $-\infty < z < \infty$.
Since shape of curve is fixed under time evolution, the area enclosed by the loop is integral of motion.
The integral below will give the area enclosed by the loop our curve:

$$Area = A = 2\int_{y(0)}^{y_0} x\, dy,$$

where

$$dy = \frac{-2}{a} \quad (\sinh z/\cosh^2 z)dz,$$

and

$$A = \frac{-4}{a^2}\int_0^{z_0}(z - 2\tanh z)(\sinh z/\cosh^2 z)dz.$$

Splitting this integral in two parts we have,

$$A = I + II = \frac{-4}{a^2}\int z \quad (\sinh z/\cosh^2 z) \quad dz + \frac{8}{a^2}\int \quad (\sinh^2 z/\cosh^3 z) \quad dz,$$

where

$$I: \quad z = u \quad \Rightarrow \quad dz = du, \quad (\sinh z/\cosh^2 z)dz = dv \quad \Rightarrow \quad (-1/\cosh z) = v,$$

and

$$I = \frac{4}{a^2} \quad z(1/\cosh z) - \frac{4}{a^2} \quad \int (1/\cosh z)dz,$$

while

$$II : \frac{8}{a^2}\int (1/\cosh z)dz - \frac{8}{a^2}\int (1/\cosh^3 z)dz.$$

Then we obtain

$$A = \frac{4}{a^2} \quad z(1/\cosh z) + \frac{4}{a^2}\int (1/\cosh z)dz - \frac{8}{a^2}\int (1/\cosh^3 z)dz.$$

By changing variable z to l,

$$\sinh z = l, \quad \cosh z dz = dl, \quad \cosh^2 z = 1 + l^2,$$

we have

$$A = \frac{4}{a^2} \quad z(1/\cosh z) + \frac{4}{a^2}\int (1/(1+l^2))dl - \frac{8}{a^2}\int (1/(1+l^2)^2)dl,$$

The second integral in this expression is of the table one, while to calculate the last one we will use the transformation below:

$$l = \tan u, \quad dl = (1+\tan^2 u)du.$$

Taking integrals and changing all variables back to z, we have:

$$A = \frac{4}{a^2} \quad z(1/\cosh z) + \frac{4}{a^2}\arctan(\sinh z) - \frac{4}{a^2}\arctan(\sinh z) - \frac{4}{a^2}(\sinh z/\cosh^2 z),$$

After simplification it gives

$$A = \frac{4}{a^2}\frac{1}{\cosh z}(z - \tanh z)|_0^{z_0}. \tag{4.25}$$

where we take the boundary of the integral from 0 to z_0.Since at the point z_0 coordinate $x(z_0) = 0$ then from (4.22) we have constraint

$$\tanh z_0 = \frac{z_0}{2},$$

After substituting to (4.25) finally we obtain the area enclosed by the loop of the soliton curve:

$$A = \frac{z_0 y(z_0)}{a} = \frac{z_0}{a^2}\sqrt{4 - z_0^2},$$

or with the approximated value of the z_0,

$$A = \frac{1.93}{a^2}.$$

As easy to see this area is completely determined by parameter a representing amplitude of soliton.

4.5 The Angle Characteristic of MKdV One Soliton Curve

Since shape of the soliton curve is fixed, the angle at intersection point is also integral of the motion. In this section, we will find the angle α between tangent lines to the loop soliton at the intersection point. We will use the parametrization of the loop curve (4.22). The slope of the tangent line to the curve at point z is:

$$\frac{dy}{dx} = \frac{dy/dz}{dx/dz} = \frac{y'(z)}{x'(z)} = \tan\alpha(z).$$

so that at point z_0

$$\tan\alpha(z_0) = \frac{y'(z_0)}{x'(z_0)}.$$

Using parametric equations (4.22) we get the angle between tangent line and axis x at point z_0:

$$\alpha(z_0) = \arctan\frac{-2\sinh z_0}{\cosh^2 az_0 - 2}. \tag{4.26}$$

When z_0 changes into $-z_0$ we get:

$$\alpha(-z_0) = \arctan\frac{2\sinh z_0}{\cosh^2 z_0 - 2}.$$

Then, the angle between the tangent lines at the intersection point z_0 is

$$\alpha(-z_0) - \alpha(z_0) = -\arctan\frac{4\sinh z_0}{\cosh^2 z_0 - 2}.$$

Approximate value of this angle is 110°. It is worth to note that the angle is independent of amplitude of soliton a.

4.6 MKdV Two Soliton Solution by Hirota Method

By the similar procedure used in Section 4.2.2 we can obtain two soliton solution for MKdV equation. In the equation of order ϵ^3

$$(\partial_t + \partial_s^3)G_3 = 0$$

instead of trivial solution we choose:

$$G_3 = e^{\eta_2},$$

where,

$$\eta_2 = k_2 s - k_2^3 t + \eta_2^{(0)}. \tag{4.27}$$

Continuing the procedure we find truncation of the series in the form,

$$G = e^{\eta_1} + e^{\eta_2} + \alpha_1 e^{2\eta_1+\eta_2} + \alpha_2 e^{2\eta_2+\eta_1},$$

$$F = 1 + \frac{e^{\eta_1}}{16{k_1}^2} + \frac{1}{2}\frac{e^{\eta_1+\eta_2}}{(k_1+k_2)^2} + \frac{e^{2\eta_2}}{16{k_1}^2} + \beta e^{2\eta_1+2\eta_2},$$

where

$$\alpha_1 = \frac{(k_1-k_2)^2}{16{k_1}^2(k_1+k_2)^2}, \qquad \alpha_2 = \frac{(k_1-k_2)^2}{16{k_2}^2(k_1+k_2)^2},$$

$$\beta = \frac{(k_1-k_2)^4}{256{k_1}^2{k_2}^2(k_1+k_2)^4}.$$

It gives 2-soliton solution of MKdV equation in the form

$$\kappa = \frac{G}{F} = \frac{(e^{\eta_1} + e^{\eta_2} + \alpha_1 e^{2\eta_1+\eta_2} + \alpha_2 e^{\eta_1+2\eta_1})}{1 + \frac{e^{2\eta_1}}{16{k_1}^2} + \frac{1}{2}\frac{e^{\eta_1+\eta_2}}{(k_1+k_2)^2} + \frac{e^{2\eta_2}}{16{k_1}^2} + \beta e^{2\eta_1+2\eta_2}}, \tag{4.28}$$

or after some transformations

$$\kappa = \frac{2k_1k_2|k_1^2-k_2^2|[k_2\cosh(\eta_1+\psi+\phi_1) + k_1\cosh(\eta_2+\psi+\phi_2)]}{(k_1-k_2)^2\cosh(\eta_1+\eta_2+2\psi+\phi_1+\phi_2) + (k_1+k_2)^2\cosh(\eta_1-\eta_2+\phi_1-\phi_2) + 4k_1k_2} \tag{4.29}$$

where

$$\eta_1 + \psi + \phi_1 = k_1(x - k_1^2\tau - X_0^1), \qquad X_0^1 = \frac{1}{k_1}[\eta_1{}^{(0)} + \ln\frac{|k_1-k_2|}{|k_1+k_2|} - \frac{1}{2}\ln 16k_1^2],$$

$$\eta_2 + \psi + \phi_2 = k_2(x - k_2^2\tau - X_0^2), \qquad X_0^2 = \frac{1}{k_1}[\eta_2{}^{(0)} + \ln\frac{|k_1-k_2|}{|k_1+k_2|} - \frac{1}{2}\ln 16k_2^2].$$

When we analyse asymptotic behaviour of this solution we get two interacting solitons, preserving their shapes under collision [Appendix]. But as it was shown in Section 4.3 to every soliton corresponds the loop curve. Therefore our solution (4.28) describes elastic scattering of two loops with parameters (amplitudes) $2/k_1$ and $2/k_2$, where k_1, k_2 are real numbers. It illustrated in figures (4.3) and (4.4).

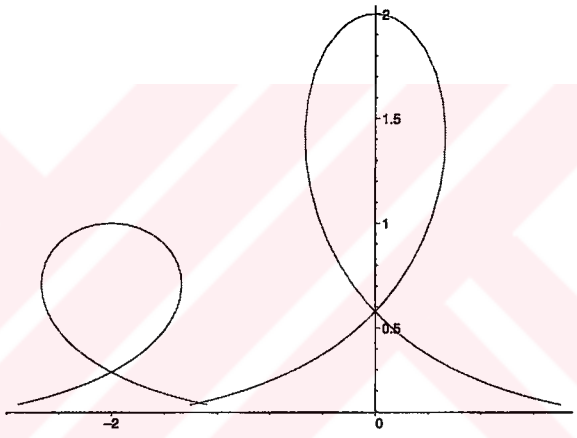

Figure 4.2: MKdV two soliton solution curve at time $t_1 \to -\infty$

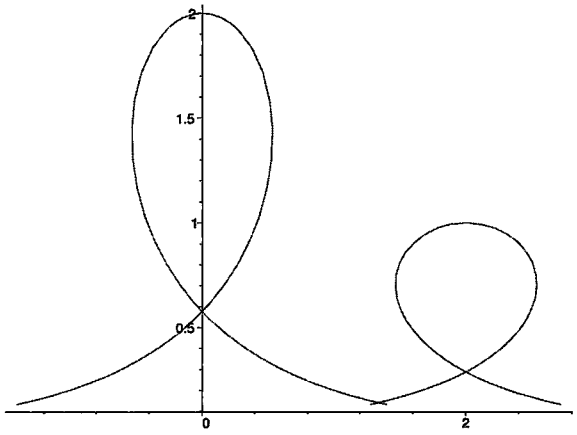

Figure 4.3: MKdV two soliton solution curve at time $t_1 \to +\infty$

4.7 Complex Miura Transformation Between MKdV and KdV

In this section we show that a combination of MKdV-squared curvature and its derivative satisfies the Korteweg de-Vries (KdV) equation. Let MKdV equation be in the general form,

$$\kappa_t + a\kappa^2\kappa_s + \kappa_{sss} = 0,$$

and let KdV equation be in the general form,

$$u_t + buu_s + u_{sss} = 0.$$

Miura's fundamental discovery is [12, 56] that if $\kappa(s,t)$ satisfies the MKdV equation, then

$$u(s,t) = \alpha\kappa(s,t)^2 + \beta\kappa_s(s,t), \tag{4.30}$$

satisfies the KdV equation. Furthermore, this transformation relates these two equations according to expression:

$$KdV(u) = (2\alpha\kappa + \beta\frac{\partial}{\partial s})(mKdV(\kappa)). \tag{4.31}$$

where parameters a and b are related with α and β as

$$a = -\frac{b^2\beta^2}{6} \quad , \quad \alpha = -\frac{b\beta^2}{6}.$$

From these formulas we have several choices for parameters a and b.

$$\begin{aligned} b > 0 \quad &, \quad \beta = i\gamma \quad , \quad a > 0, \\ b > 0 \quad &, \quad \beta > 0, \beta < 0 \quad , \quad a < 0, \\ b < 0 \quad &, \quad \beta = i\gamma \quad , \quad a > 0, \\ b < 0 \quad &, \quad \beta > 0, \beta < 0 \quad , \quad a < 0, \end{aligned}$$

where $\alpha, \beta, \gamma, a, b$ are real numbers. To reproduce MKdV equation (4.2) we need $a = \frac{3}{2}$. Let $b = 3$, then we find that corresponding values of $\alpha = \frac{1}{2}$ and $\beta = i$. So, for this special case,

$$u(s,t) = \frac{1}{2}\kappa^2 + i\kappa_s,$$

and

$$KdV(u) = (\kappa + i\frac{\partial}{\partial s})(mKdV(\kappa)).$$

For the one loop solution (4.15) the related complex solution of KdV equation is,

$$u = \frac{2a^2}{\cosh^2 \eta_1}(1 - i\sinh \eta_1). \tag{4.32}$$

Indeed, for partial derivatives we have,

$$u_t = \frac{2a^5}{\cosh^3 \eta_1}(2\sinh \eta_1 + i(1 - \sinh^2 \eta_1)),$$

$$u_s = \frac{2a^3}{\cosh^3 \eta_1}(-2\sinh \eta_1 + i(\sinh^2 \eta_1 - 1)),$$

$$u_{ss} = \frac{2a^4}{\cosh^4 \eta_1}((-2 + 4\sinh^2 \eta_1) + i(5\sinh^2 \eta_1 - \sinh^3 \eta_1)),$$

$$u_{sss} = \frac{2a^5}{\cosh^5 \eta_1}((16\sinh \eta_1 - 8\sinh^3 \eta_1) + i(5 + \sinh^4 \eta_1 - 18\sinh^2 \eta_1)),$$

then

$$u_t + 3uu_s + u_{sss} =$$

$$\frac{2a^5}{\cosh^5 \eta_1}((-2\sinh\eta_1\cosh^2\eta_1 + 2\sinh\eta_1\cosh^2\eta_1) + i(\cosh^2\eta_1 - 1 - \sinh^2\eta_1)).$$

So this solution satisfies the KdV equation in the form

$$u_t + 3uu_s + u_{sss} = 0.$$

The real part of the solution (4.32) is the one soliton solution of KdV equation:

$$u = \frac{2a^2}{\cosh^2 \eta_1}.$$

Chapter 5

SINE-GORDON EQUATION

5.1 Negative Power Reduction and Sine-Gordon Hierarchy

In previous section we have considered MKdV hierarchy generated by the recursion operator R. Since R is integro-differential operator this is why it is natural to consider not only positive powers but also negative powers of it [18]. This is why let us choose in Eq. (3.32) $U = R^{-2}\kappa_s$. Then,

$$RU = \kappa_t \quad \Rightarrow \quad \kappa_t = R^{-1}\kappa_s \quad \Rightarrow \quad R\kappa_t = \kappa_s. \tag{5.1}$$

In explicit form,

$$R\kappa_t = \kappa_{sst} + \kappa^2\kappa_t + \kappa_s \int_{-\infty}^{s} \kappa\kappa_t ds',$$

or due to (5.1),

$$\kappa_s = \kappa_{sst} + \left(\kappa \int_{-\infty}^{s} \kappa\kappa_t ds'\right)_s.$$

Integrating the above equation once with respect to s we get,

$$\kappa = \kappa_{ts} + \kappa \int_{-\infty}^{s} \kappa\kappa_t ds' + C, \tag{5.2}$$

where C is constant of integration. For simplicity we choose $C = 0$.

Then we define,

$$\theta(s,t) = \int_{-\infty}^{s} \kappa ds', \tag{5.3}$$

which means $\kappa = \frac{\partial \theta}{\partial s}$ and θ has meaning of angle for curvature similar to Eq. (2.13). It is convenient to introduce new function $F(s,t)$ according to formula,

$$\kappa_t = \sin\theta + F(s,t). \tag{5.4}$$

From (5.3) and (5.4) it yields that,

$$\kappa_{st} = (\cos\theta)\theta_s + F_s \quad , \quad \theta_s = \kappa.$$

Then, if we substitute these into (5.2), we get:

$$(\cos\theta)\theta_s + F_s + \theta_s \int_{-\infty}^{s} \theta_s(\sin\theta + F)ds' = \theta_s,$$

or

$$(\cos\theta)\theta_s + F_s - \theta_s\cos\theta(s,t) + \theta_s\cos\theta(-\infty,t) + \int_{-\infty}^{s} \theta_s F ds' = \theta_s,$$

and

$$(\cos\theta)\theta_s + F_s - \theta_s\cos\theta + \theta_s + \int_{-\infty}^{s} \theta_s F ds' = \theta_s.$$

Then we obtain the equality:

$$F_s + \int_{-\infty}^{s} \theta_s F ds' = 0. \tag{5.5}$$

By the chain rule,

$$\frac{\partial F}{\partial s} = \frac{\partial \theta}{\partial s}\frac{\partial F}{\partial \theta} = \kappa\frac{\partial F}{\partial \theta},$$

then,

$$\theta_s \frac{\partial F}{\partial \theta} + \theta_s \int_0^s \theta'_s F ds' = 0,$$

and finally we get:

$$\frac{\partial F}{\partial \theta} + \int_0^\theta F d\theta' = 0. \tag{5.6}$$

Differentiating this equation in θ we find:

$$F_{\theta\theta} + F = 0.$$

Solving this equation we have:

$$F = A\cos\theta + B\sin\theta, \tag{5.7}$$

where A, B are arbitrary constants. The equation (5.4) implies:

$$\theta_{st} = \sin\theta + F.$$

When we substitute (5.7) into the above equation we get,

$$\theta_{st} = (1+B)\sin\theta + A\cos\theta.$$

Let us choose

$$1 + B = \cos\theta_0 \quad , \quad A = \sin\theta_0,$$

then,

$$\theta_{st} = \sin(\theta + \theta_0).$$

Combining

$$\theta + \theta_0 = \varphi,$$

we finally get the Sine-Gordon Equation:

$$\varphi_{st} = \sin\varphi. \tag{5.8}$$

5.2 One Soliton Solution of Sine-Gordon Equation by Hirota Method

We consider a solution of Eq. (5.8) as the dependent variable transformation,

$$\varphi = 2i \ln \frac{F^*}{F}, \tag{5.9}$$

where F is a complex function and F^* is complex conjugate of F. Then we use,

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i},$$

to rewrite

$$\sin\varphi = \frac{1}{2i}\left(\left(\frac{F}{F^*}\right)^2 - \left(\frac{F^*}{F}\right)^2\right). \tag{5.10}$$

Now, we write Eq. (5.8) in terms of F, F^* and s, t derivatives:

$$F^2[2(2F^*_{st}F^* - 2F^*_sF^*_t + F^2) + F^2] - (F^*)^2[2(2F_{st}F - 2F_sF_t + (F^*)^2] = 0, \tag{5.11}$$

to simplify this equation let us suppose that,

$$D_sD_t(F \cdot F) = 2F^*_{st}F^* - 2F^*_sF^*_t = \alpha(F^2 - (F^*)^2),$$

where $\alpha \in R$. Then substituting to (5.11), we get:

$$2\alpha^*((F^*)^2 - F^2)F^2 - 2\alpha(F^2 - (F^*)^2)F^*)^2 + F^4 - (F^*)^4 = 0,$$

which is valid only if

$$\alpha = \alpha^* = \frac{1}{2}.$$

Now with this result we obtain the bilinear representation of Sine-Gordon equation:

$$D_sD_t(F \cdot F) = \frac{1}{2}(F^2 - F^{*2}). \tag{5.12}$$

Writing the formal expansion of F and F^* in powers of ϵ such that:

$$F = 1 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 ...,$$
$$F^* = 1 + \epsilon F_1^* + \epsilon^2 F_2^* + \epsilon^3 F_3^* ...,$$

and substituting this expansions to (5.12), we have

$D_s D_t(1 \cdot 1 + 2\epsilon(F_1 \cdot 1) + \epsilon^2(F_1 \cdot F_1 + 2F_2 \cdot 1) + ...) = \frac{1}{2}(2\epsilon(F_1 - F_1^*) + \epsilon^2(F_1^2 - (F_1^*)^2 + 2(F_2 - F_2^*)) + ...)$,

For ϵ order zero we have identically,

$$D_x D_t(1 \cdot 1) = 0.$$

For ϵ order one we get,

$$D_s D_t(2F_1 \cdot 1) = (F_1 - F_1^*),$$

or

$$2\partial_s \partial_t(F_1) = (F_1 - F_1^*),$$

and

$$2\partial_s \partial_t(F_1^*) = (F_1^* - F_1).$$

Then from this couple of equations we have,

$$(F_1 - F_1^*)_{st} = F_1 - F_1^*. \tag{5.13}$$

Let us choose,

$$F_1 - F_1^* = 2\beta i e^{\eta_1},$$

so that substituting this to (5.13) we find:

$$F_1 = i\beta e^{\eta_1}, F_1^* = -i\beta e^{\eta_1}, \quad \eta_1 = k_1 s + \frac{1}{k_1} t + \eta_1^{(0)},$$

where $k_1, \eta_1^{(0)}, \beta$ are real numbers.

For ϵ order two,

$$D_s D_t(F_1 \cdot F_1 + 2F_2 \cdot 1) = \frac{1}{2}(F_1^2 - (F_1^*)^2 + 2(F_{12} - F_2^*)).$$

Since

$$D_s D_t(F_1 \cdot F_1) = 0$$

and

$$F_1 + F_1^* = 0,$$

the above equation reduces to the form:

$$D_s D_t(F_2 \cdot 1) = \frac{1}{2}(F_2 - F_2^*),$$

or

$$\partial_s \partial_t(F_2) = \frac{1}{2}(F_2 - F_2^*).$$

We will choose $F_2 = F_2^* = 0$. This choice with following conditions,

$$F_n = F_n^* = 0 \quad , \quad n \geq 2,$$

truncates infinite series and provides exact one soliton solution of the Sine-Gordon equation in the form:

$$F = 1 + i\beta e^{\eta_1}, \quad F^* = 1 - i\beta e^{\eta_1},$$

$$\varphi = 2i \ln \frac{1 - i\beta e^{\eta_1}}{1 + i\beta e^{\eta_1}}. \tag{5.14}$$

5.3 Recovering Sine-Gordon One Soliton Curve

In previous section we found one soliton solution of Eq. (5.8) in the form:

$$\varphi = 2i \ln \frac{1 - i\beta e^{\eta_1}}{1 + i\beta e^{\eta_1}},$$

where $\eta_1 = k_1 s + \frac{1}{k_1} t + \eta_1^{(0)}$ and $k_1, \eta_1^{(0)}, \beta$ are real numbers.

Now we are going to recover the curve corresponding to the one soliton solution.The parametric form of the curve is given by Eq. (4.17) such that

$$\mathbf{r}(s,t) = \mathbf{e}_1 \int^s \cos\varphi(s')ds' + \mathbf{e}_2 \int^s \sin\varphi(s')ds'.$$

By Eq. (5.8), the second projection after integration is

$$\int^s \sin\varphi(s')ds' = \varphi_t,$$

and for the one soliton solution (5.13), we have

$$\int^s \sin\varphi(s')ds' = \frac{4\beta}{k_1}\left(\frac{e^{\eta_1}}{1+\beta^2 e^{2\eta_1}}\right).$$

Using identity,

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2},$$

and definition of φ (5.9) we have,

$$\cos\varphi = \frac{1}{2}\frac{(F^4 + (F^*)^4)}{F^2(F^*)^2}.$$

Then,

$$\int \cos\varphi(s')ds' = \frac{1}{2}\int^s \frac{(F^4 + (F^*)^4)}{F^2(F^*)^2} ds',$$

and substituting F, F^* from Eq. (5.14) we have,

$$\int \cos\varphi(s')ds' = \int \frac{1-6\beta^2e^{2\eta_1}+\beta^4e^{4\eta_1}}{1+2\beta^2e^{2\eta_1}+\beta^4e^{4\eta_1}}ds'.$$

Simplifying, this expression

$$\int \cos\varphi(s')ds' = \int (1-\frac{8\beta^2e^{2\eta_1}}{(1+\beta^2e^{2\eta_1})^2})ds',$$

after changing of variable, in the second integral

$$\beta^2e^{2\eta_1}+1 \equiv u \quad , \quad 2\beta^2e^{2\eta_1}k_1ds = du,$$

we have,

$$\int \cos\varphi(s')ds' = \int (1-\frac{4}{k_1u^2})du,$$

so that

$$\int^s \cos\varphi(s')ds' = s + \frac{4}{k_1}\frac{1}{\beta^2e^{2\eta 1}+1}.$$

Then we find the parametric form of the Sine-Gordon one soliton curve as,

$$\mathbf{r}(s,t) = \mathbf{e}_1\left[s+\frac{4}{k_1}\frac{1}{\beta^2e^{2\eta_1}+1}\right] + \mathbf{e}_2\left[\frac{4\beta}{k_1}\left(\frac{e^{\eta_1}}{\beta^2e^{2\eta_1}+1}\right)\right]. \tag{5.15}$$

where $\eta_1 = k_1s + \frac{1}{k_1}t + \eta_1^{(0)}$ and $k_1, \eta_1^{(0)}, \beta$ are real numbers.

Let $\beta = e^{\phi_0}$, so that $\beta e^{\eta_1} \equiv e^z$, choosing $\beta > 0$. (The negative value of β gives the same curve reflected under axis x), where parameter of the curve is $z = \eta_1 + \phi_0$.

For x projection we have

$$x(s,t) = s + \frac{4}{k_1}\frac{1}{e^{2z}+1}.$$

Using the definition of tanh function, from the last expression we get:

$$x(z,t) = s + \frac{2}{k_1}(1-\tanh z),$$

After simplifications and changing variable to z we obtain

$$x(z,t) = \frac{1}{k_1}(z-\tanh z) - \frac{1}{k_1^2}t + \frac{2-\eta_1^{(0)}-\phi_0}{k_1},$$

Shifting coordinate system in x direction by constant term, we find

$$x(z,t) = \frac{1}{k_1}(z - \tanh z) - \frac{1}{k_1^2}t.$$

From another side for y coordinate we have parametric form

$$y(s,t) = \frac{2}{k_1}\frac{e^z}{e^{2z+1}}.$$

Using the definition of cosh function and changing variable to z, we get:

$$y(z,t) = \frac{2}{k_1}\frac{1}{\cosh z}.$$

Finally, the parametric form of the Sine-Gordon curve can be written as,

$$\mathbf{r}(z,t) = \mathbf{e}_1\left[\frac{1}{k_1}(z - \tanh z) - \frac{1}{k_1^2}t\right] + \mathbf{e}_2\left[\frac{2}{k_1}\frac{1}{\cosh z}\right]. \tag{5.16}$$

where $z = k_1 s + \frac{1}{k_1}t + \eta_1^{(0)} + \phi_0$.

From this expression we can see that at any fixed time t our curve represent the loop soliton. But in contrast with MKdV case, now our curve in addition is moving in negative direction with constant speed $1/k_1^2$. In Figure 5.1 we reproduce the motion of one-soliton curve.

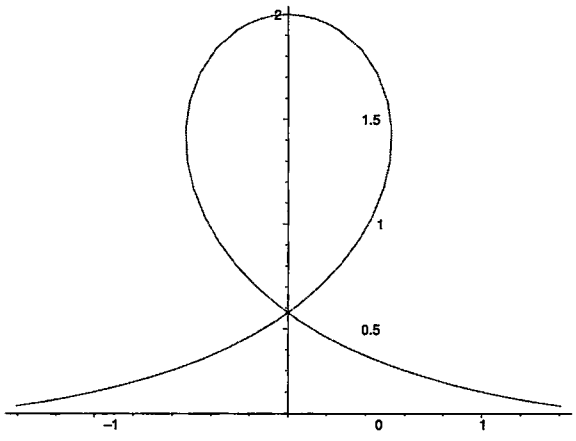

Figure 5.1: Sine-Gordon one soliton curve

Chapter 6

THREE DIMENSIONAL CURVE MOTION

In the present chapter we study the general motion of curve in three dimensional space [17, 52, 30]. In this case we have to use the full set of Serret-Frenet equations (2.8)-(2.10).

6.1 Equations of Time Evolution

Now we will specify time evolution equations for the vectors $\mathbf{t}(s,t), \mathbf{n}(s,t), \mathbf{b}(s,t)$ and for curvature and torsion $\kappa(s,t)$ and $\tau(s,t)$ respectively. As we know from Eq. (3.4) the time evolution of $\mathbf{r}(\alpha,t)$ for three dimensional motion is characterized by equation:

$$\frac{\partial \mathbf{r}}{\partial t} = \dot{\mathbf{r}} = U\mathbf{n} + V\mathbf{b} + W\mathbf{t}.$$

From equations (3.4) and (3.9),

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{r}(\alpha,t)}{\partial t} = g^{1/2}\frac{\partial}{\partial s}\frac{\partial \mathbf{r}(\alpha,t)}{\partial t} = g^{1/2}\frac{\partial}{\partial s}\left(U\mathbf{n} + V\mathbf{b} + W\mathbf{t}\right).$$

Then, using Serret-Frenet equations (2.8)-(2.10) we get

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{r}}{\partial t} = g^{1/2}\left(\frac{\partial W}{\partial s} - U\kappa\right)\mathbf{t} + \left(\frac{\partial U}{\partial s} + W\kappa - V\tau\right)\mathbf{n} + \left(\frac{\partial V}{\partial s} + U\tau\right)\mathbf{b}. \quad (6.1)$$

From another side using again (3.9),

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{r}}{\partial \alpha} = \frac{\partial}{\partial t}\left(g^{1/2}\frac{\partial \mathbf{r}}{\partial s}\right),$$

and using equation (3.3) we get,

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{r}}{\partial \alpha} = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial g}{\partial t}\mathbf{t} + g^{1/2}\frac{\partial \mathbf{t}}{\partial t}. \tag{6.2}$$

Compatibility condition (3.5) for projection of Eqs. (6.1) and (6.2) in direction of $\mathbf{t}$ gives the time evolution of the metric g:

$$\frac{\partial g}{\partial t} = 2g\left(\frac{\partial W}{\partial s} - U\kappa\right).$$

Then,

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\left(\frac{\partial U}{\partial s} - V\tau + W\kappa\right)\mathbf{n} + \left(\frac{\partial V}{\partial s} + U\tau\right)\mathbf{b}\right).$$

Taking α derivative and applying (3.9) to this expression,

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{t}(\alpha,t)}{\partial \alpha} = \frac{\partial}{\partial t}\left(g^{1/2}\frac{\partial \mathbf{t}}{\partial s}\right) = \frac{\partial}{\partial t}\left(g^{1/2}\kappa\mathbf{n}\right),$$

and using above time evolution equation for g, we find,

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{t}(\alpha,t)}{\partial \alpha} = g^{1/2}\left(\left(\frac{\partial W}{\partial s}\kappa - U\kappa^2 + \frac{\partial \kappa}{\partial t}\right)\mathbf{n} + \kappa\frac{\partial \mathbf{n}}{\partial t}\right). \tag{6.3}$$

From another side,

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{t}(\alpha,t)}{\partial t} = g^{1/2}\frac{\partial}{\partial s}\left(\frac{\partial \mathbf{t}}{\partial t}\right),$$

and by using the Serret-Frenet equations we get,

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{t}(\alpha,t)}{\partial t} = g^{1/2}\left(\frac{\partial^2 U}{\partial s^2} + \frac{\partial W}{\partial s}\kappa + W\frac{\partial \kappa}{\partial s} - \frac{\partial V}{\partial s}\tau - V\frac{\partial \tau}{\partial s} - U\tau^2\right)\mathbf{n}$$

$$-\kappa\left(\frac{\partial U}{\partial s} + W\kappa - V\tau\right)\mathbf{t} + \left(\frac{\partial^2 V}{\partial s^2} + 2\frac{\partial U}{\partial s}\tau + U\frac{\partial \tau}{\partial s} + W\kappa\tau - V\kappa^2\right)\mathbf{b}. \tag{6.4}$$

By using Eqs. (3.6) and (3.17), projected (6.5) on direction of $\mathbf{n}$, we find evolution equation for curvature κ:

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 U}{\partial s^2} + U(\kappa^2 - \tau^2) + \frac{\partial \kappa}{\partial s}\int \kappa U ds' - 2\tau \frac{\partial V}{\partial s} - \frac{\partial \tau}{\partial s} V.$$

Then, the time evolution of vector $\mathbf{n}$ is as follows:

$$\frac{\partial \mathbf{n}}{\partial t} = \left(-\frac{\partial U}{\partial s} + \tau V - \kappa W\right)\mathbf{t} + \frac{1}{\kappa}\left[\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \tau\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right]\mathbf{b}.$$

The vector $\mathbf{n}$ also satisfies the compatibility condition such that:

$$\frac{\partial}{\partial \alpha}\frac{\partial \mathbf{n}(\alpha, t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{n}(\alpha, t)}{\partial \alpha}. \tag{6.5}$$

From this condition we get evolution equations for torsion τ and binormal vector $\mathbf{b}$:

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s}\left(\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V\right) + \tau\int \kappa U ds'\right) + \kappa\tau U + \kappa\frac{\partial V}{\partial s}.$$

$$\frac{\partial \mathbf{b}}{\partial t} = -\left(\frac{\partial V}{\partial s} + \tau U\right)\mathbf{t} - \left(\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right)\mathbf{n}.$$

As a result, time evolution equations for variables $g, s, \mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau$ in three dimension are given below:

$$\frac{\partial g}{\partial t} = 2g\left(\frac{\partial W}{\partial s} - U\kappa\right), \tag{6.6}$$

$$\frac{\partial s}{\partial t} = W - \int_0^s U\kappa ds'. \tag{6.7}$$

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial U}{\partial s} - V\tau + W\kappa\right)\mathbf{n} + \left(\frac{\partial V}{\partial s} + U\tau\right)\mathbf{b}, \tag{6.8}$$

$$\frac{\partial \mathbf{n}}{\partial t} = \left(-\frac{\partial U}{\partial s} + \tau V - \kappa W\right)\mathbf{t} + \frac{1}{\kappa}\left(\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \tau\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right)\mathbf{b}, \tag{6.9}$$

$$\frac{\partial \mathbf{b}}{\partial t} = -\left(\frac{\partial V}{\partial s} + \tau U\right)\mathbf{t} - \left(\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right)\mathbf{n}, \tag{6.10}$$

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 U}{\partial s^2} + U(\kappa^2 - \tau^2) + \frac{\partial \kappa}{\partial s}\int \kappa U ds' - 2\tau\frac{\partial V}{\partial s} - \frac{\partial \tau}{\partial s}V, \tag{6.11}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s}\left(\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V\right) + \tau\int \kappa U ds'\right) + \kappa\tau U + \kappa\frac{\partial V}{\partial s}. \tag{6.12}$$

It is easy to see that these formulas are 3-dimensional generalizations of equations (3.22)-(3.26) and reduce to them when $\tau = 0$ and $V = 0$.

6.2 Relation With Nonlinear Evolution Equations

To establish relation of above equations with nonlinear evolution equations like in section 3.3, we first reformulate our problem as 2×2 matrix problem. We show that the Serret-Frenet equations, (2.8), (2.9), (2.10) are equivalent to the matrix system of equations:

$$\begin{pmatrix} \frac{\partial \omega_1}{\partial s} \\ \\ \frac{\partial \omega_2}{\partial s} \end{pmatrix} = \begin{pmatrix} i\zeta & q(s,t) \\ \\ r(s,t) & -i\zeta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \\ \omega_2 \end{pmatrix}, \tag{6.13}$$

at $\zeta = 0$, with reduction $r = -q^*$.
The last system is called the Zakharov-Shabat problem [3] According to the Serret-Frenet equations each set of components of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ satisfies,

$$\frac{\partial \mathbf{t}_j}{\partial s} = \kappa \mathbf{n}_j, \quad \frac{\partial \mathbf{n}_j}{\partial s} = -\kappa \mathbf{t}_j + \tau \mathbf{b}_j, \quad \frac{\partial \mathbf{b}_j}{\partial s} = -\tau \mathbf{n}_j \quad , \quad (j = 1, 2, 3). \tag{6.14}$$

If we multiply the first equation with $\mathbf{t}_j$ (j. component), the second one with $\mathbf{n}_j$ and the third one with $\mathbf{b}_j$ then sum all of them, we obtain as a result that these equations admit an integral, which we normalize to one,

$$\mathbf{t}_j^2 + \mathbf{n}_j^2 + \mathbf{b}_j^2 = 1, \quad (j = 1, 2, 3).$$

We define complex function,

$$\phi(s,t) = \kappa(s,t)\epsilon, \tag{6.15}$$

such that the modulus of function is given by curvature of the curve and the phase is given by integral of the torsion.

$$\epsilon = exp \quad (i \int^s \tau(s',t) ds'),$$

Let us denote the phase

$$(i \int^s \tau(s',t) ds') \equiv a.$$

Then introduce complex vector

$$N_j \equiv (\mathbf{n}_j + i\mathbf{b}_j)\epsilon, (j = 1, 2, 3).$$

For derivative we have equation

$$\frac{\partial N_j}{\partial s} = (\mathbf{n}_j + i\mathbf{b}_j)\frac{\partial \epsilon}{\partial s} + \left(\frac{\partial \mathbf{n}_j}{\partial s} + i\frac{\partial \mathbf{b}_j}{\partial s}\right)\epsilon,$$

or

$$\frac{\partial N_j}{\partial s} = (\mathbf{n}_j + i\mathbf{b}_j)i\tau\epsilon + (-\kappa\mathbf{t}_j + \tau\mathbf{b}_j - i\tau\mathbf{n}_j)\epsilon,$$

and after simplification

$$\frac{\partial N_j}{\partial s} = -\kappa\epsilon\mathbf{t}_j = -\phi\mathbf{t}_j.$$

Since combination

$$\phi N_j^* + \phi^* N_j = \kappa\epsilon(\mathbf{n}_j - i\mathbf{b}_j)exp(-ia) + \kappa\epsilon(\mathbf{n}_j + i\mathbf{b}_j)exp(-ia) = 2\kappa\mathbf{n}_j,$$

is given by the normal vector $\mathbf{n}$, from Eq. (6.14) we obtain relation

$$\frac{\partial \mathbf{t}_j}{\partial s} = \frac{1}{2}(\phi N_j^* + \phi^* N_j).$$

So Eqs. (6.14) turns into the form,

$$\frac{\partial N_j}{\partial s} = -\phi\mathbf{t}_j \quad , \quad \frac{\partial \mathbf{t}_j}{\partial s} = \frac{1}{2}(\phi N_j^* + \phi^* N_j). \tag{6.16}$$

Then we introduce new functions ω_1, ω_2 by the additional transformation,

$$\omega_1 = N_j \quad exp\left(\frac{1}{2}\int^s \left(\frac{\phi N_j^*}{1 - \mathbf{t}_j}\right) ds'\right),$$

$$\omega_2 = (1 - \mathbf{t}_j) exp\left(\frac{1}{2}\int^s \left(\frac{\phi N_j^*}{1-\mathbf{t}_j}\right) ds'\right),$$

For simplicity we denote

$$\left(\frac{1}{2}\int^s \left(\frac{\phi N_j^*}{1-\mathbf{t}_j}\right) ds'\right) = \alpha.$$

Then, we have equation

$$\frac{\partial \omega_1}{\partial s} = \frac{\partial N_j}{\partial s} e^\alpha + N_j \frac{1}{2} \frac{\phi N_j^*}{1-\mathbf{t}_j} e^\alpha,$$

or

$$\frac{\partial \omega_1}{\partial s} = \left(-\phi \mathbf{t}_j + \frac{1}{2}\frac{\phi N_j^* N_j}{1-\mathbf{t}_j}\right) e^\alpha.$$

Since $N_j^* N_j = \mathbf{n}_j^2 + \mathbf{b}_j^2$, it simplifies to the form

$$\frac{\partial \omega_1}{\partial s} = \frac{1}{2}\phi \omega_2,$$

For the second function ω_2 we have

$$\frac{\partial \omega_2}{\partial s} = \frac{1}{2}\frac{\phi N_j^*}{1-\mathbf{t}_j} e^\alpha - \frac{\partial \mathbf{t}_j}{\partial s} e^\alpha - \mathbf{t}_j \frac{1}{2}\frac{\phi N_j^*}{1-\mathbf{t}_j} e^\alpha.$$

Excluding $\partial \mathbf{t}_j / \partial s$ by the second Eq.(6.16) we have

$$\frac{\partial \omega_2}{\partial s} = -\frac{1}{2}\phi^* N_j e^\alpha,$$

or simply

$$\frac{\partial \omega_2}{\partial s} = -\frac{1}{2}\phi^* \omega_1.$$

Finally, under this transformations the Serret-Frenet equations (6.14) turns into the below system,

$$\frac{\partial \omega_1}{\partial s} = \frac{1}{2}\phi \omega_2 \quad , \quad \frac{\partial \omega_2}{\partial s} = -\frac{1}{2}\phi^* \omega_1.$$

These equations are equivalent to the system (6.13) at $\zeta = 0$ with $q = -\phi/2$, $r = -q^*$:

$$\begin{pmatrix} \frac{\partial \omega_1}{\partial s} \\ \\ \frac{\partial \omega_2}{\partial s} \end{pmatrix} = \begin{pmatrix} 0 & -\phi/2 \\ \\ -\phi^*/2 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \\ \omega_2 \end{pmatrix}. \tag{6.17}$$

Under the same transformations time evolution of the Serret-Frenet basis turns into the below linear system,

$$\frac{\partial \omega_1}{\partial t} = \frac{i}{4} a_1 \omega_1 - \frac{1}{4}(a_2 - i a_3)\omega_2,$$

$$\frac{\partial \omega_2}{\partial t} = \frac{1}{4}(a_2 + i a_3)\omega_1 - \frac{i}{4} a_1 \omega_2.$$

These are equivalent to (6.13) at $\zeta = a_1/4$ with $q = -\frac{1}{4}(a_2 - i a_3)$, $r = -q$ and matrix representation is:

$$\begin{pmatrix} \frac{\partial \omega_1}{\partial t} \\ \\ \frac{\partial \omega_2}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{i}{4} a_1 & -\frac{1}{4}(a_2 - i a_3) \\ \\ \frac{1}{4}(a_2 + i a_3) & -\frac{i}{4} a_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \\ \omega_2 \end{pmatrix}, \tag{6.18}$$

where,

$$a_1 = \frac{dU}{ds} - V\tau + W\kappa \quad , \quad a_2 = \frac{dV}{ds} + U\tau,$$

$$a_3 = \frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right).$$

Now we will substitute (6.15) into evolution equations (6.11),

$$e^{-ia}\left(\frac{\partial \phi}{\partial t} - i\phi \int^s \tau_t ds'\right)$$

$$= \frac{\partial^2 U}{\partial s^2} + (\phi^2 e^{-2ia} - \tau^2)U + e^{-ia}\left(\frac{\partial \phi}{\partial s} - i\phi\tau\right)\int^s \phi e^{-ia} U ds' - 2\tau\frac{\partial V}{\partial s} - \frac{\partial \tau}{\partial s}V,$$

Then multiplying both sides with e^{ia}, substituting Eq. (6.12) into the integral and using the definition $\phi^* = \kappa e^{-ia}$, we get equation for the time evolution

$$\frac{\partial \phi}{\partial t} = \left(\frac{\partial^2}{\partial s^2} + |\phi|^2 + i\phi \int^s ds' \tau \phi^* + \frac{\partial \phi}{\partial s} \int^s ds' \phi^* \right) (U\epsilon)$$

$$+ \left(i\frac{\partial^2}{\partial s^2} + i|\phi|^2 + i\phi \int^s ds' \tau \phi^* - i\phi \int^s ds' \frac{\partial \phi^*}{\partial s'} \right) (V\epsilon) \quad . \tag{6.19}$$

6.3 Nonlinear Schrödinger Equation

Let $U = 0$ and $V = \kappa$, then applying these to (6.19) we get

$$\frac{\partial \phi}{\partial t} = \left(\frac{\partial^2}{\partial s^2} + |\phi|^2 + i\phi \int^s ds' \tau \phi^* + \frac{\partial \phi}{\partial s} \int^s ds' \phi^* \right) (\kappa\epsilon).$$

By definition (6.15) this equation can be written in terms of function ϕ only

$$i\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial s^2} + |\phi|^2 \phi + \phi \int^s ds' \left(i\tau \phi \phi^* + \frac{\partial \phi^*}{\partial s'} \phi \right) = 0,$$

or

$$\phi \int^s ds' \left(i\tau \phi \phi^* + \frac{\partial \phi^*}{\partial s'} \phi \right) = \frac{1}{2} \phi \int^s ds' \frac{\partial |\phi|^2}{\partial s'} = \frac{1}{2} |\phi|^2.$$

Finally we get the Nonlinear Schrödinger Equation for complex function ϕ:

$$i\frac{\partial \phi}{\partial t} + \phi_{ss} + \frac{1}{2}|\phi|^2 \phi = 0. \tag{6.20}$$

6.4 One Soliton Solution by Hirota Method

To find a solution of equation (6.20) we suppose

$$\phi = \frac{G}{F},$$

where G is complex function while F is real function. Then we write (6.20) in terms of Hirota derivatives,

$$\frac{1}{F^2}\left[iD_t(G\cdot F)+D_s^2(G\cdot F)\right]-\frac{G}{F^3}\left[D_s^2(F\cdot F)-\frac{1}{2}GG^*\right]=0.$$

Since instead of one complex function ϕ we introduced one complex function G and one real function F we have freedom to fix the real one by additional constraints. Thus separating terms multiplied by $1/F^2$, G/F^3, we obtain the bilinear equations below:

$$(iD_t+D_s^2)(G\cdot F)=0, \tag{6.21}$$

$$D_s^2(F\cdot F)=\frac{1}{2}GG^*. \tag{6.22}$$

Now following the same Hirota strategy as before we expand:

$$\begin{aligned} F&=1+\epsilon^2F_2+\epsilon^4F_4+\epsilon^6F_6\ldots \quad ,\\ G&=\epsilon G_1+\epsilon^3G_3+\epsilon^5G_5\ldots \quad ,\\ G^*&=\epsilon G_1^*+\epsilon^3G_3*+\epsilon^5G_5^*\ldots \quad . \end{aligned}$$

Substituting expansions to Eqs. (6.21), (6.22) we get:

$$(iD_t+D_s^2)(\epsilon G_1\cdot 1+\epsilon^3(G_3\cdot 1+G_1\cdot F_2)+\epsilon^5(F_4\cdot G_1+F_2\cdot G_3)\ldots)=0,$$

$$D_s^2(1\cdot 1+2\epsilon^2F_2\cdot 1+\epsilon^4(F_3\cdot F_3+2F_4\cdot 1)+\ldots)=\frac{1}{2}(\epsilon^2G_1G_1^*+\epsilon^4(G_1G_3^*+G_1^*G_3)+\ldots).$$

From this system at ϵ order zero we have identically

$$D_s^2(1\cdot 1)=0.$$

At ϵ order one we have equation:

$$(D_t+D_s^2)(G_1\cdot 1)=0,$$

or

$$(\partial_t + \partial_s^2)G_1 = 0.$$

The simplest nontrivial solution of this equation is,

$$G_1 = e^{\eta_1}, \quad \eta_1 = k_1 s + i k_1^2 t + \eta_1^{(0)}. \tag{6.23}$$

where k_1 and $\eta_1^{(0)}$ are complex constants.
At ϵ order two we have

$$D_s^2(F_2 \cdot 1) = \frac{1}{4} G_1 G_1^*,$$

or

$$(\partial_s^2)F_2 = \frac{1}{4} G_1 G_1^*.$$

To integrate this equation we expand complex parameters

$$k_1 = \xi + i\zeta, \quad \eta_1^{(0)} = a + ib,$$

in terms of real numbers ξ, ζ, a, b. Then,

$$\eta_1 + \eta_1^* = (k_1 + k_1^*)s + (k_1^2 - k_1^2)t + \eta_1^{(0)} + (\eta_1^{(0)})^*,$$

or using above definitions,

$$\eta_1 + \eta_1^* = 2\xi s - 4\xi\zeta t + 2a.$$

Then,

$$\partial s^2 F_2 = \frac{1}{4} e^{\eta_1 + \eta_1^*}.$$

Integrating this equation twice we get:

$$F_2 = \frac{1}{16\xi^2} e^{\eta_2}, \eta_2 = 2\xi s - 4\xi\zeta t + 2a, \tag{6.24}$$

At ϵ order three, we have:

$$(iD_t + D_s^2)(G_1 \cdot F_2 + G_3 \cdot 1) = 0,$$

By direct calculations, we find

$$(iD_t + D_s^2)(G_1 \cdot F_2) = 0,$$

then we conclude that,

$$(iD_t + D_s^2)(G_3 \cdot 1) = 0.$$

The simplest choice is if we suppose $G_3 = 0$ and truncate the series:

$$F_n = 0, \quad n = 2, 4, \ldots \quad , \quad G_m = 0, \quad m = 3, 5, \ldots$$

Then we obtain a one soliton solution of NLS equation:

$$\phi(s,t) = \frac{e^{\eta_1}}{(1 + e^{2\xi s - 4\xi\zeta t + 2a})/16\xi^2},$$

where

$$\eta_1 = (\xi + i\zeta)s + (i(\xi^2 - \zeta^2) - 2\xi\zeta)t + a + ib.$$

To simplify this expression we denote $\frac{1}{4\xi} = e^{-\tilde{\varphi}}$ and multiply numerator and dominator with $4\xi e^{-\xi s + 2\xi\zeta t - a}$ then we get

$$\phi(s,t) = 2\xi \frac{e^{i(\xi s + (\xi^2 - \zeta^2)t + b)}}{\cosh(\xi s - 2\xi\zeta t + a - \tilde{\varphi})},$$

or finally

$$\phi(s,t) = 2\xi \frac{e^{i(\xi s + (\xi^2 - \zeta^2)t - \varphi_0)}}{\cosh \xi(s - 2\zeta t - s_0)}, \tag{6.25}$$

where s_0, φ_0 are real numbers. This expression describes envelope soliton of NLS equation.

Now we can derive curvature κ and the torsion τ of a curve. Corresponding to this soliton solution, according to definition (6.15) is

$$\phi(s,t) = \kappa(s,t)exp(i\int^s \tau(s',t)ds').$$

Then for our solution (6.25) curvature of a curve is travelling wave

$$\kappa(s,t) = \frac{2\xi}{\cosh\xi(s-2\zeta t-s_0)}. \tag{6.26}$$

This is exactly curvature of the one loop soliton curve like MKdV case (4.15). But in contrast to the planar character of MKdV loop curve, now

1. the velocity of the loop is arbitrary constant parameter,
2. the torsion of curve is nonzero in general.

For torsion of the curve we have

$$\tau = \xi. \tag{6.27}$$

As we can see this curve has constant torsion proportional to the soliton's amplitude.

6.5 The Complex Modified Korteweg-de Vries Equation

Let in evolution (6.18) $U = -\kappa_s$ and $V = -\kappa\tau$, then we obtain,

$$\frac{\partial\phi}{\partial t} = -\frac{\partial^2}{\partial s^2}(\kappa_s\epsilon + i\tau\phi) - |\phi|^2(\kappa_s\epsilon + i\tau\phi)$$

$$-\phi\int ds'\left(i\kappa_s\epsilon\tau\phi^* - \phi\phi^*\tau^2 - i\frac{\partial\phi^*}{\partial s'}\phi\tau\right) - \frac{\partial\phi}{\partial s}\int^s ds'\phi^*\kappa_s\epsilon$$

since

$$i\kappa_s\epsilon\tau\phi^* - \phi\phi^*\tau^2 - i\frac{\partial\phi^*}{\partial s'}\phi\tau = 0,$$

and

$$\int^s ds' \phi^* \kappa_s \epsilon = \frac{1}{2} |\phi|_s^2.$$

Then the time evolution is described by the Complex Modified Korteweg-de Vries Equation:

$$\frac{\partial \phi}{\partial t} + \phi_{sss} + \frac{3}{2} |\phi|^2 \phi_s = 0. \tag{6.28}$$

So as we see the motion of curve in three dimension is related with two important equations from soliton theory, The Nonlinear Schrödinger Equation and complex MKdV Equation. In both cases since natural equations of curve include two real functions, curvature and torsion, the resulting wave function is complex function.

Chapter 7

CONCLUSIONS

The simple idea of a smooth curve moving in space, recently becomes one of the basic objects in understanding of fundamental physical laws and applied science. Developments of the high energy physics in the second half of twenty century, attempting to unify all fundamentals forces of nature, including gravity, lead to the new concept of the space-time. Instead of collection of points, the space of modern unifying theories consists of fundamental strings. The different motions of the string, taking place in multi-dimensional space, lead to the spectrum of elementary particles-the building blocks of the Nature. Development of this, so called string theory, takes the origin from simple mathematical concepts of nineteenth century, describing curve in terms of curvature and torsion. But the recent progress mainly is related with quantization of this curve motions. To have consistent quantization one has to use the hidden symmetries in the motion of string. These symmetries are infinite dimensional and can be systematically treated by theory of integrable systems with infinite hierarchy of soliton equations. This is why in the modern fundamental physics the mathematical theory of solitons become the crucial topic. From another site, strings also appear in description of strong interaction of nucleons, in relativistic formulation of point particle motion, in the theory of magnetic monopoles. More generally, we can speak about motion of line defects in condensed matter and cosmology, considered as an dynamical interface problem with many applications. Remarkably these problems are connected with results of classical differential geometry and the soliton theory of modern times.

In the present thesis we analyzed some of these deep relations between geometry and solitons. Starting from basic curve theory and studying problem of motion of the curve in plane we found that natural evolution of curve is described by infinite MKdV hierarchy of soliton equations with infinite number of

integrals of motion. Solving MKdV equation and recovering curve from natural equations we found geometrical image of one soliton solution as a loop curve. We found geometrical characteristics of this curve, as the area characteristic and the angle characteristic representing integrals of motion of the system. By using direct Hirota method, we constructed two soliton solution of MKdV and found that asymptotically it describes elastic scattering of two solitons. The corresponding curve dynamics consists from collision of two loop curves, preserving they geometrical characteristics. By the negative powers of the recursion operator we found description of the curve motion in terms the Sine-Gordon equation. From the one soliton solution of this equation, we recovered the loop curve moving with constant speed. Considering more general motion of curve in three dimensional space, we found relations with Nonlinear Schrödinger equation and complex MKdV equations. We showed that the loop soliton in this case has fixed torsion during the motion. We expect that the rich variety of soliton equations, their exact solvability and relations with different fields of mathematics will have deep impact on understanding of curves as a fundamental objects of fundamental and applied sciences.

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APPENDIX A

MKdV Two Soliton Solution and Asymptotic Analysis

In the Hirota bilinear representation for MKdV equation (4.2), the solution of equation supposed in the form:

$$\kappa = \frac{G}{F},$$

where F and G are real functions of s and t. The second Eq. (4.11) gives

$$2\frac{D_s^2(F\cdot F)}{F^2} = \frac{1}{2}\frac{G^2}{F^2} = \frac{1}{2}\kappa^2.$$

Then, we have expression for curvature, directly in terms of function F only,

$$\kappa^2 = 2\frac{D_s^2(F\cdot F)}{F^2} = 4\frac{FF_{ss}-F_s^2}{F^2} = 4(\ln F)_{ss}. \tag{A.1}$$

For our two soliton solution,

$$F = 1 + \frac{e^{\eta_1}}{4{k_1}^2} + \frac{1}{2}\frac{e^{\eta_1+\eta_2}}{(k_1+k_2)^2} + \frac{e^{2\eta_2}}{4{k_1}^2} + \beta e^{2\eta_1+2\eta_2},$$

where

$$\beta = \frac{(k_1-k_2)^4}{256{k_1}^2{k_2}^2(k_1+k_2)^4}.$$

We choose

$$0 < k_1 < k_2,$$

$$\eta_i = k_i s - k_i^3 t + \eta_i^{(0)} = k_i(s - v_i t) = k_i \xi_i, i = 1, 2,$$

where $v_i = k_i^2$. To analyse asymptotic behaviour of this solution we will choose two frames.

1) In the first frame, let $\xi_1 = s - v_1 t =$ constant, then

$$\eta_2 = k_2 \xi_2 = k_2 \xi_1 + k_2[(v_1 - v_2)t + (s_{0_1} - s_{0_2})], \quad (v_1 - v_2) < 0.$$

(a) When $s \to +\infty, \quad t \to +\infty$ we have $e^{\eta_2} \to 0$ then

$$F \approx 1 + \frac{e^{2\eta_1}}{16{k_1}^2},$$

and one soliton solution due to the (A.1) is

$$\kappa = \frac{2k_1}{\cosh k_1(s - k_1^2 t - s_{0_1} - \frac{\ln 4k_1}{k_1})}.$$

(b) When $s \to -\infty, \quad t \to -\infty$ we have $e^{\eta_2} \to +\infty$ then

$$F \approx \frac{e^{2\eta_2}}{16k_1^2}(1 + 16{k_2}^2 \beta e^{2\eta_1}),$$

and one soliton solution is

$$\kappa = \frac{2k_1}{\cosh k_1(s - k_1^2 t - s_{0_1} + \frac{\ln 4k_2\sqrt{\beta}}{k_1})}.$$

2) In the second frame, let $\xi_2 = s - v_2 t =$ constant, then

$$\eta_1 = k_1 \xi_1 = k_1 \xi_2 + k_1[(v_2 - v_1)t + (s_{0_2} - s_{0_1})], \quad (v_2 - v_1) > 0.$$

(a) When $s \to +\infty, \quad t \to +\infty$ we have $e^{\eta_1} \to +\infty$ then

$$F \approx \frac{e^{2\eta_1}}{16k_1^2}(1 + 16{k_1}^2 \beta e^{2\eta_2}),$$

and one soliton solution

$$\kappa = \frac{2k_2}{\cosh k_2(s - k_2^2 t - s_{0_2} + \frac{\ln 4k_1\sqrt{\beta}}{k_2})}.$$

(b) When $s \to -\infty, \quad t \to -\infty$ we have $e^{\eta_1} \to 0$ then

$$F \approx 1 + \frac{e^{2\eta_2}}{16{k_2}^2},$$

finally one soliton solution due to the (A.1) is

$$\kappa = \frac{2k_2}{\cosh k_2(s - k_2^2 t - s_{0_2} - \frac{\ln 4k_2}{k_2})}.$$

Phase shift for the first solution is

$$s_{0_1}^+ - s_{0_1}^- = \frac{-2\ln 2 + \ln k_1 k_2 + \ln\sqrt{\beta}}{k_1}$$

Phase shift for the second solution is

$$s_{0_2}^+ - s_{0_2}^- = \frac{2\ln 2 + \ln k_1 k_2 + \ln\sqrt{\beta}}{k_2}$$

Then

$$k_1 \Delta s_{0_1} = -k_2 \Delta s_{0_2}. \tag{A.2}$$

Analysis above shows that two soliton solution of (4.2) describes collision of two solitons with phase shift. By Eq. (A.2) we conclude that the solitons have phase shift in opposite directions. Since for single soliton we have obtained one soliton curve as a loop curve, for every asymptotic solution we will have a loop soliton curve. Since solitons preserve shape and size after collision, geometrically we have collision of two loops with parameter k_1 and k_2.