INTEGRABLE VORTEX DYNAMICS AND COMPLEX BURGERS' EQUATION

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by
Zeynep Nilhan GÜRKAN

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İZMİR

We approve the thesis of **Zeynep Nilhan GÜRKAN**

Date of Signature

. . . . **19 October 2005**
Prof. Dr. Oktay PASHAEV
Supervisor
Department of Mathematics
İzmir Institute of Technology

. . . . **19 October 2005**
Prof. Dr. F. Acar SAVACI
Department of Electrical and Electronics Engineering
İzmir Institute of Technology

. . . . **19 October 2005**
Assist. Prof. Dr. Ali İhsan NESLİTÜRK
Department of Mathematics
İzmir Institute of Technology

. . . . **19 October 2005**
Prof. Dr. Oğuz YILMAZ
Head of Department
İzmir Institute of Technology

. . . .
Assoc. Prof. Dr. Semahat ÖZDEMİR
Head of the Graduate School

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ABSTRACT

Integrable dynamical models of the point magnetic vortex interactions in the plane are studied. Reformulating the Euler equations for vorticity in the Helmholtz form, the Hamiltonian and Lax representations are found. Reduction of these equations for the point vortices to the Kirchhoff equations, and non-integrability of the system of $N \geq 4$ hydrodynamical vortices are discussed. As an integrable model of planar motion with given vorticity for the stationary flow, the Liouville equation and its solutions are given. For non-stationary flows, exactly solvable case of point planar vortex diffusion and exactly solvable Initial Value Problem for the one dimensional Burgers equation are solved. By the complexified Cole-Hopf transformation, the complex Burgers equation with integrable N vortex dynamics is introduced and linearization of this equation in terms of the complex Schrödinger equation is found. This allows us to construct N vortex configurations in terms of the complex Hermite polynomials, the vortex chain lattices and study their mutual dynamics. Mapping of our vortex problem to N-particle problem, the complexified Calogero-Moser system, showing its integrability and Hamiltonian structure is given.

As an application of the general results, we consider the problem of magnetic vortices in a magnetic fluid model. The holomorphic reduction of topological magnetic system to the linear complex Schrodinger equation, allows us to apply all results on integrable vortex dynamics in the complex Burgers equation to the magnetic vortex evolution, including magnetic vortex lattices and the bound states of vortices.

ÖZET

Tezimde düzlemde noktasal manyetik vorteks etkileşimlerinin integrallenebilir dinamik modellerini çalıştık. Vorteks özelliğinin bulunabilmesi için Euler denklemleri Helmholtz formunda yeniden yazılarak Hamilton ve Lax gösterimlerini bulduk. Bu denklemlerin noktasal vorteksler için Kirşof denklemlerine indirgenmesi ve $N \geq 4$ hidrodinamik vorteks sisteminin integrallenebilir olmadığını tartıştık. Sabit akım için verilen vorteks özelliği ile düzlemsel hareketin integrallenebilir bir modeli olarak Liouville denklemi ve çözümlerini inceleyerek sabit olmayan akımlar için noktasal vorteks difuzyonunun tam çözümlü durumu ve bir boyutlu Burgers denkleminin tam çözümlü başlangıç değer problemini çözdük. Karmaşık Cole-Hopf dönüşümünü kullanarak N-vorteks dinamiği ile Karmaşık Burgers denklemi tanıtıp, karmaşık Schrödinger denklemi cinsinden linearizasyonunu bulduk. Bunu bulmamız karmaşık Hermite polinomları ve vorteks zincir latisleri cinsinden N-vorteks konfigurasyonunu kurmamıza ve ortak dinamiklerini çalışmamıza izin vermiştir. Problemimizi N parça problemine çevirerek karmaşık Calogero-Moser sisteminin integrallenebilirliği ve Hamiltonyen yapısını bulduk.

Genel sonuçların bir uygulaması olarak manyetik akışkan modelindeki manyetik vorteks problemini ele aldık. Topolojik manyetik sistemin doğrusal karmaşık Schrödinger denklemine indirgenmesi, Karmaşık Burgers' denklemindeki integrallenebilir vorteks dinamiği üzerindeki tüm sonuçları manyetik vorteks evrimine, manyetik vorteks kafeslerine ve vortekslerin sınır durumlarına uygulamamıza izin vermiştir.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

"Strepsiades: But is it not Zeus who forced them to move?

Socrates: Not at all; it's the aerial Whirlwind.

Strepsiades: The Whirlwind! Ah! I did not know that. So Zeus, it seems,has no existence, and its the Whirlwind that reigns in his stead?"

Aristophanes "The Clouds" (419 BC)

The concept of vortex motion has a long history starting from ancient times and includes many famous names (Kozlov 1998). Main ideas of Descartes vortex theory are represented in "Discours de la methode" (1637) and in a capital work "Principia Philosophiae" (1644). Cartesian cosmology is based on primordial chaos, which by motion according to fixed laws is ordering to cosmos. According to Descartes the Universe is filled by thin all-penetrable fluid (similar to the ether) which is in a permanent rotational motion. The term vortex itself, "tourbillon", is coming from comparison with turbulent motion of a river. The vortex model of gravity was proposed by Descartes, Bernoulli, Stokes and Huygens. But soon it was displaced by the Newton's gravity theory for a long time. Only in the middle of XIX century the interest to vortex theory revives with works of Helmholtz (1821-1894) (Helmholtz 1858), Thompson (Lord Kelvin) (Kelvin 1869), Kirchhoff (1824-1887) (Kirchhoff 1876) on the vortex motion of an ideal fluid. The mathematical description of processes related with the motion of vortex in a liquid is starting from Helmholtz's paper "Uber Integrale der Hydrodynamischen Gleichungen Welche den Wirbelbewegungen Entsprechen" (1858) in which he formulated his theorem on conservation of vorticity in the rotational motion of a fluid. He also notices an analogy between the fluid motion and the magnetic action of electric fields. General equations of motion for N point vortices (Kirchhoff's equations) have been introduced by Kirchhoff in his lectures in mathematical physics (Kirchhoff 1876). He derived corresponding Hamiltonian form of equations and found all possible integrals of the motion. In contrast with Newton's equation for N point masses, having the second order, the Kirchhoff equations are the first order of vortex coordinates.

Walter Gröbli (Gröbli 1877) in his thesis "Specille Probleme uber die Bewegung Geradliniger Parallerer Wirbelfaden" in 1877 analyzed the integrable problem of motion of three vortices in the plane. He obtained the system of three nonlinear equations possessing two integrals of motion and allowing to get explicit quadrature. He also considered particular case of the problem of four vortices under condition of symmetry axis and more general problem of $2N$ vortices with N symmetry axes.

The interest to vortex theory increases with Kelvin's vortex theory of atoms "On Vortex Atoms" , (Kelvin 1867). But soon his model was dismissed by quantum mechanical model of atoms (Lomonaco 1996). Kelvin also posed the problem of stability under the stationary rotation of the system of N point vortices located at N polygon vertices. He noticed that the problem is similar to the problem of stability for the system of equal magnets floating in external magnetic field. Experiments with floating magnets performed by Mayer, leaded him to the conclusion that for number of vortices (magnets) exceeding 5, rotating polygon becomes unstable (in fact the case $N = 6$ is stable). The linear stability of polygon has been studied by J.J. Thompson (who discovered the electron). He found that for $n \leq 6$ the linear stability takes place, while for $n \geqslant 8$ is not. Stability of the case $n = 7$ needs nonlinear analysis and after several failed attempts has been proved only recently (Meleshko and Konstantinov 1993). Non-integrability of the four vortex problem in plane, indicating on chaotization of the vortex motion, was found recently (Ziglin 1982).

Many problems involving interfacial motion (Lushnikov 2004) can be cast in the form of vortex sheet dynamics (Saffman 1992). The discovery of coherent structures in turbulence, increases expectations that the study of vortices will lead to models and an understanding of turbulent flow, one of the great unsolved problems of classical physics. Vortex dynamics is a natural paradigm for the field of chaotic motion and modern dynamical system theory (Poincare 1893). The theory of line vortices and vortex rings is a part of modern theory of liquid Helium II. Interaction of vortex structures essentially influences on processes in atmosphere and the ocean. In techniques complete understanding of friction to the motion, noise generation, shock waves, is impossible without clear theory of the vortex motion. In a wide and important class of motions of ideal inviscid fluid (Meleshko and Van Heijst 1994), the vortex dynamics provides physically profound examples of nonlinear Hamiltonian systems of infinite dimensions, attracting much interest in relation with chaotic phenomena in dynamical systems (Borisov and Mamaev 2003).

Theoretical progress in the study of vortex motions is essentially related with the development of computational techniques and effective numerical methods of computation. They allow numerical modelling of vortex interactions in three dimensions.

Applications of vortices extend from liquid crystals and ferromagnets (Kuratsuji and Yabu 1998, Komineas and Papanicolaou 1998) to superfluids (Ho and Mermin 1980, Mermin and Ho 1976), and from non-equilibrium patterns to Quantum Hall effect (Ezawa 2000) cosmic strings (Thess et al. 1999, Correa et al. 2001), (Pismen 1999). A superfluid (^4He) is characterized by flow without dissipation. The order parameter of this flow is represented by a complex scalar field. Vortices are readily formed in moving superfluids. The idea of superflow dominated by vortex filaments has been proposed by (Feynman 1955). A superfluid vortex is a topological defect of a complex scalar field and has quantized circulation. This is the fundamental distinction between superfluid and classical vortex.

Nano-scale magnets can have, according to their shape and size, ordered domain structures such as magnetic vortices and single domains. Experimental and theoretical studies of static and dynamical magnetic properties of nano-scale magnets as expected allow better understanding of the quantum behavior associated with domain wall displacement and magnetization reversal.

The goal of the present thesis is to study integrable dynamics of vortices in the plane and its applications to the planar ferromagnetic fluids.

In Chapter 2 we formulate the Euler equations in two dimensions (Section 2.1) and relations between incompressible and irrotational flow and analytic functions theory, Section 2.2. The vortices and sources of the flow are interpreted as the simple pole singularities of the complex velocity in Section 2.3.

In Chapter 3 we present some integrable models of planar motion in the fluid. Reformulating the Euler equations for vorticity in the Helmholtz form (Section 3.1), first we discuss the vortex-source reciprocity relations in Section 3.2. The Hamiltonian and Lax representations of the Euler equation are found in Section 3.3.

Reduction of these equations for the point vortices to the Kirchhoff equations, and non-integrability of the system of $N \geq 4$ hydrodynamical vortices are discussed in Section 3.4. As an integrable model of planar motion with given vorticity for the stationary flow, Section 3.6, the Liouville equation and its solutions as distributed finite set of vortices and as the periodic lattice of vortices are given in Section 3.7.

In Chapter 4 we consider non-stationary viscous flow described by the Navier-Stokes equation Section 4.1 and in vorticity form by the Helmholtz equations , Section 4.2. Exactly solvable case of diffusion of point planar vortex is the subject of Section 4.3. Then we treat in details the one-dimensional Burgers' equation, Section 4.4, its linearization, Section 4.5, the Initial Value Problem, Section 4.6, shock soliton structure and asymptotic shock generation from initial step function, Sections 4.7 and 4.8 correspondingly.

In Chapter 5, by interpreting the relation between complex velocity and complex potential, as the complexified Cole-Hopf transformation, the complex Burgers equation, Section 5.1, with integrable N vortex dynamics is introduced. Linearization of this equation in terms of the complex Schrödinger equation is given in Section 5.2. The identification of vortices with complex zeroes of this equation allows us to construct N vortex configurations and represent them in terms of the complex Hermite polynomials. Solutions in the form of the vortex chain lattices, and their interaction with vortices are considered in Sections 5.3 and 5.4.

In Chapter 6 we establish relation of N vortex equations with the Calogero-Moser multiparticle systems, Section 6.1, showing integrability and the Hamiltonian structure for N vortex, Section 6.2 and N-vortex lattices, Section 6.3.

As an application of the general results, in Chapter 7 we consider the problem of magnetic vortices in a magnetic fluid model. We formulate the topological magnet model in Section 7.1 and its stereographic projection representation in Section 7.2. The anti-holomorphic reduction of topological magnetic system to the linear complex Schrödinger equation in considered in Section 7.3. In Section 7.4 we study special form of topological magnet as the Ishimori model. Applying all results on integrable vortex dynamics in the complex Burgers equation to the magnetic vortex evolution, we construct N magnetic vortices in Section 7.5, and study their dynamics in Section 7.6. By time dependent Schrödinger problem in harmonic potential, Section 7.7 , we construct the bound state of N vortices in Section 7.8.

In Conclusions, Chapter 8, we discuss main results obtained in this thesis. In Appendices we calculate in details some results of the main text. In Appendices A.1 and A.2 we survey Cauchy Integral Representation, Argument principle and Rouche Theorem. Vorticity form of Euler equation is derived in Appendix B. The Lax representation is subject of Appendix B.2. Green Function solution for Laplace Operator, Appendix B.3, and the point vortex solution in polar coordinates, Appendix B.4, are found. The integrals of motion for the Kirchhoff equations are studied in Appendix B.5. In Appendix B6 system of equations describing N vortex system is derived. We review properties of Hermite polynomials with real and complex arguments in Appendix C. I.V.P for the Burgers equation and particular solution for the initial function in the form of the step function is solved in details in Appendix D. Finally, in Appendices E.1 and E.2 the Euler's Homogeneous Function Theorem and Buckingham's Pi Theorem are introduced.

CHAPTER 2

EULER EQUATIONS IN TWO DIMENSIONS

2.1 Mathematical Models of Fluids in Two Dimensions. Euler Equations.

Definition 2.1.0.1 *The Euler Equations of an ideal fluid in R^2 are the following evolution equations on the velocity field $\vec{u}(x, y, t)$ and the density $\rho(x, y, t)$ of the fluid:*

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \vec{f} \tag{2.1}$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho = 0 \tag{2.2}$$

The first system of equations (2.1) is just the Newton's Second Law for fluid particles, where p means pressure in the fluid and $\vec{f}$ is an external field (Chorin and Marsden 1992, Arnold et al. 1998). The second equation (2.2) is the continuity equation of the flow. Here the material derivative is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \tag{2.3}$$

and $\nabla = (\partial/\partial x, \partial/\partial y)$ is the gradient operator.

These equations describe the special case of parallel layer flow in three dimensions, when exists such a direction $\vec{N}$ that the whole velocity field $\vec{u}$ is orthogonal to this vector. Moreover, in all planes orthogonal to $\vec{N}$, the picture of the flow is identical. Then, such a flow is completely characterized by the flow in one of the planes orthogonal to $\vec{N}$. In our consideration we choose a coordinate system with direction z along the vector $\vec{N}$, and components of velocity field along other two directions x and y denote as u_1 and u_2.

2.2 Incompressible and Irrotational Stationary Flow

For the stationary flow (Lavrentiev and Shabat 1973), $\partial_t \vec{u} = 0$, the velocity field is time independent $\vec{u} = (u_1(x,y), u_2(x,y))$. If the flow is irrotational

$$rot\,\vec{u} = 0 \tag{2.4}$$

then

$$\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x} = 0 \tag{2.5}$$

implies existence of a real function $\varphi(x,y)$, called the *velocity potential*, such that $\vec{u} = \nabla\varphi$ or in components

$$u_1 = \frac{\partial \varphi}{\partial x} \quad , u_2 = \frac{\partial \varphi}{\partial y}. \tag{2.6}$$

Indeed Eq.(2.5) is integrability condition showing that the differential form

$$u_1 dx + u_2 dy = d\varphi$$

is exact.

For incompressible flow ($g = const.$) the continuity Eq. (2.2) is reduced to

$$div\,\vec{u} = 0 \tag{2.7}$$

or

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0. \tag{2.8}$$

It implies existence of a real function $\psi(x,y)$ called the *stream function*, such that

$$u_1 = \frac{\partial \psi}{\partial y} \quad , u_2 = -\frac{\partial \psi}{\partial x}. \tag{2.9}$$

Indeed, Eq.(2.7) is exactness condition for the differential form

$$-u_2 dx + u_1 dy = d\psi.$$

The direction of tangent line to the curve $\psi(x,y) = const$ which is determined by equation

$$-u_2 dx + u_1 dy = d\psi = 0. \tag{2.10}$$

coincides with direction of the velocity vector $dy/dx = u_2/u_1$. This is why the integral curves $\psi(x,y) = const$ are the stream lines of the flow.

2.3 Complex Potential and Complex Velocity

Eqs. (2.6) and (2.9) imply Cauchy-Riemann equations

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \ , \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{2.11}$$

which means that the velocity potential and the stream function are harmonically conjugate functions (Saff and Snider 2003). In fact Eqs.(2.11) imply that these functions satisfy the Laplace equation

$$\Delta \varphi = 0 \quad \text{and} \quad \Delta \psi = 0 \tag{2.12}$$

and determine the real and imaginary parts of analytic function $f(z)$ of complex variable $z = x + iy$,

$$\frac{\partial}{\partial \bar{z}} f(z) = 0 \tag{2.13}$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

This function is called the *complex potential*

$$f(z) = \varphi(x, y) + i\psi(x, y). \tag{2.14}$$

From the above consideration we can see that the stationary incompressible and irrotational flow is described by this analytic function. Conversely, any analytic function of $f(z)$ represents complex potential for the steady flow of an ideal incompressible and irrotational fluid. The real part of this function $\Re f = \varphi$ has meaning of the velocity potential and $\Im f = \psi$ is the stream function of the flow. The one parameter families of curves $\varphi(x, y) = \alpha$ and $\psi(x, y) = \beta$ where α and β are constants, are orthogonal families called the equipotential lines and streamlines of the flow respectively. In steady motion, streamlines represent the actual paths of fluid particles in the flow pattern. The velocity vector $\vec{u} = (u_1, u_2)$ at any point $z = x+iy$ can be represented by one complex function u, called the complex velocity:

$$u = u_1 + iu_2 = \frac{\partial \varphi}{\partial x} - i\frac{\partial \psi}{\partial x} = \overline{f'(z)} \tag{2.15}$$

This shows that complex velocity u is an anti-analytic function of $z = x + iy$.

2.3.1 The Complex Electrostatic Potential

Let r be the distance between two point electric charges q_1 and q_2. Then the force between them in the vacuum is given in magnitude by *Coulomb's law* which states that

$$F = \frac{q_1 q_2}{r^2} \tag{2.16}$$

and is one of the repulsion or attraction according as the charges are like (both positive or negative) or unlike (one positive and the other is negative).

Suppose we are given a charge distribution which may be continuous, discrete, or a combination (Spiegel 1964). This charge distribution sets up an electric field. If a unit positive charge (small enough so as not to affect the field appreciably) is placed at any point A not already occupied by charge, the force acting on this charge is called the *electric field intensity* at A and is denoted by $\vec{E}$. For the static field $rot\vec{E} = 0$. This implies existence of *electrostatic potential* φ

$$\vec{E} = -grad\,\varphi = -\nabla\varphi. \tag{2.17}$$

If the charge distribution is two dimensional,then we have complex function

$$\mathcal{E} = E_1 + iE_2 = -\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}. \tag{2.18}$$

where

$$E_1 = -\frac{\partial\varphi}{\partial x}, \quad E_2 = -\frac{\partial\varphi}{\partial y} \tag{2.19}$$

In such case if E_t denotes the component of the electric field intensity tangential to any simple closed curve C in the z plane,

$$\oint_C E_t ds = \oint_C E_1 dx + E_2 dy = 0 \tag{2.20}$$

Theorem 2.3.1.1 (Gauss' Theorem) *Let us confine ourselves to charge distributions which can be considered two dimensional. If C is any simple closed curve in the z plane having a net charge q in its interior (actually an infinite cylinder enclosing a net charge q) and E_n is the normal component of the electric field intensity , then* Gauss' theorem *states that*

$$\oint_C E_n ds = 4\pi q \tag{2.21}$$

If C does not enclose any net charge , this reduces to

$$\oint_C E_n ds = \oint_C E_1 dy - E_2 dx = 0 \tag{2.22}$$

It follows that in any region not occupied by charge,

$$\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} = 0 \tag{2.23}$$

From (2.19)and (2.23), we have

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{2.24}$$

i.e. φ is harmonic at all points not occupied by charge.

In addition, Eq.(2.23) means $div\vec{E} = 0$ and implies the existence of stream function $\psi(x, y)$ such that From the above Eqs. (2.19) and (2.23) it is evident that a harmonic function ψ is conjugate to φ so that

$$f(z) = \varphi(x, y) + i\psi(x, y) \tag{2.25}$$

is analytic in any region not occupied by charge. We call $f(z)$ the *complex electrostatic potential* or , briefly, *complex potential.* In terms of this, (2.19) becomes

$$\mathcal{E} = -\frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} + i\frac{\partial \psi}{\partial y} = -\overline{\frac{df}{dz}} = -\overline{f'(z)} \tag{2.26}$$

and the magnitude of $\mathcal{E}$ is given by

$$E = |\mathcal{E}| = |f'(z)| \tag{2.27}$$

The curves (cylindrical surfaces in three dimensions)

$$\varphi(x, y) = \alpha, \quad \psi(x, y) = \beta \tag{2.28}$$

are called *equipotential lines* and *flux lines* respectively. The analogy of the above electrostatic picture with fluid flow is quite apparent. The electric field in electrostatic problems corresponds to the velocity field in fluid flow problems, the only difference being a change of sign in the corresponding complex potentials.

The complex (electrostatic) potential due to a line charge q per unit length at z_0 (in a vacuum) is given by

$$f(z) = -2q \ln(z - z_0) \tag{2.29}$$

and represents a source or sink according to $q < 0$ or $q > 0$. The idea of sources and sinks for electrostatics have corresponding analogs for the fluid flow.

2.3.2 Sources and Vortices as Singular Points of Analytic Function

If the flow takes part not in a disk $|z| < R$ but in annular domain $R_1 < |z| < R_2$, then the complex potential has Laurent series expansion around every singular point. According to the type of singularities we have next classification of the flow configurations (Lavrentiev and Shabat 1973).

Definition 2.3.2.1 *The logarithmic singularity of complex potential*

$$f(z) = \frac{N}{2\pi} Log z \tag{2.30}$$

corresponding to

$$f'(z) = \frac{N}{2\pi}\frac{1}{z} \tag{2.31}$$

and to the pole singularity of the complex velocity

$$u = \overline{f'(z)} = \frac{N}{2\pi}\frac{1}{\bar{z}} \tag{2.32}$$

at $\bar{z} = 0$ *is called a source* $(N > 0)$ *or sink* $(N < 0)$*. Parameter* N *is the strength of the source/sink.*

The velocity field components in this case are:

$$u_1 + iu_2 = \frac{N}{2\pi}\frac{z}{|z|^2} = \frac{N}{2\pi}\frac{x + iy}{x^2 + y^2} \tag{2.33}$$

$$u_1 = \frac{N}{2\pi}\frac{x}{r^2}, \quad u_2 = \frac{N}{2\pi}\frac{y}{r^2} \tag{2.34}$$

$$\vec{u} = (u_1, u_2) = \frac{N}{2\pi}(\frac{x}{r^2}, \frac{y}{r^2}) = \frac{N}{2\pi}\frac{\vec{r}}{r^2} \tag{2.35}$$

and the stream lines are radial rays, while the equipotential lines are concentric circles.

Definition 2.3.2.2 *The logarithmic singularity of complex potential*

$$f(z) = -\frac{\Gamma i}{2\pi} Log z \tag{2.36}$$

corresponding to

$$f'(z) = -\frac{\Gamma i}{2\pi}\frac{1}{z} \tag{2.37}$$

and to the pole singularity of complex velocity

$$u = \overline{f'(z)} = \frac{\Gamma i}{2\pi}\frac{1}{\bar{z}} \tag{2.38}$$

at $\bar{z} = 0$ *is called a vortex.*

The velocity field components in this case are:

$$u_1 + iu_2 = \frac{\Gamma i}{2\pi}\frac{z}{|z|^2} = \frac{i\Gamma}{2\pi}\frac{x+iy}{x^2+y^2} \tag{2.39}$$

$$u_1 = \frac{\Gamma}{2\pi}\frac{-y}{r^2}, \qquad u_2 = \frac{\Gamma}{2\pi}\frac{x}{r^2} \tag{2.40}$$

$$u = (u_1, u_2) = \frac{\Gamma}{2\pi}(\frac{-y}{r^2}, \frac{x}{r^2}) \tag{2.41}$$

It is easy to see that vectors $\vec{u}$ and $\vec{r}$ are orthogonal

$$\vec{u} \cdot \vec{r} = 0 \tag{2.42}$$

and the stream lines are concentric circles, while the equipotential lines are radial rays. Parameter Γ has meaning of the vortex strength (circulation) since:

$$\oint \vec{u}d\vec{s} = \frac{\Gamma}{2\pi}\oint \frac{-ydx + xdy}{x^2+y^2} = \frac{\Gamma}{2\pi}\int_0^{2\pi} d\phi = \Gamma \tag{2.43}$$

Definition 2.3.2.3 *For the complex potential and velocity*

$$\begin{aligned}
C &= N - i\Gamma && (2.44)\\
f(z) &= \frac{C}{2\pi}Logz && (2.45)\\
f'(z) &= \frac{C}{2\pi}\frac{1}{z}\\
u &= \overline{f'(z)} = \frac{\overline{C}}{2\pi}\frac{1}{\bar{z}} = \frac{N+i\Gamma}{2\pi}\frac{1}{\bar{z}}
\end{aligned}$$

we have mixed source-vortex configuration.

Definition 2.3.2.4 *n-stationary vortices/sources are given by complex potential and velocity*

$$\begin{aligned}
f(z) &= \sum_{k=1}^{n}\frac{C_k}{2\pi}Log(z - z_k) && (2.46)\\
C_k &= N_k - i\Gamma_k && (2.47)\\
u &= \sum_{k=1}^{n}\frac{\overline{C_k}}{2\pi}\frac{1}{\bar{z} - \bar{z}_k} && (2.48)
\end{aligned}$$

For n-vortices of the same strength Γ *we have*

$$\begin{aligned} f(z) &= -i\sum_{k=1}^{n}\frac{\Gamma}{2\pi}Log(z-z_k) \\ &= -\frac{i\Gamma}{2\pi}\sum_{k=1}^{n}Log(z-z_k) \\ &= -\frac{i\Gamma}{2\pi}Log\prod_{k=1}^{n}(z-z_k) \end{aligned} \quad (2.49)$$

For n sources of the same strength N *we have*

$$f(z) = \frac{N}{2\pi}Log\prod_{k=1}^{n}(z-z_k) \quad (2.50)$$

Points $z_1, ..., z_n$ *correspond to the position of vortices/sources.*

Function

$$\phi(z) = \prod_{k=1}^{n}(z-z_k) \quad (2.51)$$

for a vortex/source at point z_k has simple zero $\sim (z-z_k)$. Thus, the set of vortices/sources on the plane of the same intensity C is characterized by the set of zeros of function $\phi(z)$, such that complex potential is

$$f(z) = \frac{C}{2\pi}Log\,\phi(z)$$

and complex velocity is

$$u(z) = \overline{f'(z)} = \frac{\overline{C}}{2\pi}\frac{\partial}{\partial\bar{z}}\overline{Log\phi(z)} = \frac{\overline{C}}{2\pi}\overline{(\frac{\phi_z}{\phi})} \quad (2.52)$$

Then, to every zero of $\phi(z)$ corresponds to the pole of $u(z)$.

$$u(z) = -2\nu\overline{(\frac{\phi_z}{\phi})} \quad (2.53)$$

where $\nu = -\dfrac{\overline{C}}{4\pi}$ is constant.

For the real values of z this representation $u = -2\nu\dfrac{\phi_x}{\phi}$ has form of the Cole-Hopf transformation (Cole 1951 , Hopf 1950). For $u = u(x,t)$ and $\phi = \phi(x,t)$ it relates the nonlinear Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad (2.54)$$

with the linear heat equation:

$$\phi_t = \nu\phi_{xx}. \quad (2.55)$$

Therefore, representation (2.53) for the complex flow can be considered as generalized complex Cole-Hopf transformation (CCH). If we have Complex Burgers' Equation for complex velocity $u = u(\bar{z}, t)$:

$$iu_t + uu_{\bar{z}} = \nu u_{\bar{z}\bar{z}} \tag{2.56}$$

then by CCH transformation

$$u = -2\nu \frac{\partial}{\partial \bar{z}} Log\Phi(\bar{z}, t) = -2\nu \frac{\Phi_{\bar{z}}}{\Phi} \tag{2.57}$$

we have the linear complex Schrödinger (Heat) equation,

$$i\Phi_t = \nu \Phi_{\bar{z}\bar{z}}, \tag{2.58}$$

where $\Phi(\bar{z}, t) \equiv \overline{\phi(z, t)}$. Every solution of (2.58) determines solution of the complex Burgers' equation (2.56). Moreover, every simple zero of Schrödinger equation (2.58) determines simple pole of the complex velocity and the complex Burgers' equation. Let $\bar{z}_k$ is simple zero of Φ:

$$\Phi(\bar{z}_k) = 0 \tag{2.59}$$

so that

$$\Phi(\bar{z}) = F(\bar{z})(\bar{z} - \bar{z}_k) \tag{2.60}$$

where $F(\bar{z}_k) \neq 0$. Then

$$u(\bar{z}) = (-2\nu) \left[\frac{1}{\bar{z} - \bar{z}_k} + \frac{F'(\bar{z})}{F(\bar{z})} \right] \tag{2.61}$$

has simple pole at $\bar{z} = \bar{z}_k$.

For one simple zero at point $\bar{z}_0$:

$$\Phi(\bar{z}) = \bar{z} - \bar{z}_0(t) \tag{2.62}$$

from equation (2.58) follows that $\Phi_t = 0$, and $\frac{d\bar{z}_0}{dt} \Rightarrow \bar{z}_0 = const.$ and corresponding velocity $u = -2\nu \frac{1}{\bar{z} - \bar{z}_0}$ describes the stationary vortex/source located at point $\bar{z}_0$ with the strength $N = -4\pi\nu_R$ and circulation $\Gamma = -4\pi\nu_I$, where $\nu = \nu_R + i\nu_I$. Thus, for Burgers Equation one-vortex is always stationary. Situation is different if we have more then one $n > 1$ vortices. In this case vortices are moving in the plane and undergo mutual collisions. In Sec.5 we will describe in details the dynamics of vortices for complex Burgers equation.

CHAPTER 3

INTEGRABLE MODELS OF ROTATIONAL MOTION IN TWO DIMENSIONS

In previous section we discussed incompressible and irrotational flows in terms of analytic function theory. In the present section we will consider incompressible but rotational motion of fluid in two space dimensions.

3.1 The Vorticity Form of Euler Equations

If we consider the Euler equations (2.1) for zero pressure and without external forces, then it reduces to the form (Lavrentiev and Shabat 1973)

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = 0 \tag{3.1}$$

To characterize rotation properties of a flow one introduces vorticity Ω of the flow according formula

$$\vec{\Omega} = rot\, \vec{u} \tag{3.2}$$

For the planar motion $\vec{u} = \vec{u}(x, y) = (u_1(x, y), u_2(x, y), 0)$ the vorticity vector $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is orthogonal to the plane, $\vec{\Omega} = (0, 0, \Omega_3)$, so that only one nontrivial component of this vector $\Omega_3 \equiv \Omega$ is

$$\Omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{3.3}$$

Written in terms of the stream function (2.9)

$$u_1 = \frac{\partial \psi}{\partial y} \quad , u_2 = -\frac{\partial \psi}{\partial x} \tag{3.4}$$

it has the form

$$\Omega = -(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}) = -\Delta \psi \tag{3.5}$$

As we have seen in Section 2, to analytic function in a disk corresponds irrotational and incompressible flow. This flow can be characterized by complex potential

$f(z)$ (2.14). Then the stream function for such flow is given by the imaginary part of this potential $\psi = \Im f(z)$, and vorticity function is

$$\Omega = -\Delta\psi = -\Delta\Im f(z) \tag{3.6}$$

Since imaginary part of any analytic function is harmonic function, then vorticity, $\Omega = 0$, is vanishing everywhere in the disk. But if the function (complex potential) has some singular point and is analytic in annular domain around this point, then it admits Laurent series representation. As we have seen in Section 2.3.2, logarithmic singularities of stream function, leading to the pole singularity of the complex velocity have hydrodynamic interpretation as sources and vortices. This is why in this case the vorticity function vanishes in the annular domain except the singular points.

3.2 The Vortex-Source Reciprocity Relations

If we consider the point vortex located at the beginning of coordinates according (2.40) we have velocity components

$$u_1 = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}, \quad u_2 = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{3.7}$$

The corresponding stream function is

$$\psi = -\frac{\Gamma}{2\pi}\ln r \tag{3.8}$$

where $r = \sqrt{x^2+y^2}$. Then vorticity of the point vortex is

$$\Omega = \frac{\Gamma}{2\pi}\Delta\ln r = \Gamma\delta(\vec{r}) \tag{3.9}$$

where we have used property of the Green's function for two-dimensional Laplace equation (Appendix B.3)

$$\Delta\ln|\vec{r}| = 2\pi\delta(\vec{r}). \tag{3.10}$$

So it shows that vorticity of the point vortex is zero everywhere except position of the vortex. If we calculate total vorticity in the plane

$$\int_{R^2}\int \Omega d^2x = \Gamma\int\int\delta(\vec{r})d^2x = \Gamma \tag{3.11}$$

then we can see that Γ characterizes the magnitude (circulation) of the vortex.

If the point source is located at the beginning of coordinates, then according to (2.34) we have components of the velocity

$$u_1 = \frac{N}{2\pi}\frac{x}{x^2+y^2}, \quad u_2 = \frac{N}{2\pi}\frac{y}{x^2+y^2}. \tag{3.12}$$

The corresponding stream function is

$$\psi = -\frac{N}{2\pi}\tan^{-1}\frac{x}{y} \tag{3.13}$$

and vorticity of the source

$$\Omega = -\Delta\psi = \frac{N}{2\pi}\Delta\tan^{-1}\frac{x}{y} \tag{3.14}$$

Easy to check that it is zero at every point except $x = 0$, $y = 0$.

Proposition 3.2.0.5 Reciprocity Relations: *Complex potentials for the point source with strength N_s and the point vortex with strength Γ_v, located at the same point z_0 are connected by the relation (Pashaev and Gurkan 2005)*

$$N_s f_v(z) = -i\Gamma_v f_s(z) \tag{3.15}$$

Corollary 3.2.0.6 *Complex velocity u_s for the point source with strength N_s and the point vortex u_v with strength Γ_v, at the same point z_0 are connected by the relation*

$$N_s u_v = i\Gamma_v u_s \tag{3.16}$$

Corollary 3.2.0.7 *The stream function for the point source and the velocity potential for the point vortex located at the same position are linearly dependent and vice versa.*

$$N_s\varphi_v - \Gamma_v\psi_s = 0 \tag{3.17}$$

$$N_s\psi_v + \Gamma_v\varphi_s = 0 \tag{3.18}$$

Corollary 3.2.0.8 Duality Relations: *The stream function and the velocity potential for the point vortex and the point source and vice versa located at the same position with the equal strength satisfy the duality relations*

$$\varphi_v(x,y) = +\psi_s(x,y) \tag{3.19}$$

$$\psi_v(x,y) = -\varphi_s(x,y) \tag{3.20}$$

Corollary 3.2.0.9 *Complex velocities for the point vortex and point source with equal strengths $N_s = \Gamma_v$ located at the same position satisfy the duality relation*

$$u_s(\bar{z}) = iu_v(\bar{z}) \tag{3.21}$$

Due to this relation we can restrict ourselves with the study of vortices only since for sources we need just multiply results with i which means rotate the picture at every point on angle $\pi/2$.

For the system of n vortices (2.46) of the same strength Γ we have the complex potential

$$f(z) = \sum_{k=1}^{n} \frac{-i\Gamma_k}{2\pi} Log(z - z_k) \tag{3.22}$$

The corresponding stream function is

$$\psi = \Im f = \frac{-1}{2\pi} \sum_{k=1}^{n} \Gamma_k Log|z - z_k| \tag{3.23}$$

and the vorticity function (3.6)

$$\Omega = \frac{1}{2\pi} \sum_{k=1}^{n} \Gamma_k \Delta Log|z - z_k| = \sum_{k=1}^{n} \Gamma_k \delta(\vec{r} - \vec{r}_k) \tag{3.24}$$

describes the static distribution of point vortices in the plane. The total vorticity in the plane in this case is

$$\int_{R^2}\int \Omega d^2x = \int\int \Gamma_k \sum_{k=1}^{n} \delta(\vec{r} - \vec{r}_k) d^2x = \Gamma_1 + \Gamma_2 + ... + \Gamma_n \tag{3.25}$$

This example shows that vorticity is not zero for the rotational flow. Now if point vortices are moving in the plane then vorticity in addition to z becomes a function of time. For the set of n vortices we have

$$f(z,t) = \sum_{k=1}^{n} \frac{-i\Gamma_k}{2\pi} Log(z - z_k(t)) \tag{3.26}$$

and

$$\Omega = \sum_{k=1}^{n} \Gamma_k \delta(\vec{r} - \vec{r}_k(t)) \tag{3.27}$$

It shows that vorticity in this case becomes also function of time.

3.3 The Helmholtz form of Euler Equation

Let us now determine evolution equation for vorticity in the rotational fluid. Using the Euler equations (3.1) supplied with the incompressibility condition

$$\nabla\,\vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \tag{3.28}$$

we have it in the form

$$\frac{\partial \Omega}{\partial t} + u_1 \frac{\partial \Omega}{\partial x} + u_2 \frac{\partial \Omega}{\partial y} = 0 \tag{3.29}$$

and it is called the Euler equation in the Helmholtz form (Appendix B.1). If for velocity u_1, u_2 due to (3.28) we substitute expression (3.4) in terms of the stream function ψ, then we rewrite the last equation as

$$\frac{\partial \Omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \Omega}{\partial y} = 0 \tag{3.30}$$

This evolution equation for vorticity of the planar flow has the Hamiltonian form.

3.3.1 The Euler Equation in the Hamiltonian form

To represent the Euler equation (3.30) in the Hamiltonian form we first introduce the Poisson brackets (Arnold 1998).

Definition 3.3.1.1 *The bilinear form*

$$\{\psi, \Omega\} = \frac{\partial \psi}{\partial x}\frac{\partial \Omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Omega}{\partial x} \tag{3.31}$$

defined on the space of functions of two variables x, y is called the Poisson bracket.

This Poisson bracket satisfies all properties of standard Poisson bracket.

1. Skew-symmetry

$$\{\psi, \Omega\} = -\{\Omega, \psi\} \tag{3.32}$$

2. Linearity

$$\{\psi + \Omega, \Upsilon\} = \{\psi, \Upsilon\} + \{\Omega, \Upsilon\} \tag{3.33}$$

3. Jacobi Identity

$$\{\psi, \{\Omega, \Upsilon\}\} + \{\Omega, \{\Upsilon, \psi\}\} + \{\Upsilon, \{\psi, \Omega\}\} = 0 \tag{3.34}$$

4. Leibnitz Rule

$$\{\psi \cdot \Omega, \Upsilon\} = \psi \cdot \{\Omega, \Upsilon\} + \{\psi, \Upsilon\} \cdot \Omega \tag{3.35}$$

Then using this definition it is easy to see that the Euler equation (3.30) has the next Hamiltonian form

$$\frac{\partial \Omega}{\partial t} = \{\psi, \Omega\} \tag{3.36}$$

where the stream function can be considered as Hamiltonian function of the problem. It is related with Ω according to Eq. (3.5). so integral operator (Appendix B.3) inversing Eq. (3.5)

$$\psi(x_1, y_1) = \frac{1}{2\pi} \int\int dx_2 dy_2 \ln|\vec{r}_1 - \vec{r}_2| \Omega(x_2, y_2) = \Delta^{-1}\Omega \tag{3.37}$$

from (3.36) we have equation

$$\frac{\partial \Omega}{\partial t} = \{\Delta^{-1}\Omega, \omega\} \tag{3.38}$$

3.3.2 The Lax Representation

Given eigenvalue problem (Ablowitz and Segur 1981)

$$L\varphi = \lambda\varphi \tag{3.39}$$

is called *isospectral* , $\partial\lambda/\partial t = 0$, if eigenfunctions evolution

$$\frac{\partial \varphi}{\partial t} = A\varphi \tag{3.40}$$

implies an operator equation

$$L_t = [A, L] = AL - LA \tag{3.41}$$

called the Lax equation. Indeed, differentiating (3.39) according time and using (3.40) we have identity (Appendix B.2)

$$(L_t - [A, L])\varphi = \lambda_t \varphi \tag{3.42}$$

which gives

$$\lambda_t = 0 \quad \Leftrightarrow \quad L_t = [A, L] \tag{3.43}$$

When the Lax operator L and evolution operator A depend on some function $f(x, t)$, the Lax operator equation (Lax 1968) is equivalent to a nonlinear evolution equation for function $f(x,t)$. In this case one says that the nonlinear evolution equation is representable in the Lax form. In general, existence of the Lax representation means that the corresponding nonlinear evolution equation is completely integrable system and can be solved by the Inverse Scattering or Spectral Transform Method (Novikov et al. 1984) or by some other direct methods (Hirota 1980).

3.3.3 The Lax Representation of the Euler Equation

Written in the Hamiltonian form the Euler equation for vorticity (3.36) can be interpreted as the Lax equation (Li 2001)

$$L_t = [A, L]. \tag{3.44}$$

We define the following Lax operators by action on function φ:

$$L\varphi = \{\Omega, \varphi\} = \Omega_x \varphi_y - \Omega_y \varphi_x \tag{3.45}$$

$$A\varphi = \{\psi, \varphi\} = \psi_x \varphi_y - \psi_y \varphi_x \tag{3.46}$$

where ψ is the stream function with vorticity Ω. Then the linear system (3.39), (3.40) can be written as

$$\{\Omega, \varphi\} = \lambda\varphi \tag{3.47}$$

$$\varphi_t = \{\psi, \varphi\} \tag{3.48}$$

Differentiating the first equation of the system and using the second one we have:

$$\{\Omega_t, \varphi\} + \{\Omega, \varphi_t\} = \lambda_t \varphi + \lambda \varphi_t \tag{3.49}$$

$$\{\Omega_t, \varphi\} + \{\Omega, \{\psi, \varphi\}\} = \lambda_t \varphi + \lambda\{\psi, \varphi\} \tag{3.50}$$

or

$$\{\Omega_t, \varphi\} + \{\{\Omega, \psi\}, \varphi\} = \lambda_t \varphi \tag{3.51}$$

where in the second term we have used the Jacobi Identity (3.34). For isospectral flow we obtain the Euler equation for vorticity

$$\{\partial_t \Omega - \{\psi, \Omega\}, \varphi\} = 0 \tag{3.52}$$

but in a weak sense.

3.4 The Kirchhoff Equations

In this section we derive the system of dynamical equations describing N point vortices in the plane. This system was derived first time by Kirchhoff (Kirchhoff 1876) and called by his name. Since vorticity of a point vortex is described by the Dirac delta function (3.9) which has mathematical meaning of the generalized function or the distribution (Schwartz 1997), we need to consider the generalized function solution of the vorticity equation (3.36).

3.4.1 Generalized Function Solution of Vorticity Equation

Let us consider vorticity equation (3.36) in the class of generalized functions $\Omega(x, y, t)$. Then we multiply this equation with sufficiently smooth function $f(x, y)$ having finite support in R^2 and integrate in the whole plane

$$\int\int \left(\frac{\partial\Omega}{\partial t} - \{\psi, \Omega\}\right) f(x, y)dxdy = 0 \tag{3.53}$$

or

$$\int\int \left(\frac{\partial\Omega}{\partial t} - \{\psi, \Omega\}\right) f(x, y)dxdy = \int\int \left(\frac{\partial\Omega}{\partial t} - \frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x}\right) f(x, y)dxdy = 0 \tag{3.54}$$

Using vorticity function for the set of point vortices intensity $\Gamma_1, \Gamma_2, ..., \Gamma_N$ has the form

$$\Omega = \sum_{k=1}^{N} \Gamma_k \delta(x - x_k(t))\delta(y - y_k(t)) \tag{3.55}$$

we calculate different terms as

$$\begin{aligned} \int\int \frac{\partial\Omega}{\partial t} f(x, y)dxdy &= \int\int f(x, y) \sum_{k=1}^{N} \Gamma_k \delta'(x - x_k(t))(-\dot{x}_k)\delta(y - y_k) \\ &+ \int\int f(x, y) \sum_{k=1}^{N} \Gamma_k \delta(x - x_k(t))(-\dot{y}_k)\delta'(y - y_k) \\ &= \sum_{k=1}^{N} \Gamma_k \left[\dot{x}_k \frac{\partial f(x, y)}{\partial x} + \dot{y}_k \frac{\partial f(x, y)}{\partial y}\right]\Bigg|_{x=x_k, y=y_k} \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} \int\int \frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y} f(x, y)dxdy &= \int\int \frac{\partial\psi}{\partial x} f(x, y) \sum_{k=1}^{N} \Gamma_k \delta(x - x_k)\delta'(y - y_k)dxdy \\ &= -\sum_{k=1}^{N} \Gamma_k \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x} f(x, y)\right)\Bigg|_{x=x_k, y=y_k} \end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\int\int \frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x}f(x,y)dxdy &= \int\int\frac{\partial\psi}{\partial y}f(x,y)\sum_{k=1}^{N}\Gamma_k\delta'(x-x_k)\delta(y-y_k)dxdy \\ &= -\sum_{k=1}^{N}\Gamma_k\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}f(x,y)\right)\Bigg|_{x=x_k,y=y_k} \qquad (3.58)\end{aligned}$$

Combining together it gives

$$\begin{aligned}\int\int\left(\frac{\partial\Omega}{\partial t}-\{\psi,\Omega\}\right)f(x,y)dxdy &= \sum_{k=1}^{N}\Gamma_k\left(\dot{x}_k-\frac{\partial\psi}{\partial y}\right)\frac{\partial f(x,y)}{\partial x}\Bigg|_{x=x_k,y=y_k} \qquad (3.59)\\ &+ \sum_{k=1}^{N}\Gamma_k\left(\dot{y}_k-\frac{\partial\psi}{\partial x}\right)\frac{\partial f(x,y)}{\partial y}\Bigg|_{x=x_k,y=y_k} \\ &= 0\end{aligned}$$

Since this equation must be valid for arbitrary smooth function f we have the next set of $2N$ ordinary differential equations

$$\dot{x}_k=\frac{\partial\psi}{\partial y}\Bigg|_{x=x_k,y=y_k}\quad \dot{y}_k=-\frac{\partial\psi}{\partial x}\Bigg|_{x=x_k,y=y_k},(k=1,...,N) \qquad (3.60)$$

The stream function for the set of point vortices (3.55) has the form

$$\psi=\frac{-1}{2\pi}\sum_{n=1}^{N}\Gamma_n\ln|z-z_n(t)|=\frac{-1}{4\pi}\sum_{n=1}^{N}\Gamma_n\ln[(x-x_n)^2+(y-y_n)^2] \qquad (3.61)$$

and corresponding partial derivatives are

$$\frac{\partial\psi}{\partial y} = \frac{-1}{2\pi}\sum_{n=1}^{N}\Gamma_n\frac{y-y_n}{(x-x_n)^2+(y-y_n)^2} \qquad (3.62)$$

$$\frac{\partial\psi}{\partial x} = \frac{-1}{2\pi}\sum_{n=1}^{N}\Gamma_n\frac{x-x_n}{(x-x_n)^2+(y-y_n)^2} \qquad (3.63)$$

Substituting to (3.60) we have the system

$$\dot{x}_k = \frac{-1}{2\pi}\sum_{n=1,\,n\neq k}\Gamma_n\frac{y_k-y_n}{(x_k-x_n)^2+(y_k-y_n)^2} \qquad (3.64)$$

$$\dot{y}_k = -\frac{1}{2\pi}\sum_{n=1,\,n\neq k}\Gamma_n\frac{x_k-x_n}{(x_k-x_n)^2+(y_k-y_n)^2} \qquad (3.65)$$

or in more symmetrical form

$$\Gamma_k\dot{x}_k = \frac{-1}{2\pi}\sum_{n\neq k}\Gamma_k\Gamma_n\frac{y_k-y_n}{(x_k-x_n)^2+(y_k-y_n)^2} \qquad (3.66)$$

$$\Gamma_k\dot{y}_k = -\frac{1}{2\pi}\sum_{n\neq k}\Gamma_k\Gamma_n\frac{x_k-x_n}{(x_k-x_n)^2+(y_k-y_n)^2} \qquad (3.67)$$

This system can be rewritten in terms of complex coordinates $x_k + iy_k = z_k$ as a system of N complex ODE

$$\Gamma_k \dot{z}_k = \frac{-1}{2\pi} \sum_{n \neq k} \Gamma_k \Gamma_n \frac{(y_k - y_n) - i(x_k - x_n)}{(x_k - x_n)^2 + (y_k - y_n)^2} \tag{3.68}$$

$$= \frac{i}{2\pi} \sum_{n \neq k} \Gamma_k \Gamma_n \frac{z_k - z_n}{| z_k - z_n |^2} \tag{3.69}$$

$$= \frac{i}{2\pi} \sum_{n \neq k} \Gamma_k \Gamma_n \frac{z_k - z_n}{\bar{z}_k - \bar{z}_n} \tag{3.70}$$

or in a more compact form

$$i\dot{z}_k = -\frac{1}{2\pi} \sum_{n \neq k} \frac{\Gamma_n}{\bar{z}_k - \bar{z}_n} \tag{3.71}$$

Finally we arrive with the set of $2N$ ordinary differential equations known as the Kirchhoff equations (Kirchhoff 1876).

3.4.2 The Kirchhoff Equations for N point Vortices

Consider N vortices with strengths Γ_i , $i = 1, ..., N$ in the plane R^2. Then the vorticity at any moment will be concentrated at N points , and the circulations at each of them will remain constant forever. Denote the cartesian coordinates of the vortices in the plane by $z_j = x_j + iy_j$ $j = 1, ..., N$. We will write down the evolution of vortices as a dynamical system in the configuration space for the N vortex system, the space R^{2N} with coordinates $(x_1, y_1, ..., x_N, y_N)$ (Kirchhoff 1876).

Proposition 3.4.2.1 *The vortex evolution is given by the following system of Hamiltonian canonical equations*

$$\Gamma_i \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial H}{\partial x_i} \tag{3.72}$$

$1 \leq i \leq N$ *where the Hamiltonian function is*

$$H = -\frac{1}{2\pi} \sum_{i<j} \Gamma_i \Gamma_j \ln |z_i - z_j| \tag{3.73}$$

and

$$|z_i - z_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{3.74}$$

According to Helmholtz (Helmholtz 1858), in the case of $N = 2$, the two vortices rotate uniformly in the plane R^2 about their common mass center(center of vorticity)$z = \dfrac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}$. In particular, if the circulations Γ_1 and Γ_2 are of the same sign, then the mass center is situated between the vortices, while if they are of opposite signs , then the mass center lies on the continuation of the line joining the vortices. If $\Gamma_1 = -\Gamma_2$, then the point vortices travel with equal velocity in parallel directions perpendicular to the line joining them. The three-vortex problem ($N = 3$) also turns out to be integrable. This has already been pointed out by (Kirchhoff 1876) and illuminated in the dissertation of (Gröbli 1877), where one can find equations for evolution of the sides of the vortex triangle and explicit formulas for several special cases.An elaborate treatment of the history of three vortices can be found in (Aref et al. 1992).

3.4.3 Poisson Structure and Integrals of Motion

Let us consider R^{2N} space of vortex coordinates $x_1, ..., x_N, y_1, ..., y_N$ as the phase space. Then for arbitrary functions $A(x_1, ..., x_N, y_1, ..., y_N), B(x_1, ..., x_N, y_1, ..., y_N)$ on this space we introduce the Poisson bracket

$$\{A, B\} = \sum_{n=1}^{N} \frac{1}{\Gamma_n} \left(\frac{\partial A}{\partial x_n} \frac{\partial B}{\partial y_n} - \frac{\partial A}{\partial y_n} \frac{\partial B}{\partial x_n} \right) \tag{3.75}$$

which satisfies all properties for the Poisson brackets (3.31). Then the phase space equipped with the Poisson bracket becomes a Lie algebra (Perelomov 1990).

By this Poisson brackets the Kirchhoff equations (3.66) and (3.67) can be rewritten as

$$\dot{x}_k = \{x_k, H\}, \qquad \dot{y}_k = \{y_k, H\}, \quad k = 1, ..., N \tag{3.76}$$

where the Hamiltonian function has form

$$H = -\frac{1}{4\pi} \sum_k \sum_n \Gamma_k \Gamma_n \ln[(x_k - x_n)^2 + (y_k - y_n)^2] \tag{3.77}$$

Indeed, using the Poisson bracket (3.75) equations (3.76) can be rewritten as

$$\begin{aligned} \dot{x}_k &= \{x_k, H\} = \sum_{n=1}^{N} \frac{1}{\Gamma_n} \left(\frac{\partial x_k}{\partial x_n} \frac{\partial H}{\partial y_n} - \frac{\partial x_k}{\partial y_n} \frac{\partial H}{\partial x_n} \right) && (3.78) \\ &= \frac{1}{\Gamma_k} \frac{\partial H}{\partial y_k} && (3.79) \end{aligned}$$

$$\dot{y}_k = \{y_k, H\} = \sum_{n=1}^{N} \frac{1}{\Gamma_n} \left(\frac{\partial y_k}{\partial x_n} \frac{\partial H}{\partial y_n} - \frac{\partial y_k}{\partial y_n} \frac{\partial H}{\partial x_n} \right) \tag{3.80}$$

$$= -\frac{1}{\Gamma_k} \frac{\partial H}{\partial x_k} \tag{3.81}$$

which coincide with the Kirchhoff system (3.66)and (3.67).

Now for evolution of a smooth function $f = (x_1(t), ..., x_N(t), y_1(t), ..., y_N(t))$ on the phase space we have

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial f}{\partial y_i} \frac{dy_i}{dt} \right) \tag{3.82}$$

$$= \sum_i \frac{1}{\Gamma_i} \frac{\partial f}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{1}{\Gamma_i} \frac{\partial f}{\partial y_i} \frac{\partial H}{\partial x_i} \tag{3.83}$$

$$= \sum_i \frac{1}{\Gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial H}{\partial x_i} \right) \tag{3.84}$$

or

$$\dot{f} = \{f, H\} \tag{3.85}$$

Definition 3.4.3.1 *A function $F(x_i(t), y_i(t))$ is a first integral of Hamiltonian system with Hamiltonian function $H(x_i, y_i)$ if and only if the Poisson bracket* $\{H, F\} = 0$

Theorem 3.4.3.2 *(***Poisson's Theorem)** *If F and G are two integrals of motion then their Poisson bracket $\{F, G\}$ is an integral of the motion.*

Definition 3.4.3.3 *Two functions $F_1(x_i, y_i)$ and $F_2(x_i, y_i)$ are in involution if their Poisson bracket is equal zero, $\{F_1, F_2\} = 0$.*

Theorem 3.4.3.4 *(***Liouville)** *If a system with N degrees of freedom (in $2N$ dimensional phase space) admits N independent first integrals of motion in involution then the system is integrable by quadratures.*

If we consider the Kirchhoff equations as a Hamiltonian system, important question is if this system is integrable or not. It is known that for a general N, the Kirchhoff equations of motion have the following four first integrals:

$$I_1 = H \quad I_2 = \sum_{i=1}^{N} \Gamma_i x_i \quad I_3 = \sum_{i=1}^{N} \Gamma_i y_i \quad I_4 = \sum_{i=1}^{N} \Gamma_i (x_i^2 + y_i^2) \tag{3.86}$$

In fact one can easily check it by direct calculations of the Poisson brackets (Appendix B.5)

$$\dot{I}_k = \{I_k, H\} = 0, \;\; k = 1, ..., 4 \tag{3.87}$$

The calculations can be simplified if we use the Fundamental Poisson brackets

$$\{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0 \tag{3.88}$$

$$\{x_i, y_j\} = \frac{1}{\Gamma_i}\delta_{ij}, \;\; (i, j = 1, ..., N) \tag{3.89}$$

Now if we calculate the mutual brackets (Appendix B.5)

$$\{I_2, I_3\} = \sum_{k=1}^{N} \Gamma_k \tag{3.90}$$

$$\{I_2, I_4\} = 2I_3 \tag{3.91}$$

$$\{I_3, I_4\} = -2I_2 \tag{3.92}$$

then we can see that these integrals are not in involution; that is , their Poisson brackets are not zero, and the system with four vortices or more is , generally speaking, nonintegrable (Ziglin 1982). It is proved that the dynamical system of four vortices with finite circulations has no analytic independent integral and thus has to be chaotic. Chaotic behavior of systems with four vortices was already hinted at by Poincaré (1893). Numerical evidence of it was discussed by E. Novikov (1975). In spite of the fact that the 4-vortex system is generally nonintegrable, the KAM theory (Arnold and Khesin 1998) guarantees that for any number of vortices there is a set of positive measure in the space of initial conditions for which the motion is quasiperiodic (Khanin 1982). Such vortex configurations are organized in the following way: The set of all vortices is split into several groups. In this case the vortex groups interact approximately as single vortices possessing the total circulation. The actual vortex motion is obtained as a superposition of the group motion and the independent vortex motion within the groups.

As we show in Chapter 7 the system of N vortices for a magnetic fluid model which is slightly different from the Kirchhoff equations (3.66),(3.67) has N integrals of motion and is an integrable system.

3.5 Planar Motion with Given Vorticity

To study planar motion with given vorticity (Lavrentiev and Shabat 1973) we suppose that the liquid is incompressible and external forces acting on fluid are derived by potential U: $\vec{f} = -\nabla U$. If we introduce the so called Lamb function

$$H = \frac{u^2}{2} + \frac{P}{\rho} - U \tag{3.93}$$

where $u = |\vec{u}|$, P - is the pressure and ρ is the density, then applying formula

$$(\vec{u}\nabla)\vec{u} = [\vec{\Omega}, \vec{u}] + grad\frac{u^2}{2} \tag{3.94}$$

to the Euler equation (3.1) we can rewrite it in the form

$$\frac{\partial \vec{u}}{\partial t} + [\vec{\Omega}, \vec{u}] = -grad\, H \tag{3.95}$$

which is called the Euler equation in the Lamb form. For the planar motion when

$$rot\, \vec{u} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\vec{k} \tag{3.96}$$

is orthogonal to the motion plane, the flow is characterized by one scalar vorticity function,

$$\Omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{3.97}$$

If we consider now the steady flow ($\partial u/\partial t = 0$) then the Lamb equation (3.95) can be rewritten as the couple of scalar equations

$$\Omega u_2 = \frac{\partial H}{\partial x}, \quad -\Omega u_1 = \frac{\partial H}{\partial y} \tag{3.98}$$

By the stream function ψ (3.4) equating the mixed derivatives of H we get identity

$$\frac{\partial \Omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \Omega}{\partial y}\frac{\partial \psi}{\partial x} = 0 \tag{3.99}$$

from which follows that vorticity Ω must be constant on the stream lines $\psi = const$, which means it depends only on ψ:

$$\Omega = \Omega(\psi). \tag{3.100}$$

In applications form of this dependence usually is known. Introducing the stream function to Eq. (3.97) we find that it satisfies partial differential equation

$$\Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega(\psi) \tag{3.101}$$

This equation is the nonlinear Poisson equation if function $\Omega(\psi)$ is nonlinear.

3.6 The Liouville Model

If dependence of the vorticity is $\Omega(\psi) = 8e^{-\psi}$ then Eq.(3.101) has the form of the Liouville equation

$$\Delta\psi = 8e^{-\psi}. \tag{3.102}$$

This equation is appear in the theory of surfaces (Dubrovin et al. 1984). When the Riemannian metric of a surface is written in the conformal form, then conformal factor for the constant curvature surfaces is satisfy the Liouville equation. As it was shown by Liouville (1853) the equation admits the general solution which can be written in the form

$$\psi = -\ln\frac{|\zeta_z|^2}{(1+|\zeta|^2)^2} \tag{3.103}$$

where $\zeta(z)$ is an arbitrary analytic function. If one chooses it in the form

$$\zeta(z) = \prod_{i=1}^{N}(z - z_i) \tag{3.104}$$

with N simple zeroes in the complex plane, then it determines N vortices located in the plane at position of zeroes of this function. As Stuart found in (Stuart 1967), the Liouville equation admits solution consisting of an infinite periodic array of vortices (Fig. 3.1) described by the stream function

$$\psi = 2\ln(C\cosh 2y + \sqrt{C^2-1}\cos 2x) \tag{3.105}$$

where C is a real parameter satisfying $1 \le C < \infty$

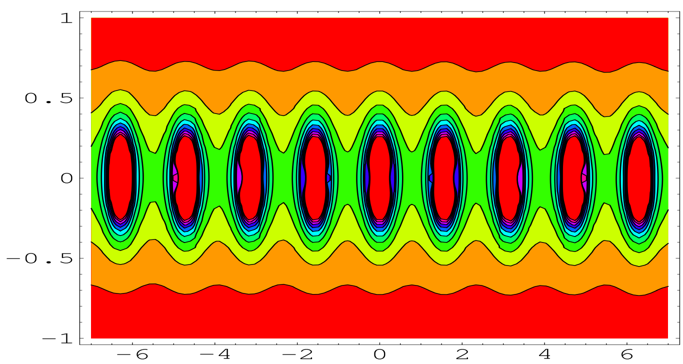


Figure 3.1. Stuart Vortex Lattice

When $C = 1$ the solution

$$\psi = 2\ln(\cosh 2y) \tag{3.106}$$

represents a homogeneous shear layer profile in which all streamlines are parallel to the x- axis and the horizontal velocity varies like a hyperbolic tangent function with vertical distance y. In the opposite limit , $C \to \infty$, the solution reduces to an infinite row of identical point vortices separated by distance 2π, each of circulation -4π. For all intermediate values $1 < C < \infty$, the solution has the structure of an infinite row of Kelvin cat's- eyes (Lamb 1932) with a smooth vorticity distribution. The parameter C governs the steepness of the vorticity profile.

Stuart's solution is one of a small group of known exact solutions of the planar Euler equations for distributed vortical equilibria. Several others are known (Saffman 1992) and (Newton 2001) but the majority of these are weak solutions in the sense that they involve vortex patches. Examples include the celebrated Kirchhoff ellipse (Lamb 1932), the Moore-Saffman vortices (Moore 1975) and the recently derived equilibria found in (Crowdy 1999, Crowdy 2002). Apart from Stuart's solution, another famous solution with a smooth vorticity distribution is the Lamb dipole (Lamb 1932). Meleshko and van Heijst survey a number of related smooth-vorticity solutions (Meleshko 1994). In Section 7.4.3 we construct infinite periodic in x lattice of vortices, different from the Stuart's one, and corresponding to the set of magnetic vortices.

CHAPTER 4

INTEGRABLE MODELS OF VISCOUS FLOW

4.1 The Navier- Stokes Equation of Viscous Flow

If we consider the viscous flow then the Euler equations should be modified by adding the viscosity term which contains the second space derivatives of the velocity field (Chorin and Marsden 1992). For incompressible viscous flow it gives the system of equations

$$\nabla \cdot \vec{u} = 0 \tag{4.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu \Delta \vec{u} \tag{4.2}$$

where coefficient ν has meaning of kinematic viscosity constant of the liquid. Eq.(4.2)is called the Navier-Stokes equation. The Navier-Stokes equation is much more complicated for solving than the Euler equation. It describes transition to the turbulent flow and even numerical approximation for some problems creates difficulties to solve it.

4.2 The Helmholtz Equation for Vorticity

If we apply *rot* operation to both sides of the Navier-Stokes equation, we will have

$$\frac{\partial \vec{\Omega}}{\partial t} + rot(\vec{u} \cdot \nabla)\vec{u} = \nu \Delta \vec{\Omega} \tag{4.3}$$

where $\vec{\Omega} = rot\,\vec{u}$. Using vector analysis identity (3.94)

$$(\vec{u} \cdot \nabla)\vec{u} = [\vec{\Omega}, \vec{u}] + grad\frac{\vec{u}^2}{2} \tag{4.4}$$

we have

$$rot\,(\vec{u}\,\nabla)\vec{u} = rot[\vec{\Omega}, \vec{u}]. \tag{4.5}$$

The right hand side can be expanded according to the double vector product formula and the Leibnitz's rule for ∇ acting on the product

$$[\vec{u}, [\vec{\Omega}, \vec{u}]] = (\vec{u}\,\nabla)\vec{\Omega} + \vec{\Omega}(\nabla \vec{u}) - \vec{u}\,(\nabla \vec{\Omega}) - (\vec{\Omega}\nabla)\vec{u} \tag{4.6}$$

Now, due to the incompressibility condition (4.1) and the identity $div\, rot \equiv 0$, this formula simplifies to

$$rot[\vec{\Omega}, \vec{u}] = (\vec{u}\,\nabla)\vec{\Omega} - (\vec{\Omega}\nabla)\vec{u}. \tag{4.7}$$

Substituting to Eq.(4.6) and then to Eq.(4.3) we arrive with the Helmholtz equation

$$\frac{D\vec{\Omega}}{Dt} = \frac{\partial\vec{\Omega}}{\partial t} + (\vec{u}\,\nabla)\vec{\Omega} = (\vec{\Omega}\nabla)\vec{u} + \nu\Delta\vec{\Omega}. \tag{4.8}$$

For the planar motion, when vorticity has only one nonzero component $\vec{\Omega} = \Omega\vec{k}$ the first term in the right hand side is vanishing and equation becomes

$$\frac{\partial\Omega}{\partial t} + u_1\frac{\partial\Omega}{\partial x} + u_2\frac{\partial\Omega}{\partial y} = \nu\left(\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2}\right) \tag{4.9}$$

4.3 Diffusion of Vortex Filament

In this section we consider exactly soluble problem of Navier-Stokes equation for the planar vorticity (4.9)(Lavrantiev and Shabat 1973). In a viscous liquid at time $t = 0$ given velocity distribution in the form of straight line vortex filament, find distribution of the velocities at any time (The Cauchy Problem or I.V.P.).

Let us choose the vortex filament in x direction and introduce cylindrical coordinates (x, r, θ); coordinates of velocity vector in this system we denote as u_x, u_r, u_θ. At initial time in all planes orthogonal to x axis, the velocity field has the same form (Appendix B.4)

$$u_x = 0, \quad u_r = 0, \quad u_\theta = \frac{\Gamma}{2\pi r}, \tag{4.10}$$

where Γ is the constant characterizing intensity of vortex filament. From symmetry reason it is clear that character of the velocity field will be preserved during the motion: u_r and u_x remain equal to zero and u_θ would depend on r and t.

It is convenient instead of velocity field to introduce the vorticity field, which in this problem would be characterized by scalar function $\Omega(r, t)$. In fact, according to Stokes formula applied to the disk of radius r with center on the x-axis, we have velocity in terms of this function

$$u_\theta = \frac{1}{r}\int_0^r \Omega(\rho,t)\rho d\rho \tag{4.11}$$

where ρ is integration variable.

Evolution of vorticity is described by the Helmholtz equation (4.9) which in our case becomes

$$\frac{\partial \Omega}{\partial t} = \nu\left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r}\right) \tag{4.12}$$

and the following initial condition: at $t = 0$ function $\Omega(r, 0)$ is equal 0 everywhere, except point $r = 0$, where it is infinite, so that

$$2\pi \int_0^\infty \Omega(r,0) r dr = \Gamma \tag{4.13}$$

Dimensional analysis of the problem shows that in addition to variables r and t , the vorticity is dependent of another two parameters ν and Γ so that

$$\Omega = \Omega(r, t, \nu, \Gamma) \tag{4.14}$$

The dimensionality of these variables is

$$[\Omega] = \frac{1}{T}, \quad [r] = L, \quad [t] = T, \quad [\nu] = \frac{L^2}{T}, \quad [\Gamma] = \frac{L^2}{T}. \tag{4.15}$$

From four variables r, t, ν, and Γ, only two have independent dimensionality, for example t and ν. With such choice we can construct three dimensionless combinations: $\pi_1 = \Omega\, t$, $\pi_2 = \dfrac{r}{\sqrt{\nu t}}, \pi_3 = \dfrac{\Gamma}{\nu}$. Then, according to the Buckingham π-theorem (Buckingham 1914, Buckingham 1915)(Appendix E), the physical laws are independent of the form of units and as follows, acceptable laws of physics are homogeneous in all dimensions:

$$F(\pi_1, \pi_2, \pi_3) = 0 \tag{4.16}$$

or

$$F\left(\Omega t, \frac{r}{\sqrt{\nu t}}, \frac{\Gamma}{\nu}\right) = 0. \tag{4.17}$$

The last equation determines a surface in three dimensional space, which admits the local form

$$\pi_1 = \omega(\pi_2, \pi_3). \tag{4.18}$$

This means that we have vorticity dependence (4.14) in the next form

$$\Omega = \frac{1}{t}\,\omega\left(\frac{r}{\sqrt{\nu t}}, \frac{\Gamma}{\nu}\right). \tag{4.19}$$

Introducing dimensionless variable $\xi = \dfrac{r}{\sqrt{\nu t}}$ and writing partial derivatives

$$\begin{aligned}
\frac{\partial \Omega}{\partial t} &= -\frac{1}{t^2}\left[\omega(\xi) + \frac{\xi}{2}\omega'(\xi)\right], && (4.20)\\
\frac{\partial \Omega}{\partial r} &= \frac{r}{\nu t^2 \xi}\omega'(\xi), && (4.21)\\
\frac{\partial^2 \Omega}{\partial r^2} &= \frac{1}{\nu t^2}\omega''(\xi) && (4.22)
\end{aligned}$$

where prime means derivative according to variable ξ, after substituting to Eq.(4.12) we get the ordinary differential equation for function ω

$$\omega'' + \left(\frac{1}{\xi} + \frac{\xi}{2}\right)\omega' + \omega = 0. \tag{4.23}$$

It is easy to check that condition (4.13) is valid at any time t (to show this one integrates Eq.(4.12)) in the whole plane (r,t), supposing that Ω and corresponding derivatives decay at infinity sufficient rapidly). Then in variable ξ this condition has the form

$$2\pi \int_0^\infty \omega(\xi)\,\xi\,d\xi = \frac{\Gamma}{\nu} \tag{4.24}$$

The solution of Eq.(4.23) satisfying the last condition (4.24) is

$$\omega = \frac{\Gamma}{4\pi\nu}\,\exp\left[-\frac{\xi^2}{4}\right] \tag{4.25}$$

and as follows the solution of our problem is

$$\Omega = \frac{\Gamma}{4\pi\nu t}\,\exp\left[-\frac{r^2}{4\nu t}\right]. \tag{4.26}$$

We note that equation (4.12) is the heat equation and above solution is just the fundamental solution of this equation. This solution describes diffusive evolution of vorticity in our problem. The corresponding evolution of the velocity field we can find according to Eq.(4.11)

$$u_\theta = \frac{\Gamma}{2\nu r}\left(1 - \exp\left[\frac{r^2}{4\nu t}\right]\right). \tag{4.27}$$

Presented solution describes exact diffusion of the point vortex in the plane.

4.4 Burgers' Equation for One Dimensional Flow

In previous section we considered exact solution for the problem of point vortex diffusion in the plane. But in general, due to nonlinearity of the Navier-Stokes equation it is difficult task to solve it exactly. This is why some exactly solvable models in the form very close to the equations have attracted lot of attention. Particularly, in one dimension the Navier-Stokes equation (4.2) is reduced to the equation

$$u_t + uu_x = \nu u_{xx} \tag{4.28}$$

but if we do not restrict our flow to be incompressible ($div\, u \neq 0$). This equation is called the Burgers' equation (Burgers 1948) and it is exactly solvable by the direct linearization.

4.5 Cole - Hopf Transformation and Heat Equation

Cole- Hopf transformation (Cole 1951, Hopf 1950)

$$u = -2\nu \frac{\phi_x}{\phi} = -2\nu (\ln \phi)_x \tag{4.29}$$

reduce the nonlinear Burgers' equation (4.28) to the linear Diffusion (Heat) equation

$$\phi_t = \nu \phi_{xx} \tag{4.30}$$

4.6 Initial Value Problem for Burgers' Equation

IVP for the Burgers' equation (Whitham 1974)

$$\begin{cases} u_t + uu_x = \nu u_{xx} \\ u(x,0) = F(x), \quad -\infty < x < \infty \end{cases}$$

can be transformed to IVP for the Diffusion equation (Appendix D)

$$\begin{cases} \phi_t = \nu \phi_{xx} \\ \phi(x,0) = \Phi(x) = e^{-\frac{1}{2\nu}\int^x F(\eta)d\eta} \end{cases}$$

The Direct Problem (Pashaev 2000): $u(x,0) \to \phi(x,0)$:

$$F(x) = -2\nu \frac{\phi_x}{\phi} = -2\nu (\ln \phi)_x$$

$$\Rightarrow \ \ln \phi = -\frac{1}{2\nu} \int^x F(\eta) d\eta$$

$$\Rightarrow \ \phi(x,0) = e^{-\frac{1}{2\nu}\int^x F(\eta)d\eta}$$

The Linear evolution for ϕ (the Heat equation)(Appendix D) : $\phi(x,0) \rightarrow \phi(x,t)$:

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \phi(\eta,0) \exp(-\frac{(x-\eta)^2}{4\nu t}) d\eta.$$

The Inverse Problem: $\phi(x,t) \rightarrow u(x,t)$:

$$u(x,t) = -2\nu \frac{\phi_x}{\phi} = \frac{\int_{-\infty}^{+\infty} \frac{x-\eta}{t} e^{-G/2\nu} d\eta}{\int_{-\infty}^{+\infty} e^{-G/2\nu} d\eta}$$

$$G(\eta; x, t) = \int^{\eta} F(\eta')d\eta' + \frac{(x-\eta)^2}{2t}$$

$$\begin{array}{ccccc} & & \textit{Direct problem} & & \\ & u(x,0) & \longrightarrow & \phi(x,0) & \\ \textit{Nonlinear} & & & & \textit{Linear} \\ & \downarrow & & \downarrow & \\ \textit{evolution} & & & & \textit{evolution} \\ & u(x,t) & \longleftarrow & \phi(x,t) & \\ & & \textit{Inverse problem} & & \end{array}$$

4.7 Shock Solitons and Their Dynamics

We can integrate Burgers' equation directly, supposing solution in the form of the travelling wave ansatz: $u(x,t) = u(x - vt) = u(\xi)$. In the moving frame coordinate $\xi = x - vt$

$$\partial_t = -v\partial_\xi, \quad \partial_x = \partial_\xi.$$

Substituting to Burgers' equation

$$-vu_\xi + uu_\xi = \nu u_{\xi\xi}$$

$$\Rightarrow \quad (-vu + \frac{u^2}{2})_\xi = (\nu u_\xi)_\xi$$

and integrating once we get

$$\Rightarrow \quad -vu + \frac{u^2}{2} + C = \nu u_\xi$$

where C - integration constant.

The Boundary Conditions:

$$\begin{cases} u(\xi) \to a_1, & x \to +\infty \\ u(\xi) \to a_2, & x \to -\infty \end{cases}$$

completely fix two constants:

$$v = \frac{a_1 + a_2}{2}, \quad C = \frac{a_1 a_2}{2}$$

Indeed:

$$-va_1 + \frac{a_1^2}{2} + C = 0,$$

$$-va_2 + \frac{a_2^2}{2} + C = 0,$$

$$\Rightarrow \ \frac{1}{2}(a_1^2 - a_2^2) - v(a_1 - a_2) = 0 \ \ \Rightarrow v = \frac{a_1 + a_2}{2}$$

$$\Rightarrow \ C = va_1 - \frac{a_1^2}{2} = \frac{a_1 a_2}{2}$$

Then

$$2\nu u_\xi = (u - a_1)(u - a_2)$$

$$\Rightarrow \ \frac{du}{(u - a_1)(u - a_2)} = \frac{d\xi}{2\nu}.$$

Suppose $a_2 > a_1$

$$\frac{1}{a_2 - a_1}(\frac{du}{u - a_1} + \frac{du}{a_2 - u}) = -\frac{d\xi}{2\nu}$$

$$\Rightarrow \ \frac{1}{a_2 - a_1} \ln \frac{u - a_1}{a_2 - u} = -\frac{\xi - \xi_0}{2\nu}$$

we get shock (kink soliton) solution:

$$u = a_1 + \frac{a_2 - a_1}{1 + e^{\frac{(a_2 - a_1)}{2\nu}(x - vt - x_0)}}, \quad v = \frac{a_1 + a_2}{2}$$

Particular form:

Let $a_1 = 0$, then

$$u = \frac{a_2}{2}[1 - \tanh \frac{a_2}{4\nu}(x - \frac{a_2}{2}t - x_0)]$$

- the *Taylor shock* profile,

$$(velocity) \ v = \frac{1}{2} a_2 \ (amplitude)$$

4.8 I.V.P for the Step Function

Consider the initial condition as the step function (Appendix D)

$$u(x,0) = F(x) = \begin{cases} a_1, & x > 0, \\ a_2 \ (> a_1) & x < 0 \end{cases} \quad at \ \ t = 0$$

Then corresponding initial condition $\phi(x,0)$ determines evolution according to the heat equation. The Linear evolution for ϕ (the Heat equation) : $\phi(x,0) \to \phi(x,t)$:

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \phi(\eta,0) \exp(-\frac{(x-\eta)^2}{4\nu t}) d\eta$$

The Inverse Problem: $\phi(x,t) \to u(x,t)$:

$$u(x,t) = -2\nu \frac{\phi_x}{\phi} = \frac{\int_{-\infty}^{+\infty} \frac{x-\eta}{t} e^{-G/2\nu} d\eta}{\int_{-\infty}^{+\infty} e^{-G/2\nu} d\eta}$$

Then at time $t > 0$:

$$u(x,t) = a_1 + \frac{a_2 - a_1}{1 + h(x,t) e^{\frac{(a_2-a_1)}{2\nu}(x - vt - x_0)}}, \quad v = \frac{a_1 + a_2}{2}$$

When $x \to \infty$, $t \to \infty$, so that x/t is fixed we have

$$h(x,t) = \frac{\displaystyle\int_{-\frac{(x-a_1 t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi}{\displaystyle\int_{\frac{(x-a_2 t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi} \to 1 \tag{4.31}$$

and it gives one shock soliton solution.

CHAPTER 5

COMPLEX BURGERS' EQUATION

5.1 Burgers' Equation for Complex Velocity in Two Dimensions

In Section 3 we have considered the evolution equation for vorticity in the planar Euler case. Despite of Poisson structure existence and the Lax type representation, the vortex dynamics for Euler equation is not known to be integrable for $N > 4$ number of point vortices. From another side, in Sections 4.4-4.8 we considered contribution of viscosity term to the Euler equation in one dimension, when simplified version of the Navier-Stokes equation in the form of the Burgers equation is linearizable (Cole- Hopf) in terms of diffusion equation, admitting N shock wave solitons. Now we consider complex extension of the Burgers equation and show that it describes integrable evolution of arbitrary N vortices in the plane (Pashaev and Gurkan 2005). We have the next 2+1 dimensional evolution equation (2.56)

$$iu_t + uu_{\bar{z}} = \nu u_{\bar{z}\bar{z}} \tag{5.1}$$

The complex velocity $u(\bar{z}, t)$ is related to the dual complex velocity $U(\bar{z}, t)$ by simple rotation on angle $\pi/2$:

$$U(\bar{z}, t) = i\, u(\bar{z}, t) = \overline{F'(z, t)} \tag{5.2}$$

$$u(\bar{z}, t) = \overline{f'(z, t)} \tag{5.3}$$

where $f(z, t) = \varphi + i\psi$ is the complex potential of a two dimensional flow $u(\bar{z}, t)$, while $F(z, t)$ is complex potential for the dual flow. Evolution equation for $U(\bar{z}, t)$ is

$$U_t - UU_{\bar{z}} = -i\nu U_{\bar{z}\bar{z}} \tag{5.4}$$

while for the complex potential $f(z, t)$ we have the *Complex Potential Burgers Equation*(CPBE)

$$-if_t + \frac{1}{2}(f_z)^2 = \nu f_{zz} \tag{5.5}$$

5.2 Complex Cole - Hopf Transformation and the Time-dependent Schrödinger Problem

By complex analog of the Cole-Hopf transformation (2.57)

$$u = -2\nu \frac{\partial}{\partial \bar{z}} Log \Phi(\bar{z}) = -2\nu \frac{\Phi_{\bar{z}}}{\Phi} \tag{5.6}$$

we can transform our nonlinear equation (5.1) to the linear complex Schrödinger (Heat) equation

$$i\Phi_t = \nu \Phi_{\bar{z}\bar{z}}. \tag{5.7}$$

Then, every solution of (5.7) determines solution of the complex Burgers' equation (5.1).

5.3 Polynomial Solutions and Planar Vortices

Suppose that function $\Phi(\bar{z}, t)$

$$\Phi(\bar{z}, t) = \prod_{k=1}^{n} (\bar{z} - \bar{z}_k(t)) \tag{5.8}$$

has n - simple zeroes moving in the plane $\bar{z}_1(t), \bar{z}_2(t), ..., \bar{z}_n(t)$. Then substituting this form to (5.7) we have (Appendix B.6) the system of first order ODEs

$$\frac{d\bar{z}_k}{dt} = 2\nu i \sum_{l(\neq k)}^{n} \frac{1}{\bar{z}_k - \bar{z}_l} \tag{5.9}$$

where $k = 1, ..., n$. It describes finite dimensional dynamical system of particles corresponding to our n moving vortices/sources. (We will discuss it in Chapter 6.) Indeed, from the form of function $\Phi(\bar{z}, t)$ (5.8) we have the complex potential of the flow in the form

$$f(z, t) = -2\bar{\nu} \ln \overline{\Phi(\bar{z}, t)} = -2\bar{\nu} \ln \prod_{k=1}^{n} (z - z_k(t)) \tag{5.10}$$

For the real $\nu = \bar{\nu}$ (in Section 7 for ferromagnetic vortices we will have $\nu = -2$ and it gives the set of sources while for the dual flow, the set of vortices)

$$F(z, t) = -if(z, t) = 2i\nu \ln \prod_{k=1}^{n} (z - z_k(t)) \tag{5.11}$$

The stream function for the last one

$$\psi = \Im F = 2\nu \ln|\Phi| = 2\nu \ln|z - z_k(t)| \tag{5.12}$$

implies vorticity

$$\Omega = -\Delta\psi = -2\nu\Delta \sum_{k=1}^{n} \ln|z - z_k(t)| \tag{5.13}$$

as the set of points moving in the plane

$$\Omega = -4\nu\pi \sum_{k=1}^{n} \delta(\vec{r} - \vec{r}_k(t)) \tag{5.14}$$

so that the total vorticity of the full plane is fixed as

$$\int_{R^2}\int \Omega d^2x = (-4\nu\pi)n \tag{5.15}$$

The last formula means that we have the set of point vortices with equal strength $\Gamma = -4\nu\pi$.

Now we are going to study in details the system of vortices as zeroes of Eq.(5.7).

5.3.1 Complex Hermite Polynomial and Vortex Expansion

First we consider anti-analytic solution of

$$i\Phi_t = \nu\Phi_{\bar{z}\bar{z}} \tag{5.16}$$

in the form of complex plane wave. It is entire function of $\bar{z}$:

$$\begin{aligned} \Phi(\bar{z}, t) &= e^{k\bar{z} - i\nu k^2 t} && (5.17) \\ &= \sum_{n=0}^{\infty} \frac{k^n \bar{z}^n}{n!} \sum_{m=0}^{\infty} \frac{(i\nu t)^m k^{2m}}{m!} && (5.18) \\ &= \sum_{N=0}^{\infty} k^N \sum_{n+2m=N} \frac{1}{n!m!} \bar{z}^n (-i\nu t)^m \end{aligned}$$

If we introduce

$$\Psi_N(\bar{z}, t) = \sum_{n+2m=N} \bar{z}^n \frac{(-i\nu t)^m}{n!m!} \tag{5.19}$$

then

$$\Phi(\bar{z}, t) = \sum_{n=0}^{\infty} k^n \Psi_n(\bar{z}, t) \tag{5.20}$$

This expansion can be rewritten in terms of the Hermite Polynomials of complex argument $\bar{z}$. Indeed, the Generating Function of Hermite polynomials is defined as (Appendix C)

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{5.21}$$

Our solution (5.17)

$$\Phi(\bar{z},t) = e^{k\bar{z}-i\nu k^2 t} \tag{5.22}$$

after identification $\alpha^2 \equiv i\nu^2 t$ and $k = \frac{\alpha}{\sqrt{i\nu t}}$ becomes

$$\begin{aligned}
\Phi(\bar{z},t) &= e^{\frac{\alpha}{\sqrt{i\nu t}}\bar{z}-i\nu^2 t} \\
&= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(\frac{\bar{z}}{2\sqrt{i\nu t}}) \\
&= \sum_{n=0}^{\infty} \frac{k^n}{n!} (i\nu t)^{\frac{n}{2}} H_n(\frac{\bar{z}}{2\sqrt{i\nu t}}) \\
&= \sum_{n=0}^{\infty} \frac{k^N}{N!} \sum_{n+2m=N}^{N} \frac{(n+2m)!}{n!m!} \bar{z}^n (-i\nu t)^m
\end{aligned} \tag{5.23}$$

or

$$\Phi(\bar{z},t) = \sum_{N=0}^{\infty} \frac{k^N}{N!} (i\nu t)^{\frac{N}{2}} H_N(\frac{\bar{z}}{2\sqrt{i\nu t}}) \tag{5.24}$$

where Hermite polynomials of complex argument are defined by the expansion

$$H_N(x) = \sum_{s=0}^{[N/2]} (-1)^s (2x)^{N-2s} \frac{N!}{(N-2s)!s!}. \tag{5.25}$$

Due to arbitrariness of parameter k in expansion (5.23), function Ψ_N

$$\Psi_N(\bar{z},t) = (i\nu t)^{N/2} H_N(\frac{\bar{z}}{2\sqrt{i\nu t}}) \tag{5.26}$$

for every integer N determines a solution of Eq.(5.16).

5.3.2 N-Vortex Dynamics. Basic case

For complex zeroes of this function, by identification

$$\begin{aligned}
\Psi_N(\bar{z},t) &= \prod_{k=1}^{N} (\bar{z} - \bar{z}_k(t)) \\
&= (i\nu t)^{N/2} H_N(\frac{\bar{z}}{2\sqrt{i\nu t}})
\end{aligned} \tag{5.27}$$

we find that $\bar{z} = \bar{z}_k(t)$ implies

$$(i\nu t)^{N/2} H_N(\frac{\bar{z}_k}{2\sqrt{i\nu t}}) = 0 \tag{5.28}$$

It means that for any $t \neq 0$, $H_N(w_k) = 0$ where w_k are complex zeroes of Hermite polynomials. Then from

$$w_k = \frac{\bar{z}_k(t)}{2\sqrt{i\nu t}} \tag{5.29}$$

we have time dependence for vortex positions in terms of these zeros

$$\bar{z}_k(t) = 2w_k\sqrt{i\nu t}. \tag{5.30}$$

First few Hermite Polynomials have the form

$$\begin{aligned} H_1(w) &= 2w \\ H_2(w) &= 4w^2 - 2 \\ H_3(w) &= 8w^3 - 12w \\ H_4(w) &= 16w^4 - 48w^2 + 12 \end{aligned} \tag{5.31}$$

and provide next solutions for $z_k(t) = 2w_k\sqrt{i\nu t}$:

1. $N = 1$ and $H_1(w) = 2w = 0$ then $w_1 = 0$ implies

$$\bar{z}_1(t) = 0 \tag{5.32}$$

2. $N = 2$ and $H_2(w) = 4w^2 - 2 = 0$ then $w_1 = \sqrt{1/2}$, $w_2 = -\sqrt{1/2}$ imply

$$\bar{z}_1(t) = \sqrt{i\nu t} \tag{5.33}$$

$$\bar{z}_2(t) = -\sqrt{i\nu t} \tag{5.34}$$

The position of the center of mass is

$$\bar{z}_1(t) + \bar{z}_2(t) = 0 \tag{5.35}$$

3. $N = 3$ and $H_3(w) = 8w^3 - 12w = 0$ then $w_1 = 0$,$w_2 = -\sqrt{3/2}$ and $w_3 = \sqrt{3/2}$ imply

$$\bar{z}_1(t) = 0 \tag{5.36}$$

$$\bar{z}_2(t) = \sqrt{6i\nu t} \tag{5.37}$$

$$\bar{z}_3(t) = -\sqrt{6i\nu t} \tag{5.38}$$

The position of the center of mass is

$$\bar{z}_1(t) + \bar{z}_2(t) + \bar{z}_3(t) = 0 \tag{5.39}$$

4. $N = 4$ and $H_4(w) = 16w^4 - 48w^2 + 12 = 0$ then $w_{1,2} = \pm\sqrt{\frac{3+\sqrt{6}}{2}}$, $w_{3,4} = \pm\sqrt{\frac{3-\sqrt{6}}{2}}$ imply

$$\bar{z}_{1,2}(t) = \pm\sqrt{(6+2\sqrt{6})i\nu t} \tag{5.40}$$

$$\bar{z}_{3,4}(t) = \pm\sqrt{(6-2\sqrt{6})i\nu t} \tag{5.41}$$

The position of the center of mass is

$$\bar{z}_1(t) + \bar{z}_2(t) + \bar{z}_3(t) + \bar{z}_4(t) = 0. \tag{5.42}$$

Due to reality of roots w_k, the form of our solution $\bar{z}_k(t) = 2w_k\sqrt{i\nu t}$ implies that all vortices are located on diagonal lines of complex plane:

$$\bar{z}_k(t) = |2w_k\sqrt{\nu t}|e^{i\pi/4}$$

We note that since the time dependence includes square root of time variable t, then under time reflection, when t is replaced by $-t$, position of vortices will rotate $\bar{z}_k \to e^{i\pi/2}\bar{z}_k$ on angle $\pi/2$. It means that under collision our vortices change velocity in orthogonal direction and from one diagonal line would be displaced to the orthogonal one. Moreover, sum of vortex positions, representing the center of mass of the system, is integral of motion located at the beginning of coordinates. We illustrate dynamics of $N = 2, 3$ and 4 vortices in Fig. 5.1, 5.2, 5.3 correspondingly.

In Fig.5.1 we showed $N = 2$ vortex dynamics for three different times $t = -1$,$t = 0$ and $t = 1$.

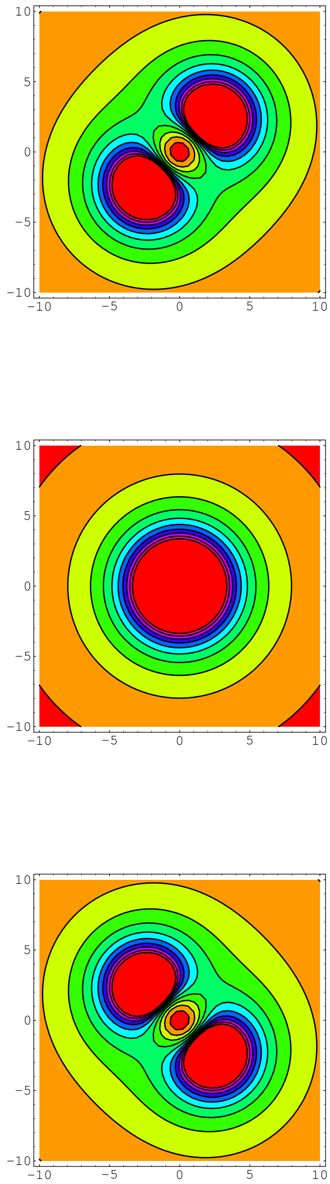


Figure 5.1. $N = 2$ Vortex Dynamics

in Fig.5.2 we show $N = 3$ vortex dynamics for three different times $t = -1$,$t = 0$ and $t = 1$.

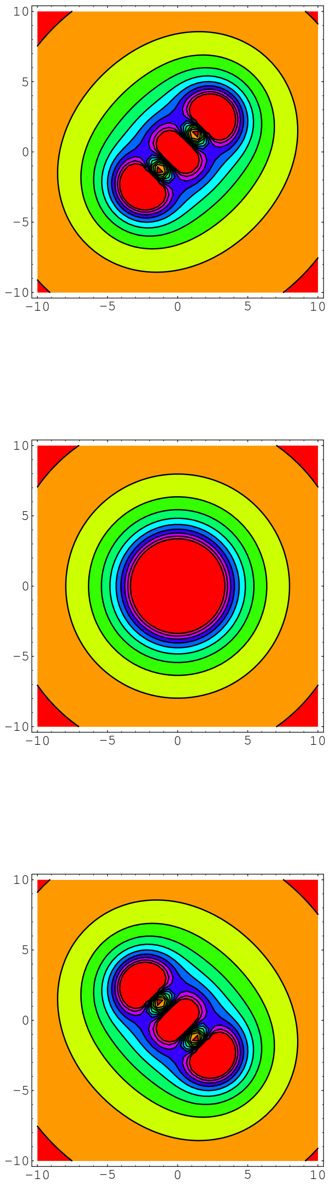


Figure 5.2. $N = 3$ Vortex Dynamics

in Fig.5.3 the case $N = 4$ vortex dynamics for three different times $t = -1$,$t = 0$ and $t = 1$.

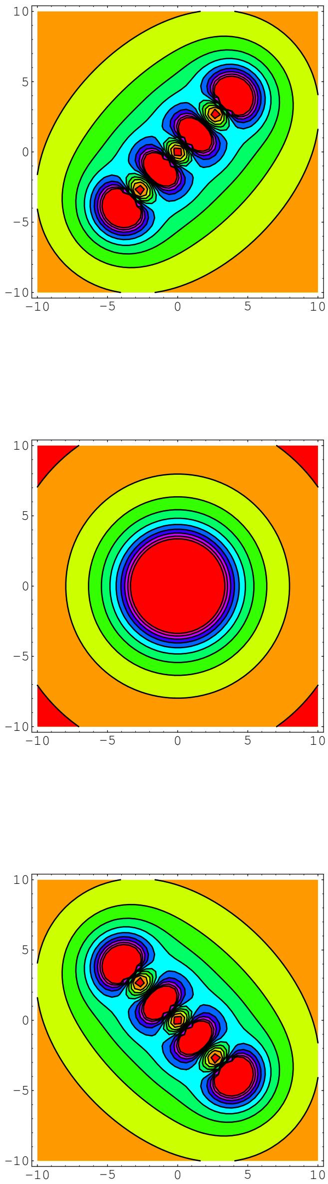


Figure 5.3. $N = 4$ Vortex Dynamics

5.3.3 N Vortex Dynamics. The General Case

Since function Ψ_N in (5.26) is solution of linear Eq.(5.16) for any integer N, then any linear combination of these functions

$$\Phi_N(\bar{z},t) = a_N\Psi_N(\bar{z},t) + a_{N-1}\Psi_{N-1}(\bar{z},t) + ... + a_0\Psi_0(\bar{z},t) \tag{5.43}$$

$a_N \neq 0$ is also a solution. This solution is determined by N complex constants $a_1, a_2, ..., a_N$, which are integrals of motion of the system. The value of higher order coefficient $a_N \neq 0$ is not essential and could be put to one. Below we represent particular cases for $N = 3$ and $N = 4$. 1) For $N = 3$ we have general solution

$$\Phi(\bar{z},t) = (\bar{z}^3 - 6\bar{z}i\nu t) + a_2(\bar{z}^2 - 2i\nu t) + a_1\bar{z} + a_0 \tag{5.44}$$

or

$$\begin{aligned}\Phi(\bar{z},t) &= \bar{z}^3 + a_2\bar{z}^2 + (a_1 - 6\bar{z}i\nu t)\bar{z} + (a_0 - 2a_2 i\nu t)\\ &= (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t))(\bar{z} - \bar{z}_3(t))\end{aligned} \tag{5.45}$$

This cubic in $\bar{z}$ polynomial has 3 complex roots $\bar{z}_1(t), \bar{z}_2(t), \bar{z}_3(t)$ moving in plane according the systems (5.9). Instead of solving that system of differential equations we will find roots of cubic equation according Cardano formulas.The coefficient

$$-a_2 = \bar{z}_1(t) + \bar{z}_2(t) + \bar{z}_3(t)$$

is integral of motion having meaning of the center of mass for three vortices. Without loss of generality we can always choose coordinate system with the beginning at this center of mass. So we will put $a_2 = 0$. Then our cubic equation has reduced Cardano form

$$\bar{z}^3 + p\bar{z} + q = 0 \tag{5.46}$$

where $p = a_1 - 6\bar{z}i\nu t$, $q = a_0 - 2a_2 i\nu t$. Solution of this equation is

$$\bar{z}_1 = \alpha_1 + \beta_1 \tag{5.47}$$

$$\bar{z}_2 = \alpha_1\omega_1 + \beta_1\omega_2 \tag{5.48}$$

$$\bar{z}_3 = \alpha_1\omega_2 + \beta_1\omega_1 \tag{5.49}$$

where α_1, β_1 is one of the couple of roots

$$\alpha = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} \tag{5.50}$$

$$\beta = \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} \tag{5.51}$$

and

$$\omega_1 = -1/2 + i\sqrt{3}/2 = e^{i2\pi/3}$$

$$\omega_2 = -1/2 - i\sqrt{3}/2 = e^{-i2\pi/3}$$

are cubic roots of 1.

For particular values $a_0 = a_2 = 0$ when $q = 0$ our roots become

$$\bar{z}_1 = \sqrt{-2i\nu t}(1 + e^{i\pi/3}) \tag{5.52}$$

$$\bar{z}_2 = \sqrt{-2i\nu t}(\omega_1 + \omega_2 e^{i\pi/3}) \tag{5.53}$$

$$\bar{z}_3 = \sqrt{-2i\nu t}(\omega_2 + \omega_1 e^{i\pi/3}) \tag{5.54}$$

and coincide with particular cases (5.36), (5.37), (5.38) when one of the vortices, $\bar{z}_2 = 0$, is static at the beginning of coordinates.

Analyzing dynamics of three vortices for different values of parameters a_1, a_0, we can see that in general case due to nonvanishing orbital momentum no one of three vortices crosses beginning of coordinates.

2) For N = 4 case we have

$$\Phi(\bar{z}, t) = (\bar{z}^4 - 12\bar{z}^2 i\nu t + 12(i\nu t)^2) + a_3(\bar{z}^3 - 6\bar{z}i\nu t) + a_2(\bar{z}^2 - 2i\nu t) + a_1\bar{z} + a_0 \tag{5.55}$$

or like in previous case if we choose the center of mass at the beginning of the coordinate system, the coefficient

$$-a_3 = \bar{z}_1(t) + \bar{z}_2(t) + \bar{z}_3(t) + \bar{z}_4(t) = 0$$

and we have

$$\Phi(\bar{z}, t) = \bar{z}^4 + \bar{z}^2(-12i\nu t + a_2) + a_1\bar{z} + (a_0 + 12(i\nu t)^2 - 2a_2 i\nu t) \tag{5.56}$$

Explicit form of the roots in this case can be done in radicals but formulas are long. This is why we will write them only for special case $a_1 = 0$. Solving bi-quadratic equation

$$\bar{z}^4 + \bar{z}^2[a_2 - 12i\nu t] + [a_0 - 2a_2 i\nu t + 12(i\nu t)^2] = 0 \tag{5.57}$$

we have four roots:

$$\bar{z}_{1,2} = \pm\sqrt{-\left[\frac{a_2}{2} - 6i\nu t\right] + \sqrt{\left(\frac{a_2}{2} - 6i\nu t\right)^2 - [a_0 - 2a_2 i\nu t + 12(i\nu t)^2]}} \qquad (5.58)$$

$$\bar{z}_{3,4} = \pm\sqrt{-\left[\frac{a_2}{2} - 6i\nu t\right] - \sqrt{\left(\frac{a_2}{2} - 6i\nu t\right)^2 - [a_0 - 2a_2 i\nu t + 12(i\nu t)^2]}} \qquad (5.59)$$

From dynamics of 3 and 4 vortices we can see that differences appear for the vortex motion at finite times. But for large time the behavior of vortices is similar to the particular case (5.30). Explanation of this fact can be done for general N-vortex configuration. Indeed, if we consider asymptotic form of the general N-vortex solution (5.43), when $t \to \infty$ and $|\bar{z}| \to \infty$ such that $|z|^2/t \to const$, then we can see that all terms of the function Ψ_N for any N have the same order. Moreover, the dominant role in (5.43) plays the function Ψ_N with highest order of N. But it is exactly solution (5.26) which we found before. It shows that asymptotically our vortices will follow diagonal lines according to the law (5.30): $\bar{z}_k(t) = 2w_k\sqrt{i\nu t}$. If we calculate complex velocity of the to k- vortex center

$$v_k = \frac{d\bar{z}_k}{dt} = w_k\sqrt{\frac{i\nu}{t}} \qquad (5.60)$$

then we can see that at large times $t \to \infty$ the velocity of the vortex asymptotically vanishes as $|v_k| \approx \frac{1}{\sqrt{t}}$.

5.3.4 Vortex Lattice Dynamics

Now in Eq. (5.16) instead of motion of simple zeroes we consider the periodic set of zeroes - which represent the periodic chain lattice in the plane. For one chain lattice in the x direction we can choose

$$\phi = \sin\bar{z} = \bar{z}\prod_{k=1}^{\infty}(1 - \frac{\bar{z}}{n^2\pi^2}) \qquad (5.61)$$

which shows that this function has periodic in x set of simple zeroes. For two chain lattices we are looking for solution of equation

$$i\Phi_t - \nu\Phi_{\bar{z}\bar{z}} = 0 \qquad (5.62)$$

in the form

$$\Phi(\bar{z},t) = \sin(\bar{z}-\bar{z}_1(t))\sin(\bar{z}-\bar{z}_2(t)) \tag{5.63}$$

Differentiating

$$\begin{aligned}
\Phi_t &= -\dot{\bar{z}}_1\cos(\bar{z}-\bar{z}_1(t))\sin(\bar{z}-\bar{z}_2(t)) - \dot{\bar{z}}_2\sin(\bar{z}-\bar{z}_1(t))\cos(\bar{z}-\bar{z}_2(t)) \\
\Phi_{\bar{z}} &= \cos(\bar{z}-\bar{z}_1(t))\sin(\bar{z}-\bar{z}_2(t)) + \sin(\bar{z}-\bar{z}_1(t))\cos(\bar{z}-\bar{z}_2(t)) \\
\Phi_{\bar{z}\bar{z}} &= -2\sin(\bar{z}-\bar{z}_1(t))\sin(\bar{z}-\bar{z}_2(t)) + 2\cos(\bar{z}-\bar{z}_1(t))\cos(\bar{z}-\bar{z}_2(t))
\end{aligned}$$

and substituting to the equation we have the system

$$\dot{\bar{z}}_1 = -4i\cot(\bar{z}_1-\bar{z}_2) \tag{5.64}$$

$$\dot{\bar{z}}_2 = 4i\cot(\bar{z}_1-\bar{z}_2) \tag{5.65}$$

Adding these two equations gives the first integral of motion

$$\bar{z}_1(t)+\bar{z}_2(t) = C_0 \tag{5.66}$$

describing the center of mass of the system. Subtracting the second equation from the first one we have following differential equation

$$\frac{d\cos(\bar{z}_1-\bar{z}_2)}{\cos(\bar{z}_1-\bar{z}_2)} = -4i\nu dt \tag{5.67}$$

solution of which can be represented in the form

$$\cos(\bar{z}_1-\bar{z}_2) = re^{-4i\nu t} \tag{5.68}$$

where r is the real constant. This constant represents the modulus of complex integration constant, the phase factor of this constant can always be absorbed to the beginning of time shifted on a constant value.

Without loss of generality we can choose coordinate system such that the center of mass of the system would be at the beginning of the coordinate. In this case $C_0 = 0$ and as follows $\bar{z}_2 = -\bar{z}_1$. Then, our solution acquires a simple form

$$\cos 2\bar{z}_1(t) = re^{-4i\nu t} \tag{5.69}$$

Substituting to (5.63) we have

$$\Phi(\bar{z},t) = \sin(\bar{z}-\bar{z}_1(t))\sin(\bar{z}+\bar{z}_1(t)) \tag{5.70}$$

$$= \sin^2 \bar{z} \cos^2 \bar{z}_1 - \cos^2 \bar{z} \sin^2 \bar{z}_1 \tag{5.71}$$

$$= \sin^2 \bar{z}(\frac{1 + \cos 2\bar{z}_1}{2}) - \cos^2 \bar{z}(\frac{1 - \cos 2\bar{z}_1}{2}) \tag{5.72}$$

$$= \sin^2 \bar{z}(\frac{1 + re^{-4i\nu t}}{2}) - \cos^2 \bar{z}(\frac{1 + re^{-4i\nu t}}{2}) \tag{5.73}$$

$$= -\frac{1}{2}\cos 2\bar{z} + \frac{r}{2}e^{-4i\nu t} \tag{5.74}$$

Using expansion

$$\cos 2\bar{z} = \cos 2(x - iy) \tag{5.75}$$

$$= \cos(2x - i2y) \tag{5.76}$$

$$= \cos 2x \cosh 2y + i \sin 2x \sinh 2y \tag{5.77}$$

finally we get two vortex chain lattice solution in the form

$$\Phi(\bar{z}, t) = (-\frac{1}{2}\cos 2x \cosh 2y - \frac{r}{2}\cos 4\nu t) \tag{5.78}$$

$$+ \quad i(-\frac{1}{2}\sin 2x \sinh 2y - \frac{r}{2}\sin 4\nu t) \tag{5.79}$$

To draw graph corresponding to this solution we notice that a zero of function Φ relates to the maximum value of function

$$F = \frac{1}{1 + |\Phi|^2}. \tag{5.80}$$

As we show in Chapter 7 this function appears in projection of spin vector

$$S_3 = \frac{1 - |\Phi|^2}{1 + |\Phi|^2} \tag{5.81}$$

$$= \frac{2}{1 + |\Phi|^2} - 1 \tag{5.82}$$

and the maximum value $S_3 = 1$ corresponds to the center of magnetic vortex. In Fig.5.4 interaction of two vortex lattices for function

$$F = \frac{4}{4 + (r\cos 8t - \cos 2x \cosh 2y)^2 + (r \sin 8t - \sin 2x \sinh 2y)^2} \tag{5.83}$$

for parameters $\nu = 2$, $r = 6$ at three different times $t = -1$, $t = 0$, $t = 1$ is given.

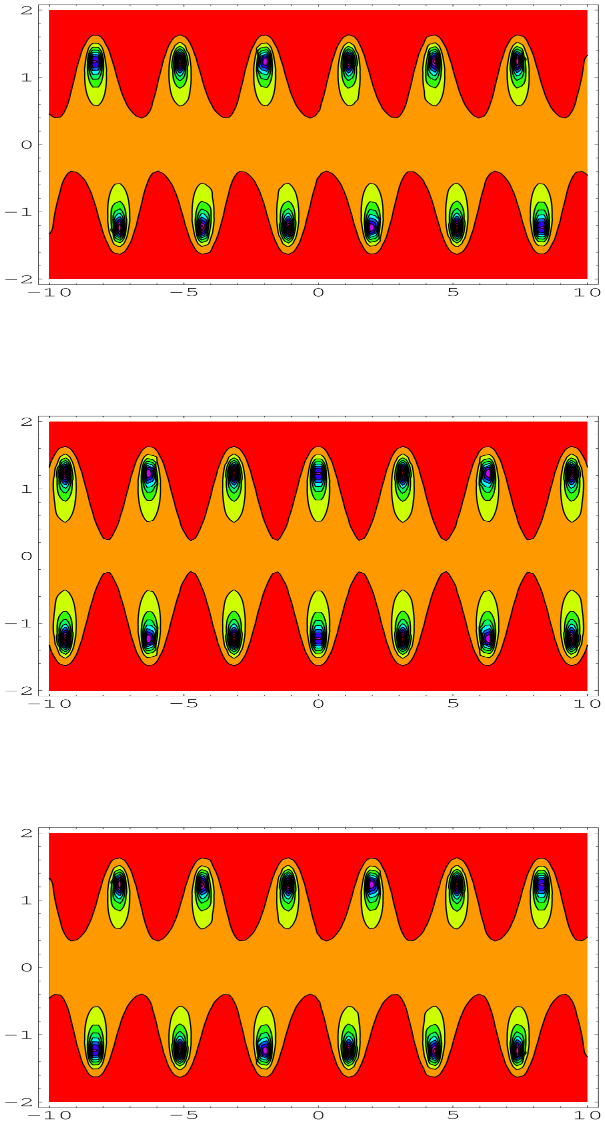


Figure 5.4. $N = 2$ Vortex Lattice Dynamics

In a similar way we can consider N vortex chain lattices in horizontal x direction by a solution

$$\Phi(\bar{z}, t) = \prod_{k=1}^{N} \sin(\bar{z} - \bar{z}_k(t)) \tag{5.84}$$

Then vortex positions are subject to the system of equations

$$\dot{\bar{z}}_j = -4i \sum_{k} \cot(\bar{z}_j - \bar{z}_k) \tag{5.85}$$

In Section 6 we show that this system allows mapping to the complexified Calogero-Moser system type III. This is why the N vortex chain lattice dynamics is Hamiltonian and integrable. Finally, if instead of $\sin \bar{z}$ function we consider solution in the form

$$\Phi(\bar{z},t) = \prod_{k=1}^{N} \sinh(\bar{z} - \bar{z}_k(t)) \tag{5.86}$$

then due to the relation

$$\sinh \bar{z} = -i \sin i\bar{z} \tag{5.87}$$

every zero of Φ undergoes a rotation in complex plane on angle $\pi/2$. It means we will have N vortex chain lattice directed in vertical y direction. In this case the system of equations on vortex positions is mapped to the complexified Calogero-Moser system type II (6.2).

5.3.5 Vortex Solutions Generating Techniques

In this section we propose a general method allowing us to create an arbitrary number of vortices on given background solution. In general, N vortex configuration is described by complex polynomial function degree N

$$\Psi(\bar{z},t) = (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t))...(\bar{z} - \bar{z}_N(t)) = \tag{5.88}$$

$$\bar{z}^N - (\bar{z}_1 + ... + \bar{z}_N)\bar{z}^{N-1} + ... + (-1)^N \bar{z}_1...\bar{z}_N \tag{5.89}$$

where complex roots $\bar{z}_1(t), ..., \bar{z}_N(t)$, are functions of time subject to the system (5.9). Eq. (5.89) has form

$$\Psi(\bar{z},t) = \sum_{s=0}^{\infty} (-1)^{N-s} P_{N-s}(t) \bar{z}^s \tag{5.90}$$

where coefficients are represented according to the Viete theorem in terms of symmetric polynomials

$$P_0 = 1 \tag{5.91}$$

$$P_1 = \bar{z}_1 + ... + \bar{z}_N \tag{5.92}$$

$$P_2 = \bar{z}_1\bar{z}_2 + ... + \bar{z}_{N-1}\bar{z}_N \tag{5.93}$$

$$... \tag{5.94}$$

$$P_N = \bar{z}_1...\bar{z}_N. \tag{5.95}$$

Adding all equations in the system (5.9), it is easy to see that the polynomial P_1 is integral of motion having meaning of the center of mass for N vortices. However, all other polynomials are not integrals of motion. Only proper combinations of these polynomials provide integrals of motion of the system. Moreover, to find solutions $\bar{z}_1(t), ..., \bar{z}_N(t)$ in terms of these polynomials means solving algebraic equation degree N, which as known to be solvable in radicals only for $N \leq 4$ (Abel, Galois).

This is why we follow another, so called vortex generation approach (Pashaev and Gurkan 2005), adding a new zero or a vortex to the system. In Section 5.3.2 we have constructed basic solutions in terms of Hermite polynomials of complex argument

$$\Psi_N(\bar{z}, t) = (i\nu t)^{N/2} H_N(\frac{\bar{z}}{2\sqrt{i\nu t}}). \tag{5.96}$$

Motivated by the following operator representation of the standard Hermite polynomials (Appendix C.3)

$$H_N(x) = (2x - \frac{d}{dx})^N 1 \tag{5.97}$$

we define operator representation for Hermite polynomials of complex argument

$$H_N(z) = (2z - \frac{d}{dz})^N 1 \tag{5.98}$$

and will write

$$H_N(\frac{\bar{z}}{2\sqrt{i\nu t}}) = (\frac{\bar{z}}{\sqrt{i\nu t}} - 2\sqrt{i\nu t}\frac{d}{d\bar{z}})^N 1 \tag{5.99}$$

which implies

$$\Psi_N(\bar{z}, t) = (i\nu t)^{N/2} (\frac{\bar{z}}{\sqrt{i\nu t}} - 2\sqrt{i\nu t}\frac{d}{d\bar{z}})^N 1 \tag{5.100}$$

and as a result we have simple operator representation for our basic solution

$$\Psi_N(\bar{z}, t) = (\bar{z} - 2i\nu t\frac{d}{d\bar{z}})^N 1. \tag{5.101}$$

Below we construct few first solutions

1. N=1

$$\begin{aligned} \Psi_1(\bar{z}, t) &= (\bar{z} - 2i\nu t\frac{d}{d\bar{z}})1 \\ &= \bar{z} \end{aligned} \tag{5.102}$$

2. N=2

$$\begin{aligned} \Psi_2(\bar{z}, t) &= (\bar{z} - 2i\nu t\frac{d}{d\bar{z}})(\bar{z} - 2i\nu t\frac{d}{d\bar{z}})1 \\ &= (\bar{z} - 2i\nu t\frac{d}{d\bar{z}})\bar{z} \qquad (5.103) \\ &= \bar{z}^2 - 2i\nu t \qquad (5.104) \end{aligned}$$

3. N=3

$$\begin{aligned}\Psi_3(\bar{z},t) &= (\bar{z}-2i\nu t\frac{d}{d\bar{z}})(\bar{z}-2i\nu t\frac{d}{d\bar{z}})(\bar{z}-2i\nu t\frac{d}{d\bar{z}})1 \\ &= (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\bar{z}^2-2i\nu t \qquad (5.105)\\ &= \bar{z}^3-6\bar{z}i\nu t \qquad (5.106)\end{aligned}$$

This representation suggests the generating operator method for finding solutions of our equation

$$i\Phi_t = \nu\Phi_{\bar{z}\bar{z}}. \tag{5.107}$$

We can prove the next, *vortex generation* lemma .

Lemma 5.3.5.1 *If $\Phi(\bar{z},t)$ is a solution of equation (5.107). Then function*

$$\Psi(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\Phi(\bar{z},t) \tag{5.108}$$

is also a solution of equation (5.107).

Proof 5.3.5.2 *By direct substitution*

$$\begin{aligned}i\Psi_t(\bar{z},t) &= i\frac{d}{dt}(\bar{z}-2i\nu t\frac{d}{d\bar{z}})\Phi(\bar{z},t) \qquad (5.109)\\ &= i(\bar{z}\Phi_t-2i\nu t\Phi_{\bar{z}}-2i\nu t\Phi_{\bar{z}t}) \qquad (5.110)\\ &= \bar{z}\nu\Phi_{\bar{z}\bar{z}}+2\nu\Phi_{\bar{z}}-2i\nu^2 t\Phi_{\bar{z}\bar{z}\bar{z}} \qquad (5.111)\end{aligned}$$

$$\begin{aligned}\nu\Psi_{\bar{z}\bar{z}}(\bar{z},t) &= \nu\frac{d}{d\bar{z}}(\Phi+\bar{z}\Phi_{\bar{z}}-2i\nu t\Phi_{\bar{z}\bar{z}}) \qquad (5.112)\\ &= 2\nu\Phi_{\bar{z}}+\nu\bar{z}\Phi_{\bar{z}\bar{z}}-2i\nu^2 t\Phi_{\bar{z}\bar{z}} \qquad (5.113)\\ &\Rightarrow\ i\Psi_t-\nu\Psi_{\bar{z}\bar{z}}=0. \qquad (5.114)\end{aligned}$$

Equation (5.107) has evident solution $\Phi = 1$. Then according to the Lemma 5.3.5.1

$$\Psi_1(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\Phi(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\cdot 1 \tag{5.115}$$

is also a solution of equation (5.107). As a next step we can construct the following solution

$$\Psi_2(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\Psi_1 = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})^2 1. \tag{5.116}$$

Continuing this procedure we have solution of (5.107) for an arbitrary positive integer N

$$\Psi_N(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})\Psi_{N-1} = ... = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})^N 1 \tag{5.117}$$

This way we derived operator representation for particular N vortex solution

$$\Psi_N(\bar{z},t) = \prod_{i=1}^{N}(\bar{z}-\bar{z}_i(t)) \quad (5.118)$$

$$= (\bar{z}-2i\nu t\frac{d}{d\bar{z}})^N 1. \quad (5.119)$$

Now let us consider the operator form for the basic solution Ψ_N (5.117). Representing complex $\bar{z}=\gamma x$ where x is real, and γ is complex variable to be determined, we have

$$\Psi_N(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})^N 1 = (\gamma x - \frac{2i\nu t}{\gamma}\frac{d}{dx})^N 1 \quad (5.120)$$

$$= (\frac{2i\nu t}{\gamma})^N(\frac{\gamma^2}{2i\nu t}x - \frac{d}{dx})^N. \quad (5.121)$$

If we choose $\gamma = \pm 2\sqrt{i\nu t}$ then

$$\Psi_N(\bar{z},t) = (\frac{2i\nu t}{\gamma})^N(2x-\frac{d}{dx})^N = (i\nu t)^{N/2}(2x-\frac{d}{dx})^N. \quad (5.122)$$

At a zero $\bar{z}_k$ of function Ψ we have equation

$$\Psi(\bar{z}_k,t) = 0 = (i\nu t)^{N/2}(2x_k - \frac{d}{dx_k})^N 1 = (i\nu t)^{N/2}H_N(x_k). \quad (5.123)$$

which implies

$$H_N(x_k) = 0 \quad (5.124)$$

It means that $x_k = w_k$ is a root of Hermite Polynomial $H_N(x)$. Then solution of vortex equation can be written in terms of this root

$$\bar{z}_k(t) = \pm 2\sqrt{i\nu t}\,w_k. \quad (5.125)$$

Using the Lemma 5.3.5.1 and the linearity of Eq.(5.107) we have the next

Corollary 5.3.5.3 *If $\Phi(\bar{z},t)$ is a solution of equation (5.107) then function*

$$\Psi(\bar{z},t) = \sum_{n=0}^{N} a_n(\bar{z}-2i\nu t\frac{d}{d\bar{z}})^n\,\Phi(\bar{z},t) \quad (5.126)$$

where $a_0,...,a_N$ are arbitrary constants, is also a solution of equation (5.107).

As a particular case of this Corollary, when $\Phi(\bar{z},t)=1$ and $(\bar{z}-2i\nu t\frac{d}{d\bar{z}})^k \cdot 1 = \Psi_k(\bar{z},t)$ we have result (5.43).

As easy to see from this Corollary, adding to the system a new vortex in a proper way, we add an additional integral of motion. Let us suppose that we have solution $\Psi(\bar{z},t)$ of Eq.(5.107) with N - simple zeroes at points $\bar{z}_1,...,\bar{z}_N$,

$$\Psi(\bar{z}_n,t)=0,\ \ n=1,...,N \tag{5.127}$$

or

$$\Psi(\bar{z},t)=\prod_{n=1}^{N}(\bar{z}-\bar{z}_n(t)) \tag{5.128}$$

which means we have N vortices located at those points. Denoting $\Psi_n(\bar{z},t) = (\bar{z}-2i\nu t\frac{d}{d\bar{z}})^n \cdot 1$ we have

$$\Psi(\bar{z},t)=\prod_{n=1}^{N}(\bar{z}-\bar{z}_n(t))=\sum_{n=0}^{N}a_n\Psi_n(\bar{z},t) \tag{5.129}$$

Using conditions (5.127) we have the system of N linear algebraic equations

$$\sum_{n=0}^{N}a_n\Psi_n(\bar{z}_k,t)=0,\ \ k=1,...,N. \tag{5.130}$$

Extracting $n=0$ term and dividing on a_0 it can be rewritten in the form of inhomogeneous system of N algebraic equations

$$\sum_{n=1}^{N}b_n\Psi_n(\bar{z}_k,t)=-1,\ \ k=1,...,N \tag{5.131}$$

on N variables $b_n=a_n/a_0$. Then, N integrals of motion in terms of $\bar{z}_1,...,\bar{z}_N$ can be found by Crammers formulas

$$b_k=\frac{\Delta_k}{\Delta},\ \ k=1,...,N. \tag{5.132}$$

From the above Corollary we have another

Corollary 5.3.5.4 *If $\Phi(\bar{z},t)$ is a solution of equation (5.107) and $F(\bar{z})$ is anti-analytic function in some domain $D_0=\{|z|<R\}$, then function*

$$\Psi(\bar{z},t)=F(\bar{z}-2i\nu t\frac{d}{d\bar{z}})\,\Phi(\bar{z},t) \tag{5.133}$$

is also a solution of equation (5.107).

This results from Corollary 5.3.5.3 if we expand

$$F(\bar{z})=\sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\bar{z}^n$$

and identify $a_n\equiv\frac{F^{(n)}(0)}{n!}$. Since coefficients a_n are integrals of motion, the function F can be considered as the generating function of integrals of motion.

5.4 Single Vortex Collision with Vortex Lattice

In this section we demonstrate efficiency of Lemma 5.3.5.1 by constructing a new class of solutions describing single vortex collision with the vortex chain lattices (Pashaev and Gurkan 2005). If in the Lemma we consider solution Φ in the double lattice form (5.74) then we have another solution of Eq.(5.7) in the form

$$\Psi(\bar{z},t) = (\bar{z} - 2i\nu t\frac{\partial}{\partial\bar{z}})\frac{1}{2}(re^{4i\nu t} - \cos 2\bar{z}) \quad (5.134)$$

$$= \frac{1}{2}[re^{4i\nu t}\bar{z} - \bar{z}\cos 2\bar{z} - 4i\nu t\sin 2\bar{z}] \quad (5.135)$$

Using properties of trigonometric function of complex argument

$$\sin 2\bar{z} = \sin 2x\cosh 2y - i\cos 2x\sinh 2y \quad (5.136)$$

we have for the real and imaginary parts of Ψ following expressions correspondingly

$$\Re\Psi = \frac{r}{2}x\cos 4\nu t - \frac{r}{2}y\sin 4\nu t \quad (5.137)$$

$$- \frac{1}{2}x\cos 2x\cosh 2y - \frac{1}{2}y\sin 2x\sinh 2y \quad (5.138)$$

$$- 4\nu t\cos 2x\sinh 2y \quad (5.139)$$

$$\Im\Psi = -\frac{r}{2}x\sin 4\nu t - \frac{r}{2}y\cos 4\nu t \quad (5.140)$$

$$+ \frac{1}{2}y\cos 2x\cosh 2y - \frac{1}{2}x\sin 2x\sinh 2y \quad (5.141)$$

$$- 4\nu t\sin 2x\cosh 2y \quad (5.142)$$

The zeroes of function Ψ are located at maximum points of the real function f:

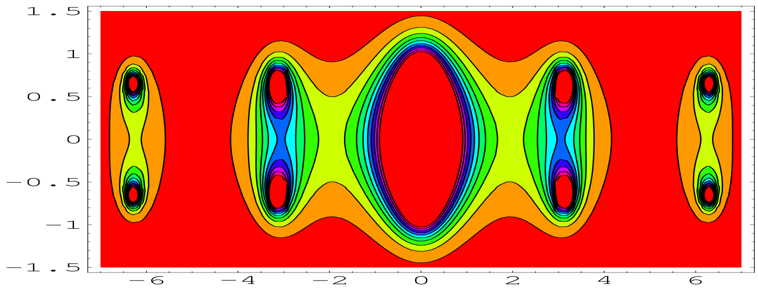
$$f = \frac{1}{1 + (\Re\Psi)^2 + (\Im\Psi)^2} \quad (5.143)$$

which provides position of magnetic vortices in Chapter 7. In Fig. 5.5 we illustrate dynamics of a single vortex with 2 vortex lattices. From these figures we can see that role of the single vortex is merge two vortex lattices, which propagates in positive and negative directions of x axis. In this sense single vortex in the rest initiate the process of fusion of two vortex lattices.

Following the same idea we can consider solution describing interaction of N- vortices with M- vortex chain lattices in the form

$$\Psi(\bar{z},t) = (\bar{z} - 2i\nu t\frac{\partial}{\partial\bar{z}})^N \prod_{k=1}^{M} \sin(\bar{z} - \bar{z}_k(t)) \quad (5.144)$$

where $\bar{z}_k(t)$ is a solution of the system (5.85).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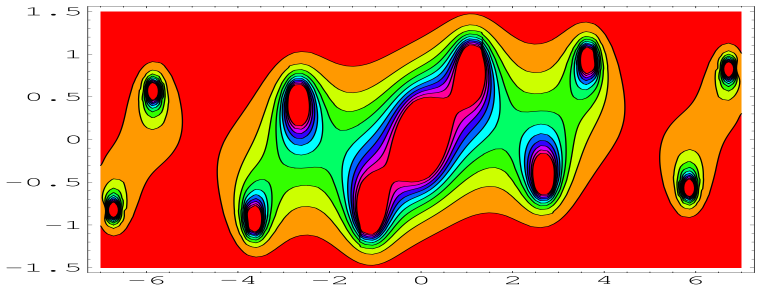
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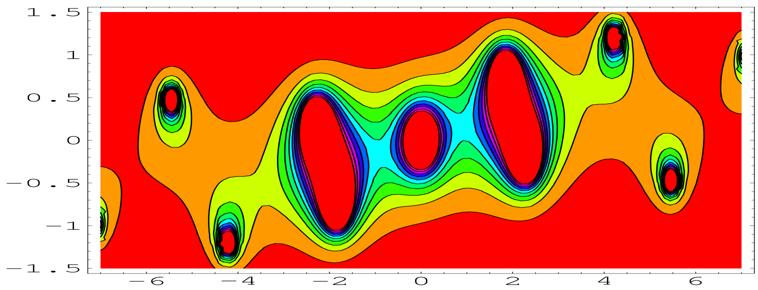
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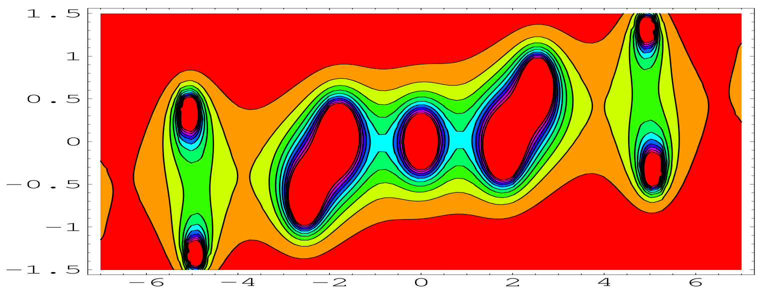
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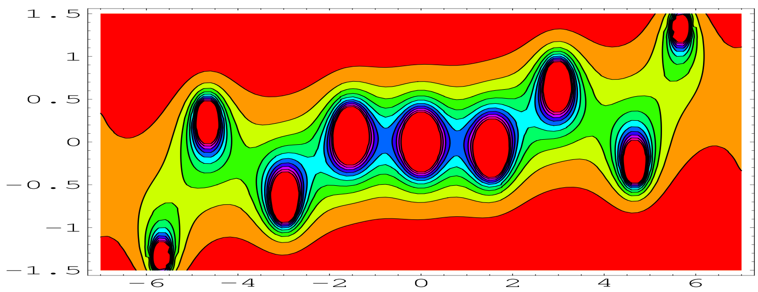
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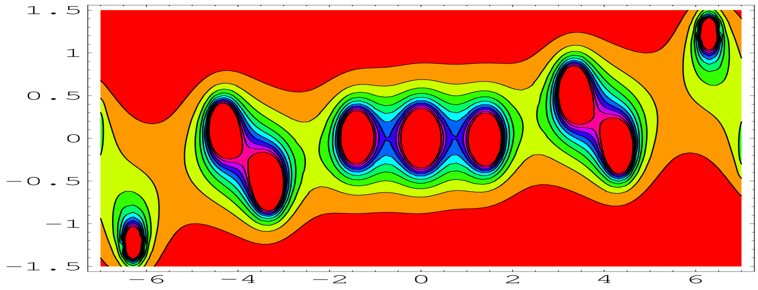
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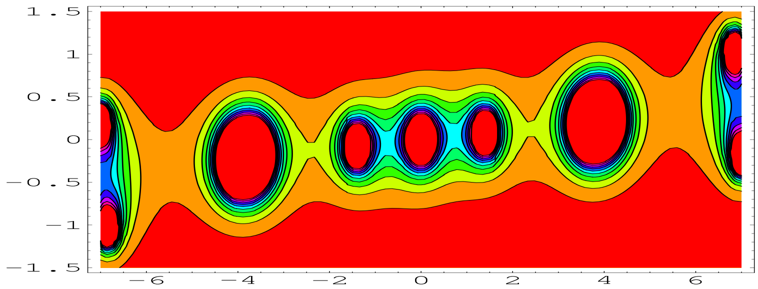
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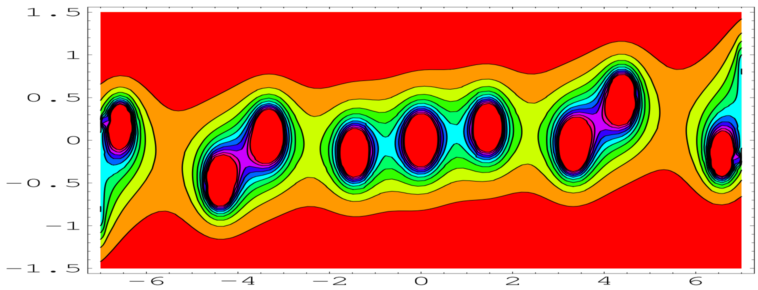
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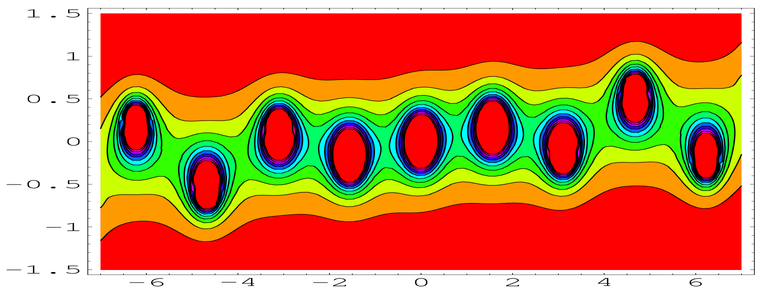
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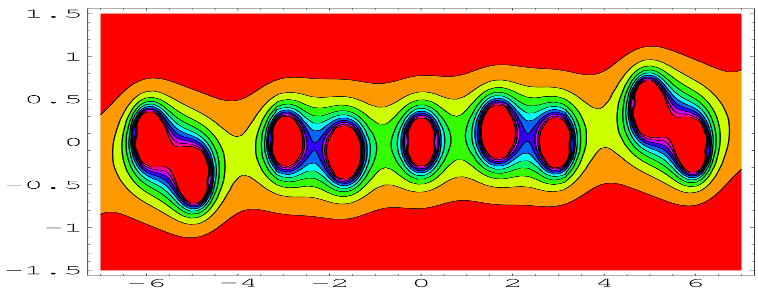
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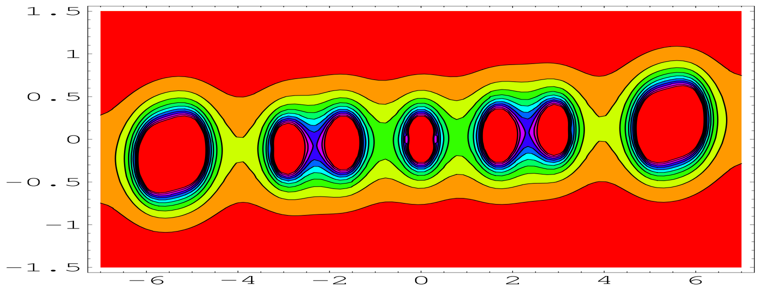
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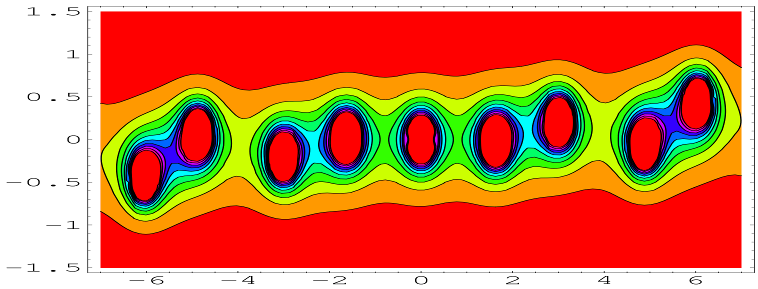
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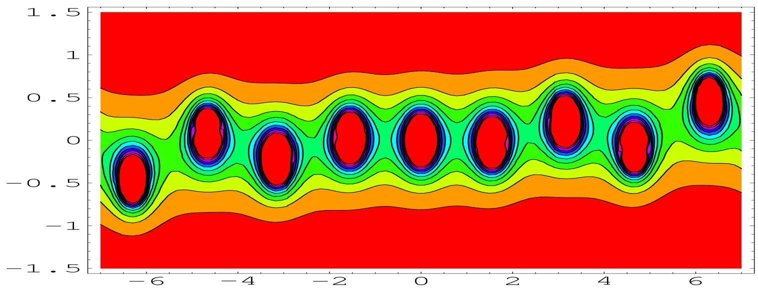
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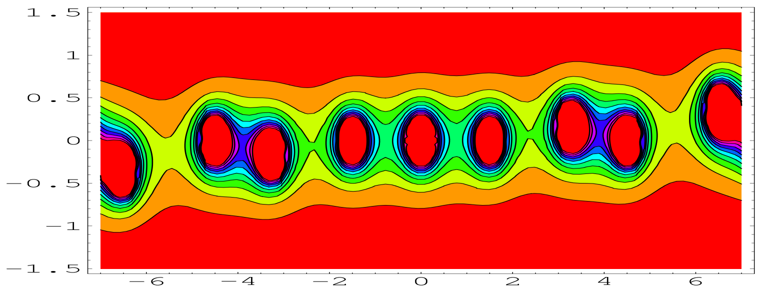
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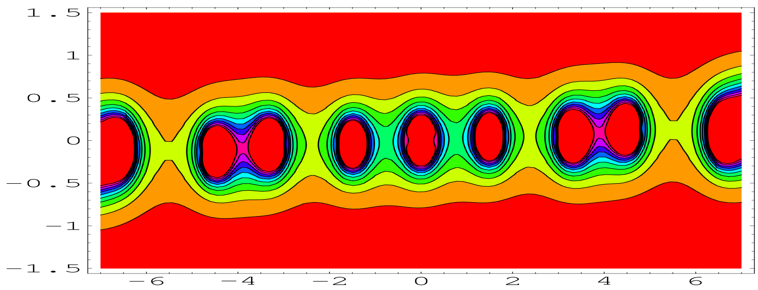
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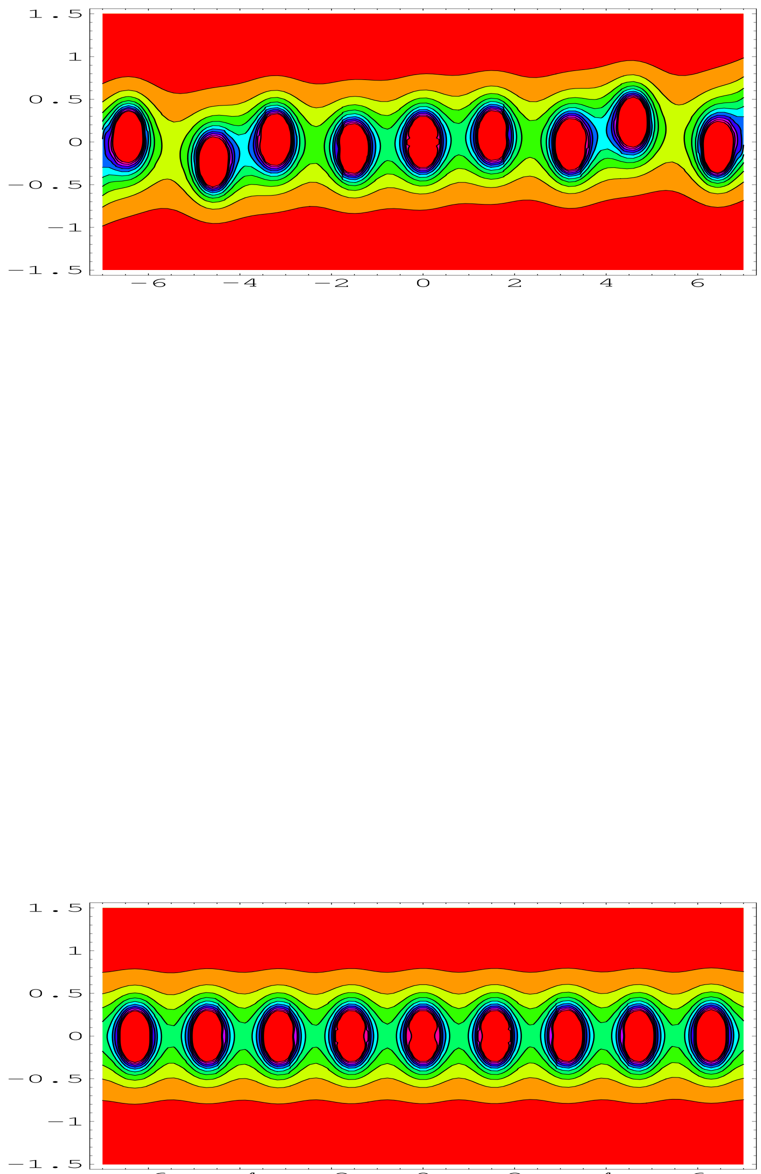

Figure 5.5. Single Vortex - 2 Vortex Lattice Dynamics

CHAPTER 6

INTEGRABLE VORTEX DYNAMICS AND MULTI-PARTICLE PROBLEM

In the present chapter we study the mapping of the point vortex equations to the integrable multiparticle problem - the complexified Calogero-Moser problem.

6.1 Calogero-Moser System

One dimensional problem of N- interacting particles admits the Lax representation and is integrable (Calogero 1978) if in the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{j<k} v(q_j - q_k) \tag{6.1}$$

the pair interaction potential $v(q_j - q_k)$ has the one of the next forms

$$v(\xi) = \begin{cases} \xi^{-2}, & \text{I;} \\ a^2 \sinh^{-2}(a\xi), & \text{II;} \\ a^2 \sin^{-2}(a\xi), & \text{III;} \\ a^2 \mathcal{P}(a\xi), & \text{IV.} \end{cases} \tag{6.2}$$

where a is an arbitrary parameter, and $\mathcal{P}(\xi) = \mathcal{P}(\xi, \omega_1, \omega_2)$ is the Weierstrass function , which is a double periodic function of the complex variable ξ with periods $2\omega_1$ and $2\omega_2$ and with second order poles at the points $2(m\omega_1 + m\omega_2)$ (Perelomov 1990). In the limit as one of the periods goes to infinity , the potential of type IV goes over into the potentials of type II or III. The potential of type I results by letting both periods go to infinity. Therefore the system of type IV is the most general one. Nevertheless the systems of type I,II and III have certain specific features that make it reasonable to treat them separately.

Then the Hamiltonian equations for the above potentials

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = p_j, \ j = 1, ..., N \tag{6.3}$$

are equivalent to the Lax matrix equation (Perelomov 1990)

$$i\dot{L} = AL - LA \tag{6.4}$$

Explicit form of the Lax operators for the Case I is

$$L_{jk} = \delta_{jk} p_j + ig(1-\delta_{jk})\frac{1}{q_j - q_k}, \tag{6.5}$$

$$A_{jk} = g\left[\delta_{jk}\sum_{l\neq j}\frac{1}{(q_j-q_l)^2} - (1-\delta_{jk})\frac{1}{(q_j-q_k)^2}\right]. \tag{6.6}$$

As we discussed before in Section 3.5 (Appendix B.2) the Lax equation (6.4) is the isospectrality condition ($\lambda_t = 0$) for the next linear problem

$$LU = \lambda U \tag{6.7}$$

$$iU_t = AU \tag{6.8}$$

From this Lax representation follows that under time evolution $L(t)$ undergoes a similarity transformation

$$L(t) = U(t)L(0)U^{-1}(t). \tag{6.9}$$

Due to this similarity transformation the eigenvalues of $L(t)$ are time independent and so are integrals of motion. Equivalently we can say that matrix $L(t)$ is isospectrally deformed with time. Instead of the eigenvalues it is often more convenient to take their symmetric functions as integrals of motion, for example ,

$$I_k = trL^{k+1} \tag{6.10}$$

If in such a way one can find N functionally independent integrals of motion and show that they are in involution, then the system is completely integrable in the Liouville sense (Section 3.4.3). It is the case for the Calogero -Moser model (6.1) of all four types I, II, II, IV.

6.2 Integrable N-particle Problem for N-Vortex Motion

In this section we show that the problem of N-point vortices in the plane can be reduced to the complexified version of the Calogero-Moser model (6.1) type I. As we have seen in Section 5.3 the system of N point vortices is described by function

$$\Phi(\bar{z}, t) = \prod_{j=1}^{N}(\bar{z} - \bar{z}_j(t)) \tag{6.11}$$

satisfying the complex Schrodinger equation (5.7). Then positions of vortices in the complex plane, $\bar{z}_1, ..., \bar{z}_N$, are subject to the first order system

$$\frac{d}{dt}\bar{z}_j = 2\nu i \sum_{k\neq(j)}^{N} \frac{1}{(\bar{z}_j - \bar{z}_k)}. \tag{6.12}$$

If we differentiate once and use the system again (Appendix B.6), then we have the second order Newton's equations of motion

$$\frac{d^2}{dt^2}\bar{z}_j = \sum_{k\neq(j)}^{N} \frac{4\nu^2}{(\bar{z}_j - \bar{z}_k)^3} \tag{6.13}$$

These equations have (complex) Hamiltonian form

$$\dot{\bar{z}}_j = \frac{\partial H}{\partial p_j} = p_j, \quad \dot{p} = -\frac{\partial H}{\partial \bar{z}_j} \tag{6.14}$$

with the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + 2\nu^2 \sum_{j<k} \frac{1}{(\bar{z}_j - \bar{z}_k)^2}. \tag{6.15}$$

The system (6.13) implies the complex conjugate one

$$\frac{d^2}{dt^2}z_j = 4\nu^2 \sum_{k} \frac{1}{(z_j - z_k)^3} \tag{6.16}$$

with Hamiltonian

$$\bar{H} = \frac{1}{2}\sum_{j=1}^{N} \bar{p}_j^2 + 2\nu^2 \sum_{j<k} \frac{1}{(z_j - z_k)^2}. \tag{6.17}$$

Then the real Hamiltonian for these systems is given by $H + \bar{H}$.

As easy to see, the system (6.13) is complexified version of the Calogero-Moser system discussed in the previous Section 6.1 with the Hamiltonian function (6.1) type I, where N-particle positions, $q_1, ..., q_N$ should be replaced by complex vortex positions $\bar{z}_1, ..., \bar{z}_N$, as in Eq.(6.15).

The Lax representation from Section 6.1 can be transformed to the complex case in a straightforward way. The complexified Hamiltonian equations (6.14) are equivalent to the Lax matrix equation

$$i\dot{L} = AL - LA \tag{6.18}$$

where

$$L_{jk} = \delta_{jk}p_j + ig(1 - \delta_{jk})\frac{1}{\bar{z}_j - \bar{z}_k} \tag{6.19}$$

$$A_{jk} = g\left[\delta_{jk}\sum_{l\neq j}\frac{1}{(\bar{z}_j-\bar{z}_l)^2} - (1-\delta_{jk})\frac{1}{(\bar{z}_j-\bar{z}_k)^2}\right] \tag{6.20}$$

and the coupling constant $g = \sqrt{2}\nu$. Since matrix $L(t)$ is isospectrally deformed with time, the corresponding (complex) eigenvalues are time independent integrals of motion. If one takes their symmetric functions as integrals of motion, then they are given by

$$I_k = trL^{k+1} \tag{6.21}$$

It shows that complexified Calogero-Moser system is an integrable system and as a consequence, the N-vortex system (6.12), which has been mapped to Calogero-Moser system, is also integrable.

6.3 Integrable N-particle Problem for N-Vortex Lattices

Similar to the previous case now we consider mapping of the N-vortex chain lattices to the complexified Calogero-Moser system of type II and III . For simplicity first we consider the system of two vortex chain lattices described by function (5.63)

$$\Phi(\bar{z},t) = \sin(\bar{z}-\bar{z}_1(t))\sin(\bar{z}-\bar{z}_2(t)) \tag{6.22}$$

so that position of lattices is subject to the first order system

$$\dot{\bar{z}}_1 = 2\nu i\cot(\bar{z}_1-\bar{z}_2) \tag{6.23}$$

$$\dot{\bar{z}}_2 = -2\nu i\cot(\bar{z}_1-\bar{z}_2) \tag{6.24}$$

Differentiating this system once in time we get the second order equations of motion in the Newton's form

$$\ddot{\bar{z}}_1 = 2i\nu\left(-\frac{1}{\sin^2(\bar{z}_1-\bar{z}_2)}\right)(\dot{\bar{z}}_1-\dot{\bar{z}}_2) \tag{6.25}$$

$$= -8\nu^2\frac{\cot(\bar{z}_1-\bar{z}_2)}{\sin^2(\bar{z}_1-\bar{z}_2)} \tag{6.26}$$

$$\ddot{\bar{z}}_2 = 2i\nu\left(\frac{1}{\sin^2(\bar{z}_1-\bar{z}_2)}\right)(\dot{\bar{z}}_1-\dot{\bar{z}}_2) \tag{6.27}$$

$$= 8\nu^2\frac{\cot(\bar{z}_1-\bar{z}_2)}{\sin^3(\bar{z}_1-\bar{z}_2)} \tag{6.28}$$

These equations are Hamiltonian

$$\dot{\bar{z}}_1 = \frac{\partial H}{\partial p_1} = p_1 \tag{6.29}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial \bar{z}_1} = 8\nu^2 \frac{\cot(\bar{z}_1 - \bar{z}_2)}{\sin^3(\bar{z}_1 - \bar{z}_2)} \tag{6.30}$$

$$\dot{\bar{z}}_2 = \frac{\partial H}{\partial p_2} = p_2 \tag{6.31}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial \bar{z}_2} = 8\nu^2 \frac{\cot(\bar{z}_2 - \bar{z}_1)}{\sin^3(\bar{z}_2 - \bar{z}_1)} \tag{6.32}$$

with the Hamiltonian function

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{4\nu^2}{\sin^2(\bar{z}_1 - \bar{z}_2)} \tag{6.33}$$

Comparing this Hamiltonian of two vortex lattices with the Calogero-Moser system, we realize that it corresponds to complexified version of the model type III.

We can generalize this result considering N vortex chain lattices periodic in the horizontal direction x. Positions of lattices are subject to the first order system

$$\dot{\bar{z}}_j = 2\nu i \sum_{k \neq j}^{N} \cot(\bar{z}_j - \bar{z}_k) \tag{6.34}$$

Differentiating once we get

$$\ddot{\bar{z}}_j = -8\nu^2 \sum_{k \neq j}^{N} \frac{\cot(\bar{z}_j - \bar{z}_k)}{\sin^2(\bar{z}_j - \bar{z}_k)} \tag{6.35}$$

which is complexified Calogero-Moser system type III with Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j<k} \frac{4\nu^2}{\sin^2(\bar{z}_j - \bar{z}_k)} \tag{6.36}$$

If instead of horizontal x direction, we consider N chain lattices periodic in the vertical y direction then as we have shown in Section 5.3.4, Eq.(5.87), it results in rotation of every zero of Φ (6.22) on angle $\pi/2$, which means replacement of complex function $\sin \bar{z}$ by $\sinh \bar{z}$. As a result, the corresponding Calagero-Moser system would be of type II. This consideration shows equivalence of complexified Calogero-Moser systems of type II and III.

CHAPTER 7

CLASSICAL FERROMAGNETIC MODELS IN CONTINUOUS MEDIA

7.1 Topological Magnet Models

The classical Heisenberg spin model (Makhankov and Pashaev 1992) determines evolution of the classical spin vector

$$\vec{S} = (S_1(x,y,t), S_2(x,y,t), S_3(x,y,t)) \tag{7.1}$$

valued on two dimensional sphere S^2,

$$S_1^2 + S_2^2 + S_3^2 = 1 \tag{7.2}$$

according to the Landau-Lifshitz equation

$$\vec{S}_t = \vec{S} \times \Delta \vec{S}. \tag{7.3}$$

In the spin liquid (ferromagnetic fluid) one have in addition to magnetic variables $\vec{S} = \vec{S}(x,y,t)$ the hydrodynamic variable $\vec{v}(x,y,t)$ (Volovik 1987) and time derivative $\partial/\partial t$ would be replaced by the material derivative (Martina et al. 1994a)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}\nabla). \tag{7.4}$$

Between hydrodynamic and spin variables exists relation called the Mermin-Ho relation (Ho and Mermin 1980, Mermin and Ho 1976). It relates vorticity of the flow with the topological charge density (or winding number),

$$rot\, \vec{v} = \vec{S}(\partial_x \vec{S} \times \partial_y \vec{S}). \tag{7.5}$$

Then we have simple model of ferromagnetic fluid - the so called Topological Magnet model (Martina et al. 1994b),

$$\vec{S}_t + \upsilon^a \partial_a \vec{S} = \vec{S} \times \partial^a \partial_a \vec{S} \tag{7.6}$$

$$\partial_a \upsilon_b - \partial_b \upsilon_a = 2\vec{S}(\partial_a \vec{S} \times \partial_b \vec{S}) \tag{7.7}$$

where the scalar product $A^aB_a = A^a g_{ab} B^b$, $a = 1, 2$is determined by the metric tensor $g_{ab} = diag(1, \alpha^2)$, $\alpha^2 = \pm 1$. For particular case of the metric $g_{ab} = (1, -1)$ we have the system

$$\vec{S}_t + v_1 \partial_1 \vec{S} - v_2 \partial_2 \vec{S} = \vec{S} \times (\partial_1^2 - \partial_2^2) \vec{S} \tag{7.8}$$

$$\partial_1 v_2 - \partial_2 v_1 = 2\vec{S}(\partial_1 \vec{S} \times \partial_2 \vec{S}). \tag{7.9}$$

For this system we have the next

Lemma 7.1.0.5 *The following identities hold*

$$-v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} = -\frac{1}{2} \partial_1 [v_1 (\partial_1 \vec{S})^2] + \frac{1}{2} (\partial_1 v_1)(\partial_1 \vec{S})^2 \tag{7.10}$$

$$v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} = +\frac{1}{2} \partial_2 [v_2 (\partial_2 \vec{S})^2] - \frac{1}{2} (\partial_2 v_2)(\partial_2 \vec{S})^2 \tag{7.11}$$

Proof 7.1.0.6

$$-v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} = -\partial_1 [v_1 (\partial_1 \vec{S})^2] + \partial_1 \vec{S} \cdot (v_1 \partial_1 \vec{S}) \tag{7.12}$$

$$= -\partial_1 [v_1 (\partial_1 \vec{S})^2] + (\partial_1 \vec{S})^2 \partial_1 v_1 + v_1 \partial_1 \vec{S} \cdot \partial_1^2 \vec{S} \tag{7.13}$$

$$= -\partial_1 [v_1 (\partial_1 \vec{S})^2] + (\partial_1 \vec{S})^2 \partial_1 v_1 \tag{7.14}$$

$$-v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} = -\partial_2 [v_2 (\partial_2 \vec{S})^2] + \partial_2 \vec{S} \cdot (v_2 \partial_2 \vec{S}) \tag{7.15}$$

$$= -\partial_2 [v_2 (\partial_1 \vec{S})^2] + (\partial_2 \vec{S})^2 \partial_2 v_2 + v_2 \partial_2 \vec{S} \cdot \partial_2^2 \vec{S} \tag{7.16}$$

$$= -\partial_2 [v_2 (\partial_2 \vec{S})^2] + (\partial_2 \vec{S})^2 \partial_2 v_2 \qquad \spadesuit \tag{7.17}$$

Lemma 7.1.0.7 *The following identities hold*

$$-v_1 \partial_2 \vec{S} \cdot \partial_1 \partial_2 \vec{S} = \frac{1}{2} \partial_1 [v_1 (\partial_2 \vec{S})^2] - \frac{1}{2} \partial_1 v_1 (\partial_2 \vec{S})^2 \tag{7.18}$$

$$-v_2 \partial_1 \vec{S} \cdot \partial_1 \partial_2 \vec{S} = -\frac{1}{2} \partial_2 [v_2 (\partial_1 \vec{S})^2] + \frac{1}{2} \partial_2 v_2 (\partial_1 \vec{S})^2 \tag{7.19}$$

Proof 7.1.0.8

$$-v_1 \partial_2 \vec{S} \cdot \partial_1 \partial_2 \vec{S} = \partial_1 [v_1 (\partial_2 \vec{S})^2] - \partial_1 v_1 (\partial_2 \vec{S})^2 - v_1 \partial_1 \partial_2 \vec{S} \cdot \partial_2 \vec{S} \tag{7.20}$$

$$= \frac{1}{2} \partial_1 [v_1 (\partial_2 \vec{S})^2] - \frac{1}{2} \partial_1 v_1 (\partial_2 \vec{S})^2 \tag{7.21}$$

$$-v_2 \partial_1 \vec{S} \cdot \partial_1 \partial_2 \vec{S} = -\partial_2 [v_2 (\partial_1 \vec{S})^2] + \partial_2 v_2 (\partial_1 \vec{S})^2 + v_2 \partial_1 \partial_2 \vec{S} \cdot \partial_1 \vec{S} \tag{7.22}$$

$$= -\frac{1}{2} \partial_2 [v_2 (\partial_1 \vec{S})^2] + \frac{1}{2} \partial_2 v_2 (\partial_1 \vec{S})^2 \qquad \spadesuit \tag{7.23}$$

Theorem 7.1.0.9 *For the system (7.8)(7.9) with the flow constrained by the incompressibility condition*

$$\partial_1 v_1 + \partial_2 v_2 = 0, \tag{7.24}$$

the conservation law

$$\partial_t J_0 + \partial_2 J_2 - \partial_1 J_1 = 0 \tag{7.25}$$

holds, where

$$\begin{aligned} J_0 &= (\partial_1 \vec{S})^2 + (\partial_2 \vec{S})^2, \qquad (7.26)\\ J_1 &= -2\partial_1 \vec{S} \cdot \vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S} + v_1 J_0 + 2\vec{S} \cdot (\partial_1 \vec{S} \times \partial_2^2 \vec{S} - \partial_1 \partial_2 \vec{S} \times \partial_2 \vec{S})\\ J_2 &= 2\partial_2 \vec{S} \cdot \vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S} + v_2 J_0 - 2\vec{S} \cdot (\partial_1^2 \vec{S} \times \partial_1 \partial_2 \vec{S} - \partial_1 \vec{S} \times \partial_2 \vec{S}).,\end{aligned}$$

Proof 7.1.0.10

$$\begin{aligned} \partial_t J_0 &= \partial_t[(\partial_1 \vec{S})^2 - \alpha^2 (\partial_2 \vec{S})^2]\\ &= 2[\partial_1 \vec{S} \cdot \partial_1 \partial_t \vec{S} + \partial_2 \vec{S} \cdot \partial_2 \partial_t \vec{S}]\\ &= 2[\partial_1(\partial_1 \vec{S} \cdot \partial_t \vec{S}) - \partial_1^2 \vec{S} \cdot \partial_t \vec{S} + \partial_2^2 \vec{S} \cdot \partial_t \vec{S} - \partial_2(\partial_2 \vec{S} \cdot \partial_t \vec{S})] \qquad (7.27)\end{aligned}$$

$$\partial_t J_0 - 2\partial_1(\partial_1 \vec{S} \cdot \partial_t \vec{S}) - 2\partial_2(\partial_2 \vec{S} \cdot \partial_t \vec{S}) = -2[\partial_1^2 \vec{S} \cdot \partial_t \vec{S} + \partial_2^2 \vec{S} \cdot \partial_t \vec{S}] \tag{7.28}$$

Using equations of motion (7.8) we estimate expression in the r.h.s.

$$\begin{aligned} \partial_1^2 \vec{S} \cdot \partial_t \vec{S} + \partial_2^2 \vec{S} \cdot \partial_t \vec{S} &= (\partial_1^2 \vec{S} + \partial_2^2 \vec{S})[-v_1 \partial_1 \vec{S} + v_2 \partial_2 \vec{S}]\\ &+ (\partial_1^2 \vec{S} + \partial_2^2 \vec{S})[\vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S}] \qquad (7.29)\\ &= -v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} - v_1 \partial_2^2 \vec{S} \cdot \partial_1 \vec{S} + v_2 \partial_1^2 \vec{S} \cdot \partial_2 \vec{S}\\ &+ v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} - (\partial_1^2 \vec{S} + \partial_2^2 \vec{S}) \cdot (\vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S})\\ &= -\frac{1}{2}\partial_1[v_1(\partial_1 \vec{S})^2] - \frac{1}{2}(\partial_1 v_1)(\partial_1 \vec{S})^2 + \frac{1}{2}\partial_2[v_2(\partial_2 \vec{S})^2]\\ &+ \frac{1}{2}(\partial_2 v_2)(\partial_2 \vec{S})^2 + \partial_1[v_2 \partial_1 \vec{S} \partial_2 \vec{S}] - \partial_2[v_1 \partial_1 \vec{S} \partial_2 \vec{S}]\\ &- 2\vec{S}(\partial_1 \vec{S} \times \partial_2 \vec{S})(\partial_1 \vec{S} \cdot \partial_2 \vec{S})\\ &+ \frac{1}{2}\partial_1(v_1(\partial_2 \vec{S})^2) - \frac{1}{2}\partial_2(v_2(\partial_1 \vec{S})^2) - \frac{1}{2}\partial_1 v_1((\partial_2 \vec{S})^2)\\ &+ \frac{1}{2}\partial_2 v_2((\partial_1 \vec{S})^2) - (\partial_1^2 \vec{S} + \partial_2^2 \vec{S}) \cdot (\vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S}) \qquad (7.30)\end{aligned}$$

$$\begin{aligned} \partial_t J_0 &= 2\partial_1(\partial_1 \vec{S} \cdot \partial_t \vec{S}) + 2\partial_2(\partial_2 \vec{S} \cdot \partial_t \vec{S})\\ &- 2\partial_1[-\frac{1}{2}v_1[(\partial_1 \vec{S})^2 - (\partial_2 \vec{S})^2] + v_2 \partial_1 \vec{S} \cdot \partial_2 \vec{S} + (\partial_2 \vec{S})^2 \cdot (\vec{S} \times \partial_1 \vec{S})\end{aligned}$$

$$
\begin{aligned}
&+ \quad \partial_1\partial_2\vec{S}\cdot(\vec{S}\times\partial_2\vec{S})] \\
&- \quad 2\partial_2[\frac{1}{2}v_2[(\partial_1\vec{S})^2-(\partial_2\vec{S})^2]-v_1\partial_1\vec{S}\cdot\partial_2\vec{S}+(\partial_1\vec{S})^2\cdot(\vec{S}\times\partial_2\vec{S})] \\
&- \quad \partial_1\partial_2\vec{S}\cdot(\vec{S}\times\partial_1\vec{S})] \qquad \spadesuit \qquad (7.31)
\end{aligned}
$$

Due to the above Theorem the functional

$$E = \int\int J_0 d^2x$$

(the energy functional), or

$$E = \int\int\{(\partial_1\vec{S})^2+(\partial_2\vec{S})^2\}d^2x \qquad (7.32)$$

is conserved quantity. From another side there exist another integral of motion, the topological charge of a spin configuration, defined as

$$Q = \frac{1}{4\pi}\int\int\vec{S}\cdot(\partial_1\vec{S}\times\partial_2\vec{S})d^2x. \qquad (7.33)$$

These two conserved quantities are related by the Bogomolnyi type Inequality

$$E \geq |Q|$$

- which means that the energy is bounded below by topological charge (Makhankov and Pashaev 1992).

Proof: We do several transformations of the evident inequality

$$\int\int(\partial_i\vec{S}\pm\epsilon_{ij}\vec{S}\times\partial_j\vec{S})^2d^2x \geq 0 \qquad (7.34)$$

$$
\begin{aligned}
\int\int(\partial_i\vec{S}\pm\epsilon_{ij}(\vec{S}\times\partial_j\vec{S})\cdot(\partial_i\vec{S}\pm\epsilon_{ik}\vec{S}\times\partial_k\vec{S})d^2x \quad &\geq \quad 0 \qquad (7.35) \\
\int\int[(\partial_i\vec{S})^2+\epsilon_{ij}\epsilon_{ik}(\vec{S}\times\partial_j\vec{S})\cdot(\vec{S}\times\partial_k\vec{S}) & \\
\pm\epsilon_{ij}(\vec{S}\times\partial_j\vec{S})\partial_i\vec{S}\pm\epsilon_{ik}\partial_i(\vec{S}\vec{S}\times\partial_k\vec{S})]d^2x \quad &\geq \quad 0 \qquad (7.36)
\end{aligned}
$$

$$\int\int[(\partial_i\vec{S})^2+\delta_{jk}\partial_j\vec{S}\cdot\partial_k\vec{S}\pm\epsilon_{ij}\partial_i\vec{S}(\vec{S}\times\partial_j\vec{S})\pm\epsilon_{ik}\partial_i\vec{S}(\vec{S}\times\partial_k\vec{S})]d^2x \geq 0 \qquad (7.37)$$

where $\epsilon_{ij}\epsilon_{ik} = \delta_{jk}$

$$\int\int[(\partial_i\vec{S})^2+(\partial_j\vec{S})^2\pm\epsilon_{ij}\partial_i\vec{S}(\vec{S}\times\partial_j\vec{S})\pm\epsilon_{ik}\partial_i\vec{S}(\vec{S}\times\partial_k\vec{S})]d^2x \geq 0. \qquad (7.38)$$

By cyclic permutation

$$\int\int[(\partial_i\vec{S})^2+(\partial_j\vec{S})^2\pm\epsilon_{ij}\vec{S}(\partial_j\vec{S}\times\partial_i\vec{S})\pm\epsilon_{ik}\vec{S}(\partial_k\vec{S}\times\partial_i\vec{S})]d^2x \geq 0 \qquad (7.39)$$

$$\int\int 2[(\partial_1\vec{S})^2 + (\partial_2\vec{S})^2 \mp 4\vec{S}\cdot(\partial_1\vec{S}\times\partial_2\vec{S})]d^2x \geq 0 \qquad (7.40)$$

$$E \mp 8\pi Q \geq 0 \Longrightarrow E \geq \pm 8\pi Q \qquad (7.41)$$

For $Q > 0$

$$E \geq 8\pi Q = 8\pi|Q| \qquad (7.42)$$

while for $Q < 0$

$$E \geq -8\pi Q = 8\pi|Q| \qquad (7.43)$$

Combining together we have

$$E \geq 8\pi|Q| \qquad (7.44)$$

This inequality is saturated for spin configurations satisfying the first order system

$$\partial_i\vec{S} \pm \epsilon_{ij}\vec{S}\times\partial_j\vec{S} = 0 \qquad (7.45)$$

called the self-duality equations (Martina et al. 1994c).

7.2 Stereographic Projection Representation

If we consider the spin phase space, the 2-dimensional sphere, we consider as a Riemann sphere for a complex plane, we can project points on this sphere to that plane. The stereographic projections are given by formulas

$$S_1 + iS_2 = \frac{2\zeta}{1+|\zeta|^2} \qquad (7.46)$$

$$S_3 = \frac{1-|\zeta|^2}{1+|\zeta|^2} \qquad (7.47)$$

where ζ is complex valued function. Now we will rewrite the self-duality equations (7.45) in the stereographic projection form:

$$\partial_i\vec{S} \pm \epsilon_{ij}\vec{S}\times\partial_j\vec{S} = 0 \qquad (7.48)$$

$$\partial_1\vec{S} \pm \vec{S}\times\partial_2\vec{S} = 0 \qquad (7.49)$$

$$\partial_2\vec{S} \mp \vec{S}\times\partial_1\vec{S} = 0 \qquad (7.50)$$

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \qquad (7.51)$$

Multiplying (7.50) by i and then adding to (7.49) we have

$$\partial_{\bar{z}}\vec{S} \mp i\vec{S} \times \partial_{\bar{z}}\vec{S} = 0 \quad (7.52)$$

$$\partial_{z}\vec{S} \pm i\vec{S} \times \partial_{z}\vec{S} = 0 \quad (7.53)$$

$$1)\ \ \partial_{\bar{z}}\vec{S} - i\vec{S} \times \partial_{\bar{z}}\vec{S} = 0 \quad (7.54)$$

$$\partial_{\bar{z}}S_1 - i(\vec{S} \times \partial_{\bar{z}}\vec{S})_1 = 0 \quad (7.55)$$

$$\partial_{\bar{z}}S_2 - i(\vec{S} \times \partial_{\bar{z}}\vec{S})_2 = 0 \quad (7.56)$$

$$\partial_{\bar{z}}S_1 - i(S_2\partial_{\bar{z}}S_3 - S_3\partial_{\bar{z}}S_2) = 0 \quad (7.57)$$

$$\partial_{\bar{z}}S_2 - i(S_3\partial_{\bar{z}}S_1 - S_1\partial_{\bar{z}}S_3) = 0 \quad (7.58)$$

Multiplying (7.58) by i and then adding to (7.57) we have

$$\partial_{\bar{z}}S_+ + [S_3\partial_{\bar{z}}S_+ - \partial_{\bar{z}}S_3S_+] = 0 \quad (7.59)$$

$$S_+ = S_1 + iS_2, \quad S_3 = \frac{1-|\zeta|^2}{1+|\zeta|^2} = -1 + \frac{2}{1+|\zeta|^2} \quad (7.60)$$

Substituting S_3 and S_+ in (7.60) we have the analyticity condition:

$$\zeta_{\bar{z}} = 0 \quad (7.61)$$

$$2)\ \ \partial_{z}\vec{S} - i\vec{S} \times \partial_{z}\vec{S} = 0 \quad (7.62)$$

$$\partial_{z}S_1 - i(\vec{S} \times \partial_{z}\vec{S})_1 = 0 \quad (7.63)$$

$$\partial_{z}S_2 - i(\vec{S} \times \partial_{z}\vec{S})_2 = 0 \quad (7.64)$$

$$\partial_{z}S_1 - i(S_2\partial_{z}S_3 - S_3\partial_{z}S_2) = 0 \quad (7.65)$$

$$\partial_{z}S_2 - i(S_3\partial_{z}S_1 - S_1\partial_{z}S_3) = 0 \quad (7.66)$$

Multiplying (7.66) by i and then adding to (7.65) we have

$$\partial_{z}S_+ + [S_3\partial_{z}S_+ - \partial_{z}S_3S_+] = 0 \quad (7.67)$$

Substituting S_3 and S_+ in (7.67)we have the anti-analyticity condition:

$$\zeta_{z} = 0 \quad (7.68)$$

The above consideration shows that the self- duality equations in the stereographic projection form are just the analyticity conditions while for the anti-self-duality equations they are anti-analyticity conditions. In both cases the energy (7.32) reaches its minima.

7.3 Anti-Holomorphic Reduction

As we have seen analytic/anti-analytic configurations saturate Bogomolny inequality and have minimal energy. This suggest to search solutions of topological magnet (7.9) with holomorphic/anti-holomorphic stereographic projections. For this reason we first rewrite equations in the stereographic form

$$i(\zeta_t + v_1\partial_1\zeta - v_2\partial_2\zeta) + \partial_1^2\zeta - \partial_2^2\zeta - 2\frac{(\partial_1\zeta)^2 - (\partial_2\zeta)^2}{1+|\zeta|^2}\bar{\zeta} = 0 \tag{7.69}$$

$$\partial_1 v_2 - \partial_2 v_1 = -4i\frac{\partial_1\bar{\zeta}\partial_2\zeta - \partial_2\bar{\zeta}\partial_1\zeta}{(1+|\zeta|^2)^2}. \tag{7.70}$$

In complex coordinates we have

$$\begin{aligned} i\zeta_t + iv_1(\zeta_z+\zeta_{\bar z}) + v_2(\zeta_z - \zeta_{\bar z}) &+ (\partial_z+\partial_{\bar z})^2\zeta + (\partial_z - \partial_{\bar z})^2\zeta \\ &- 2\frac{(\zeta_z+\zeta_{\bar z})^2 - (\zeta_z-\zeta_{\bar z})^2}{1+|\zeta|^2}\bar{\zeta} = 0 \end{aligned} \tag{7.71}$$

For $v_+ = v_1 + iv_2$ and $v_- = v_1 - iv_2$ (7.71) becomes

$$i(\zeta_t + v_-\zeta_z + v_+\zeta_{\bar z}) + 2(\partial_z^2\zeta + \partial_{\bar z}^2\zeta) - 4\frac{\bar{\zeta}}{1+|\zeta|^2}(\zeta_z^2 + \zeta_{\bar z}^2) = 0 \tag{7.72}$$

$$\begin{aligned} \partial_1 v_2 - \partial_2 v_1 &= (\partial_z + \partial_{\bar z})v_2 - i(\partial_z - \partial_{\bar z})v_1 \\ &= \partial_z(v_2 - iv_1) + \partial_{\bar z}(v_2 + iv_1) \\ &= i[-\partial_z(v_1+iv_2) + \partial_{\bar z}(v_1 - iv_2)] \\ &= i[\partial_{\bar z}v_- - \partial_z v_+] \end{aligned} \tag{7.73}$$

or in complex coordinates

$$\begin{aligned} i[\partial_{\bar z}v_- - \partial_z v_+] &= \frac{-4i}{(1+|\zeta|^2)^2}(\partial_1\bar{\zeta}\partial_2\zeta - \partial_2\bar{\zeta}\partial_1\zeta) \\ &= \frac{8}{(1+|\zeta|^2)^2}[\zeta_z\bar{\zeta}_{\bar z} - \bar{\zeta}_z\zeta_{\bar z}] \end{aligned} \tag{7.74}$$

If ζ is anti-holomorphic $\zeta_z = 0$, then the system (7.72)(7.74) is reduced to

$$i\zeta_t + iv_+\zeta_{\bar z} + 2\zeta_{\bar z\bar z} - 4\frac{\zeta_{\bar z}^2}{1+|\zeta|^2}\bar{\zeta} = 0 \tag{7.75}$$

$$\partial_z v_+ - \partial_{\bar z}v_- = \frac{-8i}{(1+|\zeta|^2)^2}\bar{\zeta}_z\zeta_{\bar z}. \tag{7.76}$$

To be consistent, the anti-holomorphicity constraint must be compatible with the time evolution. So that

$$\frac{\partial\zeta(\bar z, t)}{\partial z} = 0 \Longrightarrow \forall t' \;\; \frac{\partial\zeta(\bar z, t')}{\partial z} = 0 \tag{7.77}$$

$$
\begin{aligned}
\frac{\partial\zeta(\bar{z},t+dt)}{\partial z} &= \frac{\partial}{\partial z}[\zeta(\bar{z},t+dt)] \\
&= \frac{\partial}{\partial z}[\zeta(\bar{z},t)+\frac{\partial\zeta}{\partial t}dt] \\
&= \frac{\partial\zeta(\bar{z},t)}{\partial z}+\frac{\partial}{\partial z}\frac{\partial\zeta}{\partial t}dt \\
&= \frac{\partial}{\partial z}\frac{\partial\zeta}{\partial t} \\
&= 0 \qquad (7.78)
\end{aligned}
$$

Proposition 7.3.0.11 *For incompressible flow*

$$
v_{1x}+v_{2y}=0 \Longrightarrow div\,\vec{v}=0 \qquad (7.79)
$$

the anti-holomorphic constraint $\zeta_z = 0$ is compatible with the time evolution

$$
\frac{\partial}{\partial t}\zeta_z = 0. \qquad (7.80)
$$

Proof 7.3.0.12 *Differentiating (7.75) with respect to z*

$$
\frac{\partial}{\partial z}\left(i\zeta_t + iv_+\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\zeta_{\bar{z}}^2}{1+|\zeta|^2}\bar{\zeta}\right) = 0 \qquad (7.81)
$$

we get

$$
v_{+_z} = -4i\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \qquad (7.82)
$$

and complex conjugate of it

$$
v_{-_{\bar{z}}} = 4i\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}. \qquad (7.83)
$$

Adding (7.82) to (7.83) implies incompressibility condition

$$
v_{+_z} + v_{-_{\bar{z}}} = 0 \qquad (7.84)
$$

and subtracting implies

$$
v_{+_z} - v_{-_{\bar{z}}} = -8i\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \qquad (7.85)
$$

which coincides with the second equation (7.76) ♠

Under the above constraint we have the reduced system

$$
i\zeta_t + iv_+\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\zeta_{\bar{z}}^2}{1+|\zeta|^2}\bar{\zeta} = 0 \qquad (7.86)
$$

$$
i\zeta_t + \zeta_{\bar{z}}\left[iv_+ + 2\left(\ln\frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\right)_{\bar{z}}\right] = 0 \qquad (7.87)
$$

For function

$$F \equiv v_+ - 2i \left[\ln \frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \right]_{\bar{z}} \tag{7.88}$$

Eq. (7.87) becomes

$$\zeta_t + F\zeta_{\bar{z}} = 0 \tag{7.89}$$

where $F_z = 0$, due to Eq. (7.82).

7.4 Ishimori Model

Now we consider the topological magnet model (7.8)(7.9)

$$\vec{S}_t + v_1\partial_1\vec{S} - v_2\partial_2\vec{S} = \vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S} \tag{7.90}$$

$$\partial_1 v_2 - \partial_2 v_1 = 2\vec{S}(\partial_1\vec{S} \times \partial_2\vec{S}) \tag{7.91}$$

with incompressibility condition (7.24), which allows simplification of the equations. Equation $\vec{\nabla} \cdot \vec{v} = 0$ can be solved in terms of a real function ψ, the stream function of the flow,

$$v_1 = \partial_2\psi, \;\; v_2 = -\partial_1\psi.$$

If we replace v_1 and v_2 in equations (7.90)and(7.91) respectively, we get the so called Ishimori Model (Ishimori 1984).

$$\vec{S}_t + \partial_2\psi\partial_1\vec{S} + \partial_1\psi\partial_2\vec{S} = \vec{S} \times (\partial_1^2\vec{S} - \partial_2^2\vec{S}) \tag{7.92}$$

$$(\partial_1^2 + \partial_2^2)\psi = -2\vec{S} \cdot (\partial_1\vec{S} \times \partial_2\vec{S}) \tag{7.93}$$

where we have used

$$\partial_1 v_2 - \partial_2 v_1 = -\Delta\psi. \tag{7.94}$$

The Ishimori model is the first example of integrable classical spin model in 2+1 dimensions (Konopelchenko 1987). It was shown to be gauge equivalent to the Davey-Stewartson equation, representing the 2+1 dimensional generalization of the Nonlinear Schrodinger equation (Makhankov and Pashaev 1992, Lepovskiy and Shirokov 1989 , Pashaev 1996).

In terms of complex variables

$$v_+ \;=\; v_1 + iv_2 = -2i\psi_{\bar{z}} \tag{7.95}$$

$$v_- \;=\; v_1 - iv_2 = 2i\psi_z \tag{7.96}$$

and the stereographic projection (7.46) (7.47) we have it in the form

$$i\zeta_t - 2\psi_z\zeta_z + 2\psi_{\bar{z}}\zeta_{\bar{z}} + 2(\zeta_{zz} + \zeta_{\bar{z}\bar{z}}) - 4\frac{\bar{\zeta}}{1+|\zeta|^2}(\zeta_z^2 + \zeta_{\bar{z}}^2) = 0 \tag{7.97}$$

$$\psi_{z\bar{z}} = -2\frac{\zeta_z\bar{\zeta}_z - \bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.98}$$

7.4.1 Anti-holomorphic Reduction of Ishimori Model

The Ishimori model appears from the topological magnet model for the incompressible flow. But according to Proposition we have seen that such flow preserves anti(holomorphicity) constraint. This is why we consider now anti(holomorphicity) constrained Ishimori model. Under constraint $\zeta_z = 0$ we have dependence $\zeta = \zeta(\bar{z}, t)$ and the model reduces to

$$i\zeta_t + 2\psi_{\bar{z}}\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\bar{\zeta}}{1+|\zeta|^2}\zeta_{\bar{z}}^2 = 0 \tag{7.99}$$

$$\psi_{z\bar{z}} = 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.100}$$

We can rearrange the first equation as follows

$$i\zeta_t + 2\zeta_{\bar{z}}\left[\psi_{\bar{z}} + \frac{\zeta_{\bar{z}\bar{z}}}{\zeta_{\bar{z}}} - 4\frac{\bar{\zeta}\zeta_{\bar{z}}}{1+|\zeta|^2}\right] = 0 \tag{7.101}$$

$$i\zeta_t + 2\zeta_{\bar{z}}\left(\psi + \ln\frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\right)_{\bar{z}} = 0 \tag{7.102}$$

$$i\zeta_t + 2\zeta_{\bar{z}}\left[\psi - 2\ln(1+|\zeta|^2) + \ln\zeta_{\bar{z}}\right]_{\bar{z}} = 0 \tag{7.103}$$

7.4.2 Static N-Vortex Configuration

If we choose

$$\psi = 2\ln(1+|\zeta|^2) - \ln\zeta_{\bar{z}} - \ln\bar{\zeta}_z \tag{7.104}$$

then from Eq. (7.103)

$$\zeta_t = 0 \tag{7.105}$$

which means that our configurations are static. Differentiating (7.104) we find that Eq. (7.100) is satisfied automatically

$$\psi_{z\bar{z}} = [2\ln(1+|\zeta|^2)]_{z\bar{z}} = 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.106}$$

Then from (7.100)

$$\begin{aligned} e^{\psi} &= e^{2\ln(1+|\zeta|^2)} e^{-\ln\zeta_{\bar z}} e^{-\ln\bar\zeta_z} && (7.107)\\ &= \frac{(1+|\zeta|^2)^2}{\zeta_{\bar z}\bar\zeta_z} && (7.108)\\ e^{-\psi} &= \frac{\zeta_{\bar z}\bar\zeta_z}{(1+|\zeta|^2)^2} && (7.109)\end{aligned}$$

Using Eq. (7.100) we see that function ψ (7.108) is the general solution of the Liouville equation

$$\psi_{z\bar z} = 2e^{-\psi} \qquad (7.110)$$

It means that any solution of the Liouville equation is a static solution of the Ishimori Model. In chapter 3.6 we discussed N-vortex solution of Liouville model and the Stuart periodic array of vortices. Now we consider solution of model (7.110) in the form (7.108) where function

$$\begin{aligned} \zeta &= \sin(\bar z - \bar z_1) && (7.111)\\ \zeta_{\bar z} &= \cos(\bar z - \bar z_1) && (7.112)\\ \bar\zeta_z &= \cos(z - z_1) && (7.113)\end{aligned}$$

Then the corresponding stream fuction

$$\begin{aligned} \psi &= 2\ln(1+|\sin(\bar z - \bar z_1)|^2) - \ln\cos(\bar z - \bar z_1) - \ln\cos(\bar z - \bar z_1) && (7.114)\\ &= 2\ln(1+|\sin(\bar z - \bar z_1)|^2) - \ln|\cos(\bar z - \bar z_1)|^2 && (7.115)\\ &= \ln\frac{(1+|\sin\bar z|^2)^2}{|\cos\bar z|^2} && (7.116)\\ &= \ln\frac{[1+(\sin x\cosh y)^2+(\cos x\sinh y)^2]^2}{(\cos x\cosh y)^2+(\sin x\sinh y)^2} && (7.117)\end{aligned}$$

describes periodic in x lattice of vortices .

7.4.3 Single Vortex and Vortex Lattice

Now if in Eq. (7.108) for function ζ we choose

$$\zeta = \bar{z}\sin\bar{z} \tag{7.118}$$

$$\zeta_{\bar{z}} = \sin\bar{z} + \bar{z}\cos\bar{z} \tag{7.119}$$

then we find the stream function descriptive of the single vortex and the vortex lattice.

$$\begin{aligned} \psi &= 2\ln(1+|\bar{z}|^2|\sin\bar{z}|^2) - \ln(\sin\bar{z}+\bar{z}\cos\bar{z}) - \ln(\sin z + z\cos z) && (7.120)\\ &= \ln\frac{[1+|\bar{z}|^2|\sin\bar{z}|^2]^2}{|\sin\bar{z}+\bar{z}\cos\bar{z}|^2} && (7.121)\\ &= \ln\frac{1+(x^2+y^2)[(\sin x\cosh y)^2+(\cos x\sinh y)^2]^2}{|\sin z|^2+|z|^2|\cos z|^2+z\cos z\sin\bar{z}+\bar{z}\cos\bar{z}\sin z} && (7.122)\end{aligned}$$

7.4.4 Complex Time Dependent Schrödinger Equation

If we choose

$$\psi = 2\ln(1+|\zeta|^2) \tag{7.123}$$

then

$$\psi_{z\bar{z}} = 2[\frac{\bar{\zeta}_z\zeta}{1+|\zeta|^2}]_{\bar{z}} \tag{7.124}$$

$$= 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.125}$$

and Eq.(7.100) is satisfied automatically. Then from equation (7.103) for function ζ we have complex time dependent Schrödinger equation (5.7) with $\nu = -2$

$$i\zeta_t + 2\zeta_{\bar{z}\bar{z}} = 0 \tag{7.126}$$

In Chapter 5 we have studied a motion of zeroes of this equation and there relations with vortices of the complex Burgers equation. All these results can be interpreted now in terms of the magnetic vortices. Particularly, to find generating function of the basic vortex solutions of this equation we consider solution in the form

$$\zeta(\bar{z},t) = e^{k\bar{z}+\omega t} \tag{7.127}$$

where dispersion $\omega = 2ik^2$. Then

$$\zeta(\bar{z},t) = e^{k\bar{z}+2ik^2 t} \tag{7.128}$$

Let $x \equiv k\sqrt{\frac{2t}{i}}$, then we rewrite it as the Generating Function for the Hermite Polynomials of complex argument

$$\begin{aligned} e^{k\bar{z}+2ik^2t} &= e^{-x^2+2(\bar{z}\sqrt{\frac{i}{8t}})x} \\ &= \sum_{n=0}^{\infty} H_n(\bar{z}\sqrt{\frac{i}{8t}})\frac{x^n}{n!} \end{aligned} \tag{7.129}$$

or

$$\begin{aligned} \zeta(\bar{z},t) &= \sum_{n=0}^{\infty} \frac{k^n}{n!}(-2it)^{n/2} H_n(\bar{z}\sqrt{\frac{i}{8t}}) \\ &= \sum_{n=0}^{\infty} \frac{k^n}{n!}\Psi_n(\bar{z},t) \end{aligned} \tag{7.130}$$

where at every power k^n we have a polynomial solution of order n:

$$\Psi_n(\bar{z},t) = \left(\frac{2t}{i}\right)^{n/2} H_n(\bar{z}\sqrt{\frac{i}{8t}}) \tag{7.131}$$

This polynomial has n complex roots $\bar{z}_1(t), ..., \bar{z}_n(t)$ describing positions of vortices which we have studied in Section 5.3.2.

7.5 N Vortex System

As we have seen in Chapter 5 for N vortex system in general, we can choose

$$\zeta(\bar{z},t) = \prod_{j=1}^{N}(\bar{z} - \bar{z}_j(t)). \tag{7.132}$$

Then positions of vortices are subject to the system

$$\frac{d}{dt}\bar{z}_j = \frac{4}{i}\sum_{k\neq(j)} \frac{1}{(\bar{z}_j - \bar{z}_k)}. \tag{7.133}$$

This system admits 2N integrals of motion. The first 5 integrals are of the form

$$\sum_{j=1}^{N} \bar{z}_j = I_1 - iI_2 \tag{7.134}$$

$$\sum_{j=1}^{N} z_j^2 + \bar{z}_j^2 = I_3 \tag{7.135}$$

$$\sum_{j=1}^{N} \bar{z}_j^3 + 3\sum_{j<k<l} \bar{z}_j\bar{z}_k\bar{z}_l = I_4 - iI_5 \tag{7.136}$$

This is why in contrast with Kirchhoff equations the dynamics of vortices in Ishimori model is integrable. In fact as we have shown in Chapter 6, the system (7.133) admits mapping to the complexified Calogero-Moser N particle problem. We differentiate it once and use the system again (Appendix B.6) to have Newton's equations

$$\frac{d^2}{dt^2}\bar{z}_j = \sum_k \frac{16}{(\bar{z}_j - \bar{z}_k)^3} \tag{7.137}$$

These equations have the Hamiltonian form

$$\dot{\bar{z}}_j = \frac{\partial H}{\partial p_j} = p_j, \;\; \dot{p} = -\frac{\partial H}{\partial \dot{\bar{z}}_j} \tag{7.138}$$

with the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + 8\sum_{j<k} \frac{1}{(\bar{z}_j - \bar{z}_k)} \tag{7.139}$$

As we discussed before, the Calogero-Moser model is finite-dimensional integrable system admitting the Lax representation, from which follows the hierarchy of constants of motion in involution.

7.6 Dynamics of Magnetic Vortices in the Plane

In Chapter 5 we already have discussed in details dynamics of N vortices and vortex lattices in the plane. All these results are valid also for the magnetic system, under restriction of constant $\nu = -2$. By stereographic projection formulas

$$S_1 + iS_2 = \frac{2\zeta}{1 + |\zeta|^2} \tag{7.140}$$

$$S_3 = \frac{1 - |\zeta|^2}{1 + |\zeta|^2} \tag{7.141}$$

we can see that at every zero of function $\zeta(\bar{z}_k, t) = 0$

$$(S_1 + iS_2)(\bar{z}_k, t) = 0, \;\; , S_3(\bar{z}_k, t) = 1 \tag{7.142}$$

From another site for N degree polynomial ζ_N at infinity $|z| \to \infty$

$$(S_1 + iS_2)(\bar{z}_k, t) = 0, \;\; S_3(\bar{z}_k, t) = -1 \tag{7.143}$$

It shows that our zeroes correspond to the magnetic vortices located at that zeroes with the spin vector $\vec{S}$ directed up, while at infinity it is directed down (ferromagnetic type order). If we calculate the topological charge

$$\begin{aligned} Q &= \frac{1}{4\pi}\int\int \vec{S}\cdot(\partial_1\vec{S}\times\partial_2\vec{S})d^2x && (7.144)\\ &= -\frac{1}{8\pi}\int\int(\Delta\psi)d^2x && (7.145)\end{aligned}$$

where

$$\psi = 2\ln(1+|\zeta|^2) \tag{7.146}$$

$$Q = -\frac{1}{4\pi}\int\int(\Delta\ln(1+|\zeta|^2))d^2x \tag{7.147}$$

By Green's theorem then integral transforms

$$\int\int\left[\frac{\partial}{\partial x}\left(\underbrace{\frac{\partial}{\partial x}\ln(1+|\zeta|^2)}_{Q}\right)+\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial}{\partial y}\ln(1+|\zeta|^2)}_{P}\right)\right] = \oint Pdx + Qdy \tag{7.148}$$

$$\begin{aligned} &= \oint\left[-\frac{\partial}{\partial y}\ln(1+|\zeta|^2)\right]dx+\left[\frac{\partial}{\partial x}\ln(1+|\zeta|^2)\right]dy && (7.149)\\ &= \oint -\frac{(|\zeta|^2)_y}{1+|\zeta|^2}dx+\frac{(|\zeta|^2)_x}{1+|\zeta|^2}dy && (7.150)\\ &= \oint_{R\to\infty}\frac{-(|\zeta|^2)_y dx+(|\zeta|^2)_x dy}{1+|\zeta|^2} && (7.151)\end{aligned}$$

For N zeroes solution

$$\zeta(\bar{z},t) = \prod_{k=1}^{N}(\bar{z}-\bar{z}_k(t)) \tag{7.152}$$

asymptotically $|z|\to\infty$, $z = Re^{i\theta}$, $R\to\infty$, $|\zeta|^2\to\infty$, $\zeta\simeq\bar{z}^N$, $|\zeta|^2 = |z|^{2N}$

$$\begin{aligned} |\zeta|^2_x &= [(x^2+y^2)^N]_x = N(x^2+y^2)^{N-1}2x && (7.153)\\ |\zeta|^2_y &= N(x^2+y^2)^{N-1}2y && (7.154)\\ & && (7.155)\end{aligned}$$

and integral is equal

$$\begin{aligned} \oint_{R\to\infty}\frac{-N(x^2+y^2)^{N-1}2ydx+N(x^2+y^2)^{N-1}2xdy}{(x^2+y^2)^N} &= 2N\oint\frac{-ydx+xdy}{x^2+y^2}\\ &= 2N\cdot 2\pi\\ &= 4\pi N && (7.156)\end{aligned}$$

Then we find that topological charge is integer valued and equal to the number of vortices

$$Q = -N \tag{7.157}$$

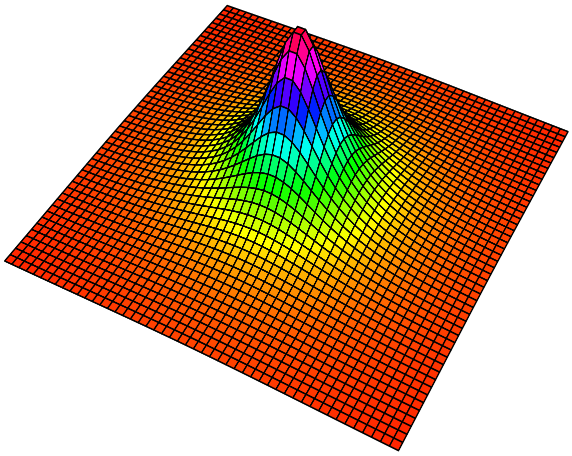

Figure 7.1. $N = 1$ Static Magnetic Vortex

In Fig. 7.1 and Fig. 7.2 we reproduce S_3 component for N = 1 and N = 2 vortices.

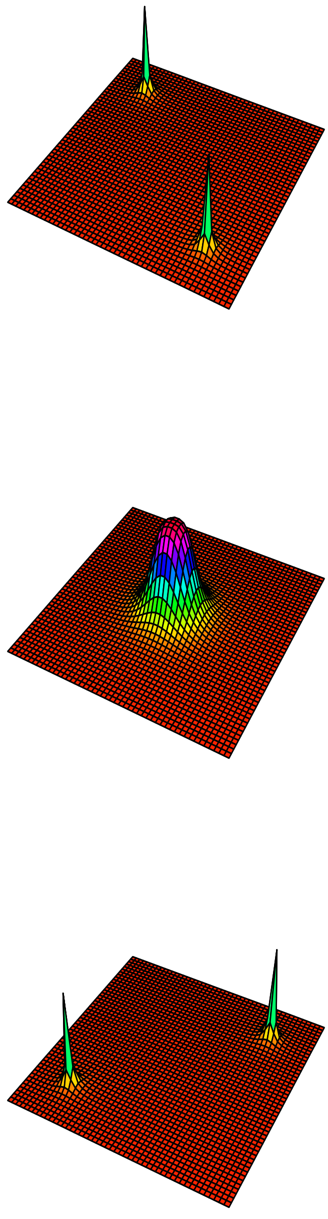

Figure 7.2. $N = 2$ Magnetic Vortex Dynamics

If we consider solution

$$\zeta(\bar{z},t) = \prod_{k=1}^{N} \sin(\bar{z} - \bar{z}_k(t)) \tag{7.158}$$

then it describes N magnetic vortex chain lattices periodic in the x direction. In Fig. 7.3 we reproduce S_3 component of these lattices for $N = 2$. Applying results of Chapter 5 on vortex generating techniques we can also generate arbitrary number of magnetic vortices interacting with the vortex lattices.

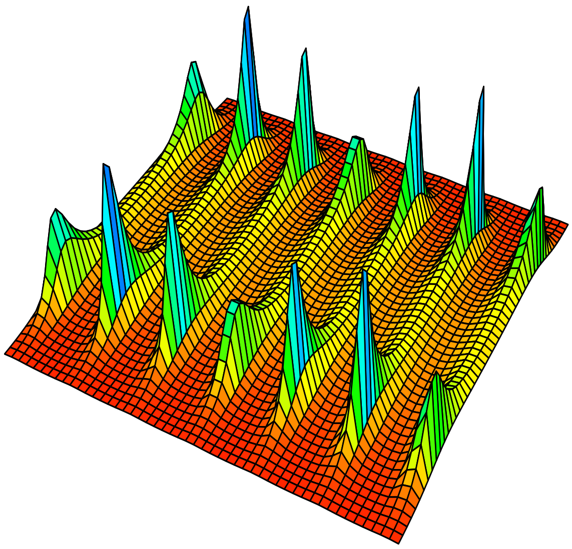

Figure 7.3. Two Magnetic Vortex Lattice Dynamics

7.7 Time Dependent Schrodinger problem in Harmonic Potential

The vorticity equation (7.100) is invariant under substitution

$$\psi \to \psi + U \tag{7.159}$$

where U is an arbitrary harmonic function:

$$\Delta U = 0$$

If we choose

$$\psi = 2\ln(1+|\zeta|^2) + U(\bar{z},t) + \bar{U}(\bar{z},t) \quad (7.160)$$

then substituting to Eq.(7.103) we have complex Schrödinger equation with additional potential term

$$i\zeta_t + \zeta_{\bar{z}\bar{z}} + \zeta_{\bar{z}} U_{\bar{z}} = 0 \quad (7.161)$$

7.8 Bound State of Vortices

Here we choose particular form

$$U(\bar{z},t) = \frac{1}{2}\bar{z}^2 \quad (7.162)$$

so that

$$\psi = 2\ln(1+|\zeta|^2) + \frac{1}{2}(\bar{z}^2 + z^2). \quad (7.163)$$

Then we have the time evolution subject to the equation

$$i\zeta_t + 2\zeta_{\bar{z}\bar{z}} + \bar{z}\zeta_{\bar{z}} = 0. \quad (7.164)$$

Looking for solution in the form

$$\zeta = \sum_n e^{int} u_n(\bar{z}) \quad (7.165)$$

we find that functions $u_n(\bar{z})$ satisfy the complex Hermite equation

$$2u''_n + \bar{z}u'_n - nu_n = 0. \quad (7.166)$$

It gives time dependent vortex solution in the form

$$\zeta = \sum_n e^{int} H_n(\bar{z}). \quad (7.167)$$

For particular value $N = 2$ we have solution

$$\zeta = H_0(\bar{z}) + e^{it}H_1(\bar{z}) + e^{2it}H_2(\bar{z}) \quad (7.168)$$

or

$$\zeta = \Re\zeta + i\Im\zeta$$

where

$$\Re\zeta = 1 + 2x\cos t + 2y\sin t + [4(x^2 - y^2) - 2]\cos 2t + 8xy\sin 2t \quad (7.169)$$

$$\Im\zeta = -2y\cos t + 2x\sin t - 8xy\cos 2t + [4(x^2 - y^2) - 2]\sin 2t \tag{7.170}$$

This solution is periodic in time with period $T = 2\pi$ and it describes the bound state of two magnetic vortices. In Fig. 7.4 we demonstrate oscillation of vortices in this bound state for function

$$f = \frac{1}{1 + (\Re\zeta)^2 + (\Im\zeta)^2}$$

which characterizes projection of spin vector S_3.

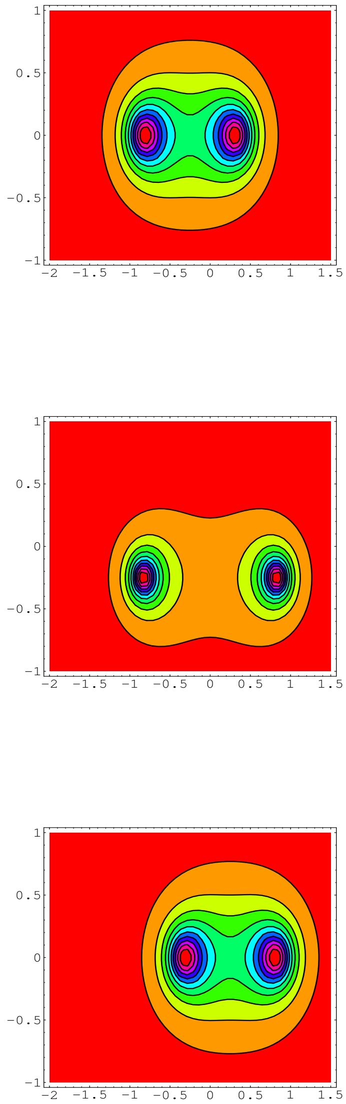

Figure 7.4. Bound State of Two Magnetic Vortices

CHAPTER 8

CONCLUSIONS

In the present thesis we have studied integrable dynamical models of the point magnetic vortex interactions in the plane. We started from general formulation of incompressible and irrotational hydrodynamical flow in terms of analytic function theory. Then, the simple pole singularities of the complex velocity we interpreted as the vortices and sources of the flow. The problem of vortex motion in this language corresponds to the motion of zeroes or poles of some complex functions. Reformulating the Euler equations for vorticity function in the Helmholz form, we found the Hamiltonian structure, where the stream function plays the role of the Hamiltonian function. We showed formal equivalence of this Hamiltonian equations with the Lax equations, which suggest that the equations are integrable in the Liouville sense. But, by reducing these equations for the point vortices to the Kirchhoff equations, we obtained that the system of $N \geq 4$ vortices has no sufficient number of integrals of motion and this is why it can't be integrable.

As an integrable model of planar motion with given vorticity for the stationary flow, we considered the Liouville equation and its solutions as distributed finite set of vortices and as the periodic lattice of vortices.

For non-stationary flows we studied exactly solvable case of point planar vortex diffusion and exactly solvable Initial Value Problem for the one dimensional Burgers equation. In the last case we found that the initial step function, asymptotically creates shock soliton for the Burgers equation. Linearizability of the Burgers equation in one dimension and analogy of the linearization formulae, the Cole-Hopf transformation, with the relation between complex velocity and complex potential, leaded us to formulation of the complex Burgers equation with integrable N vortex dynamics. We found that the complex Burgers equation is linearizable in terms of the complex Schrödinger equation and vortices correspond to zeroes of the last equation. This allowed us to construct N vortex configurations in terms of the complex Hermite polynomials, vortex chain lattices and their mutual dynamics. By mapping our vortex problem to N-particle problem, the complexified Calogero-Moser system, we showed its integrability and Hamiltonian structure.

Then we applied our general results to the problem of magnetic vortices in a magnetic fluid. First we found holomorphic reductions of topological magnetic system and showed that the evolution equation at this reduction becomes the linear complex Schrödinger equation, which we found by linearization of the complex Burgers equation. This allowed us apply all results on integrable vortex dynamics in the complex Burgers equation to the magnetic vortex evolution, including magnetic vortex lattices and the bound states of vortices.

The richness and beauty of the integrable structure of Hamiltonian systems suggests that our results can clarify extension of integrability property for 2+1 dimensional systems, which is an actual problem in mathematics. From another site, wide variety of vortex phenomena in nature, from nano-structures to hydrodynamics and cosmology, gives us hope that our results could have applications in such phenomena.

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APPENDIX A

Complex Analysis

A.1 Cauchy Integral Representation.

Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense.If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0} \tag{A.1}$$

is called the Cauchy integral Formula. It says that if a function f is to be analytic within and on on a simple closed contour C, then the values of f on C. Any change in the value of f at a point within C must, therefore, be accompanied by a change in its values on the boundary C.

A.2 Argument Principle and Rouche's Theorem.

A function f is said to be *meromorphic* in a domain D if at every point of D it is either analytic or has a pole. Suppose now that we are given a function f that is analytic and nonzero at each point of a simple closed contour C and is meromorphic inside C. Under these conditions it can be shown that f has at most a finite number of poles inside C. Then the number of poles of f inside C is to be interpreted as

$$N_p(f) := \sum_{poles\ \ inside\ \ C} (order\ \ of\ \ each\ \ pole) \tag{A.2}$$

while the number of its zeros inside C is

$$N_0(f) := (order\ \ of\ \ the\ \ zero\ \ at\ \ z = 0) \tag{A.3}$$

Theorem A.2.0.1 *Argument Principle*

If f is analytic and nonzero at each point of a simple closed positively oriented contour C and is meromorphic inside C, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N_0(f) - N_p(f) \tag{A.4}$$

where $N_0(f)$ and $N_p(f)$ are , respectively, the number of zeros and poles of f inside C.

If f is analytic inside and on a simple closed positively oriented contour C and if f is nonzero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) \tag{A.5}$$

where $N_0(f)$ is the number of zeros of f inside C.

Theorem A.2.0.2 *Rouche's Theorem*

If f and h are each functions that are analytic inside and on a simple closed contour C and if the strict inequality

$$|h(z)| < |f(z)| \tag{A.6}$$

holds at each point on C, then f and $f + h$ must have the same total number of zeros inside C.

APPENDIX B

Vortices in Euler Equations

B.1 Vorticity form of Euler Equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = 0 \tag{B.1}$$

$$\frac{\partial u_1}{\partial t} + (u_1\partial_x + u_2\partial_y)u_1 = 0 \tag{B.2}$$

$$\frac{\partial u_2}{\partial t} + (u_1\partial_x + u_2\partial_y)u_2 = 0 \tag{B.3}$$

Taking derivative of (B.2) according to y and (B.3) according to x we get

$$\frac{\partial^2 u_1}{\partial t \partial y} + \partial_y u_1 \partial_x u_1 + u_1 \partial_y \partial_x u_1 + \partial_y u_2 \partial_y u_1 + u_2 \partial_y^2 u_1 = 0 \tag{B.4}$$

$$\frac{\partial^2 u_2}{\partial t \partial x} + \partial_x u_1 \partial_x u_2 + u_1 \partial_x^2 u_2 + \partial_x u_2 \partial_y u_2 + u_2 \partial_x \partial_y u_2 = 0 \tag{B.5}$$

Subtracting (B.5) from (B.4) we get

$$\frac{\partial}{\partial t}[\partial_x u_2 - \partial_y u_1] + u_1[\partial_x^2 u_2 - \partial_x \partial_y u_1] + u_2[\partial_x \partial_y u_2 - \partial_y^2 u_1]$$
$$+\partial_x u_1 \partial_x u_2 + \partial_x u_2 \partial_y u_2 - \partial_y u_1 \partial_x u_1 - \partial_y u_2 \partial_y u_1 = 0 \tag{B.6}$$

$div\vec{u} = 0$ implies the existence of the real function $\psi(x, y)$ such that

$$u_1 = \partial_y \psi, \quad u_2 = -\partial_x \psi \tag{B.7}$$

Vorticity Ω is in the form:

$$\Omega = \partial_x u_2 - \partial_y u_1 \tag{B.8}$$

$$= -(\partial_x^2 \psi + \partial_y^2 \psi) \tag{B.9}$$

$$= -\Delta \psi \tag{B.10}$$

Substituting (B.7) and (B.10) in the equation (B.6) we have

$$\frac{\partial \Omega}{\partial t} - \partial_x \psi \partial_y \Omega + \partial_y \psi \partial_x \Omega = 0 \tag{B.11}$$

B.2 Lax Representation

$$L\Psi = \lambda\Psi \tag{B.12}$$
$$\Psi_t = A\Psi \tag{B.13}$$

Differentiating (B.12) according to t and using (B.13)gives:

$$\begin{aligned} L_t\Psi + L\Psi_t &= \lambda_t\Psi + \lambda\Psi_t & \text{(B.14)} \\ L_t\Psi + LA\Psi &= \lambda_t\Psi + \lambda A\Psi & \text{(B.15)} \\ &= \lambda_t\Psi + A\lambda\Psi & \text{(B.16)} \\ &= \lambda_t\Psi + AL\Psi & \text{(B.17)} \\ & & \text{(B.18)} \end{aligned}$$

$$\begin{aligned} L_t\Psi + LA\Psi_t - AL\Psi &= \lambda_t\Psi & \text{(B.19)} \\ L_t\Psi + (LA - AL)\Psi &= \lambda_t\Psi & \text{(B.20)} \\ L_t\Psi - [A, L]\Psi &= \lambda_t\Psi & \text{(B.21)} \\ (L_t - [A, L])\Psi &= \lambda_t\Psi & \text{(B.22)} \end{aligned}$$

$$\lambda_t = 0 \quad \Leftrightarrow \quad L_t = [A, L] \tag{B.23}$$

B.3 Green Function Solution for Laplace Operator

$$G(\vec{r}_1, \vec{r}_2) = -\frac{1}{2\pi}\ln|\vec{r}_1 - \vec{r}_2| \tag{B.24}$$
$$\Delta_1 G(\vec{r}_1, \vec{r}_2) = -\delta(\vec{r}_1 - \vec{r}_2) \tag{B.25}$$

Substituting (B.24) to (B.25) we have:

$$\Delta_1(-\frac{1}{2\pi}\ln|\vec{r}_1 - \vec{r}_2|) = -\delta(\vec{r}_1 - \vec{r}_2) \tag{B.26}$$
$$\Delta_1 \ln|\vec{r}_1 - \vec{r}_2| = 2\pi\delta(\vec{r}_1 - \vec{r}_2) \tag{B.27}$$

For $r_2 = 0$

$$\Delta \ln |\vec{r}| = 2\pi\delta(\vec{r}) \tag{B.28}$$

1. $r \neq 0 \quad \Delta \ln r = 0$

$$\begin{aligned}
\Delta \ln r &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln \sqrt{x^2+y^2} = 0 && \text{(B.29)}\\
&= \frac{\partial^2}{\partial x^2} \ln \sqrt{x^2+y^2} + \frac{\partial^2}{\partial y^2} \ln \sqrt{x^2+y^2} && \text{(B.30)}\\
&= \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} && \text{(B.31)}\\
&= 0 && \text{(B.32)}
\end{aligned}$$

2. $\Delta \ln r = 2\pi\delta(r) \qquad \int \delta(r) dr = 1$

$$\int\int \Delta \ln r d^2x = 2\pi \tag{B.33}$$

Green's Theorem

$$\int_A\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C (Pdx + Qdy) \tag{B.34}$$

$$\begin{aligned}
\int\int \Delta \ln r d^2x &= \int\int \left(\frac{\partial^2}{\partial x^2} \ln r + \frac{\partial^2}{\partial y^2} \ln r\right) dxdy && \text{(B.35)}\\
&= \int\int \left[\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)\right] dxdy && \text{(B.36)}\\
&= \oint_R \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) && \text{(B.37)}\\
&= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta) d\theta && \text{(B.38)}\\
&= 2\pi && \text{(B.39)}
\end{aligned}$$

where $x = R\cos\theta$ and $y = R\sin\theta$.

Eq. (B.25) determines integral operator Δ^{-1} :

$$\Delta^{-1} = \frac{1}{2\pi}\int\int dx_2 dx_2 \ln |\vec{r}_1 - \vec{r}_2| \tag{B.40}$$

such that

$$\Delta\Delta^{-1} = 1 \tag{B.41}$$

B.4 Point Vortex in Polar Coordinates

In this part we will find point vortex solution in polar coordinate form. Let us consider xy- plane and polar coordinate representation

$$x = \cos\theta \tag{B.42}$$

$$y = \sin\theta \tag{B.43}$$

then arbitrary vector field $\vec{r}$ in the plane is

$$\vec{r}(t) = x(t)\vec{e}_1 + y(t)\vec{e}_2 \tag{B.44}$$

$$= r(t)\cos\theta\vec{e}_1 + r(t)\sin\theta\vec{e}_2 \tag{B.45}$$

Taking derivatives according to t we have velocity field

$$\frac{\partial\vec{r}}{\partial t} = \vec{u} = (\dot{r}\cos\theta - r\sin\theta\dot{\theta})\vec{e}_1 + (\dot{r}\sin\theta - r\cos\theta\dot{\theta})\vec{e}_2 \tag{B.46}$$

The polar coordinate basis determined by vectors $\vec{e}_r$,$\vec{e}_\theta$ is related with basis $\vec{e}_1$,$\vec{e}_2$ by simple rotation

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \end{pmatrix} \tag{B.47}$$

Substituting to expression (B.46) we have velocity vector in polar coordinates as

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 = u_r\vec{e}_r + u_\theta\vec{e}_\theta \tag{B.48}$$

where

$$\vec{u}_r = \dot{r} \tag{B.49}$$

$$\vec{u}_\theta = r\dot{\theta} \tag{B.50}$$

Let us consider point vortex solution with velocity components

$$u_1 = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}, \qquad u_2 = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{B.51}$$

then using formulas relating with u_1 , u_2 and solving linear equations we have $\dot{r}$ and $\dot{\theta}$ for our point vortex in the form

$$\dot{r} = 0, \quad \dot{\theta} = \frac{\Gamma}{2\pi r^2} \tag{B.52}$$

$$u_1 = -\frac{\Gamma}{2\pi}\frac{\sin\theta}{r} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \tag{B.53}$$

$$u_2 = \frac{\Gamma}{2\pi}\frac{\cos\theta}{r} = \dot{r}\sin\theta + r\cos\theta\dot{\theta} \tag{B.54}$$

Applying Cramer's rule for the linear algebraic system

$$a_1 x + b_1 y = c_1 \quad x = \frac{\Delta_1}{\Delta} \tag{B.55}$$

$$a_2 x + b_2 y = c_2 \quad y = \frac{\Delta_2}{\Delta} \tag{B.56}$$

$$\tag{B.57}$$

we have

$$\Delta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{B.58}$$

$$\Delta_1 = \begin{vmatrix} -\frac{\Gamma}{2\pi}\frac{\sin\theta}{r} & -r\sin\theta \\ \frac{\Gamma}{2\pi}\frac{\cos\theta}{r} & r\cos\theta \end{vmatrix} = 0 \tag{B.59}$$

$$\Delta_2 = \begin{vmatrix} \cos\theta & -\frac{\Gamma}{2\pi}\frac{\sin\theta}{r} \\ \sin\theta & \frac{\Gamma}{2\pi}\frac{\cos\theta}{r} \end{vmatrix} = \frac{\Gamma}{2\pi r} \tag{B.60}$$

and

$$\dot{r} = \frac{\Delta_1}{\Delta} = 0 \tag{B.61}$$

$$\dot{\theta} = \frac{\Delta_2}{\Delta} = \frac{\Gamma}{2\pi r^2} \tag{B.62}$$

It implies

$$u_r = \dot{r} = 0 \tag{B.63}$$

$$u_\theta = r\dot{\theta} = \frac{\Gamma}{2\pi r} \tag{B.64}$$

As we found in Eq. (3.9) vorticity for the point vortex is $\Omega = \Gamma\delta(\vec{r})$. It means that the total vorticity in the plane is fixed by equations (3.11)

$$\int_{R^2}\int \Omega d^2x = \Gamma \int\int \delta(\vec{r}) d^2x = \Gamma \tag{B.65}$$

Let us rewrite this vorticity condition in polar coordinate representation

$$\int_{R^2}\int \Omega(x,y) d^2x = \int_0^{2\pi} d\theta \int_0^{\infty} \Omega(r,\theta) r dr = \Gamma \tag{B.66}$$

Since for the point vortex function $\Omega(r,\theta)$ is independent of θ we have

$$2\pi \int_0^{\infty} \Omega(r) r dr = \Gamma \tag{B.67}$$

From

$$\Omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{B.68}$$

$$u_1 = u_r \cos\theta - \sin\theta u_\theta \tag{B.69}$$

$$u_2 = u_r \sin\theta + \cos\theta u_\theta \tag{B.70}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \tag{B.71}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \tag{B.72}$$

we obtain vorticity

$$\Omega = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial}{\partial \theta} u_r\right) \tag{B.73}$$

Velocity for point vortex is

$$u_\theta = \frac{1}{r}\int_0^r (r\Omega) dr + c \tag{B.74}$$

B.5 Kirchhoff Equations

In this Appendix we study integrals of motion for the Kirchhoff equations (3.66),(3.67). As it is well known the equations admit four integrals of motion written below

$$I_1 = H \quad I_2 = \sum_{i=1}^{N} k_i x_i \quad I_3 = \sum_{i=1}^{N} k_i y_i \quad I_4 = \sum_{i=1}^{N} k_i (x_i^2 + y_i^2) \tag{B.75}$$

It is easy to check by direct calculations using Poisson bracket definition (3.31) and the Kirchhoff equations (3.66) , (3.67). The first one is evident from the skew-symmetry

$$\dot{I}_1 = \{I_1, H\} = \{H, H\} = 0 \tag{B.76}$$

For the second, the third and the fourth one we have respectively

$$\dot{I}_2 = \{I_2, H\} = \sum_{i=1}^{N} \Gamma_i \{x_i, H\} = \sum_{i=1}^{N} \frac{\partial H}{\partial y_i} \tag{B.77}$$

$$= \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{n \neq i}^{N} \Gamma_i \Gamma_n \frac{y_i - y_n}{(x_i - x_n)^2 + (y_i - y_n)^2} = 0 \tag{B.78}$$

$$\Rightarrow \quad I_2 = const \tag{B.79}$$

and

$$\dot{I}_3 = \{I_3, H\} = \sum_{i=1}^{N} \Gamma_i \{y_i, H\} = \sum_{i=1}^{N} -\frac{\partial H}{\partial x_i} \tag{B.80}$$

$$= -\frac{1}{2\pi} \sum_{i=1}^{N} \sum_{n \neq i}^{N} \Gamma_i \Gamma_n \frac{x_i - x_n}{(x_i - x_n)^2 + (y_i - y_n)^2} = 0 \tag{B.81}$$

$$\Rightarrow \quad I_3 = const. \tag{B.82}$$

and

$$\dot{I}_4 = \{I_4, H\} = \sum_{i=1}^{N} \Gamma_i \{x_i^2 + y_i^2, H\} \tag{B.83}$$

$$= \sum_{i=1}^{N} \Gamma_i \left(\{x_i^2, H\} + \{y_i^2, H\}\right) \tag{B.84}$$

$$= 2 \sum_{i=1}^{N} \Gamma_i \left(x_i\{x_i, H\} + y_i\{y_i, H\}\right) \tag{B.85}$$

$$= \frac{1}{\pi} \sum_{i=1}^{N} \sum_{n \neq i}^{N} \Gamma_i \Gamma_n \frac{-x_i y_n + x_n y_i}{(x_i - x_n)^2 + (y_i - y_n)^2} = 0 \tag{B.86}$$

$$\Rightarrow \quad I_4 = const. \tag{B.87}$$

The sums in above equations are vanishing due to symmetry properties: for every (ij) term exists term (ji) with opposite sign. Below we illustrate this property for $N = 2$

$$\dot{I}_2 = \frac{1}{2\pi} \Gamma_1 \Gamma_2 \frac{y_1 - y_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{B.88}$$

$$+ \frac{1}{2\pi} \Gamma_2 \Gamma_1 \frac{y_2 - y_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.89}$$

$$= 0 \tag{B.90}$$

$$\dot{I}_3 = -\frac{1}{2\pi}\Gamma_1\Gamma_2\frac{x_1 - x_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{B.91}$$
$$- \frac{1}{2\pi}\Gamma_2\Gamma_1\frac{x_2 - x_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.92}$$
$$= 0 \tag{B.93}$$

$$\dot{I}_4 = \frac{1}{\pi}\Gamma_1\Gamma_2\frac{-x_1 y_2 + x_2 y_1}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{B.94}$$
$$+ \frac{1}{\pi}\Gamma_2\Gamma_1\frac{-x_2 y_1 + x_1 y_2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.95}$$
$$= 0 \tag{B.96}$$

For $N = 3$ we have

$$\dot{I}_2 = \frac{1}{2\pi}\Gamma_1\Gamma_2\frac{y_1 - y_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{B.97}$$
$$+ \frac{1}{2\pi}\Gamma_2\Gamma_1\frac{y_2 - y_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.98}$$
$$+ \frac{1}{2\pi}\Gamma_2\Gamma_3\frac{y_2 - y_3}{(x_2 - x_3)^2 + (y_2 - y_3)^2} \tag{B.99}$$
$$+ \frac{1}{2\pi}\Gamma_3\Gamma_2\frac{y_3 - y_2}{(x_3 - x_2)^2 + (y_3 - y_2)^2} \tag{B.100}$$
$$+ \frac{1}{2\pi}\Gamma_1\Gamma_3\frac{y_1 - y_3}{(x_1 - x_3)^2 + (y_1 - y_3)^2} \tag{B.101}$$
$$+ \frac{1}{2\pi}\Gamma_3\Gamma_1\frac{y_3 - y_1}{(x_3 - x_1)^2 + (y_3 - y_1)^2} \tag{B.102}$$
$$= 0 \tag{B.103}$$

$$\dot{I}_3 = -\frac{1}{2\pi}\Gamma_1\Gamma_2\frac{x_1 - x_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{B.104}$$
$$- \frac{1}{2\pi}\Gamma_2\Gamma_1\frac{x_2 - x_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.105}$$
$$- \frac{1}{2\pi}\Gamma_2\Gamma_3\frac{x_2 - x_3}{(x_2 - x_3)^2 + (y_2 - y_3)^2} \tag{B.106}$$
$$- \frac{1}{2\pi}\Gamma_3\Gamma_2\frac{x_3 - x_2}{(x_3 - x_2)^2 + (y_3 - y_2)^2} \tag{B.107}$$
$$- \frac{1}{2\pi}\Gamma_1\Gamma_3\frac{x_1 - x_3}{(x_1 - x_3)^2 + (y_1 - y_3)^2} \tag{B.108}$$
$$- \frac{1}{2\pi}\Gamma_3\Gamma_1\frac{x_3 - x_1}{(x_3 - x_1)^2 + (y_3 - y_1)^2} \tag{B.109}$$
$$= 0 \tag{B.110}$$

$$\begin{aligned}
\dot{I}_4 &= \frac{1}{\pi}\Gamma_1\Gamma_2\frac{-x_1y_2+x_2y_1}{(x_1-x_2)^2+(y_1-y_2)^2} && \text{(B.111)}\\
&+ \frac{1}{\pi}\Gamma_2\Gamma_1\frac{-x_2y_1+x_1y_2}{(x_2-x_1)^2+(y_2-y_1)^2} && \text{(B.112)}\\
&+ \frac{1}{\pi}\Gamma_1\Gamma_3\frac{-x_1y_3+x_3y_1}{(x_1-x_3)^2+(y_1-y_3)^2} && \text{(B.113)}\\
&+ \frac{1}{\pi}\Gamma_2\Gamma_1\frac{-x_3y_1+x_1y_3}{(x_3-x_1)^2+(y_3-y_1)^2} && \text{(B.114)}\\
&+ \frac{1}{\pi}\Gamma_2\Gamma_3\frac{-x_2y_3+x_3y_2}{(x_2-x_3)^2+(y_2-y_3)^2} && \text{(B.115)}\\
&+ \frac{1}{\pi}\Gamma_2\Gamma_1\frac{-x_3y_2+x_2y_3}{(x_3-x_2)^2+(y_3-y_2)^2} && \text{(B.116)}\\
&= 0 && \text{(B.117)}
\end{aligned}$$

...

Then for $\forall N$ we can rewrite our sums in the form

$$\begin{aligned}
\dot{I}_2 &= \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_i\Gamma_n\frac{y_i-y_n}{(x_i-x_n)^2+(y_i-y_n)^2} && \text{(B.118)}\\
&+ \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_n\Gamma_i\frac{y_n-y_i}{(x_n-x_i)^2+(y_n-y_i)^2} = 0 && \text{(B.119)}
\end{aligned}$$

$$\begin{aligned}
\dot{I}_3 &= \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_i\Gamma_n\frac{x_i-x_n}{(x_i-x_n)^2+(y_i-y_n)^2} && \text{(B.120)}\\
&+ \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_n\Gamma_i\frac{x_n-x_i}{(x_n-x_i)^2+(y_n-y_i)^2} = 0 && \text{(B.121)}
\end{aligned}$$

$$\begin{aligned}
\dot{I}_4 &= \frac{1}{\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_i\Gamma_n\frac{-x_iy_n+x_ny_i}{(x_i-x_n)^2+(y_i-y_n)^2} && \text{(B.122)}\\
&+ \frac{1}{\pi}\sum_{i=1}^{N}\sum_{n\neq i}^{N}\Gamma_n\Gamma_i\frac{-x_ny_i+x_iy_n}{(x_n-x_i)^2+(y_n-y_i)^2} && \text{(B.123)}
\end{aligned}$$

According to the Poisson Theorem (Thm 3.4.3.2) the Posson bracket of two integrals of motion is also integral of motion. This is why we need to find all possible Poisson brackets between above integrals. All calculations could be done in a pure algebraic way if we find first the fundamental Poisson brackets

$$\{x_i, x_j\} = \sum_{k=1}^{N}\frac{1}{\Gamma_k}\left(\frac{\partial x_i}{\partial x_k}\frac{\partial x_j}{\partial y_k} - \frac{\partial x_i}{\partial y_k}\frac{\partial x_j}{\partial x_k}\right) = 0 \qquad \text{(B.124)}$$

$$\{y_i, y_j\} = \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left(\frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial y_k} - \frac{\partial y_i}{\partial y_k} \frac{\partial y_j}{\partial x_k} \right) = 0 \tag{B.125}$$

$$\{x_i, y_j\} = \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left(\frac{\partial x_i}{\partial x_k} \frac{\partial y_j}{\partial y_k} - \frac{\partial x_i}{\partial y_k} \frac{\partial y_j}{\partial x_k} \right) \tag{B.126}$$

$$= \sum_{k=1}^{N} \frac{1}{\Gamma_k} \delta_{ik} \delta_{jk} = \frac{1}{\Gamma_i} \delta_{ij} \tag{B.127}$$

or combining together

$$\{x_i, x_j\} = 0 = \{y_i, y_j\}, \quad \{x_i, y_j\} = \frac{1}{\Gamma_i} \delta_{ij} \tag{B.128}$$

Then using properties of Poisson brackets we have

$$\{I_2, I_3\} = \{\sum_{i=1}^{N} \Gamma_i x_i, \sum_{j=1}^{N} \Gamma_j y_j\} \tag{B.129}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_i \Gamma_j \{x_i, x_j\} \tag{B.130}$$

$$= \sum_{i=1}^{N} \Gamma_i \tag{B.131}$$

$$\{I_2, I_4\} = \{\sum_{i=1}^{N} \Gamma_i x_i, \sum_{j=1}^{N} \Gamma_j (x_j^2 + y_j^2)\} \tag{B.132}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_i \Gamma_j \{x_i, x_j^2 + y_j^2\} \tag{B.133}$$

$$= 2 \sum_{i=1}^{N} \Gamma_i y_i = 2 I_3 \tag{B.134}$$

$$\{I_3, I_4\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_i \Gamma_j \{y_i, x_j^2 + y_j^2\} \tag{B.135}$$

$$= -2 \sum_{i=1}^{N} \Gamma_i x_i = -2 I_2 \tag{B.136}$$

This allows us to write all commutation relations between four integrals of motion

$$\{I_2, I_3\} = \sum_{i=1}^{N} \Gamma_i, \quad \{I_2, I_4\} = 2 I_3, \quad \{I_3, I_4\} = -2 I_2 \tag{B.137}$$

As we can see these integrals are not in involution thus they are linearly dependent.

B.6 N Vortex System

In this Appendix we derived system of equations describing evolution of N vortices. Let us consider solution of complex Schrödinger equation (5.7)

$$i\Phi_t = \nu\Phi_{\bar{z}\bar{z}} \tag{B.138}$$

having N simple roots

$$\Phi(\bar{z}, t) = \prod_{k=1}^{N}(\bar{z} - \bar{z}_k(t)). \tag{B.139}$$

For simplicity we start with $N = 2$ case

$$\Phi(\bar{z}, t) = (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t)). \tag{B.140}$$

Substituting to the equation we have

$$-i\dot{\bar{z}}_1(\bar{z} - \bar{z}_2) - i\dot{\bar{z}}_2(\bar{z} - \bar{z}_1) = 2\nu. \tag{B.141}$$

This equation considered at points $\bar{z} = \bar{z}_1$ and $\bar{z} = \bar{z}_2$ gives the system

$$\dot{\bar{z}}_1 = \frac{2\nu i}{(\bar{z}_1 - \bar{z}_2)}, \quad \dot{\bar{z}}_2 = \frac{-2\nu i}{(\bar{z}_1 - \bar{z}_2)}. \tag{B.142}$$

For $N = 3$ case

$$\Phi(\bar{z}, t) = (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t))(\bar{z} - \bar{z}_3(t)) \tag{B.143}$$

Substituting to the equation we have

$$-i\dot{\bar{z}}_1(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_3) - i\dot{\bar{z}}_2(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_3) - i\dot{\bar{z}}_3(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2) = 2\nu[3\bar{z} - (\bar{z}_1 + \bar{z}_2 + \bar{z}_3)]$$

This equation considered at points $\bar{z} = \bar{z}_1$, $\bar{z} = \bar{z}_2$ and $\bar{z} = \bar{z}_3$ gives the system

$$\dot{\bar{z}}_1 = 2\nu i \left[\frac{1}{(\bar{z}_1 - \bar{z}_2)} + \frac{1}{(\bar{z}_1 - \bar{z}_3)}\right] \tag{B.144}$$

$$\dot{\bar{z}}_2 = 2\nu i \left[\frac{1}{(\bar{z}_2 - \bar{z}_1)} + \frac{1}{(\bar{z}_2 - \bar{z}_3)}\right] \tag{B.145}$$

$$\dot{\bar{z}}_3 = 2\nu i \left[\frac{1}{(\bar{z}_3 - \bar{z}_1)} + \frac{1}{(\bar{z}_3 - \bar{z}_2)}\right] \tag{B.146}$$

Following the same procedure, in general case of arbitrary N zeroes (B.139) we obtain the system of first order equations

$$\dot{\bar{z}}_j = 2\nu i \sum_{k \neq j}^{N} \frac{1}{(\bar{z}_j - \bar{z}_k)} \tag{B.147}$$

Differentiating this system once more in time we get the system of Newton's equations:

$$\ddot{\bar{z}}_j = 2\nu i \sum_{k\neq j}^{n} \frac{-(\dot{\bar{z}}_j - \dot{\bar{z}}_k)}{(\bar{z}_j - \bar{z}_k)^2} = 8\nu^2 \sum_{j<k}^{n} \frac{1}{(\bar{z}_j - \bar{z}_k)^3} \tag{B.148}$$

For $N = 2$ case we have two equations

$$\ddot{\bar{z}}_1 = 8\nu^2 \sum_{k\neq j}^{n} \frac{1}{(\bar{z}_1 - \bar{z}_2)^3}, \quad \ddot{\bar{z}}_2 = -8\nu^2 \sum_{k\neq j}^{n} \frac{1}{(\bar{z}_1 - \bar{z}_2)^3} \tag{B.149}$$

For $N = 3$ case we have the following equations

$$\ddot{\bar{z}}_1 = 2\nu i \left[\frac{-(\dot{\bar{z}}_1 - \dot{\bar{z}}_2)}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{-(\dot{\bar{z}}_1 - \dot{\bar{z}}_3)}{(\bar{z}_1 - \bar{z}_3)^2}\right] \tag{B.150}$$

$$= 8\nu^2 \left[\frac{1}{(\bar{z}_1 - \bar{z}_2)^3} + \frac{1}{(\bar{z}_1 - \bar{z}_3)^3}\right] \tag{B.151}$$

$$\ddot{\bar{z}}_2 = 2\nu i \left[\frac{-(\dot{\bar{z}}_2 - \dot{\bar{z}}_1)}{(\bar{z}_2 - \bar{z}_1)^2} + \frac{-(\dot{\bar{z}}_2 - \dot{\bar{z}}_3)}{(\bar{z}_2 - \bar{z}_3)^2}\right] \tag{B.152}$$

$$= 8\nu^2 \left[\frac{1}{(\bar{z}_2 - \bar{z}_1)^3} + \frac{1}{(\bar{z}_2 - \bar{z}_3)^3}\right] \tag{B.153}$$

(B.154)

$$\ddot{\bar{z}}_3 = 2\nu i \left[\frac{-(\dot{\bar{z}}_3 - \dot{\bar{z}}_1)}{(\bar{z}_3 - \bar{z}_2)^2} + \frac{-(\dot{\bar{z}}_3 - \dot{\bar{z}}_2)}{(\bar{z}_3 - \bar{z}_2)^2}\right] \tag{B.155}$$

$$= 8\nu^2 \left[\frac{1}{(\bar{z}_3 - \bar{z}_1)^3} + \frac{1}{(\bar{z}_3 - \bar{z}_2)^3}\right] \tag{B.156}$$

(B.157)

APPENDIX C

Hermite Polynomials of Complex Argument

C.1 Generating Function

The Hermite polynomials, $H_n(x)$, may be defined by the generating function (Arfken 1995)

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{C.1}$$

C.2 Recurrence Relations

From the generating function (C.1) we find that the Hermite polynomials satisfy the recurrence relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{C.2}$$

and

$$H_n'(x) = 2nH_{n-1}(x) \tag{C.3}$$

Equation (C.2) may be obtained by differentiating the generating function with respect to t; differentiation with respect to x leads to Eq.(C.3). From above two relations follow differential equation for Hermite Polynomials

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{C.4}$$

Direct expansion of the generating function (C.1) in t easily gives first two terms $H_0(x) = 1$ and $H_1(x) = 2x$. Then Eq.(C.2) allows us the construction $H_n(x)$ for $\forall N$. The first five Hermite polynomials are

$$H_0(x) = 1 \tag{C.5}$$

$$H_1(x) = 2x \tag{C.6}$$

$$H_2(x) = 4x^2 - 2 \tag{C.7}$$

$$H_3(x) = 8x^3 - 12x \tag{C.8}$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \tag{C.9}$$

Special values of the Hermite polynomials follow from the generating function; that is,

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!} \tag{C.10}$$

$$H_{2n+1}(0) = 0 \tag{C.11}$$

We also obtain from the generating function the important parity relation

$$H_n(x) = (-1)^n H_n(-x) \tag{C.12}$$

C.3 Alternate Representations

Differentiation of the generating function (C.1) n times with respect to t and then setting $t = 0$ yields

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{C.13}$$

This gives us a Rodrigues representation of $H_n(x)$. Hermite polynomial $H_n(x)$ in series form is

$$H_n(x) = \sum_{s=0}^{[n/2]} (-1)^s (2x)^{n-2s} \frac{n!}{(n-2s)!s!} \tag{C.14}$$

Another operator representation of Hermite polynomial is

$$H_n(x) = e^{x^2/2} \left(x - \frac{d}{dx} \right)^n e^{-x^2/2} \tag{C.15}$$

$$= \left(2x - \frac{d}{dx} \right)^n 1 \tag{C.16}$$

The last relation is easy to prove by mathematical induction.

C.4 Hermite Polynomials of Complex Argument

If in generating function (C.1) we consider analytic continuation of real variable x to complex variable $z = x + iy$ then we have analytic function

$$g(z,t) = e^{-t^2+2tz} = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} \tag{C.17}$$

which determines Hermite polynomials of complex argument z. Recursion formulas

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z) \tag{C.18}$$

and

$$H_n'(z) = 2nH_{n-1}(z) \tag{C.19}$$

are the same as in the real case, where complex derivative is defined as

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{C.20}$$

As alternative representations we have

$$H_n(z) = (-1)^n e^{z^2}\frac{\partial^n}{\partial z^n}(e^{-z^2}). \tag{C.21}$$

$$H_n(z) = \sum_{s=0}^{[n/2]}(-1)^s(2z)^{n-2s}\frac{n!}{(n-2s)!s!} \tag{C.22}$$

and

$$H_n(z) = e^{z^2/2}\left(z - \frac{\partial}{\partial z}\right)^n e^{-z^2/2} \tag{C.23}$$

$$= \left(2z - \frac{\partial}{\partial z}\right)^n 1 \tag{C.24}$$

It shows that our Hermite polynomials of complex argument z are analytic functions $\frac{\partial}{\partial \bar{z}}H_n(z) = 0$. Moreover according to the main theorem of algebra the polynomial $H_n(z)$ has N complex roots $z_1, ..., z_n$:

$$H_n(z_i) = 0, \;\; i = 1, ..., n \tag{C.25}$$

From Rodrigues formula follows symmetry property of Hermite polynomials

$$H_n(-z) = (-1)^n H(z) \tag{C.26}$$

or

$$H_{2n}(-z) = H_{2n}(z), \;\; H_{2n+1}(-z) = -H_{2n+1}(z) \tag{C.27}$$

APPENDIX D

I. V. P. for the Burgers' Equation

Here we solve the Initial Value Problem for the Burgers Equation (Pashaev 2000).

$$\begin{cases} u_t + uu_x = \nu u_{xx} \\ u(x,0) = F(x), \quad -\infty < x < \infty \end{cases}$$

Using the Cole- Hopf transformation

$$u = -2\nu \frac{\phi_x}{\phi} \tag{D.1}$$

and corresponding partial derivatives

$$u_t = -2\nu(\ln\phi)_{xt} \tag{D.2}$$

$$u_x = -2\nu(\ln\phi)_{xx} = -2\nu\frac{\phi_{xx}\phi - \phi_x^2}{\phi^2} \tag{D.3}$$

$$u_{xx} = -2\nu(\ln\phi)_{xxx} \tag{D.4}$$

the nonlinear Burgers' equation (4.28)

$$u_t + uu_x = \nu u_{xx} \tag{D.5}$$

can be reduced to the form

$$\left(\frac{-\phi_t + \nu\phi_{xx}}{\phi}\right)_x = 0 \tag{D.6}$$

or integrating once

$$\frac{-\phi_t + \nu\phi_{xx}}{\phi} = \alpha(t) \tag{D.7}$$

For the simplest case $\alpha(t) = 0$ this is the linear heat(diffusion) equation

$$\phi_t = \nu\phi_{xx}. \tag{D.8}$$

So any solution of this equation determines a solution of the Burgers equation (4.28) according formula (D.1). Due to linearity the IVP for the heat equation

$$\begin{cases} \phi_t = \nu\phi_{xx} \\ \phi(x,0) = \Phi(x) = e^{-\frac{1}{2\nu}\int^x F(\eta)d\eta} \end{cases}$$

can be solved exactly.

Let $\phi(x,t)$ is solution of the heat equation (D.8) such that at initial time

$$\phi(x,0) = f(x), \qquad \infty < x < \infty \quad t > 0 \tag{D.9}$$

By the Fourier Transform

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk F(k,t) e^{ikx} \tag{D.10}$$

and the corresponding derivatives

$$\phi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk (ik) F(k,t) e^{ikx} \tag{D.11}$$

$$\phi_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, (ik)^2 F(k,t) e^{ikx} \tag{D.12}$$

$$\phi_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, (ik) \frac{\partial F(k,t)}{\partial t} e^{ikx} \tag{D.13}$$

we have

$$\phi_t - \nu\phi_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx} \left(\frac{\partial F(k,t)}{\partial t} - \nu (ik)^2 F(k,t) \right) = 0 \tag{D.14}$$

By the inverse Fourier transform it implies equation

$$\frac{\partial F(k,t)}{\partial t} = -\nu k^2 F(k,t) \tag{D.15}$$

Integrating once we have

$$\int d\ln F(k,t) = -\int \nu k^2 dt \tag{D.16}$$

$$\ln F(k,t) = -\nu k^2 t + \ln C \tag{D.17}$$

$$F(k,t) = C(k) e^{-\nu k^2 t}. \tag{D.18}$$

It allows us to write the general solution of the heat equation (D.8) in the form

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} C(k) e^{-\nu k^2 t}. \tag{D.19}$$

The arbitrary function $C(k)$ is fixed by the initial value for $t = 0$

$$f(x) = \phi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} C(k). \tag{D.20}$$

By the inverse transform

$$C(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \tag{D.21}$$

It implies

$$F(k,t) = \int_{-\infty}^{\infty} dx'\, e^{-ikx'-\nu k^2 t} f(x') \tag{D.22}$$

and

$$\begin{aligned} \phi(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, e^{-ikx'-\nu k^2 t} f(x') && \text{(D.23)}\\ &= \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')-\nu k^2 t}. && \text{(D.24)} \end{aligned}$$

The integral in k is the Gaussian form which can be integrated explicitly. By rearranging quadratic form

$$\begin{aligned} ik(x-x') - \nu k^2 t &= -\nu t\left(k^2 - \frac{2}{2}\frac{ik(x-x')}{\nu t}\right) && \text{(D.25)}\\ &= -\nu t\left[\Bigg(\underbrace{k - \frac{i(x-x')}{\nu t}}_{k'}\Bigg)^2 + \frac{(x-x')^2}{4\nu^2 t^2}\right] && \text{(D.26)} \end{aligned}$$

and using

$$dk = dk' \tag{D.27}$$

integration in k' gives

$$\begin{aligned} \phi(x,t) &= \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi}\int_{-\infty}^{\infty} dk' e^{-\nu t (k')^2} \exp\left[\frac{-(x-x')^2}{4\nu t}\right] && \text{(D.28)}\\ &= \int_{-\infty}^{\infty} dx' f(x') \frac{1}{\sqrt{4\pi\nu t}} \exp\left[\frac{-(x-x')^2}{4\nu t}\right] && \text{(D.29)} \end{aligned}$$

where we have used the Gaussian formula

$$\int e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} \tag{D.30}$$

By substitution $x' \to \eta$ and $dx' \to d\eta$, $\phi(x,0) = f(x) \quad \phi(x',0) = f(x') = f(\eta)$ it gives us solution of the IVP for the heat equation

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \phi(\eta,0)\, \exp\left[\frac{-(x-\eta)^2}{4\nu t}\right] d\eta. \tag{D.31}$$

Using this solution now we construct solution of IVP for the Burgers equation. First we have the initial value function $u(x,0) = F(x)$ or in terms of $f(x)$

$$F(x) = u(x,0) = -2\nu\frac{\phi_x(x,0)}{\phi(x,0)} = -2\nu(\ln\phi)_x = -2\nu\frac{f_x(x)}{f(x)} \tag{D.32}$$

Integrating once we have $f(x)$ in terms of $F(x)$

$$\begin{aligned}
(\ln f(x))_x &= -\frac{1}{2\nu}F(x) && \text{(D.33)}\\
\ln f(x) &= -\frac{1}{2\nu}\int^x F(\eta)d\eta && \text{(D.34)}\\
f(x) &= e^{-\frac{1}{2\nu}\int^x F(\eta)d\eta} = \phi(x,0) && \text{(D.35)}
\end{aligned}$$

Let us substitute solution (D.35) in the Cole-Hopf transform. First differentiating in x

$$\begin{aligned}
\phi(x,t) &= \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\phi(\eta,0)\exp\left[\frac{-(x-\eta)^2}{4\nu t}\right]d\eta && \text{(D.36)}\\
\phi_x(x,t) &= \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\phi(\eta,0)\frac{-2(x-\eta)}{4\nu t}\exp\left[\frac{-(x-\eta)^2}{4\nu t}\right]d\eta
\end{aligned}$$

we have solution of Burgers equation

$$\begin{aligned}
u(x,t) &= -2\nu\frac{\phi_x(x,t)}{\phi(x,t)} && \text{(D.37)}\\
&= -2\nu\frac{\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\phi(\eta,0)\frac{-2(x-\eta)}{4\nu t}\exp\left[\frac{-(x-\eta)^2}{4\nu t}\right]d\eta}{\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\phi(\eta,0)\exp\left[\frac{-(x-\eta)^2}{4\nu t}\right]d\eta} && \text{(D.38)}
\end{aligned}$$

Using $\phi(\eta,0) = f(\xi) = \exp\left[\frac{-1}{2\nu}\int^{\eta}F(\xi)d\xi\right]$ we rewrite it as

$$u(x,t) = \frac{\int_{\infty}^{\infty}\frac{(x-\eta)}{t}\exp\left[\frac{-(x-\eta)^2}{4\nu t}-\frac{1}{2\nu}\int^{\eta}F(\xi)d\xi\right]d\eta}{\int_{\infty}^{\infty}\exp\left[\frac{-(x-\eta)^2}{4\nu t}-\frac{1}{2\nu}\int^{\eta}F(\xi)d\xi\right]d\eta}. \quad \text{(D.39)}$$

This formula solves the IVP for the Burgers equation.

Now we consider particular solution of IVP for Burgers equation with the initial condition as the step function ($a_2 > a_1$):

$$F(x) = \begin{cases} a_1, & x > 0; \\ a_2, & x < 0. \end{cases} \quad \text{(D.40)}$$

Substituting this form to Eq.(D.39) we have

$$u(x,t) = \frac{\overbrace{\int_{-\infty}^{0} \frac{(x-\eta)}{t} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_2\eta} d\eta}^{I} + \overbrace{\int_{0}^{\infty} \frac{(x-\eta)}{t} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_1\eta} d\eta}^{II}}{\underbrace{\int_{-\infty}^{0} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_2\eta} d\eta}_{III} + \underbrace{\int_{0}^{\infty} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_1\eta} d\eta}_{IV}} \tag{D.41}$$

If we replace $\eta \to -\eta$, $d\eta \to -d\eta$ in integrals (I) and (III) then we obtain

$$u(x,t) = \frac{\overbrace{\int_{0}^{\infty} \frac{(x+\eta)}{t} e^{\frac{-(x+\eta)^2}{4\nu t} + \frac{1}{2\nu}a_2\eta} d\eta}^{1} + \overbrace{\int_{0}^{\infty} \frac{(x-\eta)}{t} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_1\eta} d\eta}^{2}}{\underbrace{\int_{0}^{\infty} e^{\frac{-(x+\eta)^2}{4\nu t} + \frac{1}{2\nu}a_2\eta} d\eta}_{3} + \underbrace{\int_{0}^{\infty} e^{\frac{-(x-\eta)^2}{4\nu t} - \frac{1}{2\nu}a_1\eta} d\eta}_{4}} \tag{D.42}$$

We rearrange quadratic forms in the exponentials

$$\begin{aligned} \exp\left[\frac{-(x+\eta)^2}{4\nu t} + \frac{1}{2\nu}a_2\eta\right] &= \exp\left[\frac{x^2 + 2\eta x + \eta^2 - 2a_2 t\eta}{-4\nu t}\right] \\ &= \exp\left[\frac{x^2 + 2(x - a_2 t)\eta + \eta^2 \mp (x - a_2 t)^2}{-4\nu t}\right] \\ &= \exp\frac{-1}{4\nu t}\left[\eta + (x - a_2 t)\right]^2 + \frac{1}{4\nu t}\left[(x - a_2 t)^2 - x^2\right] \end{aligned} \tag{D.43}$$

$$\begin{aligned} \exp\left(\frac{-(x+\eta)^2}{4\nu t} - \frac{1}{2\nu}a_1\eta\right) &= \exp\frac{-1}{4\nu t}\left[x^2 - 2\eta x + \eta^2 + 2a_1 t\eta\right] \\ &= \exp\left(\frac{-1}{4\nu t}\left[x^2 + 2(x - a_1 t)\eta + \eta^2 \mp (x - a_1 t)^2\right]\right) \\ &= \exp\left(\frac{-1}{4\nu t}\left[\eta - (x - a_1 t)\right]^2 + \frac{1}{4\nu t}\left[(x - a_1 t)^2 - x^2\right]\right) \end{aligned} \tag{D.44}$$

then we substitute (D.43) in the first and third integrals of (D.42) and (D.44) in the second and fourth integrals of (D.42). As a result we have

$$u(x,t) = \frac{A(x,t)}{B(x,t)} \tag{D.45}$$

where

$$\begin{aligned} A(x,t) &= \int_0^\infty \frac{x+\eta}{t} \exp\left(\frac{-[\eta+(x-a_2t)]^2}{4\nu t}\right) \exp\left(\frac{[(x-a_2t)-x^2]^2}{4\nu t}\right) d\eta \\ &+ \int_0^\infty \frac{x-\eta}{t} \exp\left(\frac{-[\eta-(x-a_1t)]^2}{4\nu t}\right) \exp\left(\frac{[(x-a_1t)-x^2]^2}{4\nu t}\right) d\eta \end{aligned}$$

and

$$\begin{aligned} B(x,t) &= \int_0^\infty \exp\left(\frac{-[\eta+(x-a_2t)]^2}{4\nu t}\right) \exp\left(\frac{[(x-a_2t)-x^2]^2}{4\nu t}\right) d\eta \\ &+ \int_0^\infty \exp\left(\frac{-[\eta-(x-a_1t)]^2}{4\nu t}\right) \exp\left(\frac{[(x-a_1t)-x^2]^2}{4\nu t}\right) d\eta \end{aligned}$$

To simplify expressions in integrals we introduce new variables:

$$\xi_1 \equiv \frac{\eta+(x-a_2t)}{\sqrt{4\nu t}}, \quad d\xi_1 = \frac{d\eta}{\sqrt{4\nu t}} \tag{D.46}$$

$$\eta = \sqrt{4\nu t}\xi_1 - (x-a_2t), \quad d\eta = \sqrt{4\nu t}d\xi_1 \tag{D.47}$$

$$\eta \rightarrow 0 \Rightarrow \quad \xi_1 \rightarrow \frac{(x-a_2t)}{\sqrt{4\nu t}} \tag{D.48}$$

$$\eta \rightarrow \infty \Rightarrow \quad \xi_1 \rightarrow \infty \tag{D.49}$$

and

$$\xi_2 \equiv \frac{\eta-(x-a_1t)}{\sqrt{4\nu t}}, \quad d\xi_2 = \frac{d\eta}{\sqrt{4\nu t}} \tag{D.50}$$

$$\eta = \sqrt{4\nu t}\xi_2 - (x-a_1t), \quad d\eta = \sqrt{4\nu t}d\xi_2 \tag{D.51}$$

$$\eta \rightarrow 0 \Rightarrow \quad \xi_1 \rightarrow \frac{-(x-a_1t)}{\sqrt{4\nu t}} \tag{D.52}$$

$$\eta \rightarrow \infty \Rightarrow \quad \xi_1 \rightarrow \infty \tag{D.53}$$

and rewriting the $u(x,t)$

$$u(x,t) = \frac{C}{D} \tag{D.54}$$

$$\begin{aligned} C &= \exp\left[\frac{(x-a_2t)^2-x^2}{4\nu t}\right] \int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^\infty [4\nu\xi_1 + a_2\sqrt{4\nu t}\exp(-\xi_1^2)]d\xi_1 \\ &+ \exp\left[\frac{(x-a_1t)^2-x^2}{4\nu t}\right] \int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^\infty [-4\nu\xi_1 + a_1\sqrt{4\nu t}\exp(-\xi_2^2)]d\xi_2 \end{aligned}$$

and

$$\begin{aligned} D &= \exp\left[\frac{(x-a_2t)^2-x^2}{4\nu t}\right] \int_{\frac{(x-a_2t)^2}{\sqrt{4\nu t}}}^\infty \exp(-\xi_1^2)d\xi_1 \\ &+ \exp\left[\frac{(x-a_1t)^2-x^2}{4\nu t}\right] \int_{\frac{-(x-a_1t)^2}{\sqrt{4\nu t}}}^\infty \exp(-\xi_2^2)d\xi_2 \end{aligned}$$

Using substitutions

$$-\xi_1^2 = u, \quad -2\xi_1\, d\xi_1 = du \tag{D.55}$$

$$-\xi_2^2 = w, \quad -2\xi_2\, d\xi_2 = dw \tag{D.56}$$

$$\begin{aligned} C &= 2\nu \exp\left[\frac{-x}{4\nu t}\right] \\ &+ a_2\sqrt{4\nu t}\exp\left[\frac{(x-a_2t)^2 - x^2}{4\nu t}\right]\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi_1^2)d\xi_1 \\ &- 2\nu \exp\left[\frac{-x}{4\nu t}\right] \\ &+ a_1\sqrt{4\nu t}\exp\left[\frac{(x-a_1t)^2 - x^2}{4\nu t}\right]\int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi_2^2)d\xi_2 \end{aligned}$$

we have

$$u(x,t) = \frac{E}{F} \tag{D.57}$$

$$\begin{aligned} E &= a_2 \exp\left[\frac{(x-a_2t)^2}{4\nu t}\right]\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \\ &+ a_1 \exp\left[\frac{(x-a_1t)^2}{4\nu t}\right]\int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \\ &= a_2 \exp\left[\frac{(x-a_2t)^2}{4\nu t}\right]\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \\ &+ a_1 \exp\left[\frac{(x-a_1t)^2}{4\nu t}\right]\int_{\frac{-(x-a_2t)^2}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \end{aligned} \tag{D.58}$$

and

$$\begin{aligned} F &= \int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)\exp\left[\frac{(x-a_2t)^2}{4\nu t}\right]d\xi \\ &+ \int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)\exp\left[\frac{(x-a_1t)^2}{4\nu t}\right]d\xi \\ &= \exp\left[\frac{(x-a_2t)^2}{4\nu t}\right]\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \\ &+ \exp\left[\frac{(x-a_1t)^2}{4\nu t}\right]\int_{\frac{-(x-a_2t)^2}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2)d\xi \end{aligned}$$

Dividing denominator and numerator with the same expression we have

$$u(x,t) = \frac{a_1 \exp\left[\frac{(x-a_1t)^2}{4\nu t} - \frac{(x-a_2t)^2}{4\nu t}\right] \dfrac{\int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi}{\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi} + a_2 + a_1 - a_1}{1 + \exp\left[\frac{(x-a_1t)^2}{4\nu t} - \frac{(x-a_2t)^2}{4\nu t}\right] \dfrac{\int_{\frac{-(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi}{\int_{\frac{x-a_2t}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi}} \tag{D.59}$$

or

$$u(x,t) = a_1 + \frac{a_2 - a_1}{1 + h(x,t) \exp\left(\frac{a_2 - a_1}{2\nu}\left[x - \left(\frac{a_2 + a_1}{2}\right) t\right]\right)} \tag{D.60}$$

where

$$h(x,t) = \frac{\int_{-\frac{(x-a_1t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi}{\int_{\frac{(x-a_2t)}{\sqrt{4\nu t}}}^{\infty} \exp(-\xi^2) d\xi} \tag{D.61}$$

In asymptotic region when $x \to +\infty, \quad t \to +\infty$ in such a way that $x/t \to v$ where $v = \dfrac{a_1 + a_2}{2}$

$$\frac{x - a_1 t}{\sqrt{4\nu t}} = \frac{a_2 - a_1}{4} \sqrt{\frac{t}{\nu}} \tag{D.62}$$

$$\frac{x - a_2 t}{\sqrt{4\nu t}} = \frac{a_1 + a_2}{4} \sqrt{\frac{t}{\nu}} \tag{D.63}$$

So that $h(x,t) \to 1$. It shows that initial profile in the form of the step function (D.40) creates shock soliton

$$u(x,t) = a_1 + \frac{a_2 - a_1}{1 + \exp\left(\frac{a_2 - a_1}{2\nu}\left[x - \left(\frac{a_2 + a_1}{2}\right) t\right]\right)} \tag{D.64}$$

moving with velocity $v = \dfrac{a_1 + a_2}{2}$ in the right direction.

APPENDIX E

Buckingham's Pi Theorem

E.1 Eulers' Homogeneous Function Theorem

Let f(x,y) be a homogeneous function of order n so that

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \tag{E.1}$$

Then define $x' \equiv \lambda x$ and $y' \equiv \lambda y$. Differentiating once according to λ

$$n\lambda^{n-1} f(x, y) = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial \lambda} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \lambda} \tag{E.2}$$

$$= x\frac{\partial f}{\partial(\lambda x)} + y\frac{\partial f}{\partial(\lambda y)} \tag{E.3}$$

for $\lambda = 1$, then we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) \tag{E.4}$$

This can be generalized to an arbitrary number N of variables $x_1, ..., x_N$. Let

$$f(\lambda^{a_1} x_1, ..., \lambda^{a_N} x_N) = \lambda^n f(x_1, ..., x_N) \tag{E.5}$$

then

$$\sum_{i=1}^{N} a_i x_i \frac{\partial f}{\partial x_i} = nf(x_1, ..., x_N) \tag{E.6}$$

E.2 Buckingham's Pi Theorem

Buckingham's pi theorem (Buckingham 1914, Buckingham 1915) states that if we have a physically meaningful equation involving a certain number (n) of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of $p = n - k$ dimensionless variables constructed from the original variables.

In mathematical terms, if we have a physically meaningful equation such as

$$f(x_1, x_2, ..., x_n) = 0$$

where the x_i are the n physical variables and they are expressed in terms of k independent physical units, then the above equation can be restated as

$$F(\pi_1, ..., \pi_p) = 0$$

where the π_i are dimensionless parameters constructed from the x_i by $p = n - k$ equations of the form

$$\pi_i = x_1^{m_{1_i}} x_2^{m_{2_i}} ... x_n^{m_{n_i}}$$

If the units for each variable are

$$[x_i] = [length]^{a_{i,L}}[time]^{a_{i,T}}[mass]^{a_{i,M}}... \tag{E.7}$$

Changing units by a factor of λ_L cannot affect the answer, so

$$f(\lambda_L^{a_{1,L x_1}}, \lambda_L^{a_{2,L x_2}}, ..., \lambda_L^{a_{n,L x_n}}) \tag{E.8}$$

Applying Euler's homogeneous function theorem (differentiation with respect to λ_L and setting $\lambda_L = 1$),

$$a_{1,L}\frac{\partial f}{\partial x_1} + a_{2,L}\frac{\partial f}{\partial x_2} + ... + a_{n,L}\frac{\partial f}{\partial x_n} = 0 \tag{E.9}$$

or

$$a_{1,L}\frac{\partial f}{\partial \ln x_1} + a_{2,L}\frac{\partial f}{\partial \ln x_2} + ... + a_{n,L}\frac{\partial f}{\partial \ln x_n} = 0 \tag{E.10}$$

Similarly, for changing units of [time] by λ_T and [mass] by λ_M gives

$$a_{1,T}\frac{\partial f}{\partial \ln x_1} + a_{2,T}\frac{\partial f}{\partial \ln x_2} + ... + a_{n,T}\frac{\partial f}{\partial \ln x_n} = 0 \tag{E.11}$$

$$a_{1,M}\frac{\partial f}{\partial \ln x_1} + a_{2,M}\frac{\partial f}{\partial \ln x_2} + ... + a_{n,M}\frac{\partial f}{\partial \ln x_n} = 0 \tag{E.12}$$

If x_i has k independent units, there are therefore k independent equations of this type. These equations can be considered as k constraints on n variables x_i. Therefore, number of independent variables is $n - k$ and function f must be a function of $n - k$ dimensionless variables, $\pi_1, ..., \pi_p$.