ANALYTIC FUNCTIONS ON TIME SCALES

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ABSTRACT

ANALYTIC FUNCTIONS ON TIME SCALES

The concept of analyticity for complex functions on time scale complex plane was introduced by Bohner and Guseinov in 2005. They developed completely delta differentiability, delta analytic functions on products of two time scales, and Cauchy-Riemann equations for delta case.

In this thesis, beside the paper of Bohner and Guseinov, we worked on continuous, discrete and semi-discrete analytic functions and developed completely nabla differentiability, nabla analytic functions on products of two time scales, and Cauchy-Riemann equations for nabla case.

ÖZET

ZAMAN SKALALARINDA ANALİTİK FONKSİYONLAR

Zaman skalası kompleks düzleminde kompleks fonksiyonların analitikliği kavramı Bohner ve Guseinov tarafından 2005 yılında ortaya konulmuştur. Tam delta diferensiyellenebilme, iki zaman skalası çarpımı üzerinde delta analitik fonksiyonlar ve Cauchy-Riemann denklemlerinin delta versiyonu onlar tarafından geliştirilmiştir.

Bu tezde, Bohner ve Guseinov'un makalesi yanında, sürekli, kesikli ve yarı kesikli analitik fonksiyonlar üzerine çalıştık ve tam nabla diferensiyellenebilme, iki zaman skalası çarpımı üzerinde nabla analitik fonksiyonlar ve Cauchy-Riemann denklemlerinin nabla versiyonunu geliştirdik.

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CHAPTER 1

INTRODUCTION

Time Scale Calculus was developed earlier in (Hilger 1997, Guseinov 2003, Bohner and Peterson 2001). Gusein Sh. Guseinov and Martin Bohner gave the differential calculus, integral calculus and developed line integration along time scale curves (Bohner and Guseinov 2004 and Guseinov 2003). They also developed the concept of analytic functions on Time Scales in (Bohner and Guseinov 2005).

This thesis is organized as follows. In chapter two, we gave the review of the analytic functions on complex plane, Cauchy-Riemann equations and their properties. In the same chapter we gave discrete and semi-discrete analogs of analytic functions. We also presented the basic time scale calculus concepts and their classical results in chapter two.

In chapter three, we worked on the paper of Gusein Sh. Guseinov and Martin Bohner (Bohner and Guseinov 2005) on analytic functions on products of two time scales, functions of two time scales variables, Δ and ∇ differentiable functions on products of two time scales, Cauchy-Riemann equations on products of two time scales and Δ-analytic functions.

In addition, we have developed completely ∇-differentiability, ∇-analytic functions on products of two time scales, Cauchy-Riemann equations for nabla case.

CHAPTER 2

PRELIMINARIES

2.1. Continuous Analytic Functions

Suppose that $D \subseteq \mathbb{C}$ is a domain. A function $f : D \to \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{2.1}$$

and call $f'(z_0)$ the derivative of f at z_0.

If $z \neq z_0$, then

$$f(z) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) + f(z_0).$$

It follows from (2.1) that if $f'(z_0)$ exists, then $f(z) \to f(z_0)$ as $z \to z_0$, so that f is continuous at z_0. In other words, differentiability at z_0 implies continuity at z_0.

If the functions $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are both differentiable at $z_0 \in D$, then both $f + g$ and fg are differentiable at z_0, and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \quad \text{and} \quad (fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

If the extra condition $g'(z_0) \neq 0$ holds, then f/g is differentiable at z_0, and

$$\left(\frac{f}{g} \right)' (z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

Example 1. *Consider the function $f(z) = \bar{z}$, where for every $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z. Suppose that $z_0 \in \mathbb{C}$. Then*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}. \tag{2.2}$$

If $z - z_0 = h$ is real and non-zero, then (2.2) takes the value 1. On the other hand, if $z - z_0 = ik$ is purely imaginary, then (2.2) takes the value -1. It follows that this function is not differentiable anywhere in $\mathbb{C}$, although its real and imaginary parts are rather well behaved.

2.1.1. Cauchy-Riemann Equations

If we use the notation

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

then in Example 1, we have examined the behaviour of the ratio

$$\frac{f(z+h) - f(z)}{h}$$

first as $h \to 0$ through real values and then through imaginary values. Indeed, for the derivative to exist, it is essential that these two limiting processes produce the same limit $f'(z)$. Suppose that $f(z) = u(x, y) + iv(x, y)$, where, $z = x + iy$, and u and v are real valued functions. If h is real, then the two limiting processes above correspond to

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

and

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} &= \lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{h \to 0} \frac{v(x, y+h) - v(x, y)}{ih} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

respectively. Equating real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.3}$$

Definition 2. *The partial differential equations (2.3) are called the Cauchy-Riemann equations.*

Theorem 3. *Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that $f'(z)$ exists. Then the four partial derivatives in (2.3) exist, and the Cauchy-Riemann equations (2.3) hold. Furthermore, we have*

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \textit{and} \quad f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{2.4}$$

Theorem 4. *Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that the four partial derivatives in (2.3) are continuous and satisfy the Cauchy-Riemann equations (2.3) at z_0. Then f is differentiable at z_0, and the derivative $f'(z_0)$ is given by the equations (2.4) evaluated at z_0.*

Definition 5. *A function f is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some ϵ-neighbourhood of the point z_0. The function f is said to be analytic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in $\mathbb{C}$.*

2.2. Discrete and Semi-Discrete Analytic Functions

The theory of discrete and semi-discrete functions is obtained by deriving satisfactory analogues of classical results in the theory of analytic functions as well as finding results which have no direct analog in the classic theory. The study of analytic functions on $\mathbb{Z}^2$ has a history of more than sixty years. The pioneer in that field is Rufus Isaacs (Isaacs 1941), who introduced two difference equations, both of which are discrete counterparts of the Cauchy-Riemann equation in one complex variable.

The discrete analogues of analytic functions have been called by several names. Duffin (Duffin 1956) calls them discrete analytic functions, Ferrand (Ferrand 1944) calls them preholomorphic functions, and Isaacs (Isaacs 1941) calls them monodiffric functions. In this thesis we will use the definitions given by Isaacs.

2.2.1. Discrete Analytic Functions

Definition 6. *A complex-valued function f defined on a subset A of $\mathbb{Z} + i\mathbb{Z}$ is said to be holomorphic in the sense of Isaacs or monodiffric (discrete analytic) of the first kind if the equation*

$$\frac{f(z+1) - f(z)}{1} = \frac{f(z+i) - f(z)}{i} \tag{2.5}$$

holds for all $z \in A$ such that also $z+1$ and $z+i$ belong to A.

It should be noted that this definition is not the one used by Duffin or Ferrand in their works on discrete analytic functions.

In (Isaacs 1941) Isaacs defined also monodiffric functions of the second kind, in which the condition

$$\frac{f(z_0 + 1 + i) - f(z_0)}{1 + i} = \frac{f(z_0 + i) - f(z_0 + 1)}{i - 1} \tag{2.6}$$

is required instead of (2.5).

Here, we propose a new kind of monodiffric function which we call backward monodiffric function. To avoid confusion we call the monodiffric function of first kind as forward monodiffric function.

Definition 7. *A complex-valued function f defined on a subset A of $\mathbb{Z} + i\mathbb{Z}$ is said to be backward monodiffric if the equation*

$$\frac{f(z) - f(z-1)}{1} = \frac{f(z) - f(z-i)}{i} \tag{2.7}$$

holds for all $z \in A$ such that also $z - 1$ and $z - i$ belong to A.

There are many papers which concern discrete analogues for analytic functions by Duffin and Isaacs. In each of these, either a discrete analogue to the Cauchy-Riemann equations or in the case of Ferrand, a discrete version of Morera's theorem is used to define discrete analytic functions (as (2.5) and (2.6)).

2.2.2. Semi-Discrete Analytic Functions

Semi-discrete analytic functions are single-valued functions of one continuous and one discrete variable defined on a semi-lattice, a uniformly spaced sequence of lines parallel to the real axis.

The appropriate semi-discrete analogues of analytic functions are defined in (Kurowski 1963) from the classic Cauchy-Riemann equations on replacing the y-derivative by either a nonsymmetric difference

$$\frac{\partial f(z)}{\partial x} = [f(z + ih) - f(z)]/ih, \quad z = x + ikh, \tag{2.8}$$

or a symmetric difference

$$\frac{\partial f(z)}{\partial x} = [f(z + ih/2) - f(z - ih/2)]/ih, \quad z = x + ikh/2. \tag{2.9}$$

Definition 8. *Semi-discrete functions which satisfy (2.8) or (2.9) are called, respectively, semi-discrete analytic functions of the first, second kind.*

In this thesis we will deal with the semi-discrete analytic functions of the first kind and because we will propose the backward version of semi-discrete analytic functions, to avoid confusion we call the first kind, forward semi-discrete analytic functions.

Definition 9. *The function defined on $\mathbb{R} + ih\mathbb{Z}$ that satisfies the condition*

$$\frac{\partial f(z)}{\partial x} = [f(z) - f(z - ih)]/ih, \quad z = x + ikh, \tag{2.10}$$

is called backward semi-discrete analytic function.

Let $f(z) = u(x, y) + iv(x, y)$ be semi-discrete analytic function on $D \subset \mathbb{R} + ih\mathbb{Z}$, where u and v are real valued semi-discrete functions, equating real and imaginary parts of (2.8) and (2.9) respectively yields the semi-discrete Cauchy-Riemann equations. For Type I

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{1}{h}[v(x, y + h) - v(x, y)], \\ \frac{\partial v(x,y)}{\partial x} &= \frac{1}{h}[u(x, y) - u(x, y + h)], \end{aligned}$$

and for Type II

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{1}{h}[v(x, y + \frac{h}{2}) - v(x, y - \frac{1}{2})], \\ \frac{\partial v(x,y)}{\partial x} &= \frac{1}{h}[u(x, y - \frac{h}{2}) - u(x, y + \frac{h}{2})]. \end{aligned}$$

Helmbold (Helmbold) considers functions on a semi-lattice which satisfies the following semi-discrete analogue of Laplace's equation:

$$\frac{d^2u(x,k)}{dx^2} + [u(x, k + 1) - 2u(x, k) + u(x, k - 1)] = 0. \tag{2.11}$$

He calls this function semi-discrete harmonic.

Let us introduce following operators on D which are defined by Kurowski in (Kurowski 1963).

(a) $\Delta_1 f(z) = f(z + ih) - f(z)$,

(b) $\Delta_2 f(z) = f(z + ih/2) - f(z - ih/2)$,

(c) $\Delta_j^{n+1} = \Delta_j[\Delta_j^n f(z)], \quad n \geq 1$,

(d) $\nabla_j f(z) = \frac{\partial^2 f(z)}{\partial x^2} + \Delta_j^2 f(z)$,

(e) $2S_j f(z) = \frac{\partial f(z)}{\partial x} - \frac{i}{h}\Delta_j f(z)$,

(f) $2\bar{S}_j f(z) = \frac{\partial f(z)}{\partial x} + \frac{i}{h}\Delta_j f(z)$,

(g) $2\bar{S}_B f(z) = \frac{\partial f(z)}{\partial x} + \frac{i}{h}\Delta_1 f(z - ih)$.

Since

$$4S_j[\bar{S}_j(f)] = 4\bar{S}_j[S_j(f)] = \nabla_j(f),$$

if $f(z)$ is semi-discrete analytic on D, then $\nabla_j(f) = 0$ for all $z \in D^0$ and consequently

$$\nabla_j(u) = \nabla_j(v) = 0.$$

Semi-discrete functions g such that $\nabla_j(g) = 0$ are called semi-discrete harmonic functions of the first or second kind. The semi-discrete functions of the second kind are the semi-discrete harmonic functions considered by Helmbold (Helmbold), see equation (2.11), who called such functions $1/2$-harmonic.

2.3. Basic Calculus on Time Scales

By a time scale $\mathbb{T}$, we mean an arbitrary nonempty closed subset of real numbers. The set of the real numbers, the integers, the natural numbers, and the Cantor set are examples of time scales. But the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales.

The calculus of time scales was introduced by Stefan Hilger in his 1988 Ph.D. dissertation (Hilger 1997) in order to create the theory that can unify the discrete and continuous analysis. After discovery of time scales in 1988 almost every result obtained in the theory of differential and difference equations are carried into time scales.

In order to define Δ derivative (∇ derivative) we need to define forward jump operator σ, (backward jump operator ρ), graininess function μ (ν), and the region of differentiability $\mathbb{T}^\kappa$ ($\mathbb{T}_\kappa$) which is derived from $\mathbb{T}$.

Definition 10. *Let $\mathbb{T}$ be a time scale. The* ***forward jump operator*** $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}\ ,\ \forall t \in \mathbb{T}$$

*and the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}\ ,\ \forall t \in \mathbb{T}.$$

*Also $\sigma(\max(\mathbb{T})) = \max(\mathbb{T})$ and $\rho(\min(\mathbb{T})) = \min(\mathbb{T})$. t is called **right dense** if $\sigma(t) = t$, **left dense** if $\rho(t) = t$ and **right scattered** if $\sigma(t) > t$, **left scattered** if $\rho(t) < t$. t is called **dense** point if t is both left and right dense, and t is **isolated** if it is both left and right scattered. The **graininess functions** $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$.*

If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess functions μ and ν turn out to be $\mu(t) = \nu(t) = 0$ for all $t \in \mathbb{R}$.
If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, \cdots\} = t + 1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess functions μ and ν turn out to be $\mu(t) = \nu(t) = 1$ for all $t \in \mathbb{Z}$.
We define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

2.3.1. Differentiation and Integration on Time Scales

Let $\mathbb{T}$ be a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Let $\sigma(t)$ and $\rho(t)$ be the forward and backward jump operators in $\mathbb{T}$, respectively. We introduce the sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has right-scattered minimum t_2, then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that for each

$\varepsilon > 0$ there is neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

If $t \in \mathbb{T}_\kappa$, then we define the nabla derivative of f at the point t to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then Cauchy Δ-integral from a to t of f is defined by

$$\int_a^t f(s)\Delta s = F(t) - F(a) \quad \text{for all } t \in \mathbb{T}.$$

A function $\Phi : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $\Phi^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. Then Cauchy ∇-integral from a to t of f is defined by

$$\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a) \quad \text{for all } t \in \mathbb{T}.$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have

$$f^{\Delta}(t) = f^{\nabla}(t) = f'(t), \qquad \int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$, we have

$$f^{\Delta}(t) = f(t+1) - f(t), \quad f^{\nabla}(t) = f(t) - f(t-1),$$
$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k), \quad \int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k),$$

where $a, b \in \mathbb{T}$ with $a \leq b$.

The following theorems give several properties of the delta and nabla derivatives; they are found in (Bohner and Peterson 2001 and Merdivenci and Guseinov 2002).

Theorem 11. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ the following hold.*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) *If t is right-dense, then f is Δ-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exist as a finite number. In the case $f^\Delta(t)$ is equal to this limit.

(iv) *If f is Δ-differentiable at t, then*

$$f(\sigma(t)) = f(t) + [\sigma(t) - t]f^\Delta(t).$$

Theorem 12. *Assume that $f,\ g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^\kappa$. Then for $\alpha,\ \beta \in \mathbb{R}$*

(i) *The linear sum $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t)$$

(ii) *The product $(fg) : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

(iii) *If $g(t)g(\sigma(t)) \neq 0$ then $\dfrac{f}{g}$ is Δ-differentiable at t with*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Theorem 13. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$ the following hold.*

(i) *If f is ∇-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is left-scattered, then f is ∇-differentiable at t and*

$$f^\nabla(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}.$$

(iii) *If t is left-dense, then f is ∇-differentiable at t if and only if the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exist as a finite number. In the case $f^{\nabla}(t)$ is equal to this limit.

(iv) *If f is ∇-differentiable at t, then*

$$f(\rho(t)) = f(t) + [\rho(t)-t]f^{\nabla}(t).$$

Theorem 14. *Assume that $f,\ g:\mathbb{T}\to\mathbb{R}$ are ∇-differentiable at $t\in\mathbb{T}_{\kappa}$. Then for $\alpha,\ \beta\in\mathbb{R}$*

(i) *The linear sum $\alpha f+\beta g:\mathbb{T}\to\mathbb{R}$ is ∇-differentiable at t with*

$$(\alpha f+\beta g)^{\nabla}(t) = \alpha f^{\nabla}(t) + \beta g^{\nabla}(t)$$

(ii) *The product $(fg):\mathbb{T}\to\mathbb{R}$ is ∇-differentiable at t with*

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\rho(t)).$$

(iii) *If $g(t)g(\rho(t))\neq 0$ then $\dfrac{f}{g}$ is ∇-differentiable at t with*

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t)-f(t)g^{\nabla}(t)}{g(t)g(\rho(t))}.$$

Example 15. *We consider the function $f(t)=t^2$ on an arbitrary time scale.*
Let $\mathbb{T}$ be an arbitrary time scale and $t\in\mathbb{T}^{\kappa}$. The Δ-derivative of f is given by

$$f^{\Delta}(t) = \lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s} = \lim_{s\to t}\frac{(\sigma(t))^2-s^2}{\sigma(t)-s} = \lim_{s\to t}\sigma(t)+s = \sigma(t)+t.$$

1. *If $\mathbb{T}=\mathbb{R}$ then $\sigma(t)=t$. Therefore $f^{\Delta}(t)=f'(t)=2t$*

2. *If $\mathbb{T}=\mathbb{Z}$ then $\sigma(t)=t+1$. Therefore $f^{\Delta}(t)=\Delta f(t)=2t+1$.*

3. *If $\mathbb{T}=\mathbb{N}_0^{\frac{1}{2}}=\{\sqrt{n}:n\in\mathbb{N}_0\}$ then $\sigma(t)=\sqrt{t^2+1}$. Therefore $f^{\Delta}(t)=t+\sqrt{t^2+1}$.*

Similarly, to find ∇-derivative, let us take $t\in\mathbb{T}_{\kappa}$. The ∇-derivative of f is given by

$$f^{\nabla}(t) = \lim_{s\to t}\frac{f(\rho(t))-f(s)}{\rho(t)-s} = \lim_{s\to t}\frac{(\rho(t))^2-s^2}{\rho(t)-s} = \lim_{s\to t}\rho(t)+s = \rho(t)+t.$$

1. *If* $\mathbb{T}=\mathbb{R}$ *then* $\rho(t)=t$. *Therefore* $f^{\nabla}(t)=f'(t)=2t$

2. *If* $\mathbb{T}=\mathbb{Z}$ *then* $\rho(t)=t-1$. *Therefore* $f^{\nabla}(t)=\nabla f(t)=2t-1$.

3. *If* $\mathbb{T}=\mathbb{N}_0^{\frac{1}{2}}=\{\sqrt{n}:n\in\mathbb{N}_0\}$ *then* $\rho(t)=\sqrt{t^2-1}$. *Therefore* $f^{\nabla}(t)=t-\sqrt{t^2-1}$.

Theorem 16. *Assume* $a,b,c\in\mathbb{T}$, *then*

(i) $\int_a^b[f(t)g(t)]\Delta t=\int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t$,
$\int_a^b[f(t)g(t)]\nabla t=\int_a^b f(t)\nabla t+\int_a^b g(t)\nabla t$

(ii) $\int_a^b kf(t)\Delta t=k\int_a^b f(t)\Delta t,\quad \int_a^b kf(t)\nabla t=k\int_a^b f(t)\nabla t$

(iii) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t,\quad \int_a^b f(t)\nabla t=-\int_b^a f(t)\nabla t$

(iv) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t,\quad \int_a^b f(t)\nabla t=\int_a^c f(t)\nabla t+\int_c^b f(t)\nabla t$

(v) $\int_a^b f^{\Delta}(t)g(t)\Delta t=f(t)g(t)|_a^b-\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t$,
$\int_a^b f^{\nabla}(t)g(t)\nabla t=f(t)g(t)|_a^b-\int_a^b f(\rho(t))g^{\nabla}(t)\Delta t$

CHAPTER 3

ANALYTIC FUNCTIONS ON PRODUCTS OF TWO TIME SCALES

3.1. Functions of Two Real Time Scale Variables

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. Let us set $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1,\ y \in \mathbb{T}_2\}$. The set $\mathbb{T}_1 \times \mathbb{T}_2$ is a complete metric space with the metric (distance) d defined by

$$d((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2} \quad \text{for} \quad (x, y), (x', y') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

For a given $\delta > 0$, the δ-neighborhood $U_\delta(x_0, y_0)$ of a given point $(x_0, y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((x_0, y_0), (x, y)) < \delta$. Let σ_1 and σ_2 be the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Further, let ρ_1 and ρ_2 be the backward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Let $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a function. The first order delta derivatives of u at a point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ are defined to be

$$\frac{\partial u(x_0, y_0)}{\Delta_1 x} = \lim_{x \to x_0, x \neq \sigma_1(x_0)} \frac{u(\sigma_1(x_0), y_0) - u(x, y_0)}{\sigma_1(x_0) - x}$$

and

$$\frac{\partial u(x_0, y_0)}{\Delta_2 y} = \lim_{y \to y_0, y \neq \sigma_2(y_0)} \frac{u(x_0, \sigma_2(y_0)) - u(x_0, y)}{\sigma_2(y_0) - y}.$$

Similarly, we define nabla derivatives of u at a point $(x_0, y_0) \in \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa}$ as

$$\frac{\partial u(x_0, y_0)}{\nabla_1 x} = \lim_{x \to x_0, x \neq \rho_1(x_0)} \frac{u(x, y_0) - u(\rho_1(x_0), y_0)}{x - \rho_1(x_0)}$$

and

$$\frac{\partial u(x_0, y_0)}{\nabla_2 y} = \lim_{y \to y_0, y \neq \rho_2(y_0)} \frac{u(x_0, y) - u(x_0, \rho_2(y_0))}{y - \rho_2(y_0)}.$$

3.1.1. Completely Delta Differentiable Functions

Before the definition of completely delta differentiability on products of two time scales, we first give the definition for one-variable case.

Definition 17. *A function $u : \mathbb{T} \to \mathbb{R}$ is called completely delta differentiable at a point $x_0 \in \mathbb{T}^\kappa$ if there exists a number A such that*

$$u(x_0) - u(x) = A(x_0 - x) + \alpha(x_0 - x) \quad for\, all \quad x \in U_\delta(x_0) \tag{3.1}$$

and

$$u(\sigma(x_0)) - u(x) = A[\sigma(x_0) - x] + \beta[\sigma(x_0) - x] \quad for\, all \quad x \in U_\delta(x_0) \tag{3.2}$$

where $\alpha = \alpha(x_0, x)$ and $\beta = \beta(x_0, x)$ are equal zero at $x = x_0$ and

$$\lim_{x \to x_0} \alpha(x_0, x) = 0 \quad and \quad \lim_{x \to x_0} \beta(x_0, x) = 0.$$

Now we can give the definition for two variable case:

Definition 18. *We say that a function $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is **completely delta differentiable** at a point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ if there exist numbers A_1 and A_2 independent of $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ (but, in general, dependent on (x_0, y_0)) such that*

$$u(x_0, y_0) - u(x, y) = A_1(x_0 - x) + A_2(y_0 - y) + \alpha_1(x_0 - x) + \alpha_2(y_0 - y), \tag{3.3}$$

$$u(\sigma_1(x_0), y_0) - u(x, y) = A_1[\sigma_1(x_0) - x] + A_2(y_0 - y) + \beta_{11}[\sigma_1(x_0) - x] + \beta_{12}(y_0 - y), \tag{3.4}$$

$$u(x_0, \sigma_2(y_0)) - u(x, y) = A_1(x_0 - x) + A_2[\sigma_2(y_0) - y] + \beta_{21}[x_0 - x] + \beta_{22}[\sigma_2(y_0) - y] \tag{3.5}$$

for all $(x, y) \in U_\delta(x_0, y_0)$, where $\delta > 0$ is sufficiently small, $\alpha_j = \alpha_j(x_0, y_0; x, y)$ and $\beta_{jk} = \beta_{jk}(x_0, y_0; x, y)$ are defined on $U_\delta(x_0, y_0)$ such that they are equal to zero at $(x, y) = (x_0, y_0)$ and

$$\lim_{(x,y) \to (x_0, y_0)} \alpha_j(x_0, y_0; x, y) = \lim_{(x,y) \to (x_0, y_0)} \beta_{jk}(x_0, y_0; x, y) = 0 \quad for \quad j, k \in \{1, 2\}$$

Note that in case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the neighborhood $U_\delta(x_0, y_0)$ contains the single point (x_0, y_0) for $\delta < 1$. Therefore, in the case, the condition (3.3) disappears, while the conditions (3.4) and (3.5) hold with $\beta_{jk} = 0$ and with

$$A_1 = u(x_0 + 1, y_0) - u(x_0, y_0) = \frac{\partial u(x_0, y_0)}{\Delta_1 x} \tag{3.6}$$

and

$$A_2 = u(x_0, y_0 + 1) - u(x_0, y_0) = \frac{\partial u(x_0, y_0)}{\Delta_2 y}. \tag{3.7}$$

Lemma 19. *Let the function $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be completely delta differentiable at the point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$, then it is continuous at that point and has at (x_0, y_0) the first order partial delta derivatives equal to A_1 and A_2, namely*

$$\frac{\partial u(x_0, y_0)}{\Delta_1 x} = A_1 \quad \textit{and} \quad \frac{\partial u(x_0, y_0)}{\Delta_2 y} = A_2.$$

Proof. The continuity of u follows, in fact, from any one of (3.3), (3.4) and (3.5). Indeed (3.3) obviously yields the continuity of u at (x_0, y_0). Let now (3.4) holds. In the case $\sigma_1(x_0) = x_0$, (3.4) immediately gives the continuity of u at (x_0, y_0). If $\sigma_1(x_0) > x_0$, except of $u(x, y)$, each term in (3.4) has a limit as $(x, y) \to (x_0, y_0)$. Therefore $u(x, y)$ also has a limit as $(x, y) \to (x_0, y_0)$, and we have

$$u(\sigma_1(x_0), y_0) - \lim_{(x,y)\to(x_0,y_0)} u(x, y) = A_1[\sigma_1(x_0) - x_0].$$

Further, letting $(x, y) = (x_0, y_0)$ in (3.4), we get

$$u(\sigma_1(x_0), y_0) - u(x_0, y_0) = A_1[\sigma_1(x_0) - x_0].$$

Comparing the last two relations gives

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0)$$

which shows the continuity of u at (x_0, y_0). Next, setting $y = y_0$ in (3.4) and dividing both sides by $\sigma_1(x_0) - x$ and passing to limit as $x \to x_0$ we get $\frac{\partial u(x_0,y_0)}{\Delta_1 x} = A_1$. By similar approach $\frac{\partial u(x_0,y_0)}{\Delta_2 y} = A_2$ can be obtained. $\square$

3.1.2. Completely Nabla Differentiable Functions

Before the definition of completely nabla differentiability on products of two time scales, we first give the definition for one-variable case.

Definition 20. *A function $u : \mathbb{T} \to \mathbb{R}$ is called completely nabla differentiable at a point $x_0 \in \mathbb{T}_\kappa$ if there exists a number B such that*

$$u(x) - u(x_0) = B(x - x_0) + \alpha(x - x_0) \quad for\, all \quad x \in U_\delta(x_0) \tag{3.8}$$

and

$$u(x) - u(\rho(x_0)) = B[x - \rho(x_0)] + \beta[x - \rho(x_0)] \quad for\, all \quad x \in U_\delta(x_0) \tag{3.9}$$

where $\alpha = \alpha(x_0, x)$ *and* $\beta = \beta(x_0, x)$ *are equal zero at* $x = x_0$ *and*

$$\lim_{x \to x_0} \alpha(x_0, x) = 0 \quad and \quad \lim_{x \to x_0} \beta(x_0, x) = 0.$$

Now we can give the definition for two variable case:

Definition 21. *We say that a function* $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ *is* ***completely nabla differentiable*** *at a point* $(x_0, y_0) \in \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa}$ *if there exist numbers* B_1 *and* B_2 *independent of* $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ *(but, in general, dependent on* (x_0, y_0)*) such that*

$$u(x, y) - u(x_0, y_0) = B_1(x - x_0) + B_2(y - y_0) + \alpha_1(x - x_0) + \alpha_2(y - y_0), \quad (3.10)$$

$$u(x, y) - u(\rho_1(x_0), y_0) = B_1[x - \rho_1(x_0)] + B_2(y - y_0) + \beta_{11}[x - \rho_1(x_0)] + \beta_{12}(y - y_0), \quad (3.11)$$

$$u(x, y) - u(x_0, \rho_2(y_0)) = B_1(x - x_0) + B_2[y - \rho_2(y_0)] + \beta_{21}[x - x_0] + \beta_{22}[y - \rho_2(y_0)] \quad (3.12)$$

for all $(x, y) \in U_\delta(x_0, y_0)$*, where* $\delta > 0$ *is sufficiently small,* $\alpha_j = \alpha_j(x_0, y_0; x, y)$ *and* $\beta_{jk} = \beta_{jk}(x_0, y_0; x, y)$ *are defined on* $U_\delta(x_0, y_0)$ *such that they are equal to zero at* $(x, y) = (x_0, y_0)$ *and*

$$\lim_{(x,y) \to (x_0, y_0)} \alpha_j(x_0, y_0; x, y) = \lim_{(x,y) \to (x_0, y_0)} \beta_{jk}(x_0, y_0; x, y) = 0 \quad for \quad j, k \in \{1, 2\}$$

With $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, similarly we have

$$B_1 = u(x_0, y_0) - u(x_0 - 1, y_0) = \frac{\partial u(x_0, y_0)}{\nabla_1 x}$$

and

$$B_2 = u(x_0, y_0) - u(x_0, y_0 - 1) = \frac{\partial u(x_0, y_0)}{\nabla_2 y}.$$

This results and (3.6) and (3.7) show that each function $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is completely delta and nabla differentiable at every point.

Lemma 22. *Let the function* $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ *be completely nabla differentiable at the point* $(x_0, y_0) \in \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa}$*, then it is continuous at that point and has at* (x_0, y_0) *the first order partial nabla derivatives equal to* B_1 *and* B_2*, namely*

$$\frac{\partial u(x_0, y_0)}{\nabla_1 x} = B_1 \quad and \quad \frac{\partial u(x_0, y_0)}{\nabla_2 y} = B_2.$$

Proof. The proof is similar with the Lemma 19. □

Theorem 23. *If the function $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is continuous and have the first order partial nabla derivatives $\frac{\partial u(x,y)}{\nabla_1 x}$, $\frac{\partial u(x,y)}{\nabla_2 y}$ in some δ-neighborhood $U_\delta(x_0, y_0)$ of the point $(x_0, y_0) \in \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa}$ and if these derivatives are continuous at (x_0, y_0), then u is completely ∇-differentiable at (x_0, y_0).*

To prove this theorem we first give the mean value theorem for one-variable case.

Theorem 24. *Let a and b be two arbitrary points in $\mathbb{T}$ and let us set $\alpha = min\{a, b\}$ and $\beta = max\{a, b\}$. Let, further, f be a continuous function on $[\alpha, \beta]$ that has a nabla derivative at each point of $(\alpha, \beta]$. Then there exist $\xi, \xi' \in (\alpha, \beta]$ such that*

$$f^\nabla(\xi)(a - b) \leq f(a) - f(b) \leq f^\nabla(\xi')(a - b).$$

Proof. (for Theorem 23) For better clearness of the proof we first consider the single variable case. So, let $u : \mathbb{T} \to \mathbb{R}$ be a function that has a nabla derivative $u^\nabla(x)$ in some δ-neighbourhood $U_\delta(x_0)$ of the point $x_0 \in \mathbb{T}_\kappa$ (note that, in contrast to the multivariable case, in the single variable case existence of the derivative at a point implies continuity of the function at that point). The relation (3.9) with $B = u^\nabla(x_0)$ follows immediately from the definition of the nabla derivative

$$u(x) - u(\rho(x_0)) = u^\nabla(x_0)[x - \rho(x_0)] + \beta[x - \rho(x_0)], \tag{3.13}$$

where $\beta = \beta(x_0, x)$ and $\beta \to 0$ as $x \to x_0$. In order to prove (3.8), we consider all possible cases separately.

(i) If the point x_0 is isolated in $\mathbb{T}$, then (3.8) is satisfied independent of B and α, since in this case $U_\delta(x_0)$ consists of the single point x_0 for sufficiently small $\delta > 0$.

(ii) Let x_0 be left-dense. Regardless whether x_0 is right-scattered or right-dense, we have in this case $\rho(x_0) = x_0$ and (3.13) coincides with (3.8).

(iii) Finally, let x_0 be right-dense and left-scattered. Then for sufficiently small $\delta > 0$, any point $x \in U_\delta(x_0) - \{x_0\}$ must satisfy $x > x_0$. Applying Theorem 24, we obtain

$$u^\nabla(\xi)(x - x_0) \leq u(x) - u(x_0) \leq u^\nabla(\xi')(x - x_0),$$

where $\xi, \xi' \in (x, x_0]$. Since $\xi \to x_0$ and $\xi' \to x_0$ as $x \to x_0$, we get from the latter inequalities by the assumed continuity of the nabla derivative

$$\lim_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0} = u^\nabla(x_0).$$

Therefore

$$\frac{u(x) - u(x_0)}{x - x_0} = u^{\nabla}(x_0) + \alpha,$$

where $\alpha = \alpha(x_0, x)$and $\alpha \to 0$ as $x \to x_0$. Consequently, in the considered case we obtain (3.8) with $B = u^{\nabla}(x_0)$ as well.

Now we consider the two-variable case as it is stated in the theorem. To prove (3.10), we take the difference

$$u(x, y) - u(x_0, y_0) = [u(x, y) - u(x, y_0)] + [u(x, y_0) - u(x_0, y_0)]. \tag{3.14}$$

By the one-variable case considered above, we have

$$u(x, y_0) - u(x_0, y_0) = \frac{\partial u(x_0, y_0)}{\nabla_1 x}(x - x_0) + \alpha_1(x - x_0) \quad \text{for} \quad (x, y_0) \in U_\delta(x_0, y_0), \tag{3.15}$$

where $\alpha_1 = \alpha_1(x_0, y_0; x)$ and $\alpha_1 \to 0$ as $x \to x_0$. Further, applying the one-variable mean value result, Theorem 24, for fixed x and variable y, we have

$$\frac{\partial u(x, \xi)}{\nabla_2 y}(y - y_0) \leq u(x, y) - u(x, y_0) \leq \frac{\partial u(x, \xi')}{\nabla_2 y}(y - y_0), \tag{3.16}$$

where $\xi, \xi' \in (\alpha, \beta]$ and $\alpha = min\{y_0, y\}$, $\beta = max\{y_0, y\}$. Since $\xi \to y_0$ and $\xi' \to y_0$ as $y \to y_0$, by the assumed continuity of the partial derivatives at (x_0, y_0) we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\partial u(x, \xi')}{\nabla_2 y} = \lim_{(x,y)\to(x_0,y_0)} \frac{\partial u(x, \xi)}{\nabla_2 y} = \frac{\partial u(x_0, y_0)}{\nabla_2 y}.$$

Therefore from (3.16) we obtain

$$u(x, y) - u(x, y_0) = \frac{\partial u(x_0, y_0)}{\nabla_2 y}(y - y_0) + \alpha_2(y - y_0), \tag{3.17}$$

where $\alpha_2 = \alpha_2(x_0, y_0; x, y)$ and $\alpha_2 \to 0$ as $(x, y) \to (x_0, y_0)$. Substituting (3.15) and (3.17) in (3.14), we get a relation of the form (3.10) with $B_1 = \frac{\partial u(x_0,y_0)}{\nabla_1 x}$ and $B_2 = \frac{\partial u(x_0,y_0)}{\nabla_2 y}$. To prove (3.11) we take the difference

$$u(x, y) - u(\rho_1(x_0), y_0) = [u(x, y) - u(x, y_0)] + [u(x, y_0) - u(\rho_1(x_0), y_0)]. \tag{3.18}$$

By the definition of the partial nabla derivative we have

$$u(x, y_0) - u(\rho_1(x_0), y_0) = \frac{\partial u(x_0, y_0)}{\nabla_1 x}[x - \rho_1(x_0)] + \beta_{11}[x - \rho_1(x_0)], \tag{3.19}$$

where $\beta_{11} = \beta_{11}(x_0, y_0; x)$ and $\beta_{11} \to 0$ as $x \to x_0$. Now substituting (3.19) and (3.17) into (3.18), we obtain a relation of the form (3.11) with $B_1 = \frac{\partial u(x_0,y_0)}{\nabla_1 x}$ and $B_2 = \frac{\partial u(x_0,y_0)}{\nabla_2 y}$. The equality (3.12) can be proved similarly by considering the difference

$$u(x,y) - u(x_0, \rho_2(y_0)) = [u(x,y) - u(x_0,y)] + [u(x_0,y) - u(x_0, \rho_2(y_0))].$$

□

3.2. Cauchy-Riemann Equations on Time Scale Complex Plane

For given time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, let us set

$$\mathbb{T}_1 + i\mathbb{T}_2 = \{z = x + iy : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}, \tag{3.20}$$

where $i = \sqrt{-1}$ is the imaginary unit. The set $\mathbb{T}_1 + i\mathbb{T}_2$ is called the **time scale complex plane** and is a complete metric space with the metric d defined by

$$d(z, z') = |z - z'| = \sqrt{(x - x')^2 + (y - y')^2} \tag{3.21}$$

where $z = x + iy, z' = x' + iy' \in \mathbb{T}_1 + i\mathbb{T}_2$.
Any function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ can be represented in the form

$$f(z) = u(x,y) + iv(x,y) \quad \text{for} \quad z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2,$$

where $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is the real part of f and $v : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is the imaginary part of f.
Let σ_1 and σ_2 be the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For $z = x+iy \in \mathbb{T}_1 + i\mathbb{T}_2$, let us set

$$z^{\sigma_1} = \sigma_1(x) + iy \quad \text{and} \quad z^{\sigma_2} = x + i\sigma_2(y).$$

Let ρ_1 and ρ_2 be the backward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For $z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2$, we set

$$z^{\rho_1} = \rho_1(x) + iy \quad \text{and} \quad z^{\rho_2} = x + i\rho_2(y).$$

3.2.1. Delta Analytic Functions

Definition 25. *A complex-valued function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ is* ***delta differentiable*** *(or* ***delta analytic****) at a point $z_0 = x_0 + iy_0 \in \mathbb{T}_1^\kappa + i\mathbb{T}_2^\kappa$ if there exists a complex number A (depending in general on z_0) such that*

$$f(z_0) - f(z) = A(z_0 - z) + \alpha(z_0 - z) \tag{3.22}$$

$$f(z_0^{\sigma_1}) - f(z) = A(z_0^{\sigma_1} - z) + \beta(z_0^{\sigma_1} - z) \tag{3.23}$$

$$f(z_0^{\sigma_2}) - f(z) = A(z_0^{\sigma_2} - z) + \gamma(z_0^{\sigma_2} - z) \tag{3.24}$$

for all $z \in U_\delta(z_0)$, where $U_\delta(z_0)$ is a δ-neighborhood of z_0 in $\mathbb{T}_1 + i\mathbb{T}_2$, $\alpha = \alpha(z_0, z)$, $\beta = \beta(z_0, z)$ and $\gamma = \gamma(z_0, z)$ are defined for $z \in U_\delta(z_0)$, they are equal to zero at $z = z_0$, and

$$\lim_{z \to z_0} \alpha(z_0, z) = \lim_{z \to z_0} \beta(z_0, z) = \lim_{z \to z_0} \gamma(z_0, z) = 0.$$

Then the number A is called the ***delta derivative*** *(or* ***Δ-derivative****) of f at z_0 and is denoted by $f^\Delta(z_0)$.*

Theorem 26. *Let the function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ have the form*

$$f(z) = u(x, y) + iv(x, y) \quad for \quad z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2.$$

Then a necessary and sufficient condition for f to be Δ-differentiable (as a function of the complex variable z) at the point $z_0 = x_0 + iy_0 \in \mathbb{T}_1^\kappa + i\mathbb{T}_2^\kappa$ is that the functions u and v be completely Δ-differentiable (as a function of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) at the point (x_0, y_0) and satisfied the Cauchy-Riemann equations

$$\frac{\partial u}{\Delta_1 x} = \frac{\partial v}{\Delta_2 y} \quad and \quad \frac{\partial u}{\Delta_2 y} = -\frac{\partial v}{\Delta_1 x} \tag{3.25}$$

at (x_0, y_0). If these equations are satisfied, then $f^\Delta(z_0)$ can be represented in any of the forms

$$f^\Delta(z_0) = \frac{\partial u}{\Delta_1 x} + i\frac{\partial v}{\Delta_1 x} = \frac{\partial v}{\Delta_2 y} - i\frac{\partial u}{\Delta_2 y} = \frac{\partial u}{\Delta_1 x} - i\frac{\partial u}{\Delta_2 y} = \frac{\partial v}{\Delta_2 y} + i\frac{\partial v}{\Delta_1 x}, \tag{3.26}$$

where the partial derivatives are evaluated at (x_0, y_0).

Proof. First we show necessity. Assume that f is Δ-differentiable at $z_0 = x_0 + iy_0$ with $f^\Delta(z_0) = A$. Then (3.22)-(3.24) are satisfied. Letting

$$f = u + iv, \quad A = A_1 + iA_2, \quad \alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2, \quad \gamma = \gamma_1 + i\gamma_2,$$

we get from (3.22)-(3.24), equating the real and imaginary parts of both sides in each of these equations,

$$\begin{cases} u(x_0, y_0) - u(x, y) = A_1(x_0 - x) - A_2(y_0 - y) + \alpha_1(x_0 - x) - \alpha_2(y_0 - y) \\ u(\sigma_1(x_0), y_0) - u(x, y) = A_1[\sigma_1(x_0) - x] - A_2(y_0 - y) + \beta_1[\sigma_1(x_0) - x] - \beta_2(y_0 - y) \\ u(x_0, \sigma_2(y_0)) - u(x, y) = A_1(x_0 - x) - A_2[\sigma_2(y_0) - y] + \gamma_1(x_0 - x) - \gamma_2[\sigma_2(y_0) - y] \end{cases}$$

and

$$\begin{cases} v(x_0, y_0) - v(x, y) = A_2(x_0 - x) - A_1(y_0 - y) + \alpha_2(x_0 - x) - \alpha_1(y_0 - y) \\ v(\sigma_1(x_0), y_0) - v(x, y) = A_2[\sigma_1(x_0) - x] - A_1(y_0 - y) + \beta_2[\sigma_1(x_0) - x] - \beta_1(y_0 - y) \\ v(x_0, \sigma_2(y_0)) - v(x, y) = A_2(x_0 - x) - A_1[\sigma_2(y_0) - y] + \gamma_2(x_0 - x) - \gamma_1[\sigma_2(y_0) - y] \end{cases}$$

Hence, taking into account that $\alpha_j \to 0$, $\beta_j \to 0$, and $\gamma_j \to 0$ as $(x, y) \to (x_0, y_0)$, we get that the functions u and v are completely Δ-differentiable (as functions of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) and that

$$A_1 = \frac{\partial u(x_0, y_0)}{\Delta_1 x}, \quad -A_2 = \frac{\partial u(x_0, y_0)}{\Delta_2 y}, \quad A_2 = \frac{\partial v(x_0, y_0)}{\Delta_1 x}, \quad A_1 = \frac{\partial v(x_0, y_0)}{\Delta_2 y}.$$

Therefore the Cauchy-Riemann equations (3.25) hold and we have the formulas (3.26).

Now we show sufficiency. Assume that the functions u and v, where $f = u + iv$, are completely Δ-differentiable at the point (x_0, y_0) and that the Cauchy-Riemann equations (3.25) hold. Then we have

$$\begin{cases} u(x_0, y_0) - u(x, y) = A_1'(x_0 - x) - A_2'(y_0 - y) + \alpha_1'(x_0 - x) - \alpha_2'(y_0 - y) \\ u(\sigma_1(x_0), y_0) - u(x, y) = A_1'[\sigma_1(x_0) - x] - A_2'(y_0 - y) + \beta_{11}'[\sigma_1(x_0) - x] - \beta_{12}'(y_0 - y) \\ u(x_0, \sigma_2(y_0)) - u(x, y) = A_1'(x_0 - x) - A_2'[\sigma_2(y_0) - y] + \beta_{21}'(x_0 - x) - \beta_{22}'[\sigma_2(y_0) - y] \end{cases}$$

and

$$\begin{cases} v(x_0, y_0) - v(x, y) = A_1''(x_0 - x) - A_2''(y_0 - y) + \alpha_1''(x_0 - x) - \alpha_2''(y_0 - y) \\ v(\sigma_1(x_0), y_0) - v(x, y) = A_1''[\sigma_1(x_0) - x] - A_2''(y_0 - y) + \beta_{11}''[\sigma_1(x_0) - x] - \beta_{12}''(y_0 - y) \\ v(x_0, \sigma_2(y_0)) - v(x, y) = A_1''(x_0 - x) - A_2''[\sigma_2(y_0) - y] + \beta_{21}''(x_0 - x) - \beta_{22}''[\sigma_2(y_0) - y] \end{cases}$$

where α_j', β_{ij}' and α_j'', β_{ij}'' tend to zero as $(x, y) \to (x_0, y_0)$ and

$$A_1' = \frac{\partial u(x_0, y_0)}{\Delta_1 x} = \frac{\partial v(x_0, y_0)}{\Delta_2 y} = A_2'' =: A_1$$

and

$$-A_2' = -\frac{\partial u(x_0, y_0)}{\Delta_2 y} = \frac{\partial v(x_0, y_0)}{\Delta_1 x} = A_1'' =: A_2.$$

Therefore

$$f(z_0) - f(z) = (A_1 + iA_2)(z_0 - z) + \alpha(z_0 - z),$$

$$f(z_0^{\sigma_1}) - f(z) = (A_1 + iA_2)(z_0^{\sigma_1} - z) + \beta(z_0^{\sigma_1} - z),$$

$$f(z_0^{\sigma_2}) - f(z) = (A_1 + iA_2)(z_0^{\sigma_2} - z) + \gamma(z_0^{\sigma_2} - z),$$

where

$$\alpha = (\alpha_1' + i\alpha_1'')\frac{x_0 - x}{z_0 - z} + (\alpha_2' + i\alpha_2'')\frac{y_0 - y}{z_0 - z},$$

$$\beta = (\beta_{11}' + i\beta_{11}'')\frac{\sigma_1(x_0) - x}{z_0^{\sigma_1} - z} + (\beta_{12}' + i\beta_{12}'')\frac{y_0 - y}{z_0^{\sigma_1} - z},$$

$$\gamma = (\beta_{21}' + i\beta_{21}'')\frac{x_0 - x}{z_0^{\sigma_2} - z} + (\beta_{22}' + i\beta_{22}'')\frac{\sigma_2(y_0) - y}{z_0^{\sigma_2} - z}.$$

Since

$$|\alpha| \leq |\alpha_1' + i\alpha_1''|\left|\frac{x_0 - x}{z_0 - z}\right| + |\alpha_2' + i\alpha_2''|\left|\frac{y_0 - y}{z_0 - z}\right|$$
$$\leq |\alpha_1' + i\alpha_1''| + |\alpha_2' + i\alpha_2''| \leq |\alpha_1'| + |\alpha_1''| + |\alpha_2'| + |\alpha_2''|,$$

we have $\alpha \to 0$ as $z \to z_0$. Similarly, $\beta \to 0$ and $\gamma \to 0$ as $z \to z_0$. Consequently, f is Δ-differentiable at z_0 and $f^{\Delta}(z_0) = A_1 + iA_2$. □

Remark 27. *(See (Bohner and Guseinov 2004)). If the functions $u, v : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ are continuous and have the first order partial delta derivatives $\frac{\partial u(x,y)}{\Delta_1 x}$, $\frac{\partial u(x,y)}{\Delta_2 y}$, $\frac{\partial v(x,y)}{\Delta_1 x}$, $\frac{\partial v(x,y)}{\Delta_2 y}$ in some δ-neighborhood $U_\delta(x_0, y_0)$ of the point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ and if these derivatives are continuous at (x_0, y_0), then u and v are completely Δ-differentiable at (x_0, y_0). Therefore in this case, if in addition the Cauchy-Riemann equations* (3.25) *are satisfied, then $f(z) = u(x, y) + iv(x, y)$ is Δ-differentiable at $z_0 = x_0 + iy_0$.*

Example 28. *(i) The function $f(z) =$constant on $\mathbb{T}_1 + i\mathbb{T}_2$ is Δ-analytic everywhere and $f^{\Delta}(z) = 0$.*
(ii) The function $f(z) = z$ on $\mathbb{T}_1 + i\mathbb{T}_2$ is Δ-analytic everywhere and $f^{\Delta}(z) = 1$.
(iii) Consider the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy \quad on \quad \mathbb{T}_1 + i\mathbb{T}_2.$$

Hence $u(x, y) = x^2 - y^2, v(x, y) = 2xy$, and

$$\frac{\partial u(x,y)}{\Delta_1 x} = x + \sigma_1(x), \quad \frac{\partial u(x,y)}{\Delta_2 y} = -y - \sigma_2(y), \quad \frac{\partial v(x,y)}{\Delta_1 x} = 2y, \quad \frac{\partial v(x,y)}{\Delta_2 y} = 2x.$$

Therefore the Cauchy-Riemann equations become

$$x + \sigma_1(x) = 2x \quad and \quad -y - \sigma_2(y) = -2y,$$

which hold simultaneously if and only if $\sigma_1(x) = x$ and $\sigma_2(y) = y$ simultaneously. It follows that the function $f(z) = z^2$ is not Δ-analytic at each point of $\mathbb{Z} + i\mathbb{Z}$. So, the product of two Δ-analytic functions need not be Δ-analytic.

(iv) The function $f(z) = x^2 - y^2 + i(2xy + x + y)$ is Δ-analytic everywhere on $\mathbb{Z} + i\mathbb{Z}$. Since, Cauchy-Riemann equations are satisfied:

$$\frac{\partial u(x,y)}{\Delta_1 x} = x + \sigma_1(x) = 2x + 1, \qquad \frac{\partial u(x,y)}{\Delta_2 y} = -y - \sigma_2(y) = -2y - 1,$$

$$\frac{\partial v(x,y)}{\Delta_1 x} = 2y + 1, \qquad \frac{\partial v(x,y)}{\Delta_2 y} = 2x + 1.$$

This function is not analytic anywhere on $\mathbb{R} + i\mathbb{R} = \mathbb{C}$.

(v) The function $f(z) = x^2 - y^2 + i3xy$ is Δ-analytic everywhere on $\mathbb{T} + i\mathbb{T}$ where $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$.

Since, Cauchy-Riemann equations are satisfied:

$$\frac{\partial u(x,y)}{\Delta_1 x} = x + \sigma_1(x) = 3x = \frac{\partial v(x,y)}{\Delta_2 y},$$

$$\frac{\partial u(x,y)}{\Delta_2 y} = -y - \sigma_2(y) = -3y = -\frac{\partial v(x,y)}{\Delta_1 x}.$$

Remark 29. *(i) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{R} + i\mathbb{R} = \mathbb{C}$ is the usual complex plane and the three condition (3.22)-(3.24) of Definition 25 coincide and reduce to the classical definition of analyticity (differentiability) of functions of a complex variable.*

(ii) Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$. Then $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{Z} + i\mathbb{Z} = \mathbb{Z}[i]$ is the set of Gaussian integers. The neighborhood $U_\delta(z_0)$ of z_0 contains the single point z_0 for $\delta < 1$. Therefore, in this case, the condition (3.22) disappears, while the conditions (3.23) and (3.24) reduce to the single condition

$$\frac{f(z_0 + 1) - f(z_0)}{1} = \frac{f(z_0 + i) - f(z_0)}{i} \tag{3.27}$$

with $f^\Delta(z_0)$ equal to the left (and hence also to the right) hand side of (3.27). The condition (3.27) coincides with the condition of forward monodiffric functions in (2.5).

(iii) If $\mathbb{T}_1 = \mathbb{R}$ and $\mathbb{T}_2 = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ where $h > 0$, then (3.22) and (3.23) coincide and in any of them, dividing both sides by $(z_0 - z)$ where $z = x + ikh$ and

$z_0 = x_0 + ik_0h$, and taking limit as $z \to z_0$ (which just means $x \to x_0$), with $k = k_0$ we have

$$\lim_{x \to x_0} \frac{f(x_0 + ik_0h) - f(x + ik_0h)}{x_0 - x} = A.$$

Similarly by (3.24) we get

$$\frac{f(z_0 + ih) - f(z)}{ih} = A.$$

Equating this two results gives the condition of forward semi-discrete analytic functions:

$$\frac{\partial f(z)}{\partial x} = [f(z + ih) - f(z)]/ih, \quad z = x + ikh.$$

Remark 30. *We can combine Cauchy-Riemann equations for delta analytic functions in one complex equation as follows:*

$$\frac{\partial f}{\Delta_1 x} = \frac{1}{i} \frac{\partial f}{\Delta_2 y} \tag{3.28}$$

(i) *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then we have*

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{i} \frac{\partial f}{\partial y} \\ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= \frac{1}{i} \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \end{aligned}$$

Equating real and imaginary parts, we get the usual Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(ii) *If we take $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ then the equation (3.28) becomes*

$$\Delta_x f = \frac{1}{i} \Delta_y f$$

which is exactly the condition for forward monodiffric functions

$$\frac{f(z + 1) - f(z)}{1} = \frac{f(z + i) - f(z)}{i}.$$

(iii) *With $\mathbb{T}_1 = \mathbb{R}$ and $\mathbb{T}_2 = h\mathbb{Z}$, the equation (3.28) becomes*

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{f(z + ih) - f(z)}{h}.$$

which is the equation for forward semi-discrete analytic functions.

3.2.2. Nabla Analytic Functions

Definition 31. *A complex-valued function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ is **nabla differentiable** (or **nabla analytic**) at a point $z_0 = x_0 + iy_0 \in \mathbb{T}_{1\kappa} + i\mathbb{T}_{2\kappa}$ if there exists a complex number B (depending in general on z_0) such that*

$$f(z) - f(z_0) = B(z - z_0) + \alpha(z - z_0) \tag{3.29}$$

$$f(z) - f(z_0^{\rho_1}) = B(z - z_0^{\rho_1}) + \beta(z - z_0^{\rho_1}) \tag{3.30}$$

$$f(z) - f(z_0^{\rho_2}) = B(z - z_0^{\rho_2}) + \gamma(z - z_0^{\rho_2}) \tag{3.31}$$

for all $z \in U_\delta(z_0)$, where $U_\delta(z_0)$ is a δ-neighborhood of z_0 in $\mathbb{T}_1 + i\mathbb{T}_2$, $\alpha = \alpha(z_0, z)$, $\beta = \beta(z_0, z)$ and $\gamma = \gamma(z_0, z)$ are defined for $z \in U_\delta(z_0)$, they are equal to zero at $z = z_0$, and

$$\lim_{z\to z_0} \alpha(z_0, z) = \lim_{z\to z_0} \beta(z_0, z) = \lim_{z\to z_0} \gamma(z_0, z) = 0.$$

*Then the number B is called the **nabla derivative** (or ∇-**derivative**) of f at z_0 and is denoted by $f^\nabla(z_0)$.*

Theorem 32. *Let the function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ have the form*

$$f(z) = u(x, y) + iv(x, y) \quad for \quad z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2.$$

Then a necessary and sufficient condition for f to be ∇-differentiable (as a function of the complex variable z) at the point $z_0 = x_0 + iy_0 \in \mathbb{T}_{1\kappa} + i\mathbb{T}_{2\kappa}$ is that the functions u and v be completely ∇-differentiable (as a function of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) at the point (x_0, y_0) and satisfied the Cauchy-Riemann equations

$$\frac{\partial u}{\nabla_1 x} = \frac{\partial v}{\nabla_2 y} \quad and \quad \frac{\partial u}{\nabla_2 y} = -\frac{\partial v}{\nabla_1 x} \tag{3.32}$$

at (x_0, y_0). If these equations are satisfied, then $f^\nabla(z_0)$ can be represented in any of the forms

$$f^\nabla(z_0) = \frac{\partial u}{\nabla_1 x} + i\frac{\partial v}{\nabla_1 x} = \frac{\partial v}{\nabla_2 y} - i\frac{\partial u}{\nabla_2 y} = \frac{\partial u}{\nabla_1 x} - i\frac{\partial u}{\nabla_2 y} = \frac{\partial v}{\nabla_2 y} + i\frac{\partial v}{\nabla_1 x}, \tag{3.33}$$

where the partial derivatives are evaluated at (x_0, y_0).

Proof. Let us first show necessity. Assume that f is ∇-differentiable at $z_0 = x_0 + iy_0$ with $f^\nabla(z_0) = B$. Then (3.29)-(3.31) are satisfied. Letting

$$f = u + iv, \quad B = B_1 + iB_2, \quad \alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2, \quad \gamma = \gamma_1 + i\gamma_2,$$

we get from (3.29)-(3.31), equating the real and imaginary parts of both sides in each of these equations,

$$\begin{cases} u(x,y)-u(x_0,y_0)=B_1(x-x_0)-B_2(y-y_0)+\alpha_1(x-x_0)-\alpha_2(y-y_0) \\ u(x,y)-u(\rho_1(x_0),y_0)=B_1[x-\rho_1(x_0)]-B_2(y-y_0)+\beta_1[x-\rho_1(x_0)]-\beta_2(y-y_0) \\ u(x,y)-u(x_0,\rho_2(y_0))=B_1(x-x_0)-B_2[y-\rho_2(y_0)]+\gamma_1(x-x_0)-\gamma_2[y-\rho_2(y_0)] \end{cases}$$

and

$$\begin{cases} v(x,y)-v(x_0,y_0)=B_2(x-x_0)-B_1(y-y_0)+\alpha_2(x-x_0)-\alpha_1(y-y_0) \\ v(x,y)-v(\rho_1(x_0),y_0)=B_2[x-\rho_1(x_0)]-B_1(y-y_0)+\beta_2[x-\rho_1(x_0)]-\beta_1(y-y_0) \\ v(x,y)-v(x_0,\rho_2(y_0))=B_2(x-x_0)-B_1[y-\rho_2(y_0)]+\gamma_2(x-x_0)-\gamma_1[y-\rho_2(y_0)] \end{cases}$$

Hence, taking into account that $\alpha_j \to 0$, $\beta_j \to 0$, and $\gamma_j \to 0$ as $(x,y)\to(x_0,y_0)$, we get that the functions u and v are completely ∇-differentiable (as functions of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) and that

$$B_1=\frac{\partial u(x_0,y_0)}{\nabla_1 x}, \quad -B_2=\frac{\partial u(x_0,y_0)}{\nabla_2 y}, \quad B_2=\frac{\partial v(x_0,y_0)}{\nabla_1 x}, \quad B_1=\frac{\partial v(x_0,y_0)}{\nabla_2 y}.$$

Therefore the Cauchy-Riemann equations (3.32) hold and we have the formulas (3.33).

Now we show sufficiency. Assume that the functions u and v, where $f=u+iv$, are completely ∇-differentiable at the point (x_0,y_0) and that the Cauchy-Riemann equations (3.32) hold. Then we have

$$\begin{cases} u(x,y)-u(x_0,y_0)=B_1'(x-x_0)-B_2'(y-y_0)+\alpha_1'(x-x_0)-\alpha_2'(y-y_0) \\ u(x,y)-u(\rho_1(x_0),y_0)=B_1'[x-\rho_1(x_0)]-B_2'(y-y_0)+\beta_{11}'[x-\rho_1(x_0)]-\beta_{12}'(y-y_0) \\ u(x,y)-u(x_0,\rho_2(y_0))=B_1'(x-x_0)-B_2'[y-\rho_2(y_0)]+\beta_{21}'(x-x_0)-\beta_{22}'[y-\rho_2(y_0)] \end{cases}$$

and

$$\begin{cases} v(x,y)-v(x_0,y_0)=B_1''(x-x_0)-B_2''(y-y_0)+\alpha_1''(x-x_0)-\alpha_2''(y-y_0) \\ v(x,y)-v(\rho_1(x_0),y_0)=B_1''[x-\rho_1(x_0)]-B_2''(y-y_0)+\beta_{11}''[x-\rho_1(x_0)]-\beta_{12}''(y-y_0) \\ v(x,y)-v(x_0,\rho_2(y_0))=B_1''(x-x_0)-B_2''[y-\rho_2(y_0)]+\beta_{21}''(x-x_0)-\beta_{22}''[y-\rho_2(y_0)] \end{cases}$$

where α_j', β_{ij}' and α_j'', β_{ij}'' tend to zero as $(x,y)\to(x_0,y_0)$ and

$$B_1'=\frac{\partial u(x_0,y_0)}{\nabla_1 x}=\frac{\partial v(x_0,y_0)}{\nabla_2 y}=B_2'' =: B_1$$

and

$$-B_2'=-\frac{\partial u(x_0,y_0)}{\nabla_2 y}=\frac{\partial v(x_0,y_0)}{\nabla_1 x}=B_1'' =: B_2.$$

Therefore

$$f(z) - f(z_0) = (B_1 + iB_2)(z - z_0) + \alpha(z - z_0),$$

$$f(z) - f(z_0^{\rho_1}) = (B_1 + iB_2)(z - z_0^{\rho_1}) + \beta(z - z_0^{\rho_1}),$$

$$f(z) - f(z_0^{\rho_2}) = (B_1 + iB_2)(z - z_0^{\rho_2}) + \gamma(z - z_0^{\rho_2}),$$

where

$$\alpha = (\alpha_1' + i\alpha_1'')\frac{x - x_0}{z - z_0} + (\alpha_2' + i\alpha_2'')\frac{y - y_0}{z - z_0},$$

$$\beta = (\beta_{11}' + i\beta_{11}'')\frac{x - \rho_1(x_0)}{z - z_0^{\rho_1}} + (\beta_{12}' + i\beta_{12}'')\frac{y - y_0}{z - z_0^{\rho_1}},$$

$$\gamma = (\beta_{21}' + i\beta_{21}'')\frac{x - x_0}{z - z_0^{\rho_2}} + (\beta_{22}' + i\beta_{22}'')\frac{y - \rho_2(y_0)}{z - z_0^{\rho_2}}.$$

Since

$$|\alpha| \le |\alpha_1' + i\alpha_1''|\left|\frac{x - x_0}{z - z_0}\right| + |\alpha_2' + i\alpha_2''|\left|\frac{y - y_0}{z - z_0}\right|$$

$$\le |\alpha_1' + i\alpha_1''| + |\alpha_2' + i\alpha_2''| \le |\alpha_1'| + |\alpha_1''| + |\alpha_2'| + |\alpha_2''|,$$

we have $\alpha \to 0$ as $z \to z_0$. Similarly, $\beta \to 0$ and $\gamma \to 0$ as $z \to z_0$. Consequently, f is ∇-differentiable at z_0 and $f^{\nabla}(z_0) = B_1 + iB_2$. □

Remark 33. *(See Theorem 23) If the functions $u, v : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ are continuous and have the first order partial delta derivatives $\frac{\partial u(x,y)}{\nabla_1 x}$, $\frac{\partial u(x,y)}{\nabla_2 y}$, $\frac{\partial v(x,y)}{\nabla_1 x}$, $\frac{\partial v(x,y)}{\nabla_2 y}$ in some δ-neighborhood $U_\delta(x_0, y_0)$ of the point $(x_0, y_0) \in \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa}$ and if these derivatives are continuous at (x_0, y_0), then u and v are completely ∇-differentiable at (x_0, y_0). Therefore in this case, if in addition the Cauchy-Riemann equations (3.32) are satisfied, then $f(z) = u(x, y) + iv(x, y)$ is ∇-differentiable at $z_0 = x_0 + iy_0$.*

Example 34. *(i) Similar to the delta case, the functions $f(z)$ =constant and $g(z) = z$ on $\mathbb{T}_1 + i\mathbb{T}_2$ are ∇-analytic everywhere with $f^{\nabla}(z) = 0$ and $g^{\nabla}(z) = 1$.*
(ii) It is obvious that the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy \quad on \quad \mathbb{T}_1 + i\mathbb{T}_2$$

is not ∇-analytic at each point of $\mathbb{Z} + i\mathbb{Z}$ by the same reason as the delta case.
(iii) The function $f(z) = x^2 - y^2 + i(2xy - x - y)$ is ∇-analytic everywhere on $\mathbb{Z} + i\mathbb{Z}$. Since, Cauchy-Riemann equations are satisfied:

$$\frac{\partial u(x, y)}{\nabla_1 x} = x + \rho_1(x) = 2x - 1, \qquad \frac{\partial u(x, y)}{\nabla_2 y} = -y - \rho_2(y) = -2y + 1,$$

$$\frac{\partial v(x,y)}{\nabla_1 x} = 2y-1, \qquad \frac{\partial v(x,y)}{\Delta_2 y} = 2x-1.$$

Note that this function is not delta analytic on $\mathbb{Z}+i\mathbb{Z}$

(iv) The function $f(z) = x^2 - y^2 + i3xy$ *is not* ∇*-analytic on* $\mathbb{T}+i\mathbb{T}$ *where* $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$.

Since, Cauchy-Riemann equations are not satisfied:

$$\frac{\partial u(x,y)}{\nabla_1 x} = x + \rho_1(x) = \frac{3x}{2} \neq 3x = \frac{\partial v(x,y)}{\nabla_2 y},$$
$$\frac{\partial u(x,y)}{\nabla_2 y} = -y - \rho_2(y) = -\frac{3y}{2} \neq -3y = \frac{\partial v(x,y)}{\nabla_2 y}.$$

Note that this function is delta analytic everywhere on $\mathbb{T}+i\mathbb{T}$ *with the same* $\mathbb{T}$.

Remark 35. *(i) If* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$*, then* $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{R}+i\mathbb{R} = \mathbb{C}$ *is the usual complex plane and the three condition* (3.29)*-*(3.31) *of Definition* 31 *coincide and reduce to the classical definition of analyticity (differentiability) of functions of a complex variable.*

(ii) Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$*. Then* $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{Z}+i\mathbb{Z} = \mathbb{Z}[i]$ *is the set of Gaussian integers. The neighborhood* $U_\delta(z_0)$ *of* z_0 *contains the single point* z_0 *for* $\delta < 1$*. Therefore, in this case, the condition* (3.29) *disappears, while the conditions* (3.30) *and* (3.31) *reduce to the single condition*

$$\frac{f(z_0) - f(z_0 - 1)}{1} = \frac{f(z_0) - f(z_0 - i)}{i} \tag{3.34}$$

with $f^\nabla(z_0)$ *equal to the left (and hence also to the right) hand side of* (3.34)*. The condition* (3.34) *coincides with the definition of backward monodiffric functions we defined in (2.7).*

(iii) If $\mathbb{T}_1 = \mathbb{R}$ *and* $\mathbb{T}_2 = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ *where* $h > 0$*, then* (3.29) *and* (3.30) *coincide and in any of them, dividing both sides by* $(z - z_0)$ *where* $z = x + ikh$ *and* $z_0 = x_0 + ik_0h$*, and taking limit as* $z \to z_0$ *(which just means* $x \to x_0$*), with* $k = k_0$ *we have*

$$\lim_{x \to x_0} \frac{f(x_0 + ik_0h) - f(x + ik_0h)}{x_0 - x} = B.$$

Similarly by (3.31) *we get*

$$\frac{f(z) - f(z - ih)}{ih} = B.$$

Equating this two results gives the condition of backward semi-discrete analytic functions defined in (2.10).

Remark 36. *We can combine Cauchy-Riemann equations for nabla analytic functions in one equation as follows:*

$$\frac{\partial f}{\nabla_1 x} = \frac{1}{i}\frac{\partial f}{\nabla_2 y} \tag{3.35}$$

(i) *If* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$*, then we have*

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y} \\ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \end{aligned}$$

Equating real and imaginary parts, we get the usual Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(ii) *If we take* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ *then the equation (3.35) becomes*

$$\nabla_x f = \frac{1}{i}\nabla_y f$$

which is exactly the condition for backward monodiffric functions

$$\frac{f(z) - f(z-1)}{1} = \frac{f(z) - f(z-i)}{i}.$$

(iii) *With* $\mathbb{T}_1 = \mathbb{R}$ *and* $\mathbb{T}_2 = h\mathbb{Z}$*, the equation (3.35) becomes*

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{f(z) - f(z-ih)}{h}.$$

which is the equation for backward semi-discrete analytic functions.

CHAPTER 4

CONCLUSION

The history of the theory of complex functions begins at the end of sixteen century. The study of discrete counterparts of complex functions on $\mathbb{Z}^2$ has a history of more than sixty years. Further, the semi-discrete analog of complex functions was firstly introduced by Kurowski in 1963.

After calculus of time scales was introduced by Hilger in 1988, many concepts in continuous and discrete analysis was rapidly unified. Bohner and Guseinov were the first who made the unification of the continuous and discrete complex analysis by time scales. They defined completely delta differentiable functions, Cauchy-Riemann equations and delta analytic functions on products of two time scales. They also introduced complex line delta and nabla integrals along time scales curves, and obtain a time scale version of the classical Cauchy integral theorem on union of horizontal and vertical line segments.

In this thesis, we worked on continuous, discrete and semi-discrete analytic functions and developed completely ∇-differentiability, ∇-analytic functions on products of two time scales, and Cauchy-Riemann equations for nabla case.

Our further study will be on complex integration and we will investigate the generalization of time scale version of Cauchy integral theorem.

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