

DYNAMICAL SYSTEMS ON TIME SCALES

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**by
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ABSTRACT

DYNAMICAL SYSTEMS ON TIME SCALES

In this thesis, we have studied dynamical systems on time scales. Firstly, we give basic definitions and theorems about the time scales and dynamical systems. We present Floquet theory and stability criterion on periodic discrete Hamiltonian systems. We introduce the Hilger complex plane and exponential function on time scales. This exponential function is shown to satisfy an initial value problem involving a first order linear dynamic equation. Uniqueness and existence theorems are presented. And then we give stability criterion, Lyapunov transformations and a unified Floquet theory for periodic time scales. We try to collect studies (Bohner and Peterson 2001), (Dacunha 2005), (Ahlbrandt and Ridenhour 2003) about Dynamical Systems On Time Scales .

ÖZET

ZAMAN SKALALARINDA DİNAMİK SİSTEMLER

Bu tezde Zaman Skalalarında Dinamik Sistemleri çalıştık. Öncelikle, zaman skalasında ve dinamik sistemlerde temel tanım ve teoremleri verdik. Periyodik diskrit Hamilton sistemlerindeki Floquet teoremini ve kararlılık kriterini sunduk. Zaman skalasında Hilger kompleks düzlemini ve üstel fonksiyonu tanıttık. Bu üstel fonksiyonun, birinci mertebe lineer dinamik denklem içeren başlangıç değer problemini sağladığı gösterildi. Varlık ve teklik teoremleri sunuldu. Sonrasında periyodik zaman skalaları için Lyapunov dönüşümleri, kararlılık kriteri ve Floquet teoremi verildi. Zaman Skalalarında Dinamik Sistemlerle ilgili olarak (Bohner and Peterson 2001), (Dacunha 2005), (Ahlbrandt and Ridenhour 2003) çalışmalarını derlemeye çalıştık.

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CHAPTER 1

TIME SCALES

1.1. Basic Definitions

A time scale is an arbitrary nonempty closed subset of the real numbers. It is denoted by the symbol \mathbb{T} . It was first introduced by Stefan Hilger in his Ph.D thesis in 1988 in order to unified continuous and discrete analysis.

Definition 1. Let \mathbb{T} be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are both right-scattered and left-scattered are isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are both right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We need below the set \mathbb{T}^k is derived from the time scale \mathbb{T} as follows : If \mathbb{T} has a left scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. In summary

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

Finally, if $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f^\sigma(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}.$$

Example 2. Let us briefly consider the two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) = 0 \quad \text{for all } t \in \mathbb{T};$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t + 1, t + 2, \dots\} = t + 1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ is the case is

$$\mu(t) = 1 \quad \text{for all } t \in \mathbb{T}.$$

Now we consider a function $f : \mathbb{T} \rightarrow \mathbb{R}$ and define delta derivative of f at a point $t \in \mathbb{T}^k$.

1.2. Differentiation

Definition 3. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define delta derivative $f^\Delta(t)$ to be the number with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t .

Theorem 4. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following properties.

(i) If f is differentiable at t , then f is continuous at t ;

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exist as a finite number, in this case

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s};$$

(iv) If f is differentiable at t , then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Example 5. We consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$ then Theorem 4 (iii) yields that $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e, iff f is differentiable (in the ordinary sense) at t . In this case we then have

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 4 (iii);

(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 4 (ii) yields that $f : \mathbb{Z} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t)$$

where Δ is the usual forward difference operator defined by the last equation above.

Theorem 6. Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then :

(i) The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t);$$

(ii) For any constant $\alpha \in \mathbb{R}$, function $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t);$$

(iii) The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t));$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))};$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

We now define functions that are integrable an arbitrary time scales \mathbb{T} .

1.3. Integration

Definition 7. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in \mathbb{T} and its left-sided limits exist (finite) at all left-dense points in \mathbb{T} .

Definition 8. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called right-dense continuous (denote rd-continuous) provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

It follows naturally that the set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ whose first n delta derivatives exist and are rd-continuous on \mathbb{T} is denoted by

$$C_{rd}^n = C_{rd}^n(\mathbb{T}) = C_{rd}^n(\mathbb{T}, \mathbb{R}).$$

Definition 9. The functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is called piecewise right-dense continuous (denoted prd-continuous) provided it is piecewise continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . The set of prd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by

$$C_{prd} = C_{prd}(\mathbb{T}) = C_{prd}(\mathbb{T}, \mathbb{R}).$$

It follows naturally that the set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ whose first n delta derivatives exist and rd-continuous on \mathbb{T} is denoted

$$C_{prd}^n = C_{prd}^n(\mathbb{T}) = C_{prd}^n(\mathbb{T}, \mathbb{R}).$$

Some results concerning rd-continuous and regulated functions are contained in the following theorem.

Theorem 10. For a function $f : \mathbb{T} \rightarrow \mathbb{R}$;

- (i) If f is continuous, then f is rd-continuous,
- (ii) If f is rd-continuous, then f is regulated,
- (iii) The forward jump operator σ is rd-continuous,
- (iv) If f is regulated or rd-continuous, then so is f^σ ,
- (v) Assume f is continuous. If $g : \mathbb{T} \rightarrow \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ is also regulated or rd-continuous, respectively.

Definition 11. A continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is pre-differentiable with (region of differentiation) D , provided $D \subset \mathbb{T}^k, \mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of \mathbb{T} , and f is differentiable at each $t \in D$.

Theorem 12. Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^\Delta(t) = f(t)$$

holds for all $t \in D$.

Definition 13. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a regulated function. Any function F as in Theorem 12 is called a pre-antiderivative of f .

We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C$$

where C is an arbitrary constant and F is a pre-antiderivative of f .

We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an anti-derivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds for all } t \in \mathbb{T}^k.$$

1.4. Hilger's Complex Plane

Definition 14. For $h > 0$, we define the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle as

$$\begin{aligned} C_h &:= \{z \in \mathbb{C} : z \neq -\frac{1}{h}\} \\ R_h &:= \{z \in \mathbb{R} : z > -\frac{1}{h}\} \\ A_h &:= \{z \in \mathbb{R} : z < -\frac{1}{h}\} \\ I_h &:= \{z \in \mathbb{C} : |z + \frac{1}{h}| = \frac{1}{h}\} \end{aligned}$$

respectively. For $h = 0$, let $C_0 := \mathbb{C}$, $R_0 := \mathbb{R}$, $A_0 := \emptyset$, and $I_0 := i\mathbb{R}$.

Let $z \in C_h$. The Hilger real part of z is defined by

$$Re_h(z) := \frac{|zh + 1| - 1}{h}$$

and the Hilger imaginary part of z is defined by

$$Im_h(z) := \frac{Arg(zh + 1)}{h}$$

where $Arg(z)$ denotes the principal argument of z ($-\pi < Arg(z) \leq \pi$).

Definition 15. Let $\frac{-\pi}{h} < \omega \leq \frac{\pi}{h}$. We define the Hilger purely imaginary number $\mathring{i}\omega$ as

$$\mathring{i}\omega = \frac{e^{i\omega h} - 1}{h}$$

Theorem 16. For $z \in \mathbb{C}_h$ we have

$$z = Re_h z \oplus \mathring{i}Im_h z.$$

Proof. Let $z \in \mathbb{C}_h$. Then

$$\begin{aligned}
\operatorname{Re}_h z \oplus i \operatorname{Im}_h z &= \frac{|zh+1|-1}{h} \oplus i \frac{\operatorname{Arg}(zh+1)}{h} \\
&= \frac{|zh+1|-1}{h} \\
&+ \frac{\exp(i \operatorname{Arg}(zh+1))-1}{h} + \frac{|zh+1|-1}{h} \frac{\exp(i \operatorname{Arg}(zh+1))-1}{h} h \\
&= \frac{1}{h} \{ |zh+1|-1 + \exp(i \operatorname{Arg}(zh+1))-1 \} \\
&+ [|zh+1|-1][\exp(i \operatorname{Arg}(zh+1))-1] \\
&= \frac{1}{h} |zh+1| \exp(i \operatorname{Arg}(zh+1)) - 1 \\
&= \frac{(zh+1)-1}{h} \\
&= z
\end{aligned}$$

□

Definition 17. The function $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive if

$$1 + \mu(t)p(t) \neq 0 \quad t \in \mathbb{T}^k.$$

From this point, all regressive and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted as

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Circle plus addition \oplus is defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all } t \in \mathbb{T}^k, p, q \in \mathcal{R}.$$

Theorem 18. $(\mathcal{R}(\mathbb{T}, \mathbb{R}), \oplus)$ is an Abelian group.

The set of all positively regressive elements of \mathcal{R} defined by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \quad \text{for all } t \in \mathbb{T}^k\}$$

is a subgroup of \mathcal{R} .

Definition 19. The function $\ominus p$ is defined by

$$(\ominus p) = -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all } t \in \mathbb{T}^k, p \in \mathcal{R}.$$

It follows that if $p, q \in \mathcal{R}$, then $\ominus p, \ominus q, p \oplus q, p \ominus q \in \mathcal{R}$.

1.5. The Time Scale Exponential Function

We use a cylinder transformation, defined below, to define a generalized time scale exponential function for an arbitrary time scale \mathbb{T} .

Definition 20. For $h > 0$. Let Z_h be the strip

$$Z_h = \{z \in \mathbb{C} : \frac{-\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h}\}$$

and for $h = 0$, let $Z_0 := \mathbb{C}$.

Definition 21. For $h > 0$ the cylinder transformation $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + hz)$$

where Log is the principle logarithm function. When $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$. The inverse cylinder transformation $\xi_h^{-1} : \mathbb{Z}_h \rightarrow \mathbb{C}_h$ is

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}.$$

By using this cylinder transformation we now construct the generalized exponential function.

Definition 22. If $p \in \mathcal{R}$, we define the generalized time scale exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right), \quad \text{for all } s, t \in \mathbb{T}.$$

Example 23. Consider the time scale

$$\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$$

We have $\sigma(n^2) = (n + 1)^2$ for $n \in \mathbb{N}_0$ and

$$\mu(n^2) = \sigma(n^2) - n^2 = (n + 1)^2 - n^2 = 2n + 1.$$

Hence $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$ for $t \in \mathbb{T}$.

For this time scale, we claim that

$$e_1(t, 0) = 2^{\sqrt{t}}(\sqrt{t})! \quad \text{for } t \in \mathbb{T}.$$

Let y be defined by the this equation. It is clear that $y(0) = 1$ and for $t \in \mathbb{T}$.

$$\begin{aligned}
y(\sigma(t)) &= 2\sqrt{\sigma(t)}(\sqrt{\sigma(t)})! \\
&= 2^{1+\sqrt{t}}(1+\sqrt{t})! \\
&= 2 \cdot 2^{\sqrt{t}}(1+\sqrt{t})(\sqrt{t})! \\
&= 2(1+\sqrt{t})y(t) \\
&= (1+\mu(t))y(t) \\
&= y(t) + \mu(t)y(t)
\end{aligned}$$

so that

$$y^\Delta(t) = y(t).$$

Theorem 24. For all $p, q \in \mathcal{R}$, the generalized exponential function is satisfied the following properties :

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$,

(ii) $e_p(\sigma(t), s) = [1 + \mu(t)p(t)]e_p(t, s)$,

(iii) $\frac{1}{e_p(s, t)} = e_{\ominus p}(t, s)$,

(iv) $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$,

(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,

(vi) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$,

(vii) $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$,

(viii) If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$,

(ix) If $1 + \mu(t)p(t) < 0$ for some $t \in \mathbb{T}$, then $e_p(t, t_0)e_p(\sigma(t), t_0) < 0$,

(x) If $\mathbb{T} = \mathbb{R}$, then $e_p(t, s) = e^{\int_s^t p(\tau) d\tau}$. If p is constant then

$$e_p(t, s) = e^{p(t-s)},$$

(xi) If $\mathbb{T} = \mathbb{Z}$, then $e_p(t, s) = \prod_{\tau=s}^{t-1} (1 + p(\tau))$. If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$ and p is constant, then $e_p(t, s) = (1 + hp)^{\frac{t-s}{h}}$.

Proof. We will give only the proofs of (x) and (xi). For the others proofs see (Bohner and Peterson 2001). Since $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$ and

$$\begin{aligned} e_p(t, s) &= \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= \exp\left(\int_s^t \left[\frac{1}{\mu(\tau)} \text{Log}(1 + \mu(\tau)p(\tau))\right]\Delta\tau\right) \end{aligned}$$

By the L'Hospital rule

$$\begin{aligned} e_p(t, s) &= \exp\left(\int_s^t \left[\frac{p(\tau)}{1 + \mu(\tau)p(\tau)}\right]\Delta\tau\right) \\ &= e^{\int_s^t p(\tau)d\tau} \end{aligned}$$

While $\mathbb{T} = \mathbb{Z}$, $\mu(t) = 1$ and $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, ($a < b$)

$$\begin{aligned} e_p(t, s) &= \exp\left(\int_s^t \left[\frac{1}{\mu(\tau)} \text{Log}(1 + \mu(\tau)p(\tau))\right]\Delta\tau\right) \\ &= \exp\left(\int_s^t \text{Log}(1 + p(\tau))\Delta\tau\right) \\ &= \exp\left(\sum_{\tau=s}^{t-1} \text{Log}(1 + p(\tau))\right) \\ &= \exp\left(\text{Log}\Pi_{\tau=s}^{t-1}(1 + p(\tau))\right) \\ &= \Pi_{\tau=s}^{t-1}(1 + p(\tau)) \end{aligned}$$

Second part of proof is similar. □

CHAPTER 2

DYNAMICAL SYSTEMS

In this chapter we will give some key well known results for linear systems. The general form for a first order linear system is

$$\dot{x} = A(t)x + g(t) \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix function of t , and $g(t)$ is a vector function of t . We assume that $A(t)$ and $g(t)$ are continuous functions of t on the closed interval I , $a \leq t \leq b$. If we denote the right-hand side of (2.1) by $f(x, t)$, then $f(x, t)$ is a continuous function of x and t for all x and $t \in I$. Since $A(t)$ is continuous function of t on the closed interval I , there exist a constant M such that $|A(t)| \leq M$ for $t \in I$. Indeed

$$\begin{aligned} |f(x, t) - f(y, t)| &= |A(t)x + g(t) - A(t)y - g(t)| \\ &\leq |A(t)||x - y| \\ &\leq M|x - y|. \end{aligned}$$

It means $f(x, t)$ satisfies the Lipschitz condition. If $g(t) = 0$ then (2.1) is called a first order homogeneous linear systems. Consider

$$\dot{x} = A(t)x \tag{2.2}$$

is first order homogeneous linear system. Let $x_1(t), \dots, x_n(t)$ be n solutions of (2.2) on an interval $[a, b]$ and put

$$\Phi(t) = [x_1(t), \dots, x_n(t)]$$

where $x(t)$ is an $n \times n$ matrix solution of

$$\dot{\Phi}(t) = A(t)\Phi(t).$$

If $x_1(t), \dots, x_n(t)$ are also linearly independent then $\Phi(t)$ is a fundamental matrix and if $\Phi(t_0) = I$, the unit matrix, then $\Phi(t)$ is called the principal fundamental matrix. Further $W(t) = \det\Phi(t)$ is called the Wronskian. Since the elements of Φ are differentiable, we can compute Φ' .

$$\begin{aligned} \Phi'(t) &= [x'_1(t), x'_2(t), \dots, x'_n(t)] \\ &= A(t)[x_1(t), x_2(t), \dots, x_n(t)] \\ &= A(t)\Phi(t). \end{aligned}$$

That is, Φ satisfies

$$\Phi'(t) = A(t)\Phi(t). \quad (2.3)$$

Theorem 25. (Abel's Formula) *Let $A(t)$ be an $n \times n$ matrix of continuous functions on $I = [a, b]$ and let $\Phi(t)$ be a matrix of differentiable functions such that*

$$\Phi'(t) = A(t)\Phi(t).$$

Then for $t, t_0 \in I$,

$$\det \Phi(t) = \det \Phi(t_0) \exp\left(\int_{t_0}^t \text{tr} A(s) ds\right)$$

where $\text{tr} A(s)$ is the trace of $A(s)$ (the sum of the element of its principal diagonal).

2.1. The Constant Coefficient Matrix

The linear autonomous systems do not explicitly on t . So the coefficient matrix of system A is a constant matrix.

Theorem 26. *Let A be a constant matrix. A fundamental matrix Φ for*

$$\dot{x} = Ax \quad (2.4)$$

is given by

$$\Phi = e^{At} \quad (2.5)$$

where $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$.

Theorem 27. *If A is a constant matrix, λ_i an eigenvalue of A , and v_i a corresponding eigenvector, then $y = e^{\lambda_i t} v_i$ is a solution of (2.4).*

Since A has n eigenvalues, we can find n such solutions, and it would seem then that we have found the columns for a fundamental matrix. The difficulty, however, is that the eigenvalues are not necessarily distinct and the eigenvectors corresponding to a repeated eigenvalue may not be linearly independent. (Eigenvectors corresponding to distinct eigenvalues are always linearly independent.) If this occurs, we have not found n linearly independent column vectors to make a fundamental matrix. However, it is the

case that if all of the eigenvalues of A are distinct, then A is similar to a diagonal matrix, so the n solutions obtained actually are linearly independent, and a fundamental matrix has been found.

Theorem 28. *Let A be a constant $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and v_1, \dots, v_n be corresponding eigenvectors. Then a fundamental matrix for (2.4) is given by*

$$\Phi(t) = [e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2, \dots, e^{\lambda_n t} v_n].$$

If some of the eigenvalues λ_i turn out to be complex numbers, then the corresponding eigenvectors, v_i , will contain complex entries, but $e^{\lambda_i t} v_i$ will still be a solution.

Theorem 29. *If $\varphi(t)$ is a solution of (2.4) where A is a constant matrix with real-valued entries, then the real part of $\varphi(t)$ (written $\text{Re}\varphi(t)$) and the imaginary part of $\varphi(t)$ (written $\text{Im}\varphi(t)$) are both solutions of (2.4). If λ_i is a complex eigenvalue then $\bar{\lambda}_i$ is an eigenvalue, too.*

2.2. Structure of n -Dimensional Nonhomogeneous Linear Systems

The general nonhomogeneous linear system is

$$\dot{x} = A(t)x + f(t) \tag{2.6}$$

where $f(t)$ is a continuous column vector, $A(t)$ is an $n \times n$ continuous matrix. For notational purposes, let $L[x]$ denote $y' - Ay$. Then, as noted before, (2.6) can be written as

$$L[x] = f. \tag{2.7}$$

Theorem 30. *The solution of the system with initial conditions*

$$\begin{aligned} L[x] &= f \\ x(t_0) &= x_0 \end{aligned} \tag{2.8}$$

is given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t) \int_{t_0}^t \Phi^{-1}(s)f(s)ds \tag{2.9}$$

where $\Phi(t)$ is any fundamental solution matrix of the corresponding homogeneous system $\dot{\Phi} = A(t)\Phi$.

2.3. Stability of Time Solutions: Lyapunov Stability

Consider the linear system

$$x'(t) = Ax \tag{2.10}$$

where A is an $n \times n$ constant matrix and x is a vector in \mathbb{R}^n . Equation (2.10) always has the trivial solution, the function $x(t) = 0$, and this solution will play the role of “present state” in the intuitive description above. The trivial solution is said to be stable if for every $\varepsilon > 0$ there is a δ such that if $x(t)$ is any solution of (2.10) with $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t > 0$. We are using norm, $\| \ \|$, to measure how close a solution is to the trivial solution. Think of the trivial solution as the present state of the system and $x(t)$ as a solution that represents a deviation from the present state. If the trivial solution is stable $x(t)$ will remain arbitrarily close (this is the ε) to the present state (the trivial solution) for all future time if the initial condition $x(0)$ is sufficiently close (this is δ) to zero. The trivial solution is said to be unstable if it is not stable. The trivial solution of (2.10) said to be asymptotically stable if a) it is stable, and b) there is an $\varepsilon > 0$ such that if $\|x(0)\| < \varepsilon$, then $\lim_{t \rightarrow \infty} \|x(t)\| = 0$.

Theorem 31.

- (i) *The trivial solution of (2.10) is asymptotically stable if and only if all of the eigenvalues of A have negative real parts.*
- (ii) *If one eigenvalue of A has a positive real part, then the trivial solution of (2.10) is unstable.*
- (iii) *If the eigenvalues of A with zero real parts are simple and all other eigenvalues have negative real parts, then the trivial solution of (2.10) is stable.*

Definition 32. *Let $x^k(t)$ be a given real or complex solution vector of the n -dimensional system, not necessarily autonomous, $\dot{x} = X(x, t)$ in component form*

$$\begin{aligned}
\dot{x}_1 &= X(x_1, x_2, \dots, x_n, t) \\
\dot{x}_2 &= X(x_1, x_2, \dots, x_n, t) \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_n &= X(x_1, x_2, \dots, x_n, t)
\end{aligned}$$

then

- (i)** $x^k(t)$ is Lyapunov stable for $t \geq t_0$ if and only if, to each value of $\epsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ϵ and t_0) such that

$$\|x(t_0) - x^k(t_0)\| < \delta \Rightarrow \|x(t) - x^k(t)\| < \epsilon \quad (2.11)$$

for all $t \geq t_0$, where $x(t)$ represents any other neighbour is solution.

- (ii)** If the given system is autonomous, the reference to t_0 in (i) may be disregarded, the solution $x^k(t)$ is either Lyapunov stable, or not, for all t_0 .

- (iii)** Otherwise the solution $x^k(t)$ is unstable in the sense of Lyapunov.

In other words, (2.11) says that no matter how small is the permitted deviation, measured by ϵ , there still exists a non-zero tolerance, measured by δ , in the initial conditions when the system is activated, allowing it to run satisfactorily.

Definition 33. If the solution is stable for $t \geq t_0$ and the δ of Definition 32 is independent of t_0 , the solution is uniformly stable on $t \geq t_0$.

It is clear that any solutions of an autonomous system are uniformly stable, since the system is invariant with respect to time translation.

Definition 34. Let x^* be a stable (or uniformly stable) solution for $t \geq t_0$. If additionally there exists $\delta > 0$ such that

$$\|x(t_0) - x^*(t_0)\| \leq \delta \Rightarrow \lim_{t \rightarrow \infty} \|x(t) - x^*(t)\| = 0 \quad (2.12)$$

then the solution is said to be asymptotically stable (or uniformly and asymptotically stable).

The most general linear system is the nonautonomous and nonhomogeneous equation in n variables given by

$$\dot{x} = A(t)x + f(t) \quad (2.13)$$

where $A(t)$ is an $n \times n$ matrix. Let $x(t)$ represent any other solution and define $\xi(t)$ by

$$\xi(t) = x(t) - x^*(t). \quad (2.14)$$

Then $\xi(t)$ tracks the difference between the test solution and a solution having a different initial value at time t_0 . The initial condition for ξ is

$$\xi(t_0) = x(t_0) - x^*(t_0) \quad (2.15)$$

also, ξ satisfies the homogeneous equation derived from

$$\dot{\xi} = A(t)\xi. \quad (2.16)$$

By comparison of (2.14), (2.15), and (2.16) with Definition 32, it can be seen that the stability property of $x^*(t)$ is the same as the stability of the zero solution of (2.16). $\xi(t)$ is called perturbation of the solution $x^*(t)$. Since this new formulation of the problem is independent of the solution of (2.13) initially chosen, we can make the following statement.

Theorem 35. *All solutions of the regular linear system $\dot{x} = A(t)x + f(t)$ have the same Lyapunov stability property (unstable, stable, uniformly stable, asymptotically stable, uniformly and asymptotically stable). This is the same as that of the zero (or any other) solution of the homogeneous equation $\dot{\xi} = A(t)\xi$.*

2.4. Equations With Periodic Coefficients and Floquet Theory

Let

$$\dot{x} = A(t)x \quad (2.17)$$

be a n -dimensional first order system, where $A(t)$ is periodic with minimal period T ; that is, T is the smallest positive number for which

$$A(t + T) = A(t), \quad -\infty < t < \infty \quad (2.18)$$

$A(t)$, of course, also has periods $2T, 3T, \dots$). The solutions are not necessarily periodic. We shall assume that $A(t)$ is continuous on \mathbb{R} . In particular, all solutions of (2.18) can be obtained in the form

$$x(t) = \Phi(t)c$$

where Φ is a fundamental matrix and c is a constant vector. Knowledge of properties of a fundamental matrix then yields properties of solutions. A basic theorem in the theorem of ordinary differential equations, known as Floquet theorem, gives an important representation of a fundamental matrix when the coefficients are periodic. Floquet theorem contains the fundamental results for equations with periodic coefficients, that the fundamental matrix of (2.17) can be written as the product of a T -periodic matrix and a (generally) non-periodic matrix.

Theorem 36 (Floquet Theorem). *Consider the equation (2.17) with $A(t)$ a continuous T -periodic $n \times n$ matrix. Each fundamental matrix $\Phi(t)$ of equation (2.17) can be written as the product of two $n \times n$ matrices $\Phi(t) = p(t)e^{Bt}$ with $p(t)$ T -periodic and B a constant $n \times n$ -matrix.*

Proof. The fundamental matrix $\Phi(t)$ is composed of n independent solutions; $\Phi(t + T)$ is also a fundamental matrix. To show this, put $\tau = t + T$, then

$$\begin{aligned} \frac{dx}{d\tau} &= A(\tau - T)x \\ &= A(\tau)x \end{aligned}$$

so $\Phi(\tau)$ is also fundamental matrix. The fundamental matrices $\Phi(t)$ and $\Phi(\tau) = \Phi(t + T)$ are linearly independent, which means there exist a nonsingular $n \times n$ matrix C such that

$$\Phi(t + T) = \Phi(t)C.$$

There exists a constant matrix B such that

$$C = e^{BT}.$$

We shall prove that $\Phi(t)e^{-Bt}$ is T -periodic. Put $\Phi(t)e^{-Bt} = p(t)$, then

$$\begin{aligned} p(t + T) &= \Phi(t + T)e^{-B(t+T)} \\ &= \Phi(t)Ce^{-BT}e^{-Bt} \\ &= \Phi(t)e^{-Bt} \\ &= p(t) \end{aligned}$$

□

Remark 37. The matrix C which has been introduced is called the monodromy-matrix of equation (2.17). The eigenvalues of ρ_i of C are called characteristic multipliers. Each complex number λ such that

$$\rho = e^{\lambda T}$$

is called a characteristic exponent (or Floquet exponent). The imaginary part of the characteristic exponents are not determined uniquely. We can add $2\pi i/T$ to them. The characteristic multipliers determined uniquely.

Remark 38. The Floquet theorem implies that the solutions of the equation (2.17) consist of a product of polynomials in t multiplied with $e^{\lambda t}$ and T -periodic terms.

$$\dot{x} = A(t)x$$

can be transformed by $x = p(t)y$ so that

$$\dot{p}(t)y + p(t)\dot{y} = A(t)p(t)y$$

or

$$\dot{y} = p^{-1}(Ap - \dot{p})y.$$

The differentiation of $p(t) = \Phi(t)e^{-Bt}$ produces

$$\begin{aligned} \dot{p} &= \dot{\Phi}e^{-Bt} + \Phi e^{-Bt}(-B) \\ &= Ap - pB \end{aligned}$$

so we find $\dot{y} = By$. In other words, the transformation $x = p(t)y$ carries equation (2.17) over into an equation with constant coefficients, the solutions which are vector-polynomials in t multiplied with $e^{\lambda t}$. This possibility of reduction of the linear part of the system to the case of constant coefficients will play a part in the theory of stability.

2.5. Discrete Dynamical Systems

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty five years of the twentieth century, following the publication of the seminal paper “Period Three Implies Chaos”, by J. Yorke and Y. Li

in 1975. In 1987 R. Devancy published “An Introduction to Chaotic Dynamical Systems” the first book on the subject. Applications of difference equations also experienced enormous growth in many areas. In this section our goal is to present an overview of the various facts of stability theory for autonomous systems of difference equations. This section covers many of the fundamental stability results for linear systems.

2.5.1. Linear Difference Equations

Definition 39. *An equation of the form*

$$x_{n+1} = ax_n + b, \quad n = 0, 1, \dots$$

where $a \in R \setminus 0$ and $b \in R$ is called *first order linear difference equations (LDE) with constant coefficients*.

$$x_{n+1} = a_n x_n + b_n, \quad n = 0, 1, \dots \quad (2.19)$$

is called *LDE with variable coefficients*.

$$x_{n+1} = Ax_n + b_n, \quad x_0 = d, \quad n = 0, 1, \dots \quad (2.20)$$

where $k \in 1, 2, \dots$, $d \in R^k$, A is a real $k \times k$ matrix and $b_n \in R^k$ for $n = 0, 1, \dots$ the IVP (2.20) has the unique solution given by

$$x_n = A^n d + \sum_{i=0}^{n-1} A^{n-i-1} b_i, \quad n = 0, 1, \dots$$

In particular, if $b_n = b$, a is a constant vector, then we obtain

$$x_n = A^n d + \sum_{i=0}^{n-1} A^{n-i-1} b, \quad n = 0, 1, \dots$$

2.6. Initial Value Problems for Linear Systems

We consider systems of the form

$$u(t+1) = A(t)u(t) + f(t) \quad (2.21)$$

where

$$u(t) = \begin{bmatrix} u_1(t) \\ \cdot \\ \cdot \\ u_n(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & \cdot & \cdot & \cdot & a_{1n}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}(t) & \cdot & \cdot & \cdot & a_{nn}(t) \end{bmatrix}$$

$$f(t) = \begin{bmatrix} f_1(t) \\ \cdot \\ \cdot \\ f_n(t) \end{bmatrix}$$

for $t = a, a + 1, a + 2, \dots$

Theorem 40. For each $t_0 \in a, a + 1, \dots$ and each n -vector u_0 , equation (2.21) has a unique solution $u(t)$ defined for $t = t_0, t_0 + 1, \dots$, so that $u(t_0) = u_0$. If A is constant matrix and $f(t) = 0$, then the solution $u(t)$ of

$$u(t + 1) = Au(t) \tag{2.22}$$

satisfying the initial condition $u(0) = u_0$, is

$$\begin{aligned} u(t) &= A^t u_0 \quad (t = 0, 1, 2, \dots) \\ &= c_1 \lambda_1^t u^1 + \dots + c_k \lambda_k^t u^k. \end{aligned}$$

Hence the solutions of equation (2.22) can be found by calculating powers of A .

Example 41. Solve $u(t + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} u(t)$, $u_0 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

-2 and -1 are eigenvalues and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are corresponding

eigenvectors.

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} -u_1 - u_2 \\ 2u_1 + u_2 \end{bmatrix} \quad \text{with initial condition} \\ u(t) &= -(u_1 + u_2)(-2)^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (2u_1 + u_2)(-1)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Theorem 42. *The solution of*

$$u(t+1) = Au(t) + f(t)$$

satisfying the initial condition $u(0) = u_0$ is

$$u(t) = A^t u_0 + \sum_{i=0}^{t-1} A^{t-i-1} f(i) \quad (2.23)$$

Proof. By Theorem 40, it is enough to show that (2.23) satisfies the initial value problem.

First we have

$$\sum_{i=0}^{t-1} A^{-i-1} f(i) = 0$$

by the usual condition, so $u(0) = u_0$. For $t \geq 1$

$$\begin{aligned} u(t+1) &= A^{t+1} u_0 + \sum_{i=0}^t A^{t-i} f(i) \\ &= A^{t+1} u_0 + \sum_{i=0}^{t-1} A^{t-i} f(i) + f(t) \\ &= A \left[A^t u_0 + \sum_{i=0}^{t-1} A^{t-i-1} f(i) \right] + f(t) \\ &= Au(t) + f(t) \end{aligned}$$

□

2.7. Stability of Linear Systems

Theorem 43. *Let A be an $n \times n$ matrix with $r(A) < 1$; where $r(A)$ is the spectral radius of A , $r(A) = \max |\lambda_i|$; then every solution $u(t)$ of*

$$u(t+1) = Au(t)$$

satisfies $\lim_{t \rightarrow \infty} u(t) = 0$. Furthermore, if $r(A) < \delta < 1$, then there is a constant $c > 0$ so that $|u(t)| \leq c\delta^t |u_0|$ for $t \geq 0$ and every solution of u of equation (2.22).

When all solutions of the system go to the origin as t goes to infinity, the origin is said to be “asymptotically stable”.

2.8. Discrete Hamiltonian Systems

Consider the discrete Hamiltonian system

$$\begin{aligned}\Delta x(t) &= H_u(t, x(t+1), u(t)) \\ \Delta u(t) &= -H_x(t, x(t+1), u(t))\end{aligned}\tag{2.24}$$

where $t \in \mathbb{Z}$; $x, u \in \mathbb{R}^n$, $H(t, x, u)$ is the corresponding real Hamiltonian function and has continuous derivatives in x, u ; $H_x = (H_{x_1}, H_{x_2}, \dots, H_{x_n})^T$, H_x is the partial derivative of H in x_i , Δ denotes the forward difference.

If the Hamiltonian function H is of the quadratic form

$$H(t, x, u) = \frac{1}{2}(x^T, u^T)S(t) \begin{pmatrix} x \\ u \end{pmatrix}$$

where

$$S(t) = \begin{pmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}$$

is a $2n \times 2n$ symmetric matrix, then (2.24) is the discrete linear Hamiltonian system.

2.9. Discrete Hamiltonian Systems with Periodic Coefficients

Consider the following system

$$\begin{aligned}\Delta x(t) &= a(t)x(t+1) + b(t)u(t) \\ \Delta u(t) &= -c(t)x(t+1) - a(t)u(t)\end{aligned}\tag{2.25}$$

a, b, c are real-valued functions and $1 - a(t) \neq 0$, for $T \geq 2$, $a(t+T) = a(t)$, $b(t+T) = b(t)$, $c(t+T) = c(t) \forall t \in \mathbb{Z}$.

Let $\Phi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ then we may write (2.25) as

$$\Phi(t+1) = M(t)\Phi(t) \quad (2.26)$$

which

$$M(t) = \begin{bmatrix} \frac{1}{1-a(t)} & \frac{b(t)}{1-a(t)} \\ \frac{-c(t)}{1-a(t)} & 1 - a(t) - \frac{b(t)c(t)}{1-a(t)} \end{bmatrix}. \quad (2.27)$$

M has the following properties :

i) $\det M(t) = 1, \quad \forall t \in \mathbb{Z};$

ii) $M^T(t)JM(t) = J$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

If a matrix M satisfies $M^TJM = J$ then we say M is “symplectic” matrix and the discrete Hamiltonian system (2.26) is of symplectic structure. The discrete linear Hamiltonian system

$$\Delta x(t) = A(t)x(t+1) + B(t)u(t)$$

$$\Delta u(t) = C(t)x(t+1) - A^T(t)u(t)$$

is expressed as

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = M(t) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

where $E(t) = (I_n - A(t))^{-1}$ and

$$M(t) = \begin{bmatrix} E(t) & E(t)B(t) \\ C(t)E(t) & I_n - A^T(t) + C(t)E(t)B(t) \end{bmatrix}.$$

Since the transition matrix $M(t)$ is symplectic, the discrete linear Hamiltonian system is of symplectic structure.

Definition 44. Suppose $\Phi_1(t) = \begin{bmatrix} x_1(t) \\ u_1(t) \end{bmatrix}$ and $\Phi_2(t) = \begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix}$ be solutions of (2.26). The Wronskian of Φ_1 and Φ_2 is defined as follows.

$$W_t(\Phi_1, \Phi_2) = \begin{vmatrix} x_1(t) & x_2(t) \\ u_1(t) & u_2(t) \end{vmatrix} = \Phi_1^T(t) J \Phi_2(t).$$

Proposition 45. The Wronskian of (2.25) is constant.

Proof.

$$\begin{aligned} \Delta W_t(\Phi_1, \Phi_2) &= \Delta[x_1(t)u_2(t) - x_2(t)u_1(t)] \\ &= x_1(t+1)\Delta u_2(t) + u_2(t)\Delta x_1(t) - x_2(t+1)\Delta u_1(t) - u_1(t)\Delta x_2(t) \\ &= x_1(t)[-c(t)x_2(t+1) - a(t)u_2(t)] + u_2(t)[a(t)x_1(t) + b(t)u_1(t+1)] \\ &\quad - x_2(t+1)[-c(t)x_1(t+1) - a(t)u_1(t)] - u_1(t)[a(t)x_2(t+1) + b(t)u_2(t)] \\ &= 0 \end{aligned}$$

then $W_t(\Phi_1, \Phi_2)$ is constant. □

Proposition 46. Suppose Φ_1 and Φ_2 be two solutions of (2.26). Φ_1 and Φ_2 are linearly independent if and only if $W_t(\Phi_1, \Phi_2) \neq 0$ and the linear combination of Φ_1 and Φ_2 is also the solution of (2.26).

Proposition 47. There are two linearly independent solutions of system (2.25) and any solution can be written as linear combination of these solutions.

Definition 48. If all the solutions (x, u) of (2.25) on \mathbb{Z} are bounded then we say that this system is stable. If there is at least one non trivial and bounded solution of (2.25) then we say that this system is conditionally stable. If all nontrivial solutions of (2.25) are unbounded then we say that, this system is unstable.

2.9.1. Floquet Theory

Let

$$\begin{aligned} \varphi(t) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \varphi^\sigma(t) = \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ H(t) &= \begin{bmatrix} c(t) & a(t) \\ a(t) & b(t) \end{bmatrix} \end{aligned}$$

where $a(t)$, $b(t)$, and $c(t)$ are periodic. Now consider the system

$$\Delta\varphi(t) = JH(t)\varphi^\sigma(t), \quad t \in \mathbb{Z}. \quad (2.28)$$

Since a, b, c are periodic then we can get $H(t+T) = H(t)$. For a $\rho \in \mathbb{C}$, we are looking the nontrivial solution $\varphi(t)$ such that,

$$\varphi(t+T) = \rho\varphi(t), \quad t \in \mathbb{Z}. \quad (2.29)$$

Let $\varphi_1(t) = \begin{bmatrix} x_1(t) \\ u_1(t) \end{bmatrix}$ and $\varphi_2(t) = \begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix}$ be solutions of (2.28) with the following initial conditions,

$$\varphi_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \varphi_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $W_t(\varphi_1, \varphi_2) = W_0(\varphi_1, \varphi_2) = \begin{vmatrix} x_1(0) & x_2(0) \\ u_1(0) & u_2(0) \end{vmatrix} = 1.1 - 0.0 = 1 \neq 0$ then φ_1 and φ_2 are linearly independent solutions of (2.28) and $\varphi(t) = c_1\varphi_1 + c_2\varphi_2$.

Let $\varphi(t) = \Phi(t)c$ where

$$\Phi(t) = [\varphi_1(t) \quad \varphi_2(t)] = \begin{bmatrix} x_1(t) & x_2(t) \\ u_1(t) & u_2(t) \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (2.30)$$

For matrix function $\Phi(t)$ the equations

$$\Delta\Phi(t) = JH(t)\Phi^\sigma(t), \quad t \in \mathbb{Z} \quad (2.31)$$

$$\Phi(0) = I \quad (2.32)$$

are satisfied. Here $\Phi^\sigma(t) = \begin{bmatrix} x_1(t+1) & x_2(t+1) \\ u_1(t) & u_2(t) \end{bmatrix}$.

Now substitute (2.30) into (2.29), then we obtain

$$\Phi(t+T)c = \rho\Phi(t)c. \quad (2.33)$$

Now we will show that

$$\Phi(t+T) = \Phi(t)\Phi(T). \quad (2.34)$$

Since $H(t+T) = H(t)$, the left hand side of (2.34) satisfies (2.31). On the other hand the right hand side of (2.34) satisfies (2.31) too. Both sides of (2.34) is equal for $t = 0$.

Since the uniqueness of the solution, (2.34) is verified for every $t \in \mathbb{Z}$. If we write (2.34) into the left hand side of (2.33)

$$\Phi(t)\Phi(T)c = \rho\Phi(t)c. \quad (2.35)$$

Since $\det\Phi(t) = W_t(\varphi_1, \varphi_2) = 1 \neq 0$, $\Phi(t)$ is invertible for every $t \in \mathbb{Z}$. Multiplying both sides with $\Phi^{-1}(t)$,

$$\Phi(T)c = \rho c. \quad (2.36)$$

The inverse of these operations are also true. If the equation (2.36) is verified for a complex number ρ and the vector c , the function $\varphi(t) = \Phi(t)c$ is solution which has property (2.29). We must find the vector $c \neq 0$ which satisfies (2.36). ρ is eigenvalue of $\varphi(x)$ and c is corresponding eigenvector. From the linear algebra, for nonzero solution of (2.36),

$$\det[\Phi(T) - \rho I] = 0 \quad (2.37)$$

$$\begin{vmatrix} x_1(T) - \rho & x_2(T) \\ u_1(T) & u_2(T) - \rho \end{vmatrix} = 0$$

$$\rho^2 - [x_1(T) + u_2(T)]\rho + x_1(T)u_2(T) - u_1(T)u_2(T) = 0$$

$$D =: x_1(T) + u_2(T) \quad (2.38)$$

$$\rho^2 - D\rho + 1 = 0 \quad (2.39)$$

$$\rho_{1,2} = \frac{1}{2}(D \mp \sqrt{D^2 - 4}).$$

For $D^2 \neq 4$, $\Phi(T)$ has different two eigenvalues ρ_1, ρ_2 and different eigenvectors $c^{(1)}$ and $c^{(2)}$. For solutions $\varphi^{(1)}(t) = \Phi(t)c^{(1)}$ and $\varphi^{(2)}(t) = \Phi(t)c^{(2)}$ of (2.28), there are linearly independent solutions $\varphi(t)$ and $\psi(t)$ since $c^{(1)}$ and $c^{(2)}$ are linearly independent.

Proposition 49. *If $|D| > 2$, system (2.28) is unstable. Since system is real-valued and initial conditions of $\varphi_1(t)$ and $\varphi_2(t)$ are real numbers, D is a real number defined by (2.38).*

Proof. $|D| > 2 \Rightarrow 4 < D^2$.

In that case, there are linearly independent solutions $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$ of (2.28) such that

$$\begin{aligned} \varphi^{(1)}(t+T) &= \rho_1\varphi^{(1)}(t) \\ \varphi^{(2)}(t+T) &= \rho_2\varphi^{(2)}(t) \\ \rho_{1,2} &= \frac{1}{2}(D \mp \sqrt{D^2 - 4}). \end{aligned}$$

It is clear that; $\rho_1 > 1$, $0 < \rho_2 < 1$ and $\rho_1 \cdot \rho_2 = 1$. The general solution of (2.28) is

$$\varphi(t) = c_1 \varphi^{(1)}(t) + c_2 \varphi^{(2)}(t).$$

Hence,

$$\begin{aligned} \varphi(t + kT) &= c_1 \varphi^{(1)}(t + kT) + c_2 \varphi^{(2)}(t + kT) \\ &= c_1 \rho_1^k \varphi^{(1)}(t) + c_2 \rho_2^k \varphi^{(2)}(t), \quad \forall k \in \mathbb{Z}. \end{aligned}$$

Obviously; for $k \rightarrow \infty$, it is clear that $\rho_1^k \rightarrow \infty$, and $\rho_2^k \rightarrow 0$. On the other hand, for $k \rightarrow -\infty$, we see that $\rho_1^k \rightarrow 0$, and $\rho_2^k \rightarrow \infty$. As a result, every nontrivial solution $\varphi(t)$ is unbounded, so (2.25) is unstable. \square

Proposition 50. *If $|D| < 2$, the system is stable.*

Proof. $|D| < 2 \Rightarrow D^2 < 4 \Rightarrow \rho_2 = \bar{\rho}_1$

Since $\rho_2 = \bar{\rho}_1$ and $\rho_1 \cdot \rho_2 = 1$, then $|\rho_1| = |\rho_2| = 1$.

$$\begin{aligned} \varphi^{(1)}(t + T) &= \rho_1 \varphi^{(1)}(t) \\ \varphi^{(2)}(t + T) &= \rho_2 \varphi^{(2)}(t) \\ |\varphi^{(1)}(t + T)| &= |\varphi^{(1)}(t)| \\ |\varphi^{(2)}(t + T)| &= |\varphi^{(2)}(t)|. \end{aligned}$$

$|\varphi^{(1)}(t)|$ and $|\varphi^{(2)}(t)|$ are periodic. Every periodic function is bounded on \mathbb{Z} . Since $\varphi(t)$ is the linear combination of $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$, it is bounded. Thus the system (2.25) is stable. \square

Theorem 51. *Assume that system (2.30) satisfies the following conditions,*

$$i) \quad 1 - a(t) > 0, \quad b(t) \geq 0, \quad c(t) \leq 0$$

$$ii) \quad \prod_{t=1}^T \frac{1}{1-a(t)} \geq 1, \quad \prod_{t=1}^T \left\{ 1 - a(t) - \frac{b(t)}{1-a(t)} \right\} > 1.$$

Then, the system (2.25) is unstable.

Proof. To prove, we must show that

$$D = x_1(T) + u_2(T) > 2.$$

It is enough to show that $x_1(T) \geq 1$ and $u_2(T) > 1$.

We know that

$$\Phi(T) = \begin{bmatrix} x_1(T) & x_2(T) \\ u_1(T) & u_2(T) \end{bmatrix}$$

and

$$\Delta\Phi(t) = JH(t)\Phi^\sigma(t), \quad \forall t \in \mathbb{Z}.$$

It can be obtained that

$$\Phi(t+1) = M(t)\Phi(t), \quad t \in \mathbb{Z}$$

$$\Phi(0) = I$$

such that

$$\begin{aligned} M(t) &= \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a(t)} & \frac{b(t)}{1-a(t)} \\ \frac{-c(t)}{1-a(t)} & 1-a(t) - \frac{b(t)c(t)}{1-a(t)} \end{bmatrix} \\ \Phi(T) &= M(T-1)M(T-2)\dots M(0)\Phi(0) \\ &= \begin{bmatrix} M_{11}(T-1) & M_{12}(T-1) \\ M_{21}(T-1) & M_{22}(T-1) \end{bmatrix} \begin{bmatrix} M_{11}(T-2) & M_{12}(T-2) \\ M_{21}(T-2) & M_{22}(T-2) \end{bmatrix} \dots \begin{bmatrix} M_{11}(0) & M_{12}(0) \\ M_{21}(0) & M_{22}(0) \end{bmatrix}. \end{aligned}$$

Because of given conditions of the theorem

$$M_{11}(t) > 0, \quad M_{12}(t) \geq 0, \quad M_{21}(t) \geq 0, \quad M_{22}(t) > 0$$

and

$$M_{ij}(t+T) = M_{ij}(t) \quad i, j = 1, 2$$

$$\begin{aligned} x_1(T) &\geq \prod_{t=0}^{T-1} M_{11}(t) = \prod_{t=1}^T M_{11}(t) = \prod_{t=1}^T \frac{1}{1-a(t)} \geq 1 \\ u_2(T) &\geq \prod_{t=0}^{T-1} M_{22}(t) = \prod_{t=1}^T M_{22}(t) = \prod_{t=1}^T \left\{ 1-a(t) - \frac{b(t)c(t)}{1-a(t)} \right\} > 1 \end{aligned}$$

then $x_1(T) + u_2(T) > 2$ and $D > 2$.

Therefore the system (2.25) is unstable. □

CHAPTER 3

DYNAMICAL SYSTEMS ON TIME SCALES

3.1. Structure of Dynamical Systems on Time Scales

Definition 52. *The first order linear dynamic equation*

$$y^\Delta(t) = p(t)y(t) \quad (3.1)$$

is called regressive if $p \in \mathcal{R}$.

Theorem 53. *Suppose (3.1) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$y^\Delta(t) = p(t)y(t) \quad y(t_0) = y_0 \quad (3.2)$$

is given by

$$y(t) = e_p(t, t_0)y_0.$$

Definition 54. *If $p \in \mathcal{R}$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous, then the dynamic equation*

$$y^\Delta(t) = p(t)y(t) + f(t) \quad (3.3)$$

is called regressive.

Theorem 55. *Suppose (3.3) is regressive. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solutions of the initial value problems*

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0 \quad (3.4)$$

and

$$y^\Delta(t) = p(t)y(t) + f(t), \quad y(t_0) = y_0 \quad (3.5)$$

are given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau$$

and

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau. \quad (3.6)$$

Proof. First, it is easily verified that x given above solves the initial value problem (3.4).

We multiply both sides of (3.4) by integrating factor $e_p(t, t_0)$ and obtain

$$\begin{aligned} [e_p(t, t_0)x]^\Delta(t) &= e_p(t, t_0)x^\Delta(t) + p(t)e_p(t, t_0)x^\sigma(t) \\ &= e_p(t, t_0)[x^\Delta(t) + p(t)x^\sigma(t)] \\ &= e_p(t, t_0)f(t) \end{aligned}$$

and we integrate both sides from t_0 to t

$$e_p(t, t_0)x(t) - e_p(t_0, t_0)x(t_0) = \int_{t_0}^t e_p(\tau, t_0)f(\tau)\Delta\tau.$$

Hence we obtain

$$e_p(t, t_0)x(t) = x_0 + \int_{t_0}^t e_p(\tau, t_0)f(\tau)\Delta\tau.$$

We solve for x and apply Theorem 24 (iii) to arrive at

$$x(t) = e_{\ominus p}(t, t_0) + \int_{t_0}^t \frac{e_p(\tau, t_0)}{e_p(t, t_0)} f(\tau)\Delta\tau$$

Since $e_p(t, \tau)e_p(\tau, t_0) = e_p(t, t_0)$ according to Theorem 24 (v), and by (iii), $x(t)$ can be written as

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau$$

or

$$x(t) = e_{\ominus p}(t, t_0)[x_0 + \int_{t_0}^t e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau].$$

For uniquely for solution $x(t)$, similar proof will be given at Theorem 68. \square

The second proof is similar. We now introduce the concept of an rd-continuous matrix, a regressive matrix, and circle plus addition on matrix-valued functions. A is differentiable on \mathbb{T} provided each entry of A is differentiable on \mathbb{T} and

$$A^\Delta = (a_{ij}^\Delta)_{1 \leq i \leq m, 1 \leq j \leq n} \quad \text{where} \quad A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$$

Theorem 56. *If A is differentiable at $t \in \mathbb{T}^k$, then $A^\sigma(t) = A(t) + \mu(t)A^\Delta(t)$.*

Proof.

$$\begin{aligned} A^\sigma &= (a_{ij}^\sigma) \\ &= (a_{ij} + \mu a_{ij}^\Delta) \\ &= A + \mu A^\Delta \end{aligned}$$

\square

Theorem 57. Suppose A and B are differentiable $n \times n$ -matrix-valued functions. Then

$$i) (A + B)^\Delta = A^\Delta + B^\Delta;$$

$$ii) (\alpha A)^\Delta = \alpha A^\Delta \text{ if } \alpha \text{ is constant};$$

$$iii) (AB)^\Delta = A^\Delta B^\sigma + AB^\Delta = A^\sigma B^\Delta + A^\Delta B;$$

$$iv) (A^{-1})^\Delta = -(A^\sigma)^{-1} A^\Delta A^{-1} = -A^{-1} A^\Delta (A^\sigma)^{-1} \text{ if } AA^\sigma \text{ is invertible};$$

$$v) (AB^{-1})^\Delta = (A^\Delta - AB^{-1}B^\Delta)(B^\sigma)^{-1} = (A^\Delta - (AB^{-1})^\sigma B^\Delta)B^{-1} \text{ if } BB^\sigma \text{ is invertible.}$$

Definition 58. Let A be $n \times n$ matrix-valued function on a time scale \mathbb{T} . We say that A is rd-continuous on \mathbb{T} if each entry of A is rd-continuous, and the class of all such rd-continuous $m \times n$ matrix-valued function on \mathbb{T} is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathcal{R}^{m \times n}).$$

We say that $n \times 1$ -vector-valued system

$$y^\Delta(t) = A(t)y(t) + f(t) \quad (3.7)$$

is regressive provided $A \in \mathcal{R}$ and $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is a rd-continuous vector-valued function.

The next lemma provides a fact about the relationship between the $n \times n$ -matrix-valued function A and the eigenvalues $\lambda_i(t)$ of $A(t)$.

Lemma 59. The $n \times n$ -matrix-valued function A is regressive if and only if the eigenvalues of $\lambda_i(t)$ of $A(t)$ are regressive for all $1 \leq i \leq n$.

Definition 60. Assume that A and B are regressive $n \times n$ -matrix-valued functions on \mathbb{T} . Then we define $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t),$$

$\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1},$$

and $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t)$$

for all $t \in \mathbb{T}^k$.

Theorem 61. $(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}), \oplus)$ is a group.

From this theorem, whenever $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ then $A \oplus B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$. We now state two properties of the regressive matrix-valued functions A and B . Let A^* be the conjugate transpose of A . If $A \in \mathbb{R}^{n \times n}$, then $A^* = A^T$.

Property 62. Suppose that A and B are regressive matrix-valued functions taking on complex values. Then we have the following properties.

(i) A^* is regressive

(ii) $A^* \oplus B^* = (A \oplus B)^*$

Now the generalized matrix exponential function is defined. We consider the matrix-valued IVP

$$Y^\Delta(t) = A(t)Y(t), \quad Y(t_0) = I_n, \quad (3.8)$$

where I_n is the $n \times n$ -identity matrix.

Definition 63. The fundamental matrix is defined to be the general solution to the matrix dynamic equation (3.8) and is denoted by $\Phi_A(t, t_0)$.

We note that Φ_A as a transition matrix can be replaced with e_A in the following lemma and theorem. The next lemma lists some properties of the matrix exponential function.

Lemma 64. If $A, B \in \mathcal{R}$ is a matrix-valued function on \mathbb{T} , then

- i) $\Phi_0(t, s) \equiv I$ and $\Phi_A(t, t) \equiv I$;
- ii) $\Phi_A(\sigma(t), s) = (I + \mu(t)A(t))\Phi_A(t, s)$;
- iii) $\Phi_A^{-1}(t, s) = \Phi_{\ominus A^*}^*(t, s)$;
- iv) $\Phi_A(t, s) = \Phi_A^{-1}(s, t) = \Phi_{\ominus A^*}^*(s, t)$;
- v) $\Phi_A(t, s)\Phi_A(s, r) = \Phi_A(t, r)$;
- vi) $\Phi_A(t, s)\Phi_B(t, s) = \Phi_{A \oplus B}(t, s)$;
- vii) If $\mathbb{T} = \mathbb{R}$ and A is constant then $\Phi_A(t, t_0) = e^{A(t, t_0)}$;

viii) If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ and A is constant then $\Phi_A(t, s) = (I + hA)^{\frac{t-s}{h}}$.

Theorem 65. If λ_0, ξ is an eigenpair for the $n \times n$ matrix A , then $x(t) = e_{\lambda_0}(t, t_0)\xi$ is solution of

$$x^\Delta = Ax, \quad A \in \mathcal{R}$$

on \mathbb{T} .

Proof. Let λ_0, ξ be an eigenpair for A . Since A is regressive on \mathbb{T} , $\lambda_0 \in \mathcal{R}$ and so

$$x(t) = e_{\lambda_0}(t, t_0)\xi$$

is well defined on \mathbb{T} . Then

$$\begin{aligned} x^\Delta(t) &= \lambda_0 e_{\lambda_0}(t, t_0)\xi \\ &= e_{\lambda_0}(t, t_0)\lambda_0\xi \\ &= e_{\lambda_0}(t, t_0)A\xi \\ &= Ae_{\lambda_0}(t, t_0)\xi \\ &= Ax(t) \end{aligned}$$

for $t \in \mathbb{T}^k$. □

Example 66. Solve the vector dynamic equation

$$x^\Delta = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} x.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = -4$ and $\lambda_2 = -1$. This equations is regressive for any time scale such that $1 - 4\mu(t) \neq 0$ for all $t \in \mathbb{T}^k$. Eigenvectors corresponding to λ_1 and λ_2 are

$$\xi_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

respectively. So

$$x(t) = c_1 e_{-4}(t, t_0) \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e_{-1}(t, t_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Theorem 67. *If $x(t) = u(t) + iv(t)$ is complex vector-valued solution of regressive dynamic equation $x^\Delta = A(t)x$ on \mathbb{T} , then u and v are real vector-valued solutions of $x^\Delta = A(t)x$ on \mathbb{T} .*

Proof. Let $x(t) = u(t) + iv(t)$ be a complex vector-valued solution of $x^\Delta = A(t)x$ on \mathbb{T} .

Then

$$u^\Delta(t) + iv^\Delta(t) = x^\Delta(t) = A(t)x(t) = A(t)u(t) + iA(t)v(t)$$

for $t \in \mathbb{T}^k$. Consequently

$$u^\Delta(t) = A(t)u(t), \quad v^\Delta(t) = A(t)v(t) \quad \text{for } t \in \mathbb{T}^k.$$

□

We now present a theorem that guarantees a unique solution to the regressive $n \times 1$ -vector-valued dynamic IVP

$$y^\Delta(t) = A(t)y(t) + f(t), \quad y(t_0) = y_0. \quad (3.9)$$

Theorem 68. *Let $t_0 \in \mathbb{T}$ and $y(t_0) = y_0 \in \mathbb{R}^n$. Then the regressive IVP (3.9) has a unique solution $y : \mathbb{T} \rightarrow \mathbb{R}^n$ given by*

$$y(t) = \Phi_A(t, t_0)y_0 + \int_{t_0}^t \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau. \quad (3.10)$$

Proof. First, y given by (3.10) is well defined can be written because of properties of exponential function as

$$y(t) = \Phi_A(t, t_0)\{y_0 + \int_{t_0}^t \Phi_A(t_0, \sigma(\tau))f(\tau)\Delta\tau\}.$$

We use the product rule to differentiate y :

$$\begin{aligned} y^\Delta(t) &= A(t)\Phi_A(t, t_0)\{y_0 + \int_{t_0}^t \Phi_A(t_0, \sigma(\tau))f(\tau)\Delta\tau\} \\ &\quad + \Phi_A(\sigma(t), t_0)\Phi_A(t_0, \sigma(t))f(t) \\ &= A(t)y(t) + f(t). \end{aligned}$$

Obviously $y(t_0) = y_0$. Therefore y is a solution of (3.9).

Now we show that y is the only solution of (3.9). Assume u is another solution of (3.9) and put $v(t) = \Phi_A(t_0, t)u(t)$. So we have $u(t) = \Phi_A(t, t_0)v(t)$ and therefore

$$\begin{aligned} A(t)\Phi_A(t, t_0)v(t) + f(t) &= A(t)u(t) + f(t) \\ &= u^\Delta(t) \\ &= A(t)\Phi_A(t, t_0)v(t) + \Phi_A(\sigma(t), t_0)v^\Delta(t). \end{aligned}$$

So $v^\Delta(t) = \Phi_A(t_0, \sigma(t))f(t)$. Since $v(t_0)$ must be equal to y_0 , this yields

$$v(t) = y_0 + \int_{t_0}^t \Phi_A(t_0, \sigma(\tau))f(\tau)\Delta\tau$$

and therefore $u = y$, where y is given by (3.10). \square

Theorem 69. *Let $A \in \mathcal{R}$ be an $n \times n$ -matrix valued function on \mathbb{T} and suppose that $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the initial value problem*

$$x^\Delta = -A^*(t)x^\sigma + f(t), \quad x(t_0) = x_0 \quad (3.11)$$

has unique solution $x : \mathbb{T} \rightarrow \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \Phi_{\ominus A^*}(t, \tau)f(\tau)\Delta\tau. \quad (3.12)$$

Proof.

$$\begin{aligned} x^\Delta &= -A^*(t)x^\sigma + f(t) \\ &= -A^*(t)[x + \mu(t)x^\Delta] + f(t) \\ &= -A^*(t)x - \mu(t)A^*(t)x^\Delta + f(t) \end{aligned}$$

i.e.,

$$[I + \mu(t)A^*(t)]x^\Delta = -A^*(t)x + f(t)$$

$$\begin{aligned} x^\Delta &= -[I + \mu(t)A^*(t)]^{-1}A^*(t)x + [I + \mu(t)A^*(t)]^{-1}f(t) \\ &= (\ominus A^*(t))x + [I + \mu(t)A^*(t)]^{-1}f(t). \end{aligned}$$

We can obtain solution of (3.11) as

$$\begin{aligned} x(t) &= \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \Phi_{\ominus A^*}(t, \sigma(\tau))[1 + \mu(\tau)A^*(\tau)]f(\tau)\Delta\tau \\ &= \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \Phi_{A^*}(\sigma(\tau), t)[I + \mu(\tau)A^*(\tau)]^{-1}f(\tau)\Delta\tau \\ &= \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \{[I + \mu(\tau)A^*(\tau)]^{-1}\Phi_A(\sigma(\tau), t)\}^*f(\tau)\Delta\tau \\ &= \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \{\Phi_A(\tau, t)\}^*f(\tau)\Delta\tau \\ &= \Phi_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^t \Phi_{\ominus A^*}(t, \tau)f(\tau)\Delta\tau. \end{aligned}$$

\square

3.2. The Lyapunov Transformation and Stability

We begin by analyzing the stability preserving property associated with a change of variables using a Lyapunov transformation on the regressive time varying linear dynamic system

$$x^\Delta(t) = A(t)x(t), \quad x(t_0) = x_0. \quad (3.13)$$

Definition 70. *The Euclidean norm of an $n \times 1$ vector $x(t)$ is defined to be a real-valued function of t and is denoted by*

$$\|x(t)\| = \sqrt{x^T(t)x(t)}.$$

The induced norm of an $m \times n$ matrix A is defined to be

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

The norm of A induced by the Euclidean norm above is equal to the nonnegative square root of the absolute value of the largest eigenvalue of the symmetric matrix $A^T A$. Thus, we define this norm next. The spectral norm of $m \times n$ matrix A is defined to be

$$\|A\| = [\max_{\|x\|=1} x^T A^T A x]^{\frac{1}{2}}.$$

This will be the matrix norm that used in the sequel and will be denoted by $\|\cdot\|$. The notation that is used for an interval intersected with a time scale is $(a, b) \cap \mathbb{T} = (a, b)_{\mathbb{T}}$

Definition 71. *A Lyapunov transformation is an invertible matrix $L(t) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$ with the property that, for some positive $\eta, \rho \in \mathbb{R}$,*

$$\|L(t)\| \leq \rho \quad \text{and} \quad \det L(t) \geq \eta \quad (3.14)$$

for all $t \in \mathbb{T}$.

Lemma 72. *Suppose that $A(t)$ is an $n \times n$ matrix such that $A^{-1}(t)$ exists for all $t \in \mathbb{T}$.*

(i) If there exists a constant $\alpha > 0$ such that $\|A^{-1}(t)\| \leq \alpha$ for each t ,

(ii) There exists a constant β such that $|\det A(t)| \geq \beta$ and

$$\|A^{-1}(t)\| \leq \frac{\|A(t)\|^{n-1}}{|\det A(t)|} \quad (3.15)$$

for all $t \in \mathbb{T}$.

Proof. (Aitken 1962). □

A consequence of Lemma 72 is that the inverse of a Lyapunov transformation is also bounded. An equivalent condition to (3.14) is that there exists a $\rho > 0$ such that

$$\|L(t)\| \leq \rho \quad \text{and} \quad \|L^{-1}(t)\| \leq \rho \quad (3.16)$$

for all $t \in \mathbb{T}$.

Definition 73. *The time varying linear dynamic equation (3.13) is called uniformly stable if there exists a finite positive constant γ such that for any t_0 , $x(t_0)$ the corresponding solution satisfies*

$$\|x(t)\| \leq \gamma \|x(t_0)\|, \quad t \geq t_0. \quad (3.17)$$

Uniform stability can also be characterized using the following theorem.

Theorem 74. *The time varying linear dynamic equation (3.13) is uniformly stable if and only if there exists a $\gamma > 0$ such that the transition matrix Φ_A satisfies*

$$\|\Phi_A(t, t_0)\| \leq \gamma$$

for all $t \geq t_0$ with $t, t_0 \in \mathbb{T}$.

Proof. Suppose that (3.13) is uniformly stable. Then, there is a $\gamma > 0$ such that for any t_0 , $x(t_0)$, the solutions satisfy

$$\|x(t)\| \leq \gamma \|x(t_0)\|, \quad t \geq t_0.$$

Given any t_0 and $t_a \geq t_0$, let x_a be a vector such that

$$\|x_a\| = 1, \quad \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \|x_a\| = \|\Phi_A(t_a, t_0)\|$$

so the initial state $x(t_a) = x_0$ gives a solution of (3.13) that at time t_a satisfies

$$\|x(t_a)\| = \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \|x_a\| \leq \gamma \|x_a\|.$$

Since $\|x_a\| = 1$, we see that $\|\Phi_A(t_a, t_0)\| \leq \gamma$. Since x_a can be selected for any t_0 and $t_a \geq t_0$, we see that $\|\Phi_A(t, t_0)\| \leq \gamma$ for all $t, t_0 \in \mathbb{T}$. Now suppose that there exists a γ such that $\|\Phi_A(t, t_0)\| \leq \gamma$ for all $t, t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (3.13) satisfies

$$\|x(t)\| = \|\Phi_A(t, t_0)x_0\| = \|\Phi_A(t, t_0)\| \|x_0\| \leq \gamma \|x_0\|, \quad t \geq t_0.$$

Thus, uniform stability of (3.13) is established. □

Definition 75. *The time varying linear dynamic equation (3.13) is called uniformly exponentially stable if there exists finite positive constants γ, λ with $-\lambda \in \mathcal{R}^+$ such that for any $t_0, x(t_0)$ the corresponding solution satisfies*

$$\|x(t)\| \leq \|x(t_0)\| \gamma e^{-\lambda(t, t_0)}, \quad t \geq t_0. \quad (3.18)$$

Uniform exponential stability can also be characterized using the following theorem.

Theorem 76. *The time varying linear dynamic equation (3.13) is uniformly exponentially stable if and only if there exists an $\lambda, \gamma > 0$ with $-\lambda \in \mathcal{R}^+$ such that the transition matrix Φ_A satisfies*

$$\|\Phi_A(t, t_0)\| \leq \gamma e^{-\lambda(t, t_0)}$$

for all $t \geq t_0$ with $t, t_0 \in \mathbb{T}$.

Proof. First suppose that (3.13) is exponentially stable. Then there exist $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x_0 = x(t_0)$, the solution of (3.13) satisfies

$$\|x(t)\| \leq \|x_0\| \gamma e^{-\lambda(t, t_0)}$$

so for any t_0 and $t_a \geq t_0$, let x_a be a vector such that

$$\|x_a\| = 1, \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \|x_a\| = \|\Phi_A(t_a, t_0)\|.$$

Then the initial state $x(t_a) = x_0$ gives a solution of (3.13) that at time t_a satisfies

$$\|x(t_a)\| = 1, \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \|x_a\| \leq \|x_a\| \gamma e^{-\lambda(t, t_0)}.$$

Since $\|x_a\| = 1$ and $-\lambda \in \mathcal{R}^+$, we have $\|\Phi_A(t_a, t_0)\| \leq \gamma e^{-\lambda(t, t_0)}$. Since x_a can be selected for any t_0 and $t_a \geq t_0$, we see that $\|\Phi_A(t_a, t_0)\| \leq \gamma e^{-\lambda(t, t_0)}$ for all $t, t_0 \in \mathbb{T}$.

Now suppose there exist $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $\|\Phi_A(t, t_0)\| \leq \gamma e^{-\lambda(t, t_0)}$ for all $t, t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (3.13) satisfies

$$\|x(t)\| = 1, \|\Phi_A(t, t_0)x_0\| = \|\Phi_A(t, t_0)\| \|x_0\| \leq \|x_0\| \gamma e^{-\lambda(t, t_0)}, \quad t \geq t_0,$$

and thus uniform exponential stability is attained. \square

Definition 77. *The linear state equation (3.13) is defined to be uniformly asymptotically stable if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies*

$$\|x(t)\| \leq \delta \|x(t_0)\|, \quad t \geq t_0 + T. \quad (3.19)$$

It is noted that the time T that must pass before the norm of the solution satisfies (3.19) and the constant $\delta > 0$ is independent of the initial time t_0 .

Theorem 78. *Suppose there exists a constant α such that for all $t \in \mathbb{T}$, $\|A(t)\| \leq \alpha$. Then the linear state equation (3.13) is uniformly exponentially stable if and only if there exists a constant β such that*

$$\int_{\tau}^t \|\Phi_A(t, \sigma(s))\| \Delta s \leq \beta \quad (3.20)$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \sigma(\tau)$.

Proof. Suppose that the state equation (3.13) is uniformly exponentially stable. By Theorem 76, there exist $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ so that

$$\|\Phi_A(t, \tau)\| \leq \gamma e_{-\lambda}(t, \tau)$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$. So we now see that

$$\begin{aligned} \int_{\tau}^t \|\Phi_A(t, \sigma(s))\| \Delta s &\leq \int_{\tau}^t \gamma e_{-\lambda}(t, \tau) \Delta s \\ &= \frac{\gamma}{\lambda} [e_{-\lambda}(t, t) - e_{-\lambda}(t, \tau)] \\ &= \frac{\gamma}{\lambda} [1 - e_{-\lambda}(t, \tau)] \leq \frac{\gamma}{\lambda} \end{aligned}$$

for all $t \geq \sigma(\tau)$. Thus, we have established (3.20) with $\beta = \frac{\gamma}{\lambda}$. Now suppose that (3.20) holds. We see that we can represent the state transition matrix as

$$\Phi_A(t, \tau) = I - \int_{\tau}^t [\Phi_A(t, s)]^{\Delta s} \Delta s = I + \int_{\tau}^t \Phi_A(t, \sigma(s)) A(s) \Delta s,$$

so that, with $\|A(t)\| \leq \alpha$,

$$\|\Phi_A(t, \tau)\| \leq 1 + \int_{\tau}^t \|\Phi_A(t, \sigma(s))\| \|A(s)\| \Delta s \leq 1 + \alpha \beta$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \sigma(\tau)$.

To complete the proof,

$$\begin{aligned} \|\Phi_A(t, \tau)\| (t - \tau) &= \int_{\tau}^t \|\Phi_A(t, \tau)\| \Delta s \\ &\leq \int_{\tau}^t \|\Phi_A(t, \sigma(s))\| \|\Phi_A(\sigma(s), \tau)\| \Delta s \\ &\leq \beta (1 + \alpha \beta) \end{aligned} \quad (3.21)$$

for all $t \geq \sigma(\tau)$. Now choosing T with $T \geq 2\beta(1 + \alpha\beta)$ and $t = \tau + T \in \mathbb{T}$ we obtain

$$\|\Phi_A(t, \tau)\| \leq \frac{1}{2}, \quad t, \tau \in \mathbb{T}. \quad (3.22)$$

Using the bound from equations (3.21) and (3.22), we have the following set of inequalities on intervals in the time scale of the form $[\tau + kT, \tau + (k + 1)T]_{\mathbb{T}}$, with arbitrary τ ;

$$\begin{aligned} \|\Phi_A(t, \tau)\| &\leq 1 + \alpha\beta, \quad t \in [\tau, \tau + T]_{\mathbb{T}}, \\ \|\Phi_A(t, \tau)\| &= \|\Phi_A(t, \tau + T)\Phi_A(\tau + T, \tau)\| \\ &\leq \|\Phi_A(t, \tau + T)\| \|\Phi_A(\tau + T, \tau)\| \\ &\leq \frac{1 + \alpha\beta}{2}, \quad t \in [\tau + T, \tau + 2T]_{\mathbb{T}} \\ \|\Phi_A(t, \tau)\| &= \|\Phi_A(t, \tau + 2T)\Phi_A(\tau + 2T, \tau + T)\Phi_A(\tau + T, \tau)\| \\ &\leq \|\Phi_A(t, \tau + 2T)\| \|\Phi_A(\tau + 2T, \tau + T)\| \|\Phi_A(\tau + T, \tau)\| \\ &\leq \frac{1 + \alpha\beta}{2^2}, \quad t \in [\tau + 2T, \tau + 3T]_{\mathbb{T}}. \end{aligned}$$

In general, for any $\tau \in \mathbb{T}$, we have

$$\|\Phi_A(t, \tau)\| \leq \frac{1 + \alpha\beta}{2^k}, \quad t \in [\tau + kT, \tau + (k + 1)T]_{\mathbb{T}}.$$

We now choose the bounds to obtain a decaying exponential bound. Let $\gamma = 2(1 + \alpha\beta)$ and define the positive function $\lambda(t)$ (with $-\lambda(t) \in \mathcal{R}^+$) as the solution to $e_{-\lambda}(t, \tau) \geq e_{-\lambda}(\tau + (k + 1)T, \tau) = \frac{1}{2^{k+1}}$ for $t \in [\tau + 2T, \tau + 3T]_{\mathbb{T}}$ with $k \in \mathbb{N}_0$. Then for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$, we obtain the decaying exponential bound

$$\|\Phi_A(t, \tau)\| \leq \gamma e_{-\lambda}(t, \tau).$$

Therefore by Theorem 76, we have uniform exponential stability. \square

Theorem 79. *The linear state equation (3.13) is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Proof. Suppose that system (3.13) is uniformly exponentially stable. This implies that there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ so that $\|\Phi_A(t, \tau)\| \leq \gamma e_{-\lambda}(t, \tau)$ for $t \geq \tau$. Clearly, this implies uniform stability. Now, given a $\delta > 0$, we choose a sufficiently large positive constant $T \in \mathbb{T}$ so that $t_0 + T \in \mathbb{T}$ and $e_{-\lambda}(t_0 + T, t_0) \leq \frac{\delta}{\gamma}$. Then for any t_0 and

x_0 , and $t \geq T + t_0$ with $t \in \mathbb{T}$,

$$\begin{aligned}\|x(t)\| &= \|\Phi_A(t, t_0)x_0\| \\ &\leq \|\Phi_A(t, t_0)\|\|x_0\| \\ &\leq \gamma e_{-\lambda}(t_0 + T, t_0)\|x_0\| \\ &\leq \delta\|x_0\|, \quad t \geq t_0 + T.\end{aligned}$$

Thus, (3.13) is uniformly asymptotically stable.

Now suppose the converse. By definition of uniform asymptotic stability, (3.13) is uniformly stable. Thus, there exists a constant $\gamma > 0$ so that

$$\|\Phi_A(t, \tau)\| \leq \gamma \quad \text{for all } t \geq \tau. \quad (3.23)$$

Choosing $\delta = \frac{1}{2}$, let T be a positive constant so that $t = t_0 + T \in \mathbb{T}$ and (3.19) satisfied.

Given a t_0 and letting x_a be so that $\|x_a\| = 1$, we have

$$\|\Phi_A(t_0 + T, t_0)x_a\| = \|\Phi_A(t_0 + T, t_0)\|.$$

When $x_0 = x_a$, the solution $x(t)$ of (3.13) satisfies

$$\begin{aligned}\|x(t)\| &= \|x(t_0 + T)\| = \|\Phi_A(t_0 + T, t_0)x_a\| \\ &= \|\Phi_A(t_0 + T, t_0)\|\|x_a\| \leq \frac{1}{2}\|x_a\|.\end{aligned}$$

From this, we obtain

$$\|\Phi_A(t_0 + T, t_0)\| \leq \frac{1}{2}. \quad (3.24)$$

It is easy to see that for any t_0 there exists an x_a as claimed. Therefore, the above inequality holds for any t_0 . Thus, by using (3.23) and (3.24) exactly as in Theorem 78 uniform exponential stability is obtained. \square

Theorem 80. *Suppose that $L(t) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$, with $L(t)$ invertible for all $t \in \mathbb{T}$ and $A(t)$ is from the dynamic linear system (3.13). Then the transition matrix for the system*

$$Z^\Delta(t) = G(t)Z(t), \quad Z(\tau) = I \quad (3.25)$$

where

$$G(t) = L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^\Delta(t) \quad (3.26)$$

is given by

$$\Phi_G(t, \tau) = L^{-1}\Phi_A(t, \tau)L(\tau) \quad (3.27)$$

for any $t, \tau \in \mathbb{T}$.

Proof. First we see that by definition, $G(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^{n \times n})$. For any $\tau \in \mathbb{T}$, define

$$x(t) = L^{-1}(t)\Phi_A(t, \tau)L(\tau). \quad (3.28)$$

Observe that for $t = \tau$, $x(\tau) = I$. By rearranging (3.28) and differentiating $L(t)x(t)$ with respect to t , we obtain

$$L^\Delta(t)x(t) + L^\sigma(t)x^\Delta(t) = \Phi_A^\Delta(t, \tau)L(\tau) = A(t)\Phi_A(t, \tau)L(\tau)$$

and

$$\begin{aligned} L^\sigma(t)x^\Delta(t) &= A(t)\Phi_A(t, \tau)L(\tau) - L^\Delta(t)x(t) \\ &= A(t)\Phi_A(t, \tau)L(\tau) - L^\Delta(t)L^{-1}(t)\Phi_A(t, \tau)L(\tau) \\ &= [A(t) - L^\Delta(t)L^{-1}(t)]\Phi_A(t, \tau)L(\tau). \end{aligned}$$

Multiplying both sides by $L^{\sigma^{-1}}(t)$

$$\begin{aligned} x^\Delta(t) &= [L^{\sigma^{-1}}A(t) - L^{\sigma^{-1}}(t)L^\Delta(t)L^{-1}(t)]\Phi_A(t, \tau)L(\tau) \\ &= [L^{\sigma^{-1}}A(t)L(t) - L^{\sigma^{-1}}(t)L^\Delta(t)]L^{-1}(t)\Phi_A(t, \tau)L(\tau) \\ &= G(t)x(t). \end{aligned}$$

This is valid for any $\tau \in \mathbb{T}$. Thus, the transition matrix of $x^\Delta(t) = G(t)x(t)$ is $\Phi_G(t, \tau) = L^{-1}(t)\Phi_A(t, \tau)L(\tau)$. If the initial value specified in (3.25) was not the identity, i.e $z(t_0) = z_0 \neq I$, then the solution is $x(t) = \Phi_G(t, \tau)z_0$. \square

Theorem 81 (Preservation of Uniform Stability). *Suppose that $z(t) = L^{-1}(t)x(t)$ is a Lyapunov transformation. Then the system (3.13) is uniformly stable if and only if*

$$z^\Delta(t) = [L^{\sigma^{-1}}A(t)L(t) - L^{\sigma^{-1}}L^\Delta(t)]z(t), \quad z(t_0) = z_0 \quad (3.29)$$

is uniformly stable.

Proof. Equation (3.13) and (3.29) are related by $z(t) = L^{-1}(t)x(t)$. By the Theorem 80, the relationship between the two transition matrices is

$$\Phi_G = L^{-1}(t)\Phi_A(t, t_0)L(t_0).$$

Suppose that (3.13) is uniformly stable. Then there exists a $\gamma > 0$ such that $\|\Phi_A(t, t_0)\| \leq \gamma$ for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$. Then by Lemma 72 and Theorem 74,

we have

$$\begin{aligned}\|\Phi_G(t, t_0)\| &= \|L^{-1}(t)\Phi_A(t, t_0)L(t_0)\| \\ &\leq \|L^{-1}(t)\|\|\Phi_A(t, t_0)\|\|L(t_0)\| \\ &\leq \frac{\gamma\rho^n}{\eta} = \gamma_G\end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$. By Theorem 74, since $\|\Phi_G(t, t_0)\| \leq \gamma_G$, the system (3.29) is uniformly stable. The converse is similar. \square

Theorem 82 (Preservation of Uniform Exponential Stability). *Suppose that $z(t) = L^{-1}(t)x(t)$ is a Lyapunov transformation. Then the system (3.13) is uniformly exponentially stable if and only if*

$$z^\Delta(t) = [L^{\sigma^{-1}}A(t)L(t) - L^{\sigma^{-1}}L^\Delta(t)]z(t), \quad z(t_0) = z_0 \quad (3.30)$$

is uniformly exponentially stable.

Proof. Equations (3.13) and (3.30) are related by the change of variables $z(t) = L^{-1}(t)x(t)$. By Theorem 80, the relationship between the two transition matrices

$$\Phi_G(t, t_0) = L^{-1}\Phi_A(t, t_0)L(t_0)$$

Suppose that (3.13) is uniformly exponentially stable. Then there exists on $\lambda, \gamma > 0$ with $-\lambda \in \mathcal{R}^+$ such that $\|\Phi_A(t, t_0)\| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$ with $t, t_0 \in \mathbb{T}$. Then by Lemma 72 and Theorem 76, we have

$$\begin{aligned}\|\Phi_G(t, t_0)\| &= \|L^{-1}(t)\Phi_A(t, t_0)L(t_0)\| \\ &\leq \|L^{-1}(t)\|\|\Phi_A(t, t_0)\|\|L(t_0)\| \\ &\leq \frac{\gamma\rho^n}{\eta}e_{-\lambda}(t, t_0) = \gamma_G e_{-\lambda}(t, t_0)\end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$.

By Theorem 76, since $\|\Phi_G(t, t_0)\| \leq \gamma_G e_{-\lambda}(t, t_0)$, the system (3.29) is uniformly exponentially stable. The converse is similar. \square

3.3. Floquet Theory On Time Scales

In this section we assume the regressive time varying linear dynamic initial value problem

$$x^\Delta(t) = A(t)x(t), \quad x(t_0) = x_0 \quad (3.31)$$

where $A(t)$ is regressive and p -periodic. We note that in general, it is only necessary that the period of $A(t)$ is a multiple of the period of the time scale that is being analyzed. We let the period of the time scale and period of $A(t)$ be equal for simplicity.

Definition 83. Let $p \in [0, \infty)$. Then the time scale \mathbb{T} is p -periodic if we have the following

$$(i) \quad t \in \mathbb{T} \text{ implies that } t + p \in \mathbb{T},$$

$$(ii) \quad \mu(t) = \mu(t + p),$$

for all $t \in \mathbb{T}$.

Definition 84. Suppose \mathbb{T} is a p -periodic time scale. An $n \times n$ matrix-valued function $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is p -periodic if $A(t) = A(t + p)$ for all $t \in \mathbb{T}$.

Theorem 85. Suppose that \mathbb{T} is a p -periodic time scale and $R \in \mathcal{R}(\mathbb{T}, \mathbb{C}^{n \times n})$. Then the solution of the dynamic matrix initial value problem

$$Z^\Delta(t) = RZ(t), \quad Z(t_0) = z_0 \tag{3.32}$$

is unique up to a period p . That is, $e_R(t, t_0) = e_R(t + kp, t_0 + kp)$ for all $t \in \mathbb{T}$ and $k \in \mathbb{N}_0$.

Proof. The unique solution of (3.32) is $e_R(t, t_0)z_0$. Observe

$$e_R^\Delta(t, t_0)z_0 = Re_R(t, t_0)z_0$$

$$e_R(t, t_0)|_{t=t_0}z_0 = e_R(t_0, t_0)z_0 = z_0.$$

Now we show that $e_R(t, t_0) = e_R(t + kp, t_0 + kp)$. We show this by observing that $e_R(t + kp, t_0 + kp)z_0$ also solves the matrix initial value problem (3.32). We see that

$$e_R^\Delta(t + kp, t_0 + kp)z_0 = Re_R(t + kp, t_0 + kp)z_0 \tag{3.33}$$

$$e_R(t + kp, t_0 + kp)|_{t+kp=t_0+kp} = e_R(t + kp, t_0 + kp)|_{t=t_0} \tag{3.34}$$

$$= e_R(t_0 + kp, t_0 + kp)z_0 \tag{3.35}$$

$$= z_0. \tag{3.36}$$

The solution of the matrix initial value problem (3.32) is unique. Thus we have that

$$e_R(t + kp, t_0 + kp) = e_R(t, t_0) \quad \text{for all } t \in \mathbb{T} \text{ and } k \in \mathbb{N}_0. \tag{3.37}$$

Therefore, e_R can be shifted by integer multiples of p . □

Theorem 86 (The Unified Floquet Theorem for Time Scales). *Suppose that there exists an $n \times n$ constant matrix R such that $e_R(p + t_0, t_0) = \Phi_A(p + t_0, t_0)$, where Φ_A is the transition matrix for (3.31). Then the transition matrix for a p -periodic $A(t)$ can be written in the form*

$$\Phi_A(t, \tau) = L(t)e_R(t, \tau)L^{-1}(\tau) \quad \text{for all } t, \tau \in \mathbb{T} \quad (3.38)$$

where $R \in \mathbb{C}^{n \times n}$ is a constant matrix, and $L(t) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$ is p -periodic and invertible at each $t \in \mathbb{T}$. We refer to (3.38) as the Floquet decomposition for Φ_A .

Proof. We begin by defining the constant matrix R as the solution of the equation

$$e_R(p + t_0, t_0) = \Phi_A(p + t_0, t_0) \quad (3.39)$$

which may require either taking the natural logarithm or obtaining the invertible p^{th} root of the real-valued invertible constant matrix $\Phi_A(p + t_0, t_0)$. Thus, it is possible that a complex R is obtained. Define the matrix $L(t)$ by

$$L(t) = \Phi_A(t, t_0)e_R^{-1}(t, t_0). \quad (3.40)$$

It follows by definition that $L(t) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$ and is invertible at each $t \in \mathbb{T}$. By taking inverse of (3.40)

$$\Phi_A(t, t_0) = L(t)e_R(t, t_0)$$

yields

$$\Phi_A(t_0, t) = e_R^{-1}(t, t_0)L^{-1}(t) = e_R(t_0, t)L^{-1}(t)$$

which proves the claim

$$\Phi_A(t, \tau) = L(t)e_R(t, \tau)L^{-1}(\tau).$$

We conclude the proof by showing that $L(t)$ is p -periodic. By (3.40) and Theorem 85,

$$\begin{aligned} L(t+p) &= \Phi_A(t+p, t_0)e_R^{-1}(t+p, t_0) \\ &= \Phi_A(t+p, t_0+p)\Phi_A(t_0+p, t_0)e_R(t_0, t+p) \\ &= \Phi_A(t+p, t_0+p)\Phi_A(t_0+p, t_0)e_R(t_0, t_0+p)e_R(t_0+p, t+p) \\ &= \Phi_A(t+p, t_0+p)\Phi_A(t_0+p, t_0)e_R^{-1}(t_0+p, t_0)e_R(t_0+p, t+p) \\ &= \Phi_A(t+p, t_0+p)e_R^{-1}(t+p, t_0+p) \\ &= \Phi_A(t+p, t_0+p)e_R^{-1}(t, t_0). \end{aligned}$$

Letting $t' = t + p$, we see that $\Phi_A(t', t_0 + p)$ is a solution to the matrix dynamic equation

$$\begin{aligned}\Phi_A^\Delta(t', t_0 + p) &= A(t')\Phi_A(t', t_0 + p) = A(t + p)\Phi_A(t + p, t_0 + p) \\ &= A(t)\Phi_A(t + p, t_0 + p)\end{aligned}$$

with initial conditions

$$\Phi_A(t', t_0 + p)|_{t'=t_0+p} = \Phi_A(t + p, t_0 + p)|_{t=t_0} = \Phi_A(t_0 + p, t_0 + p) = I.$$

But now $\Phi_A(t, t_0)$ is another solution to the same matrix dynamic initial value problem.

Since the solutions to initial value problems are unique, we have

$$\Phi_A(t + p, t_0 + p) = \Phi_A(t, t_0).$$

Thus,

$$L(t + p) = \Phi_A(t + p, t_0 + p)e_R^{-1}(t, t_0) = \Phi_A(t, t_0)e_R^{-1}(t, t_0) = L(t).$$

□

Theorem 87. *Let $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$ as in Theorem 86. Then $x(t) = \Phi_A(t, t_0)x_0$ is a solution of the p -periodic system (3.31) if and only if $z(t) = L^{-1}(t)x(t)$ is a solution of the autonomous system*

$$z^\Delta(t) = Rz(t), \quad z(t_0) = x_0.$$

Proof. Assume $x(t)$ is a solution of (3.31). Then

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)x_0.$$

If we define

$$z(t) = L^{-1}(t)x(t) = L^{-1}(t)L(t)e_R(t, t_0)x_0 = e_R(t, t_0)x_0,$$

then $z(t)$ is a solution of $z^\Delta(t) = Rz(t)$ and satisfies the initial condition $z(t_0) = x_0$.

Suppose that $z(t) = L^{-1}(t)x(t)$ is a solution of the system $z^\Delta(t) = Rz(t)$, $z(t_0) = x_0$.

The solution is $z(t) = e_R(t, t_0)x_0$. Define $x(t) = L(t)z(t)$. It follows that

$$x(t) = L(t)e_R(t, t_0)x_0 = \Phi_A(t, t_0)x_0$$

so $x(t)$ is a solution of (3.31).

□

Theorem 88. *Given any $t_0 \in \mathbb{T}$, there exists an initial state $x(t_0) = x_0 \neq 0$ such that the solution of (3.31) is p -periodic if and only if at least one of the eigenvalues of $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$ is 1.*

Proof. Suppose that given an initial time t_0 with $x(t_0) = x_0 \neq 0$, the $x(t)$ is p -periodic. By Theorem 86, there exist a Floquet decomposition of x given by

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0.$$

Furthermore

$$x(t + p) = L(t + p)e_R(t + p, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t + p, t_0)L^{-1}(t_0)x_0;$$

since $x(t) = x(t + p)$ and $L(t) = L(t + p)$ for each $t \in \mathbb{T}$, we have

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0)L^{-1}(t_0)x_0$$

which implies

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0 + p)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0.$$

Since $e_R(t + p, t_0 + p) = e_R(t, t_0)$,

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t, t_0)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0,$$

and thus

$$L^{-1}(t_0)x_0 = e_R(t_0 + p, t_0)L^{-1}(t_0)x_0.$$

Since $L^{-1}(t_0)x_0 \neq 0$, we see that $L^{-1}(t_0)x_0 \neq 0$ is an eigenvector of the matrix $e_R(t_0 + p, t_0)$ corresponding to an eigenvalue of 1. Now suppose 1 is an eigenvalue of $e_R(t + p, t_0)$ with corresponding eigenvector z_0 . Then z_0 is real-valued and nonzero. For any $t_0 \in \mathbb{T}$ $z(t) = e_R(t, t_0)z_0$ is p -periodic. Since 1 is an eigenvalue of $e_R(t_0 + p, t_0)$ with corresponding eigenvector z_0 and $e_R(t + p, t_0 + p) = e_R(t, t_0)$,

$$\begin{aligned} z(t + p) &= e_R(t + p, t_0)z_0 \\ &= e_R(t + p, t_0 + p)e_R(t_0 + p, t_0)z_0 \\ &= e_R(t + p, t_0 + p)z_0 \\ &= e_R(t, t_0)z_0 = z(t) \end{aligned}$$

Using the Floquet decomposition from Theorem 86 and setting $x_0 = L(t_0)z_0$ we obtain the nontrivial solution of (3.31). Then

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t, t_0)z_0 = L(t)z(t)$$

which is p -periodic since $L(t)$ and $z(t)$ are p -periodic. \square

We now consider the nonhomogeneous regressive time varying linear dynamic initial value problem

$$x^\Delta(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \quad (3.41)$$

where $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$, $f(t) \in C_{prd}(\mathbb{T}, \mathbb{R}^{n \times 1}) \cap \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times 1})$ and both are p -periodic for all $t \in \mathbb{T}$.

Lemma 89. *A solution $x(t)$ of equation (3.41) is p -periodic if and only if $x(t_0 + p) = x(t_0)$.*

Proof. Suppose that $x(t)$ is p -periodic. Then by definition of a periodic function $x(t_0 + p) = x(t_0)$.

Now suppose that there exist a solution of (3.41) such that $x(t_0 + p) = x(t_0)$. Define $z(t) = x(t+p) - x(t)$. By assumption and construction of $z(t)$, we have $z(t_0) = 0$. Furthermore,

$$\begin{aligned} z^\Delta(t) &= [A(t+p)x(t+p) + f(t+p)] - [A(t)x(t) + f(t)] \\ &= A(t)[x(t+p) - x(t)] \\ &= A(t)z(t). \end{aligned}$$

By uniqueness of solutions, we see that $z(t) \equiv 0$ for all $t \in \mathbb{T}$. Thus, $x(t) = x(t+p)$ for all $t \in \mathbb{T}$. \square

The next theorem uses Lemma 89 to develop criteria for the existence of p -periodic solutions for any p -periodic vector-valued function $f(t)$.

Theorem 90. *For all $t_0 \in \mathbb{T}$ and for all p -periodic $f(t)$, there exists an initial state $x(t_0) = x_0$ such that the solution of (3.41) is p -periodic if and only if there does not exist a nonzero $z(t_0) = z_0$ and $t_0 \in \mathbb{T}$ such that the homogenous initial value problem*

$$z^\Delta(t) = A(t)z(t), \quad z(t_0) = z_0 \quad (3.42)$$

(where $A(t)$ is p -periodic) has a p -periodic solution.

Proof. For any t_0 , $x(t_0) = x_0$ and p -periodic vector-valued function $f(t)$, we know that the solution of (3.41) is

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^t \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

By Lemma 89, $x(t)$ is p -periodic if and only if $x(t_0) = x(t_0 + p)$ which is equivalent to

$$[I - \Phi_A(t_0 + p, t_0)]x_0 = \int_{t_0}^{t_0+p} \Phi_A(t_0 + p, \sigma(\tau))f(\tau)\Delta\tau. \quad (3.43)$$

By Theorem 88, we must show that this algebraic equation has a solution for x_0 , for any t_0 and any p -periodic $f(t)$ if and only if $e_R(t_0 + p, t_0)$ has no eigenvalues equal to one.

Let $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$ and suppose that there are no eigenvalues equal to 0. This is equivalent to

$$\det[I - \Phi_A(t_0 + p, t_0)] \neq 0. \quad (3.44)$$

Since Φ_A is invertible, (3.44) is equivalent to

$$\begin{aligned} 0 &\neq \det[\Phi_A(t_0 + p, p)(I - \Phi_A(p, 0))(\Phi_A(0, t_0))] \\ &= \det[\Phi_A(t_0 + p, p)(\Phi_A(0, t_0) - \Phi_A(t_0 + p, t_0))]. \end{aligned} \quad (3.45)$$

Since $\Phi_A(t_0 + p, p) = \Phi_A(t_0, 0)$, (3.44) is equivalent to the invertibility of $[I - \Phi_A(t_0 + p, t_0)]$. Thus (3.43) has a solution x_0 for any t_0 and for p -periodic $f(t)$.

Now suppose that (3.43) has a solution x_0 for any t_0 and for any p -periodic $f(t)$. Given an arbitrary $t_0 \in \mathbb{T}$ corresponding to any $n \times 1$ vector f_0 , we define a regressive p -periodic vector-valued function $f(t) \in C_{prd}(\mathbb{T}, \mathbb{R}^{n \times 1})$ by

$$f(t) = \Phi_A(\sigma(t), t_0 + p)f_0, \quad t \in [t_0, t_0 + p]_{\mathbb{T}} \quad (3.46)$$

extending this to the entire time scale \mathbb{T} using the periodicity. By construction of $f(t)$ we have

$$\int_{t_0}^{t_0+p} \Phi_A(t_0 + p, \sigma(\tau))f(\tau)\Delta\tau = \int_{t_0}^{t_0+p} f_0\Delta\tau = pf_0.$$

Thus (3.43) becomes

$$[I - \Phi_A(t_0 + p, t_0)]x_0 = pf_0. \quad (3.47)$$

For any vector-valued function $f(t)$ that is constructed as in (3.46) and thus for any corresponding f_0 , (3.47) has a solution for x_0 by assumption. Therefore,

$$\det[I - \Phi_A(t_0 + p, t_0)] \neq 0.$$

Thus, $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$ has no eigenvalue of equal to 1. By Theorem 88, (3.42) has no periodic solution. \square

CHAPTER 4

CONCLUSION

The aim of this study is to present structure of solution of linear systems of dynamic equations on time scales; stability criterions of periodic discrete Hamiltonian systems and periodic systems on periodic time scales. For these purpose, we began with expending three basic papers (Bohner and Peterson 2001), (Dacunha 2005), (Ahlbrandt and Ridenhour 2003) on this subject. We have composed these studies and presented fundamental theorems and concepts. After basic definitions and theorems of time scales and dynamical real case; stability of discrete case, basic theorems and properties of periodic discrete Hamiltonian systems were presented. Uniqueness and existence theorem was given for time scales and finally Lyapunov stability and Floquet theory was presented. By this study, the basic concepts of dynamical systems in real case have similar structure with discrete case and time scale case. It is easily seen that, structure of dynamical systems and stability criterions are similar for these cases.

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