# HYDRODYNAMIC INTERACTION IN ROTATIONAL FLOW

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**by**
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# ABSTRACT

## HYDRODYNAMIC INTERACTION IN ROTATIONAL FLOW

The interaction of water waves with arrays of vertical cylinders problem is studied using diffraction of water waves and addition theorem for bessel functions.

The linear boundary value problem which is derived from physical assumptions is used as the approximate mathematical model for time-harmonic waves. Linearization procedure is described for the nonlinear boundary conditions on the free surface. The problem is solved by using Addition theorem for Bessel functions. Limiting case, $k \to 0$, known as long wave approximation, is analysed using limiting forms of Bessel functions.

Vortex-cylinder interaction is analyzed using a similar technique involving Laurent series expansions of complex velocity and the Circle Theorem. But this method failed to work. Further analysis is necessary. Vortex dynamics is analysed in annular domains, which can conformally be mapped into infinite domain with two cylinders, using the terminology of q-calculus.

Finally, the result of vortex-cylinder interaction in annular domain is transformed into the infinite domain with two cylinders using conformal mapping. Image representation clearly shows the mechanism of inverse images which accumulate at zero and infinity in the $w$-plane and $a$ and $1/a$ in the $z$-plane.

# ÖZET


## ROTASYONEL AKIŞTA HİDRODİNAMİK ETKİLEŞİM

Dikey yerleştirilmiş silindirler arası su dalgası etkileşimi problemi, su dalgalarının kırınımı ve Bessel fonksiyonlarının toplam teoremi (Addition theorem) kullanılarak çalışıldı.

Fiziksel kabullerden elde edilen, linear sınır değer problemi, zaman uyumlu dalgalar için yaklaşık matematik modeli gibi kullanıldı. Serbest yüzeyde doğrusal olmayan sınır şartları için doğrusallaştırma prosedürü tanımlandı. Problem Bessel fonksiyonları için toplam teoremi kullanılarak çözüldü. Uzun dalga yaklaşımı olarak da bilinen $k \to 0$ limit durumu Bessel fonksiyonlarının limit formları kullanılarak incelenmiştir.

Girdap (vortex)-silindir etkileşimi, karmaşık hızın Laurent seri açılımı ve Çember teoremi (Circle Theorem) içeren benzer yöntem kullanılarak incelenebilir. Fakat henüz geliştirilemedi. Halka bölgesindeki, konformal olarak iki silindirli sonsuz bölgeye eşlenebilen girdap dinamikleri, $q$- hesaplama terminolojisi kullanılarak incelenmiştir.

Son olarak, halka bölgesindeki girdap-silindir etkileşim probleminin sonucu, uygun dönüşüm kullanılarak, iki silindirli sonsuz bölgeye dönüstürülür. Görüntü temsili, ters görüntü işleyişinin(mekanızmasının), $w$-düzlemi için sıfır ve sonsuzda, $z$-düzlemi içinse $a$ ve $1/a$'da yığıldığını açıkça gösterir.

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# CHAPTER 1

# INTRODUCTION

The history of water wave theory is almost as old as that of partial differential equations. Their founding fathers are the same: Euler, Lagrange, Cauchy, Poisson. Further contributions were made by Stokes, Lord Kelvin, Kirchhoff, and Lamb who constructed a number of explicit solutions.

There are several expositions of the classical theory (Crapper$(1984)$, Lamb$(1932)$, Lighthill$(1978)$, Sretensky$(1977)$, Stoker$(1957)$, Wehausen $\&$ Laitone$(1960)$ and Whitham$(1979)$). Various aspects of the linear water waves have been considered in works of Havelock and Ursell. Other works are focused on various applied aspects of the theory. In particular, Haskind, Mei, Newman and Wehausen (Haskind$(1973)$, Mei$(1983)$, Newman$(1977, 1978)$ and Wehausen$(1971)$)consider the wave-body interaction. Also, there is the very recent monography by Linton and McIver $(2001)$ on the mathematical methods used in the theory of such interactions, but it mainly discusses mathematical techniques from the point of view of their applications in ocean engineering.

The goal of present thesis is to study interaction in rotational flow. There are several studies in interaction theory carried out by Linton & Evans, Spring & Monkmeyer and Yilmaz(Linton & Evans (1990), Spring & Monkmeyer (1974) and Yilmaz (1994)). Vortex dynamics was analysed by Pashaev & Gurkan and Gurkan & Pashaev (Pashaev & Gurkan (2007) and Gurkan & Pashaev (2007)).

In Chapter 2 we solved the linear boundary value problem of diffraction of water waves by arrays of vertical cylinders using Addition theorem for Bessel functions and analysed long wave approximation using limiting forms of Bessel functions.

In Chapter 3 using Laurent series expansions of complex velocity and the Circle Theorem we studied Vortex-cylinder interaction

In Chapter 4 we investigated vortex dynamics in annular domains using the terminology of q-calculus.

Finally, Chapter 5, using the conformal mapping we transformed annular domain onto infinite domain with two cylinders. We sketched the images of vortices.

# CHAPTER 2

# BACKGROUND INFORMATION ABOUT HYDRODYNAMICS

## 2.1. Equations of Motion

### 2.1.1. Euler's Equations

Let $D$ be a region in two or three dimensional space filled with a fluid and bounded by one or more moving or fixed surfaces that separate water from some other medium. Our object is to describe the motion of such a fluid. Let $\mathbf{x}$ be a point in $D$ and consider the particle of fluid moving through $\mathbf{x}$ at time $t$. Relative to standard Euclidean coordinates in space, we write $\mathbf{x} = (x, y, z)$. Imagine a particle (think of particle of dust suspended) in the fluid; this particle traverses a well-defined trajectory. Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity of the particle of fluid that is moving through $\mathbf{x}$ at time $t$. Thus, for each fixed time, $\mathbf{u}$ is a vector field on $D$. We call $\mathbf{u}$ the (spatial) *velocity field of the fluid*. For each time $t$, assume that the fluid has a well-defined *mass density* $\rho(\mathbf{x}, t)$. Thus, if $W$ is any subregion of $D$, the mass of fluid in $W$ at time $t$ is given by

$$m(W, t) = \int_W \rho(\mathbf{x}, t) dV, \tag{2.1}$$

where $dV$ is the volume element in the plane or in space. The assumption that $\rho$ exists is a *continuum assumption*. Clearly, it does not hold if the molecular structure of matter is taken into account. For most macroscopic phenomena occurring in nature, it is believed that this assumption is extremely accurate.

Our derivation of the equations is based on three basic principles:

*i) Mass is neither created nor destroyed;*

*ii) The rate of change of momentum of a portion of the fluid equals the force applied to it(Newton's Second Law.);*

*iii) Energy is neither created nor destroyed.*

**i) Conservation of Mass:** The integral form of the conservation of mass is

$$\frac{d}{dt} \int_W \rho dV = - \int_{\partial W} \rho \mathbf{u}.\mathbf{n} dA, \tag{2.2}$$

where $\partial W$ is boundary of $W$. By the Divergence theorem (See Appendix A), equation $(2.2)$ is equivalent to

$$\int_W [\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u})] dV = 0. \tag{2.3}$$

Because this holds for all $W$, $(2.3)$ is equivalent to

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0. \tag{2.4}$$

The equation $(2.4)$ is the *differential form of the law of conservation of mass*, also known as the *continuity equation*.

**ii) Balance of Momentum:** If $W$ is region in the fluid at a particular instant of time $t$, the total force exerted on the fluid inside $W$ by means of stress on its boundary $\partial W$ is

$$\mathbf{S}_{\partial W} = (\text{Force on } W) = -\int_{\partial W} p \, \mathbf{n} \, dA \tag{2.5}$$

where $p$ is pressure. Divergence theorem gives

$$\mathbf{S}_{\partial W} = -\int_W \mathbf{grad} \, p \, dV. \tag{2.6}$$

If $\mathbf{b}(\mathbf{x}, t)$ denotes the given body force per unit mass, the total body force is

$$\mathbf{B} = \int_W \rho \, \mathbf{b} \, dV. \tag{2.7}$$

Thus, on any piece of fluid material,

$$\text{Force per unit volume} = -\mathbf{grad} \, p + \rho \, \mathbf{b}. \tag{2.8}$$

By Newton's Second Law (force=mass$\times$acceleration) we are lead to the differential form of the law of $balance \, of \, momentum$:

$$\rho \, \frac{D\mathbf{u}}{Dt} = -\mathbf{grad} \, p + \rho \, \mathbf{b}, \tag{2.9}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\nabla$ is *material derivative*; it takes into account the fact that the fluid is moving and that the position of fluid particles change with time.

**iii) Conservation of Energy:** For fluid moving in a domain $D$, with velocity field $\mathbf{u}$, the kinetic energy contained in a region $W \subset D$ is

$$E_{kinetik} = \frac{1}{2} \int_W \rho ||\mathbf{u}||^2 dV. \tag{2.10}$$

We assume that total energy of the fluid can be written as

$$E_{total} = E_{kinetik} + E_{internal} \tag{2.11}$$

$$\frac{1}{2} \frac{D}{Dt} ||\mathbf{u}||^2 = \mathbf{u}.\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.(\mathbf{u}.\nabla)\mathbf{u}. \tag{2.12}$$

### 2.1.1.1. Incompressible Flows

**Definition 2.1** *A flow is incompressible (Chorin and Marsden 1992) if for any fluid subregion $W$,*

$$\mathrm{div}\mathbf{u} = 0. \tag{2.13}$$

### 2.1.1.2. Euler's Equations for Incompressible Flows

Here we assume all the energy is kinetic and that the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work;

$$\frac{d}{dt}E_{kinetic} = -\int_{\partial W_t} p\,\mathbf{u}.\mathbf{n}dA + \int_{W_t} \rho\,\mathbf{u}.\mathbf{b}\,dV. \tag{2.14}$$

Using the divergence theorem and since flow is incompressible, $\mathrm{div}\mathbf{u} = 0$ equation (2.14) becomes,

$$\int_{W_t} \rho\{\mathbf{u}.(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u})\}dV = -\int_{W_t} (\mathbf{u}.\nabla p - \rho\mathbf{u}.\mathbf{b})dV. \tag{2.15}$$

This argument, in addition, shows that if we assume $E = E_{kinetic}$, then the fluid must be incompressible (unless $p = 0$). In summary, in this incompressible case, the *Euler equations* are

$$\begin{aligned} \rho\,\frac{D\mathbf{u}}{Dt} &= -\mathbf{grad}\,p + \rho\,\mathbf{b}, && (2.16)\\ \frac{D\rho}{Dt} &= 0, && (2.17)\\ \mathrm{div}\mathbf{u} &= 0, && (2.18) \end{aligned}$$

with the boundary conditions

$$\mathbf{u}.\mathbf{n} = 0 \quad \text{on } \partial D. \tag{2.19}$$

### 2.1.1.3. Isentropic Fluids

A compressible flow will be called $isentropic$ if there is a function $\omega$, called the enthalpy, such that

$$\mathbf{grad}\omega = \frac{1}{\rho}\mathbf{grad}p. \tag{2.20}$$

Given a fluid flow with velocity field $\mathbf{u}(\mathbf{x}, t)$, a *streamline* at a fixed time is an integral curve of $\mathbf{u}$; that is, if $\mathbf{x}(s)$ is a streamline at the instant $t$, it is curve parametrized by a variable, say $s$, that satisfies

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(s), t), \quad t \text{ fixed.}$$

We define a fixed *trajectory* to be the curve traced out by a particle as time progresses, as explained at the beginning of this section. Thus a trajectory is a solution of the differential equation

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(t), t)$$

with suitable initial conditions. If $\mathbf{u}$ is independent of $t$ streamlines and trajectory coincide. In this case, the flow is called *stationary*.

**Theorem 2.1 (Bernoulli's Theorem)** *(Chorin and Marsden 1992) In stationary isentropic flows and in the absence of external forces, the quantity*

$$\frac{1}{2}||\mathbf{u}||^2 + \omega \tag{2.21}$$

*is constant along streamlines. The same holds for homogeneous incompressible flow with* $\omega$ *replaced by* $p/p_0$.

## 2.2. Rotation and Vorticity

The vector $\nabla \times \mathbf{u} = curl\mathbf{u} \equiv \boldsymbol{\xi}$, called the *vorticity vector*. A *vortex line* is a line drawn tangent to at each point in the direction of the vorticity vector. When the vorticity vector is different from zero the motion is said to be *rotational*. A portion of the fluid at every point of which the vorticity is zero is said to be *irrotational motion*. In such a portion of the fluid there are no vortex lines.

Let $C$ be a simple closed contour in the fluid at $t = 0$. Let $C_t$ be the contour carried along by the flow at time $t$. In other words,

$$C_t = \varphi_t(C), \tag{2.22}$$

where $\varphi_t$ is the fluid flow map. The *circulation* around $C_t$ is defined to be the line integral

$$\Gamma_{C_t} = \oint_{C_t} \mathbf{u}.\mathbf{ds}. \tag{2.23}$$

**Theorem 2.2 (Kelvin's Circulation Theorem)** *(Chorin and Marsden 1992) For isentropic flow without external forces, the circulation,* $\Gamma_{C_t}$ *is constant in time.*

## 2.3. Potential Flow

An inviscid, irrotational flow is called a *potential flow*. A domain $D$ is called *simply connected* if any continuous closed curve in $D$ can be continuously shrunk to a point without leaving $D$. For example, in space, the exterior of a solid sphere is simply connected, whereas in the plane the exterior of a solid disc is not simply connected. For irrotational flow in a simply connected region $D$, there is a scalar function $\phi(x,t)$ on $D$ called *velocity potential* such that for each $t$, $\mathbf{u} = \mathbf{grad}\,\phi$. It follows that the circulation around any closed curve $C$ in $D$ is zero. In particular, if the flow is stationary,

$$\frac{1}{2}\| \mathbf{u} \|^2 + w = \text{constant in space.} \tag{2.24}$$

For the last equation to hold, simple connectivity of $D$ is unnecessary. For potential flow in nonsimply connected domains, it can occur that the circulation $\Gamma_C$ around closed curve $C$ is nonzero. For instance, consider

$$\mathbf{u} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}) \tag{2.25}$$

outside the origin. If the contour $C$ can be deformed within $D$ to another contour $C'$, then $\Gamma_C = \Gamma_{C'}$. The reason is that basically $C \cup C'$ forms the boundary of a surface $\sum$ in $D$. Stokes' theorem then gives

$$\int_{\sum} \boldsymbol{\xi}.d\mathbf{A} = \int_C \mathbf{u}.\mathbf{ds} - \int_{C'} \mathbf{u}.\mathbf{ds} = \Gamma_C - \Gamma_{C'} \tag{2.26}$$

and because $\boldsymbol{\xi} = 0$ in $D$, we get $\Gamma_C = \Gamma_{C'}$. From Kelvin's Circulation theorem, the circulation around a curve is constant in time. Thus, the circulation around an obstacle in the plane is well-defined and constant in time. Next, consider incompressible potential flow in a simply connected domain $D$. From $\mathbf{u} = \mathbf{grad}\,\phi$ and $div\,\mathbf{u} = 0$, we have

$$\Delta\phi = 0. \tag{2.27}$$

Let the velocity of $\partial D$ be specified as $\mathbf{V}$, so

$$\mathbf{u}\,\mathbf{n} = \mathbf{V}\,\mathbf{n}. \tag{2.28}$$

Thus, $\phi$ solves the Neumann Problem:

$$\Delta\phi = 0, \quad \frac{\partial\phi}{\partial n} = \mathbf{V}.\mathbf{n}. \tag{2.29}$$

If $\phi$ is a solution of the equation (2.29), then $\mathbf{u} = \mathbf{grad}\phi$ is a solution of the stationary homogeneous Euler equations, $i.e.$,

$$\begin{aligned} \rho(\mathbf{u}.\nabla)\mathbf{u} &= -\mathbf{grad}p, & (2.30)\\ div\,\mathbf{u} &= 0, & (2.31)\\ \mathbf{u}.\mathbf{n} &= \mathbf{V}.\mathbf{n} \quad \text{on } \partial D, & (2.32) \end{aligned}$$

where $p = \rho \| u \|^2 / 2$.

**Theorem 2.3** *Let $D$ be a simply connected, bounded region with prescribed velocity $V$ on $\partial D$. Then*

*i. there is a exactly one potential homogeneous incompressible flow satisfying* $(2.26)$ *in $D$ if and only if $\int_{\partial D} \mathbf{V}.\mathbf{n} \, dA = 0$;*

*ii. this flow is the minimizer of the kinetic energy function*

$$E_{kinetic} = \frac{1}{2} \int_D \rho \| u \|^2 dV,$$

*among all divergence-free vector fields $\mathbf{u}'$ on $D$ satisfying $\mathbf{u}'.\mathbf{n} = \mathbf{V}.\mathbf{n}$.*

# CHAPTER 3

# DIFFRACTION OF WATER WAVES BY MULTIPLE CYLINDERS

## 3.1. Nonlinear Problem

### 3.1.1. Equations of Motion

The conservation of mass implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0 \quad \text{in } W. \tag{3.1}$$

Under the assumption that the fluid is incompressible (which is usual in the water wave theory), equation $(3.1)$ becomes

$$\mathrm{div}\mathbf{u} = 0 \quad \text{in } W. \tag{3.2}$$

The conservation of momentum in inviscid fluid leads to the so called *Euler Equations*. Taking into account the gravity force, one can write last two equations in the following vector form:

$$\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} = -\rho^{-1}\nabla p + \mathbf{g}. \tag{3.3}$$

Here g is the vector of the gravity force having zero horizontal components and the vertical one equal to $-g$ where $g$ denotes the acceleration caused by gravity. An Irrotational character of motion is another usual assumption in the water-wave theory; that is

$$\mathrm{rot}\,\mathbf{u} = 0 \quad \text{in } W. \tag{3.4}$$

Equation $(3.4)$ guarantees the existence of a velocity potential $\phi$, so that

$$u = \nabla\phi \quad \text{in } W. \tag{3.5}$$

From $(3.2)$ and $(3.5)$ one obtains the Laplace equation

$$\Delta\phi = 0 \quad \text{in } W. \tag{3.6}$$

This greatly facilitates the theory but, in general solutions of $(3.6)$ do not manifest wave character. Waves are created by the boundary conditions on the free surface.

### 3.1.2. Boundary Conditions

We consider boundaries of two types: free surface separating water from the atmosphere and rigid surfaces of bodies floating in and/or beneath the free surface.

Let $y = \eta(x,t)$ be the equation of the free surface. The pressure is prescribed to be equal to the constant atmospheric pressure $p_0$ on $y = \eta(x,t)$. From (3.3) and (3.5) we can obtain Bernoulli's equation,

$$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} = -\frac{p}{\rho} - gy + c(t) \quad \text{in } \widetilde{W}. \tag{3.7}$$

On the free surface $y = \eta(x,t), p = p_0$ and $c(t) = \frac{p_0}{\rho}$. We get the *dynamic boundary condition* on the free surface:

$$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + g\eta(x,t) = 0 \quad \text{for } y = \eta(x,t),\ x \in F. \tag{3.8}$$

Another boundary condition holds on every "physical" surface $S$ bounding the fluid domain W. Let $S(x,y,t) = 0$ be equation of $S$, then

$$\begin{aligned}
\frac{DS}{Dt} &= 0 \quad \text{on } S && (3.9)\\
\frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \mathbf{u}\,\nabla S = 0 \quad \text{on } S && (3.10)\\
\frac{\partial S}{\partial t} &= -\mathbf{u}\nabla S, \quad \nabla S = |\nabla S|\,\mathbf{n} && (3.11)\\
-\frac{\partial S}{\partial t} &= \mathbf{u}\,|\nabla S|\mathbf{n} && (3.12)\\
\mathbf{un} &= \frac{1}{|\nabla S|}\frac{\partial S}{\partial t} = \boldsymbol{v_n} && (3.13)
\end{aligned}$$

where $v_n$ denotes the *normal velocity* of $S$. Thus the *kinematic boundary condition* means that the normal velocity of particles is continuous across a physical boundary. On the other hand,

$$\begin{aligned}
\nabla S &= \frac{\partial S}{\partial x}i + \frac{\partial S}{\partial y}j && (3.14)\\
0 &= \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial t}dt && (3.15)\\
\frac{\partial S}{\partial t} &= -\frac{\partial S}{\partial x}\frac{dx}{dt} - \frac{\partial S}{\partial y}\frac{dy}{dt} && (3.16)\\
\frac{\partial S}{\partial t} &= -\nabla S\,\boldsymbol{v} && (3.17)\\
\mathbf{u}\,\mathbf{n} &= -\frac{1}{|\,\nabla S\,|}\,(\nabla S\,\boldsymbol{v}) && (3.18)\\
\boldsymbol{v}\mathbf{n} &= \mathbf{un}. && (3.19)
\end{aligned}$$

We obtained the kinematic boundary condition on $S$,

$$\mathbf{u}\,\mathbf{n} = \boldsymbol{v}\,\mathbf{n} \quad \text{on } S. \tag{3.20}$$

On the fixed part of $S$, $(3.20)$ takes the form of,

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } S. \tag{3.21}$$

On the free surface, condition $(3.10)$, written as follows,

$$\eta_t + \phi_x.\eta_x - \phi_y = 0 \quad y = \eta(x,t),\ x \in F, \tag{3.22}$$

complements the dynamic condition $(3.8)$. Thus, in the present approach, two nonlinear conditions $(3.8)$ and $(3.22)$ on the unknown boundary are responsible for waves, which constitute the main characteristic feature of water-surface wave theory.

In the water-wave problem one seeks the velocity potential $\phi(x,y,t)$ and the free surface elevation $\eta(x,t)$ satisfying

$$\begin{aligned}
\nabla^2\phi &= 0 \quad \text{in } W, && (3.23)\\
\frac{\partial \phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + g\eta(x,t) &= 0 \quad \text{for } y = \eta(x,t),\ x \in F, && (3.24)\\
\eta_t + \phi_x, \eta_x - \phi_y &= 0 \quad y = \eta(x,t),\ x \in F, && (3.25)\\
\frac{\partial \phi}{\partial n} &= u_n \quad \text{on } \widetilde{W}. && (3.26)
\end{aligned}$$

The initial values of $\phi$ and $\eta$ should also be prescribed, as well as the conditions at infinity (for unbounded $W$) to complete the problem, which is known as *Cauchy-Poisson Problem*.

## 3.2. Linearization of the Problem

There is a mathematical evidence that the linearized problem provides an approximation to the nonlinear one. More precisely, under the assumption that the undisturbed water occupies a layer of constant depth, the followings are proved:

1. The nonlinear problem is solvable for sufficiently small values of the linearization parameter.

2. As this parameter tends to zero, solutions of the nonlinear problem do converge to the solution of the linearized problem in the norm of some suitable function space.

To be in a position to describe water waves in the presence of bodies, the equations should be approximated by more tractable ones. The usual and rather reasonable simplification consist of a linearization of the problem under certain assumptions concerning the motion of a floating body. An example of such assumptions (there are other ones leading to the same conclusions) suggest that a body's motion near the equilibrium position is so small that it produces only waves having a small amplitude and small wavelength. There are three characteristic geometry parameters:

1. A typical value of the wave height $H$.
2. A typical wavelength $L$.
3. The water depth $D$.

They give the three characteristic quotients : $H/L, H/D$, and $L/D$. The relative importance of these quotients is different in different situations. If

$$\frac{H}{D} \ll 1 \quad and \quad \frac{H}{L}\left(\frac{L}{D}\right)^3 \ll 1, \tag{3.27}$$

then the linearization can be justified by some heuristic considerations. The last parameter $(H/L)(L/D)^3 = (H/D)(H/D)^2$ is usually referred to as Ursell's number.

### 3.2.1. Equations for Small Amplitude Waves

Let assume that the velocity potential $\phi$ and free surface elevation $\eta$ admit expansions with respect to a certain small parameter $\epsilon$:

$$\phi(x,y,t) = \epsilon\phi^{(1)}(x,y,t) + \epsilon^2\phi^{(2)}(x,y,t) + \epsilon^3\phi^{(3)}(x,y,t) + \ldots, \tag{3.28}$$

$$\eta(x,t) = \eta^{(0)}(x,t) + \epsilon\eta^{(1)}(x,t) + \epsilon^2\eta^{(2)}(x,t) + \ldots, \tag{3.29}$$

where $\phi^{(1)}, \phi^{(2)}, \ldots, \eta^{(0)}, \eta^{(1)}, \ldots$ and all their derivatives are bounded. If we substitute $(3.28)$ and $(3.29)$ into laplace equation $(3.6)$ we get,

$$\nabla^2\phi^{(k)} = 0 \quad \text{in } W. \quad k = 1, 2, \ldots \tag{3.30}$$

When expansion for $\phi$ and $\eta$ are substituted into dynamic boundary condition $(3.8)$ and grouped according to powers of $\epsilon$, for $y = 0$, we get,

$$\eta^{(0)} = 0 \quad x \in F, \tag{3.31}$$

$$g\eta^{(1)} + {\phi_t}^{(1)} = 0 \quad x \in F, \tag{3.32}$$

$$g\eta^{(2)} + {\phi_t}^{(2)} = -\eta^{'}{\phi_{yt}}^{(1)} - \frac{1}{2}\left|\ \nabla\phi^{'}\ \right|^2, \tag{3.33}$$

and so on; that is all these conditions hold on the mean position of the free surface at rest. Similarly, the kinematic condition $(3.22)$ leads to

$$\begin{aligned}(\epsilon{\eta_t}^{(1)}+\epsilon^2{\eta_t}^{(2)}) &+ (\epsilon{\eta_x}^{(1)}+\epsilon^2{\eta_x}^{(2)}).(\epsilon{\phi_x}^{(1)}+\epsilon^2{\phi_{yx}}^{(1)}\eta^{(1)}+\epsilon^2{\phi_x}^{(2)}) \\ &- (\epsilon{\phi_y}^{(1)}+\epsilon^2{\phi_{yy}}^{(1)}\eta^{(1)}+\epsilon^2{\phi_y}^{(2)})=0. \qquad (3.34)\end{aligned}$$

After grouping, we obtain,

$$\begin{aligned}{\phi_y}^{(1)}-{\eta_t}^{(1)} &= 0 \quad x\in F, \qquad (3.35)\\ {\phi_y}^{(2)}-{\eta_t}^{(2)} &= {\phi_x}^{(1)}{\eta_x}^{(1)}-{\phi_{yy}}^{(1)}\eta^{(1)}. \qquad (3.36)\end{aligned}$$

By eliminating $\eta^{(1)}$ between $(3.32)$ and $(3.35)$, one finds the classical first order linear free surface condition,

$$g\,{\phi_y}^{(1)}+{\phi_{tt}}^{(1)}=0 \quad \text{for } y=0\,, x\in F. \qquad (3.37)$$

In the same way, one obtains from $(3.33)$ and $(3.36)$ the following,

$$g\,{\phi_y}^{(2)}+{\phi_{tt}}^{(2)}=-{\phi_t}^{(1)}(\phi_x^{(1)})^2-\frac{1}{g^2}[{\phi_t}^{(1)}{\phi_{ttt}}^{(1)}+\mid\nabla_x\phi^{(1)}\mid^2] \quad \text{for } y=0,\ x\in F. \qquad (3.38)$$

All these conditions have the same operator on the left-hand side and the right-hand term depends nonlinearly on terms of smaller orders.

### 3.2.2. Boundary Condition on an Immersed Rigid Surface

The homogeneous Neumann condition $(3.21)$ is linear on fixed surfaces. The situation reverses for the nonhomogeneous Neumann condition $(3.13)$ on a moving surface $S$. It is convenient to carry out linearization locally. If linearization is done as we did before, we get linearized boundary condition:

$$\partial\phi^{(1)}/\partial n=v_n^{(1)}\,\text{on}\,S, \qquad (3.39)$$

where

$$v_n^{(1)}=\zeta_t^{(1)}/[1+\mid\nabla_\xi\zeta^{(0)}\mid^2]^{1/2},$$

is the first-order approximation of the normal velocity of $S(t,\epsilon)$.

## 3.3. Linearized Problem

As a result we can write boundary value problem for the first-order velocity potential $\phi^{(1)}(x,y,t)$. It is defined in $W$ occupied by water at rest with a boundary consisting of the free surface $F$, the bottom $B$, and the wetted surface of immersed bodies $S$ and it must satisfy

$$\nabla^2\phi^{(1)} = 0 \quad \text{in } W, \tag{3.40}$$

$$\phi_{tt}{}^{(1)} + g\phi_y{}^{(1)} = 0 \quad x \in F, \tag{3.41}$$

$$\partial\phi^{(1)}/\partial n = v_n^{(1)} \quad \text{on } S, \tag{3.42}$$

$$\partial\phi^{(1)}/\partial n = 0 \quad \text{on } B, \tag{3.43}$$

$$\phi^{(1)}(x,0,0) = \phi_0(x), \tag{3.44}$$

$$\phi_t^{(1)}(x,0,0) = -g\eta_0(x), \tag{3.45}$$

where $\phi_0, v_n^{(1)}\, and\, \eta_0$ are given functions and $\eta_0(x) = \eta^{(1)}(x,0)$.

## 3.4. Linear Time-Harmonic Waves (The Water-Wave Problem)

Since our study is concerned with the steady-state problem of scattering of water waves by bodies floating in/or beneath the free surface, we assume all motions to be simple harmonic in time. The corresponding frequency is denoted by $\omega$. Thus, the right-hand-side (3.39) is

$$v_n^{(1)} = Re\{e^{-i\omega t} f\} \quad \text{on } S, \tag{3.46}$$

where $f$ is complex function independent of $t$ and fist-order velocity potential $\phi^{(1)}$ can be written in the form,

$$\phi^{(1)}(x,y,t) = Re\{e^{-i\omega t}\widetilde{\phi}(x,y)\}. \tag{3.47}$$

A complex function $\widetilde{\phi}$ in (3.47) is also referred to as velocity potential (this does not lead to confusion, because it will always be clear what kind of time dependence is considered). $u$ is defined in the fixed domain $W$ occupied by water at rest outside any bodies present.

The boundary $\partial W$ consists of three disjoint sets:

i. $S$, is the union of the wetted surfaces of bodies in equilibrium;

ii. $F$, denotes the free surface at rest that is the part of $y = 0$ outside all the bodies;

iii. $B$, denotes the bottom positioned below $F \cup S$.

Sometimes we will consider $W$ unbounded below and corresponding to infinitely deep water. In this case $\partial W = F \cup S$.

## 3.5. Time Independent Problem

Substituting $(3.46)$ and $(3.47)$ into $(3.40) - (3.45)$ gives the boundary value problem for $\widetilde{\phi}$

$$\begin{aligned} \nabla^2 \widetilde{\phi} &= 0 \quad \text{in } W, && (3.48) \\ g\widetilde{\phi}_y - \omega^2 \widetilde{\phi} &= 0 \quad \text{on } F, && (3.49) \\ \partial \widetilde{\phi} / \partial n &= f \quad \text{on } S, && (3.50) \\ \partial \widetilde{\phi} / \partial n &= 0 \quad \text{on } B. && (3.51) \end{aligned}$$

Normal $n$ to a surface always directed into the water domain $W$. For deep water ($B = \emptyset$), condition ($\partial \widetilde{\phi} / \partial n = 0$) should be replaced by the following one,

$$|\nabla \widetilde{\phi}| \to 0 \quad \text{as} \quad y \to -\infty \tag{3.52}$$

that is, the water motion decays with depth. A natural requirement that a solution to $(3.48) - (3.52)$ should be unique also imposes a certain restriction on the behavior of $\widetilde{\phi}$ as $|x| \to \infty$. This is known as $Radiation\ Condition$.

### 3.5.1. Waves With No Bodies Present

**Example 3.1 (Waves With No Bodies Present)** *As an example we can consider waves existing in the absence of bodies. For a layer $W$ of constant depth $d$,*

$$F = \{\mathbf{x} \in \mathbb{R}^2, y = 0\}$$

*is the free surface and*

$$B = \{\mathbf{x} \in \mathbb{R}^2, y = -d\}$$

*is the bottom.*

**Solution :** We can write the problem,

$$\begin{aligned} \nabla^2\widetilde{\phi} &= 0 \quad \text{in } W, \\ \widetilde{\phi}_y - \nu\widetilde{\phi} &= 0 \quad \text{on } F, \\ \frac{\partial\widetilde{\phi}}{\partial y} &= 0 \quad \text{on } B. \end{aligned} \tag{3.53}$$

The corresponding potential can be easily obtained by separation of variables (see Appendix B)

$$\widetilde{\phi} = \frac{\cosh k(y+d)}{\cosh(kd)}(c^1 e^{i\mathbf{kx}} + c^2 e^{-i\mathbf{kx}}) \tag{3.54}$$

$$\phi^1 = \Re\left\{\frac{\cosh k(y+d)}{\cosh(kd)}(c^1 e^{i\mathbf{kx}-i\omega t} + c^2 e^{-i\mathbf{kx}-i\omega t})\right\} \tag{3.55}$$

in which $c^1$, $c^2$ any constants and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$. If $c^2 = 0$, we get *plane progressive wave*. A plane progressive wave propagating in the direction of a wave vector $\mathbf{k} = (k_1, k_2)$ has the velocity potential

$$\widetilde{\phi} = \frac{\cosh k(y+d)}{\cosh kd}\big(c^1 e^{i\mathbf{kx}}\big), \tag{3.56}$$

$$\phi^{(1)} = \Re\Big\{\frac{\cosh k(y+d)}{\cosh kd}(c^1 e^{i\mathbf{kx}-i\omega t})\Big\}. \tag{3.57}$$

If the coefficients in equation $(3.54)$ are identical, progressive waves propagating in opposite directions give a *standing wave*. Standing cylindrical wave in water layer of depth $d$ has the following potential (see Appendix C),

$$\begin{aligned} \phi^{(1)} &= \Re\{Ae^{i\mathbf{kx}-i\omega t}\}\cosh k(y+d) + \Re\{Ae^{-i\mathbf{kx}-i\omega t}\}\cosh k(y+d) && (3.58) \\ &= \Re\Big\{2Ae^{-i\omega t}\sum_{n=-\infty}^{\infty}\cos[n(\frac{\pi}{2}-\theta)]J_n(kr)\Big\}\cosh[k(y+d)]. && (3.59) \end{aligned}$$

Here $r = |\mathbf{x}|$, $A = c^1/\cosh kd$ and $J_0$ denotes the Bessel function of order zero (see Appendix C). Then a standing cylindrical wave in a water layer of depth $d$ has the following potential,

$$C_1\,\cos(\omega t)J_0(kr)\cosh[k(y+d)], \tag{3.60}$$

where $C_1$ is a real constant.

When $J_0$ replaced by $H_0$, which is another solution of Bessel's equation, we can obtain a cylindrical wave having arbitrary phase at infinity. This allows one to construct a potential of outgoing waves as follows,

$$\Re\big\{C_2\,\cos(\omega t){H_0}^{(1)}(kr)\cosh[k(y+d)]e^{-i\omega t}\big\}. \tag{3.61}$$

$C_2$ is a complex constant and ${H_0}^{(1)}$ denotes the first Hankel function of order zero. Outgoing behavior of this wave becomes clear from the asymptotic formula (see Appendix C).

### 3.5.2. Case of Single Cylinder

**Example 3.2 (Case of Single Cylinder)**

**Solution :** Since there is a cylinder with radius $a$ and with fixed position, only difference from previous example is boundary condition on $S$. Then we can write

$$\begin{aligned} \nabla^2 \widetilde{\phi} &= 0 \quad \text{in } W, \\ \widetilde{\phi}_y - \nu \widetilde{\phi} &= 0 \quad \text{on } F, \\ \tfrac{\partial \widetilde{\phi}}{\partial y} &= 0 \quad \text{on } B, \\ \tfrac{\partial \widetilde{\phi}}{\partial n} &= 0 \quad \text{on } S. \end{aligned} \tag{3.62}$$

For the problem we can decompose $\widetilde{\phi}$ as a combination of incident and diffracted waves, $\widetilde{\phi} = \widetilde{\phi}_d + \widetilde{\phi}_i$ .

| $\underline{Without\, Bodies}$ | $\underline{With\, Bodies}$ |
|---|---|
| $\nabla^2 \widetilde{\phi}_i = 0 \quad \text{in } W$ | $\nabla^2 \widetilde{\phi}_d = 0 \quad \text{in } W$ |
| $\widetilde{\phi}_i y - \nu \widetilde{\phi}_i = 0 \quad \text{on } F$ | $\widetilde{\phi}_d y - \nu \widetilde{\phi}_d = 0 \quad \text{on } F$ |
| $\frac{\partial \widetilde{\phi}_i}{\partial y} = 0 \quad \text{on } B$ | $\frac{\partial \widetilde{\phi}_d}{\partial y} = 0 \quad \text{on } B$ |
| | $\frac{\partial \widetilde{\phi}_i}{\partial n} = -\frac{\partial \widetilde{\phi}_d}{\partial n} \quad \text{on } S.$ |

For the case without body, there are only incident waves

$$\widetilde{\phi}_i = Re\left\{ Ae^{-i\omega t} \sum_{n=-\infty}^{\infty} J_n(kr) e^{in(\frac{\pi}{2}-\theta)} \right\} \cosh[k(y+d)]. \tag{3.63}$$

For the diffracted waves we can construct a potential of outgoing waves

$$\widetilde{\phi}_d = Re\left\{ Ae^{-i\omega t} \sum_{n=-\infty}^{\infty} C_n {H_n}^{(1)}(kr) e^{in(\frac{\pi}{2}-\theta)} \right\} \cosh[k(y+d)]. \tag{3.64}$$

If we apply boundary condition $\frac{\partial \widetilde{\phi}_i}{\partial n} = -\frac{\partial \widetilde{\phi}_d}{\partial n}$ on $r = a$, $a$ is the radius of cylinder, we can find the unknown coefficient

$$C_n = -\frac{{J_n}'(ka)}{H_n'(ka)}. \tag{3.65}$$

## 3.5.3. Multiple Cylinders in Progressive Waves

Now we assume that there are $N(\geqslant 1)$ fixed vertical cylinders each of which extends from the bottom, $z = -d$ up to the free surface. We will use $N+1$ coordinate systems: $(r, \theta)$ are polar coordinates in the $(x_1, x_2)$-plane. $(r_j, \theta_j)$, are polar coordinates and $(x_{1j}, x_{2j})$ are cartesian coordinates, $(j = 1, \ldots, N)$, centered at $(x^0_{1j}, x^0_{2j})$ which is the center of the $jth$ cylinder. (see Figure 3.1)

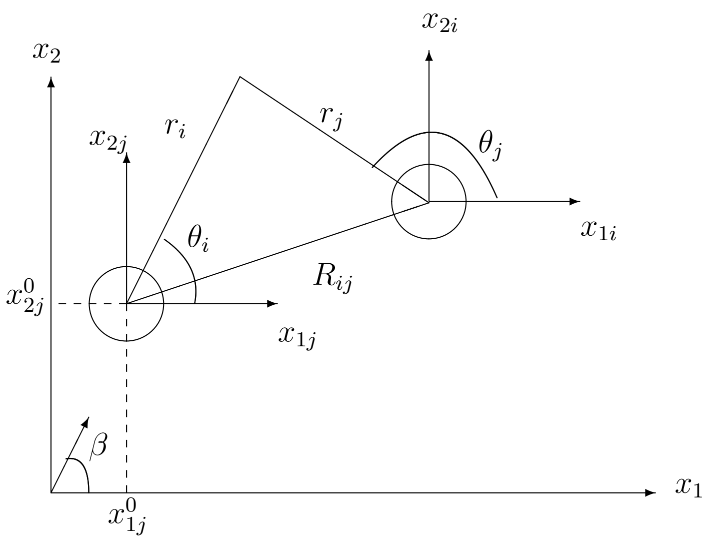


Figure 3.1. Plan View of Two Cylinders.

An incident plane wave making an angle $\beta$ with the $x$-axis is characterized by

$$\phi_I = e^{ik(x_1 \cos\beta + x_2 \sin\beta)}. \tag{3.66}$$

If we put $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$,

$$\phi_I = e^{ikr\cos(\theta-\beta)}, \tag{3.67}$$

Since

$$x_1 = x^0_{1j} + x_{1j}, \qquad x_{1j} = r_j \cos\theta_j \tag{3.68}$$

$$x_2 = x^0_{2j} + x_{2j}, \qquad x_{2j} = r_j \sin\theta_j \tag{3.69}$$

$\phi_I$ for $jth$ cylinder

$$\begin{aligned}
\phi_I^j &= e^{ik(x_1\cos\beta+x_2\sin\beta)} && (3.70)\\
&= e^{ik((x_{1j}^0+x_{1j})\cos\beta+(x_{2j}^0+x_{2j})\sin\beta)} && (3.71)\\
&= e^{ik\left((x_{1j}^0+r_j\cos\theta_j)\cos\beta+(x_{2j}^0+r_j\sin\theta_j)sin\beta\right)} && (3.72)\\
&= I_j e^{ikr_j\cos(\theta_j-\beta)} && (3.73)
\end{aligned}$$

where $I_j(=e^{ik(x_{1j}^0\cos\beta+x_{2j}^0\sin\beta)})$ is a phase factor associated with cylinder $j$.

Incident wave with respect to cylinder $j$ can be written,

$$\phi_I^j = I_j\sum_{n=-\infty}^{\infty} J_n(kr_j)e^{in(\pi/2+\beta-\theta_j)} = I_j\sum_{n=-\infty}^{\infty} J_n(kr_j)e^{in(\pi/2-\beta)}e^{in\theta_j}. \tag{3.74}$$

A general form for diffracted wave emanating from cylinder $i$,

$$\phi_d^i = I_i\sum_{n=-\infty}^{\infty} A_n^i H_n(kr_i)e^{in(\pi/2-\theta_i+\beta)}. \tag{3.75}$$

Using Graf's addition theorem for Bessel functions (see Appendix C) we can express $(3.75)$ in terms of the coordinates $(r_j,\theta_j)$

$$\phi_d^i = I_i\sum_{n=-\infty}^{\infty} A_n^i e^{in(\pi/2-\beta)}\sum_{l=-\infty}^{\infty} H_{n-l}(kR_{ij})e^{i\alpha_{ij}(n-l)}J_l(kr_j)e^{il\theta_j},$$

where $R_{ij}$ is the distance between the center of cylinder $j$ and cylinder $i$.

Total incident wave for cylinder $j$,

$$\phi_{TI}^j = \phi_I^j + \sum_{\substack{i=1\\ i\neq j}}^{N}\phi_d^i. \tag{3.76}$$

where $\phi_{TI}^j$ is the total incident wave for cylinder $j$.

$$\begin{aligned}
\phi_{TI}^j &= I_j\sum_{n=-\infty}^{\infty} J_n(kr_j)e^{in(\pi/2-\theta_j+\beta)} && (3.77)\\
&+ \sum_{\substack{i=1\\ i\neq j}}^{N} I_i\sum_{n=-\infty}^{\infty} A_{ni}e^{in(\pi/2-\beta)}\sum_{\ell=-\infty}^{\infty} H_{n-\ell}(kR_{ij})e^{i\alpha_{ij}(n-\ell)}J_\ell(kr_j)e^{i\ell\theta_j}.
\end{aligned}$$

Total velocity potential for cylinder $j$ is,

$$\phi_T^j = \phi_I^j + \sum_{\substack{i=1\\ i\neq j}}^{N}\phi_d^i + \phi_d^j. \tag{3.78}$$

If we apply the boundary condition

$$\frac{\partial \phi_T^j}{\partial r_j} = 0 \quad \text{on} \quad r_j = a_j, \quad j = 1, \ldots, N \tag{3.79}$$

$$\frac{\partial \phi_{TI}^j}{\partial r_j} = -\frac{\partial \phi_d^j}{\partial r_j} \tag{3.80}$$

For $N$ cylinder with radius $a_j$, we get,

$$A_m^j = -Z_m^j - \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{I_i}{I_j} \sum_{n=-\infty}^{\infty} A_n^i Z_m^j H_{n-m}(kR_{ij}) e^{i(n-m)(\alpha_{ij}+\pi/2-\beta)} \qquad \forall j, \forall m \tag{3.81}$$

where $Z_m^j = \frac{J'_m(ka_j)}{H'_m(ka_j)}$. If we change the indices $m$ and $n$ ,

$$A_n^j = -Z_n^j - \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{I_i}{I_j} \sum_{m=-\infty}^{\infty} A_m^i Z_n^j H_{m-n}(kR_{ij}) e^{i(m-n)(\alpha_{ij}+\pi/2-\beta)} \qquad \forall j, \forall n. \tag{3.82}$$

or

$$\widetilde{A}_n^j = -Z_n^j I_j + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{m=-\infty}^{\infty} \widetilde{A}_m^i Z_n^j H_{m-n}(kR_{ij}) e^{i(m-n)(\alpha_{ij}+\pi/2-\beta)} \qquad \forall j, \forall n. \tag{3.83}$$

Where $\widetilde{A}_n^j = A_n^j I_j$.

In order to evaluate the constants $A_n^j$, the infinite system is truncated to an $N(2M+1)$ systems of equations in $N(2M+1)$ unknowns,

$$\widetilde{A}_n^j = -Z_n^j I_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{m=-M}^{M} \widetilde{A}_m^i Z_n^j H_{m-n}(kR_{ij}) e^{i(m-n)(\alpha_{ij}+\pi/2-\beta)} \quad \forall j, n. \tag{3.84}$$

Converge is studied numerically by several authors (Linton and Evans 1990).

### 3.5.4. Limiting Value of Velocity Potential as Wave Number Approaches Zero(Long-Wave Approximation)

**Example 3.3 ( Two Cylinder Case )** *As an example we can consider two cylinder case: center of first cylinder is at $(0, R/2)$ and second one is at $(0, -R/2)$. $R$ is the distance between the center of two cylinders, $a$ is the radius of cylinders. $\alpha_{12} = -\pi/2, \alpha_{21} = \pi/2$. (see Figure 2.2)*

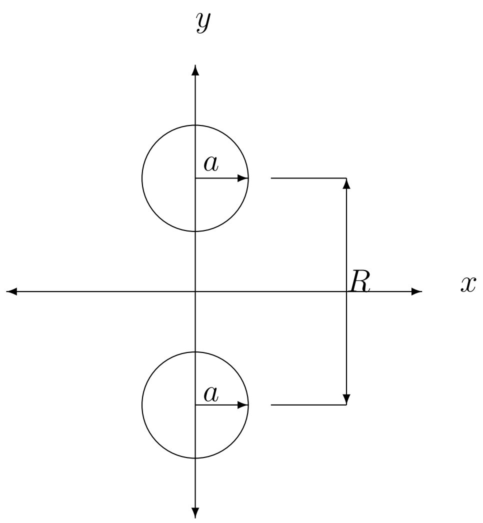


Figure 3.2. Two Cylinder Case.

**Solution:** If $j = 1, i = 2$, equation $(3.84)$ can be written as,

$$\widetilde{A}_n^1 + \sum_{m=-\infty}^{\infty} \widetilde{A}_m^2 Z_n^1 H_{m-n}(kR) e^{i(m-n)(\pi-\beta)} = -Z_n^1 I_1, \qquad \forall n. \tag{3.85}$$

If $j = 2, i = 1$ equation $(3.84)$ can be written as,

$$\widetilde{A}_n^2 + \sum_{m=-\infty}^{\infty} \widetilde{A}_m^1 Z_n^2 H_{m-n}(kR) e^{i(m-n)(-\beta)} = -Z_n^2 I_2, \qquad \forall n. \tag{3.86}$$

When we write $\widetilde{A}_n^2$ into equation $(3.85)$, we get

$$\begin{aligned} \widetilde{A}_n^1 \quad &- \quad \sum_{l=-\infty}^{\infty} \widetilde{A}_l^1 \sum_{m=-\infty}^{\infty} e^{i(l-m)(-\beta)} H_{l-m}(kR) Z_m^2 e^{i(m-n)(\pi-\beta)} H_{m-n}(kR) Z_n^1 \\ &= \quad -Z_n^1 I_1 + \sum_{m=-\infty}^{\infty} Z_m^2 I_2 e^{i(m-n)(\pi-\beta)} H_{m-n}(kR) Z_n^1 \end{aligned}$$

or

$$M_{nl}^1 \widetilde{A}_l^1 = B_n^1 \tag{3.87}$$

where

$$M_{nl}^1 = \delta_{nl} - \sum_{m=-\infty}^{\infty} e^{i(n-l)\beta} (-1)^{m-n} H_{l-m}(kR) Z_m^2 H_{m-n}(kR) Z_n^1 \tag{3.88}$$

$$B_n^1 = -Z_n^1 I_1 + \sum_{m=-\infty}^{\infty} Z_m^2 I_2 e^{i(m-n)(\pi-\beta)} H_{m-n}(kR) Z_n^1. \tag{3.89}$$

When we write $\widetilde{A}_n^1$ into equation $(3.86)$, we get

$$\begin{aligned} \widetilde{A}_n^2 \quad &- \quad \sum_{l=-\infty}^{\infty} \widetilde{A}_l^2 \sum_{m=-\infty}^{\infty} e^{i(n-l)(\beta)} (-1)^{l-m} H_{l-m}(kR) Z_m^1 H_{m-n}(kR) Z_n^2 \\ &= \quad -Z_n^2 I_2 + \sum_{m=-\infty}^{\infty} Z_m^1 I_1 e^{i(m-n)(\beta)} H_{m-n}(kR) Z_n^2 \end{aligned}$$

or

$$M_{nl}^2 \widetilde{A}_l^2 = B_n^2 \tag{3.90}$$

where

$$\begin{aligned} M_{nl}^2 &= \delta_{nl} - \sum_{m=-\infty}^{\infty} e^{i(n-l)\beta} (-1)^{l-m} H_{l-m}(kR) Z_m^1 H_{m-n}(kR) Z_n^2 &(3.91)\\ B_n^2 &= -Z_n^2 I_2 + \sum_{m=-\infty}^{\infty} Z_m^1 I_1 e^{i(m-n)(\beta)} H_{m-n}(kR) Z_n^2. &(3.92)\end{aligned}$$

Let's truncate the infinite series by taking $l = 0$ and $n = 0$,

$$\begin{aligned} M_{00}^1 \widetilde{A}_l^1 &= B_0^1 &(3.93)\\ M_{00}^2 \widetilde{A}_l^2 &= B_0^2 &(3.94)\end{aligned}$$

where

$$\begin{aligned} M_{00}^1 = M_{00}^2 &= 1 - \sum_{m=-\infty}^{\infty} (-1)^{-m} H_{-m}(kR) Z_m^1 H_m(kR) Z_n^2 &(3.95)\\ &= 1 - \sum_{m=-\infty}^{\infty} {H_m}^2(kR) Z_m^1 Z_0^2 &(3.96)\end{aligned}$$

$$\begin{aligned} B_0^1 &= -Z_0^1 I_1 + Z_0^1 I_2 \sum_{m=-\infty}^{\infty} H_m(kR) Z_m^2 e^{im\beta}\\ &\sim -i\frac{\pi(ak)^2}{4} e^{ikx_2^1 \sin\beta} - i\frac{\pi}{8}(ak)^4 e^{ikx_2^2 \sin\beta} \ln(kR) +\\ &\quad -i\pi e^{ikx_2^2 \sin\beta} \sum_{m=1}^{\infty} (\frac{ak}{2})^{m+2} \frac{\lambda^m}{m!}((-1)^m e^{-im\beta} + e^{im\beta}).\end{aligned}$$

$$\begin{aligned} B_0^2 &= -Z_0^2 I_2 + Z_0^2 I_1 \sum_{m=-\infty}^{\infty} H_m(kR) Z_m^1 e^{im\beta}\\ &\sim -i\frac{\pi(ak)^2}{4} e^{ikx_2^2 \sin\beta} - i\frac{\pi}{8}(ak)^4 e^{ikx_2^1 \sin\beta} \ln(kR) +\\ &\quad -i\pi e^{ikx_2^1 \sin\beta} \sum_{m=1}^{\infty} (\frac{ak}{2})^{m+2} \frac{\lambda^m}{m!}(e^{-im\beta} + (-1)^m e^{im\beta})\end{aligned}$$

where $\lambda = a/R$.

By using limiting forms of the Bessel functions (see Appendix C) and taking the leading order terms we get

$$\begin{aligned} \widetilde{A}_0^1 &\sim -i\frac{\pi(ak)^2}{4} e^{ikx_2^1 \sin\beta} + O(\lambda(ak)^3) &(3.97)\\ \widetilde{A}_0^2 &\sim -i\frac{\pi(ak)^2}{4} e^{ikx_2^2 \sin\beta} + O(\lambda(ak)^3). &(3.98)\end{aligned}$$

Regardless of angle of attack $\beta$, modulus of $\widetilde{A}_0^{(j)}$ is the same for both cylinders. Also, we observe that $\lambda$ does not appear in the first term, as obtained by (Linton and Evans 1990). This suggests that in the limiting case, $\lambda \to 1/2$, where the cylinders are almost touching, there should be no convergence problem. Indeed, this is verified by the numerical calculations of the general problem $(3.84)$ (Yılmaz 2004). In order to investigate the effect of $\lambda$ on convergence, we choose $n = -1, 0, 1$ and $l = -1, 0, 1$ in $(3.88, 3.89$ and $3.91, 3.92)$. After some algebra we get,

$$
\begin{aligned}
\widetilde{A}_{-1}^{(2)} &= i\frac{\pi}{4}(ak)^2(I_2 + I_1\lambda^2 e^{-2i\beta} + I_2\lambda^4 + O(\lambda^6)) \\
\widetilde{A}_{0}^{(2)} &= -i\frac{\pi}{4}(ak)^2 I_2 \\
\widetilde{A}_{1}^{(2)} &= i\frac{\pi}{4}(ak)^2(I_2 + I_1\lambda^2 e^{2i\beta} + I_2\lambda^4 + O(\lambda^6))
\end{aligned}
$$

which means

$$
\begin{aligned}
A_{-1}^{(2)} &= i\frac{\pi}{4}(ak)^2(1 + \lambda^2 e^{-2i\beta} e^{2ikb\sin\beta} + \lambda^4 + O(\lambda^6)) \\
A_{0}^{(2)} &= -i\frac{\pi}{4}(ak)^2 \\
A_{1}^{(2)} &= i\frac{\pi}{4}(ak)^2(1 + \lambda^2 e^{2i\beta} e^{2ikb\sin\beta} + \lambda^4 + O(\lambda^6))
\end{aligned}
$$

$$
\begin{aligned}
A_{-1}^{(2)} &= i\frac{\pi}{4}(ak)^2(1 + \lambda^2 e^{-2i\beta} + \lambda^4 + O(\lambda^6)) \\
A_{0}^{(2)} &= -i\frac{\pi}{4}(ak)^2 \\
A_{1}^{(2)} &= i\frac{\pi}{4}(ak)^2(1 + \lambda^2 e^{2i\beta} + \lambda^4 + O(\lambda^6))
\end{aligned}
$$

where $2b = R$. We see that the convergence is still unaffected by $\lambda$ approaching the value $1/2$.

# CHAPTER 4

# VORTEX-CYLINDER INTERACTION IN FLOWS WITH NO FREE SURFACE

## 4.1. Complex Velocity

Let $D$ be a region in the plane, the flow is incompressible , that is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

where $\mathbf{V} = (u, v)$ is velocity. Also assume that the flow is irrotational, that is,

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{4.2}$$

Let

$$\bar{V} = u - iv, \tag{4.3}$$

which is called *complex velocity*, where $\bar{V}$ denotes complex conjugation. Equations $(4.1)$ and $(4.2)$ are exactly the Caucy-Riemann equations for $\bar{V}$, and so $\bar{V}$ is an analytic function on $D$. The vector representing the complex velocity is the reflection, in the line through the point considered parallel to the $x$-axis, of the vector of the actual velocity.

## 4.2. Complex Potential

If $\bar{V}$ has a primitive, $\bar{V} = dF/dz$, then we call $F$ the *complex potential*. Write $F = \varphi + i\psi$. Then $(4.3)$ is equivalent to

$$u = \partial_x \varphi = \partial_y \psi \quad \text{and} \quad v = \partial_y \varphi = -\partial_y \psi, \tag{4.4}$$

that is, $\mathbf{V} = \mathrm{grad}\varphi$ and $\psi$ is the stream function.

Conversely, if we assume for $F$ any analytic function of $z$, the corresponding real and imaginary parts give the velocity potential and stream of a possible two-dimensional irrotational motion, for they satisfy $(4.4)$ and Laplace equation.

### 4.2.1. Complex Potential for a Vortex

Complex potential for a vortex of strength $\kappa$ at $z_0$ is

$$\omega = i\kappa \log(z - z_0).$$

### 4.2.2. Complex Velocity for a Vortex

Complex velocity for a vortex of strength $\kappa$ at $z_0$ is

$$\Omega(z) = \frac{i\kappa}{z - z_0}. \tag{4.5}$$

### 4.2.3. Streamlines of the Particle

A line drawn in the fluid so that its tangent at each point is in the direction of the fluid velocity at that point is called a *streamline*. The stream function is constant along the streamline. When the stream function is constant $u = v = 0$ and $V = 0$.

### 4.2.4. Boundary Conditions

As we analysed in Chapter 2, boundary condition, that the normal velocities are both zero, or, the fluid velocity is everywhere tangential to the fixed surface. The boundary of $c_k$ is given as a closed parametric curve

$$c_k : z = z(t) = Z_k(t), \quad Z_k(0) = Z_k(2\pi), \quad 0 \leq t \leq 2\pi,$$

(k=1,...,K). Then the tangent vector field to the curve $c_k$ is proportional to

$$T = T_x + iT_y = \dot{z} = \dot{Z}_k(t),$$

while the normal vector field is related to

$$n = n_x + in_y = i\dot{z} = i\dot{Z}_k(t).$$

On boundary $c_k$, the normal component of the velocity field must be zero. This is equivalent to the boundary condition

$$Vn|_{c_k} = \Re(\bar{V}n)|_{c_k} = \frac{1}{2}(\bar{V}n + V\bar{n})|_{c_k} = 0.$$

or

$$\text{Stream Function} = \text{Constant}.$$

For the circle

$$c_k : z(t) = z_k + re^{it},\ 0 \leq t \leq 2\pi,$$

the tangent vector is

$$T = \dot{z} = ire^{it} = i(z(t) - z_k),$$

the normal is

$$n = i\dot{z} = -(z(t) - z_k).$$

Then the boundary conditions are

$$\left[\bar{V}(z)(z - z_k) + V(\bar{z})(\bar{z} - \bar{z}_k)\right]|_{c_k} = 0,\ k = 1, ..., K.$$

## 4.3. Circle Theorem in Terms of Complex Velocity

**Theorem 4.1 (The Circle Theorem)** *(Milne-Thomson 1960) Let there be irrotational two-dimensional flow of incompressible inviscid fluid in the complex z plane. Let there be no rigid boundaries and let the complex potential of the flow be $F(z)$, where the singularities of $F(z)$ are all at the distance greater than $a$ from the origin. If a circular cylinder, typified by its cross-section the circle $C$, $|z| = a$, be introduced into the fluid of flow the complex potential becomes*

$$\omega = F(z) + \bar{F}(\frac{a^2}{z}). \tag{4.6}$$

*Where $\bar{F}$ is complex conjugate.*

**Proof:** Since $\bar{z} = a^2/z$ on the circle, we see that $\omega$ as given by $(4.6)$ is purely real on the circle $C$ and therefore stream function $\psi = 0$. Thus $C$ is a streamline. If the point $z$ is outside $C$, the point $a^2/z$ is inside $C$, and vice-versa. Since all the singularities of $F(z)$ are by hypothesis exterior to $C$, all the singularities of $\bar{F}(\frac{a^2}{z})$ are interior to $C$; in particular $\bar{F}(\frac{a^2}{z})$ has no singularity at infinity, since $F(z)$ has none at $z = 0$. Thus $\omega$ has exactly the same singularities as $F(z)$ and so all the conditions are satisfied.■

## 4.4. The Circle Theorem for Complex Velocity

**Theorem 4.2 (The Circle Theorem for Complex Velocity)** *Let $\bar{V}(z)$ be complex velocity, where $\bar{V}(z) = u - iv$, in the infinite $2 - D$ flow. All singularities of $\bar{V}(z)$ are outside $r = a$. When a cylinder of radius $a$ is introduced at the origin, $complex\ velocity$ becomes:*

$$\Omega(z) = \bar{V}(z) - \frac{a^2}{z^2} V(\frac{a^2}{z}). \tag{4.7}$$

**Proof :** Boundary condition at $r = a$ is that $\Omega(z)_n$ must be zero. To show $\Omega(z)_n = 0$ on the circle $z = ae^{i\theta}$, it is equivalent to show $\Re\{\Omega(z).e^{i\theta}\} = 0$,

$$\begin{aligned}
\Re\{\Omega(z).e^{i\theta}\} &= \Re\left\{\left[\bar{V}(z) - \frac{a^2}{z^2}V(\frac{a^2}{z})\right].e^{i\theta}\right\} \\
&= \Re\left\{\bar{V}(z).e^{i\theta}\right\} - \Re\left\{\frac{a^2}{z^2}V(\frac{a^2}{z}).e^{i\theta}\right\} \\
&= \Re\left\{(u - iv)e^{i\theta}\right\} - \Re\left\{(u + iv)e^{-i\theta}\right\} \\
&= (u\cos\theta + v\sin\theta) - (u\cos\theta + v\sin\theta) \\
&= 0.
\end{aligned}$$

This means $\Omega(z)$ satisfies the boundary condition.■

**Example 4.3 (Application of Theorem (4.2) for a Vortex of Strength $\kappa$)**

**Solution :** Let us suppose that $\bar{V}(z) = \frac{i\kappa}{z - z_0}$. Then,

$$\Omega(z) = \frac{i\kappa}{z - z_0} - \frac{a^2}{z^2}\frac{-i\kappa}{\frac{a^2}{z} - z_0} \tag{4.8}$$

$$\Omega(z) = i\kappa\frac{1}{z - z_0} + i\kappa\frac{a^2}{z}\frac{1/\bar{z}_0}{a^2/\bar{z}_0 - z} \tag{4.9}$$

$$\Omega(z) = \frac{i\kappa}{z - z_0} + \frac{i\kappa}{z} + \frac{i\kappa}{a^2/\bar{z}_0 - z}. \tag{4.10}$$

To show $\Re\{\Omega(z).e^{i\theta}\} = 0$ is equivalent to show $\Omega z + \bar{\Omega}\bar{z} = 0$.

$$z.\Omega(z) = i\kappa\left[\frac{z}{z - z_0} + 1 + \frac{\bar{z}_0 z}{a^2 - z\bar{z}_0}\right] \tag{4.11}$$

$$\bar{z}.\bar{\Omega}(z) = -i\kappa\left[\frac{\bar{z}}{\bar{z} - \bar{z}_0} + 1 + \frac{\bar{z}z_0}{a^2 - \bar{z}z_0}\right]. \tag{4.12}$$

after the summation $(4.11)$ and $(4.12)$ and putting $z\bar{z} = a^2$ we get,

$$z\Omega + \bar{z}\bar{\Omega} = i\kappa\left[\frac{z}{z - z_0} + \frac{z\bar{z}_0}{a^2 - z\bar{z}_0} - \frac{a^2}{a^2 - \bar{z}_0 z} - \frac{z_0}{z - z_0}\right]. \tag{4.13}$$

After some algebraic manipulation, the right side of $(4.13)$ vanishes.

### 4.4.1. Single Cylinder with a Vortex

*1. A Vortex at $z_0$ and a Cylinder at the Origin*: For a vortex of strength $\kappa$ at $z_0$ and a cylinder of radius $a$ at the origin complex velocity as we found above,

$$\Omega(z) = \frac{i\kappa}{z - z_0} + \frac{i\kappa}{z} + \frac{i\kappa}{a^2/\bar{z}_0 - z}. \tag{4.14}$$

*2. A Vortex at $z_0$ and a Cylinder at $z_1$,*

$$\begin{aligned}\Omega(z) &= \frac{i\kappa}{z - z_0} - \frac{a^2}{(z - z_1)^2} \frac{-i\kappa}{(\frac{a^2}{z - z_1} + \bar{z}_1) - \bar{z}_0} \\ &= \frac{i\kappa}{z - z_0} + \frac{i\kappa}{z - z_1} + \frac{i\kappa}{\frac{a^2}{\bar{z}_0 - \bar{z}_1} - (z - z_1)}.\end{aligned}$$

*3. A Vortex at $z_0$ and a Cylinder at $z_j$,*

$$\Omega(z) = \frac{i\kappa}{z - z_0} + \frac{i\kappa}{z - z_j} + \frac{i\kappa}{\frac{a^2}{\bar{z}_0 - \bar{z}_j} - (z - z_j)}. \tag{4.15}$$

### 4.4.2. Multiple Cylinders with a Vortex

*1. Laurent Series of $\frac{i\kappa}{z - z_0}$ around $z_j$,*

$$\frac{i\kappa}{z - z_0} = i\kappa \begin{cases} -(\xi_{0j})^{-1} \sum_{n=0}^{\infty} \left(\frac{\xi_j}{\xi_{0j}}\right)^n & \text{if} \quad |\xi_j| < |\xi_{0j}| \\ (\xi_{0j})^{-1} \sum_{n=0}^{\infty} \left(\frac{\xi_{0j}}{\xi_j}\right)^n & \text{if} \quad |\xi_{0j}| < |\xi_j|. \end{cases}$$

*2. Laurent Series of $\frac{i\kappa}{z - z_j} + \frac{i\kappa}{\frac{a^2}{\bar{z}_0 - \bar{z}_j} - (z - z_j)}$ around $z_j$,*

$$= i\kappa \left\{ -\sum_{n=0}^{\infty} \frac{(a^2/\bar{\xi}_{0j})^{n+1}}{(\xi_j)^{n+2}} \qquad \text{if} \quad |\ \xi_j\ |<|\ \xi_{0j}\ | \right.$$

Where $\xi_j = z - z_j$ and $\xi_{0j} = z_0 - z_j$.

### 4.4.3. Complex Velocity and Boundary Condition for Cylinder $j$

Total complex velocity around cylinder $j$ is

$$V_j^T = V_j^I + V_j^D + \sum_{\substack{i=1 \\ i \neq j}}^{n} V_i^D \tag{4.16}$$

$$= \frac{-i\kappa}{\xi_{0j}} \sum_{n=0}^{\infty} \left(\frac{\xi_j}{\xi_{0j}}\right)^n + i\kappa \sum_{n=0}^{\infty} A_n^j \frac{(\xi_{0j}{}')^{n+1}}{\xi_j^{n+1}} \tag{4.17}$$

$$+ \ i\kappa \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{n=0}^{\infty} A_n^i \frac{(\xi'_{0i})^{n+1}}{(z_j - z_i + \xi_j)^{n+2}}. \tag{4.18}$$

Boundary condition is

$$\Re\{V_j^T.e^{i\theta_j}\} = 0, \text{ when } \mid \xi_j \mid = a_j.$$

Applying the boundary condition, unknown coefficients $A_n^i$ can be found.

Changing coordinate system from $(x_i, y_i)$ to $(x_j, y_j)$: In equation $(4.16)$ to write $V_i$, we must change the coordinate system.

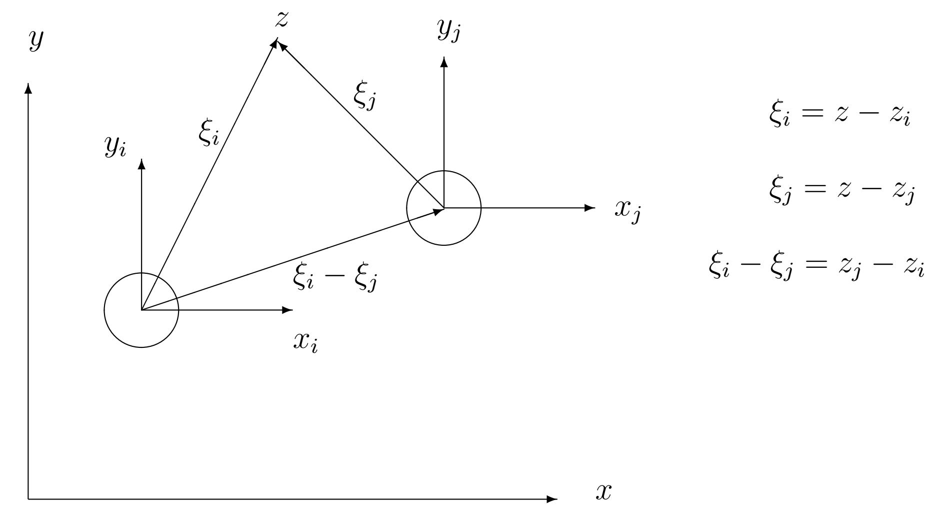


Figure 4.1. Changing Coordinate System From $\xi_i$ to $\xi_j$.

# CHAPTER 5

# VORTICES IN ANNULAR DOMAIN

## 5.1. Flow in Bounded Domain

**Problem**: Find complex velocity for incompressible irrotational flow in bounded domain $D$. Where $D$ is bounded by concentric circles. For multiply connected region $D$ with $K$ islands of shapes $c_1, ..., c_K$, the complex velocity $\bar{V} = V_x - iV_y$ is given by Laurent series (see Appendix D)

$$\bar{V}(z) = \bar{V}_0(z) + \sum_{n=0}^{\infty} a_n z^n + \sum_{\alpha=1}^{K} \sum_{n=0}^{\infty} \frac{b_{n+2}}{(z - z_\alpha)^{n+2}}$$

where points $z_1, ..., z_K$ are inside of corresponding islands and $\bar{V}_0(z)$ represents vortex part.

## 5.2. $N$ Vortices in Annular Domain

We consider problem of $N$ point vortices in annular domain $D : \{r_1 \leq |z| \leq r_2\}$, where $z_1, ..., z_N$ are positions of vortices with strength $\kappa_1, ..., \kappa_N$ correspondingly. The region is bounded by two concentric circles:

$$C_1 : z\bar{z} = r_1^2$$

and

$$C_2 : z\bar{z} = r_2^2.$$

(see Figure 4.1.)

Complex velocity is given by Laurent series (see Appendix D)

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}}. \tag{5.1}$$

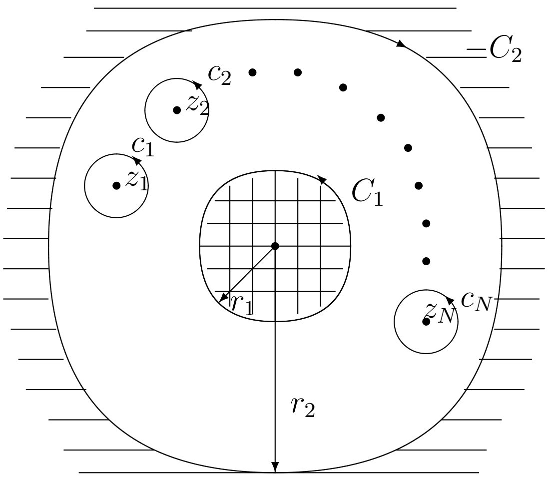


Figure 5.1. N Vortices in Annular Domain.

### 5.2.1. Algebraic System for Boundary Value Problem

To find the unknown coefficients $a_n, b_{n+2}$ we have to determine the boundary conditions,

$$\begin{aligned}\Gamma &= \bar{V}(z)z + V(\bar{z})\bar{z} && (5.2)\\ &= \sum_{k=1}^{N} \frac{i\kappa_k z}{z - z_k} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} \\ &+ \sum_{k=1}^{N} \frac{-i\kappa_k \bar{z}}{\bar{z} - \bar{z}_k} + \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^{n+1} + \sum_{n=0}^{\infty} \frac{\bar{b}_{n+2}}{\bar{z}^{n+2}}. && (5.3)\end{aligned}$$

Since $|z_k| > |z|$ and $z\bar{z} = {r_1}^2$ on boundary $C_1$, we rewrite equation $(5.3)$ as follows;

$$\sum_{k=1}^{N} \frac{i\kappa_k z}{-z_k\left(1 - \frac{z}{z_k}\right)} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + C.C. \ . \qquad (5.4)$$

using $\bar{z} = r_1^2/z$ and expanding the first term in $(5.4)$,

$$\begin{aligned}\Gamma_{|C_1} &= \sum_{k=1}^{N}(-i\kappa_k)\sum_{n=0}^{\infty}\left(\frac{z}{z_k}\right)^{n+1} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} \\ &+ \sum_{k=1}^{N} i\kappa_k \sum_{n=0}^{\infty}\left(\frac{\bar{z}}{\bar{z}_k}\right)^{n+1} + \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^{n+1} + \sum_{n=0}^{\infty} \frac{\bar{b}_{n+2}}{\bar{z}^{n+1}} && (5.5)\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=1}^{N} \frac{-i\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2}\frac{1}{r_1^{2(n+1)}}\right] z^{n+1} + C.C. = 0, && (5.6)\end{aligned}$$

where $C.C$ stands for complex conjugate. This implies the following algebraic system

$$\sum_{k=1}^{N} \frac{-i\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2}\frac{1}{r_1^{2(n+1)}} = 0, \quad n = 0, 1, \ldots \qquad (5.7)$$

Since $|z_k| < |z|$ and $z\bar{z} = {r_2}^2$ on boundary $C_2$, we rewrite equation $(5.3)$ as follows;

$$\sum_{k=1}^{N} \frac{i\kappa_k z}{z\left(1-\frac{z_k}{z}\right)} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + C.C. \ . \tag{5.8}$$

Then, expanding the first term and using $\bar{z} = r_2^2/z$, we have

$$\begin{aligned}
\Gamma_{|C_2} &= \sum_{k=1}^{N} i\kappa_k \sum_{n=0}^{\infty} \left(\frac{z_k}{z}\right)^n + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} \\
&+ \sum_{k=1}^{N} -i\kappa_k \sum_{n=0}^{\infty} \left(\frac{\bar{z}_k}{\bar{z}}\right)^n + \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^{n+1} + \sum_{n=0}^{\infty} \frac{\bar{b}_{n+2}}{\bar{z}^{n+1}} &(5.9)\\
&= \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} -i\kappa_k \frac{\bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2} \frac{1}{r_2^{2(n+1)}}\right] z^{n+1} + C.C. = 0. &(5.10)
\end{aligned}$$

This implies another algebraic system

$$\sum_{k=1}^{N} \frac{-i\kappa_k \bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2} \frac{1}{r_2^{2(n+1)}} = 0, \quad n = 0, 1, \ldots \tag{5.11}$$

### 5.2.2. Solution of Algebraic System

We have two sets of algebraic systems $(5.7)$ and $(5.11)$. By substracting $(5.11)$ from $(5.7)$, we eliminate $a_n$

$$\bar{b}_{n+2} \left[\frac{1}{r_2^{2(n+1)}} - \frac{1}{r_1^{2(n+1)}}\right] + \sum_{k=1}^{N} (-i\kappa_k) \left[\frac{\bar{z}_k^{n+1}}{r_2^{2(n+1)}} - \frac{1}{z_k^{n+1}}\right] = 0. \tag{5.12}$$

If $q \equiv r_2^2/r_1^2$ we find,

$$\bar{b}_{n+2} = \sum_{k=1}^{N} \left(\frac{i\kappa_k}{z_k^{n+1}} \frac{r_2^{2(n+1)} - |z_k|^{2(n+1)}}{q^{n+1} - 1}\right) \tag{5.13}$$

$$b_{n+2} = \sum_{k=1}^{N} \left(\frac{-i\kappa_k}{\bar{z}_k^{n+1}} \frac{r_2^{2(n+1)} - |z_k|^{2(n+1)}}{q^{n+1} - 1}\right) \tag{5.14}$$

and from $(5.7)$ we determine $a_n$,

$$a_n = \sum_{k=1}^{N} \frac{i\kappa_k}{z_k^{n+1}} - \bar{b}_{n+2} \frac{1}{r_1^{2(n+1)}}, \tag{5.15}$$

or

$$a_n = \sum_{k=1}^{N} \frac{-i\kappa_k}{z_k^{n+1}} \frac{r_1^{2(n+1)} - |z_k|^{2(n+1)}}{r_1^{2(n+1)}(q^{n+1} - 1)}. \tag{5.16}$$

### 5.2.3. Solution in Terms of q-logarithmic Function

The Taylor series part of $(5.1)$ gives the following,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty}\sum_{k=1}^{N} \frac{-i\kappa_k}{z_k^{n+1}} \frac{z^n}{q^{n+1}-1} - \sum_{n=0}^{\infty}\sum_{k=1}^{N} \frac{(-i\kappa_k)\bar{z}_k^{(n+1)} z^n}{r_1^{2(n+1)}(q^{n+1}-1)} \\ &= \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1-\frac{z}{z_k}\right) - \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1-\frac{z z_k}{r_1^2}\right) \qquad (5.17)\end{aligned}$$

and the Laurent part gives,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}} &= \sum_{n=0}^{N}\sum_{k=1}^{\infty} \frac{-i\kappa_k}{\bar{z}_k^{n+1}} \frac{r_2^{2(n+1)}}{(q^{n+1}-1)} \frac{1}{z^{n+2}} + \sum_{n=0}^{N}\sum_{k=1}^{\infty} \frac{i\kappa z_k^{n+1}}{(q^{n+1}-1)z^{n+2}} \\ &= \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1-\frac{r_2^2}{z\bar{z}_k}\right) - \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1-\frac{z_k}{z}\right) \qquad (5.18)\end{aligned}$$

where $\ln_q^{SZ}$ is q-logarithmic function given by the following definition.

**Definition 5.1 (q-logarithmic function defined by series)**

$$\ln_q^{SZ}(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{q^n - 1}, \; |x| < q, \; q > 1. \qquad (5.19)$$

*In the limiting case*

$$\lim_{q\to 1} \ln_q(1+x) = \ln(1+x). \qquad (5.20)$$

We will need the following lemma and its corollary later.

**Lemma 5.1** *Let $q$ be real and $q > 1$. Then for all integers $n \geq 0$ and $k$, with $0 < k < q^{n+1}$, the following identity holds,*

$$\sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu + k} = \frac{1}{k} \ln_q^{SZ}(1 + \frac{k}{q^n}). \qquad (5.21)$$

**Corollary 5.2** *In the particular case when $n = 0$ from the lemma $(5.1)$, we have a representation of q-logarithm*

$$\ln_q^{SZ}(1+x) = \sum_{n=1}^{\infty} \frac{x}{q^n + x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{q^n - 1} \qquad (5.22)$$

*or*

$$\ln_q(1+x) = (q-1)\sum_{n=1}^{\infty} \frac{x}{q^n + x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{[n]}. \qquad (5.23)$$

**Proof**: $n = 0$ in the lemma,

$$\sum_{\nu=1}^{\infty} \frac{1}{q^{\nu} + k} = \frac{1}{k} \ln_q^{SZ}(1+k). \tag{5.24}$$

If $n$ is replaced instead of $\nu$

$$\sum_{n=1}^{\infty} \frac{1}{q^{n} + k} = \frac{1}{k} \ln_q^{SZ}(1+k). \tag{5.25}$$

substitute $k = x$

$$\sum_{n=1}^{\infty} \frac{1}{q^{n} + x} = \frac{1}{x} \ln_q^{SZ}(1+x). \tag{5.26}$$

or

$$\ln_q^{SZ}(1+x) = \sum_{n=1}^{\infty} \frac{x}{q^{n} + x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{q^n - 1}. \blacksquare \tag{5.27}$$

### 5.2.4. Complex Velocity and $q$-logarithm

Substituting equations $(5.17)$ and $(5.18)$ in $(5.1)$ we get the following,

$$\begin{aligned} \bar{V}(z) &= \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1 - \frac{z}{z_k}\right) - \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1 - \frac{z\bar{z}_k}{r_1^2}\right) \\ &+ \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1 - \frac{r_2^2}{z\bar{z}_k}\right) - \sum_{k=1}^{N} \frac{i\kappa_k}{z} \ln_q^{SZ}\left(1 - \frac{z_k}{z}\right). \end{aligned} \tag{5.28}$$

Expanding $q$-log according to the Corollary $(5.2)$ we have

$$\begin{aligned} \bar{V}(z) &= \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} - \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - q^n \frac{r_1^2}{\bar{z}_k}} \\ &+ \sum_{k=1}^{N}\sum_{n=1}^{\infty} \left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n}\frac{r_2^2}{\bar{z}_k}}\right] - \sum_{k=1}^{N}\sum_{n=1}^{\infty} \left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n} z_k}\right]. \end{aligned} \tag{5.29}$$

$$\begin{aligned} \bar{V}(z) &= \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N} \left[\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} + \sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^{-n}}\right], \\ &- \sum_{k=1}^{N} \left[\sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{-n}} + \sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_2^2}{\bar{z}_k} q^{n}}\right]. \end{aligned} \tag{5.30}$$

### 5.2.5. Vortex Image Representation

Equation $(5.30)$ for complex velocity has countable infinite number of pole singularities. These singularities can be interpreted as vortex images in two cylindrical surfaces.

For simplicity let us consider only one vortex at position $z_0$, $r_1 < |z_0| < r_2$. Then the set of images in the cylinder $C_1$ we denote $z_I^{(1)}, z_I^{(2)}, ...$ and in the cylinder $C_2$ as $z_{II}^{(1)}, z_{II}^{(2)}, ....$

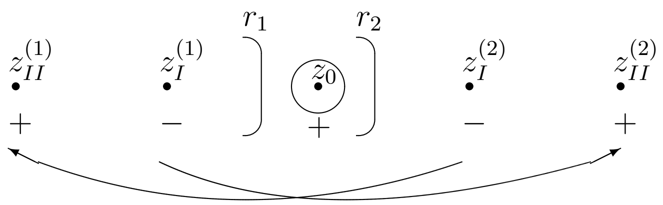


Figure 5.2. Vortex Image Representation.

Considering the above figure, we obtain the following,

$$
\begin{aligned}
z_I^{(1)} &= \frac{r_1^2}{\bar{z}_0} & z_{II}^{(1)} &= \frac{r_2^2}{\bar{z}_0} && (5.31)\\
z_I^{(2)} &= \frac{r_1^2}{\bar{z}_{II}^{(1)}} = \frac{z_0}{q} & z_{II}^{(2)} &= \frac{r_2^2}{\bar{z}_I^{(1)}} = q z_0 && (5.32)\\
z_I^{(3)} &= \frac{r_1^2}{\bar{z}_{II}^{(2)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q} & z_{II}^{(3)} &= \frac{r_2^2}{\bar{z}_I^{(2)}} = \frac{r_2^2}{\bar{z}_0} q && (5.33)\\
z_I^{(4)} &= \frac{r_1^2}{\bar{z}_{II}^{(3)}} = \frac{z_0}{q^2} & z_{II}^{(4)} &= \frac{r_2^2}{\bar{z}_I^{(3)}} = z_0 q^2 && (5.34)\\
z_I^{(5)} &= \frac{r_1^2}{\bar{z}_{II}^{(4)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q^2} & z_{II}^{(5)} &= \frac{r_2^2}{\bar{z}_I^{(4)}} = \frac{r_2^2}{\bar{z}_0} q^2. && (5.35)
\end{aligned}
$$

Combining together and taking into account alternating signs (positive for the first image and negative for the next one - image of the image ) we have the set

$$+ z_0,\ -z_I^{(1)},\ +z_{II}^{(2)},\ -z_I^{(3)},\ +z_{II}^{(4)},\ -z_I^{(5)}, ... \tag{5.36}$$

and

$$+ z_0,\ -z_{II}^{(1)},\ +z_I^{(2)},\ -z_{II}^{(3)},\ +z_I^{(4)},\ -z_{II}^{(5)}, .... \tag{5.37}$$

This shows that the set of vortex images is completely determined by the singularities of the $q$-logarithmic function.

## 5.3. Complex Potential and Elliptic Functions

In the above representation we can combine sums so that, we have

$$\bar{V}(z) = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}\frac{i\kappa_k}{z-z_kq^n}\right]-\sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\frac{i\kappa_k}{z-\frac{r_1^2}{\bar{z}_k}q^{-n}}+\sum_{n=1}^{\infty}\frac{i\kappa_k}{z-\frac{r_1^2}{\bar{z}_k}q^{n}}\right] \quad (5.38)$$

$$= \sum_{k=1}^{N} i\kappa_k\sum_{n=-\infty}^{\infty}\left[\frac{1}{z-z_kq^n}-\frac{1}{z-\frac{r_1^2}{\bar{z}_k}q^n}\right]. \quad (5.39)$$

Let us introduce the complex potential of the flow

$$\bar{V}(z) = F'(z)$$

so that by simple integration we get

$$F(z) = \sum_{k=1}^{N} i\kappa_k\sum_{n=-\infty}^{\infty} Ln\left(\frac{z-z_kq^n}{z-\frac{r_1^2}{\bar{z}_k}q^n}\right) \quad (5.40)$$

$$= \sum_{k=1}^{N} i\kappa_k Ln\prod_{n=-\infty}^{\infty}\frac{z-z_kq^n}{z-\frac{r_1^2}{\bar{z}_k}q^n} \quad (5.41)$$

$$= \sum_{k=1}^{N} i\kappa_k Ln\left(\frac{z-z_k}{z-\frac{r_1^2}{\bar{z}_k}}\right)\prod_{n=1}^{\infty}\frac{(z-z_kq^n)(z-z_kq^{-n})}{(z-\frac{r_1^2}{\bar{z}_k}q^n)(z-\frac{r_1^2}{\bar{z}_k}q^{-n})}. \quad (5.42)$$

To compare our solution with that of Johnson and McDonald(2004), we consider one vortex of strength $\kappa$ at position $z_0$,

$$F'(z) = \bar{V}(z) = \sum_{n=-\infty}^{\infty}\frac{i\kappa}{z-z_0q^n}-\sum_{n=-\infty}^{\infty}\frac{i\kappa}{z-\frac{r_1^2}{\bar{z}_0}q^n}. \quad (5.43)$$

Then for complex potential we have

$$F(z) = i\kappa Ln\underbrace{\prod_{n=-\infty}^{\infty}\frac{(z-z_0q^n)}{(z-\frac{r_1^2}{\bar{z}_0}q^n)}}_{\Phi}. \quad (5.44)$$

For comparison purpose, we fix the radius $r_2 = 1$ so that $q = \frac{r_2^2}{r_1^2} = \frac{1}{r_1^2} \equiv \frac{1}{\tilde{q}^2}$, where we introduce new parameter $\tilde{q}$. Thus all singularities of $F(z)$ are determined by zeroes and poles of the function

$$\Phi = \prod_{n=-\infty}^{\infty}\frac{(z-z_0\frac{1}{\tilde{q}^{2n}})}{(z-\frac{\tilde{q}^2}{\bar{z}_0}\frac{1}{\tilde{q}^{2n}})} = \prod_{n=-\infty}^{\infty}\frac{(z-z_0\tilde{q}^{2n})}{(z-\frac{1}{\bar{z}_0}\tilde{q}^{2n})} \quad (5.45)$$

$$= \frac{z-z_0}{z-\frac{1}{\bar{z}_0}}\prod_{n=1}^{\infty}\frac{(z-z_0\tilde{q}^{2n})(z_0-z\tilde{q}^{2n})}{(z-\frac{1}{\bar{z}_0}\tilde{q}^{2n})(\frac{1}{\bar{z}_0}-z\tilde{q}^{2n})}. \quad (5.46)$$

If

$$\frac{z}{z_0} = e^{2iu}, \quad z\bar{z}_0 = e^{2iv}, \quad z_0\bar{z}_0 = \frac{e^{2iv}}{e^{2iu}} \tag{5.47}$$

we then obtain,

$$\Phi = \frac{(1-e^{-2iu})}{(1-e^{-2iv})}\prod_{n=1}^{\infty}\frac{e^{2iv}}{e^{2iu}}\frac{(1-\tilde{q}^{2n}e^{2iu})(1-\tilde{q}^{2n}e^{-2iu})}{(1-\tilde{q}^{2n}e^{2iv})(1-\tilde{q}^{2n}e^{-2iv})} \tag{5.48}$$

$$\Phi' = \frac{2G}{2G}\frac{\tilde{q}^{1/4}\sin u}{\tilde{q}^{1/4}\sin v}\frac{\prod_{n=1}^{\infty}(1-\tilde{q}^{2n}e^{2iu})(1-\tilde{q}^{2n}e^{-2iu})}{\prod_{n=1}^{\infty}(1-\tilde{q}^{2n}e^{2iv})(1-\tilde{q}^{2n}e^{-2iv})} \tag{5.49}$$

where $\Phi'$ is a constant multiple of $\Phi$,

$$G \equiv \prod_{n=1}^{\infty}(1-\tilde{q}^{2n}), \quad \tilde{q} < 1, \tag{5.50}$$

$$\Phi' = \frac{\Theta_1(u,\tilde{q})}{\Theta_1(v,\tilde{q})} = \frac{\Theta_1(\frac{i}{2}(\tau-\tau_0),\tilde{q})}{\Theta_1(\frac{i}{2}(\tau+\tau_0),\tilde{q})} \tag{5.51}$$

where $\tau \equiv -\ln z$, $\tau_0 \equiv -\ln z_0$, $\bar{\tau}_0 \equiv -\ln \bar{z}_0$ and where $\Theta_1$ is the first Jacobi Theta function (see Appendix D). (The last transformation is the conformal mapping of annular domain to semi-infinite strip). Then

$$F(z) = i\kappa Ln\frac{\Theta_1(\frac{i}{2}(\tau-\tau_0),\tilde{q})}{\Theta_1(\frac{i}{2}(\tau+\tau_0),\tilde{q})}. \tag{5.52}$$

For the stream function, we have

$$\begin{aligned}
\Psi &= \frac{F-\bar{F}}{2i} = \frac{i\kappa}{2i}\left[Ln\frac{\Theta_1}{\Theta_1} + Ln\frac{\bar{\Theta}_1}{\bar{\Theta}_1}\right] && (5.53)\\
&= \kappa Ln\left|\frac{\Theta_1(\frac{i}{2}(\tau-\tau_0),\tilde{q})}{\Theta_1(\frac{i}{2}(\tau+\tau_0),\tilde{q})}\right| && (5.54)\\
&= \frac{-\Gamma}{2\pi}Ln\left|\frac{\Theta_1(\frac{i}{2}(\tau-\tau_0),\tilde{q})}{\Theta_1(\frac{i}{2}(\tau+\tau_0),\tilde{q})}\right| && (5.55)
\end{aligned}$$

This coincides with result of Johnson and McDonald $(2004)$.

## 5.4. Motion of a Point Vortex in Annular Domain

We use the above formulas to determine the motion of single vortex in annular domain. Complex velocity of vortex is determined by

$$\begin{aligned}
\dot{z}_0 &= \dot{x}_0 + i\dot{y}_0 = V(\bar{z})|_{z=z_0} && (5.56)\\
&= \sum_{n=-\infty}^{\infty}\frac{-i\kappa}{\bar{z}_0-\bar{z}_0q^n} - \sum_{n=-\infty}^{\infty}\frac{-i\kappa}{\bar{z}_0-\frac{r_1^2}{z_0}q^n} && (5.57)\\
&= \frac{-i\kappa}{\bar{z}_0}\sum_{n=-\infty}^{\infty}\frac{1}{1-q^n} + z_0 i\kappa\sum_{n=-\infty}^{\infty}\frac{1}{|z_0|^2-r_1^2q^n}. && (5.58)
\end{aligned}$$

Substituting this to following equation we have,

$$\bar{z}_0 \dot{z}_0 + z_0 \dot{\bar{z}}_0 = \frac{d}{dt}|z_0|^2 = 0 \rightarrow |z_0| = \text{constant}. \tag{5.59}$$

This implies that the distance of the vortex from the origin is constant. Then only the argument is time independent,

$$z_0 = |z_0| e^{i\varphi(t)}, \qquad \dot{z}_0 = z_0 e^{i\varphi} i\dot{\varphi}, \tag{5.60}$$

which satisfies

$$\dot{\varphi} = -\kappa \left[ \frac{1}{|z_0|^2} \sum_{n=-\infty}^{\infty} \frac{1}{1-q^n} - \sum_{n=-\infty}^{\infty} \frac{1}{|z_0|^2 - r_1^2 q^n} \right]. \tag{5.61}$$

The right hand side of equation $5.61$ is a constant and the solution is

$$\varphi(t) = \omega t + \varphi_0 \tag{5.62}$$

where frequency $\omega$ is dependent on modulus $|z_0|$,

$$\begin{aligned} \omega &= -\kappa \left[ \frac{1}{|z_0|^2} \sum_{n=-\infty}^{\infty} \frac{1}{1-q^n} - \sum_{n=-\infty}^{\infty} \frac{1}{|z_0|^2 - r_1^2 q^n} \right] && (5.63) \\ &= \frac{-\kappa}{|z_0|^2} \sum_{n=0}^{\infty} \frac{1}{q^{n+1}-1} \left[ \left( \frac{|z_0|^2}{r_1^2} \right)^{n+1} - \left( \frac{r_2^2}{|z_0|^2} \right)^{n+1} \right]. && (5.64) \end{aligned}$$

Finally solution of our problem can be written by using q-logarithmic function

$$\omega = \frac{\kappa}{|z_0|^2 (q-1)} \left[ \ln_q \left( 1 - \frac{|z_0|^2}{r_1^2} \right) - \ln_q \left( 1 - \frac{r_2^2}{|z_0|^2} \right) \right]. \tag{5.65}$$

So, we found that vortex is uniformly rotating around the center,

$$z_0(t) = |z_0| e^{i\omega t + i\varphi_0} = z_0(0) e^{i\omega t} \tag{5.66}$$

with frequency depending on vortex strength, initial position and geometry of annular domain in terms of q-logarithms. This reflects the fact that the motion of vortex results from interaction with infinite set of its images in the cylinders.

# CHAPTER 6

# CONFORMAL MAPPING OF ANNULAR DOMAIN ONTO THE INFINITE DOMAIN WITH TWO CYLINDERS

## 6.1. Möbius Transformation and Mappings of the Annular Domain

**Definition 6.1** *A mapping of the form*

$$t(z) = \frac{az+b}{cz+d} \tag{6.1}$$

*where $a, b, c, d \in \mathbb{C}$, is called a linear fractional transformation. If $a, b, c$ and $d$ also satisfy $ad - bc \neq 0$ then $t(z)$ is called Möbius transformation. (Churcill and Brown 1984)*

when $c \neq 0$, equation $(6.1)$ can be written

$$w = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}; \tag{6.2}$$

and it is clear how the condition $ad - bc \neq 0$ ensures that a linear fractional transformation is never a constant function.

If $c = 0\,(d \neq 0)$, then $t(z)$ in equation $(6.1)$ reduces to the entire linear transformation,

$$t(z) = \alpha z + \beta, \qquad \left(\alpha = \frac{a}{d}\ , \beta = \frac{b}{d}\ \right) \tag{6.3}$$

Möbius transformations carries generalized circles into generalized circles. In the extended plane a generalized circle is either a circle or a straight line (Circle with infinite radius).

### 6.1.1. Equation of Circle

Let $c \in \mathbb{C}$, $\rho > 0$ and $\Gamma$ be the circle with center $c$ and radius $\rho$, then the equation of the circle

$$(z-c)(\bar{z}-\bar{c}) = \rho^2. \tag{6.4}$$

### 6.1.2. Symmetry with Respect to Circle

**Definition 6.2** *If*

$$(z_1 - c)(\bar{z}_2 - \bar{c}) = \rho^2 \tag{6.5}$$

*then $z_1$ and $z_2$ are symmetric with respect to $\Gamma$.*

**Lemma 6.1** *If $a$ and $b$ are complex numbers, $\mu > 0$*

$$\left|\frac{z-a}{z-b}\right| = \mu \tag{6.6}$$

*represents a generalized circle. Number $\mu$ is related with the circle radius $\rho$, $\rho = \frac{\mu|a-b|}{(1-\mu^2)}$. Equation (6.6) becomes a straight line if and only if $\mu = 1$.*

**Proof :** First of all lets show that equation (6.6) is a circle. Expansion of $|z-c|^2 = \rho^2$ gives the following,

$$z\bar{z} - \bar{c}z - c\bar{z} + c\bar{c} - \rho^2 = 0 \tag{6.7}$$

On the other hand, expansion of (6.6) yields,

$$\begin{aligned} (1-\mu^2)z\bar{z} + (\mu^2\bar{b} - \bar{a})z + (\mu^2 b - a)\bar{z} + (a\bar{a} - \mu^2 b\bar{b}) &= 0 &(6.8)\\ z\bar{z} + \frac{(\mu^2\bar{b} - \bar{a})}{(1-\mu^2)}z + \frac{(\mu^2 b - a)}{(1-\mu^2)}\bar{z} + \frac{(a\bar{a} - \mu^2 b\bar{b})}{1-\mu^2} &= 0 &(6.9)\end{aligned}$$

If we compare equation (6.9) with equation (6.7), we find, $c = \frac{(a-\mu^2 b)}{(1-\mu^2)}$ and $\rho^2 = \frac{\mu^2(\bar{a}-\bar{b})(a-b)}{(1-\mu^2)^2}$. This implies that (6.6) represents a circle.

If generalized circle represents a straight line then we will show $\mu = 1$. For the generalized circle in equation (6.8) to be a straight line, the coefficient of $z\bar{z}$ must be zero. This means $1-\mu^2 = 0$, then since $\mu > 0$, we get $\mu = 1$. If we substitute $\mu = 1$ in equation (6.6)

$$\begin{aligned} |z-a| &= |z-b| &(6.10)\\ (z-a)(\bar{z}-\bar{a}) &= (z-b)(\bar{z}-\bar{b}) &(6.11)\\ (\bar{b}-\bar{a})z + (b-a)\bar{z} + a\bar{a} - b\bar{b} &= 0 &(6.12)\end{aligned}$$

since the last equation is in the form $Az + \bar{A}\bar{z} + C = 0$, which is equation of a line, we get the desired result.■

**Theorem 6.2** *The points $a$ and $b$ are symmetric with respect to the generalized circle (6.6).*

**Proof :** If $a$ and $b$ are symmetric point, from equation $(6.5)$ we should have

$$(a-c)(\bar{b}-\bar{c})=\rho^2. \tag{6.13}$$

If we substitute $c=\frac{(a-\mu^2 b)}{(1-\mu^2)}$ and $\rho^2=\frac{\mu^2(\bar{a}-\bar{b})(a-b)}{(1-\mu^2)^2}$ in equation $(6.13)$, then we get

$$\begin{aligned}\left(a-\frac{a-\mu^2 b}{1-\mu^2}\right)\left(\bar{b}-\frac{\bar{a}-\mu^2\bar{b}}{(1-\mu^2)}\right) &= \frac{\mu^2(b-a)}{1-\mu^2}.\frac{\bar{b}-\bar{a}}{1-\mu^2} && (6.14)\\ &= \rho^2. && (6.15)\end{aligned}$$

Hence equation $(6.13)$ is satisfied.■

**Theorem 6.3** *Let $a$ and $b$ be symmetric with respect to generalized circle $\Gamma$, $a\neq b$, then $\exists\ \mu>0$ such that $\Gamma$ is given by equation $(6.6)$*

**Proof :** Since $a$ and $b$ are symmetric we can write

$$\begin{aligned}(a-c)(\bar{b}-\bar{c}) &= \rho^2 && (6.16)\\ a\bar{b}-a\bar{c}-c\bar{b} &= \rho^2-c\bar{c}. && (6.17)\end{aligned}$$

Equation of a circle with center $c$ and radius $\rho$ is

$$\begin{aligned}(z-c)(\bar{z}-\bar{c}) &= \rho^2 && (6.18)\\ z\bar{z}-\bar{c}z-c\bar{z}+c\bar{c} &= \rho^2. && (6.19)\end{aligned}$$

In the last equation, $(\rho^2-c\bar{c})$, can be replaced by$(6.17)$ and using $c=\frac{(a-\mu^2 b)}{(1-\mu^2)}$, we get

$$\begin{aligned}z\bar{z}-\frac{(\bar{a}-\mu^2\bar{b})}{(1-\mu^2)}z-\frac{(a-\mu^2 b)}{(1-\mu^2)}\bar{z} &= a\bar{b}-a\frac{(\bar{a}-\mu^2\bar{b})}{(1-\mu^2)}-\frac{(a-\mu^2 b)}{(1-\mu^2)}\bar{b} && (6.20)\\ z\bar{z}-\mu^2 z\bar{z}-\bar{a}z+\mu^2\bar{b}z-a\bar{z}+\mu^2 b\bar{z} &= \mu^2 b\bar{b}-a\bar{a} && (6.21)\\ \frac{z\bar{z}-\bar{a}z-a\bar{z}+a\bar{a}}{b\bar{b}+z\bar{z}-\bar{b}z-b\bar{z}} &= \mu^2 && (6.22)\\ \left|\frac{z-a}{z-b}\right| &= \mu.\blacksquare && (6.23)\end{aligned}$$

**Theorem 6.4** *Let $a$ and $b$ be two points of the complex plane which are symmetric with respect to generalized circle $\Gamma$ and let $t$ be a Möbius transformation, then $a'=t(a)$ and $b'=t(b)$ are symmetric with respect to $\Gamma'$,where $\Gamma'$ is the image of $\Gamma$ under $t$.*

**Proof :** If

$$\begin{aligned}t(z) &= z+m, && t \text{ is translation;} && (6.24)\\ t(z) &= mz,\ m>0, && t \text{ is a dilation;} && (6.25)\\ t(z) &= e^{i\theta}z, && t \text{ is rotation;} && (6.26)\\ t(z) &= 1/z, && t \text{ is inversion.} && (6.27)\end{aligned}$$

Möbius transformation is a composition of translation, rotation, inversion and dilation. When the circle is translated, rotated and dilated, its symmetry is conserved. So it is sufficient to consider the inversion only, $t(z) = 1/z$. Since $a$ and $b$ are inverses of each other with respect to the circle $\Gamma$ by Theorem 6.3 equation for $\Gamma$ is,

$$\left|\frac{z-a}{z-b}\right| = \mu \quad \text{for some } \mu. \tag{6.28}$$

Applying inversion $(z = 1/w)$ to $t$, we have,

$$\left|\frac{\frac{1}{w}-a}{\frac{1}{w}-b}\right| = \mu \tag{6.29}$$

$$\left|\frac{w-a'}{w-b'}\right| = \mu\left|\frac{b}{a}\right|, \quad \text{where, } \frac{1}{a} = a',\ \frac{1}{b} = b'. \tag{6.30}$$

This shows that $a'$ and $b'$ are symmetric with respect to $\Gamma'$.■

### 6.1.3. Mapping of Disc onto Unit Disc

**Example 6.5** *Let $D$ be a disk and let $a$ be a point in $D$. Find all Möbius transformations $t$ which maps $D$ onto the unit disc such that $t(a) = 0$.*

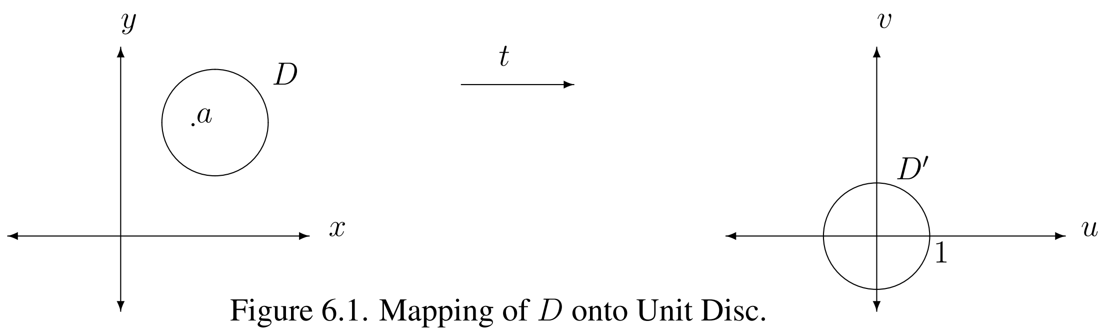


Figure 6.1. Mapping of $D$ onto Unit Disc.

**Solution :** We have two cases,

1) If $a$ is the center of $D$, by using the entire Möbius transformation $t = az + b$ we can do the mapping.

2) If $a$ is not the center, we can use Möbius transformation

$$t = c\frac{z-a}{z-b}, \tag{6.31}$$

where $b$ is the inverse of $a$, then $a\bar{b} = 1$, $a \neq b$.

Let us take any point $z_1$ on the boundary of $D$,

$$z_1{}' = c\frac{z_1-a}{z_1-b}$$

since ${z_1}'$ is on the unit circle then

$$|c\frac{z_1 - a}{z_1 - b}| = 1$$

then $|c| = |\frac{z_1 - b}{z_1 - a}|$ and $c = |c|e^{i\alpha}$, $0 < \alpha < 2\pi$. Möbius transformation for this case is

$$t = e^{i\alpha}\left|\frac{z_1 - b}{z_1 - a}\right|\frac{z - a}{z - b}, \quad 0 < \alpha < 2\pi. \tag{6.32}$$

### 6.1.4. Mapping of Unit Disc onto Unit Disc

**Example 6.6** *Find the most general Möbius transformation $t$ such that it maps unit disk onto itself.*

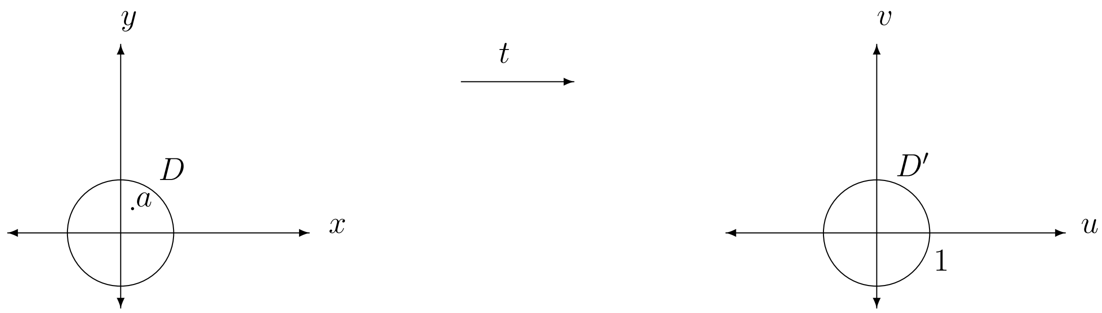


Figure 6.2. Mapping of Unit Disc onto Unit Disc.

**Solution :** This example is a special case of the previous example. Since $D$ is unit disk we can take $z_1 = 1$, in $(6.32)$,

$$t(z) = e^{i\alpha'}\frac{z - a}{1 - \bar{a}z} \tag{6.33}$$

where $\left|\frac{\bar{a}-1}{1-a}\right| = 1$ and $e^{i\alpha'} = e^{i\alpha}.\frac{\bar{a}}{|\bar{a}|}$.

**Example 6.7** *Let $c$ and $\rho$ be real and $c > \rho > b$. Map the domain by $|z - c| \leq \rho$ and left hand side of the imaginary axis onto an annular domain bounded by $|w| = 1$ and a concentric circle. Find the Möbius transformation.*

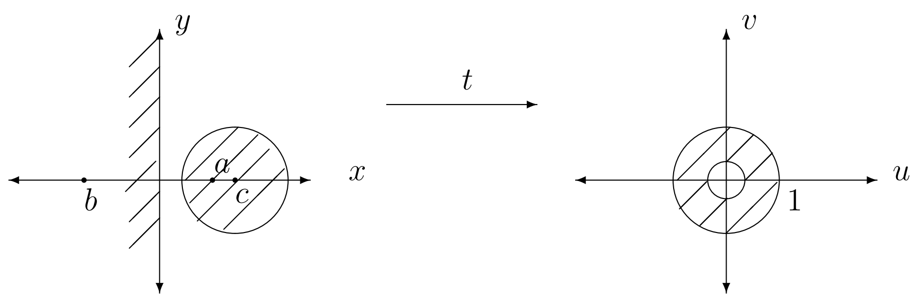


Figure 6.3. Mapping of Given $D$ onto Annular Domain.

**Solution :** In this example $a$ and $b$ are symmetric with respect to $y$-axis and $|z - c| = \rho^2$. Since $a$ and $b$ are symmetric with respect to $y$-axis then

$$b = -a.$$

Since they are symmetric with respect to $|z - c| = \rho^2$ they must satisfy

$$(b - c)(a - c) = \rho^2.$$

If we replace $b = -a$, we can find

$$\begin{aligned} a &= (c^2 - \rho^2)^{1/2} \\ b &= -(c^2 - \rho^2)^{1/2}. \end{aligned}$$

Möbius transformation is

$$t(z) = \frac{z - (c^2 - \rho^2)^{1/2}}{z + (c^2 - \rho^2)^{1/2}}. \tag{6.34}$$

**Example 6.8** *Let $\rho$ and $c$ be real and, $0 < c < 1 - \rho$. Map the domain bounded by the circles $|z| = 1$ and $|z - c| = \rho$ onto domain bounded by two concentric circles such that the outer circle is mapped onto itself. What is the radius $R_0$ of the inner circle?*

**Solution :** Using the symmetry of $a$ and $b$ with respect to unit circle and $|z - c| = \rho$ we can find

$$a = \frac{1 + c^2 - \rho^2 + \sqrt{(1 - (c + \rho)^2)(1 - (c - \rho)^2)}}{2c} \tag{6.35}$$

$$\bar{b} = \frac{2c}{1 + c^2 - \rho^2 + \sqrt{(1 + c^2 - \rho^2)^2 - 4c^2}}. \tag{6.36}$$

and Möbius transformation;

$$t(z) = w = \frac{z - a}{az - 1} \tag{6.37}$$

To find $R_0$ we can substitute $z = c - \rho$ in (6.37), then

$$R_0 = \frac{1 - c^2 + \rho^2 + \sqrt{(1 - (c + \rho)^2)(1 - (c - \rho)^2)}}{2\rho} \tag{6.38}$$

If $c = \frac{x_1 + x_2}{2}$ and $\rho = \frac{x_1 - x_2}{2}$, we get same result with that of Churchill and Brow(Churchill and Brow 1984).

### 6.1.5. Mapping of Two Cylinder onto Annular Domain

**Example 6.9 (Special Case of Example 5.8):** *Let $\rho$ and $c$ be real,* $1 < c - \rho$

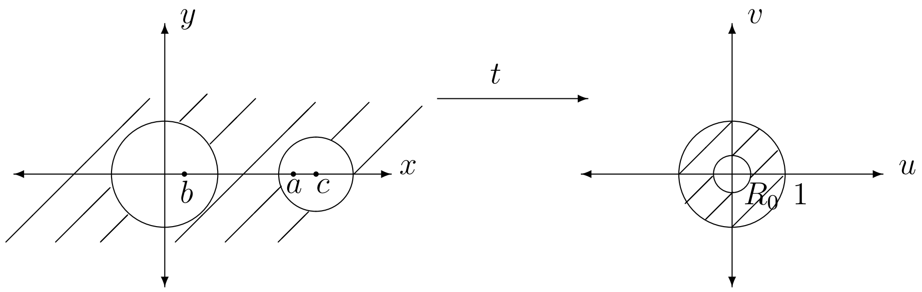


Figure 6.4. Mapping of Two Cylinder onto Annular Domain .

**Solution :** Lets take two real points $a$ and $b$ which are symmetric points with respect to both circles $|z| = 1$ and $|z - c| = \rho$.

Symmetry with respect to $|z| = 1$,

$$ab = 1 \quad \Rightarrow \quad b = \frac{1}{a}$$

symmetry with respect to $|z - c| = \rho$

$$(a - c)(b - c) = \rho^2$$

or

$$a^2 c + a\left(\rho^2 - c^2 - 1\right) + c = 0$$

when we solve last equation

$$a = \frac{1 + c^2 - \rho^2 + \sqrt{((c+\rho)^2 - 1)((c-\rho)^2 - 1)}}{2c}. \tag{6.39}$$

Using $b = \frac{1}{a}$ in equation $(6.31)$ Möbius transformation

$$t(z) = w = \frac{z - a}{az - 1} \tag{6.40}$$

where $c\,a = 1$. When we substitute $z = c + \rho$ in equation $(6.40)$

$$R_0 = \frac{c^2 - \rho^2 - 1 - \sqrt{((c+\rho)^2 - 1)((c-\rho)^2 - 1)}}{2\rho} \tag{6.41}$$

or in terms of $x_1$ and $x_2$

$$R_0 = \frac{x_1 x_2 - 1 - \sqrt{({x_1}^2 - 1)({x_2}^2 - 1)}}{x_1 - x_2} \tag{6.42}$$

$$a = \frac{1 + x_1 x_2 + \sqrt{({x_1}^2 - 1)({x_2}^2 - 1)}}{x_1 + x_2}. \tag{6.43}$$

## 6.2. Application of the Möbius Transformation to the Vortex-Cylinder Problem

Let assume that there exist a vortex at $z_0$ in infinite domain (see Figure$(6.8)$), then the image of complex potential in $w$-plane is,

$$F(z) = ln(z - z_0) \quad (6.44)$$

$$F(z(w)) = f(w) = ln\left(\frac{w-a}{aw-1} - \frac{w_0 - a}{aw_0 - 1}\right) \quad (6.45)$$

$$= ln\left(\frac{a^2-1}{a(aw_0-1)}\right) + ln(w - w_0) - ln(w - \frac{1}{a}), \quad (6.46)$$

where $w_0$ is the image of the point $z_0$. Hence if there exists a vortex $z_0$ in $z$-plane then its image vortices with opposite strength are at $w_0$ and $1/a$. In Chapter $4$, we found vortex image representation for $N$ vortices in annular domain (See equation $(5.40)$). Now we consider 2 vortices at positions $w_0$ and $\frac{1}{a}$,

$$f(w) = \sum_{k=1}^{2} i\kappa_k \sum_{n=-\infty}^{\infty} Ln\left(\frac{w - w_k q^n}{w - \frac{r_1^2}{\bar{w}_k} q^n}\right) \quad (6.47)$$

$$= i\kappa \sum_{n=-\infty}^{\infty} Ln\left(\frac{w - w_0 q^n}{w - \frac{R_0^2}{\bar{w}_0} q^n}\right) - i\kappa \sum_{n=-\infty}^{\infty} Ln\left(\frac{w - \frac{q^n}{a}}{w - aR_0{}^2 q^n}\right) \quad (6.48)$$

$$= i\kappa \sum_{n=-\infty}^{\infty} Ln\left(\frac{w - w_0 q^n}{w - \frac{R_0^2}{\bar{w}_0} q^n} \cdot \frac{w - aR_0^2 q^n}{w - \frac{q^n}{a}}\right). \quad (6.49)$$

where $r_1 = R_0,\ r_2 = 1$ and $q \equiv \frac{1}{R_0^2}$.

By applying the inverse mapping $w = \frac{z-a}{az-1}$, we can find its vortex image representation in $z$-plane,

$$F(z) = i\kappa \sum_{n=-\infty}^{\infty} Ln\left(\alpha_n \frac{z + \frac{w_0 q^n - a}{1 - a w_0 q^n}}{z + \frac{R_0^2 q^n - a\bar{w}_0}{\bar{w}_0 - aR_0^2 q^n}} \cdot \frac{z + \frac{-a + aR_0^2 q^n}{1 - a^2 R_0^2 q^n}}{z + \frac{q^n - a^2}{a - aq^n}}\right) \quad (6.50)$$

$$= i\kappa \sum_{n=-\infty}^{\infty} Ln\left(\alpha_n \frac{z - t(w_0 q^n)}{z - t(\frac{R_0^2 q^n}{\bar{w}_0})} \cdot \frac{z - t(aq^n R_0^2)}{z - t(\frac{q^n}{a})}\right) \quad (6.51)$$

where $\alpha_n = \frac{(1 - aw_0 q^n)(1 - a^2 R_0^2 q^n)}{(\bar{w}_0 - aR_0^2 q^n)(a - aq^n)}$.

By differentiating the equation $(6.51)$, we can write the corresponding complex velocity as,

$$\bar{V}(z) = i\kappa \sum_{n=-\infty}^{\infty} \left(\frac{1}{z - t(w_0 q^n)} + \frac{1}{z - t(aq^n R_0^2)} - \frac{1}{z - t(\frac{R_0^2 q^n}{\bar{w}_0})} - \frac{1}{z - t(\frac{q^n}{a})}\right). \quad (6.52)$$

From figure $(6.5)$ we can easily understand the nature of the reflection mechanism,

$$
\begin{tikzcd}
& (+)w_0(n=0) \arrow[dl, "R=R_0"'] \arrow[dr, "R=1"] & \\
(-)w^I_{R_0} = \frac{R_0^2}{\bar{w}_0}(n=0) \arrow[d, "R=1"] & & (-)w^I_{R=1} = \frac{1}{\bar{w}_0}(n=1) \arrow[d, "R=R_0"] \\
(+)w^{II}_{R=1} = \frac{1}{w^I_{R_0}} = \frac{w_0}{R_0^2}(n=1) \arrow[d, "R=R_0"] & & (+)w^{II}_{R_0} = \frac{R_0^2}{w^I_{R=1}} = R_0^2 w_0(n=-1) \arrow[d, "R=1"] \\
(-)w^{III}_{R_0} = \frac{R_0^4}{\bar{w}_0}(n=-1) \arrow[d, "R=1"] & & (-)w^{III}_{R_0} = \frac{1}{R_0^2\bar{w}_0}(n=-2). \arrow[d, "R=R_0"] \\
{} & & {}
\end{tikzcd}
$$

Figure 6.5. Inverses of $w_0$ With Respect to circles $|w| = R_0$ and $|w| = 1$.

$$
\begin{tikzcd}
& (-)\frac{1}{a}(n=0) \arrow[dl, "R=R_0"'] \arrow[dr, "R=1"] & \\
(+)w^I_{1/a} = \frac{R_0^2}{1/a} = aR_0^2(n=0) \arrow[d, "R=1"] & & (+)w^I_{R=1} = \frac{1}{1/a} = a(n=1) \arrow[d, "R=R_0"] \\
(-)w^{II}_{R=1} = \frac{1}{aR_0^2}(n=1) \arrow[d, "R=R_0"] & & (-)w^{II}_{R_0} = \frac{R_0^2}{a}(n=-1) \arrow[d, "R=1"] \\
(+)w^{III}_{R_0} = aR_0^4(n=-1) \arrow[d, "R=1"] & & (+)w^{III}_{R=1} = \frac{a}{R_0^2}(n=-2). \arrow[d, "R=R_0"] \\
{} & & {}
\end{tikzcd}
$$

Figure 6.6. Inverses of $1/a$ With Respect to Circles $|w| = R_0$ and $|w| = 1$.

As we can see from the figure $(6.5)$, we start with $w_0$ and take the inverses with respect to both circles and then take successive inverses. We see that the inverses accumulate at $0$ and $\infty$, in $w$-plane. Similarly, the vortex at $1/a$ has successive inverses in both circles and they accumulate at $0$ and $\infty$. Same idea can be applied for a vortex in $z$-plane,

$$
\begin{tikzcd}[column sep=tiny]
 & (-)z_0(n=0) \arrow[dl, "R=\rho"'] \arrow[dr, "R=1"] & \\
(+)z_\rho^I = \frac{\rho^2}{\bar{z}_0 - c} + c(n=0) \arrow[d, "R=1"] & & (+)z_{R=1}^I = \frac{1}{\bar{z}_0}(n=1) \arrow[d, "R=\rho"] \\
(-)z_{R=1}^{II} = \frac{1}{\bar{z}_\rho^I}(n=1) \arrow[d, "R=\rho"] & & (-)z_\rho^{II} = \frac{\rho^2}{\bar{z}_{R=1}^I - c} + c(n=-1) \arrow[d, "R=1"] \\
(+)w_\rho^{III} = \frac{\rho^2}{\bar{z}_{R=1}^{II}} + c(n=-1) \arrow[d, "R=1"] & & (+)w_{R=1}^{III} = \frac{1}{\bar{z}_{R_0}^{II}}(n=-2) \arrow[d, "R=\rho"] \\
{} & & {}
\end{tikzcd}
$$

Figure 6.7. Inverses of $z_0$ With Respect to Circles $|z| = 1$ and $|z - c| = \rho$.

We notice that inverses in $z$-plane accumulate at $a$ and $1/a$.

**Example 6.10** *In order to visualize the images, we consider a numerical example;* $x_1 = 4, x_2 = 2$ *and* $w_0 = 0.5i + 0.5$.

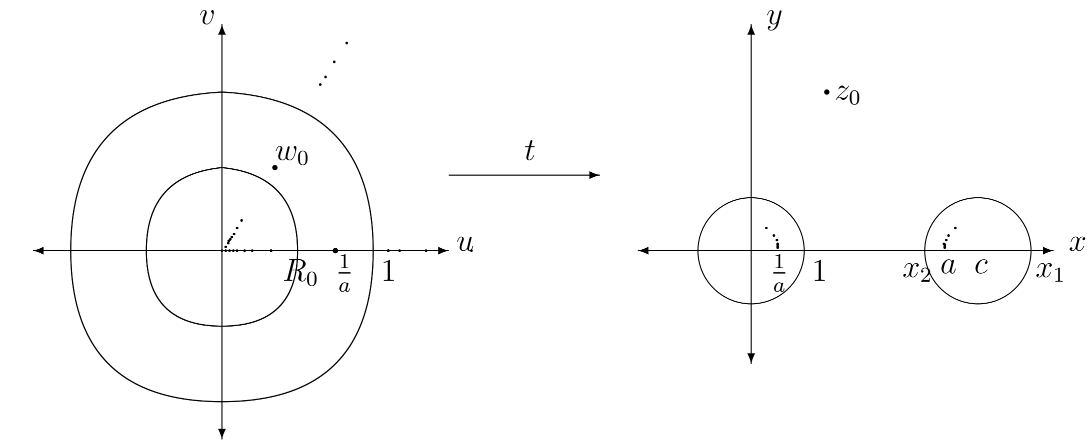


Figure 6.8. Vortex Image Approximation.

We calculate that $a = 2.61803$ and $R_0 = 0.145898$.

Vortex at $w_0$

$$w_0 = 0.5 + 0.5i$$

$$w^{I}_{R_0} = 0.2 + 0.2i \qquad w^{I}_{R=1} = 1 + i$$

$$w^{II}_{R=1} = 23.4 + 23.4i \qquad w^{II}_{R_0} = 0.01 + 0.01i$$

$$w^{III}_{R_0} = 4x10^{-4} + 4x10^{-4}i \qquad w^{I}_{R=1} = 46.9 + 46.9i$$

$$w^{IV}_{R=1} = 1103 + 1103i \qquad w^{I}_{R_0} = 0.0002 + 0.0002i$$

$$w^{V}_{R_0} = 9.6x10^{-6} + 9.6x10^{-6}i \qquad w^{V}_{R=1} = 2207 + 2207i$$

Vortex at $1/a$

$$1/a = 0.3819$$

$$w^{I}_{1/a} = 0.05 \qquad w^{I}_{R=1} = 2.6$$

$$w^{I}_{1/a} = 17.9 \qquad w^{I}_{R=1} = 0.008$$

$$w^{I}_{1/a} = 10^{-3} \qquad w^{I}_{R=1} = 122$$

$$w^{I}_{1/a} = 842 \qquad w^{I}_{R=1} = 10^{-4}$$

$$w^{I}_{1/a} = 2x10^{-5} \qquad w^{I}_{R=1} = 5778$$

Figure 6.9. Images of $w_0$ and $1/a$.

From the calculations we can see that images of both vortices accumulate at zero and infinity in $w$-plane. When we map the annular domain onto $z$-plane, there exist a vortex at $z_0$ and images of this vortex converge to $a$ and $1/a$ as you see below,

$$
\begin{tikzcd}
 & z_0 = -4.805 \times 10^{-17} + 1.6180i \arrow[dl, "R=\rho"'] \arrow[dr, "R=1"] & \\
z_\rho^I = 2.74 + 0.139i \arrow[drr] & & z_{R=1}^I = -10^{-17} + 0.6i \arrow[dll] \\
z_{R=1}^{II} = 0.36 + 0.01i \arrow[drr] & & z_\rho^{II} = 2.68 + 0.06i \arrow[dll] \\
z_\rho^{III} = 2.62 + 0.002i \arrow[drr] & & z_{R=1}^{III} = 0.37 + 0.009i \arrow[dll] \\
z_{R=1}^{IV} = 0.38 + 0.0003i \arrow[drr] & & z_{R_0}^{IV} = 2.61 + 0.001i \arrow[dll] \\
z_\rho^V = 2.61 + 0.00005i & & z_{R=1}^V = 0.38 + 0.0001i
\end{tikzcd}
$$

Figure 6.10. Images of $z_0$.

# CHAPTER 7

# CONCLUSION

Scattering of water waves and its long wave approximation and vortex-cylinder interaction were studied.

In the first part, starting from mathematical formulation of the water-wave problem we solved truncated system for velocity potential with unknown coefficients. In the limiting case which is known as Long-Wave approximation we found unknown coefficients; $\widetilde{A}_0^1$ and $\widetilde{A}_0^2$ behaves like $O((ak)^2)$ as $k \to 0$. Also $\widetilde{A}_1^1, \widetilde{A}_{-1}^1, \widetilde{A}_1^2$ and $\widetilde{A}_{-1}^2$ behave like $O((ak)^2)$ as $k \to 0$.

In the second part we tried to do the same thing for vortex-cylinder case by using the Circle Theorem. To apply boundary condition Laurent series expansion was necessary. This problem needs further attention.

Finally, complex velocity and potential in annular domain are obtained using complex analysis and $q$-logarithm. By using conformal mapping, annular domain is mapped onto the infinite $2-D$ plane with two circular cylinders. Solution in both domains involve the inverse images in both cylinders. Image representation is clearly depicted.

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# APPENDIX A

# DIVERGENCE THEOREM

## A.1. Divergence Theorem

**Theorem A.1** (**Divergence Theorem**) *( Kaplan Wilfred 1959) Let* $\mathbf{v} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ *be a vector field in a domain* $D$ *of space; let* $D, M, N$ *be continuous and have continuous derivatives in* $D$*. Let* $S$ *be a piecewise smooth surface in* $D$ *that forms the complete boundary of a bounded closed region* $R$ *in* $D$*. Let* $\mathbf{n}$ *be the outer normal of* $S$ *with respect to* $R$*. Then*

$$\int_S \int v_n d\sigma = \int \int_R \int \mathrm{div}\mathbf{v}\, dxdydz; \qquad \text{(A.1)}$$

*that is,*

$$\int_S \int Ldydz + Mdzdx + Ndxdy = \int \int_R \int \left( \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} \right). \qquad \text{(A.2)}$$

# APPENDIX B

# SEPARATION OF VARIABLES

## B.1. Solution of No Body Problem Using Separation of Variables

**Problem**

$$\begin{aligned} \nabla^2 \widetilde{\phi} &= 0 \quad \text{in } W, & \text{(B.1)} \\ \widetilde{\phi}_y - \nu \widetilde{\phi} &= 0 \quad \text{on } F \, (\text{on}\, y = 0), & \text{(B.2)} \\ \frac{\partial \widetilde{\phi}}{\partial y} &= 0 \quad \text{on } B \, (\text{on}\, y = -d). & \text{(B.3)} \end{aligned}$$

**Solution:** If we replace $\widetilde{\phi} = G(x_1)H(x_2)Y(y)$ into equation $(B.1)$ and dividing by $G(x_1)H(x_2)Y(y)$ we get,

$$\frac{G_{x_1x_1}}{G} + \frac{H_{x_2x_2}}{H} + \frac{Y_{yy}}{Y} = 0. \tag{B.4}$$

or equivalently

$$\frac{G_{x_1x_1}}{G} = \frac{-H_{x_2x_2}}{H} + \frac{-Y_{yy}}{Y} = -k_1^2 \tag{B.5}$$

$$\tag{B.6}$$

which means

$$\begin{aligned} G_{x_1x_1} + k_1^2 G &= 0 & \text{(B.7)} \\ \frac{H_{x_2x_2}}{H} + \frac{Y_{yy}}{Y} &= k_1^2 & \text{(B.8)} \end{aligned}$$

if second separation is done to last equation

$$H_{x_2x_2} + k_2^2 H = 0 \tag{B.9}$$

$$Y_{yy} - (k_1^2 + k_2^2)Y = 0 \tag{B.10}$$

for some constant $k_2$. From the solution of equations $(B.7)$, $(B.9)$ and $(B.10)$ we get,

$$\begin{aligned} G(x_1) &= c_1 e^{ik_1x_1} + c_2 e^{-ik_1x_1} & \text{(B.11)} \\ H(x_2) &= c_3 e^{ik_2x_2} + c_4 e^{-ik_2x_2} & \text{(B.12)} \\ Y(y) &= c_5 \cosh(\sqrt{k_1^2 + k_2^2})y + c_6 \sinh(\sqrt{k_1^2 + k_2^2})y & \text{(B.13)} \end{aligned}$$

If the boundary condition $(B.3)$ is applied to the equation $(B.13)$

$$Y(y) = c_5\left(\cosh(ky) + \tanh(kd)\sinh(hy)\right) \tag{B.14}$$

where $k = \sqrt{k_1^2 + k_2^2}$

$$\begin{aligned}
\widetilde{\phi} &= Y(y)G(x_1)H(x_2) \\
&= c_5\left(\cosh(ky) + \tanh(kd)\sinh(hy)\right)\left(c_1 e^{ik_1x_1} + c_2 e^{-ik_1x_1}\right)\left(c_3 e^{ik_2x_2} + c_4 e^{-ik_2x_2}\right) \\
&= c_5\left(\cosh(ky) + \tanh(kd)\sinh(hy)\right)\left(c_1 e^{ik_1x_1+ik_2x_2} + c_2 e^{-ik_1x_1-ik_2x_2}\right) \\
&= \frac{\cosh k(y+d)}{\cosh(kd)}(c^1 e^{i\mathbf{k}.\mathbf{x}} + c^2 e^{-i\mathbf{k}.\mathbf{x}})
\end{aligned}$$

where $c^1 = c_5.c_1, c^2 = c_5.c_2$ and $\mathbf{k}.\mathbf{x} = k_1x_1 + k_2x_2$.

Velocity potential

$$\phi^1 = Re\left\{\frac{\cosh k(y+d)}{\cosh(kd)}(c^1 e^{i\mathbf{k}.\mathbf{x}-i\omega t} + c^2 e^{-i\mathbf{k}.\mathbf{x}-i\omega t})\right\}. \tag{B.15}$$

# APPENDIX C

# BESSEL FUNCTIONS

## C.1. Bessel's Differential Equation

$$x^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0 \tag{C.1}$$

Solutions of equation $(C.1)$ are the Bessel functions of the first kind $J_{\pm}(z)$, of the second kind $Y_\nu(z)$ and of the third kind $H_\nu^{(1)}$, $H_\nu^{(2)}$ (also called the Hankel functions).

## C.2. Generating Function, Integral Order, $J_n(x)$

Let us introduce a function of two variables,

$$g(x,t) = e^{(x/2)(t-1/t)}. \tag{C.2}$$

Expanding this function in a Laurent series, we obtain

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{C.3}$$

The coefficient of $t^n$, $J_n(x)$, is defined to be Bessel function of the first kind of integral order $n$. Expanding the exponentials, we have a product of Maclaurin series in $xt/2$ and $-x/2t$, respectively,

$$e^{xt/2}.\, e^{-x/2t} = \sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r \frac{t^r}{r!}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^s\frac{t^{-s}}{s!}. \tag{C.4}$$

For a given $s$ we get $t^n$ $(n \geq 0)$ from $r = n + s$

$$\left(\frac{x}{2}\right)^{n+s}\frac{t^{n+s}}{(n+s)!}(-1)^s\left(\frac{x}{2}\right)^s\frac{t^{-s}}{s!}. \tag{C.5}$$

The coefficient of $t^n$ is then

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}. \tag{C.6}$$

Equation $(C.6)$ actually holds for $n < 0$, also giving

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n)!} \left(\frac{x}{2}\right)^{2s-n}, \tag{C.7}$$

which amounts to replacing $n$ by $-n$ in equation $(C.6)$. Since $n$ is an integer, $(s-n)! \to \infty$ for $s = 0, \ldots, (n-1)$. Hence the series may be considered to start with $s = n$. Replacing $s$ by $s+n$, we obtain

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s},$$

showing immediately that $J_n(x)$ and $J_{-n}(x)$ are not independent but are related by

$$J_{-n}(x) = (-1)^n J_n(x) \tag{C.8}$$

## C.3. Integral Representation

If we substitute $t = e^{i\theta}$ into the generating function $(C.3)$

$$e^{(x/2)(t-1/t)} = e^{(x/2)(e^{i\theta}-e^{-i\theta})} = e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} \tag{C.9}$$

## C.4. Standing Cylindrical Waves In terms of Bessel Functions

Standing cylindrical wave in water layer of depth $d$ has the following potential,

$$\phi^{(1)} = Re\left\{Ae^{-i\omega t}\left(e^{i\kappa x} + e^{-i\kappa x}\right)\right\}\cosh k(y+d). \tag{C.10}$$

Using the equation $(C.9)$ and $x = r\cos\theta$

$$\begin{aligned} e^{i\kappa x} &= e^{i\kappa r\cos\theta} = e^{i\kappa\sin(\frac{\pi}{2}-\theta)} = \sum_{n=-\infty}^{\infty} J_n(\kappa r) e^{in(\frac{\pi}{2}-\theta)} && \text{(C.11)} \\ e^{-i\kappa x} &= e^{-i\kappa r\cos\theta} = e^{i\kappa\sin(\theta-\frac{\pi}{2})} = \sum_{n=-\infty}^{\infty} J_n(\kappa r) e^{-in(\frac{\pi}{2}-\theta)} && \text{(C.12)} \end{aligned}$$

If equations $(C.11)$ and $(C.12)$ are replaced into the equation $(C.10)$,

$$\begin{aligned} \phi^{(1)} &= Re\left\{Ae^{-i\omega t}\sum_{n=-\infty}^{\infty} J_n(\kappa r)\left(e^{in(\frac{\pi}{2}-\theta)} + e^{-in(\frac{\pi}{2}-\theta)}\right)\right\}\cosh k(y+d) && \text{(C.13)} \\ &= Re\left\{Ae^{-i\omega t}\sum_{n=-\infty}^{\infty} J_n(\kappa r)\left(2\cos[n(\frac{\pi}{2}-\theta)]\right)\right\}\cosh k(y+d). && \text{(C.14)} \end{aligned}$$

## C.5. Asymptotic Expansion for Large Arguments

### C.5.1. Hankel's Asymptotic Expansions

When $\nu$ is fixed and $|x| \to \infty$

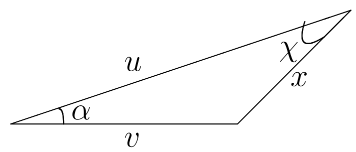
$$
\begin{aligned}
J_\nu(x) &= \sqrt{2/(\pi x)}\{P(\nu,x)\cos\chi - Q(\nu,x)\sin\chi\} \quad (|\arg x| < \pi) && \text{(C.15)}\\
Y_\nu(x) &= \sqrt{2/(\pi x)}\{P(\nu,x)\sin\chi - Q(\nu,x)\cos\chi\} \quad (|\arg x| < \pi) && \text{(C.16)}\\
H_\nu^{(1)}(x) &= \sqrt{2/(\pi x)}\{P(\nu,x) + iQ(\nu,x)\}e^{i\chi} \quad (-\pi < \arg x < 2\pi) && \text{(C.17)}\\
H_\nu^{(2)}(x) &= \sqrt{2/(\pi x)}\{P(\nu,x) - iQ(\nu,x)\}e^{-i\chi} \quad (-2\pi < \arg x < \pi) && \text{(C.18)}
\end{aligned}
$$

where $\chi = x - (\frac{1}{2}\nu + \frac{1}{4})\pi$ and, with $4\nu^2$ denoted by $\mu$,

$$
\begin{aligned}
P(\nu,x) &\sim \sum_{k=0}^{\infty}(-)^k\frac{(\nu,2k)}{(2z)^{2k}} = 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8)^4} - \dots\\
Q(\nu,x) &\sim \sum_{k=0}^{\infty}(-)^k\frac{(\nu,2k+1)}{(2x)^{2k+1}} = \frac{(\mu-1)}{(8z)} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8)^3} + \dots
\end{aligned}
$$

**Theorem C.1** *Graf's Addition Theorem For Bessel Functions*

$$
H_\nu(x)^{\cos}_{\sin}\nu\chi = \sum_{k_-\infty}^{\infty} H_{\nu+k}(u)J_k(v)^{\cos}_{\sin}k\alpha \quad (|ve^{\pm i\alpha}| < |u|) \tag{C.19}
$$




Graf's Addition Theorem

## C.6. Limiting Forms of Bessel Functions for Small Arguments

When $\nu$ is fixed and $x \to 0$

$$
\begin{aligned}
J_\nu(x) &\sim \frac{1}{\nu!}\left(\frac{1}{2}x\right)^\nu && \text{(C.20)}\\
H_\nu^{(1)}(x) &\sim -i\frac{(\nu-1)!}{\pi}\left(\frac{1}{2}x\right)^{-\nu} \quad (\Re\nu > 0) && \text{(C.21)}
\end{aligned}
$$

Using the relations

$$
\begin{aligned}
J_{-\nu}(x) &= (-)^n J_n(x) && \text{(C.22)}\\
H_{-\nu}^{(1)}(x) &= e^{\nu\pi i}H_\nu^{(1)}(x) && \text{(C.23)}
\end{aligned}
$$

$$J_{-\nu}(x) \sim \frac{(-)^n}{\nu!}\left(\frac{1}{2}x\right)^{\nu} \tag{C.24}$$

$$H^{(1)}_{-\nu}(x) \sim -i(-)^{n+1}\frac{(\nu-1)!}{\pi}\left(\frac{1}{2}x\right)^{-\nu} \quad (\Re\nu > 0) \tag{C.25}$$

For $\nu = 0$,

$$H^{(1)}_0(z) \sim \frac{2i}{\pi}\ln z. \tag{C.26}$$

# APPENDIX D

# CAUCHY INTEGRAL FORMULA

## D.1. Cauchy Integral Formula

**Theorem D.1** *(Churcill and Brown 1984) Let $f$ be analytic everywhere within and on a simple closed contour $C$, taken in the positive sense. If $z$ is any point interior to $C$, then*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)d\xi}{\xi - z}. \tag{D.1}$$

Formula $(D.1)$ is called the *Cauchy integral formula*.

## D.2. Complex Velocity Using Cauchy Integral Formula

### D.2.1. No Singularity in $C$

Since complex velocity $f(z) = \bar{V} = u - iv$, which we analysed in Chapter $3$ is an analytic function in $C$, we can apply the Theorem $D.1$. Complex velocity $\bar{V}$ can be written,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)d\xi}{\xi - z} \tag{D.2}$$

and

$$\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1 - \frac{z}{\xi}} = \sum_{n=0}^{\infty} \frac{z^n}{\xi^{n+1}}. \tag{D.3}$$

If we substitute equation $(D.3)$ into $(D.2)$

$$f(z) = \frac{1}{2\pi i} \oint_C f(\xi) \sum_{n=0}^{\infty} \frac{z^n}{\xi^{n+1}} d\xi. \tag{D.4}$$

We can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi^{n+1}} d\xi. \tag{D.5}$$

### D.2.2. One Singularity in $C$

Let there exist a singularity at $z = 0$. In this case we can apply Theorem $D.1$ in annular domain,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(\xi)d\xi}{\xi - z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)d\xi}{\xi - z}, \tag{D.6}$$

where $C_1$ is inner circle around singular point $z = 0$ and $C_2$ is outer circle.

If we expand $\frac{1}{\xi - z}$ around $C_1$ and $C_2$

$$C_1 : \frac{1}{\xi - z} = \sum_{n=0}^{\infty} -\frac{\xi^n}{z^{n+1}} \quad \text{when } |\xi| < |z|, \tag{D.7}$$

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\xi)d\xi}{\xi - z} = \sum_{n=0}^{\infty}\frac{b_{n+1}}{z^{n+1}}, \quad b_{n+1} = \frac{-1}{2\pi i}\oint f(\xi)\xi^n d\xi \tag{D.8}$$

$$C_2 : \frac{1}{\xi - z} = \sum_{n=0}^{\infty}\frac{z^n}{\xi^{n+1}} \quad \text{when } |z| < |\xi|, \tag{D.9}$$

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)d\xi}{\xi - z} = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{-1}{2\pi i}\oint \frac{f(\xi)}{\xi^{n+1}}d\xi. \tag{D.10}$$

From the equations $(D.8)$ and $(D.10)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty}\frac{b_{n+1}}{z^{n+1}} \tag{D.11}$$

### D.2.3. $N$ Singularity in $C$

If we generalize method which is applied above for $N$ singular point,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{k=0}^{N}\sum_{n=0}^{\infty}\frac{b_{n+1}^k}{(z - z_k)^{n+1}}. \tag{D.12}$$

These singular points can be interpret as a cylinder, as an example let analysis one cylinder case when there is no vorticity

### D.2.4. Complex Velocity for One Cylinder

Let there exist a cylinder at $z = 0$ with radius $a$. Complex velocity is

$$f = \bar{V} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty}\frac{b_{n+1}}{z^{n+1}} \tag{D.13}$$

which is same with equation $(D.11)$. To find unknown coefficients $a_n$ and $b_{n+1}$ we must apply boundary condition

$$\bar{V}(z).z + V(\bar{z}).\bar{z}|_{c_1} = 0. \tag{D.14}$$

Since there is no vorticity around $z = 0$, $b_1 = 0$, equation $(D.13)$ becomes

$$\bar{V} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}}. \tag{D.15}$$

When the boundary condition is applied

$$b_{n+2} = -\bar{a}_n a^{2(n+1)} \tag{D.16}$$

and

$$\bar{V}(z) = \sum_{n=0}^{\infty} a_n z^n - \frac{a^2}{z^2} \sum_{n=0}^{\infty} \bar{a}_n \left(\frac{a^2}{z}\right)^n. \tag{D.17}$$

This coincides with result of Circle Theorem (4.2) for complex velocity.

## D.2.5. Complex Velocity for One Cylinder and One Vortex

Let consider a vortex at $z_0$ with strength $\kappa$ and cylinder at $z_1$ with radius $a$ . Complex velocity

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_1)^n + \sum_{n=1}^{\infty} \frac{b_{n+1}}{(z - z_1)^{n+1}} \tag{D.18}$$

with boundary

$$c : (z - z_1)(\bar{z} - \bar{z}_1) = a^2. \tag{D.19}$$

If first term is expanded around $z_1$

$$\frac{i\kappa}{z - z_0} = -i\kappa \sum_{n=0}^{\infty} \frac{(z - z_1)^n}{(z_0 - z_1)^{n+1}} \tag{D.20}$$

and substituted into equation $(D.37)$;

$$\bar{V}(z) = -i\kappa \sum_{n=0}^{\infty} \frac{(z - z_1)^n}{(z_0 - z_1)^{n+1}} + \sum_{n=0}^{\infty} a_n (z - z_1)^n + \sum_{n=1}^{\infty} \frac{b_{n+1}}{(z - z_1)^{n+1}}. \tag{D.21}$$

Boundary condition

$$\bar{V}(z - z_1) + V(\bar{z} - \bar{z}_1) = 0 \tag{D.22}$$

is applied to complex potential $(D.21)$

$$\bar{V}(z-z_1) = -i\kappa\sum_{n=0}^{\infty}\frac{(z-z_1)^{n+1}}{(z_0-z_1)^{n+1}} + \sum_{n=0}^{\infty}a_n(z-z_1)^{n+1} + \sum_{n=1}^{\infty}\frac{b_{n+1}}{(z-z_1)^n} \tag{D.23}$$

$$V(\bar{z}-\bar{z}_1) = i\kappa\sum_{n=0}^{\infty}\frac{(\bar{z}-\bar{z}_1)^{n+1}}{(\bar{z}_0-\bar{z}_1)^{n+1}} + \sum_{n=0}^{\infty}\bar{a}_n(\bar{z}-\bar{z}_1)^{n+1} + \sum_{n=1}^{\infty}\frac{\bar{b}_{n+1}}{(\bar{z}-\bar{z}_1)^n} \tag{D.24}$$

from the equation $(D.19)$

$$\frac{1}{(z-z_1)} = \frac{(\bar{z}-\bar{z}_1)}{a^2} \quad\Rightarrow\quad \frac{1}{(\bar{z}-\bar{z}_1)} = \frac{(z-z_1)}{a^2}. \tag{D.25}$$

If the last two equation is substitute into equations $(D.23)$ and $(D.24)$

$$\begin{aligned}\bar{V}(z-z_1)+V(\bar{z}-\bar{z}_1) = & - i\kappa\sum_{n=0}^{\infty}\frac{(z-z_1)^{n+1}}{(z_0-z_1)^{n+1}} + \sum_{n=0}^{\infty}a_n(z-z_1)^{n+1} + \sum_{n=1}^{\infty}\frac{b_{n+1}(\bar{z}-\bar{z}_1)^n}{(a^2)^n} \\ & + i\kappa\sum_{n=0}^{\infty}\frac{(\bar{z}-\bar{z}_1)^{n+1}}{(\bar{z}_0-\bar{z}_1)^{n+1}} + \sum_{n=0}^{\infty}\bar{a}_n(\bar{z}-\bar{z}_1)^{n+1} + \sum_{n=1}^{\infty}\frac{\bar{b}_{n+1}(z-z_1)^n}{(a^2)^n}\end{aligned}$$

$$\bar{V}(z-z_1)+V(\bar{z}-\bar{z}_1) = \sum_{n=0}^{\infty}\left[\frac{-i\kappa}{(z_0-z_1)^{n+1}} + a_n + \frac{\bar{b}_{n+2}}{a^{2(n+1)}}(z-z_1)^{n+1}\right] + C.C. \tag{D.26}$$

where $C.C$ is complex conjugate. Then

$$\frac{-i\kappa}{(z_0-z_1)^{n+1}} + a_n + \frac{\bar{b}_{n+2}}{a^{2(n+1)}} = 0 \tag{D.27}$$

or

$$b_{n+2} = a^{2(n+1)}\left[\frac{-i\kappa}{(\bar{z}_0-\bar{z}_1)^{n+1}} - \bar{a}_n\right]. \tag{D.28}$$

Equation $(D.21)$ becomes

$$\bar{V}(z) = i\kappa\sum_{n=0}^{\infty}\frac{(z-z_1)^n}{(z_0-z_1)^{n+1}} + \sum_{n=0}^{\infty}a_n(z-z_1)^n \tag{D.29}$$

$$+ \sum_{n=0}^{\infty}(a^2)^{n+1}\left[\frac{-i\kappa}{(\bar{z}_0-\bar{z}_1)^{n+1}} - \bar{a}_n\right]\frac{1}{(z-z_1)^{n+2}} \tag{D.30}$$

Term $\sum_{n=0}^{\infty}a_n(z-z_1)^n$ represents flow without vortex and cylinder but since there is no flow, $a_n = 0$.

$$\bar{V}(z) = \frac{i\kappa}{z-z_0} - i\kappa\sum_{n=0}^{\infty}\frac{a^{2(n+1)}}{(\bar{z}_0-\bar{z}_1)^{n+1}(z-z_1)^{n+2}} \tag{D.31}$$

$$= \frac{i\kappa}{z-z_0} + \frac{i\kappa}{z-z_1} - \frac{i\kappa}{(z-z_1)-\frac{a^2}{(\bar{z}_0-\bar{z}_1)}}. \tag{D.32}$$

### D.2.6. Complex Velocity for Two Cylinder and One Vortex

We consider problem of vortex at $z = z_0$ with strength $\kappa$ in annular domain $D : \{r_1 \leq |z| \leq r_2\}$. The region is bounded by two concentric circles:

$$C_1 : z\bar{z} = r_1^2 \tag{D.33}$$

and

$$C_2 : z\bar{z} = r_2^2. \tag{D.34}$$

Complex velocity is given by Lourent series

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}}. \tag{D.35}$$

To find the unknown coefficients $a_n$ and $b_{n+2}$ we must apply the boundary conditions on $C_1$ and $C_2$.

### D.2.7. Complex Velocity for $K$ Cylinder and One Vortex

Complex velocity for $K$ cylinder with vortex is

$$\bar{V}(z) = \bar{V}_0(z) + \sum_{n=0}^{\infty} a_n z^n + \sum_{\alpha=1}^{K} \sum_{n=0}^{\infty} \frac{b_{n+2}}{(z - z_\alpha)^{n+2}} \tag{D.36}$$

where $\bar{V}_0(z)$ represents the complex velocity of vortex.

### D.2.8. Complex Velocity for Two Cylinder and $N$ Vortices

Complex velocity for two concentric cylinder at the origin, with $N$ vortices at position $z_1, \ldots, z_N$ with strength $\kappa_1, \ldots \kappa_N$ is

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}} \tag{D.37}$$

## D.3. Jacobi's Expressions for the Theta-functions as Infinite Products

For complex numbers $w$ and $q$ with $|q| < 1$ and $w \neq 0$ first order Jacobi's Theta function

$$\theta_1(w, q) = 2Gq^{1/4} \sin z \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2z + q^{4n}) \tag{D.38}$$

where

$$G = \prod_{n=1}^{\infty} (1 - q^{2n}). \tag{D.39}$$