DAMPING OSCILLATORY MODELS IN GENERAL THEORY OF RELATIVITY

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ABSTRACT

DAMPING OSCILLATORY MODELS IN GENERAL THEORY OF RELATIVITY

In my thesis we have studied the universe models as dynamical systems which can be represented by harmonic oscillators. For example, a harmonic oscillatior equation is constructed by the transformation of the Riccati differential equations for the anisotropic and homogeneous metric. The solution of the Friedman equations with the state equation satisfies both bosonic expansion and fermionic contraction in Friedman Robertson Walker universe with different curvatures is studied as a conservative system with the harmonic oscillator equations. Apart from the oscillator representations mentioned above (constructed from the universe models), we showed that the linearization of the Einstein field equations produces harmonic oscillator equation with constant frequency and the linearization of the metric on the de-Sitter background produces damped harmonic oscillator system. In addition to these, we have constructed the doublet and the Caldirola type oscillator equations with time dependent damping and frequency terms in the light of the Sturm Liouville form. The Lagrangian and Hamiltonian functions are calculated for all particular cases of the Sturm Liouville form. Finally, we have shown that zeros of the oscillator equations constructed from the particular cases can be transformed into pole singularities of the Riccati equations.

ÖZET

GENEL GÖRELİLİK TEORİSİNDE SÖNÜMLÜ OSİLASYON MODELLERİ

Tezimde harmonik salınıcı olarak betimlenebilen dinamik evren modelleri çalışıldı. Bu modellere örnek olarak homojen ve isotropik olmayan evren için Riccati diferansiyel denkleminin doğrusal dönüşümünden oluşturulan harmonik salınıcı denklemlerini elde ettik. Buna ek olarak, Friedman diferansiyel denklemlerinin ve durum denkleminin Friedman Robertson Walker evreninde ortak çözümünden bosonik olarak genişleyen ve fermionik olarak büzülen evren modellerini farklı eğrilikler için korunumlu sistem olarak harmonik salınıcı denklemler ışığında inceledik. Yukarıda ki harmonik salınıcı evren modellerinden farklı olarak, Einstein denklemlerinin Minkowski uzayında doğrusallaştırılmasının sabit frekanslı standart harmonik salınıcı denklemini ve aynı denklemlerin de-Sitter uzayında doğrusallaştırılmasıyla da sönümlü harmonik salınıcı denklemlerinin üretildiğini gösterdik. Bunlara ek olarak, zaman bağımlı sönümleme ve frekans bileşenlerine sahip olan çift ve Caldirola tipi sönümlü salınıcı denklemleri, Sturm Liouville formu yardımıyla elde edildi. Sturm Liouville formunun tüm özel durumları için Lagrangian ve Hamiltonian fonksiyonları elde edildi. Son olarak, özel durumlar için elde edilmiş salınıcı denklemlerinin sıfırlarının Riccati denklemlerinin kutup noktalarına dönüşebileceğini gösterdik.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Modern mathematical cosmology started in 1917 when Albert Einstein applied his gravity model in order to understand the large scale structure of the universe. This model was built using his general theory of relativity which was constructed in 1916 by using the Riemannian Geometry. The first problem faced by his model was that it is not static, while it was generally believed that the universe is static. The model predicted that the universe is expanding while observations, of that time, did not support this prediction. When Alexander Friedman used the modified equations of the General Relativity in 1922, he got again an expanding universe model. Einstein was enforced to modify the field equations of the General Theory of Relativity by adding a term, called the Cosmological constant. This term introduces a repulsion which expected to compensate the gravity attraction to stop expansion and to get a static model. However, Edwin Hubble by studying the red-shift of distant galaxies, in 1929 confirmed the prediction of the General Theory of Relativity that the universe is expanding. Therefore, Einstein rejected this term as it does not stop expansion. After Hubbles discovery, many researches started to build models for the expanding Universe, in the context of the General Theory of Relativity, investigating the consequences of different assumptions about the distribution of matter in the universe. Then initial simplicity of the cosmological model has been replaced by more complex models taking into account nonlinearity and dissipation.

The modern standard models of the universe are the Friedman-Lemaitre family of models, based on the Robertson-Walker (1934) spatially homogeneous and isotropic geometries. There is evidence supporting these models on the largest observable scales, but at smaller scales they are clearly not well description. The question is (Ellis 1997): On what scales and in what domains is the universe's geometry nearly Friedman-Robertson-Walker (FRW) ? What are the best fit FRW parameters in the observable domain? Why is it FRW? How did the universe come to have such an improbable geometry?

One of the approaches to this problem is the inflation theory (Linde 1982 and Guth 1981). According to this theory quantum fluctuations in the very early universe formed the seeds of inhomogeneities that could then grow. To examine these questions one needs to consider the family of cosmological solutions in the full state space of solutions, allowing to see how realistic models are related to each other and to higher symmetry models including, in particular, the Friedman-Lemaitre models.

In the present thesis we develop general techniques for examining FRW type family of models and their generalizations which could be useful in description of universe at smaller scales. First of all in FRW type approach the universe is characterized by cosmic scale parameter which is a function of global time variable. From this point of view the isotropic universe is a dynamical system with one degree of freedom. But at smaller scales anisotropy of the universe could be important this is why one can consider more general situation with three different scale parameters depending on one global time. In this case, which is in the class of so called Bianchi family of universes, we have dynamical system with three degrees of freedom and the FRW universe appears only as symmetric reduction valid for isotropic case. The general anisotropic case can be described by the Riccati equation and the last one admits transformation to the time dependent damped harmonic oscillator. This is why such models are called the oscillatory models of the universe. Using factorization properties of the oscillatory models we can introduce new characteristic of the models as in the super symmetric quantum mechanics with graded structure or fermionic and bosonic models. General oscillatory models suggest also to consider time dependent gravitational and cosmological constants.

Time dependent metric leads also to the problem what is the physics in such time dependent background. Concerning with this, particles and their interaction in expanding geometry have been considered, particularly in inflation models, where at smaller distances expansion rate of the universe is very high. Perelomov showed that inflationary universe or inflationary metrics also implies time-dependent frequency for the gravitational wave modes, and this leads one to extend the canonical quantization method for non-unitary time evolution so to include the quantization formalism for parametric oscillator (Perelomov 1986). Lemos and Natividade studied a harmonic oscillator with a time-dependent frequency and a constant mass in an expanding universe (Lemos and Natividade 1987). In the inflating case the Friedmann-Robertson-Walker (FRW) metrics is producing the damped harmonic oscillator equation for the partial waves of the field $h_{\mu\nu}$ (Grishchuk and Sidorov 1990). Vitiello et al. discussed the canonical quantization of non-unitary time evolution in inflating Universe. Vitiello considered gravitational wave modes in the FRW metrics in a de Sitter phase then proceeds with the quantization method for the damped oscillator mentioned above. The doubling of the $h_{\mu\nu}$ partial waves which was called double universe was shown (Vitiello et al. 1997).

If damped oscillatory models are very important at small scales the natural question is quantization of these models, when one can not neglect quantum fluctuations. One of the first approaches to quantize the damped harmonic oscillator is to start with the classical equation of motion, find Lagrangian and then Hamiltonian, which leads to the Hamilton equations of motion and then quantize them by canonical formalism method. This approach is called (Bateman)-Caldirola-Kanai model which derives quantum mechanics from a dissipative Hamiltonian. This Hamiltonian was actually proposed earlier by Bateman, but in a classical context (Bateman 1938). This approach has the attractiveness of providing an exact solution, in essence because the classical equation of motion has an exact solution and formal quantization merely has the effect of converting the classical variables into operators. A second approach uses an interaction Hamiltonian and applies perturbation theory. One is a rather simple system (the undamped harmonic oscillator) that we construct an environment of the damped harmonic oscillator also exist. These, in fact, close the system which make a realistic or artificial embedding in a larger system that preserves energy. This way, Hamiltonians that describe a total, conserved energy can be obtained. An example of this line of thought is the so-called doubling the degrees of freedom approach. In fact also this idea can be traced back to a Hamiltonian that was coined by Bateman, the so-called dual Hamiltonian (Bateman 1938). The idea is that the damped oscillator is coupled to its time-reversed image oscillator which absorbs the energy lost, so that the energy of the whole system is conserved or closed. In fact, since the phase space of the whole system describes the damped harmonic oscillator and its image, the degrees of freedom are effectively doubled. Another way of looking at this is that adding a time-reversed oscillator restores the breaking of the time-reversal symmetry. In the earlier attempts to elaborate this idea, difficulties arose, such as, the time evolution leads out of the Hilbert space of states, but later a satisfactory quantization could be achieved within the framework of quantum field theory (Feshbach et al. 1977, Celeghini et al. 1992, Blasone et al. 1996). The doubling of degrees of freedom approach has the conceptual disadvantage that the environment to which the damped harmonic oscillator is coupled is artificial. But the word artificial is used for just well known systems.

Because, the structure of universe is not well defined, this approach has advantage to show main form of dissipation as a system.

Apart from these approaches for damped harmonic oscillator with constant frequency and damping coefficient, the general form of time dependent Hamiltonian which describes a classical forced oscillator with a time dependent damping coefficient and frequency were studied by Havas (1957) and this kind of system was also considered by others (Khandekar and Lawande 1979, Lee and George and Um 1989). Moreover, Kim demonstrated that canonical transformations in classical mechanics correspond to unitary transformations in quantum mechanics (Kim et al. 1997). Also, Kim and Lee studied time dependent harmonic and anharmonic oscillator and found the exact Fock space and density operator for time dependent anharmonic oscillator (Kim and Lee 2000).

The goal of the first part of present thesis is to study dissipative geometry of universe models in General Theory of Relativity as in the following context.

In Chapter 2, we give the fundamental definitions of General Theory of Relativity such as Christoffel symbol, Rieman tensor, Ricci tensor and Ricci scalar (Section 2.1). The definitions of Einstein field equations both in the presence and in the absence of matter and then the definitions of the cosmological constant are given. Moreover, the one of the most important tensors of the GRT is the energy momentum tensor which tell us the energy like aspects of the system given. In Section 2.2, we discussed Universe as a dynamical system based on time and scale factor dependent metric. The solution of the Einstein field equation with this metric produces two linearly independent equation which is called the Friedman equations. The state equation with the Friedman differential equations can help us to understand universe as a dynamical system.

In Chapter 3, The construction of the universe models begin with the idea that the universe on large scales is isotropic and homogeneous. In the light of this, we consider the Friedman universe models as a dynamical models because of time and scale factor dependency. The Friedman models include four basic group of models which are the static, the empty, the non empty models with zero cosmological constant and the non empty models with non zero cosmological constant. These universe models also have sub universe models (Section 3.1). In Section 3.2, the Milne's model are discussed with its fundamental properties and we have compared the Milne's model with the Friedman models.

In Chapter 4, Anisotropic and homogeneous universe models are investigated in terms of different density and pressure functions. In Section 4.1, we have formulated the general solution of the field equations with respect for the anisotropic and homogeneous metrics. In the following subsections, we consider the particular results of the field equations in terms of constant density with zero pressure function, constant pressure with zero density function and absence of both pressure and density functions.

In Chapter 5, the Riccati equation can be derived by using the Friedman equations and the equation of state, in other words, barotropic equation. The linearization of it gives a second order differential equation. In Section 5.1, we investigate the results of the second order differential equations in terms of negative and positive curved bosonic Friedman Robertson Walker (FRW) universe models. In Section 5.2, we consider the same differential equation with respect to the fermionic Friedman Robertson Walker (FRW) universe models with hyperbolic and spherical geometries. In Section 5.3, we study the Dirac equation without a mass term in the super symmetric non-relativistic formalism in fermionic and bosonic universe models with respect to the negative and positive curvatures. In Section 5.4, similar to the previous section, we consider the Dirac equation with mass term.

In Chapter 6, we consider the cosmological constants as a function of time. In Section 6.1, we construct the field equations and in Section 6.2, we solve the field equations. In Subsection 6.2.1, we assume that the gravitational constant is proportional to the Hubble parameter. Hence, under this assumption, we investigate the inflationary and radiation dominated phase of universe. In Subsection 6.2.2, we assume that gravitational constant is inversely proportional to the Hubble parameter. Then, under this assumption, we investigate the inflationary and radiation dominated phase of universe.

In Chapter 7, we formulate the linearization of the Einstein equations which produces gravitational waves on Minkowski background and from the Fourier expansion of the field, we get the Fourier component of the field which satisfies harmonic oscillator equation with constant frequency (Section 7.1). In Section 7.2, the linearization of the same equation on the de-Sitter background produces damped harmonic oscillator systems with respect to the Bateman approach and then the double universe models can be formed with respect to this approach.

The second part of this thesis is devoted to study variational formulation of time dependent harmonic oscillator.

The Lagrangian and the Hamiltonian description are vital to understand the damped oscillator in quantum and classical theory. Hence, we give the basic knowledge in Chapter 8. In Section 8.1, we give the definitions of the generalized coordinates and the velocities. In Section 8.2, a formulation for the study at mechanical system which is called least action principle is given. In Section 8.3, we discuss the Hamiltonian and the Hamilton's equations, the Poisson brackets and the properties of the Poisson brackets. In Section 8.4, we consider the solution of damped harmonic oscillators for three different cases which are called over damping, critical damping and under damping. In Section 8.5, we give the definition of the Bateman dual description and by using this approach, we investigate the Lagrangian and the Hamiltonian functions for doublet damped oscillator systems. In Section 8.6, we consider the time dependent Hamiltonian with time dependent mass satisfying the standard damped harmonic oscillator equation which is called Caldirola-Kanai Hamiltonian. In Section 8.7, we quantize the Caldirola-Kanai Hamiltonian with constant damping coefficient and frequency.

In Chapter 9, it will be shown that two different formulations of damped oscillator are related with the self adjoint expansion of the Sturm Liouville problem (Section 9.1). In Section 9.2, we will discuss the particular representation for the time dependent frequencies and the damping coefficient functions with different special functions. In Chapter 10, Riccati representation of the special functions as oscillator type problem is considered and some particular cases are given. In conclusion, Chapter 11, we discuss main results obtained in this thesis.

In Appendices, we study the essential points mentioned in the main text in detail. In Appendix A, we give the preliminaries of tensor calculus. In Appendix B, we consider the definition and some properties of the Riccati equation. In Appendix C, we write the important properties of the Hermite differential equation and its generating function. In addition to this, the recurrence relations and the special results of the Hermite polynomials are given. In Appendix D, we obtain oscillator type equation constructed by the Friedman Equations. The oscillator equation is transformed into the Schrödinger equation with $m = 1/2,\ \hbar = 1$ and we consider this Schrödinger equation with delta potential and we investigate the condition for positive and negative cosmological constants which are interpreted as bound and unbound states in quantum mechanics.

Part I

DISSIPATIVE GEOMETRY AND GENERAL RELATIVITY THEORY

CHAPTER 2

PSEUDO-RIEMANNIAN GEOMETRY AND GENERAL RELATIVITY

Einstein's relativity theories, both the special theory of 1905 and the general theory of 1915, are the modern theories of space and time which have replaced Newton's concepts.

In the special theory of relativity it is assumed that the space of events is the pseudo-Euclidean Minkowski space which admits introduction of a coordinate system, $x^0 = t, x^1, x^2, x^3$ and possesses a pseudo-Euclidean metric with signature $(+, -, -, -)$, that is,

$$ds^2 = (dx^0)^2 - \sum_{\alpha=1}^{3} (dx^\alpha)^2 \tag{2.1}$$

However, according to the Einstein's general theory of relativity, the space of events is a four dimensional pseudo-Riemannian space and it possesses a pseudo- Riemannian metric represented by symmetric metric tensor $g_{\mu\nu}(\overrightarrow{x}, t)$ with a signature $sign[g_{\mu\nu}] = (+, -, -, -)$ which is often used as $(-, +, +, +)$. In each local coordinates ($x^0 = t, x^1, x^2, x^3$), the pseudo-Riemannian metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.2}$$

provides a complete information on how to measure physical distances and time lapses between space-time points. Here and everywhere in this thesis, we use Einstein's convention of summations for repeated indices (upper and lower). Furthermore, metric tensor $g_{\mu\nu}$ plays the role of gravitational field and specifies fully the dynamics of gravitating bodies, and thus it is of a fundamental importance for the Einstein relativistic theory of gravitation.

In the next subsections, there will be given main definitions and ingredients of the Einstein's as gravitational field equations and the energy momentum tensor will be considered. Then we consider the Friedman-Robertson-Walker metric and the Friedman equations to describe the universe at large scales. Main notations and definitions form tensor analysis are given in Appendix A.

2.1. Curvature of Space Time and Einstein Field Equations

In pseudo-Riemannian geometry the metric is symmetric and compatible with t he corresponding connection which is called the Levi Civita connection.

Definition 2.1 *The covariant derivative in the tensor formalism is defined by using the Levi- Civita connection $\Gamma^{\mu}_{\nu\beta}$, which physicists generally refer to as the Christoffel symbol named for Elwin Bruno Christoffel (1829 - 1900). In pseudo-Riemannian geometry the Levi-Civita connection is symmetrical, compatible with metric and is determined by two conditions, the covariant constancy of the metric and the absence of torsion. In the tensor notations, these conditions are*

metricity:

$$g_{\mu\nu;\alpha} = \partial_\alpha g_{\mu\nu} - \Gamma^{\lambda}{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\alpha\nu} g_{\mu\lambda} = 0$$

no torsion:

$$T^{\mu}_{\alpha\beta} = \frac{1}{2}\left(\Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\beta\alpha}\right) = 0$$

From these conditions the Christoffel symbols are then uniquely determined in terms of the metric to be

$$\Gamma^{\mu}_{\nu\beta} = \frac{1}{2} g^{\lambda\mu}\left(g_{\beta\lambda,\nu} + g_{\lambda\nu,\beta} - g_{\nu\beta,\lambda}\right) \tag{2.3}$$

where the matrix $g^{\lambda\mu}$ is an inverse of the matrix $g_{\lambda\mu}$, defined as $g^{\lambda\mu} g_{\sigma\mu} = \delta^{\lambda}_{\sigma}$. By using the Christoffel symbols the corresponding Riemann curvature tensor can be obtained.

Definition 2.2 *The Riemann curvature tensor characterizes the curvature of the space and it is defined as*

$$R^{\mu}_{\nu\sigma\beta} = \Gamma^{r}_{\nu\sigma}\Gamma^{\mu}_{r\beta} - \Gamma^{r}_{\nu\beta}\Gamma^{\mu}_{r\sigma} + \frac{\partial \Gamma^{\mu}_{\nu\sigma}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\mu}_{\nu\beta}}{\partial x^{\sigma}} \tag{2.4}$$

The definitions of Ricci tensor is symmetric, and the only nonzero contraction of the Riemann curvature tensor is given below.

Definition 2.3 *The Ricci tensor is defined by*

$$R_{\mu\nu} = R^{\beta}_{\mu\beta\nu} = g^{\sigma\beta} R_{\mu\sigma\nu\beta} \tag{2.5}$$

Definition 2.4 *We also define the Ricci scalar by*

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.6}$$

The General relativistic space time is curved and its curvature is caused by matter. The relation between curvature and the source is governed by Einstein's field equations.

2.1.1. Einstein Field Equations

Definition 2.5 *a. The Einstein equation for a gravitational field in the absence of matter and all other physical fields has the form*

$$R_{\mu\nu} = 0 \tag{2.7}$$

where $R_{\mu\nu}$ is the Ricci tensor.

b. In the presence of matter, the Einstein equation has the universal form as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.8}$$

where G is the gravitational Newton's constant ($G = 6.67 \times 10^{-11} N.m^2/kg^2$), c is the speed of light in a vacuum ($c = 2.9979 \times 10^8 m/s$). $T_{\mu\nu}$ is called the energy momentum tensor (we will discuss the energy momentum tensor in the next section in details). R is the Ricci scalar and $G_{\mu\nu}$ is called the Einstein tensor.

c. Modified Einstein equations are defined as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.9}$$

where Λ is the cosmological constant.

Definition 2.6 *The vacuum energy density is a universal number which is proportional a constant. This constant is proportional the cosmological constant Λ. To clarify the definition of cosmological constant, we need to define meaning of the vacuum: it is state of matter without any matter, radiation, and other substances. It has the lowest energy, but there is no physical reason for that energy to be equal zero. However, in General Relativity, any form of energy affects the gravitational field, so the vacuum energy becomes a potentially crucial ingredient. We believe that the vacuum is the same everywhere in the* *universe is a good approximation and the cosmological constant is proportional to the vacuum energy density* ρ_Λ

$$\Lambda = \frac{8\pi G}{3c^2}\rho_\Lambda \tag{2.10}$$

2.1.2. Energy Momentum Tensor

Definition 2.7 *The energy momentum tensor* $T^{\mu\nu}$ *is a symmetric (2, 0) tensor and this tensor contains all energy characteristics of the system: energy density, pressure, stress, and so forth. If the matter is a "perfect" fluid then the following formula holds*

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu - p g_{\mu\nu} \tag{2.11}$$

where U_μ *is the 4-velocity (it satisfies* $U_\mu U_\nu = diag\left(c^2, 0, 0, 0\right)$ *where we assume* $c = 1$*) and* ρ *is density and* p *is pressure. They are related by* $\rho = \rho(p)$.

In the general relativity all interesting types of matter from stars to electromagnetic fields and the entire universe can be thought of as a perfect fluid.

Weyl's Postulate states that the cosmological substratum in general relativity is a perfect fluid, which means the fluid with no fluid particle intersections and interactions. Therefore, we consider the perfect fluid tensor as our energy-momentum source. There are many other definitions, for example, Schutz (1970) defines a perfect fluid to be the one with no heat conduction and no viscosity, while Weinberg (1972) defines it as a fluid which looks isotropic in its rest frame. These two viewpoints turn out to be equivalent. Operationally, we should think of a perfect fluid as one which may be completely characterized by its pressure and density. Hence, (2.11) is a typical formula for applications such as stellar structure and cosmology. In addition to this, it is generally agreed that a perfect fluid with zero pressure is technically referred to as the dust. In the early universe, uniform radiation is thought to have predominated.

Besides being symmetric, $T^{\mu\nu}$ has important property of being conserved. Conservation is expressed as the vanishing of the divergence

$$\nabla_\mu T^{\mu\nu} = 0$$

2.2. Universe as a Dynamical System

Here we are making the standard assumption that on large scales, physics is dominated by gravity, which is well described by the general relativity theory, where gravitational effects result from the space-time curvature. This is why we can apply general relativity to the universe. We suppose first the existence of a universal time- a global time coordinate t. Then simplest structure of the universe is the one where metric tensor components are only time dependent. If we consider an isotropic universe which is characterized by the global time and the characteristic scale, evolution of the universe is described by the dynamical system (Rindler 2001). In such a universe, the metric is defined by the Friedman-Robertson-Walker (FRW) metric and the dynamical system is described by the Friedman equations.

2.2.1. Friedman-Robertson-Walker (FRW) Metric

Before discussing full four dimensional space time, we consider the particular case where the space is represented by a two dimensional surface in three dimensional Euclidean space. From geometry of surfaces we know that there are three classes of isotropic and homogeneous two dimensional spaces

- 2-sphere S^2 whose Gaussian curvature is positive, $\kappa > 0$
- x-y plane $(\mathbb{R}^2)$ whose Gaussian curvature is zero, $\kappa = 0$
- hyperbolic plane $(\mathfrak{H}^2)$ whose Gaussian curvature is negative, $\kappa < 0$

We will now compute what the metric for these spaces looks like. Differential distance, ds, in Euclidean space, $(\mathbb{R}^2)$

$$ds^2 = {dx_1}^2 + {dx_2}^2 \tag{2.12}$$

gives the metric tensor $g_{\mu\nu} = \delta_{\mu\nu}$. Changing to polar coordinates r', θ

$$x_1 =: r' \cos\theta, \;\; x_2 =: r' \sin\theta$$

it is easy to see that

$$ds^2 = dr'^2 + r'^2 d\theta^2$$

substituting $r' = ar$ by re-scaling, we have

$$ds^2 = a\left(dr^2 + r^2 d\theta^2\right)$$

A more complicated case occurs if the space is curved. For example, the surface of three-dimensional sphere, two sphere S^2 with radius a in $\mathbb{R}^3$ is defined by

$$x_1{}^2 + x_2{}^2 + x_3{}^2 = a^2 \tag{2.13}$$

Length element of $\mathbb{R}^3$

$$ds^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2$$

Equation (2.13) gives

$$x_3 = \sqrt{a^2 - x_1{}^2 - x_2{}^2}$$

such that

$$dx_3 = \frac{\partial x_3}{\partial x_1}dx_1 + \frac{\partial x_3}{\partial x_2}dx_2 = -\frac{x_1 dx_1 + x_2 dx_2}{\sqrt{a^2 - x_1{}^2 - x_2{}^{2^2}}}$$

Introduce again polar coordinates r', θ in $x_1 x_2$ plane

$$x_1 =: r' \cos\theta, \;\; x_1 =: r' \sin\theta$$

(note: r', θ only unique in upper or lower half-sphere). The differentials are given by

$$dx_1 = \cos\theta dr' - r' \sin\theta d\theta$$

$$dx_2 = \sin\theta dr' + r' \cos\theta d\theta \tag{2.14}$$

In cartesian coordinates, the length element on S^2 is

$$ds^2 \;\; = \;\; dx_1{}^2 + dx_1{}^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{a^2 - x_1{}^2 - x_2{}^2}$$

Inserting equation (2.14), we obtain

$$ds^2 \;\; = \;\; r'^2 d\theta^2 + \frac{a^2}{a^2 - r'^2} dr'^2 \tag{2.15}$$

after redefining $r = r'a$, we get the result

$$ds^2 \;\; = \;\; a^2 \left(\frac{dr^2}{1 - r^2} + r^2 d\theta^2\right)$$

In the spherical coordinates as

$$\begin{aligned} x_1 &= a\sin\phi\cos\theta \\ x_2 &= a\sin\phi\sin\theta \\ x_3 &= a\cos\phi \end{aligned}$$

$(\theta \in [0,\pi]\,,\ \ \phi \in [0,2\pi])$. This line element is

$$ds^2 \;=\; a^2\left(d\theta^2 + \sin^2\theta\right)$$

The hyperbolic plane ,$\mathfrak{H}^2$, is defined by

$${x_1}^2 + {x_2}^2 - {x_3}^2 = -a^2$$

If we work in Minkowski space

$$ds^2 \;=\; d{x_1}^2 + d{x_2}^2 - d{x_3}^2$$

then

$$ds^2 \;=\; d{x_1}^2 + d{x_2}^2 - \frac{(x_1 dx_1 + x_2 dx_2)^2}{a^2 + {x_1}^2 + {x_2}^2}$$

Using polar coordinates to obtain similar form of the metric as for the sphere (2.15),

$$ds^2 \;=\; a^2\left(\frac{dr^2}{1+r^2} + r^2 d\theta^2\right)$$

The analogy to spherical coordinates on hyperbolic plane are given by

$$\begin{aligned} x_1 &= a\sinh\theta\cosh\phi \\ x_2 &= a\sinh\theta\sinh\phi \\ x_3 &= a\cosh\theta \end{aligned}$$

$(\theta \in [-\infty,+\infty]\,,\ \ \phi \in [0,2\pi])$. Hence,

$$ds^2 \;=\; a^2\left(d\theta^2 + \sinh^2\theta d\phi^2\right)$$

If we summarize, the line element for sphere

$$ds^2 \;=\; a^2\left(\frac{dr^2}{1+r^2} + r^2 d\theta^2\right)$$

the line element for plane

$$ds^2 \;\; = \;\; a^2 \left(dr^2 + r^2 d\theta^2 \right)$$

and for hyperbolic plane

$$ds^2 \;\; = \;\; a^2 \left(\frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right)$$

can be written as

$$ds^2 \;\; = \;\; a^2 \left(\frac{dr^2}{1 + \kappa r^2} + r^2 d\theta^2 \right) \tag{2.16}$$

where curvature κ defines the geometry

$$\kappa = \begin{cases} +1, & \text{spherical} \\ 0, & \text{planar} \\ -1, & \text{hyperbolic} \end{cases}$$

Because of the homogeneity, we can choose the same time coordinate for each point in space, and at each time slice. Then we must have the isotropic and homogeneous three dimensional metric

$$ds^2 = a^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right), \;\; d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \;\; \kappa = 0 \pm 1 \tag{2.17}$$

The angles ϕ and θ are the usual azimuthal and polar angles of spherical coordinates, with $(\theta \in [0, \pi]\,, \;\; \phi \in [0, 2\pi])$. As before, the parameter κ can take on three different values. However, there is no constraint relating the cosmic scale factor a at different time slices, which can therefore be a function of time $a = a(t)$. Aside from isotropy and homogeneity, general relativity requires that locally the line element

$$ds^2 = c^2 dt^2 - d\tilde{\mathbf{x}}^2, \;\; d\tilde{\mathbf{x}}^2 = a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right)$$

be invariant under the Lorentz transformations. Thus we arrive at the Friedman-Robertson-Walker (FRW) metric, which is the metric (up to coordinate transformations) fulfilling the cosmological principle:

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right), \;\; d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \;\; \kappa = 0 \pm 1 \tag{2.18}$$

Coordinates t, r, θ, ϕ are called co-moving coordinates. The reason is because two objects at different spatial coordinates can remain at those coordinates at all times, provided the proper distance between them changes with time according to how the scale factor $a(t)$ changes with time.

2.2.2. Friedman Equations

Definition 2.8 *Friedman equations were derived by Alexander Friedman in 1922 and they are*

$$\frac{3\left(\dot{a}^2+c^2\kappa\right)}{a^2}-c^2\Lambda=\frac{8G\pi}{c^2}\rho(t) \tag{2.19}$$

$$\frac{\left(2a\ddot{a}+\dot{a}^2+c^2\kappa\right)}{a^2c^2}-\Lambda=-\frac{8\pi G}{c^4}p(t) \tag{2.20}$$

Proposition 2.1 *The Friedman equations derived from the modified Einstein field equations by using FRW metric.*

Proof First, the metric which describe evolution of the space is given in the form

$$ds^2=c^2dt^2-a(t)^2\left(\frac{dr^2}{1-\kappa r^2}+r^2d\Omega^2\right),\ \ d\Omega^2=d\theta^2+\sin^2\theta d\phi^2,\ \ \kappa=0\pm1$$

which we have defined before (2.18). Then, the metric tensor and its inverse are represented by using the following matrix forms

$$g_{\mu\nu}=\begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -a^2(t)(1-\kappa r^2)^{-1} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{2.21}$$

$$g^{\mu\nu}=\begin{pmatrix} c^{-2} & 0 & 0 & 0 \\ 0 & -\left(1-\kappa r^2\right)a^{-2}(t) & 0 & 0 \\ 0 & 0 & -a^{-2}(t)r^{-2} & 0 \\ 0 & 0 & 0 & -a^{-2}(t)r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{2.22}$$

which satisfies the $g^{\mu\nu}g_{\mu\sigma}=\delta^{\nu}_{\sigma}$ where δ is Kronecker Delta tensor. Then, we use the modified Einstein's equations

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=-\frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.23}$$

to find components of the Einstein's tensor $G_{\mu\nu}$ in terms of Ricci tensors $R_{\mu\nu}$ and Ricci scalar R by using the metrics (2.21) and (2.22). To obtain the Ricci tensor, we need to calculate Christoffel symbols and then the Riemann tensor. The Christoffel symbols are defined by the following equation

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2} g^{\mu\lambda}\big(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}\big) \tag{2.24}$$

where $g_{\alpha\lambda,\beta} = \frac{\partial g_{\alpha\lambda}}{\lambda x^{\beta}}$. After substituting metric tensors (2.21) and (2.22) into (2.24), the nonvanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^{1}_{01} &= \Gamma^{2}_{02} = \Gamma^{3}_{03} = \frac{\dot{a}(t)}{a(t)} \\
\Gamma^{1}_{22} &= r\left(-1 + r^2\kappa\right) \\
\Gamma^{1}_{11} &= \frac{r\kappa}{1 - \kappa r^2} \\
\Gamma^{1}_{33} &= r\left(-1 + r^2\kappa\right)\sin^2\theta \\
\Gamma^{2}_{12} &= \Gamma^{3}_{13} = \frac{1}{r} \\
\Gamma^{2}{}_{33} &= -\cos\theta \sin\theta \\
\Gamma^{0}_{11} &= \frac{\dot{a}(t)a(t)}{c^2\left(1 - \kappa r^2\right)} \\
\Gamma^{0}_{22} &= \frac{r^2\dot{a}(t)a(t)}{c^2} \\
\Gamma^{0}_{33} &= \frac{r^2\dot{a}(t)a(t)\sin^2\theta}{c^2} \\
\Gamma^{3}_{23} &= \cot\theta
\end{aligned} \tag{2.25}$$

For the Riemann tensor defined by

$$R^{\alpha}{}_{\beta\mu\nu} = -\Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\sigma}{}_{\beta\nu}\Gamma^{\alpha}{}_{\sigma\mu} - \Gamma^{\sigma}{}_{\beta\mu}\Gamma^{\alpha}{}_{\sigma\nu} \tag{2.26}$$

we obtain

$$\begin{aligned}
R_{0101} &= \frac{a\ddot{a}}{-1 + \kappa r^2} \\
R_{0202} &= -r^2 a\ddot{a} \\
R_{0303} &= -r^2 a\ddot{a}\sin^2\theta \\
R_{1212} &= -\frac{r^2a^2\left(c^2\kappa + \dot{a}^2\right)}{c^2\left(-1 + \kappa r^2\right)} \\
R_{1313} &= -\frac{r^2a^2\sin^2\theta\left(c^2\kappa + \dot{a}^2\right)}{c^2\left(-1 + \kappa r^2\right)} \\
R_{2323} &= \frac{1}{c^2}\left(r^4a^2\sin^2\theta\right)\left(c^2\kappa + \dot{a}^2\right)
\end{aligned} \tag{2.27}$$

For the Ricci tensor

$$2R_{\mu\nu} = R^{\alpha}{}_{\nu,\mu\alpha} + R_{\mu\alpha,\nu}{}^{\alpha} - R_{\mu\nu,\alpha}{}^{\alpha} - R_{,\mu\nu} \tag{2.28}$$

we have

$$\begin{aligned}
R_{00} &= \frac{3\ddot{a}}{a} \\
R_{11} &= \frac{2c^2\kappa + 2\dot{a}^2 + a\ddot{a}}{c^2\left(-1+\kappa r^2\right)} \\
R_{22} &= -\frac{1}{c^2}\{r^2\left(2c^2\kappa + 2\dot{a}^2 + a\ddot{a}\right)\} \\
R_{33} &= -\frac{1}{c^2}\{r^2\sin^2\theta\ \left(2c^2\kappa + 2\dot{a}^2 + a\ddot{a}\right)\}
\end{aligned} \tag{2.29}$$

The Ricci scalar

$$R = R_{\mu\nu}g^{\mu\nu} \tag{2.30}$$

is

$$R = R_{00}g^{00} + R_{11}g^{11} + R_{22}g^{22} + R_{33}g^{33} = \frac{\{6\left(c^2\kappa + \dot{a}^2 + a\ddot{a}\right)\}}{c^2a^2} \tag{2.31}$$

and the non-vanishing components of Einstein's tensor

$$\begin{aligned}
G_{00} &= -3\frac{\left(c^2\kappa + \dot{a}^2\right)}{a^2} \\
G_{11} &= -\frac{\left(c^2\kappa + \dot{a}^2 + 2a\ddot{a}\right)}{c^2\left(-1+r^2\kappa\right)} \\
G_{22} &= G_{33} = 0
\end{aligned} \tag{2.32}$$

Finally, the energy momentum tensor can be written as

$$\begin{aligned}
T_{00} &= c^2\rho \\
T_{11} &= \frac{a^2}{\left(1-\kappa r^2\right)}p = T_{22} = T_{33} \qquad (2.33)
\end{aligned}$$

(2.34)

Substitution equations (2.21), (2.32) and (2.33) in equation (2.23) leads to the Friedman Equations

$$\frac{3\left(\dot{a}^2 + c^2\kappa\right)}{a^2} - c^2\Lambda = \frac{8G\pi}{c^2}\rho(t)$$

$$\frac{\left(2a\ddot{a} + \dot{a}^2 + c^2\kappa\right)}{a^2c^2} - \Lambda = -\frac{8\pi G}{c^4}p(t)$$

2.2.3. Adiabatic Expansion and Friedman Differential Equation

In this subsection, we consider the universe as a thermodynamical system. For this thermodynamical system we have the next proposition

Proposition 2.2 *Universe has to expand adiabatically*

$$dE + pdV = 0 \tag{2.35}$$

where V is the volume of a co-moving ball of the substratum and is proportional to a^3 ($V \sim a^3$) and E is the total energy in the volume $E = \rho V = \rho a^3$ (Rindler 2001).

Let us discuss the First Law of Thermodynamics and the adiabatic expansion in detail so as to understand equation (2.35) physically. The first law of thermodynamics is the application of the conservation of energy principle to heat and thermodynamic processes. The first law of thermodynamics is

Proposition 2.3 *The change in internal energy of a system is equal to the change in heat added or leaved to the system minus the work done by the system.*

$$\underbrace{\Delta E}_{Changeininternalenergy} = \underbrace{\Delta Q}_{Heataddedtothesystem} - \underbrace{\Delta W}_{Workdonebythesystem} \tag{2.36}$$

or

$$\underbrace{\Delta Q}_{Heataddedtothesystem} = \underbrace{\Delta E}_{Changeininternalenergy} + \underbrace{p\Delta V}_{Workdonebythesystem} \tag{2.37}$$

where p and V are the pressure and volume.

Definition 2.9 *Adiabatic expansion occurs where no heat enters or leaves the system and occurs when the work is done fast. Therefore, there will be no change in the heat of the system which means $\Delta Q = 0$. Hence, equation (2.38) is reduced to the following form*

$$\Delta E = -p\Delta V \tag{2.38}$$

The Friedman equations (2.19) and (2.20) may be rewritten as

$$3\frac{(\dot{a}^2 + k)}{a^2} - \Lambda = 8\pi\rho(t) \tag{2.39}$$

$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} - \Lambda = -8\pi p(t) \tag{2.40}$$

where $G = c = 1$.

Proposition 2.4 *The Friedman equations can be interpreted as an energy type equation (2.39) and equation of motion type of equation (2.40)(Rindler 2001). By using equations (2.39) and (2.40), the adiabatic expansion equation (2.35) is derived.*

Proof Let us differentiate equation (2.39) in terms of time t, then we obtain

$$\frac{6\dot{a}\ddot{a}}{a^2} - \frac{6\dot{a}\left(a^2+k\right)}{a^3} = \quad 8\pi\dot{\rho} \tag{2.41}$$

after multiplying this equation by $\frac{1}{8\pi}$ it becomes,

$$\frac{6\dot{a}\ddot{a}}{8\pi a^2} - \frac{6\dot{a}\left(a^2+k\right)}{8\pi a^3} = \dot{\rho} \tag{2.42}$$

multiplying the second Friedman equation (2.40) by $\frac{-3\dot{a}}{8\pi a}$

$$-\frac{6\dot{a}\ddot{a}}{8\pi a^2} - \frac{3\dot{a}\left(a^2+k\right)}{8\pi a^3} + \frac{3\dot{a}\Lambda}{8\pi a} = 3p\frac{\dot{a}}{a} \tag{2.43}$$

then adding (2.42) and (2.43), we get

$$\frac{3\dot{a}}{8\pi a}\left[\frac{3\dot{a}+3}{a^2} - \Lambda\right] = 3p\frac{\dot{a}}{a} + \dot{\rho} \tag{2.44}$$

By using the first Friedman equation, we can write $8\pi\rho$ instead of $\left[3\frac{\left(\dot{a^2}+k\right)}{a^2} - \Lambda\right]$. Hence, equation (2.44) can be reduced as

$$-3\frac{\dot{a}}{a}\rho = \quad -3p\frac{\dot{a}}{a} + \dot{\rho} \tag{2.45}$$

Multiplying (2.45) through by a^3, we get

$$\dot{\rho}a^3 + 3\rho\dot{a}a^2 = -3p\dot{a}a^2$$

or

$$\frac{d}{dt}\left(\rho a^3\right) + p\frac{d}{dt}a^3 = 0 \tag{2.46}$$

Since the volume V of a co-moving ball of the substratum is proportional to a^3 ($V \sim a^3$). Then we can write V instead of a^3, and equation (2.46) becomes

$$\frac{d}{dt}\left(\rho a^3\right) + p\frac{d}{dt}V = 0$$

We assume the total mass energy in the volume to be E. Hence, $E = \rho V = \rho a^3$ and

$$\frac{d}{dt}E + p\frac{d}{dt}V = 0 \tag{2.47}$$

As a result, we can represent (2.47) equation in the form,

$$dE + pdV = 0$$

Finally, we have derived the adiabatic expansion equation from Friedman equations.
If we write $\rho a^3 = E$ for the total energy density, A for the surface area of the ball and r for the radius, we obtain $dE = -pdV = -pAdr$. In the absence of thermal flow, isotropy implies that the motion must be adiabatic. In general perfect fluids obeys a simple equation of state,

$$p = \omega c^2 \rho \tag{2.48}$$

at the following calculations, we choose the velocity of light as $c = 1$. Hence

$$\rho a^3 = E \tag{2.49}$$

take derivative of (2.49)

$$\dot{\rho} a^3 + 3\rho \dot{a} a^2 = -3p\dot{a}a^2$$

by using (2.48) in the last equation, we obtain

$$\dot{\rho} a^3 = -3(\omega + 1)\rho \dot{a} a^2 \tag{2.50}$$

After rearranging the equation (2.50), we get the first order differential equation

$$\frac{\dot{\rho}}{\rho} = -3(\omega + 1)\frac{\dot{a}}{a} \tag{2.51}$$

$$\ln \rho = -3(\omega + 1) \ln a$$

Hence, the result is

$$\rho = a^{-3(\omega+1)}$$

or

$$\rho a^{3(\omega+1)} = constant \tag{2.52}$$

Particularly we have

a. For pure radiation, $\omega = \frac{1}{3}$, then $\rho a^4 = constant$ (radiation/relativistic particle)

b. For pure dust, $\omega = 0$, then $\rho a^3 = constant$ (dark matter/dust)

c. For cosmological constant $\omega = -1$, then $\rho = constant$

Now assume that we have a dust-dominated universe, and accordingly set p=0. Then directly from $d\rho a^3 + pdV = 0$ where $d\rho a^3 = 0$ then $\rho a^3 = constant$ (conservation of mass). It convenient to define a constant C by the equation

$$\frac{8}{3}\pi G\rho a^3 =: C(constant). \tag{2.53}$$

Hence, the first Friedman equation becomes,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3} - \frac{\kappa}{a^2} + \frac{\Lambda}{3}, \tag{2.54}$$

or

$$\dot{a}^2 = \frac{8\pi\rho a^2}{3} - k + \frac{\Lambda a^2}{3} \tag{2.55}$$

Combining the last equation with equation (2.53) finally we obtain

$$\dot{a}^2 = \frac{C}{a} + \frac{\Lambda a^2}{3} - \kappa \tag{2.56}$$

which is called the Friedman's differential equation. In the next chapter, the Friedman universe models are constructed with respect to this Friedman differential equations.

CHAPTER 3

DYNAMICS OF UNIVERSE MODELS

Cosmological models are based on the idea that the universe is the same everywhere which known as the Copernican principle. The Copernican principle is related to two mathematical properties that the space might have isotropy and homogeneity.

Definition 3.1 *Isotropy applies at some specific point in the space, and states that the space looks the same no matter what direction you look in. The isotropy of the universe is verified by modern observations of the microwave background.*

Definition 3.2 *Homogeneity is the statement that the metric is the same throughout the space.*

We begin construction of cosmological models with the idea that the universe is homogeneous and isotropic in space. In general relativity this translates into the statement that the universe can be foliated into space-like slices such that each slice is homogeneous and isotropic. It will be very useful, if we think of isotropy as invariance under rotations, and homogeneity as invariance under translations.

3.1. Friedman Models

We shall discuss the solutions of the Friedman differential equation (2.56) to obtain and classify all General Relativity dust universes that are homogeneous and isotropic which are known as Friedman models. So that Friedman's equation can be written as

$$\dot{a}^2 = \frac{C}{a} + \frac{\Lambda a^2 c^2}{3} - \kappa c^2 =: F(a) \tag{3.1}$$

where $C = \frac{8}{3}\pi G\rho a^3$ is considered as a constant due to the adiabatic expansion, when the universe expands so quickly, the slow change of the density of universe can be neglected. There $F(a)$ is function of the scale factor and simply the abbreviation for the three terms. Hence, equation (3.1) becomes

$$\dot{a}^2 = F(a), \quad \frac{da}{dt} = \sqrt{F(a)}$$

and we can formally write down the solution at once by quadrature,

$$t = \int \frac{da}{\sqrt{F}} \tag{3.2}$$

Analyzing form of F we could proceed to the full solution by using elliptic functions. But instead of it we obtain solutions in special cases which construct the universe models (Rindler 2001) as

a. Static Models

b. Empty Models

c. Non-empty Models with $\Lambda = 0$

d. Non-empty Models with $\Lambda \neq 0$

3.1.1. Static Models

The static models have $\dot{a} \equiv 0$. This is the exceptional case in which Friedman's equation is insufficient and both its parent equations (2.39) and (2.40) must be used. $\dot{a} \equiv 0$ provided that

$$\kappa = \frac{C}{a} + \frac{\Lambda a^2}{3}$$

After substituting $C = \frac{8}{3}\pi G \rho a^3$ and rearranging the last equation, we obtain

$$\frac{3\kappa}{a^2} = 8\pi G\rho + \Lambda \tag{3.3}$$

Equation (3.3) implies constant density (ρ =constant) and under this condition we can construct two models as follows

a. Einstein Universe ($\rho > 0$ and $\kappa = 1$)

   This model is the first General Relativistic universe was proposed by Einstein in 1917.

   Note that a positive density which means the non empty universe can not stay in static equilibrium without Λ

b. Static Non-gravitating Universe ($\rho = \Lambda = \kappa = 0$ and a =constant)

   These conditions lead to the Minkowski metric and the model represents a static and non gravitating universe.

3.1.2. Empty Models

For empty models we choose the density to be zero so the constant $C = \frac{8}{3}\pi G\rho a^3$ vanishes. Hence, equation (3.1) reduces to the following form

$$\dot{a}^2 = \frac{\Lambda a^2}{3} - \kappa = F(a) \tag{3.4}$$

Substituting (3.4) into (3.2) we get,

$$t = \int \frac{da}{\sqrt{F(a)}} = \int \frac{1}{\sqrt{\frac{\Lambda a^2}{3} - \kappa}} da \tag{3.5}$$

Depending of Λ and κ we have different models:

a. $\Lambda = 0$, $\kappa = -1$, $a(t) = t$ Milne's model

b. $\Lambda > 0$, $\kappa = 0$, $a(t) = \exp(\frac{t}{b})$ de Sitter space

c. $\Lambda > 0$, $\kappa = 1$, $a(t) = b\cosh(\frac{t}{b})$ de Sitter space

d. $\Lambda > 0$, $\kappa = -1$, $a(t) = b\sinh(\frac{t}{b})$ de Sitter space

e. $\Lambda < 0$, $\kappa = -1$, $a(t) = b\sin(\frac{t}{b})$ is the analog of Milne's model in anti-de Sitter space $\tilde{D}^4$.

where $b = (\frac{\Lambda}{3})^{-\frac{1}{2}}$.

3.1.3. Three Non-Empty Models with $\Lambda = 0$

When $\Lambda = 0$ and $C \neq 0$ the Friedman's equation (3.1) becomes

$$\dot{a}^2 = \frac{C}{a} - \kappa \tag{3.6}$$

There are three models including $\kappa = -1, \ \ 0, \ \ -1$.

a. $\Lambda = 0$ and $\kappa = 0$

If we choose the cosmological constant and curvature to be zero, equation (3.1) becomes

$$\dot{a}^2 = \frac{C}{a} = F(a) \tag{3.7}$$

while (3.2) reduces to the following form

$$t = \int \frac{da}{\sqrt{F(a)}} = \int \sqrt{\frac{a}{C}} da \tag{3.8}$$

and

$$a(t) = t^{2/3} {c_0}^{-2/3} \left(C \frac{9}{4} \right)^{1/3} \tag{3.9}$$

where c_0 is integration constant. Equations (3.7) and (3.9) construct the model which is called the Einstein de-Sitter universe. Moreover, we can find the density parameter of the Einstein universe by solving equation (3.7) as follows

$$\dot{a}^2 = \frac{C}{a}$$

we mentioned $C = \frac{8G\pi\rho}{3} a^3$ before. After substituting C into the last equation, it becomes

$$\rho = \frac{3}{8\pi G} \left(\frac{\dot{a}}{a} \right)^2$$

or

$$\rho = \frac{3}{8\pi G} H^2 \tag{3.10}$$

where $H = \frac{d}{dt} \ln a(t)$ is called to the Hubble parameter.

When the curvature κ of the non empty models is different from zero, there are two models proposed as follows

b. $\Lambda = 0$ and $\kappa = 1$

Under these conditions, the equation (3.1) reduces to the form

$$\dot{a}^2 = \frac{C}{a} - 1 = F(a) \tag{3.11}$$

and (3.2) becomes

$$t = \int \frac{\sqrt{a}}{\sqrt{C - a}} da$$

If we apply trigonometric substitution $a = C \sin^2\theta$, $da = 2C \sin\theta \cos\theta d\theta$, this integral easily can be solved as follows

$$t = C \left[-\sqrt{\frac{a}{C} - \frac{a^2}{C^2}} + \arcsin \sqrt{\frac{a}{C}} \right] \tag{3.12}$$

This transcendental equation does not allow to find $a(t)$ explicitly. But instead of this, we can find parametric form of the curve $a = a(t)$. Let assume $X \equiv \frac{a}{C}$. Then,

$$t = C\left[-\sqrt{X - X^2} + \arcsin\sqrt{X}\right] \tag{3.13}$$

Proposition 3.1 *Substituting $X = \sin^2\frac{\chi}{2}$ in equation (3.13), we find the relation between time and the scale factor in the next parametric form*

$$\frac{dt}{d\chi} = \frac{1}{2}C\left[1 - \cos\chi\right] = a \tag{3.14}$$

Proof Let substitute $X = \sin^2\frac{\chi}{2}$ into (3.13). Hence we get

$$t = C\left[-\sqrt{\sin^2\frac{\chi}{2} - \sin^4\frac{\chi}{2}} + \arcsin\sqrt{\sin^2\frac{\chi}{2}}\right] \tag{3.15}$$

and it becomes

$$t = C\left[-\sin\frac{\chi}{2}\cos\frac{\chi}{2} + \arcsin\sin\frac{\chi}{2}\right] \tag{3.16}$$

where $\sin\frac{\chi}{2}\cos\frac{\chi}{2} = \frac{1}{2}\sin\chi$ and $\arcsin\sin\frac{\chi}{2} = \frac{\chi}{2}$. Then the equation (3.16) reduces to the following equation

$$t = C\left[\frac{\chi}{2} - \frac{1}{2}\sin\chi\right] \tag{3.17}$$

Differentiating this equation, we obtain

$$dt = \frac{1}{2}C\left[1 - \cos\chi\right]d\chi \tag{3.18}$$

and the result is

$$\begin{aligned} a(\chi) &= \frac{1}{2}C\left[1 - \cos\chi\right] \\ t(\chi) &= C\left[\frac{\chi}{2} - \frac{1}{2}\sin\chi\right] \end{aligned}$$

which is the parametrized curve of $a = a(t)$. The $\kappa = 1$ model includes the geometry which is the parametric equation of cycloid. Here the radius of the circle is $\frac{1}{2}C$ and χ is the angle. The full range of χ for the cycloid universe is 0 to 2π. Its maximal radius is given by $a_{max} = a(\chi = \pi) = C$ and its total duration by $t_{tot} = t(\chi = 2\pi) = \pi C$.

c. $\Lambda = 0$ and $\kappa = -1$

If the nonempty models with zero cosmological constant is defined with the negative curvature $\kappa = -1$, then equations (3.1) and (3.2) become

$$\dot{a}^2 = \frac{C}{a} + 1 \tag{3.19}$$

and

$$t = \int \frac{\sqrt{a}}{\sqrt{C+a}} da$$

Now, we can apply the hyperbolic substitution $a = C\sinh^2\theta$, $da = 2C\sinh\theta\cosh\theta d\theta$ to solve this integral. Then we obtain

$$t = C\left[\sqrt{\frac{a}{C} + \frac{a^2}{C^2}} + \sinh^{-1}\sqrt{\frac{a}{C}}\right] \tag{3.20}$$

And again, let us take X instead of $\frac{a}{C}$. Hence,

$$t = C\left[\sqrt{X + X^2} + \sinh^{-1}\sqrt{X}\right] \tag{3.21}$$

Proposition 3.2 *Substituting $X = \sinh^2\frac{\chi}{2}$ we can express this solution in the parametric form*

$$\frac{dt}{d\chi} = \frac{1}{2}C[\cosh\chi - 1] = a \tag{3.22}$$

Proof Let substitute $X = \sinh^2\frac{\chi}{2}$ into (3.21). Hence we get

$$t = C\left[\sqrt{\sinh^2\frac{\chi}{2} + \sinh^4\frac{\chi}{2}} - \sinh^{-1}\sqrt{\sinh^2\frac{\chi}{2}}\right] \tag{3.23}$$

and equation (3.23) becomes

$$t = C\left[\sinh\frac{\chi}{2}\cosh\frac{\chi}{2} - \sinh^{-1}\sinh\frac{\chi}{2}\right] \tag{3.24}$$

where $\sinh\frac{\chi}{2}\cosh\frac{\chi}{2} = \frac{1}{2}\sinh\chi$ and $\sinh^{-1}\sinh\frac{\chi}{2} = \frac{\chi}{2}$. Hence equation (3.24) becomes

$$t = \frac{1}{2}C\left[\sinh\chi - \chi\right] \tag{3.25}$$

After differentiating equation (3.25), we obtain

$$dt = \frac{1}{2}C[\cosh\chi - 1]d\chi \tag{3.26}$$

and the result is

$$
\begin{aligned}
a(\chi) &= \frac{1}{2}C[\cosh\chi - 1] \\
t(\chi) &= C\left[\sinh\frac{\chi}{2}\cosh\frac{\chi}{2} - \frac{\chi}{2}\right]
\end{aligned}
$$

which is the parametric form of the scale factor $a = a(t)$. The curvature constant is negative $\kappa = -1$, the scale factor is $a = \frac{1}{2}C[\cosh\chi - 1]$ and time is $t = \frac{1}{2}C[\sinh\chi - \chi]$. Since $\sinh\chi \gg \chi$ for large χ that $a \sim t$ and this model look like Milne's model whose details will be given later.

3.1.4. Non-Empty Models with $\Lambda \neq 0$

If we allow arbitrary value of Λ, the number of possible solutions will increase. To obtain the class of solutions of the Friedman differential equation, we begin by rewriting it in the form

$$
\dot{a}^2 + \kappa = \frac{C}{a} + \frac{\Lambda a^2}{3} =: f(a, \Lambda) \tag{3.27}
$$

where we assume $C > 0$ and function $f(a, \Lambda)$ is defined by equation (3.27). This function will serve as a kind of potential.

3.2. Milne Model

The Milne model was the first cosmological model proposed by Edward Arthur Milne in 1932. Milne derived a model of an empty universe and it is, in fact, one of the Friedman models corresponding to the limit $\rho \longrightarrow 0$ with $\Lambda = 0$. Using Friedman equation (2.56) for flat, spherical and hyperbolic universes, let us check the Milne's model, under the above constraints. The Friedman equation becomes

$$
\dot{a}^2 = -\kappa \tag{3.28}
$$

a. If the Milne's universe is flat $\kappa = 0$, the equation (3.28) becomes $\frac{da}{dt} = 0$ and then $a(t) = constant$

b. If the Milne's universe has the hyperbolic geometry $\kappa = -1$, the equation (3.28) becomes $\frac{da}{dt} = 1$ and then $a(t) \sim t$

c. Case $\kappa = 1$ where the Milne's universe has the spherical geometry does not have physical meaning.

Friedman, Lemaitre, Robertson, Walker Metric (Friedman-Robertson-Walker metric) reduces to the Milne universe for a space-time that is a pure vacuum without matter, radiation, or a cosmological constant. Milne cosmology therefore corresponds to a cosmological solution to the Einstein equations for

$$T_{\mu\nu} = 0 \tag{3.29}$$

The Milne metric follows when the scale factor of the FRLW metric is a constant over time, yielding

$$ds^2 = c^2dt^2 - dr^2 - r^2d\Omega^2 \tag{3.30}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

r gives the proper motion distance between points in the universe and appropriate constants are subsumed into relevant coordinates. The Milne metric is therefore simply a restatement of the Minkowski metric. Whereas the FLRW metric explains the expansion of space, Milne's model has no expansion of space.

CHAPTER 4

ANISOTROPIC AND HOMOGENEOUS UNIVERSE MODELS

In the previous chapter we have derived the Friedman equations under the assumption of an isotropic and homogeneous space. Because of the homogeneity in this kind of spaces we can choose the same time coordinate for each point and the scale parameters are the same for all coordinates due to isotropy. However, the situation is different for anisotropic space. An anisotropic space introduces different scales for each space coordinates with the same time coordinate. The simplest metric for a universe in which the expansion rate changes with direction is given by the family of Bianchi universes (Ellis and Mac Callum 1996).

$$dl^2 = c^2T^2(t)dt^2 - X^2(t)dx^2 - Y^2(t)dy^2 - Z^2(t)dz^2 \tag{4.1}$$

and the cosmic time τ can be given by

$$d\tau = T(t)dt$$

As a consequence of this, the metric becomes,

$$dl^2 = c^2d\tau^2 - X^2(\tau)dx^2 - Y^2(\tau)dy^2 - Z^2(\tau)dz^2 \tag{4.2}$$

and the Riemannian metric tensor and the inverse of the metric tensor can are

$$g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -X^2(\tau) & 0 & 0 \\ 0 & 0 & -Y^2(\tau) & 0 \\ 0 & 0 & 0 & -Z^2(\tau) \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} c^{-2} & 0 & 0 & 0 \\ 0 & -X^{-2}(\tau) & 0 & 0 \\ 0 & 0 & -Y^{-2}(\tau) & 0 \\ 0 & 0 & 0 & -Z^{-2}(\tau) \end{pmatrix} \tag{4.3}$$

The Christoffel symbols are defined by the following equation,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right)$$

$$
\begin{aligned}
\Gamma^0_{11} &= \frac{X\dot{X}}{c^2} \\
\Gamma^0_{22} &= \frac{Y\dot{Y}}{c^2} \\
\Gamma^0_{33} &= \frac{Z\dot{Z}}{c^2} \\
\Gamma^1_{01} &= \Gamma^1_{10} = \frac{\dot{X}}{X} \\
\Gamma^2_{02} &= \Gamma^2_{20} = \frac{\dot{Y}}{Y} \\
\Gamma^3_{03} &= \Gamma^3_{30} = \frac{\dot{Z}}{Z}
\end{aligned} \tag{4.4}
$$

The Riemannian Tensor is

$$R^{\alpha}_{\beta\mu\nu} = -\Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\mu} - \Gamma^{\sigma}_{\beta\mu}\Gamma^{\alpha}_{\sigma\nu}$$

$$
\begin{aligned}
R_{0101} &= -X\ddot{X} \\
R_{0202} &= -Y\ddot{Y} \\
R_{0303} &= -Z\ddot{Z} \\
R_{1212} &= \frac{XY\dot{X}\dot{Y}}{c^2} \\
R_{1313} &= \frac{XZ\dot{X}\dot{Z}}{c^2} \\
R_{2323} &= \frac{YZ\dot{Y}\dot{Z}}{c^2}
\end{aligned} \tag{4.5}
$$

The Ricci tensor is

$$R_{\beta\nu} = -\Gamma^{\alpha}_{\beta\alpha,\nu} + \Gamma^{\alpha}_{\beta\nu,\alpha} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\alpha} - \Gamma^{\sigma}_{\beta\alpha}\Gamma^{\alpha}_{\sigma\nu}$$

Hence,

$$
\begin{aligned}
R_{00} &= \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} \\
R_{11} &= -\frac{1}{c^2}\left[\ddot{X}X + \frac{X\dot{X}\dot{Y}}{Y} + \frac{X\dot{X}\dot{Z}}{Z}\right] \\
R_{22} &= -\frac{1}{c^2}\left[\ddot{Y}Y + \frac{Y\dot{Y}\dot{X}}{X} + \frac{Y\dot{Y}\dot{Z}}{Z}\right] \\
R_{33} &= -\frac{1}{c^2}\left[\ddot{Z}Z + \frac{Z\dot{Z}\dot{X}}{X} + \frac{Z\dot{Z}\dot{Y}}{Y}\right]
\end{aligned} \tag{4.6}
$$

The Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33}$$

Then,

$$R = \frac{2}{c^2} \left[\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{Z}}{XZ} + \frac{\dot{Y}\dot{Z}}{YZ} \right] \tag{4.7}$$

The Einstein equations are given,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{3c^4} T_{\mu\nu}$$

where the energy momentum tensor is defined $T_{\mu\nu} = (c^2\rho, -g_{ii}p)$ (Rindler 2001) and the Einstein's tensor is

$$\begin{aligned}
G_{00} &= -\left[\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{Z}}{XZ} + \frac{\dot{Y}\dot{Z}}{YZ} \right] \\
G_{11} &= \frac{X^2}{c^2} \left[\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} \right] \\
G_{22} &= \frac{Y^2}{c^2} \left[\frac{\ddot{X}}{X} + \frac{\ddot{Z}}{Z} + \frac{\dot{X}\dot{Z}}{XZ} \right] \\
G_{33} &= \frac{Z^2}{c^2} \left[\frac{\ddot{Y}}{Y} + \frac{\ddot{X}}{X} + \frac{\dot{Y}\dot{X}}{YX} \right]
\end{aligned} \tag{4.8}$$

and the energy momentum tensor components are

$$\begin{aligned}
T_{00} &= c^2 \rho \\
T_{11} &= X^2 p \\
T_{22} &= Y^2 p \\
T_{33} &= Z^2 p
\end{aligned} \tag{4.9}$$

Then, the field equations become

$$\left[\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{Z}}{XZ} + \frac{\dot{Y}\dot{Z}}{YZ} \right] = \frac{8\pi G}{3c^2} \rho \tag{4.10}$$

$$\left[\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} \right] = -\frac{8\pi G}{3c^2} p \tag{4.11}$$

$$\left[\frac{\ddot{X}}{X}+\frac{\ddot{Z}}{Z}+\frac{\dot{X}\dot{Z}}{XZ}\right]=-\frac{8\pi G}{3c^2}p \tag{4.12}$$

$$\left[\frac{\ddot{Y}}{Y}+\frac{\ddot{X}}{X}+\frac{\dot{Y}\dot{X}}{YX}\right]=-\frac{8\pi G}{3c^2}p \tag{4.13}$$

To solve the system of the equations, let us define new variables written as

$$u(\tau)\equiv\frac{\dot{X}}{X},\ \ \theta(\tau)\equiv\frac{\dot{Y}}{Y},\ \ v(\tau)\equiv\frac{\dot{Z}}{Z} \tag{4.14}$$

with the first derivatives

$$\dot{u}(\tau)=\frac{\ddot{X}}{X}-\left(\frac{\dot{X}}{X}\right)^2,\ \ \dot{\theta}(\tau)=\frac{\ddot{Y}}{Y}-\left(\frac{\dot{Y}}{Y}\right)^2,\ \ \dot{v}(\tau)=\frac{\ddot{Z}}{Z}-\left(\frac{\dot{Z}}{Z}\right)^2 \tag{4.15}$$

Inserting into the system of equations we get the system

$$\begin{aligned}
u\theta+vu+\theta v &= \frac{8\pi G}{3c^2}\rho(\tau)\\
\dot{\theta}+\theta^2+\dot{v}+v^2+\theta v &= -\frac{8\pi G}{3c^2}p(\tau)\\
\dot{u}+u^2+\dot{v}+v^2+uv &= -\frac{8\pi G}{3c^2}p(\tau)\\
\dot{\theta}+\theta^2+\dot{u}+u^2+\theta u &= -\frac{8\pi G}{3c^2}p(\tau)
\end{aligned} \tag{4.16}$$

the first equation does not include time derivatives thus can be considered as a constraint.

4.1. General Solution

Let us add the third and the fourth equations of system (4.16)

$$2\dot{u}+2u^2+uv+\theta u+\left(\dot{\theta}+\dot{v}+\theta^2+v^2\right)=-2\frac{8\pi G}{3c^2}p(\tau) \tag{4.17}$$

and subtract the second equation of the system. Then we get

$$2\dot{u}+2u^2+uv+\theta u-\theta v=-\frac{8\pi G}{3c^2}p(\tau) \tag{4.18}$$

After using the first equation of (4.16), (4.18) becomes

$$\dot{u}+uR=\frac{4\pi G}{3c^2}\left(\rho(\tau)-p(\tau)\right) \tag{4.19}$$

where $R \equiv u + v + \theta$. Hence (4.16) becomes

$$\begin{aligned} \dot{u} + uR &= \frac{4\pi G}{3c^2}\left[\rho(\tau) - p(\tau)\right] \\ \dot{\theta} + \theta R &= \frac{4\pi G}{3c^2}\left[\rho(\tau) - p(\tau)\right] \\ \dot{v} + vR &= \frac{4\pi G}{3c^2}\left[\rho(\tau) - p(\tau)\right] \end{aligned} \tag{4.20}$$

if we add them together, we obtain the Riccati equation

$$\dot{R} + R^2 = \frac{4\pi G}{c^2}\left[\rho(\tau) - p(\tau)\right] \tag{4.21}$$

we can determine explicit solution of u, v and θ for given R.

$$\begin{aligned} u &= u_0 + u_p \\ \theta &= \theta_0 + \theta_p \\ v &= v_0 + v_p \end{aligned} \tag{4.22}$$

where u_0, θ_0 and v_0 are general solutions for homogeneous equation, the particular solutions are u_p, θ_p and v_p. The general solutions are the same but u, θ and v are given by different constants in the general solutions.

The homogeneous solution of the first equation of system (4.22) is the solution of

$$\dot{u}_0 + u_0 R = 0 \tag{4.23}$$

and is given by

$$u_0 = e^{\int_{\tau_1}^{\tau} R(\xi)d\xi} \tag{4.24}$$

To obtain the particular solution, we multiply the first equation of the system (4.20) by an integration constant

$$\mu(\tau)\dot{u} + \dot{\mu}u + \mu u R - \dot{\mu}u = \frac{4\pi G}{3c^2}\left[\rho(\tau) - p(\tau)\right]\mu \tag{4.25}$$

where $\mu = e^{\int^{\tau} R(\xi)d\xi}$. Then

$$\frac{d}{d\tau}(\mu u) = \frac{4\pi G}{3c^2}\left[\rho(\tau) - p(\tau)\right]\mu \tag{4.26}$$

Hence the particular solution for u

$$u_p = \frac{4\pi G}{3c^2} e^{-\int^{\tau} R(\xi)d\xi} \int^{\tau} \left[\rho(\xi) - p(\xi)\right] e^{\int^{\xi} R(\eta)d\eta} d\xi \tag{4.27}$$

and the general solutions are

$$\begin{aligned} u(\tau) &= e^{-\int_{\tau_1}^{\tau} R(\xi)d\xi}\left(1+\frac{4\pi G}{3c^2}\int^{\tau}\left[\rho(\xi)-p(\xi)\right]e^{\int^{\xi} R(\eta)d\eta}d\xi\right) \\ v(\tau) &= e^{-\int_{\tau_2}^{\tau} R(\xi)d\xi}\left(1+\frac{4\pi G}{3c^2}\int^{\tau}\left[\rho(\xi)-p(\xi)\right]e^{\int^{\xi} R(\eta)d\eta}d\xi\right) \\ \theta(\tau) &= e^{-\int_{\tau_3}^{\tau} R(\xi)d\xi}\left(1+\frac{4\pi G}{3c^2}\int^{\tau}\left[\rho(\xi)-p(\xi)\right]e^{\int^{\xi} R(\eta)d\eta}d\xi\right) \end{aligned} \tag{4.28}$$

we can find explicit solution of u, v and θ for given R.

4.1.1. Constant Density and Zero Pressure

In this part, we define the system of equations for the dust-dominated universe. If the universe is dust-dominated, we can neglect the pressure $p = 0$. Hence, system (4.20) reduces to

$$\begin{aligned} \dot{u} + uR &= \frac{4\pi G}{3c^2}\rho(\tau) \\ \dot{\theta} + \theta R &= \frac{4\pi G}{3c^2}\rho(\tau) \\ \dot{v} + vR &= \frac{4\pi G}{3c^2}\rho(\tau) \end{aligned} \tag{4.29}$$

Adding the equations together, we have the Riccati equation (see Appendix B) as

$$\dot{R}(\tau) + R^2(\tau) = \frac{4\pi G}{c^2}\rho(\tau) \tag{4.30}$$

The Riccati equation is a first order nonlinear differential equation. To linearize equation (4.30), the transformation $R(\tau) = \frac{\dot{\phi}(\tau)}{\phi(\tau)}$ is employed. Hence, equation (4.30) becomes

$$\ddot{\phi} - \rho(\tau)\frac{4\pi G}{c^2}\phi = 0 \tag{4.31}$$

which is a second order nonlinear differential equation. It has the form of the harmonic oscillator with time dependent frequency $\omega^2(\tau) = -\frac{4\pi G}{c^2}\rho$. For $\rho < 0$ it is a standard oscillator if $p = \rho_0 = constant$. If the density ρ is chosen as a positive constant, equation (4.31) becomes simple hyperbolic oscillator equation. But density function is positive definite function. Therefore, we solve the hyperbolic oscillator equation

$$\ddot{\phi} - \frac{4\pi G}{c^2}\rho_0\phi = 0 \tag{4.32}$$

with the general solution

$$\phi(\tau) = C_1 \sinh\frac{2}{c}\sqrt{(\rho_0\pi G)}\tau + C_2\cosh\frac{2}{c}\sqrt{(\rho_0\pi G)}\tau \tag{4.33}$$

where C_1 and C_2 are integration constants. Let us substitute this solution into $R = \frac{\dot{\phi}(\tau)}{\phi(\tau)}$. Hence

$$R = \frac{\dot{\phi}(\tau)}{\phi(\tau)} = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\frac{C_1 \cosh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau + C_2 \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau}{C_1 \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau + C_2 \cosh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau} \tag{4.34}$$

In addition to this, let us multiply and divide equation (4.34) by C_2,

$$R = \frac{\dot{\phi}(\tau)}{\phi(\tau)} = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\frac{\frac{C_1}{C_2} \cosh \frac{2}{c^2}\sqrt{(\rho_0 \pi G)}\tau + \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau}{\frac{C_1}{C_2} \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau + \cosh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau} \tag{4.35}$$

we may write C instead of $\frac{C_1}{C_2}$ and then, equation (4.35) becomes

$$R = u + v + \theta = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\frac{C \cosh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau + \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau}{C \sinh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau + \cosh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau} \tag{4.36}$$

We conclude that if universe has constant density function, the sum of Hubble parameters which is given by

$$R(\tau) = u(\tau) + v(\tau) + \theta(\tau) = \frac{\dot{X}}{X} + \frac{\dot{Z}}{Z} + \frac{\dot{Y}}{Y} \tag{4.37}$$

and

$$R(\tau) = \frac{d}{d\tau}\left(\ln X + \ln Y + \ln Z\right) = \frac{d}{d\tau}\ln\left(XYZ\right) \tag{4.38}$$

where $\frac{\dot{X}}{X} = H_X, \ \frac{\dot{Y}}{Y} = H_Y, \ \frac{\dot{Z}}{Z} = H_Z$ are Hubble parameters in X, Y, Z directions. Rates of expansion in arbitrary X, Y, Z direction is given by $R(\tau)$. Equation (4.36) can be also written as

$$R(\tau) = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\frac{C + \tanh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau}{1 + C \tanh \frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau} \tag{4.39}$$

When we choose $\tau = 0$, $R(\tau)$ becomes $R(0) = \frac{2}{c}\sqrt{(\rho_0 \pi G)}C$ where $C \equiv \tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau_0\right)$. Hence, equation (4.39) is rewritten

$$R(\tau) = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\frac{\tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau_0\right) + \tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau\right)}{1 + \tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\tau_0\right)\tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\right)\tau} \tag{4.40}$$

For simplicity, we use the following identity

$$\begin{aligned} \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} &= \frac{\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha}{\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta} \\ &= \frac{\sinh\left(\alpha + \beta\right)}{\cosh\left(\alpha + \beta\right)} \\ &= \tanh\left(\alpha + \beta\right) \end{aligned} \tag{4.41}$$

then equation (4.40) is reduced to

$$R(\tau) = \frac{2}{c}\sqrt{(\rho_0 \pi G)}\left[\tanh\left(\frac{2}{c}\sqrt{(\rho_0 \pi G)}\,(\tau + \tau_0)\right)\right] \tag{4.42}$$

Rate of expansion $R(\tau)$ is considered as a kink. At time $\tau_0 = \frac{c}{2\sqrt{\pi\rho_0 G}}\tanh^{-1}C$, rate of expansion changes the sign (from contraction to expansion). Now, we can write the general solutions for u, v and θ explicitly. Hence

$$\begin{aligned} \int^{\tau} R(\xi)d\xi &= \frac{2}{c}\sqrt{(\rho_0 \pi G)}\int_{-\tau_0}^{\tau} \tanh\left(\frac{2}{c}\sqrt{\rho_0 \pi G}\,(\xi + \xi_0)\right)d\xi \\ &= \ln\cosh\frac{2}{c}\sqrt{\pi\rho_0 G}\,(\tau + \tau_0) \end{aligned} \tag{4.43}$$

and integration factor for constant density

$$\mu(\tau) = e^{\int_{-\tau_0}^{\tau} R(\xi)d\xi} = \cosh\frac{2}{c}\sqrt{\rho_0 \pi G}\,(\tau + \tau_0) \tag{4.44}$$

Let us substitute (4.43) and (4.44) in (4.28). Then the general solutions are

$$u(\tau) = \frac{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_1 + \tau_0)}{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0)} + \frac{1}{3}\sqrt{\frac{4\rho_0\pi G}{c^2}}\tanh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0) \tag{4.45}$$

$$v(\tau) = \frac{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_2 + \tau_0)}{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0)} + \frac{1}{3}\sqrt{\frac{4\rho_0\pi G}{c^2}}\tanh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0) \tag{4.46}$$

$$\theta(\tau) = \frac{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_3 + \tau_0)}{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0)} + \frac{1}{3}\sqrt{\frac{4\rho_0\pi G}{c^2}}\tanh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0) \tag{4.47}$$

Hence, the mean rate of change can be written as

$$\begin{aligned} R(\tau) &= \frac{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_1 + \tau_0) + \cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_2 + \tau_0) + \cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau_3 + \tau_0)}{\cosh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0)} \\ &+ \sqrt{\frac{4\rho_0\pi G}{c^2}}\tanh\sqrt{\frac{4\rho_0\pi G}{c^2}}\,(\tau + \tau_0) \end{aligned} \tag{4.48}$$

4.1.2. Constant Pressure and Zero Density

In this part, we define the system of equations for the radiation-dominated universe. If the universe is radiation-dominated, we can neglect the density $\rho = 0$. Hence, system (4.20) reduces to

$$\begin{aligned} \dot{u} + uR &= -\frac{4\pi G}{3c^2}p(\tau) \\ \dot{\theta} + \theta R &= -\frac{4\pi G}{3c^2}p(\tau) \\ \dot{v} + vR &= -\frac{4\pi G}{3c^2}p(\tau) \end{aligned} \tag{4.49}$$

If we add the equations of system (4.49), the system reduces to the Riccati equation

$$\dot{R} + R^2 = -\frac{4\pi G}{c^2}p(\tau) \tag{4.50}$$

Linearizing equation (4.50) as $R(\tau) = \frac{\dot{\phi}}{\phi}$ we have

$$\frac{\ddot{\phi}}{\phi} = -\frac{4\pi G}{c^2}p(\tau) \tag{4.51}$$

$$\ddot{\phi} + \frac{4\pi G}{c^2}\phi p(\tau) = 0 \tag{4.52}$$

It is a second order differential equation. If pressure is constant $p = p_0$ and $p > 0$, equation (4.52) becomes a standard harmonic oscillator. While for $p < 0$, equation (4.52) becomes a hyperbolic oscillator. Here we investigate the case

$$\ddot{\phi} + \frac{4\pi G}{c^2}p_0\phi = 0 \tag{4.53}$$

$$\Delta = b^2 - 4ac = -4\left[\frac{4\pi G}{c^2}p_0\right]$$

Hence, the roots of the equation are

$$r_{1,2} = \pm\frac{2}{c}i\sqrt{\pi G p_0} \tag{4.54}$$

the general solution of the differential equation is

$$\phi(\tau) = C_1 \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau + C_2 \cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau \tag{4.55}$$

Let us substitute this solution into $R = \frac{\dot{\phi}(\tau)}{\phi(\tau)}$,

$$R = \frac{\dot{\phi}}{\phi} = \frac{\frac{2}{c}\sqrt{(\pi G p_0)}C_1 \cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau - \frac{2}{c}\sqrt{(\pi G p_0)}C_2 \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau}{C_1 \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau + C_2 \cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau} \tag{4.56}$$

Next, multiply and divide the equation (4.56) by C_2,

$$R = \frac{\dot{\phi}}{\phi} = \frac{\frac{2}{c}\sqrt{(\pi G p_0)}\frac{C_1}{C_2} \cos\frac{2}{c^2}\sqrt{(\pi G p_0)}\tau - \frac{2}{c}\sqrt{(\pi G p_0)} \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau}{\frac{C_1}{C_2} \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau + \cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau} \tag{4.57}$$

we can write C instead of $\frac{C_1}{C_2}$ and then, the equation becomes

$$R = u + v + \theta = \frac{\dot{\phi}}{\phi} = \frac{2}{c}\sqrt{(\pi G p_0)}\frac{C\cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau - \sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau}{C\sin\frac{2}{c}\sqrt{(\pi G p_0)}\tau + \cos\frac{2}{c}\sqrt{(\pi G p_0)}\tau} \quad (4.58)$$

When $\tau = 0$, $R(0) = \frac{2}{c}\sqrt{\rho_0\pi G}C$ where $C \equiv \tan\left(\frac{2}{c}\sqrt{\rho_0\pi G}\right)\tau_0$.

$$R(\tau) = \frac{2}{c}\sqrt{(p_0\pi G)}\frac{\tan\left(\frac{2}{c}\sqrt{(p_0\pi G)}\tau_0\right) - \tan\left(\frac{2}{c}\sqrt{(p_0\pi G)}\tau\right)}{1 + \tan\left(\frac{2}{c}\sqrt{(p_0\pi G)}\tau_0\right)\tan\left(\frac{2}{c}\sqrt{(p_0\pi G)}\right)\tau} \quad (4.59)$$

For simplicity, we use the following trigonometric identity

$$\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\cos\alpha\cos\beta + \sin\alpha\sin\beta} = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \tan(\alpha-\beta) \quad (4.60)$$

Hence, equation (4.59) is reduced to the following form

$$R(\tau) = \frac{2}{c}\sqrt{(p_0\pi G)}\left[\tan\frac{2}{c}\sqrt{p_0\pi G}\,(\tau - \tau_0)\right] \quad (4.61)$$

Now, we can obtain the following terms

$$\begin{aligned}\int^{\tau} R(\xi)d\xi &= -\frac{2}{c}\sqrt{(p_0\pi G)}\int_{\tau}^{\tau_0}\tan\left(\frac{2}{c}\sqrt{p_0\pi G}\,(\xi - \xi_0)\right)d\xi \\ &= \ln\cos\frac{2}{c}\sqrt{\pi p_0 G}\,(\tau - \tau_0) \quad (4.62)\end{aligned}$$

and

$$\mu(\tau) = e^{-\int_{\tau}^{\tau_0} R(\xi)d\xi} = \cos\frac{2}{c}\sqrt{p_0\pi G}\,(\tau - \tau_0) \quad (4.63)$$

Let us substitute (4.63) and (4.62) in (4.28). Then the general solutions are

$$u(\tau) = \frac{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_1 - \tau_0)}{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0)} - \frac{1}{3}\sqrt{\frac{4p_0\pi G}{c^2}}\tan\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0) \quad (4.64)$$

$$v(\tau) = \frac{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_2 - \tau_0)}{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0)} - \frac{1}{3}\sqrt{\frac{4p_0\pi G}{c^2}}\tan\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0) \quad (4.65)$$

$$\theta(\tau) = \frac{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_3 - \tau_0)}{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0)} - \frac{1}{3}\sqrt{\frac{4p_0\pi G}{c^2}}\tan\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0) \quad (4.66)$$

The mean rate of change $R(\tau)$ is

$$\begin{aligned}R(\tau) &= \frac{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_1 - \tau_0) + \cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_2 - \tau_0) + \cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau_3 - \tau_0)}{\cos\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0)} \\ &- \sqrt{\frac{4p_0\pi G}{c^2}}\tan\sqrt{\frac{4p_0\pi G}{c^2}}\,(\tau - \tau_0) \quad (4.67)\end{aligned}$$

4.1.3. Absence of Pressure and Density

In the absence of both pressure and density the system (4.20) is reduced to the following equation

$$\begin{aligned} \dot{u} + uR &= 0 \\ \dot{\theta} + \theta R &= 0 \\ \dot{v} + vR &= 0 \end{aligned} \tag{4.68}$$

Let us add the equations together and the result is

$$\dot{R} + R^2 = 0 \tag{4.69}$$

where $R = u + v + \theta$. Similar to the previous case, we get the Riccati equation. Let us linearize (4.69) by using transformation $R(t) = \frac{\dot{\phi}(\tau)}{\phi(\tau)}$. Hence, it becomes

$$\frac{\ddot{\phi}}{\phi} = 0 \tag{4.70}$$

we can easily recognize that,

$$\dot{\phi} = C_1 \tag{4.71}$$

and we know that $R(t) = \frac{\dot{\phi}}{\phi}$. Then, the equation (4.71) becomes,

$$R(t) = \frac{\dot{\phi}(\tau)}{\phi(\tau)} = \frac{C_1}{C_1\tau + C_0} \tag{4.72}$$

we reach the last form the solution by dividing the numerator and the denominator by C_1 and let say that $C = \frac{C_0}{C_1}$

$$R(\tau) = u(\tau) + v(\tau) + \theta(\tau) = \frac{\dot{\phi}(\tau)}{\phi(\tau)} = \frac{1}{\tau + C} \tag{4.73}$$

This model has a singularity at $\tau = -C$. When cosmic time equivalent to C, the volume of the anisotropic universe goes to infinity. Thus, $\tau = -C$ can be represented as Big Bang.

CHAPTER 5

BAROTROPIC MODELS OF FRW UNIVERSE

In Chapter 4, it was shown that mean value of the rate change of cosmic scales satisfies the Riccati equation. In FRW model the rate of change $\frac{\dot{a}}{a}$ is the Hubble parameter. We are going to show in this chapter that the Hubble parameter satisfies the Riccati equation.

Barotropic cosmological zero modes are simple trigonometric and/or hyperbolic solutions of second order differential equations of the oscillator type equation and these oscillator type equations can be reduced to FRW system of equations when it is passed to the conformal time variable (Rosu and Ojeda-May 2006). In this chapter, we briefly review these differential equations in mathematical a scheme.

Barotropic FRW cosmologies in co-moving time t obey the Einstein-Friedmann dynamical equations for the scale factor $a(t)$ of the universe supplemented by the (barotropic) equation of state of the cosmological fluid

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{5.1}$$

$$H_0{}^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \tag{5.2}$$

$$p = (\gamma - 1)\rho \tag{5.3}$$

where ρ and p are the density and the pressure of the perfect fluid of which a classical universe is usually assumed to be made of $\kappa = 0, \pm 1$ are the curvature indices of flat, closed, open universes and γ is the constant adiabatic constant.

5.1. Bosonic FRW Model

Proposition 5.1 *Passing to the conformal time variable η, defined through $dt = a(\eta)d\eta$, we can combine the three equations (5.1-5.3) in a Riccati equation for the Hubble parameter $H_0(\eta)$ as follows*

$$H_0{}'(\eta) + cH_0^2(\eta) + c\kappa = 0 \tag{5.4}$$

where $H_0(\eta) = \frac{a'}{a} = \frac{d}{d\eta}\ln a$

Proof Let substitute (5.3) in (5.1) and then the equation (5.1) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[(3\gamma - 2)\rho\right] \tag{5.5}$$

and by using the equation (5.5), we get the density ρ as

$$\rho = -\frac{3}{4\pi G}\frac{\ddot{a}}{a}\frac{1}{(3\gamma - 2)} \tag{5.6}$$

insert (5.6) ρ into (5.2). Hence

$$\left(\frac{\dot{a}}{a}\right)^2 = \left[-2\frac{\ddot{a}}{a}\frac{1}{(3\gamma - 2)}\right] - \frac{\kappa}{a^2} \tag{5.7}$$

arrange equation (5.7) and then we obtain

$$\left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}}{a}\right)^2 = -\frac{\ddot{a}}{a} - \left(\frac{3}{2}\gamma - 1\right)\frac{\kappa}{a^2} \tag{5.8}$$

for simplification in notation, let us take that $c = (\frac{3}{2}\gamma - 1)$. Hence, equation (5.8) is reduced to the following form

$$-\frac{\ddot{a}}{a} = c\left(\frac{\dot{a}}{a}\right)^2 + c\frac{\kappa}{a^2} \tag{5.9}$$

and multiply equation (5.9) by a^2. Thus

$$-\ddot{a}a = c\dot{a}^2 + c\kappa \tag{5.10}$$

By using conformal time, we can construct equation (5.10) in terms of Hubble parameter:

$$dt = a(\eta)d\eta \Rightarrow \frac{dt}{d\eta} = a(\eta) \tag{5.11}$$

by using the chain rule, we get

$$\frac{da}{d\eta} = \frac{da}{dt}\frac{dt}{d\eta} \Rightarrow \acute{a} = \dot{a}a(\eta) \Rightarrow \frac{a'}{a} = \dot{a} \tag{5.12}$$

the second derivative in terms of η is given by

$$\frac{d^2a}{d\eta^2} = \frac{d^2a}{dt^2}\left(\frac{dt}{d\eta}\right)^2 + \frac{da}{dt}\frac{d^2t}{d\eta^2} \Rightarrow a'' = \ddot{a}a^2 + \dot{a}a' \tag{5.13}$$

In the light of equations (5.12) and (5.13), the second derivative of $a(t)$ in terms of time can be written as

$$\ddot{a} = \frac{a''}{a^2} - \frac{a''^2}{a^3} \tag{5.14}$$

Substitute equations (5.14) and (5.12) in (5.10). Thus, we obtain the following equation

$$c(\frac{a'}{a})^2 + c\kappa = -\left[\frac{a''}{a} - \frac{a'^2}{a^2}\right] \tag{5.15}$$

it is clear that $H_0(\eta) = \frac{a'}{a}$ and $H'{}_0(\eta) = \frac{a''}{a}\frac{a'^2}{a^2}$ consequently, (5.15) is

$$H_0{}'(\eta) + cH_0^2(\eta) + c\kappa = 0$$

Equation (5.4) for the Hubble parameter $H_0(\eta)$ is the Riccati equation (see Appendix B). To linearize this Riccati equation, the Cole-Hopft transformation should be given as $H_0(\eta) = \frac{1}{c}\frac{\omega'}{\omega}$. After substituting this transformation, the linearized equation is given

$$\frac{1}{c}\frac{\omega''}{\omega} + c\kappa = 0 \tag{5.16}$$

Let arrange (5.16) as follows

$$\omega'' - c.c_{\kappa,b}\omega = 0 \tag{5.17}$$

where $c_{\kappa,b} = -\kappa c$. Moreover, the particular Riccati solutions for the positive and negative curvature indices are discussed (Rosu and Ojeda-May, 2006) as follows.

a. For $\kappa = 1$

The positive curvature index means that the constant $c_{\kappa,b}$ becomes

$$c_{\kappa,b} = -1.c = -c \tag{5.18}$$

Hence, equation (5.17) becomes

$$\omega'' + c^2\omega = 0 \tag{5.19}$$

which is the standard oscillator equation and the solution of this second order differential equation is obtained as

$$\omega(\eta) = A\sin(c\eta) + B\cos(c\eta) \tag{5.20}$$

this solution corresponds to under damped harmonic oscillator. The solution of (5.20) can be reduced to the following equation

$$\omega_{1,b}(\eta) = C_+ \cos(c\eta + d) \tag{5.21}$$

where d is an arbitrary phase and C_+ is the integration constant. If we substitute (5.21) and its first derivatives in the Hubble parameter, the Hubble parameter becomes

$$H_0{}^+(\eta) = \frac{1}{c}\frac{\omega'}{\omega} = -\left(\frac{\sin(c\eta + d)}{\cos(c\eta + d)}\right) = -\tan(c\eta + d) \tag{5.22}$$

where the symbol $(+)$ means positive curvature. Equation (5.22) is the particular solution of Riccati equation for $\kappa = 1$. We conclude that the Hubble parameter is including a periodic motion as oscillations.

b. For $\kappa = -1$

For the negative curvature, the constant $c_{\kappa,b}$ is transformed into the following

$$c_{\kappa,b} = -(-1).c = c \tag{5.23}$$

Hence, (5.17) becomes

$$\omega'' - c^2\omega = 0 \tag{5.24}$$

The solution of this differential equation as

$$\omega(\eta) = D\sinh c\eta + E\cosh c\eta \tag{5.25}$$

which corresponds to over-damped harmonic oscillator. In addition to this, equation (5.25) can be written as

$$\omega_{-1,b}(\eta) = C_- \sinh c\eta \tag{5.26}$$

and the Hubble parameter of bosonic case for the open universe, in other word the negative curvature, is

$$H_0{}^-(\eta) = \frac{1}{c}\frac{\omega'}{\omega} = \frac{\cosh c\eta}{\sinh c\eta} = \coth c\eta \tag{5.27}$$

where the symbol $(-)$ means negative curvature. Equation (5.27) is the particular solution of the Riccati equation for $\kappa = -1$. As a result, we consider that the Hubble parameter is including the hyperbolic type motion as over damped oscillations.

Generally, the two Hubble parameter are obtained as ${H_0}^{-}(\eta) = \coth c\eta$ and ${H_0}^{+}(\eta) = -\tan(c\eta + d)$ which completely depending on the geometry of the universe. Moreover, ${H_0}^{-}(\eta) = \coth c\eta$ and ${H_0}^{+}(\eta) = -\tan(c\eta + d)$ are related to the common factorizations of equation (5.17) and then

$$\left(\frac{d}{d\eta} + cH_0\right)\left(\frac{d}{d\eta} - cH_0\right)\omega = 0 \tag{5.28}$$

and

$$\left(\frac{d^2\omega}{d\eta^2} - c\frac{dH_0}{d\eta}\omega - c^2H_0^2\omega\right) = 0 \tag{5.29}$$

by using (5.29), we get

$$\omega'' - c(H_0' + cH_0^2)\omega = 0 \tag{5.30}$$

Equations (5.30) and (5.28) are equivalent. Then we can combine them as follows,

$$\left(\frac{d}{d\eta} + cH_0\right)\left(\frac{d}{d\eta} - cH_0\right)\omega = \omega'' - c(H_0' + cH_0^2)\omega = 0 \tag{5.31}$$

Borrowing a terminology from super symmetric quantum mechanics, we call the solutions ω as bosonic zero modes in terms of scale factors. First, the Hubble parameter is defined as

$$H = \frac{1}{c}\frac{d\omega}{\omega} = \frac{da}{a} \tag{5.32}$$

the solution of this first order differential equation is

$$\omega^{\frac{1}{c}} = Ca \tag{5.33}$$

where C is integration constant. Generally, we can write

$$\omega^{\frac{1}{c}} \sim a \tag{5.34}$$

this result can be specified for both geometry for closed and open universes as follows

a. **For Spherical Geometry** $\kappa = 1$

$$\omega^{\frac{1}{c}}{}_{1,b} \sim a_{1,b} \tag{5.35}$$

and

$$\omega_{1,b}(\eta) \sim \cos(c\eta + d) \longrightarrow a_{1,b} \sim [\cos(c\eta + d)]^{\frac{1}{c}} \tag{5.36}$$

As a consequence, we can say that for bosonic zero modes, the universe with closed geometry has the scale factor which shows oscillator character as contracting and expanding.

b. For $\kappa = -1$

$$\omega_{-1,b}{}^{\frac{1}{c}} \sim a_{-1,b} \tag{5.37}$$

$$\omega_{-1,b}(\eta) \sim \sinh c\eta \longrightarrow a_{-1,b} \sim [\sinh c\eta]^{\frac{1}{c}} \tag{5.38}$$

Finally, we can conclude that the universe with open geometry has the scale factor showing over damped oscillator character as contracting and expanding universe for bosonic zero modes.

5.2. Fermionic FRW Barotropy

A class of barotropic FRW cosmologies with inverse scale factors with respect to the bosonic ones can be obtained by considering the supersymmetric partner (or fermionic) equation of equation (5.30) which is obtained by applying the factorization brackets in reverse order

$$\left(\frac{d}{d\eta} - cH_0\right)\left(\frac{d}{d\eta} + cH_0\right)\omega = 0 \tag{5.39}$$

and

$$\left(\frac{d^2\omega}{d\eta^2} + c\frac{dH_0}{d\eta}\omega - c^2H_0^2\omega\right) = 0 \tag{5.40}$$

we can write the equation (5.40) as follows

$$\omega'' - c(-H_0' + cH_0^2)\omega = 0 \tag{5.41}$$

Equation (5.41) can be rewritten as

$$\omega'' - c.c_{\kappa,f}\omega = 0 \tag{5.42}$$

where

$$c_{\kappa,f}(\eta) = -H_0' + cH_0^2 = \begin{cases} c(1 + 2\tan^2 c\eta), & \text{if } \kappa = 1 \\ c(-1 + 2\coth^2 c\eta), & \text{if } \kappa = -1 \end{cases} \tag{5.43}$$

denotes the supersymmetric partner adiabatic index of fermionic type associated trough the mathematical scheme to the constant bosonic index. Notice that the fermionic adiabatic index is time dependent. The fermionic ω solutions are

$$\omega_{1,f} = \frac{c}{\cos(c\eta + d)} \tag{5.44}$$

and

$$\omega_{-1,f} = \frac{c}{\sinh c\eta} \tag{5.45}$$

for $\kappa = 1$ and $\kappa = -1$, respectively. In addition to them relation between scale factor and ω is the similar as before

$$\omega^{\frac{1}{c}} \sim a \tag{5.46}$$

a. For $\kappa = 1$

$$\omega^{\frac{1}{c}}{}_{1,f} \sim a_{1,f} \tag{5.47}$$

$$\omega_{1,f}(\eta) \sim \frac{c}{\cos(c\eta + d)} \longrightarrow a_{1,f} \sim [\cos(c\eta + d)]^{-\frac{1}{c}} \tag{5.48}$$

b. For $\kappa = -1$

$$\omega_{-1,f}{}^{\frac{1}{c}} \sim a_{-1,f} \tag{5.49}$$

$$\omega_{-1,f}(\eta) \sim \sinh c\eta \longrightarrow a_{-1,f} \sim [\sinh c\eta]^{-\frac{1}{c}} \tag{5.50}$$

Summary, we see that the bosonic and fermionic cosmologies are reciprocal to each other, in the sense that

$$a_{1,b} = a_{+b} \sim [\cos(c\eta + d)]^{\frac{1}{c}} \tag{5.51}$$

and

$$a_{1,f} = a_{+f} \sim [\cos(c\eta + d)]^{-\frac{1}{c}} \tag{5.52}$$

Both equations have the positive curvature. From multiplication of two equations (5.51) and (5.52), the result is

$$a_{1,b}a_{1,f} = a_{+b}a_{+f} = [\cos(c\eta + d)]^{\frac{1}{c}}[\cos(c\eta + d)]^{-\frac{1}{c}} = 1 \tag{5.53}$$

For negative curvature, the relations between the scale factors of bosonic and fermionic cases are given in the following equations

$$a_{-1,b} = a_{-b} \sim [\sinh c\eta]^{\frac{1}{c}} \tag{5.54}$$

and

$$a_{-1,f} = a_{-f} \sim [\sinh c\eta]^{-\frac{1}{c}} \tag{5.55}$$

Again, the multiplication of these two quantity gives the same result

$$a_{-1,b}a_{-1,f} = a_{-b}a_{-f} = [\sinh c\eta]^{\frac{1}{c}}[\sinh c\eta]^{-\frac{1}{c}} = 1 \tag{5.56}$$

Then, we can write these results in general

$$a_{\pm b}a_{\pm f} = constant \tag{5.57}$$

Thus, bosonic expansion corresponds to fermionic contraction and viceversa

5.3. Decoupled Fermionic and Bosonic FRW Barotropies

The Dirac equation in the super symmetric non relativistic formalism has been discussed by Cooper (Cooper et.al 1988). They showed that the Dirac equation with a Lorentz scalar potential is associated with a SUSY pair of Schrödinger Hamiltonians. Rosu made an application to barotropic FRW cosmologies that he found not to be a trivial exercise except for the decoupled ”zero-mass” case (Rosu 2006).

A matrix formulation of the previous results is possible as follows. Introducing the following two Pauli matrices. The cosmological matrix equation

$$\sigma_y D_\eta W + \sigma_x(icH_0)W = 0 \tag{5.58}$$

where $W = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ is two component zero-mass spinor, D_η denotes the derivative in terms of the conformal time η. Then σ_x and σ_y are defined as the following matrices

$$\alpha = -i\sigma_y = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\beta = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Equation (5.58) is equivalent to the following decoupled equations

$$\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} D_\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(icH_0)\right]\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0 \tag{5.59}$$

then

$$\begin{pmatrix} 0 & i(cH_0 - D_\eta) \\ i(D_\eta + cH_0) & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0 \tag{5.60}$$

hence from matrix multiplication, we get the system of equations

$$i\left(cH_0 + D_\eta\right)\omega_1 = 0 \tag{5.61}$$

and

$$i(cH_0 - D_\eta)\omega_2 = 0 \tag{5.62}$$

Let us solve these equations one by one. First, equation (5.61) is reduced to

$$D_\eta \omega_1 = -cH_0\omega_1 \tag{5.63}$$

for $\kappa = 1$, the Hubble parameter is ${H_0}^+ = -\tan(c\eta + d)$. Substitute this in (5.63). Then

$$D_\eta \omega_1 = c\tan(c\eta + d)\omega_1 \tag{5.64}$$

solution of (5.64) is

$$\omega_1 = \frac{A}{\cos(c\eta + d)} \tag{5.65}$$

where A is integration constant. In general,

$$\omega_1 \sim \frac{1}{\cos(c\eta + d)} \tag{5.66}$$

For negative curvature $\kappa = -1$ and the Hubble parameter for negative curvature is defined as ${H_0}^- = \coth c\eta$. Substitute this in (5.63) and then we get

$$D_\eta \omega_1 = -c\coth c\eta\omega_1 \tag{5.67}$$

the solution of (5.67) is obtained as

$$\omega_1 = -\frac{B}{\sinh c\eta} \tag{5.68}$$

where B is integration constant and ω_1 is

$$\omega_1 \sim \frac{1}{\sinh c\eta} \tag{5.69}$$

We will use the same procedure in order to get solution of ω_2. Equation (5.62),

$$i(cH_0 - D_\eta)\omega_2 = 0 \tag{5.70}$$

and

$$D_\eta\omega_2 = cH_0\omega_2 \tag{5.71}$$

for $\kappa = 1$, the Hubble parameter is ${H_0}^+ = -\tan(c\eta + d)$. Substitute this in equation (5.71) and it becomes

$$D_\eta\omega_2 = -c\tan(c\eta + d)\omega_2 \tag{5.72}$$

then the solution is given as

$$\omega_2 = C\cos(c\eta + d) \tag{5.73}$$

where C is integration constant. Equation (5.73) can be written as

$$\omega_2 \sim \cos(c\eta + d) \tag{5.74}$$

For $\kappa = -1$, the Hubble parameter is ${H_0}^- = \coth c\eta$. Substitute this in equation (5.62) as

$$D_\eta\omega_2 = c\coth c\eta\omega_2 \tag{5.75}$$

The solution is

$$\omega_2 = D\sinh c\eta \tag{5.76}$$

where D is an integration constant and the ω_2 function is

$$\omega_2 \sim \sinh c\eta \tag{5.77}$$

In summary,

$$\kappa = 1 \begin{cases} \omega_1 \sim \frac{1}{\cos(c\eta+d)} \\ \omega_2 \sim \cos(c\eta + d) \end{cases} \tag{5.78}$$

and

$$\kappa = -1 \begin{cases} \omega_1 \sim \frac{1}{\sinh c\eta} \\ \omega_2 \sim \sinh c\eta \end{cases} \tag{5.79}$$

Thus, we obtain $W = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_f \\ \omega_b \end{pmatrix}$. This shows that the matrix equation contains two reciprocal barotropic cosmologies same as the two components of the spinor W.

5.4. Coupled Fermionic and Bosonic Cosmological Barotropies

Consider a ”massive” Dirac equation which is defined as

$$\sigma_y D_\eta W + \sigma_x (icH_0 + m)W = mW \tag{5.80}$$

where m is equivalent to the mass parameter of a Dirac spinor. Equation (5.80) can be written as

$$\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} D_\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (icH_0 + m)\right] \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = m \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

is equivalent to the following system of coupled equations as follows

$$(iD_\eta + icH_0 + m)\omega_1 = m\omega_2 \tag{5.81}$$

$$(-iD_\eta + icH_0 + m)\omega_2 = m\omega_1 \tag{5.82}$$

These two coupled first order differential equations are equivalent to second order differential equations for each of the two spinor components. First, we formulate the second order equation for bosonic case which is ω_2. To do this, multiply equation (5.82) by $\frac{1}{m}$ and then it becomes

$$\frac{1}{m}(-iD_\eta + icH_0 + m)\omega_2 = \omega_1 \tag{5.83}$$

Insert (5.83) into equation (5.81) instead of ω_1. Then

$$[iD_\eta + icH_0 + m][\frac{1}{m}(-iD_\eta + icH_0 + m)]\omega_2 = m\omega_2 \tag{5.84}$$

and

$$D^2{}_\eta \omega_2 - cH_0{}'\omega_2 - c^2 H_0{}^2 \omega_2 + 2icmH_0\omega_2 = 0 \tag{5.85}$$

after arranging (5.85), we obtain

$$D^2{}_\eta \omega_2 - c[H_0{}' + cH_0{}^2 - 2imH_0]\omega_2 = 0 \tag{5.86}$$

Hubble parameter of the bosonic case for closed universe is $H_0{}^+ = -\tan(c\eta + d)$ and $H_0{}^{+\prime} = -c.\sec^2(c\eta + d)$. Substitute them in (5.86). Hence

$$D^2{}_\eta \omega_2 - c[-c.\sec^2(c\eta + d) + c\tan^2(c\eta + d) + 2im\tan(c\eta + d)]\omega_2 = 0 \tag{5.87}$$

by using trigonometric identity $\sec^2\eta - \tan^2\eta = 1$, we find the solution as

$$D^2{}_\eta\omega_2 + c[c - 2im\tan(c\eta + d)]\omega_2 = 0 \tag{5.88}$$

we can write (5.88) in terms of different notation as

$$\omega^{+\prime\prime}{}_2 + c[c - 2im\tan(c\eta + d)]\omega^+{}_2 = 0 \tag{5.89}$$

To obtain the function ω_2, we are using Hubble parameter which belongs to the open universe for bosonic case. $H_0{}^- = \coth c\eta$ and its derivative is $H_0{}^{-\prime} = -c.csch^2 c\eta$. When we insert them into (5.86), we obtain

$$D^2{}_\eta\omega_2 - c[-c\{csch^2 c\eta - \coth^2 c\eta\} - 2im\coth c\eta]\omega_2 = 0 \tag{5.90}$$

using hyperbolic identity $\coth^2\eta - csch^2\eta = 1$, we get the solution as

$$D^2{}_\eta\omega_2 + c\left[-c + 2im\coth c\eta\right]\omega_2 = 0 \tag{5.91}$$

In different notation, equation (5.91) becomes

$$\omega^{-\prime\prime}{}_2 + c[-c + 2im\coth c\eta]\omega^-{}_2 = 0 \tag{5.92}$$

We will apply the similar procedure to find the fermionic spinor component. Equation (5.82) is divided by $\frac{1}{m}$ as follows

$$\frac{1}{m}[(iD_\eta + icH_0 + m)\omega_1] = \omega_2 \tag{5.93}$$

and insert this ω_2 function into equation (5.81) and then it becomes

$$(-iD_\eta + icH_0 + m)\frac{1}{m}[(iD_\eta + icH_0 + m)\omega_1] = K\omega_1 \tag{5.94}$$

hence

$$D^2{}_\eta\omega_1 + cH_0{}'\omega_1 - c^2H_0{}^2\omega_1 + 2icmH_0\omega_1 = 0 \tag{5.95}$$

In a more appropriate form, equation (5.95) becomes

$$D^2{}_\eta\omega_1 + c[H_0{}' - cH_0{}^2 + 2imH_0]\omega_1 = 0 \tag{5.96}$$

For the fermionic case, Hubble parameter of the closed universe ($\kappa = 1$) is similar to the bosonic case which is $H_0{}^+ = -\tan(c\eta + d)$ and its derivative $H_0{}^{+\prime} = -c.\sec^2(c\eta + d)$. Substitute them in equation (5.86). Thus,

$$D^2{}_\eta\omega_1 + c[-c.\sec^2(c\eta + d) - c\tan^2(c\eta + d) - 2im\tan(c\eta + d)]\omega_1 = 0 \tag{5.97}$$

from trigonometric substitution, we can write equation (5.97) by using the following trigonometric identity $\sec^2(c\eta+d)+\tan^2(c\eta+d)=1+2\tan^2(c\eta+d)$ as

$$D^2{}_\eta\omega_1+c[-c\{1+2\tan^2(c\eta+d)\}-2im\tan(c\eta+d)]\omega_1=0 \tag{5.98}$$

Moreover,

$$c_{\kappa,f}(\eta)=-H_0'+cH_0^2=\begin{cases} c(1+2\tan^2 c\eta) & \text{if } \kappa=1 \\ c(-1+2\coth^2 c\eta) & \text{if } \kappa=-1 \end{cases} \tag{5.99}$$

shows that $c(1+2\tan^2(c\eta+d))$ is equivalent to the fermionic index for the $\kappa=1$. Hence, we can write (5.98) as follows

$$D^2{}_\eta\omega_1-c[c_{1,f}(\eta)+2iK\tan(c\eta+d)]\omega_1=0 \tag{5.100}$$

in other word,

$$\omega^{+\prime\prime}{}_1-c[c_{1,f}(\eta)+2iK\tan(c\eta+d)]\omega^+{}_1=0 \tag{5.101}$$

for $\kappa=-1$, $H_0{}^-=\coth c\eta$ and its derivative is $H_0{}^{-\prime}=-c.csch^2 c\eta$. When we insert them into (5.96), it becomes

$$D^2{}_\eta\omega_1+c[-c\{csch^2 c\eta+\coth^2 c\eta\}+2im\coth c\eta]\omega_1=0 \tag{5.102}$$

using trigonometric substitution, we can write $csch^2 c\eta+\coth^2 c\eta=-1+2\coth^2 c\eta$ and combination with equation (5.102). The result is

$$D^2{}_\eta\omega_1+c[-c(-1+2\coth^2 c\eta)+2im\coth c\eta]\omega_1=0 \tag{5.103}$$

and

$$\omega^{-\prime\prime}{}_1-c[-c_{-1,f}(\eta)+2im\coth c\eta]\omega^-{}_1=0 \tag{5.104}$$

The solutions of the bosonic equations are expressed in terms of the Gauss hypergeometric functions ${}_2F_1$ of complex parameters that can be written in explicit form

$$\begin{aligned} z_2{}^{-k_2}\omega^+{}_2(\eta) &= Az_1{}^{k_1}{}_2F_1[k_1+k_2+1,k_1+k_2,1+2k_1;-\frac{z_1}{2}] \\ -Be^{-i(1+2k_1)\pi}&(\frac{4}{z_1})^{k_1}{}_2F_1[-k_1+k_2,-k_1+k_2+1,1-2k_1;-\frac{z_1}{2}] \end{aligned} \tag{5.105}$$

and

$$\begin{aligned} z_4{}^{-k_4}\omega^-{}_2(\eta) &= Cz_3{}^{k_3}{}_2F_1[k_3+k_4,k_3+k_4+1,1+2k_3;\frac{z_3}{2}] \\ &+D(\frac{4}{z_3})^{k_3}{}_2F_1[-k_3+k_4+1,-k_3+k_4,1-2k_3;\frac{z_3}{2}] \end{aligned} \tag{5.106}$$

where the variables z_i $(i = 1, ..., 4)$ are given in the following form

$$z_1 = i\tan(c\eta + d) - 1,\ \ z_2 = i\tan(c\eta + d) + 1 \tag{5.107}$$

$$z_3 = \coth c\eta + 1,\ \ z_4 = \coth c\eta - 1 \tag{5.108}$$

and the k parameters are

$$k_1 = \frac{1}{2}\big(1 - \frac{2m}{c}\big)^{\frac{1}{2}},\ \ k_2 = \frac{1}{2}\big(1 + \frac{2m}{c}\big)^{\frac{1}{2}} \tag{5.109}$$

$$k_3 = \frac{1}{2}\big(1 + i\frac{2m}{c}\big)^{\frac{1}{2}},\ \ k_4 = \frac{1}{2}\big(1 - i\frac{2m}{c}\big)^{\frac{1}{2}} \tag{5.110}$$

whereas A, B, C, D are constants (Rosu and Perez 2004). Based on these m zero-modes, we can introduce bosonic scale factors and Hubble parameters depending on the parameter m

$$a_{m,+} = (\omega^{+}{}_2)^{\frac{1}{c}},\ \ H_{m,+}(\eta) = \frac{1}{c}\frac{d}{d\eta}(\log\omega^{+}{}_2) \tag{5.111}$$

$$a_{m,-} = (\omega^{-}{}_2)^{\frac{1}{c}},\ \ H_{m,-}(\eta) = \frac{1}{c}\frac{d}{d\eta}(\log\omega^{-}{}_2) \tag{5.112}$$

and similarly for the fermionic components by changing $\omega^{\pm}{}_2$ to $\omega^{\pm}{}_1$ in equations (5.111) and (5.112).

CHAPTER 6

TIME DEPENDENT GRAVITATIONAL AND COSMOLOGICAL CONSTANTS

In relativistic and observational cosmology, the evolution of the universe is described by Einstein's field equations together with the equation of state (for perfect fluid)which we discussed in the second chapter. Einstein's theory of gravity contains gravitational and cosmological constant and the gravitational constant G plays the role of a coupling constant between geometry of space and matter content in Einstein's field equation (Singh 2006). In evolving universe, it appears natural to look at this constant as a function of time. Dirac (Dirac 1937a) and Dicke (Dicke 1961) have suggested a possible time varying gravitational constant. The Large Number Hypothesis (LNH) proposed by Dirac (Dirac 1937b, 1938) leads to a cosmology where G varies with the cosmic time.

In addition to this, many cosmologists believe that the value of Λ is a function of time (Canuto et al. 1977b). For example, Carvalho (Carvalho 1996) studied a spatially homogeneous and isotropic cosmological model of the universe in general relativity by using the equation of state $p = (\gamma - 1)\rho$, where the γ, varies with cosmic time. A unified description of early evolution of the universe has been presented by him in which an inflationary period is followed by a radiation-dominated period. His analysis allows one to consider both G and Λ. A spatially homogeneous and isotropic Friedman Robertson Walker line element is considered with variables G and Λ in general relativity by Singh (Singh 2006). His approach is similar to that of Carvalho (Carvalho 1996) but with time-dependent gravitational and cosmological constants. Singh (Singh 2006) applied the same gamma-law equation of state in which the parameter γ depends on scale factor $a(t)$.

6.1. Model and Field Equation

The Einstein field equations are considered in zero-curvature RobertsonûWalker cosmology with perfect fluid source and time-dependent gravitational and cosmological constants. Exact solutions of the field equations are obtained by using the gamma-law equation of state $p = (\gamma - 1)\rho$ in which γ varies continuously with cosmological time.

The functional form of $\gamma(a)$ is used to analyze a wide range of cosmological solutions at early universe for two phases in cosmic history

a. Inflationary phase

b. Radiation-dominated phase

The corresponding physical interpretations of the cosmological solutions will be discussed in this section. We consider a spatially homogeneous and isotropic Robertson-Walker line element

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{6.1}$$

where $a(t)$ is the scale factor and $\kappa = -1,\ 0$ or $+1$ is the curvature parameter for open, flat and closed universe. The universe is assumed to be filled with distribution of matter represented by energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu - pg_{\mu\nu} \tag{6.2}$$

where we choose $c = 1$ for simplicity, ρ is the energy density of the cosmic matter and p is its pressure, U_μ is the four velocity vector such that $U_\mu U^\mu = 1$. The field equations are those of Einstein but with time-dependent cosmological and gravitational constants and given by Weinberg (1971)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(t)T_{\mu\nu} + \Lambda(t)g_{\mu\nu} \tag{6.3}$$

where $R_{\mu\nu}$ is the Ricci tensor, $G(t)$ and $\Lambda(t)$ being the variable gravitational and cosmological constants. An additional equation relating the variations of G and Λ with time can be obtained by taking the covariant divergence of equation (6.3), and taking into account the Bianchi identity. This gives

$$[8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}]^{;\nu} = 0 \tag{6.4}$$

Equations (6.3) and (6.4) can be considered as the fundamental equations of gravity G and Λ coupling parameters. Using co-moving coordinates

$$U_{\mu\nu} = (1,0,0,0) \tag{6.5}$$

in equation (6.2) and with line element (6.1), Einstein's field equation (6.3) yields two independent equations which are called Friedman equations

$$\frac{2a\ddot{a} + \dot{a}^2 + \kappa}{a^2} - \Lambda(t) = -8\pi G(t)p \tag{6.6}$$

$$\frac{3(\dot{a}+\kappa)}{a^2}\Lambda(t) = 8\pi G(t)\rho \tag{6.7}$$

After arranging equations (6.6) and (6.7), the Friedman equations become

$$3\ddot{a} = -4\pi G(t)a - \left[3p + \rho - \frac{\Lambda(t)}{4\pi G(t)}\right] \tag{6.8}$$

$$3\dot{a}^2 = 8\pi G(t)a^2 \left[\rho + \frac{\Lambda(t)}{8\pi G(t)}\right] - 3\kappa \tag{6.9}$$

Proof Firstly, multiply Equation (6.6) by three and then we get

$$3\ddot{a} = -12\pi G(t)ap - \frac{3}{2}\left(\frac{\dot{a}^2 + \Lambda a^2 + \kappa}{a}\right) \tag{6.10}$$

and from equation (6.7) we obtain

$$\dot{a}^2 = \frac{8\pi G(t)a^2}{3}\left(\rho + \frac{\Lambda}{8\pi G(t)}\right) - \kappa \tag{6.11}$$

Substitute (6.11) in equation (6.8). Rearranging this equation

$$3\ddot{a} = -4\pi G(t)a\left[3p + \rho - \frac{\Lambda(t)}{4\pi G(t)}\right] \tag{6.12}$$

similarly, we rearrange equation (6.7) easily as follows,

$$3\dot{a}^2 = 8\pi G(t)a^2 \left[\rho + \frac{\Lambda(t)}{8\pi G(t)}\right] - 3\kappa$$

Proposition 6.1 *In uniform cosmology $G = G(t)$ and $\Lambda = \Lambda(t)$ so that conservation equation (6.4) should be in the form*

$$\dot{\Lambda} = -8\pi\dot{G}\rho \tag{6.13}$$

Proof

$$(8\pi G T_{\mu\nu}) + \Lambda g_{\mu\nu})^{;\nu} = 0 \tag{6.14}$$

and

$$8\pi\left[G(t)_{;\nu}T_{\mu\nu} + G(t)T_{\mu\nu_{;\nu}}\right] + \Lambda(t)_{;\nu}g_{\mu\nu} + \Lambda(t)g_{\mu\nu_{;\nu}} = 0 \tag{6.15}$$

We know that $T_{\mu\nu;\nu} = g_{\mu\nu;\nu} = 0$ due to energy conservation of energy momentum tensor (6.15) is reduced to the following form

$$8\pi G(t)_{;\nu} T_{\mu\nu} + \Lambda(t)_{;\nu} g_{\mu\nu} = 0 \tag{6.16}$$

For the time component, equation (6.16) becomes as

$$8\pi G(t)_{;0} T_{00} = \Lambda(t)_{;0} g_{00} \tag{6.17}$$

Hence, for the time component (00), energy momentum tensor becomes

$$T_{00} = -pg_{00} + (p+\rho)u_0 u_0 = \rho \tag{6.18}$$

where $g_{\mu\nu}$ is metric tensor of the Minkowski space in the matrix form

$$\mathbf{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.19}$$

Let us substitute equations (6.18) and (6.19) in equation (6.17) with respect to time. Then it becomes

$$\dot{\Lambda} = -8\pi \dot{G}\rho$$

Proposition 6.2 *Equations (6.6) and (6.7) can be rewritten in terms of the Hubble parameter H*

$$H = \frac{d}{dt}\ln a = \frac{\dot{a}}{a} \tag{6.20}$$

as follows

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(t)\left(3p+\rho\right) + \frac{1}{3}\Lambda(t) \tag{6.21}$$

and

$$H^2 = \frac{8\pi}{3}G(t)\rho + \frac{1}{3}\Lambda(t) - \frac{\kappa}{a^2} \tag{6.22}$$

Proof Take the derivative of equation (6.20) in terms of time. Hence

$$\dot{H} = \frac{d^2}{dt^2}\ln a = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{6.23}$$

and using equations (6.20) and (6.23), we obtain

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} \tag{6.24}$$

let us insert equation (6.24) into (6.8). Then, it becomes

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(t)\left(3p + \rho\right) + \frac{1}{3}\Lambda(t)$$

similarly, substitute H^2 instead of $(\frac{d}{dt}\ln a)^2 = \frac{\dot{a}^2}{a^2}$ in equation (6.9), we get

$$H^2 = \frac{8\pi}{3}G(t)\rho + \frac{1}{3}\Lambda(t) - \frac{\kappa}{a^2}$$

The system of equations (6.13), (6.21) and (6.22) may be solved by a physical assumption such as he form of the equation of state and additional explicit assumption on H, $G(t)$ and $\Lambda(t)$ in terms of t or H (which itself depends on cosmic time). The equation of state is defined as

$$p = (\gamma - 1)\,\rho \tag{6.25}$$

where γ is an adiabatic parameter varying continuously with cosmological time so that in the course of its evolution the universe goes through a transition from an inflationary phase to radiation-dominated phase. Carvalho (1996) assumed the functional form of γ depends on scale factor as

$$\gamma\,(a) = \frac{4}{3}\frac{A\left(\frac{a}{a_0}\right)^2 + \frac{b}{2}\left(\frac{a}{a_0}\right)^b}{A\left(\frac{a}{a_0}\right)^2 + \left(\frac{a}{a_0}\right)} \tag{6.26}$$

where A is a constant and b is free parameter related to the power of cosmic time and lies $0 \leq b < 1$. Here a_0 is reference value such that if $a \ll a_0$, inflationary phase of the evolution of the universe is obtained and for $a \gg a_0$, we have a radiation-dominated phase. Substituting equation (6.25) in (6.21), we obtain

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(t)\left[3\gamma - 2\right]\rho + \frac{1}{3}\Lambda(t) \tag{6.27}$$

Proposition 6.3 *We eliminate ρ between equations (6.21) and (6.27) for zero curvature $(\kappa = 0)$. Then, we get*

$$H'H + H^2 = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{G(t)\Lambda'}{G'a} + \frac{1}{3}\frac{\Lambda(t)}{a} \tag{6.28}$$

where a prime denotes differentiation with respect to the scale factor a.

Proof To obtain equation (6.28), first of all, in the light of equation (6.13) we find ρ as

$$\rho = -\frac{1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}} \tag{6.29}$$

then insert this ρ into equation (6.27). Hence,

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(t)\left[3\gamma - 2\right]\left(-\frac{\dot{\Lambda}}{8\pi\dot{G}}\right) + \frac{1}{3}\Lambda(t) \tag{6.30}$$

after arranging equation (6.30), we get

$$\dot{H} + H^2 = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{G(t)\dot{\Lambda}}{\dot{G}} + \frac{1}{3}\Lambda(t) \tag{6.31}$$

Solving equation (6.31) is easy. We need the derivatives with respect to the scale factor $a(t)$, instead of time derivation. To obtain this, we use the chain rule

$$\dot{H} = \frac{dH}{dt} = \frac{dH}{da}\frac{da}{dt} = H'\dot{a} \tag{6.32}$$

$$\dot{\Lambda} = \frac{d\Lambda}{dt} = \frac{d\Lambda}{da}\frac{da}{dt} = \Lambda'\dot{a} \tag{6.33}$$

$$\dot{G} = \frac{dG}{dt} = \frac{dG}{da}\frac{da}{dt} = G'\dot{a} \tag{6.34}$$

$$H = \frac{d}{dt}\ln a = \frac{\dot{a}}{a} \rightarrow \dot{a} = Ha \tag{6.35}$$

Substitute (6.32), (6.33), (6.34) and (6.35) in equation (6.31). As a result of this substitution, we get

$$H'Ha + H^2 = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{G(t)\Lambda'\dot{a}}{G'\dot{a}} + \frac{1}{3}\Lambda(t)$$

and then multiply the equation by $\frac{1}{a}$. Finally, we obtain the following equation

$$H'H + \frac{H^2}{a} = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{G(t)\Lambda'}{G'a} + \frac{1}{3}\frac{\Lambda(t)}{a}$$

6.2. Solution of The Field Equation

In this section, we discuss the solutions of the field equation for two different early phases which are inflationary and Radiation-dominated. Equation (6.28), involving H, $\Lambda(t)$, and $G(t)$ admits a solution for H only if $\Lambda(t)$ and $G(t)$ are specified. According to the LNH (Dirac 1937a, 1938), gravitational constant G varies linearly with the Hubble parameter. G decreases with the age of the universe. Chen and Wu built a dimensional argument to justify $\Lambda \sim a^{-2}$ (Chen and Wu 1990). Lima and Carvalho argued in favor of a new term of the type $\Lambda \sim H^2$ (Lima and Carvalho 1994). Thus, the phenomenological approach to investigate the cosmological constant is generalized to include a term proportional to H^2. Singh obtained the solution of equation (6.28) by taking certain assumption on G and Λ (Singh 2006).

6.2.1. $G(t) \sim H$

Proposition 6.4 *We assume that*

$$G(t) = \alpha H \tag{6.36}$$

and

$$\Lambda(t) = \beta H^2 \tag{6.37}$$

where α and β are dimensionless positive constants. Substituting the values $G(t)$ and $\Lambda(t)$ from equations (6.36) and (6.37) in (6.28), we obtain

$$H' + \left(\frac{\beta+3}{3} - \gamma\beta\right)\frac{H}{a} = 0 \tag{6.38}$$

Proof

$$H'H + \frac{H^2}{a} = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{G(t)\Lambda'}{G'a} + \frac{1}{3}\frac{\Lambda(t)}{a} \tag{6.39}$$

$$H'H + \frac{H^2}{a} = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{2\beta H^2}{a} + \frac{1}{3}\frac{\beta H^2}{a} \tag{6.40}$$

$$H' + \frac{H}{a} = \frac{1}{3}\left(\frac{3}{2}\gamma - 1\right)\frac{2\beta H}{a} + \frac{1}{3}\frac{\beta H}{a} \tag{6.41}$$

$$H' + \frac{H}{a} = \left(\gamma\beta - \frac{2}{3}\beta + \frac{1}{3}\beta\right)\frac{H}{a} \tag{6.42}$$

$$H' + \left(\frac{\beta+3}{3} - \gamma\beta\right)\frac{H}{a} = 0$$

Proposition 6.5 *The solution of the first order differential equation (6.38) is*

$$H = \frac{C}{a^{\frac{\beta+3}{3}}\left[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b\right]^{-\frac{2\beta}{3}}} \tag{6.43}$$

where C is the integration constant.

Proof At the first stage, to solve the equation, we use separation of variables

$$\int \frac{dH}{H} = -\int \left(\frac{\beta+3}{3} - \gamma\beta\right) \frac{da}{a} \tag{6.44}$$

by using (6.26) in equation (6.38), we formulate the following equations

$$\int \frac{dH}{H} = -\int \left[\frac{\beta+3}{3} - \beta(\frac{4}{3}\frac{A(\frac{a}{a_0})^2 + (\frac{b}{2})(\frac{a}{a_0})^b}{A(\frac{a}{a_0})^2 + (\frac{a}{a_0})})\right] \frac{da}{a} \tag{6.45}$$

$$\ln H = -\left(\frac{\beta+3}{3}\right) \ln a + \beta \int (\frac{4}{3}\frac{A(\frac{a}{a_0})^2 + (\frac{b}{2})(\frac{a}{a_0})^b}{A(\frac{a}{a_0})^2 + (\frac{a}{a_0})})] \frac{da}{a} \tag{6.46}$$

Let us define K as

$$K \equiv \beta \int \left[\frac{4}{3}\frac{A(\frac{a}{a_0})^2 + (\frac{b}{2})(\frac{a}{a_0})^b}{A(\frac{a}{a_0})^2 + (\frac{a}{a_0})}\right] \frac{da}{a} \tag{6.47}$$

and doing the substitution $\frac{a}{a_0} = z \Rightarrow da = a_0 dz$ in (6.52), we obtain

$$K = \beta\frac{4}{3} \int \left[\frac{Az^2 + (\frac{b}{2})z^b}{Az^2 + z^b}\right] \frac{dz}{z} \tag{6.48}$$

$$K = \beta\frac{4}{3} \int \frac{1}{z} dz + \beta\frac{4}{3}(\frac{b}{2} - 1) \int \frac{z^{b-3}}{[A + z^{b-2}]} dz \tag{6.49}$$

and after the second substitution $A + z^{b-2} \equiv u$, $z^{b-3}dz \equiv \frac{du}{b-2}$

$$K = \beta\frac{4}{3} \ln z + \beta\frac{4}{3}(\frac{b}{2} - 1)\frac{1}{b-2} \int \frac{du}{u}$$

$$K = \beta\frac{4}{3} \ln z + \beta\frac{2}{3} \ln u + \ln C$$

$$K = \beta\frac{4}{3} \ln z + \beta\frac{2}{3} \ln[A + z^{b-2}] + \ln C \tag{6.50}$$

Hence K follows as

$$K = \beta\frac{4}{3} \ln(\frac{a}{a_0}) + \beta\frac{2}{3} \ln[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b] + \ln C \tag{6.51}$$

Insert equation (6.51) into equation (6.46). Hence, equation (6.46) becomes

$$\ln H = -(\frac{\beta+3}{3}) \ln a + \beta\frac{2}{3} \ln[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b] + \ln C \tag{6.52}$$

and finally, we obtain

$$H = \frac{C}{a^{\frac{\beta+3}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}}}$$

If we choose $H = H_0$ and $a = a_0$ in equation (6.43), we have relation between constant A and C, is given by

$$H_0 = \frac{C}{{a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}}} \tag{6.53}$$

and the integration constant becomes

$$C = H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} \tag{6.54}$$

By using of (6.54) into (6.53), equation (6.43) is transformed into the following form

$$H = \frac{H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}}}{a^{\frac{\beta+3}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}}} \tag{6.55}$$

and

$$H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} = H a^{\frac{\beta+3}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}} \tag{6.56}$$

by using the property $H = \frac{d}{dt}\ln a = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$ in to (6.56), we get

$$H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} = \frac{1}{a}\frac{da}{dt} a^{\frac{\beta+3}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}} \tag{6.57}$$

so

$$H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} = \frac{da}{dt} a^{\frac{\beta}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}} \tag{6.58}$$

and integrating to obtain an expression for t in terms of the scale factor $a(t)$, is given by

$$\int H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} dt = \int a^{\frac{\beta}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}} da \tag{6.59}$$

and finally, we obtain

$$H_0 {a_0}^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}} t = \int a^{\frac{\beta}{3}}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{-\frac{2\beta}{3}} da \tag{6.60}$$

We can obtain not only the scale factor $a(t)$ but also Hubble, Gravitational, Cosmological constants and the energy density in terms of both inflationary and radiation-dominated regimes, by solving (6.60). We will discuss these regimes in the light of equation (6.60) as follows

6.2.1.1. Inflationary Phase

For inflationary phase $a \ll a_0$, the second term on the right hand side of integral in equation (6.60) dominates for $(b \neq 0)$ a and (6.60) is reduced to the following equation

$$H_0 {a_0}^{\frac{\beta+3}{3}} [A+1]^{-\frac{2\beta}{3}} t = \int a^{\frac{\beta}{3}} (\frac{a}{a_0})^{-\frac{2\beta}{3} b} da \tag{6.61}$$

Taking the integral, we obtain

$$H_0 {a_0}^{\frac{\beta+3}{3}} [A+1]^{-\frac{2\beta}{3}} t = \frac{a^{\frac{\beta}{3}(1-2b)+1} {a_0}^{\frac{2\beta}{3}} a}{\frac{\beta}{3}(1-2b)+1} \tag{6.62}$$

Then, scale factor a can be written as

$$a^{\frac{\beta}{3}(1-2b)+1} = H_0 {a_0}^{\frac{\beta+3}{3}} t [A+1]^{-\frac{2\beta}{3}} [\frac{\beta}{3}(1-2b)+1] \tag{6.63}$$

and

$$a = a_0 \left[\frac{(1-2b)\beta+3}{3} \frac{H_0}{(A+1)^{\frac{2\beta}{3}}} t \right]^{\frac{3}{(1-2b)\beta+3}} \tag{6.64}$$

Equation (6.64) shows that during inflation, the dimensions of the universe increase according to the formula

$$a \sim t^{\frac{3}{(1-2b)\beta+3}} \tag{6.65}$$

which is the case of power-law inflation (Singh 2006). We can easily see that the radius of the universe increases linearly with the age of universe and we can obtain the Hubble parameter,

$$H = \frac{d}{dt} \ln a = \frac{\dot{a}}{a} = \frac{3}{(1-2b)\beta+3} \frac{t^{\frac{3}{(1-2b)\beta+3}-1}}{t^{\frac{3}{(1-2b)\beta+3}}} = \frac{3}{(1-2b)\beta+3} t^{-1} \tag{6.66}$$

Gravitational constant

$$G(t) = \alpha H = \frac{3\alpha}{(1-2b)\beta+3} t^{-1} \tag{6.67}$$

The gravitational constant G varies inversely with the age of the universe.

Cosmological constant

$$\Lambda(t) = \beta H^2 = \frac{9\beta}{[(1-2b)\beta+3]^2} t^{-2} \tag{6.68}$$

and energy density

$$\rho(t) = \frac{-1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}} \tag{6.69}$$

$$\rho(t) = \frac{-3}{4\pi}\frac{\beta}{\alpha}\frac{1}{[(1-2b)\beta+3]}t^{-1} \tag{6.70}$$

For energy density to be positive definite, we must have $\beta < 3$. The energy density tends to infinity as t tends to zero. The energy density decreases as time increases and it tends to zero as t tends to infinity.

6.2.1.2. Radiation Dominated Phase

For the radiation-dominated phase $a \gg a_0$, the first term on right-hand side of the integral in equation (6.60) dominates so

$$H_0 a_0^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}}t = \int a^{\frac{\beta}{3}}\left[A(\frac{a}{a_0})\right]^{-\frac{4\beta}{3}} da \tag{6.71}$$

After taking the integral, equation (6.71) becomes

$$H_0 a_0^{\frac{\beta+3}{3}}[A+1]^{-\frac{2\beta}{3}}t = \frac{a^{-\beta+1}}{-\beta+1}A^{\frac{-2\beta}{3}}a_0^{\frac{4\beta}{3}} \tag{6.72}$$

and the scale factor is

$$a^{(1-\beta)} = H_0 a_0^{(1-\beta)}\left[\frac{A}{A+1}\right]^{\frac{2\beta}{3}}(1-\beta)t \tag{6.73}$$

and

$$a = a_0\left[(1-\beta)(\frac{A}{1+A})^{\frac{2\beta}{3}}H_0 t\right]^{\frac{1}{(1-\beta)}} \tag{6.74}$$

From equation (6.74), we find the following solution for the scale factor

$$a \sim t^{\frac{1}{(1-\beta)}} \tag{6.75}$$

Hubble parameter,

$$H = \frac{d}{dt}\ln a = \frac{\dot{a}}{a} = \frac{1}{(1-\beta)}\frac{t^{\frac{1}{(1-\beta)}-1}}{t^{\frac{1}{(1-\beta)}}} = \frac{1}{(1-\beta)}t^{-1} \tag{6.76}$$

gravitational constant,

$$G(t) = \alpha H = \frac{\alpha}{(1-\beta)}\frac{1}{t} \tag{6.77}$$

cosmological constant

$$\Lambda(t) = \beta H^2 = \frac{\beta}{(1-\beta)}\frac{1}{t^2} \tag{6.78}$$

and energy density

$$\rho(t) = \frac{-1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}} = \frac{-1}{8\pi}\frac{\frac{-2\beta}{(1-\beta)^2 t^3}}{\frac{(1-\beta)}{-\alpha t^2}} \tag{6.79}$$

$$\rho(t) = \frac{-1}{4\pi}\frac{\beta}{\alpha(1-\beta)}\frac{1}{t} \tag{6.80}$$

6.2.2. $G(t) \sim 1/H$

Proposition 6.6 *We assume that*

$$G = \frac{\alpha_1}{H} \tag{6.81}$$

where α_1 is a positive constant (Singh 2006) and

$$\Lambda(t) = \beta H^2 \tag{6.82}$$

Using equations (6.81) and (6.82) in (6.28), we obtain

$$H' + \frac{H}{a}\left[(1-\beta) + \gamma\beta\right] = 0 \tag{6.83}$$

Proof Firstly, by using the chain rule, we obtain first derivative of the cosmological and gravitational constants in terms of the scale factor as follows

$$\Lambda' = \frac{d\Lambda}{da} = \frac{d\Lambda}{dt}\frac{dt}{da} = \frac{d}{dt}(\beta H^2)\frac{1}{\dot{a}} = 2\beta H H^2 \frac{1}{\dot{a}} \tag{6.84}$$

and

$$G' = \frac{dG}{da} = \frac{dG}{dt}\frac{dt}{da} = \frac{d}{dt}(\frac{\alpha_1}{H})\frac{1}{\dot{a}} = -\alpha_1\frac{\dot{H}}{H^2}\frac{1}{\dot{a}} \tag{6.85}$$

Substituting (6.84) and (6.85) in equation (6.28) which was defined as

$$H'H + \frac{H^2}{a} = \frac{1}{3}(\frac{3}{2}\gamma - 1)\frac{G(t)\Lambda'}{G'a} + \frac{1}{3}\frac{\Lambda(t)}{a}$$

results in

$$H'H + \frac{H^2}{a} = \frac{1}{3}(\frac{3}{2}\gamma - 1)\frac{(\frac{\alpha_1}{H})}{a}\frac{(2\beta H H^2 \frac{1}{\dot{a}})}{(-\alpha_1\frac{\dot{H}}{H^2}\frac{1}{\dot{a}})} + \frac{1}{3}\frac{\beta H^2}{a} \tag{6.86}$$

so

$$H'H + \frac{H^2}{a} = \left[\frac{2}{3} + \frac{1}{3} - \gamma\right]\frac{1}{a}\beta H^2 \tag{6.87}$$

dividing equation (6.87) by H,

$$H' + \frac{H}{a} = \left[\frac{2}{3} + \frac{1}{3} - \gamma\right]\frac{1}{a}\beta H \tag{6.88}$$

arrange equation (6.88), we obtain

$$H' + \frac{H}{a}\left[(1-\beta) + \gamma\beta\right] = 0$$

Proposition 6.7 *Solution of differential equation (6.89) is*

$$H = \frac{C}{a^{(1-\beta)}\left[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b\right]^{\frac{2}{3}\beta}} \tag{6.89}$$

where C is the integration constant.

Proof Equation (6.89) can be solved by the separation of variables as follows

$$\frac{dH}{da} = -[(1-\beta) + \gamma\beta]\frac{H}{a} \tag{6.90}$$

so

$$\frac{dH}{H} = -[(1-\beta) + \gamma\beta]\frac{da}{a} \tag{6.91}$$

Substitute the adiabatic parameter in (6.91). Then the equation becomes

$$\frac{dH}{H} = -[(1-\beta) + \beta(\frac{4}{3}\frac{A(\frac{a}{a_0})^2 + (\frac{b}{2})(\frac{a}{a_0})^b}{A(\frac{a}{a_0})^2 + (\frac{a}{a_0})})]\frac{da}{a} \tag{6.92}$$

and by substituting $\frac{a}{a_0} \equiv z \Rightarrow da \equiv a_0 dz$ and integrate (6.92). Hence

$$\int \frac{dH}{H} = -\int (1-\beta)\frac{da}{a} - \int \frac{4}{3}\beta(\frac{Az^2 + (\frac{b}{2})z^b}{Az^2 + z^b})\frac{da}{a} \tag{6.93}$$

to get solution, we use the same procedure as equation (6.45). Then equation (6.93) can be written as

$$\ln H = -(1-\beta)\ln a - \frac{4}{3}\ln(\frac{a}{a_0}) - \frac{2}{3}\beta\ln(A + z^{b-2}) + \ln C \tag{6.94}$$

so (6.94) becomes

$$H = \frac{C}{a^{(1-\beta)}(\frac{a}{a_0})^{\frac{4}{3}\beta}(A + z^{b-2})^{\frac{2}{3}\beta}} \tag{6.95}$$

and

$$H = \frac{C}{a^{(1-\beta)}(\frac{a}{a_0})^{\frac{4}{3}\beta}(\frac{a_0}{a})^{-\frac{4}{3}\beta}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta}} \quad (6.96)$$

Finally, we obtain the solution of (6.89)

$$H = \frac{C}{a^{(1-\beta)}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta}}$$

To obtain the integration constant, let us assume $H = H_0$ and $a = a_0$ as the initial value problem and equation (6.89) is obtained as

$$H_0 = \frac{C}{a_0^{(1-\beta)}[A(\frac{a_0}{a_0})^2 + (\frac{a_0}{a_0})^b]^{\frac{2}{3}\beta}} \quad (6.97)$$

Then, we can obtain the integration constant

$$C = H_0 a_0^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} \quad (6.98)$$

Substituting equation (6.98) in (6.89), equation (6.89) becomes

$$H = \frac{H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta}}{a^{(1-\beta)}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta}} \quad (6.99)$$

and

$$H = \frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} = \frac{H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta}}{a^{(1-\beta)}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta}} \quad (6.100)$$

then

$$H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} dt = a^{(1-\beta)}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta}\frac{da}{a} \quad (6.101)$$

Integrating equation (6.101), we obtain

$$\int H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} dt = \int a^{-\beta}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta} da \quad (6.102)$$

and

$$H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} t = \int \frac{1}{a^\beta}[A(\frac{a}{a_0})^2 + (\frac{a}{a_0})^b]^{\frac{2}{3}\beta} da \quad (6.103)$$

Using equation (6.103), we obtain the solutions for two different early phases of the universe

6.2.2.1. Inflationary Phase

For inflationary phase $a \ll a_0$ and equation (6.103) is reduced to the following equation

$$H_0 {a_0}^{(1-\beta)} [A+1]^{\frac{2}{3}\beta} t = \int \frac{1}{a^\beta} (\frac{a}{a_0})^{\frac{2}{3}b\beta} da \qquad (6.104)$$

and after integrating (6.104), we obtain

$$H_0 {a_0}^{(1-\beta)} [A+1]^{\frac{2}{3}\beta} t = \frac{a^{\beta(\frac{2}{3}b-1)+1}}{\beta(\frac{2}{3}b-1)+1} \qquad (6.105)$$

after rearranging (6.105), it becomes

$$H_0 {a_0}^{(1-\beta)} [A+1]^{\frac{2}{3}\beta} [\beta(\frac{2}{3}b-1)+1] t = a^{\beta(\frac{2}{3}b-1)+1} \qquad (6.106)$$

and the scale factor is

$$a = \left[H_0 {a_0}^{(1-\beta)} \left[A+1\right]^{\frac{2}{3}\beta} \left(\beta(\frac{2}{3}b-1)+1 \right) t \right]^{\frac{3}{2b\beta+3(1-\beta)}} \qquad (6.107)$$

For inflationary phase we find the following solutions respectively,

$$a \sim t^{\frac{3}{2b\beta+3(1-\beta)}} \qquad (6.108)$$

Hubble parameter

$$H = \frac{d}{dt} \ln a = \frac{\dot{a}}{a} = \frac{3}{2b\beta + 3\left(1-\beta\right)} \frac{1}{t} \qquad (6.109)$$

Gravitational constant

$$G(t) = \frac{\alpha_1}{H} = \frac{\alpha_1 \left[2b\beta + 3\left(1-\beta\right)\right]}{3} t \qquad (6.110)$$

Cosmological constant

$$\Lambda(t) = \beta H^2 = \frac{9\beta}{\left[2b\beta + 3\left(1-\beta\right)\right]^2} \frac{1}{t^2} \qquad (6.111)$$

Energy density

$$\rho(t) = -\frac{1}{8\pi} \frac{\dot{\Lambda}}{\dot{G}} = -\frac{1}{8\pi} \frac{\left(\frac{-18\beta}{\left[2b\beta+3(1-\beta)\right]^2} \right) \frac{1}{t^3}}{\left(\frac{3\alpha_1}{2b\beta+3(1-\beta)} \right)} \qquad (6.112)$$

clearly

$$\rho(t) = -\frac{3}{4\pi} \frac{\beta}{\alpha_1} \frac{1}{t^3} \qquad (6.113)$$

6.2.2.2. Radiation Dominated Phase

For radiation-dominated phase $a \gg a_*$, the first term on right-hand side of the integral in equation (6.103) dominates as

$$H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} t = \int \frac{1}{a^\beta} A^{\frac{2}{3}\beta} (\frac{a}{a_0})^{\frac{4}{3}\beta} da \tag{6.114}$$

and then

$$H_0 {a_0}^{(1-\beta)}[A+1]^{\frac{2}{3}\beta} t = A^{\frac{2}{3}\beta} {a_0}^{-\frac{4}{3}\beta} \int \frac{1}{a^\beta} a^{\frac{4}{3}\beta} da \tag{6.115}$$

after rearranging (6.115), we obtain

$$H_0 {a_0}^{(1-\beta)} {a_0}^{\frac{4}{3}\beta} A^{-\frac{2}{3}\beta} [A+1]^{\frac{2}{3}\beta} t = \int a^{\frac{\beta}{3}} da \tag{6.116}$$

and take integral

$$H_0 {a_0}^{\frac{3}{\beta+3}} \left[\frac{A+1}{A}\right]^{\frac{2}{3}\beta} t = \frac{a^{\frac{\beta+3}{3}}}{\left(\frac{\beta+3}{3}\right)} \tag{6.117}$$

by constructing equation (6.117), we get the scale factor

$$\left[H_0 {a_0}^{\frac{3}{\beta+3}} \left[\frac{A+1}{A}\right]^{\frac{2}{3}\beta} \left(\frac{\beta+3}{3}\right)\right] t = a^{\frac{\beta+3}{3}} \tag{6.118}$$

and

$$a = \left[H_0 {a_0}^{\frac{3}{\beta+3}} [\frac{A+1}{A}]^{\frac{2}{3}\beta} \left(\frac{\beta+3}{3}\right) t\right]^{\frac{3}{\beta+3}} \tag{6.119}$$

for radiation-dominated phase we obtain the following solutions, For inflationary phase we find the following solutions respectively,

$$a \sim t^{\frac{3}{\beta+3}} \tag{6.120}$$

Hubble parameter

$$H = \frac{d}{dt} \ln a = \frac{\dot{a}}{a} = \left(\frac{3}{\beta+3}\right) \frac{t^{\frac{3}{\beta+3}-1}}{t^{\frac{3}{\beta+3}}} = \left(\frac{3}{\beta+3}\right) \frac{1}{t} \tag{6.121}$$

Gravitational constant

$$G(t) = \frac{\alpha_1}{H} = \frac{\alpha_1}{\frac{\beta+3}{3}} t \tag{6.122}$$

Cosmological constant

$$\Lambda(t) = \beta H^2 = \frac{9\beta}{(\beta+3)^2} \frac{1}{t^2} \tag{6.123}$$

Energy density

$$\rho(t) = -\frac{1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}} = -\frac{1}{8\pi}\left[-\frac{54\beta}{(\beta+3)}\frac{1}{t^3}\right] \tag{6.124}$$

and finally

$$\rho(t) = \frac{3}{4\pi}\frac{\beta}{\alpha_1}\frac{1}{t^3} \tag{6.125}$$

From the above solutions we observe that gravitational constant increases with the cosmic time in both phases whereas cosmological constant varies inversely as the square of the cosmic time. The energy density varies with the cube of the cosmic time inversely and hence tends ρ to infinity as t tends to zero. The expanding model has singularity at $t = 0$.

CHAPTER 7

METRIC WAVES IN NON-STATIONARY UNIVERSE AND DISSIPATIVE OSCILLATOR

The Einstein equations describe the General Relativity Theory that consist of a set of nonlinear differential equations. The application of these equations to cosmology produces metrics depending on time with exponential growth and decay which are characteristics of dissipative phenomena.

7.1. Linear Metric Waves in Flat Space Time

Solving Einstein equations is very difficult because of the nonlinearity. The linearization of the equations is a useful method to get an appropriate solution. The linearization method is known as the weak field approximation in general relativity and this method is used to describe the gravitational field very far from the source of gravity. In this approximation, we have a weak gravitational field, in the sense that for its space time, we can find quasi-Minkowskian coordinates $x^\mu = \{x, y, z, t\}$ (we shall work in units making $c = 1$) such that the metric differs from its Minkowskian form

$$\eta_{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu} \tag{7.1}$$

only by a small quantity $h_{\mu\nu}$;

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{7.2}$$

These must clearly be symmetric: $h_{\mu\nu} = h_{\nu\mu}$. We assume not only the smallness of the *h* themselves, but also that all of their derivatives, and neglect products of any of these. This corresponds to assuming that the *hs* are small multiples of regular functions $H_{\mu\nu}$,

$$h_{\mu\nu} = \epsilon H_{\mu\nu} \tag{7.3}$$

where ϵ is taken to be a non-zero real constant and we can get rid of ϵ^2. Beside, we have the inverse of the metric

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu} \tag{7.4}$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ (which is also order ϵ); for only this guarantees $g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}{}_{\sigma}$. It follows that indices on quantities of order ϵ are shifted by using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, since, for example, $g^{\mu\alpha}h_{\alpha\nu} = \eta^{\mu\nu}h_{\alpha\nu}$ to first order in ϵ. In summary, that is to say that the metric tensor is written in terms of the sum of the Minkowski metric and a correction term whose components are much less than unity. After obtaining the Christoffel symbols, Riemann and Ricci tensors and finally Ricci scalar in terms of the weak field approximation, we substitute them into the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

Then the field equation transform into the following equation

$$h^{\alpha}{}_{\nu,\alpha}{}^{\mu} + h^{\mu}{}_{\alpha,\nu}{}^{\alpha} - h^{\mu}{}_{\nu,\alpha}{}^{\alpha} - h_{,\nu}{}^{\mu} - \eta_{\mu\nu}\left(h_{\mu\beta,}{}^{\mu\beta} - h_{,\beta}{}^{\beta}\right) = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{7.5}$$

Proof The Christoffel symbol is defined as the following equation

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}g^{\mu\lambda}\left(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}\right) \tag{7.6}$$

Let us substitute (7.2) and (7.4) to the equation (7.6). Hence, it becomes,

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}(\eta^{\mu\lambda} + \epsilon h^{\mu\lambda})\{(\eta_{\alpha\lambda,\beta} + \epsilon h_{\alpha\lambda,\beta}) + (\eta_{\beta\lambda,\alpha} + \epsilon h_{\beta\lambda,\alpha}) - (\eta_{\alpha\beta,\lambda} + \epsilon h_{\alpha\beta,\lambda})\}$$

and

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}\eta^{\mu\lambda}\left(h_{\alpha\lambda,\beta} + h_{\beta\lambda,\alpha} - h_{\alpha\beta,\lambda}\right)$$

we obtain the Christoffel symbol

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}\left(h^{\mu}{}_{\alpha,\beta} + h^{\mu}{}_{\beta,\alpha} - h_{\alpha\beta,}{}^{\mu}\right) \tag{7.7}$$

Subsequently, the Riemann tensor is defined

$$R^{\alpha}{}_{\beta\mu\nu} \equiv -\Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\sigma}{}_{\beta\nu}\Gamma^{\alpha}{}_{\sigma\mu} - \Gamma^{\sigma}{}_{\beta\mu}\Gamma^{\alpha}{}_{\sigma\nu} \tag{7.8}$$

Since the Christoffel symbols are the first order quantities, the only contribution to the Riemann tensor will come from the derivatives of Γ, not the Γ^2 terms. Then the Riemann tensor becomes,

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} \tag{7.9}$$

and lowering α indices,

$$R_{\alpha\beta\mu\nu} = \Gamma_{\alpha\beta\nu,\mu} - \Gamma_{\alpha\beta\mu,\nu} \tag{7.10}$$

and by using the equation (7.7), we obtain the $\Gamma_{\alpha\beta\nu,\mu}$ and $\Gamma_{\alpha\beta\mu,\nu}$ which are necessary to get solution of (7.10). Hence

$$\Gamma_{\alpha\beta\nu,\mu} = \frac{1}{2}\big(h_{\beta\alpha,\nu\mu} + h_{\nu\alpha,\beta\mu} - h_{\beta\nu,\alpha\mu}\big) \tag{7.11}$$

$$\Gamma_{\alpha\beta\mu,\nu} = \frac{1}{2}\left(h_{\beta\alpha,\mu\nu} + h_{\mu\alpha,\beta\nu} - h_{\beta\mu,\alpha\nu}\right) \tag{7.12}$$

Substitute the Christoffel symbols (7.12) and (7.11) in $R_{\alpha\beta\mu\nu}$. Then, it becomes

$$2R_{\alpha\beta\mu\nu} = h_{\nu\alpha,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu} - h_{\mu\alpha,\beta\nu} \tag{7.13}$$

After raising the α in (7.13), (7.13) becomes

$$2R^{\alpha}{}_{\beta\mu\nu} = \eta^{\alpha\lambda}R_{\lambda\beta\mu\nu} \tag{7.14}$$

and we get

$$2R^{\alpha}{}_{\beta\mu\nu} = h^{\alpha}{}_{\nu,\beta\mu} + h_{\beta\mu,\nu}{}^{\alpha} - h_{\beta\nu,\mu}{}^{\alpha} - h^{\alpha}{}_{\mu,\beta\nu} \tag{7.15}$$

and contract the Riemann tensor (7.15) to get Ricci tensor

$$2R_{\beta\nu} = 2R^{\alpha}{}_{\beta\alpha\nu} = h^{\alpha}{}_{\nu,\beta\alpha} + h_{\beta\alpha,\nu}{}^{\alpha} - h_{\beta\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\beta\nu} \tag{7.16}$$

and

$$2R_{\mu\nu} = h^{\alpha}{}_{\nu,\mu\alpha} + h_{\mu\alpha,\nu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu} \tag{7.17}$$

using the definition $h \equiv h^{\alpha}{}_{\alpha}$, we can re-write the Ricci tensor (7.18) is

$$2R_{\mu\nu} = h^{\alpha}{}_{\nu,\mu\alpha} + h_{\mu\alpha,\nu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} \tag{7.18}$$

Raise the μ index to obtain the Ricci scalar in equation (7.18)

$$2R^{\mu}{}_{\nu} = \eta^{\mu\lambda} R_{\lambda\nu} = \eta^{\mu\lambda}\big(h^{\alpha}{}_{\nu,\lambda\alpha} + h_{\lambda\alpha,\nu}{}^{\alpha} - h_{\lambda\nu,\alpha}{}^{\alpha} - h_{,\lambda\nu}\big) \tag{7.19}$$

Then, it becomes,

$$2R^{\mu}{}_{\nu} = h^{\alpha}{}_{\nu,\alpha}{}^{\mu} + h^{\mu}{}_{\alpha,\nu}{}^{\alpha} - h^{\mu}{}_{\nu,\alpha}{}^{\alpha} - h_{,\nu}{}^{\mu} \tag{7.20}$$

If we assume that $\mu = \nu$. Hence, equation (7.20) transforms into the Ricci scalar as follows,

$$2R = 2R^{\mu}{}_{\mu} = 2h_{\mu\alpha,}{}^{\mu\alpha} - 2h_{,\alpha}{}^{\alpha} \tag{7.21}$$

and

$$R = h_{\mu\alpha,}{}^{\mu\alpha} - h_{,\alpha}{}^{\alpha} \tag{7.22}$$

Let us assume $\alpha = \beta$ in equation (7.22). Then the Ricci scalar becomes

$$R = h_{\mu\beta,}{}^{\mu\beta} - h_{,\beta}{}^{\beta} \tag{7.23}$$

The Einstein's field equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

or

$$2R_{\mu\nu} - g_{\mu\nu} R = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{7.24}$$

Substitute Ricci tensor (7.18), metric tensor (7.2) and Ricci scalar (7.23), which we have found for weak field, in the Einstein's field equation.

$$h^{\alpha}{}_{\nu,\alpha}{}^{\mu} + h^{\mu}{}_{\alpha,\nu}{}^{\alpha} - h^{\mu}{}_{\nu,\alpha}{}^{\alpha} - h_{,\nu}{}^{\mu} - \eta_{\mu\nu}\big(h_{\mu\beta,}{}^{\mu\beta} - h_{,\beta}{}^{\beta}\big) = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Finally, we obtain the Einstein equation in terms of the weak field approximation.

Equation (7.5) is invariant under the gauge-transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \psi_{\mu,\nu} + \psi_{\nu,\mu} \tag{7.25}$$

Impose the gauge condition

$$\partial^{\mu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) = 0 \tag{7.26}$$

which is called the Hilbert condition. Field equation (7.5) then reduced to

$$\partial^\alpha \partial_\alpha \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \right) = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{7.27}$$

This expression may be written as

$$c_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{7.28}$$

Equation (7.28) is the trace-reversed version of the hs. Writing $c := c_\lambda{}^\lambda$. We can show that $c = -h$ and consequently the usual inverse relation

$$c_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \Rightarrow h_{\mu\nu} = c_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c \tag{7.29}$$

Ricci tensor (7.18) can rewritten in terms of c by using relation (7.29) and the identity $c = -h$. Then the Ricci tensor becomes

$$2R_{\mu\nu} = c_{\mu\alpha\,\nu}{}^{\alpha} + c^{\alpha}_{\nu\alpha\,\mu} - \Box h_{\mu\nu} \tag{7.30}$$

where $\Box = (\partial_t{}^2 - \nabla^2)$. Let us prove this.

Proof Let us calculate the all terms of the equation (7.18). Then

- $h^{\alpha}_{\mu\alpha,\nu} = c_{\mu\alpha\,\nu}{}^{\alpha} - \frac{1}{2}\eta_{\mu\alpha} c_{,\nu}{}^{\alpha} = c_{\mu\alpha\,\nu}{}^{\alpha} - \frac{1}{2} c_{,\nu\mu}$
- $h^{\alpha}_{\nu\alpha,\mu} = c^{\alpha}_{\nu\alpha\,\mu} - \frac{1}{2}\eta_{\nu\alpha} c_{,\mu}{}^{\alpha} = c^{\alpha}_{\nu\alpha\,\mu} - \frac{1}{2} c_{,\mu\nu}$
- $h^{\alpha}_{\mu\nu,\alpha} = c^{\alpha}_{\mu\nu,\alpha} - \frac{1}{2}\eta_{\mu\nu} c^{\alpha}_{,\alpha}$
- $h_{,\mu\nu} = -c_{,\mu\nu}$.

After substituting every components in (7.18). It can be condensed to

$$2R_{\mu\nu} = c_{\mu\alpha\,\nu}{}^{\alpha} + c^{\alpha}_{\nu\alpha\,\mu} - \Box h_{\mu\nu}$$

A suitable coordinate transformation will even rid us of the last two term of equation (7.30). To this end, consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + f^{\mu}(x) \tag{7.31}$$

where the functions f^μ of position are of order ϵ, $f^\mu = \epsilon F^\mu$ just like the hs and cs. In addition to this, this equation is not an infinitesimal Lorentz formation (unless $f_{\mu,\nu}+f_{\nu,\mu} = 0$). The $g_{\mu\nu}$ transforms as a tensor, so

$$g_{\mu\nu} = g^{'}_{\mu\nu} \frac{\partial x'^{\alpha}}{\partial x^\mu} \frac{\partial x'^{\beta}}{\partial x^\nu} \tag{7.32}$$

and

$$x'^{\alpha} = x^\alpha + f^\alpha \longrightarrow \frac{\partial x'^{\alpha}}{\partial x^\mu} = \frac{\partial x^\alpha}{\partial x^\mu} + f^\alpha{}_{,\mu} = \delta^\alpha{}_{,\mu} + f^\alpha{}_{,\mu} \tag{7.33}$$

similarly,

$$x'^{\beta} = x^\beta + f^\beta \longrightarrow \frac{\partial x'^{\beta}}{\partial x^\nu} = \frac{\partial x^\beta}{\partial x^\nu} + f^\beta{}_{,\nu} = \delta^\beta{}_{,\nu} + f^\beta{}_{,\nu}. \tag{7.34}$$

Substitute partial derivatives of coordinates (7.34) and (7.33) in (7.32) as,

$$g_{\mu\nu} = g^{'}_{\alpha\beta} \left(\delta^\alpha{}_{,\mu} + f^\alpha{}_{,\mu}\right)\left(\delta^\beta{}_{,\nu} + f^\beta{}_{,\nu}\right) \tag{7.35}$$

Then, equation (7.35) is reduced to,

$$h^{'}_{\mu\nu} = h_{\mu\nu} - f_{\mu,\nu} - f_{\nu,\mu} \tag{7.36}$$

this form is in analogy to the gauge transformation $\Phi^{'}_{\mu} = \Phi_\mu - \Psi_{,\mu}$. From the $c = -h$ property, we get the relation $c^{'} - c = h - h^{'}$. To use this relation, let us multiply $h_{\mu\nu} - h^{'}_{\mu\nu} = f_{\mu,\nu} + f_{\nu,\mu}$ by $\eta^{\mu\nu}$. After these calculations, we obtain the following equation

$$h - h^{'} = 2 f_{\lambda,}{}^{\lambda} = c - c^{'}. \tag{7.37}$$

By using this result and $h_{\mu\nu} = c_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c$, find $h^{'}_{\mu\nu} = h_{\mu\nu} - f_{\mu,\nu} - f_{\nu,\mu}$ in terms of c. To construct c forms, firstly, use $h_{\mu\nu} = c_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c$ equation in appropriate notation as,

$$c^{'}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c^{'} = c_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c - f_{\mu,\nu} - f_{\nu,\mu} \tag{7.38}$$

and we obtain the following equation,

$$c^{'}_{\mu\nu} = c_{\mu\nu} + \eta_{\mu\nu} f_{\lambda}{}^{\lambda} - f_{\mu,\nu} - f_{\nu,\mu} \tag{7.39}$$

let us multiply equation (7.39) by $\eta^{\mu\nu}$. Then

$$c^{'} = c - f^\mu{}_{,\mu} \tag{7.40}$$

again multiply (7.40) with $\eta^{\mu\nu}$,

$$c^{'\mu\nu} = c^{\mu\nu} - f^{\mu\nu}_{,} \tag{7.41}$$

and take the derivative in terms of ν,

$$c^{'\mu\nu}_{,\nu} = c^{\mu\nu}_{,\nu} - f^{\mu\nu}_{\;\;\;\;,\nu} \tag{7.42}$$

and it is obvious that $f^{\mu\nu}_{\;\;\;\;,\nu} = \Box f^{\mu} = c^{\mu\nu}_{,\nu}$ under the condition $c^{'\mu\nu}_{,\nu} = 0$. Then, we know that $c^{'\mu\nu} = h^{'\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{'}$ and this multiply by $\eta^{\mu\nu}$ twice and take the derivative in terms of ν and then

$$h^{'\mu\nu}_{,\nu} - \frac{1}{2}\eta^{\mu\nu}h^{'}_{\;,\nu} = 0, \tag{7.43}$$

then we get harmonic gauge,

$$h^{'\mu\nu}_{,\nu} - \frac{1}{2}h^{'\;\mu}_{\;,} = 0. \tag{7.44}$$

The harmonic gauge is analogous to the Lorentz gauge of electromagnetism characterized by $\Phi_{,\mu}^{\;\;\;\mu} = 0$ and condensed Ricci tensor is $2R_{\mu\nu} = c^{\lambda}_{\;\;\mu,\nu\lambda} + c^{\lambda}_{\;\;\nu,\mu\lambda} - \Box h_{\mu\nu}$. Then $c^{'\mu\nu}_{,\nu} = 0$ in the gauge condition and then then Ricci tensor (7.30) becomes,

$$2R_{\mu\nu} = -\Box h_{\mu\nu}, \tag{7.45}$$

so that Einstein's vacuum field equation $R_{\mu\nu} = 0$ reduces to

$$\Box h_{\mu\nu} = 0 \;\; (c^{\mu\nu}_{,\nu} = 0) \tag{7.46}$$

and

$$\Box h = \Box \eta^{\mu\nu} h_{\mu\nu} = \eta^{\mu\nu}\Box h_{\mu\nu} = 0 \tag{7.47}$$

similarly,

$$c = -h, \;\; \Box h_{\mu\nu} = -\Box c_{\mu\nu} = 0 \rightarrow \Box c_{\mu\nu} = 0 \tag{7.48}$$

finally, $\Box h_{\mu\nu} = 0$ suggests the existence of gravitational waves. In this addition, $R_{\lambda\mu\nu\rho}$ Riemann curvature tensor becomes we will get gravitational waves

$$\Box h_{\mu\nu} = 0 \quad or \quad \Box h^{\mu\nu} = 0 \tag{7.49}$$

The field $h^{\mu\nu}$ is then decomposed in partial waves

$$h^{\mu\nu} = \sum_{\lambda} \frac{1}{2\pi^{\frac{3}{2}}} \int d^3k e^{\mu\nu}{}_{k\lambda} \{u_{k\lambda}(t) e^{i\mathbf{k}x} + u^{\dagger}{}_{k\lambda}(t) e^{-i\mathbf{k}x}\} \tag{7.50}$$

with $k \equiv \{k_0 = \omega = ck, \mathbf{k}\}$. Wave function $u_{k\lambda}(t)$ satisfies the simple harmonic oscillator equation (Vitiello et al. 1997) as follows

$$\frac{d^2}{dt^2} u_{k\lambda}(t) + \omega^2 u_{k\lambda}(t) = 0 \tag{7.51}$$

where ω is called frequency. In Minkowskian space, the frequency is time independent. In addition to this, we can obtain the line element of the Minkowski space for gravitational waves. The geodesic equations (see Appendix A) are

$$\frac{d^2 x^{\gamma}}{d\tau^2} + \Gamma^{\gamma}_{\beta\mu} \frac{dx^{\beta}}{d\tau} \frac{dx^{\mu}}{d\tau} = 0 \tag{7.52}$$

But for a slowly moving particle $\tau \approx t$, equation (7.52) is

$$\frac{d^2 x^{\gamma}}{dt^2} + \Gamma^{\gamma}_{\beta\mu} \frac{dx^{\beta}}{dt} \frac{dx^{\mu}}{dt} = 0 \tag{7.53}$$

Moreover, $\frac{dx^i}{dt}$ is very small , so we can neglect terms like $\Gamma^{\gamma}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$. Geodesic equation (7.53) reduces to

$$\frac{d^2 x^{\gamma}}{dt^2} + \Gamma^{\gamma}_{00} \frac{dx^0}{dt} \frac{dx^0}{dt} = 0 \tag{7.54}$$

so the space equation (three- acceleration) is

$$\frac{d^2 x^{i}}{dt^2} + \Gamma^{i}_{00} \frac{dx^0}{dt} \frac{dx^0}{dt} = 0 \tag{7.55}$$

Since $\frac{dx^0}{dt} = c$, equation (7.55)

$$\frac{d^2 x^{i}}{dt^2} = -c^2 \Gamma^{i}_{00} \tag{7.56}$$

Now

$$\Gamma^{i}_{00} = \frac{1}{2} \epsilon \left(h^{i}_{0,0} + h^{i}_{0,0} - h^{;i}_{0,0} \right) \approx -\frac{1}{2} \epsilon h^{;i}_{0,0}$$

The spatial geodesic equation becomes

$$\frac{d^2 x^{i}}{dt^2} = \frac{c^2}{2} \epsilon h^{;i}_{0,0} = \frac{c^2}{2} \epsilon \nabla^i h_{0,0} \tag{7.57}$$

But the Newtonian theory has

$$\frac{d^2 x^{i}}{dt^2} = \nabla^i \Phi \tag{7.58}$$

and

$$\nabla^i \Phi = \frac{c^2}{2} \epsilon \nabla^i h_{0,0} \tag{7.59}$$

where Φ is the gravitational potential. So we make the identification

$$g_{00} = \eta_{00} + \epsilon h_{00} \tag{7.60}$$

$$g_{00} = \left(1 + \frac{2\Phi}{c^2}\right) \tag{7.61}$$

This is equivalent to having space-time with the line element

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \left(dx^2 + dy^2 + dz^2\right) \tag{7.62}$$

7.2. Linear Metric Waves in Non-Stationary Universe

We consider the gravitational wave modes in the FRW metrics in a de Sitter space. In the four dimensional space-time

$$x^\mu = \{x_0 = ct, x^i\}, i = 1, 2, 3$$

In linear approximation one decomposes the metric $g_{\mu\nu}$ as before

$$g_{\mu\nu} = g^0{}_{\mu\nu} + \epsilon h_{\mu\nu}$$

When one chooses the flat background metrics for de-Sitter space as follows

$$g^0{}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \tag{7.63}$$

its inverse metric is

$$g^{0\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^{-2}(t) & 0 & 0 \\ 0 & 0 & -a^{-2}(t) & 0 \\ 0 & 0 & 0 & -a^{-2}(t) \end{pmatrix} \tag{7.64}$$

with

$$a(t) = a_0 e^{\frac{1}{3}Ht} \tag{7.65}$$

Similar to previous case, we will get gravitational waves

$$\Box h_{\mu\nu} = 0 \quad or \quad \Box h^{\mu\nu} = 0$$

where $\Box = \partial^2{}_t + 3H\partial_t - a^2(t)\nabla^2$ is the D'Alembertian operator for the de-Sitter space. Field $h^{\mu\nu}$ is then decomposed in partial waves

$$h^{\mu\nu} = \sum_\lambda \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{\mu\nu}{}_{k\lambda} u_{k\lambda}(t) e^{i\mathbf{k}x} + u^\dagger{}_{k\lambda}(t) e^{-i\mathbf{k}x}$$

with $k \equiv \{k_0 = \omega = ck, \mathbf{k}\}$. Wave function $u_{k\lambda}(t)$ satisfies the oscillator equation. However, in such a case we obtains the equation for parametric (frequency time dependent) damped harmonic oscillator.

$$\ddot{u}_{k\lambda}(t) + H\dot{u}_{k\lambda}(t) + \omega^2(t)u(t)_{k\lambda} = 0 \tag{7.66}$$

with

$$\omega^2(t) = \frac{k^3c^2}{a^2(t)} \tag{7.67}$$

7.2.1. Hyperbolic Geometry of Damped Oscillator and Double Universe

The canonical quantization in inflating universe was discussed by Vitiello, Alfinito and Manka (Vitiello and Alfinito and Manka 1997). Similarly, the quantization of the one dimensional damped harmonic oscillator with constant frequency was studied by Vitiello et. al. (Vitiello et. al. 1992). It has been shown that the canonical quantization can be properly performed by doubling the degrees of freedom of the system. The reason to choose the doubling system of the damped or dissipative system is based on the fact that one must work with closed systems as required by the canonical quantization formalism. Therefore, we consider the double oscillator system so as to perform the canonical quantization of the oscillator equation which is

$$\ddot{u}(t) + H\dot{u}(t) + \omega^2(t)u(t) = 0 \tag{7.68}$$

$$\ddot{v}(t) - H\dot{v}(t) + \omega^2(t)v(t) = 0 \tag{7.69}$$

In this sense we speak of ”Double Universe” (Vitiello et.al 1997). In the same way equation (7.68) is implied by inflating metrics, equation (7.69) for the oscillator v can be associated to the deflating metric and the metric of the deflating universe is defined by

$$\mathbf{g^0}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^{-2}(t) & 0 & 0 \\ 0 & 0 & -a^{-2}(t) & 0 \\ 0 & 0 & 0 & -a^{-2}(t) \end{pmatrix}$$

In summary,

- the oscillator u can be associated to the **inflating** metric
- the oscillator v can be associated to the **deflating** metric

The Lagrangian with respect to u and v modes from which equations (7.68) and (7.69) is obtained as

$$L = m\dot{u}\dot{v} + \frac{H}{2}(u\dot{v} - \dot{u}v) - \omega^2(t)uv \tag{7.70}$$

The canonical momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = m\dot{v} - \frac{H}{2}v \tag{7.71}$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = m\dot{u} + \frac{H}{2}u \tag{7.72}$$

and the Hamiltonian is

$$\mathcal{H} = p_u\dot{u} + p_v\dot{v} - L = p_u p_v + \frac{1}{2}H(vp_v - up_u) + \left(\omega^2(t) - \frac{H^2}{4}\right)^{\frac{1}{2}} \tag{7.73}$$

$$\Omega^2(t) \equiv \left(\omega^2(t) - \frac{H^2}{4}\right) \tag{7.74}$$

which we will get real $\Omega > 0$ for inflating universe. The reason is implied by $\omega^2(t) > \frac{H^2}{4}$ and then $0 \le t < \frac{3}{H}\ln\left(\frac{2k^3c}{a_0H}\right)$ with $\left(\frac{2k^3c}{a_0H}\right) \ge 1$. There is also an interesting relation between u and v

$$v = ue^{Ht} \tag{7.75}$$

Proof. First, let us solve the system of the equations

$$\ddot{u}(t) + H\dot{u}(t) + \omega^2(t)u(t) = 0$$

$$\ddot{v}(t) - H\dot{v}(t) + \omega^2(t)v(t) = 0$$

these equations are the second order differential equations by using the constant coefficient method, we obtain the following results for each equation

$$u(t) = Ce^{\frac{-H\pm\sqrt{H^2-4\omega^2}}{2}t} \;\; v(t) = Ce^{\frac{H\pm\sqrt{H^2-4\omega^2}}{2}t}$$

and let divide u by v. Hence,

$$u(t) = ve^{-Ht}, \;\; v(t) = ue^{Ht}$$

by substituting

$$u(t) = \frac{1}{\sqrt{2}} r(t) e^{\frac{-Ht}{2}} \tag{7.76}$$

and

$$v(t) = \frac{1}{\sqrt{2}} r(t) e^{\frac{Ht}{2}} \tag{7.77}$$

into systems of equations (7.68) and (7.69), finally the doublet oscillator system becomes

$$\ddot{r} + \Omega^2 r = 0 \tag{7.78}$$

which is the equation for parametric oscillator $r(t)$. This clarifies the meaning of the doubling of the u oscillator. Then $u - v$ oscillator is a non-inflating or non-deflating system. This is why it is now possible to set up the canonical quantization. Hence, we introduce the commutators

$$[u, p_u] = i\hbar = [v, p_v], \;\; [u, v] = 0 = [p_u, p_v] \tag{7.79}$$

and it will be more convenient to introduce the variables U and V by the transformations

$$U(t) = \frac{u(t) + v(t)}{\sqrt{2}} \tag{7.80}$$

$$V(t) = \frac{u(t) - v(t)}{\sqrt{2}} \tag{7.81}$$

which is known as the hyperbolic transformation and this transformations also preserve the commutation relation as

$$[U, p_U] = i\hbar = [V, p_V], \;\; [U, V] = 0 = [p_U, p_V] \tag{7.82}$$

In terms of the U and V we can do the decomposition of the parametric oscillator $r(t)$ on the hyperbolic plane and then the parametric oscillator becomes

$$r^2(t) = U^2(t) - V^2(t) \tag{7.83}$$

As we said before that is the decomposition of the oscillator on hyperbolic plane and Lagrangian function (7.70) is rewritten in terms of U and V as follows

$$L = L_{0,U} - L_{0,V} + \frac{H}{2}(\dot{U}V - \dot{V}U) \tag{7.84}$$

in which $L_{0,U}$ and $L_{0,V}$ are

$$L_{0,U} = \frac{1}{2}\dot{U}^2 - \frac{\omega^2(t)}{2}U^2 \tag{7.85}$$

$$L_{0,V} = \frac{1}{2}\dot{V}^2 - \frac{\omega^2(t)}{2}V^2 \tag{7.86}$$

The associate momenta for this system are

$$p_U = \dot{U} + \frac{H}{2}V \tag{7.87}$$

$$p_V = -\dot{V} - \frac{H}{2}U \tag{7.88}$$

and equations of motion for system (7.68) and (7.69) are given by the following equations

$$\ddot{U} + H\dot{V} + \omega^2(t)U = 0 \tag{7.89}$$

$$\ddot{V} + H\dot{U} + \omega^2(t)V = 0 \tag{7.90}$$

and the Hamiltonian becomes

$$\mathcal{H} = \mathcal{H}_1 - \mathcal{H}_2 = \frac{1}{2}\left(p_U - \frac{H}{2}V\right)^2 + \frac{\omega^2(t)}{2}U^2 - \frac{1}{2}\left(p_V + \frac{H}{2}U\right)^2 - \frac{\omega^2(t)}{2}V^2 \tag{7.91}$$

Equation (7.91) shows that the dissipation or inflation term H which is the Hubble constant acts as a coupling between the oscillators U and V and produces a correction to the kinetic energy for both oscillators. Finally, this system can be quantized by using this Hamiltonian which is in hyperbolic form.

Part II

VARIATIONAL PRINCIPLES FOR TIME DEPENDENT OSCILLATIONS AND DISSIPATIONS

CHAPTER 8

LAGRANGIAN AND HAMILTONIAN DESCRIPTION

8.1. Generalized Co-ordinates and Velocities

The position of a particle in space is defined by its radius vector **r**, whose components are its Cartesian coordinates x, y, z. The derivative $\mathbf{v} = d\mathbf{r}/dt$ of **r** with respect to the time t is called the velocity of the particle, and the second derivative $d^2\mathbf{r}/dt^2$ is its acceleration. To define a position of a system of N particles in space, it is necessary to specify N radius vectors. The number of independent quantities which must be specified so as to define uniquely position of any system is called the number of generalized coordinates here; this number is $3N$.

Definition 8.1 *Any s quantities $q_1, q_2, ..., q_s$ which completely define the position of a system with s degrees of freedom are called* generalized co-ordinates *of the system, and the derivatives $\dot{q}_i$ are called its* generalized velocities *(Landau and Liftshitz 1960).*

8.2. Principle of Least Action

The most general formulation of the laws governing the motion of mechanical systems is the principle of least action or the Hamilton's principle.

Definition 8.2 *According to which every mechanical systems are characterized by a definite function $L\left(q_1, q_2, ..., q_s \dot{q}_1, \dot{q}_2, ..., \dot{q}_s, t\right)$, or briefly $L\left(q, \dot{q}, t\right)$.*
Let the system occupy, at the instants t_1 and t_2, positions defined by two sets of values of the coordinates, $q^{(1)}$ and $q^{(2)}$. Then the condition is that the system moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \tag{8.1}$$

takes the least possible value. The function L is called Lagrangian of the system and the integral (8.1) is called the action.

Let us derive the differential equations that solve the problem of minimising the integral (8.1). For simplicity, we assume the system has only one degree of freedom, so that only one function $q(t)$ has to be determined.

Let $q = q(t)$ be the function for which S is a minimum. This means that S is increased when $q(t)$ is replaced by any function of the form

$$q(t) + \delta q(t) \tag{8.2}$$

where $\delta q(t)$ is a function which is small every where in the interval of time from t_1 to t_2; $\delta q(t)$ is called a variation of the function $q(t)$. Since, for $t = t_1$ and for $t = t_2$, all the functions (8.2) must take the values $q^{(1)}$ and $q^{(2)}$ respectively, it follows that

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{8.3}$$

The change in S when q is replaced by $q + \delta q$ is

$$\delta S = \int_{t_1}^{t_2} L\left(q + \delta q, \dot{q} + \delta\dot{q}, t\right) dt - \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt$$

When this difference is expanded in powers of δq and $\delta\dot{q}$ in the integrand,the leading terms are of the first order. The necessary condition for S to have a minimum is that these terms should be zero. Hence, the principle of least action can be written in the form

$$\delta S = \delta \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt = 0$$

that is,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt = 0$$

Since $\delta\dot{q} = d\delta q/dt$ and we integrate the second term by parts,

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) \delta q dt = 0 \tag{8.4}$$

The conditions (8.3) indicate that the first term in (8.4) is zero. There remains an integral which must vanish for all values of δq. This can be so only if the integrand is zero identically, Thus we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

When the system has more than one degree of freedom, the s different functions $q_i(t)$ must be varied independently in the principle of least action. We then obtain s equations of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, ...s. \tag{8.5}$$

These differential equations are called Lagrange's equations. If the Lagrangian of a given mechanical system is known, the equations (8.5) give relations between accelerations, velocities and coordinates. They are the equations of motions of the system.

8.3. Hamilton's Equations

In the Lagrangian formalism the mechanical state of a system can be described by its generalized coordinates and the velocities. At the previous section we look a mechanical system from the Lagrangian perspective. In Hamiltonian formalism, we can represent the same mechanical system in terms of the generalized coordinates and the momenta.

We will use Legendre's transformation to interpret the Hamiltonian equation in terms of Lagrangian and the passage from one set of independent variable to another can be effected by means of Legendre's transformation. The total differential of the Lagrangian as follows

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \tag{8.6}$$

this expression may be written

$$dL = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i \tag{8.7}$$

since the derivatives are given by the definition

$$p_i \equiv \frac{\partial L}{\partial q_i}, \quad \dot{p}_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{8.8}$$

where p_i are generalized momenta. Writing the second term in (8.7) as $\sum p_i d\dot{q}_i = d\left(\sum p_i \dot{q}_i\right) - \sum \dot{q}_i dp_i$, the taking the differential $d\left(\sum p_i \dot{q}_i\right)$ to the left hand side, the reversing the signs, we obtain from (8.7)

$$d\left(\sum p_i \dot{q}_i - L\right) = -\sum \dot{p}_i dq_i + \sum \dot{q}_i d\dot{p}_i \tag{8.9}$$

which is called the Hamilton's function or Hamiltonian of the system and this expressed in terms of coordinates and momenta

$$H\left(p,q,t\right)=\sum p_i\dot{q}_i-L \tag{8.10}$$

From the equation in differentials

$$dH=-\sum \dot{p}_i dq_i+\sum \dot{q}_i dp_i \tag{8.11}$$

in which the independent variables are the coordinates and momenta, we have the equations

$$\dot{q}_i=\frac{\partial H}{\partial p_i},\ \ \dot{p}_i=-\frac{\partial H}{\partial q_i} \tag{8.12}$$

These are required equations of motion in the variables p and q, and are called Hamilton's equations. They form a set of $2s$ first order differential equations for the $2s$ unknown functions $p_i(t)$ and $q_i(t)$, replacing the s second order equations in the Lagrangian treatment. Because of their simplicity and symmetry of form, they are also called canonical equations.

The total time derivative of the Hamiltonian is

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum \frac{\partial H}{\partial q_i}\dot{q}_i+\sum \frac{\partial H}{\partial p_i}\dot{p}_i \tag{8.13}$$

Substitution of $\dot{q}_i$ and $\dot{p}_i$ from the equations (8.12) shows that the last two terms cancel, and

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}-\sum \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{8.14}$$

then the result

$$\frac{dH}{dt}=\frac{\partial H}{\partial t} \tag{8.15}$$

If the Hamiltonian does not depend explicitly on time, then

$$\frac{dH}{dt}=0 \tag{8.16}$$

and we have the law of conservation of energy.

8.3.1. Poisson Brackets

Definition 8.3 *Let f is the function of coordinates, momenta and time $f\,(p,q,t)$. Its total derivative is*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) \tag{8.17}$$

Inserting of the values of $\dot{q}_i$ and $\dot{p}_i$ given by Hamilton's equations (8.12) leads to the expression

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \tag{8.18}$$

where

$$\{H, f\} = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \right) \tag{8.19}$$

This expression is called the Poisson bracket of the quantities H and f.

Definition 8.4 *Those functions of the dynamical variables which remain constant during the motion of the system are called integrals of the motion. From (8.18), the condition for the quantity f to be an integral of the motion*

$$\frac{df}{dt} = 0 \tag{8.20}$$

or

$$\frac{\partial f}{\partial t} + \{H, f\} = 0 \tag{8.21}$$

If the integral of the motion is not explicitly dependent on the time

$$\{H, f\} = 0 \tag{8.22}$$

Thus, the Poisson bracket of the integral and the Hamiltonian must be zero.

For any two quantities f and g, the Poisson bracket is defined analogously to (8.19)

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) \tag{8.23}$$

The Poisson bracket has the following properties,

a. Antisymmetric

$$\{f, g\} = -\{g, f\}$$

b. The Poisson bracket satisfies Leibniz identity

$$\{gf_1, f_2\} = g\{f_1, f_2\} + f_1\{g, f_2\}$$

c. The Poisson bracket is bilinear

$$\begin{aligned}\{f_1 + f_2, g\} &= \{f_1, g\} + \{f_2, g\} \\ \{f_1, f_2 + g\} &= \{f_1, f_2\} + \{f_1, g\} \\ \{f_1, cg\} &= c\{f_1, g\} \\ \{cf_1, g\} &= c\{f_1, g\}\end{aligned}$$

where c is constant.

d. The Jacobi's identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{8.24}$$

Taking the partial derivatives of (8.23) in terms of time

$$\frac{\partial}{\partial t}\{f, g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\}$$

If one of the functions f and g is one of the momenta or coordinates, the Poisson bracket reduces to a partial derivative

$$\{f, p_i\} = -\frac{\partial f}{\partial q_i}, \;\; \{f, q_i\} = \frac{\partial f}{\partial p_i}$$

and the canonical relations are given by the following equations

$$\{q_i, q_k\} = 0 = \{p_i, p_k\}, \;\; \{p_i, q_k\} = \delta_{ik}$$

Theorem 8.1 *An important property of the Poisson bracket is that, if g and f are two integrals of the motion, their Poisson bracket is an integral of the motion*

$$\{f, g\} = constant$$

This is called Poisson's theorem.

Putting $h = H$ in Jacobi's identity (8.24), we obtain

$$\{H, \{f, g\}\} + \{f, \{g, H\}\} + \{g, \{H, f\}\} = 0$$

Hence, if $\{H, g\} = 0$ and $\{H, f\} = 0$, then $\{H, \{f, g\}\} = 0$, which is the required results.

CHAPTER 9

DAMPED OSCILLATOR: CLASSICAL AND QUANTUM

9.1. Damped Oscillator

Definition 9.1 *The equation of motion of the* Damped Harmonic Oscillator *(DHO) is in the form*

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{9.1}$$

where m is mass, γ is damping constant and k is string constant. The Damped Harmonic Oscillator equation (9.1) is solved in the form $x = e^{qt}$ and then the characteristic equation of equation (9.1) is

$$mq^2 + \gamma q + k = 0 \tag{9.2}$$

The roots of equation (9.2) are

$$q_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \tag{9.3}$$

which give the three cases

a. $\gamma^2 - 4mk > 0$, Over Damping

b. $\gamma^2 - 4mk = 0$, Critical Damping

c. $\gamma^2 - 4mk < 0$, Under Damping

The solutions of equation (9.1) for the cases

a. Over Damping Case

$$x(t) = e^{-\frac{\gamma}{2}t}\left(C_1 \sinh \frac{\sqrt{\gamma^2 - 4mk}}{2m}t + C_2 \cosh \frac{\sqrt{\gamma^2 - 4mk}}{2m}t\right) \tag{9.4}$$

b. Critical Damping Case

$$x(t) = e^{-\frac{\gamma}{2}t}\left(C_1 + C_2 t\right) \tag{9.5}$$

c. Under Damping Case

$$x(t) = e^{-\frac{\gamma}{2}t}\left(C_1 \sin \frac{\sqrt{\gamma^2 - 4mk}}{2m}t + C_2 \cos \frac{\sqrt{\gamma^2 - 4mk}}{2m}t\right) \tag{9.6}$$

9.2. Bateman Dual Description

Definition 9.2 *A dissipative system is physically incomplete and so additional equations are to be expected when an attempt is made to derive the defining equations from a variational principle which is called Bateman Dual Description (Bateman 1931).*

It is well known that to set up the canonical formalism for dissipative systems the doubling of degrees of freedom is required in such a way to complement the given dissipative system with its time-reversed image, thus obtaining a globally closed system for which the Lagrangian formalism is well defined.

The Bateman method of finding a complementary set is illustrated by the following example. Consider for simplicity a single equation

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

in which dot denote differentiations with respect to t and the coefficients m, γ and k are constants. This equation is evidently derivable from the variational principle

$$\delta \int y\left(m\ddot{x} + \gamma\dot{x} + kx\right) = 0 \tag{9.7}$$

$$\int \delta y\left(m\ddot{x} + \gamma\dot{x} + kx\right) + y\delta\left(m\frac{d^2}{dt^2}x + \gamma\frac{d}{dt}x + kx\right) = 0 \tag{9.8}$$

$$\int \delta y\left(m\ddot{x} + \gamma\dot{x} + kx\right) + y\left(m\frac{d^2}{dt^2}\delta x + \gamma\frac{d}{dt}\delta x + k\delta x\right) = 0 \tag{9.9}$$

and

$$\int \delta y\left(m\ddot{x} + \gamma\dot{x} + kx\right) + \left(m\frac{d^2}{dt^2}y - \gamma\frac{d}{dt}y + ky\right)\delta x = 0 \tag{9.10}$$

in which both x and y are to be varied. This principle gives, moreover, the complementary equation

$$m\ddot{y} - \gamma\dot{y} + ky = 0 \tag{9.11}$$

and the dual description is given by the system of equations

$$\begin{aligned} m\ddot{x} + \gamma\dot{x} + kx &= 0 \\ m\ddot{y} - \gamma\dot{y} + ky &= 0 \end{aligned} \tag{9.12}$$

The time reversed image of the given system plays the role of reservoir or thermal path into which the energy dissipated by the original system flows and the whole system acts as a conservative system (Bateman 1931). This system is described by the Lagrangian density as follows

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}[x\dot{y} - \dot{x}y] - kxy \tag{9.13}$$

its momenta in terms of x and y are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y, \;\; p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x \tag{9.14}$$

The Hamiltonian through Legendre transformation is determined from

$$\mathcal{H} = \dot{x}p_x + \dot{y}p_y - L$$

by substituting momenta (9.14), that is

$$\mathcal{H} = (m\dot{y} - \frac{\gamma}{2}y)\dot{x} + (m\dot{x} + \frac{\gamma}{2}x)\dot{y} - m\dot{x}\dot{y} - \frac{\gamma}{2}\,[x\dot{y} - \dot{x}y] + kxy$$

so

$$\mathcal{H} = m\dot{x}\dot{y} + kxy \tag{9.15}$$

The Hamiltonian can be written in terms of the canonical momenta p_x and p_y, we get through

$$m\dot{x} = p_y - \frac{\gamma}{2}x \Rightarrow \dot{x} = \frac{1}{m}[p_y - \frac{\gamma}{2}x]$$

$$m\dot{y} = p_x + \frac{\gamma}{2}y \Rightarrow \dot{y} = \frac{1}{m}[p_x + \frac{\gamma}{2}y]$$

Substitute (9.16) and (9.16) in equation (9.15). Hence, we obtain the Hamiltonian

$$H(x, y, p_x, p_y) = \frac{p_x p_y}{m} + \frac{\gamma}{2m}(yp_y - xp_x) + xy(k - \frac{\gamma^2}{4m}) \tag{9.16}$$

9.3. Caldirola Kanai Approach for Damped Oscillator

The standard equation of motion for damped harmonic oscillator may be written in the following form

$$\ddot{q} + \Gamma(t)\dot{q} + \omega^2 q = 0 \tag{9.17}$$

where $\Gamma(t) = \frac{d}{dt} \ln m(t)$ for the case where the mass $m(t)$ is given by $m(t) = m_0 e^{\Gamma t}$. The constant frequency is ω^2 and the convenient Lagrangian of the oscillator is given by

$$L = \frac{1}{2} e^{\Gamma t} \left(m_0 \dot{q}^2 - m_0 \omega^2 q^2 \right) \tag{9.18}$$

and Lagrangian (9.18) satisfies equation (9.17) of motion. To construct the Hamiltonian of the oscillator, we need momenta which is given by

$$p = \frac{\partial L}{\partial \dot{q}} = e^{\Gamma t} m_0 \dot{q} \tag{9.19}$$

and

$$\frac{p^2 e^{-2\Gamma t}}{{m_0}^2} = \dot{q}^2 \tag{9.20}$$

Hence, the Hamiltonian is obtained by the Legendre transformation

$$H(q,p) = \dot{q}p - L(q,\dot{q})$$

If we substitute equation (9.19) into the Legendre transformation, we obtain

$$H = \dot{q} \left(e^{\Gamma t} m_0 \dot{q} \right) - \frac{1}{2} e^{\Gamma t} \left(m_0 \dot{q}^2 - m_0 \omega^2 q^2 \right) \tag{9.21}$$

By substituting equation (9.20) to (9.21), we get a harmonic oscillator with a time dependent mass and described by the Hamiltonian

$$H = \frac{p^2}{2m(t)} + \frac{1}{2} m(t) \omega^2 q^2 \tag{9.22}$$

where the mass $m(t)$ is given by $m(t) = m_0 e^{\Gamma t}$ with $\Gamma = constant$. The Hamiltonian (9.22) is called Caldirola (Caldirola 1977) and Kanai Hamiltonian (Kanai 1948).

9.4. Quantization of Caldirola-Kanai Damped Oscillator with Constant Frequency and Constant Damping

In this section, we will quantize the Caldirola-Kanai's damped harmonic oscillator with constant damping Γ and frequency Ω^2. The damped harmonic oscillator is given in the following form

$$\ddot{q} + \Gamma \dot{q} + \omega^2 q = 0$$

where Γ is constant damping term and constant frequency ω^2 and the convenient Lagrangian of the oscillator is

$$L = \frac{1}{2} e^{\Gamma t} \left(\dot{q}^2 - \omega^2 q^2 \right)$$

which is known as the Caldirola-Kanai Lagrangian and this Lagrangian satisfies the damped harmonic oscillator equation of motion. To construct the Hamiltonian of the oscillator, we define momentum and coordinate operators as

$$q \to q, \;\; p \to -i\hbar \frac{\partial}{\partial q} \tag{9.23}$$

The classical Caldirola-Kanai Hamiltonian function was given by equation (9.21) but in this section we assume that $m_0 = 1$ for simplicity and the Hamiltonian is

$$2H = e^{-\Gamma t} p^2 + e^{\Gamma t} \omega^2 q^2 \tag{9.24}$$

Then, evolution of the wave function is described by the time dependent Schrödinger equation as follows

$$i\hbar \frac{\partial}{\partial t} \Phi(q,t) = H \Phi(q,t) \tag{9.25}$$

where H is defined by equation (9.24). Then, we get the corresponding Schrodinger equation

$$i\hbar \frac{\partial \Phi}{\partial t} = \left(\frac{1}{2} e^{-\Gamma t} p^2 + \frac{1}{2} \omega^2 q^2 e^{\Gamma t} \right) \Phi \tag{9.26}$$

which is nonlinear Schrödinger equation. In terms of new coordinates, we reduce equation (9.26) to the standard Schrödinger equation for the harmonic oscillator by using the following transformation

$$q(q', t) \equiv e^{-\frac{\Gamma}{2} t} q', \;\; t \equiv t' \tag{9.27}$$

then partial derivatives become

$$\frac{\partial}{\partial q} \;\; = \;\; \frac{\partial q'}{\partial q} \frac{\partial}{\partial q'} + \frac{\partial t'}{\partial q} \frac{\partial}{\partial t'} = e^{\frac{\Gamma}{2} t} \frac{\partial}{\partial q'} \tag{9.28}$$

and

$$\frac{\partial}{\partial t} \;\; = \;\; \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial q'}{\partial t} \frac{\partial}{\partial q'} = \frac{\partial}{\partial t'} + \frac{\Gamma}{2} q' \frac{\partial}{\partial q'} \tag{9.29}$$

and the Jacobian matrix of the transformation is given by the following matrix

$$J = \begin{pmatrix} \frac{\partial q'}{\partial q} & \frac{\partial t'}{\partial q} \\ \frac{\partial q'}{\partial t} & \frac{\partial t'}{\partial t} \end{pmatrix} = \begin{pmatrix} e^{\frac{\Gamma}{2}t} & 0 \\ \frac{\Gamma}{2}q'e^{\frac{\Gamma}{2}t} & 1 \end{pmatrix} \tag{9.30}$$

the determinant of the Jacobian matrix is

$$J = \begin{vmatrix} e^{\frac{\Gamma}{2}t} & 0 \\ \frac{\Gamma}{2}q'e^{\frac{\Gamma}{2}t} & 1 \end{vmatrix} = e^{\frac{\Gamma}{2}t} \tag{9.31}$$

Since determinant (9.31) is not equivalent to zero, this transformation will not form a singular equation. After substituting (9.28) and (9.29) in (9.26), we obtain

$$i\hbar\left[\frac{\partial}{\partial t} + \frac{\Gamma}{2}q'\frac{\partial}{\partial q'}\right]\Phi = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q'^2}\Phi + \frac{1}{2}\omega^2 q'^2\Phi \tag{9.32}$$

if we arrange (9.32), it becomes

$$i\hbar\frac{\partial}{\partial t}\Phi = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q'^2}\Phi + \frac{1}{2}\omega^2 q'^2\Phi - i\hbar\frac{\Gamma}{2}q'\frac{\partial}{\partial q'}\Phi \tag{9.33}$$

and the new momentum operator defined as

$$p' = -i\hbar\frac{\partial}{\partial q'}, \quad p^2 = -\hbar^2\frac{\partial^2}{\partial q'^2} \tag{9.34}$$

and which is satisfies the following commutation relations

$$\begin{aligned} [p', q'] &= -i\hbar \\ [p', q] &= -i\hbar e^{\frac{-\Gamma}{2}t} \\ [p', p'] &= 0 = [q', q'] \end{aligned} \tag{9.35}$$

Hence, the new Hamiltonian function of equation (9.33) can be represented in terms of p' as

$$H_1 = \frac{1}{2}\left(p'^2 + \Gamma q'p'\right) + \frac{1}{2}\omega^2 q'^2 \tag{9.36}$$

By adding and subtracting $\frac{\Gamma^2}{8}q'^2$ to get the square of the parenthesis, the new Hamiltonian H_1 can be written as

$$H_1 = \frac{1}{2}\left(p' + \frac{\Gamma}{2}q'\right)^2 + \frac{1}{2}\left(\omega^2 - \frac{\Gamma^2}{4}\right)q'^2 \tag{9.37}$$

and we can define the new frequency Ω^2 as follows

$$\Omega^2 \equiv \omega^2 - \frac{\Gamma^2}{4} \tag{9.38}$$

Hence, equation (9.37) becomes

$$H_1 = \frac{1}{2}\left(p' + \frac{\Gamma}{2}q'\right)^2 + \frac{1}{2}\Omega^2 q'^2 \tag{9.39}$$

let us define momentum operator $\mathcal{P}$ in the following form

$$\mathcal{P} \equiv \left(p' + \frac{\Gamma}{2}q'\right) \tag{9.40}$$

So, we can define the second Hamiltonian with respect to new operator $\mathcal{P}$

$$H_1 = \frac{1}{2}\mathcal{P}^2 + \frac{1}{2}\Omega^2 q'^2 \tag{9.41}$$

If we insert $\mathcal{P} = p' + \alpha q'$ into the H_1. Then, we get

$$H_2 = \frac{1}{2}(p' + \alpha q')(p' + \alpha q') + \frac{1}{2}\Omega^2 q'^2 \tag{9.42}$$

and

$$H_2 = \frac{1}{2}p'^2 + \frac{\alpha}{2}\{p'q' + q'p'\} + \frac{1}{2}\left(\alpha^2 + \Omega^2\right)q'^2 \tag{9.43}$$

using commutation relation as follows

$$[q', p'] = i\hbar = [q, p]\,, \quad p'q' = q'p' - i\hbar \tag{9.44}$$

Substitute (9.44) to (9.43) and then

$$H_2 = \frac{1}{2}p'^2 + \frac{\alpha}{2}\{q'p' - i\hbar + q'p'\} + \{\frac{\alpha^2}{2} + \frac{1}{2}\Omega^2\}q'^2 \tag{9.45}$$

as a result,

$$H_2 = \frac{1}{2}p'^2 + \alpha q'p' + \{\frac{\alpha^2}{2} + \frac{1}{2}\Omega^2\}q'^2 - \frac{i\hbar}{2}\alpha \tag{9.46}$$

If we compare H_1 and H_2, we obtain the relation

$$H_1 = H_2 + \frac{i\hbar}{2}\alpha \tag{9.47}$$

where $\alpha = \frac{\Gamma}{2}$

$$H_1 = H_2 + i\hbar\frac{\Gamma}{4} \tag{9.48}$$

Under this representation, we can rewrite time dependent Schrödinger equation (9.25) as,

$$i\hbar\frac{\partial}{\partial t}\Phi = H_1\Phi = \left[H_2 + i\hbar\frac{\Gamma}{4}\right]\Phi$$

and

$$i\hbar\frac{\partial}{\partial t}\Phi - i\hbar\frac{\Gamma}{4}\Phi = \frac{1}{2}\mathcal{P}^2\Phi + \frac{1}{2}\Omega^2 {q'}^2\Phi \tag{9.49}$$

Let us substitute $\mathcal{P} = p' + \alpha q' = p' + \frac{\Gamma}{2}q'$ in equation (9.49). Then, we obtain

$$i\hbar\left[\frac{\partial}{\partial t} - \frac{\Gamma}{4}\right]\Phi = \frac{1}{2}\left(-i\hbar\frac{\partial}{\partial q'} + \frac{\Gamma}{2}q'\right)^2\Phi + \frac{1}{2}\Omega^2 {q'}^2\Phi \tag{9.50}$$

We consider the first term of the right hand side of equation (9.50) to solve the equation with respect to the Gauge transformation as

$$\left(-i\hbar\frac{\partial}{\partial q'} + \frac{\Gamma}{2}q'\right)\Phi = -i\hbar\left(\frac{\partial}{\partial q'} + \frac{i\Gamma}{2\hbar}q'\right)\Phi = -i\hbar e^{-\frac{\Gamma i}{4\hbar}{q'}^2}\frac{\partial}{\partial q'}\Phi e^{\frac{\Gamma i}{4\hbar}{q'}^2} \tag{9.51}$$

and the square of (9.51)

$$\left(-i\hbar\frac{\partial}{\partial q'} + \frac{\Gamma}{2}q'\right)^2\Phi = -\hbar^2\left(\frac{\partial}{\partial q'} + \frac{i\Gamma}{2\hbar}q'\right)^2\Phi = -\hbar^2 e^{-\frac{\Gamma i}{4\hbar}{q'}^2}\frac{\partial^2}{\partial {q'}^2}\underbrace{\Phi e^{\frac{\Gamma i}{4\hbar}{q'}^2}}_{K} \tag{9.52}$$

Let us define a new function K which is a function of q' and t as a given on the right hand side of equation (9.52),

$$K(q',t) \equiv \Phi(q',t)e^{\frac{\Gamma i}{4\hbar}{q'}^2} \tag{9.53}$$

We can easily see that

$$\Phi(q',t) = K(q',t)\left(e^{-\frac{i\Gamma}{4\hbar}{q'}^2}\right) \tag{9.54}$$

We know that the Schrödinger equation has the general solution in the following form

$$\Phi(q',t) = \underbrace{e^{\frac{-i}{\hbar}\epsilon t}\xi(q')}_{K(q',t)}\left(e^{-\frac{i\Gamma}{4\hbar}{q'}^2}\right) \tag{9.55}$$

by using the gauge transformation, we get the wave function as the following form

$$\Phi(q',t) = e^{\frac{-i}{\hbar}\epsilon t}e^{-\frac{i\Gamma}{4\hbar}{q'}^2}\xi(q') \tag{9.56}$$

After substituting (9.56) in Schrödinger equation (9.50) becomes

$$\begin{aligned} i\hbar\left[\frac{\partial}{\partial t} - \frac{\Gamma}{4}\right]e^{\frac{-i}{\hbar}\epsilon t}e^{-\frac{i\Gamma}{4\hbar}{q'}^2}\xi(q') &= -\frac{\hbar^2}{2}e^{\frac{-i}{\hbar}\epsilon t}\xi''(q')e^{-\frac{i\Gamma}{4\hbar}{q'}^2} \\ &+\frac{1}{2}\Omega^2{q'}^2 e^{\frac{-i}{\hbar}\epsilon t}e^{-\frac{i\Gamma}{4\hbar}{q'}^2}\xi(q') \end{aligned} \tag{9.57}$$

and neglecting the exponential terms, we obtain

$$\left[\epsilon - i\hbar\frac{\Gamma}{4}\right]\xi = -\frac{\hbar^2}{2}\xi'' + \frac{1}{2}\Omega^2{q'}^2\xi \tag{9.58}$$

To normalize equation (9.58), we multiply the equation with $-\frac{2m}{\hbar^2}$

$$-\frac{2}{\hbar^2}\left[\epsilon - i\hbar\frac{\Gamma}{4}\right]\xi = \xi'' - \frac{1}{\hbar^2}\Omega^2 q'^2 \xi \tag{9.59}$$

At this stage we will do one more transformation in terms of q' with respect to the first transformation as

$$q' \equiv ay \rightarrow \frac{\partial^2}{\partial q'^2} = \frac{1}{a^2}\frac{\partial}{\partial y^2} \tag{9.60}$$

then equation (9.59) becomes

$$-\frac{2}{\hbar^2}a^2\left[\epsilon - i\hbar\frac{\Gamma}{4}\right]\xi = \frac{\partial^2}{\partial y^2}\xi - \frac{1}{\hbar^2}\Omega^2 a^4 y^2 \xi \tag{9.61}$$

and we assume that

$$a \equiv \sqrt{\frac{\hbar}{\Omega}} \tag{9.62}$$

and by substituting (9.62), equation (9.61) is reduced to the following form

$$-\frac{2}{\hbar\Omega}\left[\epsilon - i\hbar\frac{\Gamma}{4}\right]\xi = \frac{\partial^2}{\partial y^2}\xi - y^2\xi \tag{9.63}$$

If we choose $s \equiv \frac{2}{\hbar\Omega}\left[\epsilon - i\hbar\frac{\Gamma}{4}\right]$, Schrödinger equation (9.63) is reduced to the following equation

$$\xi'' + \left(s - y^2\right)\xi = 0 \tag{9.64}$$

This differential equation can be transformed and solved by inspection by making the substitution

$$\xi(y) = e^{-\frac{y^2}{2}} f(y) \tag{9.65}$$

Taking the first and the second derivatives with respect to y gives

$$\begin{aligned} \xi'(y) &= \left[f' - yf\right] e^{-\frac{y^2}{2}} \\ \xi''(y) &= \left[f'' - 2yf' - f + y^2 f\right] e^{-\frac{y^2}{2}} \end{aligned} \tag{9.66}$$

so plugging the first and second derivatives into (9.64) gives

$$f'' - 2yf' + (s-1)f = 0 \tag{9.67}$$

If $s-1=2n$, equation (9.67) is called the Hermite differential equation. Before solving (9.67), we can define the energy eigenstates are

$$s=2n+1\equiv\frac{2}{\hbar\Omega}\left[\epsilon_n-i\hbar\frac{\Gamma}{4}\right]\rightarrow\epsilon_n=\hbar\Omega\left[\left(n+\frac{1}{2}\right)\right]+i\frac{\Gamma}{2\Omega}\tag{9.68}$$

We can see that the energy eigenstates includes imaginary term. It means that the eigenstates with positive imaginary eigenvalues imply growing states (Feshbach and Tikochinsky 1977).

The Hermite differential equation is solved by applying series solution as

$$\begin{aligned}f(y)&=\sum_{n=0}^{\infty}c_ny^n\\f'(y)&=\sum_{n=1}^{\infty}c_nny^{n-1}\\f''(y)&=\sum_{n=2}^{\infty}c_nn\left(n-1\right)y^{n-2}\end{aligned}\tag{9.69}$$

If we substitute them into the Hermite differential equation

$$\sum_{n=2}^{\infty}c_nn\left(n-1\right)y^{n-2}-2y\sum_{n=1}^{\infty}c_nny^{n-1}+\left(s-1\right)\sum_{n=0}^{\infty}c_ny^n\ =\ 0\tag{9.70}$$

let us choose $s-1=\lambda$

$$\sum_{n=0}^{\infty}c_{n+2}n\left(n+2\right)\left(n+1\right)y^n-2\sum_{n=1}^{\infty}c_nny^n+\lambda\sum_{n=0}^{\infty}c_ny^n\ =\ 0\tag{9.71}$$

and

$$\left(2c_2+\lambda c_0\right)+\sum_{n=1}^{\infty}\left[\left(n+1\right)\left(n+2\right)c_{n+2}-2nc_n+\lambda c_n\right]y^n\ =\ 0\tag{9.72}$$

by using recurrence relation, we get

$$c_{n+2}\ =\ \frac{2n-\lambda}{\left(n+1\right)\left(n+2\right)}c_n\tag{9.73}$$

and

$$2c_2+\lambda c_0=0,\ \ c_2=-\frac{\lambda}{2}c_0\tag{9.74}$$

this relation is just a special case of the first general recurrence relation for $n=0,1,....$ To obtain the linearly independent solutions, we are firstly choosing $c_0=0,\ \ c_1=1.$ Hence, the surviving terms are

$$\begin{aligned}c_3&=\frac{2-\lambda}{3!}c_1\\c_5&=\frac{\left(6-\lambda\right)\left(2-\lambda\right)}{5!}c_1...\end{aligned}\tag{9.75}$$

so on, but the even integers are zero because of $c_0 = 0$. In addition, the first linearly independent solution is given by

$$f_1 = c_1 \left[y + \frac{2-\lambda}{3!} y^3 + \frac{(6-\lambda)(2-\lambda)}{5!} y^5 + ... \right] \tag{9.76}$$

and the surviving coefficients for the second linearly independent solutions with respect to $c_1 = 0, \;\; c_0 = 1$

$$\begin{aligned} c_2 &= -\frac{\lambda}{2!} c_0 \\ c_4 &= -\frac{(4-\lambda)\lambda}{4!} c_0 \\ c_6 &= -\frac{(8-\lambda)(4-\lambda)\lambda}{6!} c_0 ... \end{aligned} \tag{9.77}$$

Hence,

$$f_2 = c_0 \left[1 - \frac{\lambda}{2!} y^2 - \frac{(4-\lambda)\lambda}{4!} y^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!} y^6 ... \right] \tag{9.78}$$

These can be represented by using the superposition principle as

$$\begin{aligned} f_1 &= c_0 \left[1 - \frac{\lambda}{2!} y^2 - \frac{(4-\lambda)\lambda}{4!} y^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!} y^6 ... \right] \\ &+ c_1 \left[y + \frac{2-\lambda}{3!} y^3 + \frac{(6-\lambda)(2-\lambda)}{5!} y^5 + ... \right] \end{aligned} \tag{9.79}$$

we can write the function f in terms of the Hypergeometric function as follows

$$f = c_{01} F_1 \left(-\frac{1}{4}\lambda; \frac{1}{2}; y^2 \right) + c_1 y_1 F_1 \left(-\frac{1}{4}(\lambda - 2); \frac{3}{2}; y^2 \right) \tag{9.80}$$

Then, the solution to the original differential equation is

$$\xi(y) = e^{-\frac{y^2}{2}} \left[c_0 \;\, {}_1F_1 \left(-\frac{1}{4}\lambda; \frac{1}{2}; y^2 \right) + c_1 H_{\frac{\lambda}{2}}(y) \right] \tag{9.81}$$

where c_0 and c_1 are constants, $H_{\frac{\lambda}{2}}$ is Hermite polynomial (see Appendix C) and ${}_1F_1$ is a confluent hypergeometric function. However, we are just interested in the solution for which $\xi(y) \to 0$ as $y \to \infty$. Then,

$$\xi(y) = e^{-\frac{y^2}{2}} c_1 H_{\frac{\lambda}{2}}(y) \tag{9.82}$$

where $\lambda = 2n$. Hence, the wave function becomes

$$\Phi(q', t) = e^{\frac{-i}{\hbar}\epsilon t} e^{-\frac{\Gamma i}{4\hbar} q'^2} e^{-\frac{y^2}{2}} c_1 H_n(y) \tag{9.83}$$

We take the module of the wave function so as to find the coefficient c_1

$$|\Phi|^2 = \Phi\Phi^* = {c_1}^2 e^{-y^2} {H_n}^2(y) \quad (9.84)$$

and

$$\int_{-\infty}^{\infty} |\Phi|^2 dq' = 1 \Rightarrow {c_1}^2 \int_{-\infty}^{\infty} H_n^2(y) e^{-y^2} dy = 2^n n! \sqrt{\pi} = 1 \quad (9.85)$$

which is the orthogonality property of the Hermite polynomials (see Appendix C). Then, we know that $q' = ay$ and as a result of this, we can write that $y = \frac{q}{a} e^{\frac{\Gamma}{2}t}$. If we again change the variables for simplicity in the following form

$$\sigma \equiv e^{-\frac{\Gamma}{2}t} a \quad (9.86)$$

Hence

$$y = \frac{q}{\sigma} \quad (9.87)$$

by using this transformation in (9.85), we obtain

$$\frac{{c_1}^2}{\sigma} \int_{-\infty}^{\infty} H_n^2(\frac{q}{\sigma}) e^{\frac{q^2}{\sigma^2}} dq = 2^n n! \sigma \sqrt{\pi} = 1 \quad (9.88)$$

$${c_1}^2 = 2^n n! \sigma^2 \sqrt{\pi} = 1 \quad (9.89)$$

$$c_1 = \frac{1}{\sigma\sqrt{2^n n! \pi^{1/2}}} \quad (9.90)$$

let us define

$$A_n \equiv \left(2^n n! \pi^{1/2}\right)^{-1/2} \quad (9.91)$$

hence,

$$c_1 = \frac{A_n}{\sigma} \quad (9.92)$$

Then, the wave function is given by

$$\Phi_n(q', t) = \frac{A_n}{\sigma} e^{\frac{-i}{\hbar}\epsilon_n t} e^{-\frac{i\Gamma}{4\hbar} q'^2} e^{-\frac{q^2}{2\sigma^2}} H_n\left(\frac{q}{\sigma}\right) \quad (9.93)$$

By substituting $q' = e^{\frac{\Gamma}{2}t} q$ into the last equation, we obtain

$$\Phi_n(q, t) = \frac{A_n}{\sigma} e^{\frac{-i}{\hbar}\epsilon_n t} e^{-\frac{i\Gamma}{4\sigma^2} q^2} e^{-\frac{q^2}{2\sigma^2}} H_n\left(q e^{\frac{\Gamma}{2}t} \sqrt{\frac{\Omega}{\hbar}}\right) \quad (9.94)$$

where $\sigma = e^{-\frac{\Gamma}{2}t} \sqrt{\frac{\hbar}{\Omega}}$. Regarding the Hermite polynomial in equation (9.94), the inclusion of $e^{\frac{\Gamma}{2}t}$ term in the argument makes this quite different from the case of a simple harmonic oscillator.

CHAPTER 10

STURM LIOUVILLE PROBLEM AS DAMPED PARAMETRIC OSCILLATOR

10.1. Sturm Liouville Problem in Doublet Oscillator Representation and Self-Adjoint Form

We consider the second order differential equations corresponding to linear, second order differential operators of the general form

$$\mathcal{L}u(t) = p_0(t)\frac{d^2}{dt^2}u(t) + p_1(t)\frac{d}{dt}u(t) + p_2(t)u(t) \tag{10.1}$$

and assume that the p_0, p_1 and p_2 are real functions of t in interval $a \leq t \leq b$ and the first $2-i$ derivatives of p_i are continuous. Further, if function $p_0(t)$ has singular points, then we choose our interval $[a, b]$ so that there are no singular points in the interior of the interval.

Definition 10.1 *For a linear operator $\mathcal{L}$ the bilinear form*

$$\begin{aligned} S &\equiv \langle v(t)|\mathcal{L}u(t)\rangle = \int_b^a v(t)\mathcal{L}u(t)dt \\ &= \int_b^a v\left[p_0\ddot{u} + p_1\dot{u} + p_2 u\right]dt \end{aligned} \tag{10.2}$$

is the associated action integral, where the dots on the real function $u(t)$ denote derivatives.

After integration by parts we are led to the equivalent expression

$$\begin{aligned} S = \langle \mathcal{L}v(t)|u(t)\rangle &= \left[vp_1u + v\dot{p}_0\dot{u} - \frac{d}{dt}\left(vp_0\right)u\right]_{t=a}^{b} \\ &+ \int_b^a \left[\frac{d^2}{dt^2}\left(vp_0\right)u - \frac{d}{dt}\left(vp_1\right)u + p_2vu\right]dt \end{aligned} \tag{10.3}$$

so

$$\begin{aligned} S &= \langle \mathcal{L}v(t)|u(t)\rangle = \left[vp_1u + v\dot{p}_0\dot{u} - \frac{d}{dt}\left(vp_0\right)u\right]_{t=a}^{b} \\ &+ \int_b^a u\left[\ddot{v}p_0 + \dot{v}\left(2\dot{p}_0 - p_1\right) + \left(p_2 - \dot{p}_1 + \ddot{p}_0\right)v\right]dt \end{aligned} \tag{10.4}$$

where we will skip the non integral part because of the Lagrange function theorem which satisfies

$$L_1 = L_2 + \frac{df}{dt} \Rightarrow \delta S_1 = \delta S_2 = 0$$

If $p_1 = \dot{p}_0$ in equation (10.4), we can easily see that

$$\langle v(t)|\mathcal{L}u(t)\rangle = \langle \mathcal{L}v(t)|u(t)\rangle = S \tag{10.5}$$

In the light of the condition $p_1 = \dot{p}_0$, we can write equation (10.1) and (10.4) as

$$\begin{aligned} \frac{d}{dt}\left[\dot{u}p_0\right] + p_2 u = 0 \rightarrow \mathcal{L} = \frac{d}{dt}\left[p_0 \frac{d}{dt}\right] + p_2 \\ \frac{d}{dt}\left[\dot{v}p_0\right] + p_2 v = 0 \rightarrow \overline{\mathcal{L}} = \frac{d}{dt}\left[p_0 \frac{d}{dt}\right] + p_2 \end{aligned} \tag{10.6}$$

as the adjoint operator $\overline{\mathcal{L}}$. The necessary and sufficient condition that $\overline{\mathcal{L}} = \mathcal{L}$ or $\langle v(t)|\mathcal{L}u(t)\rangle = \langle \mathcal{L}v(t)|u(t)\rangle$ is that we can write

$$\mathcal{L}u = \overline{\mathcal{L}}u = \frac{d}{dt}\left[p(t)\frac{d}{dt}u(t)\right] + q(t)u(t) \tag{10.7}$$

where p_0 is replaced by p and p_2 is replaced by q. If (10.7) is satisfied then the operator $\mathcal{L}$ is called self adjoint.

If p_1 is not equal to $\dot{p}_0$ in equation (10.4), we obtain the equations

$$\ddot{u}p_0 + \dot{u}p_1 + p_2 u = 0 \tag{10.8}$$

$$\ddot{v}p_0 + (2\dot{p}_0 - p_1)\,\dot{v} + (p_2 - \dot{p}_1 + \ddot{p}_0)\,v = 0 \tag{10.9}$$

The operators for equations (10.9) and (10.8) become

$$\ddot{u}p_0 + \dot{u}p_1 + p_2 u = 0 \rightarrow \mathcal{L} \;=\; p_0\frac{d^2}{dt^2} + p_1\frac{d}{dt} + p_2 \tag{10.10}$$

$$\begin{aligned} \ddot{v}p_0 + (2\dot{p}_0 - p_1)\,\dot{v} + (p_2 - \dot{p}_1 + \ddot{p}_0)\,v = 0 \rightarrow \overline{\mathcal{L}} \;&=\; p_0\frac{d^2}{dt^2} + (2\dot{p}_0 - p_1)\,\frac{d}{dt} \\ &+\; (p_2 - \dot{p}_1 + \ddot{p}_0) \end{aligned} \tag{10.11}$$

We can easily see that for the condition $p_1 \neq \dot{p}_0$, the operator $\mathcal{L}$ is not self adjoint.

If we divide equations (10.8) and (10.9) by p_0, we obtain

$$\ddot{u} + \frac{p_1}{p_0}\dot{u} + \frac{p_2}{p_0}u = 0 \tag{10.12}$$

$$\ddot{v} + \left(\frac{2\dot{p}_0 - p_1}{p_0}\right)\dot{v} + \left(\frac{p_2 - \dot{p}_1 + \ddot{p}_0}{p_0}\right)v = 0 \tag{10.13}$$

If we define new variables as $\Gamma \equiv \frac{p_1}{p_0}$ and $\omega^2 \equiv \frac{p_2}{p_0}$, above equations (10.12) and (10.13) become

$$\ddot{u} + \Gamma(t)\dot{u} + \omega^2(t)u = 0 \tag{10.14}$$

$$\ddot{v} + \left(-\Gamma(t) + \frac{2\dot{p}_0}{p_0}\right)\dot{v} + \left(\omega^2(t) - \dot{\Gamma} - \Gamma\frac{\dot{p}_0}{p_0} + \frac{\ddot{p}_0}{p_0}\right)v = 0 \tag{10.15}$$

We assume that $\ln p_0 \equiv \Phi(t)$. After substituting this in equation (10.15), we get

$$\ddot{v} + \left(-\Gamma(t) + 2\dot{\Phi}(t)\right)\dot{v} + \left(\omega^2(t) - \dot{\Gamma} - \Gamma\dot{\Phi}(t) + \ddot{\Phi}(t) + \dot{\Phi}^2(t)\right)v = 0 \tag{10.16}$$

If $p_0 = 1$, then $\Phi = 0$. Hence, equations (10.14) and (10.16) become

$$\ddot{u} + \Gamma\dot{u} + \omega^2 u = 0 \tag{10.17}$$

$$\ddot{v} - \Gamma(t)\dot{v} + \left(\omega^2(t) - \dot{\Gamma}\right)v = 0 \tag{10.18}$$

The following Lagrangian function is corresponding equations (10.17) and (10.18)

$$L = \dot{u}\dot{v} - \frac{1}{2}\Gamma(t)\left(v\dot{u} - \dot{v}u\right) - \left(\omega^2(t) - \frac{1}{2}\dot{\Gamma}(t)\right)uv \tag{10.19}$$

and the conjugate momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\Gamma(t)v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\Gamma(t)u \tag{10.20}$$

The Legendre transformation

$$H = \dot{u}p_u + \dot{v}p_v - L$$

results in

$$\begin{aligned} H &= p_u p_v + \frac{1}{2}\Gamma\left(vp_v - up_u\right) \\ &+ \left(\omega^2(t) - \frac{1}{4}\Gamma^2(t) - \frac{1}{2}\dot{\Gamma}(t)\right)uv \end{aligned} \tag{10.21}$$

(Pashaev and Tigrak 2007).

10.1.1. Particular Cases for Non-Self Adjoint Equation

Case I: If the damping coefficient Γ is constant and the frequency ω^2 is a generic function of time, the damped oscillator equation becomes

$$\ddot{u} + \Gamma\dot{u} + \omega^2(t)u = 0$$

$$\ddot{v} - \Gamma\dot{v} + \omega^2(t)v = 0$$

Then, we get the Bateman Dual Description for the damped harmonic oscillator with time dependent frequency and the Lagrangian function for the double oscillator system is

$$L = \dot{u}\dot{v} - \frac{1}{2}\Gamma(t)\left(v\dot{u} - \dot{v}u\right) - \omega^2(t)uv \tag{10.22}$$

its momenta in terms of u and v are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\Gamma v, \quad p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\Gamma u \tag{10.23}$$

and the Hamiltonian is

$$H = p_u p_v + \frac{1}{2}\Gamma\left(vp_v - up_u\right) + \left(\omega^2(t) - \frac{1}{4}\Gamma^2\right)uv \tag{10.24}$$

Case II: If the frequency ω^2 is constant and the damping Γ is a generic function of time, the damped oscillator equations become

$$\ddot{u} + \Gamma(t)\dot{u} + \omega^2 u = 0$$

$$\ddot{v} - \Gamma(t)\dot{v} + \left(\omega^2 - \dot{\Gamma}\right)v = 0$$

Hence, we obtain the Lagrangian equation of the damped oscillator with constant frequency

$$L = \dot{u}\dot{v} - \frac{1}{2}\Gamma(t)\left(v\dot{u} - \dot{v}u\right) - \left(\omega^2 - \frac{1}{2}\dot{\Gamma}(t)\right)uv \tag{10.25}$$

The momenta in terms of u and v are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\Gamma(t)v, \quad p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\Gamma(t)u \tag{10.26}$$

and the Hamiltonian

$$\begin{aligned} H = p_u p_v \quad &+ \quad \frac{1}{2}\Gamma(t)\left(vp_v - up_u\right) \\ &+ \quad \left(\omega^2 - \frac{1}{4}\Gamma^2(t) - \frac{1}{2}\dot{\Gamma}(t)\right)uv \end{aligned} \tag{10.27}$$

Case III: If the damping coefficient Γ and the frequency ω^2 are constants, the doublet damped oscillator system can be written as

$$\ddot{u} + \Gamma \dot{u} + \omega^2 u = 0$$

$$\ddot{v} - \Gamma \dot{v} + \omega^2 v = 0$$

This system is standard double damped harmonic oscillator with constant damping and constant frequency and the Lagrangian is given by the following equation

$$L = \dot{u}\dot{v} - \frac{1}{2}\Gamma\left(v\dot{u} - \dot{v}u\right) - \omega^2 uv \tag{10.28}$$

The momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\Gamma v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\Gamma u \tag{10.29}$$

The Hamiltonian is

$$H = p_u p_v + \frac{1}{2}\Gamma\left(v p_v - u p_u\right) + \left(\omega^2 - \frac{1}{4}\Gamma^2\right) uv \tag{10.30}$$

From above consideration, we can conclude that if the system described by a second order differential equation that is that not self adjoint, the variation description of the system includes adjoint equation which can be called as mirror image. In particular case in which the damping term Γ and the frequency ω^2 are constant. It is time reversal image of the system and the energy for the considered system is a conserved quantity (Pashaev and Tigrak 2007).

10.1.2. Variational Principle for Self Adjoint Operator

In the previous consideration, we have formulated variational action functional as a bilinear form for operator $\mathcal{L}$ that describes the doublet oscillator system. If we like to avoid doublet of degrees of freedom, we should find another Lagrangian formulation which includes only the variables of the original system. To do this, we notice the well known fact, using integration factor any linear second order system can be written in a self adjoint form which is called the Sturm Liouville form. Consider the oscillator representation as

$$\ddot{u} + \Gamma(t)\dot{u} + \omega^2(t)u = 0$$

and the corresponding linear operator is non self adjoint

$$\mathcal{L} = \frac{d^2}{dt^2} + \Gamma(t)\frac{d}{dt} + \omega^2(t) \tag{10.31}$$

To obtain the self adjoint operator, we need an integration factor. If we multiply equation (10.14) with the integration factor $\mu(t)$ as

$$\mu\left[\frac{d^2}{dt^2}u + \Gamma(t)\frac{d}{dt}u + \omega^2(t)u\right] = 0 \tag{10.32}$$

which can be written as

$$\frac{d}{dt}\left[\mu\dot{u}\right] - \dot{\mu}\dot{u} + \Gamma(t)\mu\dot{u} + \mu\omega^2(t)u = 0 \tag{10.33}$$

Hence, we choose $\dot{\mu}\dot{u} = \Gamma(t)\mu\dot{u}$ and the integration factor is

$$\mu(t) = e^{\int^t \Gamma(\tau)d\tau} \tag{10.34}$$

so the self adjoint operator is

$$\mathcal{L}_s \equiv \mu(t)\mathcal{L} = \mu(t)\frac{d^2}{dt^2} + \mu(t)\Gamma(t)\frac{d}{dt} + \mu(t)\omega^2(t) \tag{10.35}$$

The operator $\mathcal{L}_s = \mu(t)\mathcal{L}$ corresponding to differential equation (10.32) leads to the same equation of motion but it is now self adjoint.

Definition 10.2 *Action corresponding to the self adjoint linear operator $\mathcal{L}_s$ is*

$$S = \frac{1}{2}\langle u|\mathcal{L}_s u\rangle = \frac{1}{2}\int_b^a u\left[\frac{d}{dt}\left(\mu\dot{u}\right) + \mu\omega^2(t)\right]udt \tag{10.36}$$

where the coefficient $\frac{1}{2}$ comes from the symmetry between $\langle u|\mathcal{L}_s u\rangle = \langle \mathcal{L}_s u|u\rangle$

and

$$S = \frac{1}{2}\langle \mathcal{L}_s u|u\rangle = \frac{1}{2}\int_b^a \left[-\mu\dot{u}^2 + \mu\omega^2(t)u^2\right]dt \tag{10.37}$$

$$\begin{aligned} S &= \frac{1}{2}\langle u(t)|\mathcal{L}_s|u(t)\rangle \equiv \frac{1}{2}\langle u(t)|\mathcal{L}_s u(t)\rangle = \frac{1}{2}\int_b^a u(t)\mathcal{L}_s u(t)dt \\ &= \frac{1}{2}\int_b^a u\left[\frac{d}{dt}\left[e^{\int^t \Gamma(\tau)d\tau}\dot{u}\right] + e^{\int^t \Gamma(\tau)d\tau}\omega^2(t)u\right]dt \end{aligned} \tag{10.38}$$

and then

$$\begin{aligned} \frac{1}{2}\langle \mathcal{L}_s u(t)|u(t)\rangle &= \frac{1}{2}\int_b^a \mathcal{L}u(t)u(t)dt = \frac{1}{2}u\mu\dot{u}|_{t=a}^b \\ &+ \frac{1}{2}\int_b^a \left[-e^{\int^t \Gamma(\tau)}\dot{u}^2 + e^{\int^t \Gamma(\tau)d\tau}\omega^2(t)u^2\right]dt \end{aligned} \tag{10.39}$$

From action (10.39), we obtain the Lagrangian in the following form

$$L = \frac{1}{2} e^{\int^t \Gamma(\tau) d\tau} \dot{u}^2 - \frac{1}{2} \omega^2(t) e^{\int^t \Gamma(\tau) d\tau} u^2 \tag{10.40}$$

and its canonical momenta

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^{\int^t \Gamma(\tau) d\tau} \tag{10.41}$$

and the Hamiltonian for the self adjoint system is

$$H = e^{-\int^t \Gamma(\tau) d\tau} \frac{1}{2} {p_u}^2 + \frac{1}{2} e^{\int^t \Gamma(\tau) d\tau} \omega^2(t) u^2 \tag{10.42}$$

10.1.3. Particular Cases for Self Adjoint Equation

Case I: If $\Gamma(t)$ is constant and the frequency is a generic function of time, then the integration constant becomes $e^{\Gamma t}$ and the Lagrangian equation is reduced to the following form

$$L = \frac{1}{2} e^{\Gamma t} \dot{u}^2 - \frac{1}{2} \omega^2(t) e^{\Gamma t} u^2 \tag{10.43}$$

satisfies the equation of motion

$$\ddot{u} + \Gamma \dot{u} + \omega^2(t) u = 0 \tag{10.44}$$

and its momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^{\Gamma t} \dot{u} \tag{10.45}$$

By using the Legendre transformation, we obtain the Hamiltonian function

$$H = e^{-\Gamma t} \frac{1}{2} {p_u}^2 + \frac{1}{2} e^{\Gamma t} \omega^2(t) u^2 \tag{10.46}$$

Case II: If the frequency ω^2 is constant and the damping coefficient is the generic function of time, the Lagrangian function becomes

$$L = \frac{1}{2} e^{\int^t \Gamma(\tau) d\tau} \dot{u}^2 - \frac{1}{2} \omega^2 e^{\int^t \Gamma(\tau) d\tau} u^2 \tag{10.47}$$

This Lagrangian satisfies the following equation of motion

$$\ddot{u} + \Gamma(t) \dot{u} + \omega^2 u = 0$$

The momentum

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^{\int^t \Gamma(\tau)d\tau}\dot{u} \tag{10.48}$$

and the Hamiltonian is obtained as

$$H = e^{-\int^t \Gamma(\tau)d\tau}\frac{1}{2}{p_u}^2 + \frac{1}{2}e^{\int^t \Gamma(\tau)d\tau}\omega^2 u^2 \tag{10.49}$$

Case III: If the damping coefficient Γ and the frequency $\omega^2(t)$ are constants, the Lagrangian can be written as

$$L = \frac{1}{2}e^{\Gamma t}\dot{u}^2 - \frac{1}{2}\omega^2 e^{\Gamma t}u^2 \tag{10.50}$$

satisfies the equation of motion

$$\ddot{u} + \Gamma\dot{u} + \omega^2 u = 0 \tag{10.51}$$

and its momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^{\Gamma t}\dot{u} \tag{10.52}$$

by using the Legendre transformation, we obtain the Caldirola-Kanai Hamiltonian

$$H = e^{-\Gamma t}\frac{1}{2}{p_u}^2 + \frac{1}{2}e^{\Gamma t}\omega^2 u^2 \tag{10.53}$$

Variational formalism for doubled and self adjoint oscillators is valid for harmonic oscillator which has $\omega^2 > 0$ and for hyperbolic oscillator which has negative frequency $\omega^2 < 0$.

10.2. Oscillator Equation with Three Regular Singular Points

Oscillator representation is just homogeneous degree two structure of solution this equation according to Frobenius theory by singular points of frequency and damping. In this part, we will consider special case, when equation admits three singular points. If singular points are $-1, 1, \infty$, the solution of this equation is known as Gauss Hypergeometric function. When the singular points merge together, the equation becomes confluent hypergeometric form as a particular cases of many special functions such as Bessel, Hermite, Laguerre..., etc. The elements of self adjoint form of these special functions which are the coefficients $p(t)$, $q(t)$, the eigenvalue λ and the weight function $w(t)$ are given in table (10.1) (Arfken and Weber 1995). In this section, we will describe particular forms of damping for frequencies with fixed singularities and corresponding solutions in terms of the special functions. In every case, we will provide the Lagrangian formulation in terms of the doublet oscillator representation and the self adjoint form and then, using this Lagrangian, we will construct Hamiltonian description (Pashaev and Tigrak 2007).

Table 10.1. Coefficients and parameters of the special functions.

Equation	$p(t)$	$q(t)$	λ	$w(t)$
Legendre	$(1-t^2)$	0	l(l+1)	1
Shifted Legendre	$t(1-t)$	0	l(l+1)	1
Associated Legendre	$(1-t^2)$	$-m^2/(1-t^2)$	l(l+1)	1
Chebyshev I	$(1-t^2)^{1/2}$	0	n^2	$(1-t^2)^{-1/2}$
Shifted Chebyshev I	$[t(1-t)]^{1/2}$	0	n^2	$[t(1-t)]^{-1/2}$
Chebyshev II	$(1-t^2)^{3/2}$	0	$n(n+2)$	$(1-t)^{-1/2}$
Ultraspherical (Gegenbauer)	$(1-t^2)^{\alpha+\frac{1}{2}}$	0	$n(n+2\alpha)$	$(1-t^2)^{\alpha-1/2}$
Bessel	t	$-n^2/t$	a^2	t
Laguerre	te^{-t}	0	α	e^{-t}
Associated Laguerre	$t^{k+1}e^{-t}$	0	$\alpha-k$	$t^k e^{-k}$
Hermite	e^{-t^2}	0	2α	e^{-t^2}
Simple Harmonic Oscillator	1	0	n^2	1

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+q(x)u(x)=w(x)\lambda u(x) \tag{10.54}$$

and let us assume that $x\equiv t$. Hence, equation (10.54) becomes

$$p(t)\frac{d^2u(t)}{dt^2}+p'(t)\frac{du}{dt}+q(t)u(t)=w(t)\lambda u(t) \tag{10.55}$$

where $p(t)$, $q(t)$ are the coefficients, λ is the eigenvalue and $w(t)$ is the weighting function. Then

$$\frac{d^2u(t)}{dt^2}+\frac{p'(t)}{p(t)}\frac{du}{dt}+\left(\frac{q(t)}{p(t)}-\frac{w(t)\lambda}{p(t)}\right)u(t)=0 \tag{10.56}$$

Let define new variables as

$$\Gamma \equiv \frac{p'(t)}{p(t)}, \;\; \omega^2 \equiv \left(\frac{q(t)}{p(t)} - \frac{w(t)\lambda}{p(t)}\right) \tag{10.57}$$

In addition to the damping and the frequency terms, we can obtain the integration factor as follows

$$\mu(t) = e^{\int^t \Gamma(\tau)d\tau} = e^{\int^t \frac{p'}{p}d\tau} = p(t) \tag{10.58}$$

Then, equation (10.59) is reduced to the following form

$$\frac{d^2u(t)}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0 \tag{10.59}$$

The second order differential operator which satisfies equation (10.59) is given by

$$\mathcal{L} = \frac{d^2}{dt^2} + \Gamma(t)\frac{d}{dt} + \omega^2(t) \tag{10.60}$$

which is not a self adjoint operator. However by using the integration factor $\mu(t)$, we can construct the self adjoint operator. The following table (10.2), the damping Γ and frequency ω^2 values for the special functions are given. In addition, the interval $[a, b]$ for the special function is defined in table (10.3).

After introducing the new time variable in equation (10.54) given as

$$p(t)\frac{d}{dt} = \frac{d}{d\tau} \tag{10.61}$$

$$\frac{dt}{p(t)} = d\tau \tag{10.62}$$

$$\tau = \tau(t) = \int^t \frac{d\xi}{p(\xi)} \tag{10.63}$$

we obtain the harmonic oscillator equation

$$u'' + \Omega^2(\tau)u = 0 \tag{10.64}$$

where the time dependent frequency $\Omega^2(\tau)$ is

$$\Omega^2(t(\tau)) = p(t(\tau))\left[q(t(\tau)) - \lambda w(t(\tau))\right] \tag{10.65}$$

The new time variable can be written in terms of dissipation coefficient function Γ as follows

$$\tau = \int^t e^{\int^\xi \Gamma(\eta)d\eta} d\xi \tag{10.66}$$

Table 10.2. Dampings and frequencies for the special functions.

Equation	Damping $\Gamma(t)$	Frequency $\omega^2(t)$
Hypergeometric	$\frac{[c-(a+b+1)t]}{t(1-t)}$	$-\frac{ab}{t(1-t)}$
Confluent Hypergeometric	$\frac{(c-t)}{t}$	$-\frac{a}{t}$
Legendre	$-\frac{2t}{1-t^2}$	$-\frac{l(l+1)}{1-t^2}$
Shifted-Legendre	$\frac{1-2t}{t(1-t)}$	$-\frac{l(l+1)}{t(1-t)}$
Associated Legendre	$-\frac{1}{1-t^2}\left[2t+m^2\right]$	$-\frac{l(l+1)}{1-t^2}$
Chebyshev I	$-\frac{t}{1-t^2}$	$-\frac{n^2}{1-t^2}$
Shifted Chebyshev I	$\frac{1}{2}\frac{1-2t}{t(1-t)}$	$-\frac{n^2}{t(1-t)}$
Chebyshev II	$-\frac{3t}{1-t^2}$	$-\frac{n(n+2)}{1-t^2}$
Ultraspherical (Gegenbauer)	$-\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}$	$-\frac{n(n+2\alpha)}{(1-t^2)}$
Bessel	$\frac{1}{t}$	$-\left[\frac{n^2}{t^2}+a^2\right]$
Laguerre	$-\frac{2(1-t)}{t}$	$-\frac{\alpha}{t}$
Associated Laguerre	$[(k+1)t^{-1}-1]$	$-(\alpha-k)t^{-1}$
Hermite	$-2t$	-2α

Table 10.3. Intervals $[a, b]$ of the special functions.

Equation	a	b
Hypergeometric	0	1
Confluent Hypergeometric	0	∞
Legendre	-1	1
Shifted-Legendre	0	1
Associated Legendre	-1	1
Chebyshev I	-1	1
Shifted Chebyshev I	0	1
Chebyshev II	-1	1
Ultraspherical (Gegenbauer)	-1	1
Bessel	0	∞
Laguerre	0	∞
Associated Laguerre	0	∞
Hermite	$-\infty$	∞

This means that by choosing proper time variable τ damped oscillator equation (10.59) can be transformed into (10.64). Here we like to stress in general oscillatory representation of Sturm Liouville problem, the frequencies $\omega^2(t)$ and $\Omega^2(\tau)$ could change sign at some values of time. It means that in some time intervals our harmonic oscillator may become a hyperbolic and unstable oscillator. It would be interesting to analyze cosmological models in such type of oscillatory representations. Because in such models we can observe different behaviors: for positive frequency -oscillatory character and for negative frequency-hyperbolic dissipative character. According to this, we can get different dynamics of universe which include both possibilities. Below we give Lagrangian and Hamiltonian descriptions for the main equations of special functions of mathematical physics in oscillatory representation. In next chapter we will discuss related nonlinear Riccati equations and the quantization of harmonic oscillators with general time dependent frequency when ω^2 is positive. They are characterized by quasi discreet energy spectrum with wave functions written in terms of Hermite polynomials with time dependent argument. Quantization of hyperbolic unstable oscillator, when ω^2 is negative and no bound state or quasi discreet state exist, still is an unsolved problem.

10.2.1. Hypergeometric Functions

The Hypergeometric differential equation

$$t\,(1-t)\,u''(t)+\left[c-(a+b+1)\,t\right]u'(t)-abu(t)=0 \tag{10.67}$$

was introduced as a canonical form of a linear second-order differential equation with regular singularities at $t=0,1$ and ∞ and let divide equation (10.67) by $t\,(1-t)$. Hence

$$u''(t)+\frac{\left[c-(a+b+1)\,t\right]}{t\,(1-t)}u'(t)-\frac{ab}{t\,(1-t)}u(t)=0 \tag{10.68}$$

The range of convergence $|t|<1$ and $t=1$, for $c>a+b$, and $t=-1$, for $c>a+b-1$. We assume that $\Gamma\equiv\frac{[c-(a+b+1)t]}{t(1-t)}$ and $\omega^2\equiv-\frac{ab}{t(1-t)}$ and then equation (10.68) becomes

$$u''(t)+\Gamma u'(t)+\omega^2 u(t)=0 \tag{10.69}$$

which does not have a self adjoint operator. For this non self adjoint operator, we obtain the two equations which are (10.17) and (10.18) as the doublet oscillator representation

$$\ddot{u}+\Gamma\dot{u}+\omega^2 u=0$$

$$\ddot{v}-\Gamma(t)\dot{v}+\left(\omega^2(t)-\dot{\Gamma}\right)v=0$$

Let substitute the damping Γ and the frequency ω^2 to get the doublet oscillator representation of the Hypergeometric equation as

$$\ddot{u}+\frac{\left[c-(a+b+1)\,t\right]}{t\,(1-t)}\dot{u}-\frac{ab}{t\,(1-t)}u=0 \tag{10.70}$$

$$\begin{aligned}\ddot{v}\quad &-\quad \frac{\left[c-(a+b+1)\,t\right]}{t\,(1-t)}\dot{v}-\frac{ab}{t\,(1-t)}v\\ &-\quad \left[\frac{a+b+1}{t(1-t)}+\frac{(1-2t)\left[c-(a+b+1)\,t\right]}{t^2(1-t)^2}\right]v=0\end{aligned} \tag{10.71}$$

The solution of oscillator (10.70) is

$$\begin{aligned}u(t)\quad &=\quad {}_2F_1(a,b,c;t)=1+\frac{ab}{c}\frac{t}{1!}+\frac{a\,(a+1)\,b\,(b+1)}{c\,(c+1)}\frac{t^2}{2!}+...=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!},\\ &\quad c\neq 0,-1,-2,-3...\end{aligned} \tag{10.72}$$

which is known as the hypergeometric series and $(a)_n$ is Pochhammer symbol.

Definition 10.3 *The Pochhammer symbol is defined as*

$$\begin{aligned}(a)_n &= a(a+1)(a+2)...(a+n-1) = \frac{(a+n-1)!}{(a-1)!} \\ (a)_0 &= 1\end{aligned} \tag{10.73}$$

We notice that Lagrangian function is to corresponding equations (10.70) and (10.71) is

$$\begin{aligned}L &= \dot{u}\dot{v} - \frac{[c-(a+b+1)\,t]}{t\,(1-t)}\frac{1}{2}\,(v\dot{u}-\dot{v}u) \\ &+ \left(\frac{ab}{t\,(1-t)} - \frac{1}{2}\left[\frac{a+b+1}{t(1-t)} + \frac{(1-2t)\,[c-(a+b+1)\,t]}{t^2(1-t)^2}\right]\right)uv\end{aligned}$$

and the momenta in terms of u and v are

$$\begin{aligned}p_u &= \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\frac{[c-(a+b+1)\,t]}{t\,(1-t)}v \\ p_v &= \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\frac{[c-(a+b+1)\,t]}{t\,(1-t)}u\end{aligned} \tag{10.74}$$

and finally, we can write the Hamiltonian function for non self adjoint form as

$$\begin{aligned}H &= p_u p_v + \frac{1}{2}\frac{[c-(a+b+1)\,t]}{t\,(1-t)}\,(vp_v - up_u) \\ &- \left[\frac{ab}{t\,(1-t)} - \frac{1}{4}\left[\frac{[c-(a+b+1)\,t]}{t\,(1-t)}\right]^2\right]uv \\ &+ \frac{1}{2}\left[\frac{a+b+1}{t(1-t)} + \frac{(1-2t)\,[c-(a+b+1)\,t]}{t^2(1-t)^2}\right]uv\end{aligned} \tag{10.75}$$

It is possible to get self adjoint form of equation (10.69) by multiplying it with an integration factor. Hence, the integration factor of equation (10.69) to transform it into the self adjoint form is

$$\mu(t) = e^{\int^t \frac{[c-(a+b+1)\tau]}{\tau(1-\tau)}d\tau} = \frac{t^c}{(1-t)^{a+b+c+1}} \tag{10.76}$$

The Lagrangian for self adjoint case is defined as

$$L = \frac{1}{2}e^{\int^t \frac{[c-(a+b+1)\tau]}{\tau(1-\tau)}d\tau}\dot{u}^2 - \frac{1}{2}e^{\int^t \frac{[c-(a+b+1)\tau]}{\tau(1-\tau)}d\tau}\omega^2(t)u^2$$

and

$$L = \frac{1}{2}\frac{t^c}{(1-t)^{a+b+c+1}}\dot{u}^2 + \frac{1}{2}\frac{abt^{c-1}}{(1-t)^{a+b+c+2}}u^2 \tag{10.77}$$

and its momentum

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{t^c}{(1-t)^{a+b+c+1}}\dot{u} \tag{10.78}$$

and the Hamiltonian is given in the following form

$$H = e^{-\int^{t} \frac{[c-(a+b+1)\tau]}{\tau(1-\tau)} d\tau} \frac{1}{2} {p_u}^2 + \frac{1}{2} e^{\int^{t} \frac{[c-(a+b+1)\tau]}{\tau(1-\tau)} d\tau} \omega^2 u^2 \tag{10.79}$$

where ω^2 was defined as $\omega^2 \equiv -\frac{ab}{t(1-t)}$ and then

$$H = \frac{1}{2t^c} (1-t)^{a+b+c+1} {p_u}^2 - \frac{1}{2} \frac{t^c}{(1-t)^{a+b+c+1}} \frac{ab}{t\,(1-t)} u^2 \tag{10.80}$$

10.2.2. Confluent Hypergeometric Function

The confluent hypergeometric equation is

$$t\ddot{u}(t) + (c-t)\,\dot{u}(t) - au(t) = 0 \tag{10.81}$$

may be obtained from the hypergeometric equation by merging two of its singularities. The relating equation has a regular singularity at $t = 0$ and an irregular one at $t = \infty$. After dividing equation (10.81) by t

$$\ddot{u}(t) + \frac{(c-t)}{t} \dot{u}(t) - \frac{a}{t} u(t) = 0 \tag{10.82}$$

After re-arranging equation (11.8), we obtain the damped harmonic oscillator equation with time dependent frequency and damping

$$\ddot{u}(t) + \Gamma \dot{u}(t) + \omega^2 u(t) = 0 \tag{10.83}$$

We choose the damping term as $\Gamma \equiv \frac{(c-t)}{t}$ and the frequency as $\omega^2 \equiv -\frac{a}{t}$. Equation (10.83) is not self adjoint. Then, we form the doublet oscillator representation of this equation as

$$\ddot{u} + \frac{(c-t)}{t} \dot{u} - \frac{a}{t} u = 0 \tag{10.84}$$

$$\ddot{v} - \frac{(c-t)}{t} \dot{v} + \left(-\frac{a}{t} + \frac{c}{t^2} \right) v = 0 \tag{10.85}$$

We know the solution of oscillator equation (10.84) which is called Kummer's differential equation. It has a regular singular point at 0 and an irregular singularity at ∞. The solutions

$$u(a,c,t) = M(a,c,t) = A_1F_1(a;c;t) + BU(a,c,t) \tag{10.86}$$

where ${}_1F_1(a;c;t)$ is the first and $U(a,c,t)$ is the second kind confluent hypergeometric functions.

Definition 10.4 *The confluent hypergeometric function of the first kind* ${}_1F_1(a;c;t)$ *has a hypergeometric series given by*

$$ {}_1F_1(a;c;t) = 1 + \frac{a}{c}t + \frac{a(a+1)}{b(b+1)}\frac{t^2}{2!} + ... = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{t^k}{k!} \tag{10.87} $$

where $(a)_k$ *and* $(b)_k$ *are Pochhammer symbols.*

Definition 10.5 *The confluent hypergeometric function of the second kind* $U(a,c,t)$ *has a hypergeometric series given by*

$$ U(a,c,t) = \pi \csc(\pi c) \left[\frac{{}_1F_1(a;c;t)}{\Gamma(a-c+1)} - \frac{t^{1-c}{}_1F_1(a-c+1;2-c;t)}{\Gamma(a)} \right] \tag{10.88} $$

where Γ *is gamma function.*

Definition 10.6 *The gamma function* $\Gamma(n)$ *is defined to be an extension of the factorial to complex and real number arguments. It is related to the factorial by*

$$ \Gamma(n) = (n-1)! \tag{10.89} $$

The following Lagrangian function is corresponding to equations (10.84) and (10.85)

$$ L = \dot{u}\dot{v} - \frac{1}{2}\frac{(c-t)}{t}(v\dot{u} - \dot{v}u) - \left(-\frac{a}{t} + \frac{1}{2}\frac{c}{t^2} \right) uv \tag{10.90} $$

and the corresponding momenta are

$$ p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\frac{(c-t)}{t}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\frac{(c-t)}{t}u \tag{10.91} $$

and finally, we can write the Hamiltonian function as

$$ \begin{aligned} H &= p_u p_v + \frac{(c-t)}{2t}(vp_v - up_u) \\ &+ \left(-\frac{a}{t} - \frac{1}{4}\left[\frac{(c-t)}{t}\right]^2 + \frac{1}{2}\frac{c}{t^2} \right) uv \end{aligned} \tag{10.92} $$

Subsequently, to construct the self adjoint differential operator from equation (11.8), we obtain the integration factor as

$$ \mu(t) = e^{\int^t \frac{(c-\tau)}{\tau} d\tau} = t^c e^{-t} \tag{10.93} $$

The Lagrangian of the damped harmonic oscillator is defined as

$$ L = \frac{1}{2}t^c e^{-t}\dot{u}^2 - \frac{1}{2}t^c e^{-t}\omega^2(t)u^2 \tag{10.94} $$

The corresponding momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = t^c e^{-t} \dot{u} \tag{10.95}$$

and the Hamiltonian is given in the following form

$$H = \frac{1}{2t^c} e^t {p_u}^2 - \frac{1}{2} t^c e^{-t} \frac{a}{t} u^2 \tag{10.96}$$

10.2.3. Bessel Equation

$p(t) = t,\ q(t) = -\frac{n^2}{t},\ \lambda = a^2,\ \omega(t) = t$ insert into (10.55),

$$t \frac{d^2 u}{dt^2} + \frac{du}{dt} - \frac{n^2}{t} u(t) = t a^2 u(t) \tag{10.97}$$

After dividing equation (10.97) by t, we obtain

$$\frac{d^2 u}{dt^2} + \frac{1}{t} \frac{du}{dt} - \left[\frac{n^2}{t^2} + a^2 \right] u(t) = 0 \tag{10.98}$$

Then, equation (10.98) is reduced to the following form

$$\frac{d^2 u}{dt^2} + \Gamma(t) \frac{du}{dt} + \omega^2(t) u(t) = 0 \tag{10.99}$$

where we assume that the damping $\Gamma \equiv \frac{1}{t}$ and the frequency $\omega^2 \equiv -\left[\frac{n^2}{t^2} + a^2 \right]$. By using this non self adjoint equation, we can construct the Lagrangian function for double oscillator. In the light of the following equations

$$\ddot{u} + \frac{1}{t} \dot{u} - \left[\frac{n^2}{t^2} + a^2 \right] u = 0 \tag{10.100}$$

$$\ddot{v} - \frac{1}{t} \dot{v} - \left(\left[\frac{n^2}{t^2} + a^2 \right] - \frac{1}{t^2} \right) v = 0 \tag{10.101}$$

The solution of equation (10.100) is

$$u(t) = A J_n(t) + B N_n(t) \tag{10.102}$$

where J_n is the first kind Bessel and N_n is the second kind Bessel function. A Bessel function of the third kind called a Hankel function is a special combination of the first and second kinds.

Definition 10.7 *J_n is called Bessel function of the first kind of order n defined by*

$$\begin{aligned} J_n(t) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{t}{2} \right)^{n+2s} = \frac{t^n}{2^n n!} - \frac{t^{n+2}}{2^{n+2}(n+1)!} + ... \\ J_{-n}(t) &= (-1)^n J_n(t) \end{aligned} \tag{10.103}$$

Definition 10.8 *N_n is called Bessel function of the second kind or the Neumann function can be defined by linear combination of $J_{-n}(t)$ and $J_n(t)$*

$$N_n(t) = \frac{\cos n\pi J_n(t) - J_{-n}(t)}{\sin n\pi} \tag{10.104}$$

We obtain the Lagrangian function by substituting the damping and the frequency terms in the following form

$$L = \dot{u}\dot{v} - \frac{1}{2}\frac{1}{t}\left(v\dot{u} - \dot{v}u\right) + \left[\frac{1}{t^2}\left(n^2 - \frac{1}{2} + a^2\right)\right] uv \tag{10.105}$$

the momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2t}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2t}u \tag{10.106}$$

and by using the Legendre transformation, we find the Hamiltonian as

$$\begin{aligned} H &= p_u p_v + \frac{1}{2t^2}\left(vp_v - up_u\right) \\ &- \left(\left[\frac{n^2}{t^2} + a^2\right] - \frac{1}{4t^2}\right) uv \end{aligned} \tag{10.107}$$

It is possible to get self adjoint form of equation (10.69) by multiplying it with an integration factor. Hence, the integration factor of equation (10.69) to transform it into the self adjoint form is defined as

$$\mu(t) = e^{\int^t \frac{1}{\tau} d\tau} = t \tag{10.108}$$

The Lagrangian of the Bessel function for self adjoint case is defined as

$$L = \frac{1}{2}t\dot{u}^2 + \frac{1}{2}t\left[\frac{n^2}{t^2} + a^2\right] u^2 \tag{10.109}$$

the momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = t\dot{u} \tag{10.110}$$

and the Hamiltonian is

$$H = \frac{1}{2t}{p_u}^2 - \frac{1}{2}t\left[\frac{n^2}{t^2} + a^2\right] u^2 \tag{10.111}$$

10.2.4. Legendre Equation

$p(t) = 1 - t^2,\ q(t) = 0,\ \lambda = l(l+1),\ \omega(t) = 1$ insert into (10.55),

$$(1-t^2)\frac{d^2u}{dt^2} - 2t\frac{du}{dt} - l(l+1)u(t) = 0 \tag{10.112}$$

multiply equation (10.112) with $1/(1-t^2)$ and it becomes

$$\frac{d^2u}{dt^2} - \frac{2t}{1-t^2}\frac{du}{dt} - \frac{l(l+1)}{1-t^2}u(t) = 0 \tag{10.113}$$

We can re-write equation (11.48) as damped harmonic oscillator in the following form

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where we assume that the damping $\Gamma \equiv -\frac{2t}{1-t^2}$ and the frequency $\omega^2 \equiv -\frac{l(l+1)}{1-t^2}$ are time dependent.

We can easily see this equation has non self adjoint form. Hence, the doublet oscillator representation of the Legendre equation can be obtained as follows

$$\ddot{u} - \frac{2t}{1-t^2}\dot{u} - \frac{l(l+1)}{1-t^2}u = 0 \tag{10.114}$$

$$\ddot{v} + \frac{2t}{1-t^2}\dot{v} - \left(\frac{l(l+1)}{1-t^2} + \frac{2(1+t^2)}{(1-t^2)^2}\right)v = 0 \tag{10.115}$$

The solution of equation (10.114) gives the Legendre series which is defined by

$$u(t) = P_l(t) = \sum_{k=0}^{[l/2]} (-1)^l \frac{(2l-2k)!}{2^l k!\,(l-k)!\,(l-2k)!}t^{l-2k} \tag{10.116}$$

For l even, P_l has only even powers of t and even parity, and odd powers and odd parity for l.

The following Lagrangian function corresponds to equations (10.114) and (10.115)

$$L = \dot{u}\dot{v} + \frac{1}{2}\frac{2t}{1-t^2}\left(v\dot{u} - \dot{v}u\right) - \left(-\frac{l(l+1)}{1-t^2} + \frac{1}{2}\frac{2(1+t^2)}{(1-t^2)^2}\right)uv \tag{10.117}$$

The corresponding momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} + \frac{1}{2}\frac{2t}{1-t^2}v,\ \ p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} - \frac{1}{2}\frac{2t}{1-t^2}u \tag{10.118}$$

and we can write the Hamiltonian function as

$$\begin{aligned} H &= p_u p_v - \frac{1}{2}\frac{2t}{1-t^2}\left(vp_v - up_u\right) \\ &+ \left(-\frac{l(l+1)}{1-t^2} - \frac{1}{4}\left[-\frac{2t}{1-t^2}\right]^2 + \frac{1}{2}\frac{2(1+t^2)}{(1-t^2)^2}\right)uv \end{aligned} \tag{10.119}$$

Moreover, we can define the integration factor to get the self adjoint form of equation (11.48) as

$$\mu(t) = e^{\int^t -\frac{2\tau}{1-\tau^2}d\tau} = \left(1-t^2\right) \tag{10.120}$$

The Lagrangian equation of the oscillator of the Legendre function for self adjoint case is

$$L = \frac{1}{2}\left(1-t^2\right)\dot{u}^2 + \frac{1}{2}l(l+1)u^2 \tag{10.121}$$

and its momentum is defined as

$$p_u = \frac{\partial L}{\partial \dot{u}} = \left(1-t^2\right)\dot{u} \tag{10.122}$$

and the Hamiltonian is

$$H = \frac{1}{2\left(1-t^2\right)}{p_u}^2 - \frac{1}{2}l(l+1)u^2 \tag{10.123}$$

10.2.5. Shifted-Legendre Equation

$p(t) = t(1-t),\ q(t) = 0,\ \lambda = l(l+1),\ \omega(t) = 1$ insert into (10.55),

$$t(1-t)\frac{d^2u}{dt^2} + (1-2t)\frac{du}{dt} - l(l+1)u(t) = 0 \tag{10.124}$$

Dividing equation (10.125) by $t(1-t)$, we obtain

$$\frac{d^2u}{dt^2} + \frac{1-2t}{t(1-t)}\frac{du}{dt} - \frac{l(l+1)}{t(1-t)}u(t) = 0 \tag{10.125}$$

If we write equation (10.125) as damping oscillator

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where we assume that the damping $\Gamma \equiv \frac{1-2t}{t(1-t)}$ and the frequency $\omega^2 \equiv -\frac{l(l+1)}{t(1-t)}$ are time dependent. Due to existing of the non self adjoint operator, we can construct the doublet oscillator representation of the shifted Legendre equation can be obtained as follows

$$\ddot{u} + \frac{1-2t}{t(1-t)}\dot{u} - \frac{l(l+1)}{t(1-t)}u = 0 \tag{10.126}$$

$$\ddot{v} - \frac{1-2t}{t(1-t)}\dot{v} + \left(-\frac{l(l+1)}{t(1-t)} + \left[\frac{2}{t(1-t)} + \frac{(1-2t)^2}{t^2(1-t)^2}\right]\right)v = 0 \tag{10.127}$$

The solution of oscillator equation (10.126) gives the shifted Legendre series.

The following Lagrangian function is corresponding equations (10.17) and (10.18)

$$\begin{aligned} L = \dot{u}\dot{v} \quad &- \quad \frac{1}{2}\frac{1-2t}{t(1-t)}\left(v\dot{u} - \dot{v}u\right) \\ &- \quad \left(-\frac{l(l+1)}{t(1-t)} + \frac{1}{2}\left[\frac{2}{t(1-t)} + \frac{(1-2t)^2}{t^2(1-t)^2}\right]\right)uv \end{aligned} \quad (10.128)$$

the momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\frac{1-2t}{t(1-t)}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\frac{1-2t}{t(1-t)}u \quad (10.129)$$

and the Hamiltonian function is

$$\begin{aligned} H \;\; &= \;\; p_u p_v + \frac{1}{2}\frac{1-2t}{t(1-t)}\left(vp_v - up_u\right) \\ &- \;\; \left[\frac{l(l+1)}{t(1-t)} + \frac{1}{4}\left[\frac{1-2t}{t(1-t)}\right]^2\right]uv \\ &+ \;\; \frac{1}{2}\left[\frac{2}{t(1-t)} + \frac{(1-2t)^2}{t^2(1-t)^2}\right]uv \end{aligned} \quad (10.130)$$

In addition to this doublet oscillator representation, we obtain the integration factor of equation (10.125) to construct the self adjoint form of the oscillator equation of the shifted Legendre equation as

$$\mu(t) = e^{\int^t \frac{1-2\tau}{\tau(1-\tau)}d\tau} = t\,(1-t) \quad (10.131)$$

The Lagrangian function of the damped harmonic oscillator of the shifted Legendre function for self adjoint case is defined by

$$L = \frac{1}{2}t\,(1-t)\,\dot{u}^2 + \frac{1}{2}l(l+1)tu^2 \quad (10.132)$$

The momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = t\,(1-t)\,\dot{u} \quad (10.133)$$

and the Hamiltonian is defined as

$$H = \frac{1}{2t\,(1-t)}{p_u}^2 - \frac{1}{2}t\,(1-t)\,\frac{l(l+1)}{t(1-t)}u^2 \quad (10.134)$$

10.2.6. Associated-Legendre Equation

$$p(t)=1-t^2,\ q(t)=-\tfrac{m^2}{1-t^2},\ \lambda=l(l+1),\ \omega(t)=1 \text{ insert into (10.55),}$$

$$(1-t^2)\frac{d^2u}{dt^2}-2t\frac{du}{dt}-\frac{m^2}{1-t^2}u-l(l+1)u(t)=0 \tag{10.135}$$

Divide equation (10.135) by $1-t^2$. Hence,

$$\frac{d^2u}{dt^2}-\frac{2t}{1-t^2}\frac{du}{dt}-\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]u(t)=0 \tag{10.136}$$

and

$$\frac{d^2u}{dt^2}+\Gamma(t)\frac{du}{dt}+\omega^2(t)u(t)=0$$

where the damping $\Gamma\equiv-\frac{2t}{1-t^2}$ and the frequency $\omega^2\equiv-\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]$. Then, we can construct the doublet oscillator representation of equation (11.15) as

$$\ddot{u}-\frac{2t}{1-t^2}\dot{u}-\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]u=0 \tag{10.137}$$

$$\ddot{v}+\frac{2t}{1-t^2}\dot{v}+\left(-\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]+\frac{2\left(1+t^2\right)}{\left(1-t^2\right)^2}\right)v=0 \tag{10.138}$$

Equation (10.137) has associated Legendre series.

The following Lagrangian function is corresponding equations (10.137) and (10.138)

$$\begin{aligned} L &= \dot{u}\dot{v}+\frac{1}{2}\frac{2t}{1-t^2}\left(v\dot{u}-\dot{v}u\right)\\ &- \left(-\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]+\frac{\left(1+t^2\right)}{\left(1-t^2\right)^2}\right)uv \end{aligned} \tag{10.139}$$

and the momentum are

$$p_u=\frac{\partial L}{\partial\dot{u}}=\dot{v}+\frac{t}{1-t^2}v,\ \ p_v=\frac{\partial L}{\partial\dot{v}}=\dot{u}-\frac{t}{1-t^2}u \tag{10.140}$$

and in the light of the Legendre transformation, we obtain the Hamiltonian function as

$$\begin{aligned} &H=p_up_v-\frac{t}{1-t^2}\left(vp_v-up_u\right)\\ -&\left(\frac{1}{1-t^2}\left[\frac{m^2}{1-t^2}+l(l+1)\right]+\left[-\frac{t}{1-t^2}\right]^2-\frac{\left(1+t^2\right)}{\left(1-t^2\right)^2}\right)uv \end{aligned} \tag{10.141}$$

Moreover, we can define the integration factor to get the self adjoint form of equation (11.15). The integration factor is defined as

$$\mu(t)=e^{\int^t-\frac{2\tau}{1-\tau^2}d\tau}=\left(1-t^2\right) \tag{10.142}$$

The Lagrangian equation of the damped harmonic oscillator of the associated Legendre function for self adjoint case is

$$L = \frac{1}{2}\left(1-t^2\right)\dot{u}^2 + \frac{1}{2}\left[\frac{m^2}{1-t^2} + l(l+1)\right]u^2 \tag{10.143}$$

Then, the momentum

$$p_u = \frac{\partial L}{\partial \dot{u}} = \left(1-t^2\right)\dot{u} \tag{10.144}$$

and the Hamiltonian is

$$H = \frac{1}{2\left(1-t^2\right)}{p_u}^2 - \frac{1}{2}\frac{l(l+1)}{u^2} \tag{10.145}$$

10.2.7. Hermite Equation

$p(t) = e^{-t^2},\ q(t) = 0,\ \lambda = 2\alpha,\ \omega(t) = e^{-t^2}$ insert into (10.55),

$$e^{-t^2}\frac{d^2u}{dt^2} - 2te^{-t^2}\frac{du}{dt} - 2\alpha e^{-t^2}u(t) = 0 \tag{10.146}$$

multiply equation (10.146) with e^{t^2} and then

$$\frac{d^2u}{dt^2} - 2t\frac{du}{dt} - 2\alpha u(t) = 0 \tag{10.147}$$

The Hermite differential equation has the time dependent damping but frequency is constant.

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where the damping $\Gamma \equiv -2t$ and the frequency $\omega^2 \equiv -2\alpha$. Hence, the doublet oscillator representation can be written for equation (11.21).

$$\ddot{u} - 2t\dot{u} - 2\alpha u = 0 \tag{10.148}$$

$$\ddot{v} + 2t\dot{v} + 2\left(1-\alpha\right)v = 0 \tag{10.149}$$

The following Lagrangian function is corresponding equations (10.148) and (10.149)

$$L = \dot{u}\dot{v} + t\left(v\dot{u} - \dot{v}u\right) - \left(-2\alpha+1\right)uv \tag{10.150}$$

its momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} + tv,\ \ p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} - tu \tag{10.151}$$

and the Hamiltonian function can be written in the following form as

$$H = p_u p_v - t\left(vp_v - up_u\right) + \left(-2\alpha - t^2 + 1\right)uv \tag{10.152}$$

By using integration factor, we can get the self adjoint form of the Hermite differential equation. Hence, the integration factor of equation (11.21) is

$$\mu(t) = e^{\int^t -2\tau d\tau} = e^{-t^2} \tag{10.153}$$

Now, we can write the Lagrangian for the damped oscillator which is generated from the Hermite differential equation as

$$L = \frac{1}{2}e^{-t^2}\dot{u}^2 - \frac{1}{2}e^{-t^2}\omega^2 u^2 \tag{10.154}$$

and its momentum

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^{-t^2}\dot{u} \tag{10.155}$$

and the Hamiltonian which is produced from the Hermite differential equation for self adjoint form is

$$H = e^{t^2}\frac{1}{2}{p_u}^2 - \frac{1}{2}e^{-t^2}\alpha u^2 \tag{10.156}$$

10.2.8. Ultra-Spherical(Gegenbauer)Equation

$p(t) = (1-t^2)^{\alpha+\frac{1}{2}},\ q(t) = 0,\ \lambda = n(n+2\alpha),\ \ \omega(t) = (1-t^2)^{\alpha-\frac{1}{2}}$ insert into (10.55),

$$\begin{aligned}(1-t^2)^{\alpha+\frac{1}{2}}\frac{d^2u}{dt^2} &- 2t(\alpha+\frac{1}{2})(1-t^2)^{\alpha-\frac{1}{2}}\frac{du}{dt} \\ &- n(n+2\alpha)(1-t^2)^{\alpha-\frac{1}{2}}u(t) = 0\end{aligned} \tag{10.157}$$

multiply the above equation with $(1-t^2)^{-\alpha-\frac{1}{2}}$

$$\frac{d^2u}{dt^2} - \frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}\frac{du}{dt} - \frac{n(n+2\alpha)}{(1-t^2)}u(t) = 0 \tag{10.158}$$

let arrange equation (10.158) as damped harmonic oscillator,

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where the damping $\Gamma \equiv -\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}$ and the frequency $\omega^2 \equiv -\frac{n(n+2\alpha)}{(1-t^2)}$ are time dependent. Then, we can construct the doublet oscillator representation of equation (10.158) as

$$\ddot{u} - \frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}\dot{u} - \frac{n(n+2\alpha)}{(1-t^2)}u = 0 \tag{10.159}$$

$$\ddot{v}+\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}\dot{v}+\left(-\frac{n(n+2\alpha)}{(1-t^2)}+2\left(\alpha+\frac{1}{2}\right)\frac{1+t^2}{1-t^{2^2}}\right)v=0 \tag{10.160}$$

The solution of equation (10.159) has the Ultraspherical series.

The following Lagrangian function is corresponding equations (10.159) and (10.160)

$$\begin{aligned} L &= \dot{u}\dot{v}+\frac{1}{2}\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}(v\dot{u}-\dot{v}u) \\ &- \left(-\frac{n(n+2\alpha)}{(1-t^2)}+\left(\alpha+\frac{1}{2}\right)\frac{1+t^2}{1-t^{2^2}}\right)uv \end{aligned} \tag{10.161}$$

and the momenta are

$$p_u=\frac{\partial L}{\partial \dot{u}}=\dot{v}+\frac{1}{2}\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}v,\ \ p_v=\frac{\partial L}{\partial \dot{v}}=\dot{u}-\frac{1}{2}\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}u \tag{10.162}$$

and the Hamiltonian function of the Ultraspherical Gegenbauer differential equation for the non-self adjoint form is

$$\begin{aligned} &H=p_u p_v-\frac{t(\alpha+\frac{1}{2})}{(1-t^2)}(vp_v-up_u)+ \\ &\left(-\frac{n(n+2\alpha)}{(1-t^2)}-\frac{1}{4}\left[-\frac{2t(\alpha+\frac{1}{2})}{(1-t^2)}\right]^2+\left(\alpha+\frac{1}{2}\right)\frac{1+t^2}{1-t^{2^2}}\right)uv \end{aligned} \tag{10.163}$$

On the other hand, by using integration factor, we can get the self adjoint form of the Ultraspherical differential equation. Hence, the integration factor of equation (10.158) is

$$\mu(t)=e^{\int^t -\frac{2\tau(\alpha+\frac{1}{2})}{(1-\tau^2)}d\tau}=\left(1-t^2\right)^{\alpha+\frac{1}{2}} \tag{10.164}$$

The Lagrangian equation of the damped harmonic oscillator produced from the Ultra-Spherical (Gegenbauer) equation is

$$L=\frac{1}{2}\left(1-t^2\right)^{\alpha+\frac{1}{2}}\dot{u}^2+\frac{1}{2}\left(1-t^2\right)^{\alpha-\frac{1}{2}}n(n+2\alpha)u^2 \tag{10.165}$$

The momentum is

$$p_u=\frac{\partial L}{\partial \dot{u}}=\left(1-t^2\right)^{\alpha+\frac{1}{2}}\dot{u} \tag{10.166}$$

and the Hamiltonian is in the following form

$$H=\frac{1}{2(1-t^2)^{\alpha+\frac{1}{2}}}{p_u}^2-\frac{1}{2}\left(1-t^2\right)^{\alpha-\frac{1}{2}}n(n+2\alpha)u^2 \tag{10.167}$$

10.2.9. Laguerre

$p(t) = te^{-t},\ q(t) = 0,\ \lambda = \alpha,\ \omega(t) = e^{-t}$ insert into (10.55),

$$te^{-t}\frac{d^2u}{dt^2} + (1-t)e^{-t}\frac{du}{dt} - \alpha e^{-t}u(t) = 0 \tag{10.168}$$

multiply equation (10.168) with $\frac{e^t}{t}$. Hence, it becomes

$$\frac{d^2u}{dt^2} + \frac{(1-t)}{t}\frac{du}{dt} - \frac{\alpha}{t}u(t) = 0 \tag{10.169}$$

and as oscillator type equation (11.27) becomes

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where the damping $\Gamma \equiv \frac{(1-t)}{t}$ and the frequency $\omega^2 \equiv -\frac{\alpha}{t}$ are time dependent. For this non self adjoint form, we get the double oscillator representation as

$$\ddot{u} + \frac{(1-t)}{t}\dot{u} - \frac{\alpha}{t}u = 0 \tag{10.170}$$

$$\ddot{v} - \frac{(1-t)}{t}\dot{v} + \left(-\frac{\alpha}{t} + \frac{1}{t^2}\right)v = 0 \tag{10.171}$$

Equation (10.170) has power series representation as

$$u(t) = L_\alpha(t) = \sum_{s=0}^{\alpha} \frac{\alpha!}{(\alpha-s)!\,(\alpha-s)!s!} \tag{10.172}$$

The following Lagrangian function is corresponding equations (10.170) and (10.171)

$$L = \dot{u}\dot{v} - \frac{1}{2}\frac{(1-t)}{t}\left(v\dot{u} - \dot{v}u\right) - \left(-\frac{\alpha}{t} + \frac{1}{2}\frac{1}{t^2}\right)uv \tag{10.173}$$

The momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\frac{(1-t)}{t}v,\ \ p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\frac{(1-t)}{t}u \tag{10.174}$$

and by using the Legendre transformation, the Hamiltonian can be written as

$$\begin{aligned} H &= p_u p_v + \frac{1}{2}\Gamma\left(vp_v - up_u\right) \\ &+ \left(-\frac{\alpha}{t} - \frac{1}{4}\left[\frac{(1-t)}{t}\right]^2 + \frac{1}{2}\frac{1}{t^2}\right)uv \end{aligned} \tag{10.175}$$

In addition to the doublet oscillator representation, we can construct the self adjoint form of equation (11.27) by using the integration factor. Hence, the integration factor is

$$\mu(t) = e^{\int^t \frac{(1-\tau)}{\tau}d\tau} = te^{-t} \tag{10.176}$$

The Lagrangian equation of the damped harmonic oscillator produced from the Laguerre equation is given as follows

$$L = \frac{1}{2}te^{-t}\dot{u}^2 - \frac{1}{2}te^{-t}\omega^2 u^2 \tag{10.177}$$

The momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = te^{-t}\dot{u} \tag{10.178}$$

and the Hamiltonian is

$$H = e^{-\int^t \frac{(1-\tau)}{\tau}d\tau}\frac{1}{2}{p_u}^2 + \frac{1}{2}e^{\int^t \frac{(1-\tau)}{\tau}d\tau}\omega^2 u^2 \tag{10.179}$$

where ω^2 is $-\frac{\alpha}{t}$. Hence

$$H = \frac{1}{2t}e^t {p_u}^2 - \frac{1}{2}te^{-t}\frac{\alpha}{t} \tag{10.180}$$

10.2.10. Associated Laguerre Equation

$p(t) = t^{k+1}e^{-t},\ q(t) = 0,\ \ \lambda = \alpha - k,\ \ \omega(t) = t^k e^{-t}$ insert into (10.55),

$$t^{k+1}e^{-t}\frac{d^2u}{dt^2} + \left[k + 1t^{-1}e^{-t} - 1\right]t^{k+1}e^{-t}\frac{du}{dt} - (\alpha - k)t^k e^{-t}u(t) = 0 \tag{10.181}$$

divided by the $t^{k+1}e^{-t}$. Hence, let arrange equation (10.181),

$$\frac{d^2u}{dt^2} + \left[(k+1)t^{-1} - 1\right]\frac{du}{dt} - (\alpha - k)t^{-1}u(t) = 0 \tag{10.182}$$

let rewrite equation (11.33). Then

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where the damping $\Gamma \equiv [(k+1)t^{-1} - 1]$ and the frequency $\omega^2 \equiv -(\alpha - k)t^{-1}$. The doublet oscillator representation of the associated Laguerre differential equation can be represented as

$$\ddot{u} + \left[(k+1)t^{-1} - 1\right]\dot{u} - (\alpha - k)t^{-1}u = 0 \tag{10.183}$$

$$\ddot{v} - \left[(k+1)t^{-1} - 1\right]\dot{v} + \left(-(\alpha - k)t^{-1} - \left[\frac{k+1}{t} - 1\right]\right)v = 0 \tag{10.184}$$

Equation (10.183) has the associated Laguerre series solution.

The following Lagrangian function is corresponding equations (10.183) and (10.184)

$$\begin{aligned} L &= \dot{u}\dot{v} - \frac{1}{2}\left[(k+1)t^{-1} - 1\right](v\dot{u} - \dot{v}u) \\ &- \left(-(\alpha - k)t^{-1} - \frac{1}{2}\left[\frac{k+1}{t} - 1\right]\right)uv \end{aligned} \tag{10.185}$$

$$\begin{aligned} p_u &= \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{2}\left[(k+1)t^{-1} - 1\right]v \\ p_v &= \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{2}\left[(k+1)t^{-1} - 1\right]u \end{aligned} \tag{10.186}$$

and we can write the Hamiltonian function (**??**) as

$$\begin{gathered} H = p_u p_v + \frac{1}{2}\left[(k+1)t^{-1} - 1\right](vp_v - up_u) + \\ \left(-(\alpha - k)t^{-1} - \frac{1}{4}\left[(k+1)t^{-1} - 1\right]^2 - \frac{1}{2}\left[\frac{k+1}{t} - 1\right]\right)uv \end{gathered} \tag{10.187}$$

To transform the non self adjoint form (11.33) into the self adjoint form, we need integration factor which is

$$\mu(t) = e^{\int^t \left[(k+1)\tau^{-1} - 1\right] d\tau} = (k+1)\ln t - t \tag{10.188}$$

The Lagrangian equation for the self adjoint case

$$L = \frac{1}{2}(k+1)\ln t - t\dot{u}^2 - \frac{1}{2}\left[(k+1)\ln t - t\right]\omega^2 u^2 \tag{10.189}$$

The momenta

$$p_u = \frac{\partial L}{\partial \dot{u}} = \left[(k+1)\ln t - t\right]\dot{u} \tag{10.190}$$

and the Hamiltonian is

$$H = \frac{1}{2(k+1)\ln t - 2t}{p_u}^2 + \frac{1}{2}\left[(k+1)\ln t - t\right]\omega^2 u^2 \tag{10.191}$$

where ω^2 is $-(\alpha - k)t^{-1}$.

10.2.11. Chebyshev Equation I

$p(t) = (1-t^2)^{\frac{1}{2}},\ q(t) = 0,\ \lambda = n^2,\ \omega(t) = (1-t^2)^{-\frac{1}{2}}$ insert into (10.55),

$$(1-t^2)^{\frac{1}{2}}\frac{d^2u}{dt^2} - \frac{t}{\sqrt{1-t^2}}\frac{du}{dt} - (1-t^2)^{-\frac{1}{2}}n^2 u(t) = 0 \tag{10.192}$$

Equation (10.192) is divided by $(1-t^2)^{1/2}$ and then we obtain the following equation

$$\frac{d^2u}{dt^2} - \frac{t}{1-t^2}\frac{du}{dt} - \frac{n^2}{1-t^2}u(t) = 0 \tag{10.193}$$

as the damped harmonic oscillator, we can write equation (11.46)

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where we assume that damping $\Gamma \equiv -\frac{t}{1-t^2}$ and the frequency $\omega^2 \equiv -\frac{n^2}{1-t^2}$. Due to the fact that, the damped oscillator equation has non self adjoint operator, it is represented with respect to the doublet oscillator equations as

$$\ddot{u} - \frac{t}{1-t^2}\dot{u} - \frac{n^2}{1-t^2}u = 0 \tag{10.194}$$

$$\ddot{v} + \frac{t}{1-t^2}\dot{v} + \left(-\frac{n^2}{1-t^2} + \frac{1+t^2}{(1-t^2)^2}\right)v = 0 \tag{10.195}$$

The power series representation of the solution of equation (10.194) is

$$u(t) = T_n(t) = \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^n\frac{(n-m-1)!}{m!\,(n-2m)!}(2t)^{n-2m} \tag{10.196}$$

The following Lagrangian function corresponds to equations (10.194) and (10.195)

$$L = \dot{u}\dot{v} + \frac{1}{2}\frac{t}{1-t^2}(v\dot{u} - \dot{v}u) - \left(\omega^2(t) + \frac{1}{2}\frac{1+t^2}{(1-t^2)^2}\right)uv \tag{10.197}$$

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} + \frac{1}{2}\frac{t}{1-t^2}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} - \frac{1}{2}\frac{t}{1-t^2}u \tag{10.198}$$

and finally, we can write the Hamiltonian function as

$$\begin{aligned} H &= p_u p_v - \frac{1}{2}\frac{t}{1-t^2}(vp_v - up_u) + \\ &\left(\omega^2(t) - \frac{1}{4}\left[-\frac{t}{1-t^2}\right]^2 + \frac{1}{2}\frac{1+t^2}{(1-t^2)^2}\right)uv \end{aligned} \tag{10.199}$$

By using integration factor, we can get the self adjoint form of the first kind Chebyshev differential equation. Hence, the integration factor of equation (10.192) is given by

$$\mu(t) = e^{\int^t -\frac{\tau}{1-\tau^2}d\tau} = \left(1-t^2\right)^{-1/2} \tag{10.200}$$

The Lagrangian equation of the oscillator of the first Chebyshev function for the self adjoint case is given as

$$L = \frac{1}{2}\left(1-t^2\right)^{-1/2}\dot{u}^2 - \frac{1}{2}\left(1-t^2\right)^{-1/2}\omega^2u^2 \tag{10.201}$$

The momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = \left(1-t^2\right)^{-1/2}\dot{u} \tag{10.202}$$

and the Hamiltonian is

$$H = \left(1-t^2\right)^{1/2}\frac{1}{2}{p_u}^2 + \frac{1}{2}\left(1-t^2\right)^{-1/2}\omega^2u^2 \tag{10.203}$$

where ω^2 is $-\frac{n^2}{1-t^2}$.

10.2.12. Chebyshev Equation II

$$p(t) = (1-t^2)^{\frac{3}{2}},\ q(t)=0,\ \lambda = n(n+2),\ \omega(t) = (1-t^2)^{\frac{1}{2}} \text{ insert into (10.55),}$$

$$(1-t^2)^{\frac{3}{2}}\frac{d^2u}{dt^2} - 3t(1-t^2)^{\frac{1}{2}}\frac{du}{dt} - (1-t^2)^{\frac{1}{2}}n(n+2)u(t) = 0 \tag{10.204}$$

multiply equation (10.204) with $(1-t^2)^{-\frac{3}{2}}$ and we obtain

$$\frac{d^2u}{dt^2} - \frac{3t}{1-t^2}\frac{du}{dt} - \frac{n(n+2)}{1-t^2}u(t) = 0 \tag{10.205}$$

and we can write equation (11.45) as

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where we assume that damping $\Gamma \equiv -\frac{3t}{1-t^2}$ and the frequency $\omega^2 \equiv -\frac{n(n+2)}{1-t^2}$. Since, this damped harmonic oscillator equation has non self adjoint differential operator, we can get the doublet oscillator representation as

$$\ddot{u} - \frac{3t}{1-t^2}\dot{u} - \frac{n(n+2)}{1-t^2}u = 0 \tag{10.206}$$

$$\ddot{v} + \frac{3t}{1-t^2}\dot{v} + \left(-\frac{n(n+2)}{1-t^2} + 3\frac{1+t^2}{\left(1-t^2\right)^2}\right)v = 0 \tag{10.207}$$

The power series representation of the solution of equation (10.206) is

$$u(t) = U_n(t) = \sum_{m=0}^{[n/2]}(-1)^n\frac{(n-m)!}{m!\,(n-2m)!}(2t)^{n-2m} \tag{10.208}$$

The following Lagrangian function is corresponding to equations (10.206) and (10.207)

$$L = \dot{u}\dot{v} + \frac{1}{2}\frac{3t}{1-t^2}\left(v\dot{u} - \dot{v}u\right) - \left(-\frac{n(n+2)}{1-t^2} + \frac{3}{2}\frac{1+t^2}{\left(1-t^2\right)^2}\right)uv \tag{10.209}$$

The momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} + \frac{1}{2}\frac{3t}{1-t^2}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} - \frac{1}{2}\frac{3t}{1-t^2}u \tag{10.210}$$

We can write the Hamiltonian function (**??**) as

$$H = p_u p_v - \frac{1}{2}\frac{3t}{1-t^2}\left(vp_v - up_u\right) + \left(-\frac{n(n+2)}{1-t^2} - \frac{1}{4}\left[-\frac{3t}{1-t^2}\right]^2 + \frac{3}{2}\frac{1+t^2}{\left(1-t^2\right)^2}\right)uv \tag{10.211}$$

Beside the double oscillator representation, the oscillator equation can be transformed into self adjoint form. To do this, the integration factor for the second kind Chebyshev function is

$$\mu(t) = e^{\int^t -\frac{3\tau}{1-\tau^2}d\tau} = \left(1-t^2\right)^{3/2} \tag{10.212}$$

and the Lagrangian equation is for the self adjoint case is

$$L = \frac{1}{2}\left(1-t^2\right)^{3/2}\dot{u}^2 + \frac{1}{2}\left(1-t^2\right)^{1/2}n(n+2)u^2 \tag{10.213}$$

and its momentum is

$$p_u = \frac{\partial L}{\partial \dot{u}} = \left(1-t^2\right)^{3/2}\dot{u} \tag{10.214}$$

and the Hamiltonian is

$$H = \frac{1}{2\left(1-t^2\right)^{3/2}}{p_u}^2 - \frac{1}{2}\left(1-t^2\right)^{1/2}n(n+2)u^2 \tag{10.215}$$

10.2.13. Shifted Chebyshev Equation I

$p(t) = [t(1-t)]^{\frac{1}{2}}, \;\; q(t) = 0, \; \lambda = n^2, \;\; \omega(t) = [t(1-t)]^{-\frac{1}{2}}$ insert into (10.55),

$$[t(1-t)]^{\frac{1}{2}}\frac{d^2u}{dt^2} + \frac{1}{2}\frac{1-2t}{\sqrt{t(1-t)}}\frac{du}{dt} - [t(1-t)]^{-\frac{1}{2}}n^2u(t) = 0 \tag{10.216}$$

multiply equation (10.216) with $[t(1-t)]^{-\frac{1}{2}}$. Hence

$$\frac{d^2u}{dt^2} + \frac{1}{2}\frac{1-2t}{t(1-t)}\frac{du}{dt} - \frac{n^2}{t(1-t)}u(t) = 0 \tag{10.217}$$

We obtain the damped harmonic oscillator as follows

$$\frac{d^2u}{dt^2} + \Gamma(t)\frac{du}{dt} + \omega^2(t)u(t) = 0$$

where we assume that damping $\Gamma \equiv \frac{1}{2}\frac{1-2t}{t(1-t)}$ and the frequency $\omega^2 \equiv -\frac{n^2}{t(1-t)}$.

$$\ddot{u} + \frac{1}{2}\frac{1-2t}{t(1-t)}\dot{u} - \frac{n^2}{t(1-t)}u = 0 \tag{10.218}$$

$$\ddot{v} - \frac{1}{2}\frac{1-2t}{t(1-t)}\dot{v} + \left(-\frac{n^2}{t(1-t)} + \frac{1}{2}\left[\frac{2}{t\,(1-t)} + \frac{(1-2t)^2}{t^2\,(1-t)^2}\right]\right)v = 0 \tag{10.219}$$

The solution of the oscillator equation (10.218) gives the first kind shifted Chebyshev function as a series solution.

The following Lagrangian function is corresponding equations (10.218) and (10.219)

$$\begin{aligned} L &= \dot{u}\dot{v} - \frac{1}{4}\frac{1-2t}{t(1-t)}\left(v\dot{u} - \dot{v}u\right) \\ &- \left(-\frac{n^2}{t(1-t)} + \frac{1}{8}\left[\frac{2}{t\,(1-t)} + \frac{(1-2t)^2}{t^2\,(1-t)^2}\right]\right)uv \end{aligned} \tag{10.220}$$

The corresponding momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{v} - \frac{1}{4}\frac{1-2t}{t(1-t)}v, \;\; p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} + \frac{1}{4}\frac{1-2t}{t(1-t)}u \tag{10.221}$$

and we can obtain the Hamiltonian function as

$$\begin{aligned} H = p_u p_v + \frac{1-2t}{4t(1-t)}\left(vp_v - up_u\right) - \left(\frac{n^2}{t(1-t)}\right)uv \\ + \left(-\frac{1}{16}\left[\frac{1-2t}{t(1-t)}\right]^2 + \left[\frac{1}{2t\,(1-t)} + \frac{(1-2t)^2}{4t^2\,(1-t)^2}\right]\right)uv \end{aligned} \tag{10.222}$$

Then, the oscillator equation can be transformed into self adjoint form. To do this, the integration factor is

$$\mu(t) = e^{\int^t \frac{1}{2}\frac{1-2\tau}{\tau(1-\tau)}d\tau} = (t(1-t))^{1/2} \tag{10.223}$$

and the Lagrangian equation is for the self adjoint case is

$$L = \frac{1}{2}(t(1-t))^{1/2}\dot{u}^2 + \frac{1}{2}(t(1-t))^{-1/2}n^2u^2 \tag{10.224}$$

The momentum

$$p_u = \frac{\partial L}{\partial \dot{u}} = (t(1-t))^{1/2}\dot{u} \tag{10.225}$$

and the Hamiltonian is in the following form

$$H = \frac{1}{2(t(1-t))^{1/2}}{p_u}^2 - \frac{1}{2}(t(1-t))^{-1/2}n^2u^2 \tag{10.226}$$

CHAPTER 11

RICCATI REPRESENTATION OF TIME DEPENDENT DAMPED OSCILLATORS

In the first part of this thesis, we discussed the cosmological models where the size of the universe satisfies the Riccati equation and that equation was linearized in terms of the Schrödinger problem. In the second part, we studied general second order linear differential equation in oscillator representation and found several solutions in terms of special functions. In the present, we are going to construct the Riccati equations corresponding solutions for oscillatory models discussed in this chapter.

Theorem 11.1 *The damped harmonic oscillator with time dependent parameters was defined as*

$$\ddot{q} + \Gamma(t)\dot{q} + \omega^2(t)q = 0 \tag{11.1}$$

We can construct the Riccati equation as

$$\dot{\eta} + \eta^2 + \Gamma(t)\eta + \omega^2(t) = 0 \tag{11.2}$$

where $\eta = \frac{\dot{q}}{q}$.

Proof To construct the Riccati equation from the time dependent harmonic oscillator equation, we can change the variables as $q = e^{\ln q}$ in equation (11.1). The first and the second derivative of $q = e^{\ln q}$ are given as in the following equations

$$\begin{aligned} \frac{dq}{dt} &= (\ln q)_t e^{\ln q} \\ \frac{d^2q}{dt^2} &= \{(\ln q)_{tt} + [(\ln q)_t]^2\} e^{\ln q} \end{aligned} \tag{11.3}$$

After substituting the first and the second derivatives in the time dependent harmonic oscillator equation (11.3), we obtain

$$\{(\ln q)_{tt} + [(\ln q)_t]^2 + \Gamma(t)(\ln q)_t + \omega^2(t)\} e^{\ln q} = 0 \tag{11.4}$$

If we choose $\frac{\dot{q}}{q} = (\ln q)_t \equiv \eta$, we get the Riccati equation as

$$\dot{\eta} + \eta^2 + \Gamma(t)\eta + \omega^2(t) = 0$$

As a consequence, equation (11.2) is explicitly soluble system in which the damping and frequency terms are given in Table (10.2). Besides, we know q or $\frac{\dot{q}}{q} \equiv \eta$ values for every particular cases for example, the solution to the oscillator equation represented by Hermite differential equation is Hermite polynomials. In addition to this result, every zeros of $\frac{\dot{q}}{q}$ determine pole of singularity of η.

Definition 11.1 *A point z_0 is called a zero of order m for the function f if f is analytic at z_0 and f and its first $m-1$ derivatives vanish at z_0, but $f^{(m)}(z_0) \neq 0$ (Saff and Snider, 2003).*

In other words, we have

$$f(z_0) \;\; = \;\; f'(z_0) = f''(z_0) = ... = f^{(m-1)}(z_0) = 0 \neq f^{(m)}(z_0)$$

In this case the Taylor series for f around z_0 takes the form

$$f(z) \;\; = \;\; a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + a_{m+2}(z - z_0)^{m+2} + ...$$

or

$$f(z) \;\; = \;\; (z - z_0)^m \left[a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + ...\right] \qquad (11.5)$$

where $a_m = \frac{f^m(z_0)}{m!} \neq 0$.

Theorem 11.2 *Let f be analytic at z_0. Then f has a zero of order m at z_0 if and only if f can be written as*

$$f(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$ (Saff and Snider, 2003).

Lemma 11.3 *A function f has a pole of order m at z_0 if and only if in some punctured neighborhood of z_0*

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is analytic at z_0 and $g(z_0) \neq 0$ (Saff and Snider, 2003).

Lemma 11.4 *If f has a zero of order m at z_0, then $\frac{1}{f}$ has a pole of order m at z_0 (Saff and Snider, 2003).*

Proposition 11.1 *Every zero of solution of (11.1) corresponds to the poles of Riccati equation (11.2).*

Example, let $t = t_0$ which is simple zero and let $q(t)$ is analytic function and $q(t) = (t - t_0)f(t)$ where $f(t_0) \neq 0$. Then,

$$\eta = \frac{f(t) + (t - t_0)f'}{(t - t_0)f(t)} = \frac{g(t)}{(t - t_0)} \tag{11.6}$$

In this chapter, we are going to investigate this result with respect to the special functions.

11.1. Hypergeometric Equation

The oscillator representation of the Hypergeometric function is defined as

$$\ddot{q} + \frac{[c - (a + b + 1)\,t]}{t\,(1 - t)}\dot{q} - \frac{ab}{t\,(1 - t)}q = 0$$

and the Riccati representation of the oscillator equation becomes

$$\dot{\eta} + \eta^2 + \frac{[c - (a + b + 1)\,t]}{t\,(1 - t)}\eta - \frac{ab}{t\,(1 - t)} = 0 \tag{11.7}$$

11.2. Confluent Hypergeometric Equation

The damped harmonic oscillator form of the confluent hypergeometric function is written as

$$\ddot{q} + \frac{(c - t)}{t}\dot{q} - \frac{a}{t}q = 0$$

Hence, the Riccati representation can be written as

$$\dot{\eta} + \eta^2 + \frac{(c - t)}{t}\eta - \frac{a}{t} = 0 \tag{11.8}$$

All special functions can be represented as hypergeometric or confluent hyper geometric form. In the following parts, we will discuss the zeros and poles relations between oscillator type and the Riccati type of oscillations for some particular cases as examples. All examples given here have zeros related to the degree of the polynomials. As a result of this, the Riccati representations have poles related to the degree of the polynomials.

11.3. Legendre Equation

The oscillator representation of Legendre equation is defined as

$$\ddot{q} - \frac{2t}{1-t^2}\dot{q} - \frac{l(l+1)}{1-t^2}q = 0$$

and the Riccati representation of the oscillator is written as

$$\dot{\eta} + \eta^2 - \frac{2t}{1-t^2}\eta - \frac{l(l+1)}{1-t^2} = 0 \tag{11.9}$$

The Legendre polynomials, sometimes called Legendre functions of the first kind and its recurrence relation is given by

$$P_l'(t) = \frac{l}{1-t^2}P_{l-1} - \frac{lt}{1-t^2}P_l \tag{11.10}$$

(Arfken and Weber 1995) Then, $\eta = \frac{\dot{q}}{q}$ becomes

$$\eta = \frac{\dot{q}}{q} = \frac{P_l'(t)}{P_l(t)} = \frac{l}{1-t^2}\left[\frac{P_{l-1}}{P_l} - t\right] \tag{11.11}$$

The first few Legendre polynomials are

$$\begin{aligned} P_0 &= 1 \\ P_1 &= t \\ P_2 &= \frac{1}{2}\left(3t^2-1\right) \\ P_3 &= \frac{1}{2}\left(5t^3-3t\right) \\ P_4 &= \frac{1}{8}\left(35t^4-30t^2+3\right) \end{aligned} \tag{11.12}$$

For example,

$$P_1 = t, \;\; \eta = \frac{\dot{P}_1}{P_1} = \frac{1}{t} \tag{11.13}$$

where $t = 0$ is zero of P_1 but the pole of η in other words, the pole of the Riccati equation.

$$P_2 = \frac{1}{2}\left(3t^2-1\right), \;\; \eta = \frac{\dot{P}_2}{P_2} = \frac{6t}{3t^2-1} \tag{11.14}$$

where $t = \pm\sqrt{\frac{1}{3}}$ are zeros of P_2 at the same time, they are two poles of the Riccati equation. As a result of these examples, the lth Legendre's polynomial has l zeros and the Riccati equation of the Legendre oscillator has l poles.

11.4. Associated-Legendre Equation

The associated Legendre oscillator equation is

$$\ddot{q} - \frac{2t}{1-t^2}\dot{q} - \frac{1}{1-t^2}\left[\frac{m^2}{1-t^2} + l(l+1)\right]q = 0$$

Hence, under $\eta = \frac{\dot{q}}{q}$ transformation, it becomes

$$\dot{\eta} + \eta^2 - \frac{2t}{1-t^2}\eta - \frac{1}{1-t^2}\left[\frac{m^2}{1-t^2} + l(l+1)\right] = 0 \qquad (11.15)$$

Associated Legendre's polynomials satisfy the following recurrence relation

$$P_l{}^{m\prime}(t) = \frac{1}{\sqrt{1-t^2}}P_l{}^{m+1}(t) - \frac{mt}{1-t^2}P_l{}^m(t) \qquad (11.16)$$

where $-l \leq m \leq l$ (Arfken and Weber 1995). For example, if $l = 1$, $m = -1, 0, 1$. Hence, the $\eta = \frac{\dot{q}}{q}$ becomes

$$\eta = \frac{\dot{q}}{q} = \frac{P_l{}^{m\prime}(t)}{P_l{}^m(t)} = \frac{1}{\sqrt{1-t^2}}\frac{P_l{}^{m+1}}{P_l{}^m} - \frac{mt}{1-t^2} \qquad (11.17)$$

and the first few associated Legendre polynomials are

$$\begin{aligned} P_1{}^1(t) &= \sqrt{1-t^2} \\ P_2{}^1(t) &= 3t\sqrt{1-t^2} \\ P_2{}^2(t) &= 3\left(1-t^2\right) \end{aligned} \qquad (11.18)$$

As an example,

$$P_1{}^1(t) = \sqrt{1-t^2}, \;\; \eta = \frac{\dot{P}_1{}^1}{P_1{}^1} = \frac{t\left(1-t^2\right)}{\sqrt{1-t^2}} \qquad (11.19)$$

where $t = \pm 1$ are zeros of $P_1{}^1$ and at the same time, the poles of η or the pole of the Riccati equation.

$$P_2{}^1(t) = 3t\sqrt{1-t^2}, \;\; \eta = \frac{\dot{P}_2{}^1}{P_2{}^1} = \frac{1}{t} - \frac{t}{1-t^2} \qquad (11.20)$$

where $t = 0, \;\; \pm 1$ are zeros of $P_2{}^1$, in other words, three poles of η or three poles of the Riccati equation. As a result of these examples, we conclude that the Riccati representation of the associated Legendre equation has poles which are directly related to the sum of the lower and upper indices $l + m$ of the Legendre's polynomial.

11.5. Hermite Equation

The Hermite oscillator equation is

$$\ddot{q} - 2t\dot{q} - 2\alpha q = 0$$

and the Riccati representation is written as

$$\dot{\eta} + \eta^2 - 2t\eta - 2\alpha = 0 \tag{11.21}$$

The recurrence relation of the Hermite polynomials is

$$H'_\alpha(t) = 2\alpha H_{\alpha-1} \tag{11.22}$$

(Arfken and Weber 1995) and then

$$\eta = \frac{\dot{q}}{q} = \frac{H_\alpha{}'(t)}{H_\alpha(t)} = 2\alpha\frac{H_{\alpha-1}(t)}{H_\alpha(t)} \tag{11.23}$$

For the few Hermite polynomial (see Appendix C).

$$\begin{aligned} H_0(t) &= 1 \\ H_1(t) &= 2t \\ H_2(t) &= 4t^2 - 2 \\ H_3(t) &= 8t^3 - 12t \end{aligned} \tag{11.24}$$

For example,

$$H_1(t) = 2t, \;\; \eta = \frac{\dot{H}_1}{H_1} = \frac{1}{t} \tag{11.25}$$

where $t = 0$ are zeros of $H_1(t)$ and at the same time, the poles of η or the pole of the Riccati equation.

$$H_2(t) = 4t^2 - 2, \;\; \eta = \frac{\dot{H}_2}{H_2} = \frac{1}{t} - \frac{4t}{2t^2 - 1} \tag{11.26}$$

where $t = \pm\frac{1}{2}$ are zeros of $H_2(t)$, in other words, two poles of η or two poles of the Riccati equation. As a result of these examples, we conclude that the Riccati representation of the Hermite differential equation has poles related to the lower indices α of the Hermite's polynomials.

11.6. Laguerre Equation

The Laguerre oscillator equation is written as

$$\ddot{q} + \frac{(1-t)}{t}\dot{q} - \frac{\alpha}{t}q = 0$$

and the Riccati representation of the oscillator equation becomes

$$\dot{\eta} + \eta^2 + \frac{(1-t)}{t}\eta - \frac{\alpha}{t} = 0 \tag{11.27}$$

The recurrence relation for the Laguerre polynomials is

$$L'_{\alpha}(t) = \frac{\alpha}{t}L_{\alpha}(t) - \frac{\alpha}{t}L_{\alpha-1}(t) \tag{11.28}$$

(Arfken and Weber 1995). Hence, η becomes

$$\eta = \frac{\dot{q}}{q} = \frac{{L_{\alpha}}'(t)}{L_{\alpha}(t)} = \frac{\alpha}{t}\left[1 - \frac{L_{\alpha-1}(t)}{L_{\alpha}(t)}\right] \tag{11.29}$$

The first few Laguerre polynomials are

$$\begin{aligned} L_0 &= 1 \\ L_1 &= 1 - t \\ L_2 &= \frac{1}{2}\left(t^2 - 4t + 2\right) \\ L_3 &= \frac{1}{6}\left(-t^3 + 9t^2 - 18t + 6\right) \end{aligned} \tag{11.30}$$

Examples,

$$L_1 = 1 - t, \;\; \eta = \frac{\dot{L}_1}{L_1} = \frac{-1}{1-t} \tag{11.31}$$

where $t = 1$ is zero of L_1 and at the same time, the pole of η or the pole of the Riccati equation.

$$L_2 = \frac{1}{2}\left(t^2 - 4t + 2\right), \;\; \eta = \frac{\dot{L}_2}{L_2} = \frac{2t-4}{t^2 - 4t + 2} \tag{11.32}$$

where $t = 2 \pm 2\sqrt{2}$ are zeros of L_2, in other words, two poles of η or two poles of the Riccati equation. As a result of these examples, we can conclude that the Riccati representation of the Laguerre oscillator equation has poles which are related to the lower indices α of the Laguerre's polynomial.

11.7. Associated Laguerre Equation

The associated Laguerre oscillator equation is written as in the following form

$$\ddot{q} + \left[(k+1)t^{-1} - 1\right]\dot{q} - (\alpha - k)t^{-1}q = 0$$

and the Riccati representation of the associated Laguerre oscillator equation becomes

$$\dot{\eta} + \eta^2 + \left[(k+1)t^{-1} - 1\right]\eta - (\alpha - k)t^{-1} = 0 \tag{11.33}$$

The associated Laguerre L_α^m polynomials satisfy the following recurrence relation

$$L_\alpha^{m\prime}(t) = \frac{\alpha}{t}L_\alpha^m(t) - \frac{\alpha + m}{t}L_{\alpha-1}^m(t) \tag{11.34}$$

(Arfken and Weber 1995) and the η function becomes

$$\eta = \frac{\dot{q}}{q} = \frac{L_\alpha^{m\prime}(t)}{L_\alpha^m(t)} = \frac{1}{t}\left[\alpha - (m+k)\frac{L_{\alpha-1}^m(t)}{L_\alpha^m(t)}\right] \tag{11.35}$$

The few first associated Laguerre polynomials

$$\begin{aligned} L_0^0 &= 1 \\ L_1^1 &= 2 - t \\ L_1^2 &= \frac{1}{2}\left(3t - t^2\right) \\ {L_2}^2 &= \frac{1}{2}\left(12 - t^2\right) \end{aligned} \tag{11.36}$$

Examples,

$$L_1^1 = 2 - t, \ \ \eta = \frac{\dot{L}_1^1}{L_1^1} = -\frac{1}{2-t} \tag{11.37}$$

where $t = 2$ is zero of L_1^1 and the pole of the Riccati equation.

$$L_1^2 = \frac{1}{2}\left(3t - t^2\right), \ \ \eta = \frac{\dot{L}_1^2}{L_1^2} = \frac{3-2t}{3t-t^2} \tag{11.38}$$

where $t = 0, \ \ 1$ are zeros of L_1^2 and two poles of the Riccati representation of the associated Laguerre equation. As a result of these examples, we can conclude that the Riccati representation of the associated Laguerre oscillator equation has poles which are related to the upper indices m of the associated Laguerre's polynomial.

11.8. Chebyshev Equation I

The oscillator representation of the first kind of the Chebyshev equation can be defined as

$$\ddot{q} - \frac{t}{1-t^2}\dot{q} - \frac{n^2}{1-t^2}q = 0$$

and its Riccati representation is given

$$\dot{\eta} + \eta^2 - \frac{t}{1-t^2}\eta - \frac{n^2}{1-t^2} = 0 \qquad (11.39)$$

The recurrence relation for the first kind Chebyshev polynomial is

$$T_n{}'(t) = \frac{n}{1-t^2}T_{n-1}(t) - \frac{nt}{1-t^2}T_n(t) \qquad (11.40)$$

(Arfken and Weber 1995) and the η function becomes

$$\eta = \frac{\dot{q}}{q} = \frac{T_n{}'(t)}{T_n(t)} = -\frac{n}{1-t^2}\left[t - \frac{T_{n-1}(t)}{T_n(t)}\right] \qquad (11.41)$$

The first few Chebyshev polynomials of the first kind are

$$\begin{aligned} T_0 &= 1 \\ T_1 &= t \\ T_2 &= 2t^2 - 1 \\ T_3 &= 4t^3 - 3t \\ T_4 &= 8t^4 - 8t^2 + 1 \end{aligned} \qquad (11.42)$$

For example,

$$T_1 = t, \;\; \eta = \frac{\dot{T}_1}{T_1} = \frac{1}{t} \qquad (11.43)$$

where $t = 0$ is zero of T_1 and the pole of the Riccati equation.

$$T_2 = 2t^2 - 1, \;\; \eta = \frac{\dot{T}_2}{T_2} = \frac{4t}{2t^2 - 1} \qquad (11.44)$$

where $t = \pm\sqrt{\frac{1}{2}}$ are zeros of T_2 and two poles of the Riccati representation of the first kind of the Chebyshev equation. As a result of these examples, we can conclude that the Riccati representation of the first kind of Chebyshev equation oscillator equation has poles which are related to the lower indices n of the first kind of Chebyshev polynomial.

11.9. Chebyshev Equation II

The oscillator representation of the second kind of the Chebyshev equation can be written as

$$\ddot{q} - \frac{3t}{1-t^2}\dot{q} - \frac{n(n+2)}{1-t^2}q = 0$$

Hence, its Riccati representation

$$\dot{\eta} + \eta^2 - \frac{3t}{1-t^2}\eta - \frac{n(n+2)}{1-t^2} = 0 \tag{11.45}$$

The recurrence relation is

$$U_n{}'(t) = \frac{n+1}{1-t^2}U_{n-1} - \frac{nt}{1-t^2}U_n(t) \tag{11.46}$$

(Arfken and Weber 1995) and the η function becomes

$$\eta = \frac{\dot{q}}{q} = \frac{U_n{}'(t)}{U_n(t)} = -\frac{n}{1-t^2}\left[t - \frac{U_{n-1}(t)}{U_n(t)}\right] \tag{11.47}$$

The first few Chebyshev polynomials of the second kind are

$$\begin{aligned} U_0 &= 1 \\ U_1 &= 2t \\ U_2 &= 4t^2 - 1 \\ U_3 &= 8t^3 - 4t \\ U_4 &= 16t^4 - 12t^2 + 1 \end{aligned} \tag{11.48}$$

For example,

$$U_1 = 2t, \ \ \eta = \frac{\dot{U}_1}{U_1} = \frac{1}{t} \tag{11.49}$$

where $t = 0$ is zero of U_1 and the pole of the Riccati equation.

$$U_2 = 4t^2 - 1, \ \ \eta = \frac{\dot{U}_2}{U_2} = \frac{4t}{2t^2 - 1} \tag{11.50}$$

where $t = \pm\frac{1}{2}$ are zeros of U_2 and two poles of the Riccati representation of the second kind of the Chebyshev equation. As a result of these examples, Hence, the Riccati representation of the second kind of Chebyshev equation oscillator equation has poles related to the lower indices n of the second kind of Chebyshev polynomial.

CHAPTER 12

CONCLUSION

In the present thesis we have studied the universe models as oscillatory dynamical systems. These systems are constructed for Bianchi type anisotropic models by the linearization transformation of nonlinear Riccati differential equation for the mean rate of change. Using factorization properties of the oscillatory models we introduced new characteristic of universe as in the supersymmetric quantum mechanics with bosonic and fermionics structures corresponding to expanding and contracting universes.

In addition to this, we have showed that physics in non-stationary universe like inflation cosmology, string theory and gravitational waves implies the damping character of the oscillatory models and the possibility for time reflection symmetry and the dual oscillator representation. To quantize the oscillatory models the Lagrangian and the Hamiltonian formalism for damped system becomes essential this is why we construct classical and quantum theory of damped oscillator in two different approaches known as the Bateman dual and the Bateman-Caldirola-Kanai approaches. We have showed their classical equivalence as the self-adjoint extension of the Lagrangian linear operator for the dual system. For Sturm Liouville Problem represented as damped parametric oscillator and the Lagrangian and the Hamiltonain formulations with exact solutions for all special functions of the mathematical physics in other words, the Hypergeometric functions are given. Finally, we have found that the zeros of the oscillator equation transformed into the poles of the corresponding Riccati equations are important for real physical application which we discussed in cosmological Friedman type models. As a result of this, the size of universe is described by the Riccati equation and pole singularity would corresponds to singularities of universe such as Big Bang.

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APPENDIX A

PRELIMINARIES FOR TENSOR CALCULUS

A.1. Tensor Calculus

The main principle of general relativity is that the only valid physical laws are those that equate two quantities that transform in the same way under any arbitrary change of coordinates. We need to express the equations of physics in a frame independent way. We can apply this idea by introducing more general coordinate independent quantities called Tensors.

Definition A.1 *A tensor or tensor field of type (p,q) and rank $p+q$ is relative to coordinate system $x^i = (x^1, x^2, ..., x^n)$*

$$T^{i_1,i_2,...i_p}_{j_1,j_2,...j_q} \tag{A.1}$$

which is family of functions and these functions or components can be transformed under coordinate changing $z^i = (z^1, z^2, ...z^n)$ as

$$T^{i_1,i_2,...i_p}_{j_1,j_2,...j_q} = \sum_{\substack{k_1 \; . \; . \; . \; k_p \\ l_1 \; . \; . \; . \; l_q}} \widetilde{T}^{k_1,k_2,...k_p}_{l_1,l_2,...l_q} \frac{\partial x^{i_1}}{\partial z^{k_1}} ... \frac{\partial x^{i_p}}{\partial z^{k_p}} \frac{\partial z^{l_1}}{\partial x^{j_1}} ... \frac{\partial z^{l_q}}{\partial x^{j_q}} \tag{A.2}$$

where $\widetilde{T}^{k_1,k_2,...k_p}_{l_1,l_2,...l_q}$ includes components of vector in coordinate system $z^1, z^2, ..., z^n$.

In particularly, we can define the contravariant and covariant tensors and their transformation rules are given by the following definitions

Definition A.2 *An object having components $T^{i_1,i_2,...i_p}$ in the coordinate system $x^i = (x^1, x^2, ..., x^n)$ is called contravariant tensor, besides the type of the tensor field is (p,0) and its rank is p. Under the coordinate changing $z^i = (z^1, z^2, ...z^n)$ transformation rule is*

$$T^{i_1,i_2,...i_p} = \sum_{k_1 \; . \; . \; . \; k_p} \widetilde{T}^{k_1,k_2,...k_p} \frac{\partial x^{i_1}}{\partial z^{k_1}} ... \frac{\partial x^{i_p}}{\partial z^{k_p}} \tag{A.3}$$

Similarly,

Definition A.3 *An object having components $T_{i_1,i_2,...i_p}$ in the coordinate system $x^i = (x^1, x^2, ..., x^n)$ is called covariant tensor. In addition to this, the type of the covariant tensor is (0,p) and its rank is p. Under the coordinate changing $z^i = (z^1, z^2, ...z^n)$ transformation rule is*

$$T_{i_1,i_2,...i_p} = \sum_{k_1 \; . \; . \; . \; k_p} \widetilde{T}_{k_1,k_2,...k_p} \frac{\partial z^{k_1}}{\partial x^{i_1}} ... \frac{\partial z^{k_p}}{\partial x^{i_p}} \tag{A.4}$$

(Novikov and Fomenko 1990).

Next, we will find the derivative of a tensor, which should give us back a new tensor and we are going to require some additional mathematical formalism. Then we will show how this works and then describe the metric tensor, which plays a central role in the study of gravity. Next we will introduce some quantities that are important in Einsteins equation such as the Ricci tensor and the Ricci scalar,...etc.

A.2. Calculating Christoffel Symbols from Metric

Since $V^{\alpha}{}_{;\beta}$ is a tensor we can lower the index α using the metric tensor:

$$V_{\alpha;\beta} = g_{\alpha\mu} V^{\mu}{}_{;\beta} \tag{A.5}$$

But by linearity, we have:

$$V_{\alpha;\beta} = (g_{\alpha\mu} V^{\mu})_{;\beta} = g_{\alpha\mu;\beta} V^{\mu} + g_{\alpha\mu} V^{\mu}{}_{;\beta} \tag{A.6}$$

So consistency requires $g_{\alpha\mu;\beta} V^{\mu} = 0$. Since **V** is arbitrary this implies that

$$g_{\alpha\mu;\beta} = 0 \tag{A.7}$$

Thus the covariant derivative of the metric is zero in every frame. We will prove that $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$ (i.e. symmetric in β and γ). In a general frame we have a scalar field ϕ:

$$\phi_{;\beta\alpha} = \phi_{,\beta;\alpha} = \phi_{,\beta\alpha} - \Gamma^{\gamma}{}_{\beta\alpha} \phi_{,\gamma} \tag{A.8}$$

in a local inertial frame, this is just $\phi_{,\beta\alpha} \equiv \frac{\partial^2 \phi}{\partial x^{\beta} \partial x^{\alpha}}$, which is symmetric in β and α. Thus it must also be symmetric in a general frame. Hence $\Gamma^{\gamma}{}_{\alpha\beta}$ is symmetric in β and α:

$$\Gamma^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\beta\alpha} \tag{A.9}$$

We now use this to express $\Gamma^\gamma{}_{\alpha\beta}$ in terms of the metric. Since $g_{\alpha\beta,\mu} = 0$, we have:

$$g_{\alpha\beta,\mu} = \Gamma^\nu{}_{\alpha\mu} g_{\nu\mu} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} \tag{A.10}$$

By writing different permutations of the indices and using the symmetry of $\Gamma^\gamma{}_{\alpha\beta}$, we get

$$g_{\alpha\beta,\mu} + g_{\nu\mu,\beta} - g_{\beta\mu,\alpha} = 2 g_{\alpha\nu} \Gamma^\nu{}_{\beta\mu} \tag{A.11}$$

Multiplying by $\frac{1}{2} g^{\alpha\gamma}$ and using $g^{\alpha\gamma} g^{\alpha\nu} = \delta^\gamma{}_\nu$ gives

$$\Gamma^\gamma{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\nu\mu,\beta} - g_{\beta\mu,\alpha} \right) \tag{A.12}$$

Note that $\Gamma^\gamma{}_{\beta\mu}$ is not a tensor since it is defined in terms of partial derivatives. In a local inertial frame $\Gamma^\gamma{}_{\beta\mu} = 0$ since $g_{\alpha\beta,\mu} = 0$. We will see later the significance of this result.

A.3. Parallel Transport and Geodesics

A vector field **V** is parallel transported along a curve with tangent

$$\mathbf{U} = \frac{dx}{d\lambda} \tag{A.13}$$

where λ is the parameter along the curve (usually taken to be the proper time τ if the curve is time-like) if and only if

$$\frac{dV^\alpha}{d\lambda} = 0 \tag{A.14}$$

In an inertial frame this is

$$\frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial \lambda} = U^\beta V^\alpha{}_{,\beta} \tag{A.15}$$

so in a general frame the condition becomes:

$$U^\beta V^\alpha{}_{;\beta} = 0 \tag{A.16}$$

we just replace the partial derivatives (,) with a covariant derivative (;). The curve is a geodesic if it parallel transports its own tangent vector.

$$U^\beta U^\alpha{}_{;\beta} = 0 \tag{A.17}$$

This is the closest we can get to defining a straight line in a curved space. In flat space a tangent vector is everywhere tangent only for a straight line. Now

$$U^\beta U^\alpha{}_{;\beta} = 0 \Rightarrow U^\beta U^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta} U^\mu U^\beta = 0 \tag{A.18}$$

Since $U^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$ and $\frac{d}{d\lambda} = U^{\beta}\frac{\partial}{\partial x^{\alpha}}$ we can write this as

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}{}_{\mu\beta}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0 \tag{A.19}$$

This is the geodesic equation. It is a second order differential equation for $x^{\alpha}(\lambda)$, so one gets a unique solution by specifying an initial position x_0 and velocity U_0.

A.4. Variational Method For Geodesics

Definition A.4 *Geodesic is a generalization of the motion of a ”straight line” to ”curved spaces”. In presence of a metric, geodesic is defined to be locally the shortest path between points on the space.*

We apply the variational technique to compute the geodesics for a given metric. For a curved space-time, the proper time $d\tau$ is defined to be

$$d\tau^2 = -\frac{1}{c^2}ds^2 = -\frac{1}{c^2}g_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{A.20}$$

Remember in flat space-time it was just

$$d\tau^2 = -\frac{1}{c^2}\eta_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{A.21}$$

Therefore the proper time between two points A and B along an arbitrary time-like curve is

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \frac{d\tau}{d\lambda}d\lambda \tag{A.22}$$

$$= \int_A^B d\tau = \int_A^B \frac{1}{c}\left[-g_{\beta\gamma(\mathbf{x})}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda}\right]^{1/2} d\lambda \tag{A.23}$$

so we can write the Lagrangian as

$$\mathcal{L}\left(x^{\alpha}, \dot{x}^{\alpha}, \lambda\right) = \frac{1}{c}\left[-g_{\beta\gamma(\mathbf{x})}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda}\right]^{1/2} \tag{A.24}$$

and the action becomes

$$S = \tau_{AB} = \int_A^B \mathcal{L}\left(x^{\alpha}, \dot{x}^{\alpha}, \lambda\right) d\lambda \tag{A.25}$$

varying the action, we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) \tag{A.26}$$

Now

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{1}{2c}\left(-g_{\mu\nu}\frac{\partial x^\mu}{d\lambda}\frac{\partial x^\nu}{d\lambda}\right)^{-1/2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \tag{A.27}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\frac{1}{2c}\left(-g_{\mu\nu}\frac{\partial x^\mu}{d\lambda}\frac{\partial x^\nu}{d\lambda}\right)^{-1/2} 2g_{\alpha\beta}\frac{dx^\beta}{d\lambda} \tag{A.28}$$

Since

$$\frac{1}{c}\left(-g_{\mu\nu}\frac{\partial x^\mu}{d\lambda}\frac{\partial x^\nu}{d\lambda}\right)^{1/2} = \frac{d\tau}{d\lambda} \tag{A.29}$$

we get

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{1}{2}\frac{d\lambda}{d\tau}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \tag{A.30}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\frac{d\lambda}{d\tau}g_{\alpha\beta}\frac{dx^\beta}{d\lambda} = -g_{\alpha\beta}\frac{dx^\beta}{d\tau} \tag{A.31}$$

so the Euler- Lagrange equations become:

$$\frac{1}{2}\frac{d\lambda}{d\tau}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = \frac{d}{d\lambda}\left[g_{\beta\gamma}\frac{dx^\beta}{d\tau}\right] \tag{A.32}$$

Multiplying by $\frac{d\lambda}{d\tau}$ we obtain

$$\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \frac{d}{d\tau}\left[g_{\beta\gamma}\frac{dx^\beta}{d\tau}\right] \tag{A.33}$$

Using

$$\frac{dg_{\alpha\beta}}{d\tau} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\gamma}{d\tau} \tag{A.34}$$

we get

$$\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\gamma}{d\tau}\frac{dx^\beta}{d\tau} \tag{A.35}$$

Multiplying by $g^{\delta\alpha}$ gives

$$\frac{d^2x^\delta}{d\tau^2} = -g^{\delta\alpha}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \tag{A.36}$$

Now

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\gamma}{d\tau}\frac{dx^\beta}{d\tau}\right) \tag{A.37}$$

$$= \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\right)\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \tag{A.38}$$

Using the above result gives us

$$\frac{d^2x^\delta}{d\tau^2} = -\frac{1}{2}g^{\delta\alpha}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \tag{A.39}$$

$$= -\Gamma^\delta{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \tag{A.40}$$

so we get the geodesic equation again

$$\frac{d^2x^\delta}{d\tau^2} + \Gamma^\delta{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{A.41}$$

This is the equation of motion for a particle moving on a time-like geodesic in curved space-time. Note that in a local inertial frame i.e. where $\Gamma^\delta{}_{\beta\gamma} = 0$, the geodesic equation is reduced to

$$\frac{d^2x^\delta}{d\tau^2} = 0 \tag{A.42}$$

which is the equation of motion for a free particle.

A.5. Properties of Riemann Curvature Tensor

Definition A.5 *The Riemann tensor which is sometimes called the curvature tensor. In terms of the metric connection (Christoffel symbols) it is given by*

$$R^\alpha{}_{\beta\mu\nu} \equiv \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu} \tag{A.43}$$

In a local inertial frame we have $\Gamma^\alpha{}_{\mu\nu} = 0$, so in this frame

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} \tag{A.44}$$

Now

$$\Gamma^\alpha{}_{\beta\nu} = \frac{1}{2}g^{\alpha\delta}\left(g_{\delta\beta,\nu} + g_{\delta\nu,\beta} - g_{\beta\nu,\delta}\right) \tag{A.45}$$

so

$$\Gamma^{\alpha}{}_{\beta\nu,\mu} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\nu\mu} + g_{\delta\nu,\beta\mu} - g_{\beta\nu,\delta\mu} \right) \tag{A.46}$$

since $g^{\alpha\delta}{}_{,\mu} = 0$ i.e., the first derivative of the metric vanishes in a local inertial frame. Hence

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\nu\mu} + g_{\delta\nu,\beta\mu} - g_{\beta\nu,\delta\mu} - g_{\delta\beta,\mu\nu} \right) \tag{A.47}$$

Using the fact that partial derivatives always commute so that $g_{\delta\beta,\nu\mu} = g_{\delta\beta,\mu\nu}$, we get

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\nu,\beta\mu} - g_{\delta\mu,\beta\nu} + g_{\beta\mu,\delta\nu} - g_{\beta\nu,\delta\mu} \right) \tag{A.48}$$

in a local inertial frame. Lowering the index α with the metric we get

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} \tag{A.49}$$

$$= \frac{1}{2} g^{\delta}{}_{\alpha} \left(g_{\delta\nu,\beta\mu} - g_{\delta\mu,\beta\nu} + g_{\beta\mu,\delta\nu} - g_{\beta\nu,\delta\mu} \right) \tag{A.50}$$

So, in a local inertial frame the result is

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} \left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu} \right) \tag{A.51}$$

We can use this result to discover what the symmetries of $R_{\alpha\beta\mu\nu}$ are. It is easy to show from the above result that

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta} \tag{A.52}$$

and

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \tag{A.53}$$

Thus $R_{\alpha\beta\mu\nu}$ is antisymmetric on the first pair and second pair of indices, and symmetric on exchange of the two pairs. Since these last two equations are valid tensor equations, although they were derived in a local inertial frame, they are valid in all coordinate systems. We can use these two identities to reduce the number of independent components of $R_{\alpha\beta\mu\nu}$ from 256 to just 20. A flat manifold is one which has a global definition of parallelism: for example, a vector can be moved around parallel to itself on an arbitrary curve and will return to its starting point unchanged. This clearly means that

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \tag{A.54}$$

Hence, the manifold is flat. All together, in n dimensions, there are $n^2\left(n^2-1\right)/12$ independent nonzero components of the Riemann tensor. This fact together with the symmetries in tells us that in two dimensions, the possible nonzero components of the Riemann tensor are

$$R_{1212}=R_{2121}=-R_{1221}=-R_{2112} \tag{A.55}$$

while in three dimensions the possible nonzero components of the Riemann tensor are

$$R_{1212}=R_{1313}=R_{2323}=R_{1213}=R_{1232}=R_{2123},\quad R_{1323}=R_{3132} \tag{A.56}$$

Computation of these quantities using a coordinate basis is extremely tedious, especially when we begin dealing with real space-time metrics. An important use of the curvature tensor comes when we examine the consequences of taking two covariant derivatives of a vector field **V**

$$\nabla_\alpha\nabla_\beta V^\mu = \nabla_\alpha\left(V^\mu{}_{;\beta}\right) \tag{A.57}$$

$$= \left(V^\mu{}_{;\beta}\right)_{,\alpha}+\Gamma^\mu{}_{\sigma\alpha}V^\sigma{}_{;\beta}-\Gamma^\sigma{}_{\beta\alpha}V^\mu{}_{;\sigma} \tag{A.58}$$

As usual we can simplify things by working in a local inertial frame. So in this frame we get

$$\nabla_\alpha\nabla_\beta V^\mu = \left(V^\mu{}_{;\beta}\right)_{,\alpha} \tag{A.59}$$

$$= \left(V^\mu{}_{,\beta}+\Gamma^\mu{}_{\nu\beta}V^\nu\right)_\alpha \tag{A.60}$$

$$= V^\mu{}_{,\beta\alpha}+\Gamma^\mu{}_{,\beta\alpha}V^\nu+\Gamma^\mu{}_{,\nu\beta}V^\nu{}_{,\alpha} \tag{A.61}$$

The third term of this is zero in a local inertial frame, so we obtain

$$\nabla_\alpha\nabla_\beta V^\mu = V^\mu{}_{,\alpha\beta}+\Gamma^\mu{}_{\nu\beta,\alpha}V^\nu \tag{A.62}$$

Consider the same formula with the interchanged indices:

$$\nabla_\alpha\nabla_\beta V^\mu = V^\mu{}_{,\beta\alpha}+\Gamma^\mu{}_{\nu\alpha,\beta}V^\nu \tag{A.63}$$

If we subtract these we get the commutator of the covariant derivative operators ∇_α and ∇_β:

$$\left[\nabla_\alpha,\nabla_\beta\right]V^\mu = \nabla_\alpha\nabla_\beta V^\mu-\nabla_\beta\nabla_\alpha V^\mu \tag{A.64}$$

The terms involving the second derivatives of V^μ drop out because $V^\mu{}_{,\alpha\beta} = V^\nu{}_{,\beta\alpha}$ (partial derivatives commute). Since in a local inertial frame the Riemann tensor takes the form

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta} \tag{A.65}$$

we get

$$\left[\nabla_\alpha, \nabla_\beta\right] V^\mu = R^\mu{}_{\nu\alpha\beta} V^\mu \tag{A.66}$$

This is closely related to our original derivation of the Riemann tensor from parallel transport around loops, because the parallel transport problem can be thought of as computing, first the change of **V** in one direction, and then in another, followed by subtracting changes in the reverse order.

A.6. Bianchi Identities; Ricci and Einstein Tensors

In the last section we found that in a local inertial frame the Riemann tensor could be written as

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{A.67}$$

Differentiating with respect to x^λ we get

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right) \tag{A.68}$$

From this equation, the symmetry $g_{\alpha\beta} = g_{\beta\alpha}$ and the fact that partial derivatives commute, it is showed that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \tag{A.69}$$

This equation is valid in a local inertial frame, therefore in a general frame we get

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{A.70}$$

or

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{A.71}$$

This is a tensor equation, therefore valid in any coordinate system. It is called the Bianchi identities.

A.6.1. Ricci Tensor

The Riemann tensor can be used to derive two more quantities that are used to define the Einstein tensor. The first of these is the Ricci tensor, which is calculated from the Riemann tensor by contraction on the first and third indices. Before looking at the consequences of the Bianchi identities, we need to define the Ricci tensor $R_{\alpha\beta}$.

Definition A.6

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha} = 0 \tag{A.72}$$

It is the contraction of $R^{\mu}{}_{\alpha\nu\beta}$ on the first and third indices.

Other contractions would in principle also be possible: on the first and second, the first and fourth, etc. Because $R_{\alpha\beta\mu\nu}$ is antisymmetric on α and β and on μ and ν, all these contractions either vanish or reduce to $R_{\pm\alpha\beta}$. Therefore the Ricci tensor is essentially the only contraction of the Riemann tensor. Similarly, the Ricci scalar is defined as

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{\alpha\beta}g^{\mu\nu}R_{\mu\alpha\nu\beta} = 0 \tag{A.73}$$

A.6.2. Einstein Tensor

Proposition A.1 *The Einstein tensor $G_{\mu\nu}$ is constructed by applying the Ricci contraction to the Bianchi identities.*

Proof The curvature tensor, in other word the Riemann tensor $R_{\alpha\mu\beta\nu}$ satisfies the following differential equations

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{A.74}$$

They are known as the Bianchi relations. The Bianchi relation (A.74) involves five suffixes.

$$g^{\alpha\mu}\left(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}\right) = 0 \tag{A.75}$$

Since $g_{\alpha\beta;\mu} = 0$ and $g^{\alpha\beta}{}_{;\mu} = 0$, we can take $g^{\alpha\mu} = 0$ in and out of covariant derivatives. Then we obtain

$$R^{\mu}{}_{\beta\mu\nu;\lambda} + R^{\mu}{}_{\beta\lambda\mu;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0 \tag{A.76}$$

Using the antisymmetry on the indices μ and λ we get

$$R^{\mu}{}_{\beta\mu\nu;\lambda} - R^{\mu}{}_{\beta\mu\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0 \tag{A.77}$$

so

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0 \tag{A.78}$$

These equations are called the contracted Bianchi identities. Let us now contract a second time on the indices β and ν:

$$g^{\beta\nu}\left(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu}\right) = 0 \tag{A.79}$$

This gives

$$\left(R^{\nu}{}_{\nu;\lambda} - R^{\nu}{}_{\lambda;\nu} + R^{\mu\nu}{}_{\nu\lambda;\mu}\right) = 0 \tag{A.80}$$

so

$$R_{;\lambda} - 2R^{\mu}{}_{\lambda;\mu} = 0 \tag{A.81}$$

or

$$2R^{\mu}{}_{\lambda;\mu} - R_{;\lambda} = 0 \tag{A.82}$$

Since $R_{;\lambda} = g^{\mu}{}_{\lambda}R_{;\mu}$, we get

$$\left[R^{\mu}{}_{\lambda} - \frac{1}{2}g^{\mu}{}_{\lambda}R\right]_{;\mu} = 0 \tag{A.83}$$

Raising the index λ with $g^{\lambda\nu}$ we get

$$\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]_{;\mu} = 0 \tag{A.84}$$

Defining

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0 \tag{A.85}$$

we obtain

$$G^{\mu\nu}{}_{;\nu} = 0 \tag{A.86}$$

The tensor $G^{\mu\nu}$ is constructed only from the Riemann tensor and the metric, and it is automatically divergence free as an identity. Einstein's field equations for General Relativity are

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \tag{A.87}$$

The Bianchi Identities then imply

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{A.88}$$

which is known as the conservation of energy and momentum.

APPENDIX B

RICCATI DIFFERENTIAL EQUATION

Definition B.1 *The Riccati differential equation is the second order nonlinear differential equation. The standard form of this equation is*

$$y' = py^2 + qy + r \tag{B.1}$$

where p, q and r are functions of x alone, and p is not identically zero (Ince, 1946).

The Riccati differential equation may be integrated completely when any particular solution, say $y = y_1$, is known, a result by the substitution

$$y = y_1 + \frac{1}{v} \tag{B.2}$$

where v is a new variable which is function x. After substituting (B.2)

$$v' + (2py_1 + q)\, v = 0 \tag{B.3}$$

and the integration can be completed by quadratures.

Important Properties

The general integral can be expressed in terms of any three functions y_1, y_2 and y_3 which satisfy the Riccati equation. If we make the substitution $y = y_1 + \frac{1}{v}$, y_2 and y_3 will correspond to two particular values of v which are defined as v_1 and v_2

$$y_2 = y_1 + \frac{1}{v_1}, \quad y_3 = y_1 + \frac{1}{v_2} \tag{B.4}$$

and since $v = v_1$ and $v = v_2$ are particular solutions of the linear equation (B.3) and its general solution becomes

$$\frac{v - v_1}{v_2 - v_1} = c \tag{B.5}$$

where c is the integration constant. Making the reverse substitutions, we obtain

$$v = \frac{1}{y - y_1}, \quad v_1 = \frac{1}{y_2 - y_1}, \quad v_3 = \frac{1}{y_3 - y_1} \tag{B.6}$$

we obtain that

$$\frac{y - y_2}{y - y_1} = c\frac{y_3 - y_2}{y_3 - y_1} \tag{B.7}$$

(Ince, 1946).

It is well known how to get Riccati equations from the general form of homogeneous linear differential equation of the second order

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0 \tag{B.8}$$

it requires the transformation

$$y = \frac{1}{S(x)}\frac{1}{u}\frac{du}{dx} \tag{B.9}$$

where $S(x)$ is an arbitrary function and this transformation is also called the Cole-Hopft transformation. Note that we are using the same transformation to obtain the linearize the Riccati equation. Then, the first and second derivative of (B.9) become

$$u' = yuS(x) \tag{B.10}$$

and the second derivative of u in terms of x is

$$u'' = y'uS(x) + y^2S^2(x) + yuS' \tag{B.11}$$

Substitute them in the second order differential equation. Then,

$$A(x)\left[y'uS(x) + y^2S^2(x)u + yuS'\right] + B(x)\left[yuS(x)\right] + C(x)u = 0$$

and it becomes

$$y' + y^2S(x) + y\left[\frac{S'(x)}{S(x)} + \frac{B(x)}{A(x)}\right] + \frac{C}{A(x)S(x)} = 0 \tag{B.12}$$

we can choose $\left[\frac{S'(x)}{S(x)} + \frac{B(x)}{A(x)}\right]$ such that the coefficient of y be zero. Then

$$\frac{S'(x)}{S(x)} + \frac{B(x)}{A(x)} = 0 \rightarrow \quad \ln S(x) = \int^x -\frac{B(x')}{A(x')}dx'$$

$$S_0(x) = e^{\int^x -\frac{B(x')}{A(x')}dx'} \tag{B.13}$$

Hence,

$$y' + y^2S_0 = -\frac{C}{A(x)S_0} \tag{B.14}$$

Let us apply the change of independent variable

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \tag{B.15}$$

Hence, equation (B.8) becomes

$$\frac{1}{S_0}\frac{dy}{dz}\frac{dz}{dx}+y^2 \quad = \quad -\frac{C}{A(x)S^2{}_0} \tag{B.16}$$

and again let assume that

$$\frac{1}{S_0}\frac{dz}{dx}=1, \quad z(x) \quad = \quad \int^x S_0 dx' \tag{B.17}$$

under this condition, the Riccati equation can be given as

$$\frac{dy}{dz}+y^2 \quad = \quad -\frac{C(z)}{A(z)S^2{}_0} \tag{B.18}$$

APPENDIX C

HERMITE DIFFERENTIAL EQUATION

Definition C.1 *Hermite differential equation is defined as:*

$$y'' - 2xy' + 2ny = 0 \tag{C.1}$$

where n is a real number. For n is a non-negative integer, i.e., n= 0, 1, 2, 3,..., the solutions of Hermite's differential equation are often referred to as Hermite polynomials $H_n(x)$ *(Arfken and Weber 1995).*

By solving the Hermite differential equation, the series

$$H_{2k}(x) = (-1)^k 2^k (2k-1)!! \left[1 + \sum_{j=1}^{k} \frac{(-4k)(-4k+4)...(-4k+4j-4)}{(2j)!} x^{2j} \right] \tag{C.2}$$

Important Properties

Definition C.2 *The Hermite polynomials* $H_n(x)$ *can be expressed by Rodrigues' formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \tag{C.3}$$

where n= 0, 1, 2, 3,...

and the first few Hermite polynomials are

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120 \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240
\end{aligned} \tag{C.4}$$

The values $H_n(0)$ may be called Hermite numbers.

Definition C.3 *The generating function of Hermite polynomial is*

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{C.5}$$

Proof Consider $F(x+t) = \exp(-(x+t)^2)$ and the Taylor series expansion,

$$f(x) = \sum_n \frac{x^n}{n!} \frac{d^n f}{dx^n}|_{x=0} \tag{C.6}$$

yields,

$$F(x+t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} F^{(n)}(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n H_{(n)}(x) e^{-x^2} \tag{C.7}$$

Let $t' = t$. Hence,

$$e^{x^2} F(x-t') = \sum_n \frac{(-t')^n}{n!} (-1)^n H_{(n)}(x), \tag{C.8}$$

$$e^{x^2} F(x-t) = \sum_n \frac{(t)^n}{n!} H_{(n)}(x). \tag{C.9}$$

$$e^{(-t^2+2tx)} = g(x,t) = \sum_n \frac{(t)^n}{n!} H_{(n)}(x) = \sum_n \frac{(t)^n}{n!} \frac{\partial^n}{\partial t^n} e^{(-t^2+2tx)}|_{t=0}. \tag{C.10}$$

Then,

$$H_{(n)}(x) = \frac{\partial^n}{\partial t^n} e^{(-t^2+2tx)}|_{t=0}, \tag{C.11}$$

$e^{(-t^2+2tx)}$ is called the generating function for the Hermite polynomials $H_{(n)}(x)$.

C.1. Orthogonality

Hermite polynomials $H_n(x)$ form a complete orthogonal set on the interval $-\infty < x < \infty$ in terms of the weighting function e^{-x^2}. It can be shown that

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = n! 2^n \sqrt{\pi} \delta_{nm} \tag{C.12}$$

where δ_{nm} is the Kronecker delta, which equals unity when $n = m$ and zero otherwise. The same integration can be written in different way for simplicity as follows,

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = n! 2^n \sqrt{\pi} \delta_{nm} = \begin{cases} 0, & m \neq n \\ n! 2^n \sqrt{\pi}, & n = m \end{cases}$$

C.2. Even/Odd Functions

Whether a Hermite polynomial is even or odd function depends on its degree n. Based on $H_n(-x) = (-1)^n H_n(-x)$,

a. $H_n(x)$ is an even function, when n is even.

b. $H_n(x)$ is an odd function, when n is odd.

C.3. Recurrence Relation

A Hermite polynomial at one point can be expressed by neighboring Hermite polynomials

a. $H_{n+1}(x) = xH_n(x) - 2nH_{n-1}(x)$.

b. $H_n{}'(x) = 2nH_{n-1}$.

C.4. Special Results

a. $H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots$

b. $\int\limits_0^x H_n(t)dt = \frac{H_{n+1}(x)}{2(n+1)} - \frac{H_{n+1}(0)}{2(n+1)}$

c. $\int\limits_0^x e^{-t^2} H_n(t)dt = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$

d. $H_n(x+y) = \sum_{k=0}{}^n \frac{1}{2^{n/2}} \begin{pmatrix} n \\ k \end{pmatrix} H_k(x\sqrt{2})H_{n-k}(y\sqrt{2})$

e.

$$H_n(0) = \begin{cases} (-1)^{\frac{n}{2}} 2^{\frac{n}{2}}.1.3.5...(n-1), & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

f. $\dfrac{d}{dx}\{e^{(-x^2)} H_n(x)\} = -e^{(-x^2)} H_{n+1}(x)$

g. $\displaystyle\sum_{k=0}^n \frac{H_k(x)H_k(y)}{2^k k!} = \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{2^{n+1} n!(x-y)}$

APPENDIX D

NON-STATIONARY OSCILLATOR REPRESENTATION OF FRW UNIVERSE

We can determine the standard oscillator equation by combining the Friedman's equations as follows,

$$3\frac{(\dot{a}^2+k)}{a^2} - \Lambda = 8\pi\rho(t) \tag{D.1}$$

$$\frac{2a\ddot{a}+\dot{a}^2+k}{a^2} - \Lambda = -8\pi p(t) \tag{D.2}$$

where Λ is the cosmological constant, ρ is density and p is the pressure of the Friedman Robertson Walker universe. Moreover, the density and pressure are time dependent functions. Firstly we arrange the first Friedman equation (D.1)

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{D.3}$$

After substituting (D.3) in the second Friedman equation (D.2), we get the oscillator type equation with time dependent frequency

$$\ddot{a}(t) + \frac{4\pi}{3}\big[\rho(t)+3p(t)\big]a(t) = \frac{\Lambda}{3}a(t) \tag{D.4}$$

In this appendix we will interpret (D.4) as the Schrödinger equation in order to solve it for some particular cases. The scale factor $a(t)$ is considered as a wave function of the stationary Schrödinger equation. To get Schrödinger type equation with $\hbar = 1$ and $m = \frac{1}{2}$. In below consideration we will treat $a(t)$ as a complex function from which only reel or imaginary part have direct physical meaning. Let assume that the second term of the left hand side is the potential part of the Schrödinger equation

$$V(t) \equiv \frac{4\pi}{3}\left[\rho(t)+3p(t)\right] \tag{D.5}$$

Hence, the Schrödinger type equation is obtained

$$\ddot{a}(t) + V(t)a(t) = \frac{\Lambda}{3}a(t) \tag{D.6}$$

At the following subsection, we will discuss the different cases with respect to time dependent delta potential.

D.1. Time Dependent Oscillator

D.1.1. Delta Function Potential

Equation (D.6) is obtained as

$$\ddot{a}(t) + \frac{4\pi}{3}\big[\rho(t) + 3p(t)\big]a(t) = \frac{\Lambda}{3}a(t) \tag{D.7}$$

Let assume that pressure and density are in the form of the delta δ function.

$$\rho(t) = \rho_0\delta(t), \;\; p(t) = p_0\delta(t) \tag{D.8}$$

and in the oscillator equation, the potential part is given by $\frac{4\pi}{3}\big[\rho(t) + 3p(t)\big]$. Then, the potential becomes

$$V \equiv \frac{4\pi}{3}\left[\rho_0\delta(t) + 3p_0\delta(t)\right] = \frac{4\pi}{3}\left[\rho_0 + 3p_0\right]\delta(t) \tag{D.9}$$

and we assume that

$$U_0 \equiv \frac{4\pi}{3}\left[\rho_0 + 3p_0\right] \tag{D.10}$$

The goal is to get the similar form of the Schrödinger equation. In this assumption, the potential of the Schrodinger equation is $V = U_0\delta(t)$. Let us assume that is the Schrödinger equation, where $a(t)$ represents the one-dimensional wave function

$$\ddot{a}(t) + U_0\delta(t)a(t) = \frac{\Lambda}{3}a(t) \tag{D.11}$$

and multiply equation (D.11) by -1 and then

$$-\ddot{a}(t) - U_0\delta(t)a(t) = Ea(t) \tag{D.12}$$

where we assume that E is the energy and it is equivalent to $E = -\frac{\Lambda}{3}$. Now, let us firstly look at the region outside of $t = 0$ in which the delta function becomes zero

$$\ddot{a}(t) = -Ea(t) = \kappa^2 a(t) \tag{D.13}$$

where $\kappa = \pm\sqrt{-E} = \sqrt{\frac{\Lambda}{3}}$

a. If the energy is negative or $\Lambda > 0$, κ will become real.

b. If the energy is positive or $\Lambda < 0$, κ will become purely imaginary.

Case [$\Lambda > 0$] or Bound State:

If cosmological constant κ is the real, the solution of equation (D.13) is obtained as

$$a(t) = Ae^{-\kappa t} + Be^{\kappa t} \tag{D.14}$$

For negative t in other words, the left side of the potential, the wave function a will blow up as t goes to negative infinity, so A must be zero. On the right side the same thing happens and there, the constant B must be zero. Then we get the step function

$$a(t) = \begin{cases} Be^{\kappa t}, & t \leq 0 \\ Ae^{-\kappa t}, & t \geq 0 \end{cases}$$

As a consequence, at $t = 0$, the wave function must be continuous, so $A = B$ and we have

$$a(t) = \begin{cases} Ae^{\kappa t}, & t \leq 0 \\ Ae^{-\kappa t}, & t \geq 0 \end{cases}$$

Now we must use the information at $t = 0$, in the region near $t = 0$, equation (D.13) gives us,

$$-\ddot{a}(t) - U_0\delta(t)a(t) = Ea(t) \tag{D.15}$$

We integrate both sides with respect to t over infinitesimally small region around the delta potential from $-\epsilon$ to ϵ:

$$\int_{-\epsilon}^{\epsilon} - \ddot{a}(t)dt - \int_{-\epsilon}^{\epsilon} U_0\delta(t)a(t)dt = \int_{-\epsilon}^{\epsilon} Ea(t)dt \tag{D.16}$$

and

$$-\left[\frac{\partial a(t)}{\partial t}|_{-\epsilon} - \frac{\partial a(t)}{\partial t}|_{\epsilon}\right] - U_0 a(0) \quad = \quad E\int_{-\epsilon}^{\epsilon} a(t)dt \tag{D.17}$$

Now, let epsilon goes to zero. Then the right hand side of the equation will go to zero since a is finite and we integrate it over a zero width. This gives us

$$\left[\frac{\partial a(t)}{\partial t}|_{-\epsilon} - \frac{\partial a(t)}{\partial t}|_{\epsilon}\right] \quad = \quad -U_0 a(0) \tag{D.18}$$

and let $\epsilon \rightarrow 0$

a. $t > 0$, $a(t) = Ae^{-\kappa t}$, then $\frac{\partial a(t)}{\partial t}|_0 = -A\kappa$

b. $t < 0$, $a(t) = Ae^{\kappa t}$, then $\frac{\partial a(t)}{\partial t}|_{-0} = A\kappa$

so that

$$\left[\frac{\partial a(t)}{\partial t}|_{-\epsilon} - \frac{\partial a(t)}{\partial t}|_{\epsilon}\right] = -A\kappa - A\kappa = -2A\kappa \tag{D.19}$$

Now, $a(0) = A$ and then we can obtain κ as follows

$$-2A\kappa = -AU_0$$

$$\kappa = \frac{1}{2}U_0 = \pm\sqrt{-E} \tag{D.19}$$

where we defined the energy $E = -\frac{U_0}{2}$. Hence, there is one and only one allowed energy level as

$$E = -\frac{\Lambda}{3}, \;\; U_0 = \pm 2\sqrt{\frac{\Lambda}{3}}$$

To be complete, we can find the constant A via the normalization constant

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = 2A^2 \int_{0}^{\infty} e^{-2\kappa t} dt = 1 \tag{D.19}$$

Then,

$$A \;\; = \;\; \sqrt{\kappa} = \left(\sqrt{-E}\right)^{1/2} = \left(\frac{U_0}{2}\right)^{1/2} = \left(\pm\sqrt{\frac{\Lambda}{3}}\right)^{1/2} \tag{D.20}$$

Finally, the wave function $a(t) = Ae^{-\kappa t}$ becomes,

$$a(t) = \left(\frac{U_0}{2}\right)^{1/2} e^{-\frac{U_0}{2}|t|} \tag{D.21}$$

or

$$a(t) = \left(\pm\sqrt{\frac{\Lambda}{3}}\right)^{1/2} e^{-\left(\pm\sqrt{\frac{\Lambda}{3}}\right)^{1/2}|t|} \tag{D.22}$$

Case [$\Lambda < 0$] or Unbound State:

When the state is unbound, it does not vanish at infinity, it will not be possible to normalize the wave function. In addition to this, such states are called scattering states. For these states different normalization convention is used. Equation (D.13) becomes

$$-\ddot{a}(t) - U_0\delta(t)a(t) = Ea(t)$$

and

$$\ddot{a}(t) + U_0\delta(t)a(t) = \kappa^2 a(t) \tag{D.22}$$

where $\kappa = \sqrt{-E} = \sqrt{-\frac{\Lambda}{3}} = i\sqrt{\frac{\Lambda}{3}}$. Out side $t = 0$, the solution of equation (D.22) becomes

$$a(t) = Ae^{i\kappa t} + Be^{-i\kappa t} \tag{D.23}$$

This is the most general form both to the left and to the right. Thus, there are four constants involved: A_{left}, A_{right}, B_{left}, B_{right}. We need four constraints to determine these constants. But, there are only two constraints. Clearly we need a different procedure. For scattering states it is traditional to specify the direction of the incident beam of particles. This eliminates one of the four unknown constants. Since such states are not normalizable, the incident flux (amplitude) is chosen as one. Suppose we specify the beam is incident from the left. Then the solutions are taken to be,

$$a(t) = \left\{ \begin{array}{ll} 1.e^{i\kappa t} + B_{left}e^{-i\kappa t}, & t < 0 \\ A_{right}e^{i\kappa t}, & t > 0 \end{array} \right) .$$

We assume that the amplitude of the incident wave, the first term on the right-hand side of the first equation of the $a(t)$ is unity. Then the constraint relations are

$$1 + B_{left} = A_{right} \tag{D.24}$$

and note that the point $t = 0$ is singularity; we integrate equation (D.22) between $-\epsilon$ and ϵ where ϵ is small positive number, then take the limit as $\epsilon \longrightarrow 0$ and thus

$$a'(0^+) - a'(0^-) = -U_0 a(0) \tag{D.25}$$

Also at the point $t = 0$, equation (D.25) is reduced to

$$a'(0^+) = a'(0^-) \tag{D.26}$$

Hence,

$$a'(0^+) = i\kappa A_{right} \tag{D.27}$$

$$a'(0^-) = i\kappa[1 - B_{left}] \tag{D.28}$$

$$a(0) = A_{right} \tag{D.29}$$

let us substitute (D.27), (D.28) and (D.29) in equation (D.25). Hence

$$i\kappa A_{right} - i\kappa[1 - B_{left}] = -U_0 A_{right} \tag{D.30}$$

dividing equation (D.30) by κ and

$$iA_{right} - i[1 - B_{left}] = -\frac{U_0}{\kappa} A_{right} \tag{D.31}$$

we assumed that $\beta \equiv \frac{U_0}{2\kappa}$ and using by equation (D.31) and (D.24), we determine the coefficients B_{left} and A_{right} as follows

$$A_{right} = \frac{1}{1 - i\beta} \tag{D.32}$$

$$B_{left} = \frac{i\beta}{1 - i\beta} \tag{D.33}$$

Reflection and transmission coefficients

Whenever a Quantum Mechanical wave experiences a change of potential there is a probability that the wave will be reflected in terms of the coefficients of the wave function, this probability is

$$R = |\frac{B_{left}}{A_{left}}|^2 = |B_{left}|^2 \tag{D.34}$$

Hence,

$$R = |B_{left}|^2 = \left(\frac{i\beta}{1 - i\beta}\right)\left(\frac{-i\beta}{1 + i\beta}\right) \tag{D.35}$$

and the probability density of reflection term is

$$R = \frac{\beta^2}{1 + \beta^2} = \left(\frac{1}{1 + \frac{4E}{U_0^2}}\right) \tag{D.36}$$

where $U_0 = \frac{4\pi}{3}\left[\rho_0 + 3p_0\right]$. The probability density of transmission across the interface is

$$T = 1 - R = |\frac{A_{right}}{A_{left}}|^2 = |A_{right}|^2 \tag{D.37}$$

$$T = 1 - R = 1 - \left(\frac{1}{1 + \frac{4E}{U_0^2}}\right) = \frac{1}{1 + \frac{U_0^2}{4E}} \tag{D.38}$$

where $U_0 = \frac{4\pi}{3}\left[\rho_0 + 3p_0\right]$.