ANALYSIS OF STOCHASTIC DYNAMICAL SYSTEMS

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ABSTRACT

ANALYSIS OF STOCHASTIC DYNAMICAL SYSTEMS

In this thesis, analysis of stochastic dynamical systems have been considered in the sense of stochastic differential equations (SDEs). Brownian motion, which can be considered as a first example of stochastic dynamical systems, its derivation and its properties have been investigated, then the analytic and numerical solution methods of SDE have been studied with the examples from the physical world. In order to construct a random variable in a computer environment, random number generation algorithms have also been investigated. Finally a Matlab-Simulink block for numerical solutions of linear SDEs has been newly developed.

ÖZET

STOKASTİK DİNAMİK SİSTEMLERİN ANALİZİ

Bu tezde, stokastik dinamik sistemlerin analizi, stokastik diferansiyel denklemler bağlamında incelenmiştir. Stokastik dinamik sistemlerin ilk örneği olarak, Brown hareketi, türetilmesi ve özellikleri, incelendi. Daha sonra stokastik diferansiyel denklemlerin analitik ve nümerik çözüm metodları, bu metodların türetilmesi, bu metodlara ilişkin fiziksel örnekler çalışıldı. Öte yandan bir rastsal değişkenin bilgisayar ortamında oluşturulmasında yüksek öneme sahip olan rastgele sayı üreteç algoritmaları incelendi ve son olarak doğrusal stokastik diferansiyel denklemlerin nümerik çözümü için yeni bir Matlab-Simulink bloğu geliştirildi.

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CHAPTER 1

INTRODUCTION

In deterministic dynamical systems theory, system behaviours are modeled with partial differential equations (PDEs) (wave equation, heat equation, Laplace equation, electrodynamics, fluid flow etc.) or ordinary differential equations (ODEs) (classical Newtonian mechanic, lump electric circuits, etc.). When stochastic fluctuations, such as noise in electronic circuitry, fluctuations in stock exchanges, disturbances in communication systems can not be ignored then additional random terms should be included in the ODEs and PDEs to represent the stochastic dynamics as in (1.1) (Primak et al. 2004).

$$\frac{dX(t)}{dt} = f(X(t), t) + g(X(t), t)\xi(t), \quad X(t_0) = x_0 \tag{1.1}$$

or in differential form

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), \quad X(t_0) = x_0 \tag{1.2}$$

The above equation (1.2) is a general form of stochastic differential equation where $\xi(t) = \frac{dB(t)}{dt}$. $\xi(t)$ is the derivative of Brownian motion which is usually referred as white noise. Brownian motion is alternatively called Wiener Process who built the theory of the Brownian motion (Wiener 1920, 1921, 1923).

If (1.2) is integrated, one obtains

$$X(t) = \int_{t_0}^{t} f(X(\tau), \tau)d\tau + \int_{t_0}^{t} g(X(\tau), \tau)dB(\tau) \ . \tag{1.3}$$

The second integral in the right handside of (1.3) is not the usual integral in the sense of ordinary calculus since this integral involves a stochastic differential. Therefore to overcome the problems obtaining the solutions of stochastic differential equations, stochastic calculus was developed by Itô in 1944. In this thesis, related references have been investigated to understand the concepts of stochastic differential equations such as Itô formula, numerical integration methods for SDEs, Stratonovich integral, Fokker-Planck equation (Itô 1944, 1950, 1951a,b, Kannan and Lakshmikantham 2002, Oksendal 2000, McKean 1969, WEB_1 2005, Protter 2004, Risken 1989, Lasota and Mackey 1985, Friedman 1975, Kloeden and Platen 1992). Probabilistic concepts of stochastic dynamics, such as covariance of a process, Markovian property, contuinuity of a process, Markovian property, martingale property, continuity in mean square sense, differentiability of a process and differentiability in the mean square sense have been introduced in Chapter II based on the related references (Stark and Woods 2002, Kannan 1979, Ross 1997).

The main problem in obtaining the solution of (1.1) is the integration with respect to a stochastic process in (1.3) which does not have a deterministic increment element $dB(t)$. This increment element has infinite total (first) variation, which implies nondifferentiability in ordinary calculus sense that violates the fundamental theorem of calculus. This theorem states that differentiation and integration are inverse operations. If a function first integrated and then differentiated, the result is the original function. But if a Brownian motion is first integrated with this stochastic increment and then differentiated the result is not the Brownian motion but with an extra term in the result of integration. This will be shown in Section 3.2.1.2.

First variation basically measures total increasing and decreasing movement of a function. It is similar to arclength of the graph of a function. An everywhere differentiable function has bounded first variation. But Brownian motion paths have unbounded first variation. Hence it is necessary to define a higher variation that the Brownian motion have bounded variation. The definition of variation will be explained in Chapter II.

In Chapter II it has been explained how the Brownian motion is constructed and then fundamental properties of the Brownian motion have been explained. Chapter III describes the stochastic differential equation comparing it to ordinary differential equations with some applications and also the Langevin equation which decribes the Brownian motion has been introduced. In Chapter IV, existence and uniqeness problems of SDE has been introduced. Then analytic and numerical solutions have been investigated. Afterwards in order to demonstrate the analytic solution methods, in Chapter V, examples have been given and a Matlab-Simulink numerical integrator block for a linear type of stochastic differential equation has been newly developed.

CHAPTER 2

BROWNIAN MOTION

In this chapter the history of the Brownian motion has been covered based on works of Nelson, Cohen, Hanggi and Marchesoni (Nelson 1967, Cohen 2005, Hänggi and Marchesoni 2005). Properties of the Brownian Motion studied based on the related books (Kannan 1979, Situ 2005, Stark and Woods 2002).

Historically, Brownian motion is a stochastic process of the pollen particles in a viscous fluid that was observed by Robert Brown (Brown 1827). Before Brown, Jan Ingenhousz had observed carbon dust in alcohol in 1785. R.Brown in his second paper concluded that "not only organic tissues but also inorganic substances consists of animated or irritable particles" (Brown 1828). R.Brown observed the motion quantitatively but he did not make a remark on the cause of this random motion. George Gouy, an experimental physicist, he also did not make a remark on the cause of this random motion but he concluded that Brownian motion was not due to the external forces like light, vibrations, electricity etc.. Einstein in his first paper discussed Brownian motion briefly based on diffusion and kinetic theory of gases (Einstein 1905). He also emphasized it was possible that the movements of the particles described in his paper were identical with the so-called Brownian motion. In his second paper, he put the statement that the motion was caused by the irregular thermal movement of the molecules of the liquid and his third paper was published in 1907 with the title *"Theoretical Observations on the Brownian Motion "*. Further, in 1908 he published *"The Elementary Theory of Brownian Motion "*. In his papers, Einstein derived the diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \qquad f(x,t=0) = \delta(0) \tag{2.1}$$

where $f(x,t)$ is the probability density function of Brownian particle being at position x at time t and D is the diffusion coefficient. Then he solved it for the impulsive initial distribution (δ-function of position at time zero normalized to the number of small particles n) and found

$$f(x,t) = \frac{n}{\sqrt{4\pi D}}\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{t}}. \tag{2.2}$$

Then he wrote the mean displacement of the particle in time t

$$\lambda_x = \sqrt{\overline{x}^2} = \sqrt{2Dt} \tag{2.3}$$

as the standard deviation, which states that the mean displacement is proportional to the square root of time. On the other hand, interestingly Einstein found the diffusion coefficient D as

$$D = \frac{RT}{N}\frac{1}{6\pi kP} \tag{2.4}$$

where R is the gas constant, T is the temperature, k is the viscosity (is not the Boltzman constant), P is the radius of the small particle and N is the Avagadro's number $(6,02 \times 10^{-23}$ which is the number of ^{12}C atoms in 12 grams of unbound ^{12}C in its ground state). Then combining equations (2.3) and (2.4) and solving for N gives

$$N = \frac{1}{\lambda_x^2}\frac{RT}{3\pi kP} \tag{2.5}$$

This was an interesting result about finding and verifying the universal number N. The verification of Einstein's works had come from Smoluchowski (Smoluchowski 1906). Perrin confirmed the mean displacement experimentally (Perrin 1909). The first modern mathematical theory was developed by Wiener (Wiener 1920, 1921, 1923). Wiener assigned a probability measure to Brownian motion and this measure defines the physical properties of Brownian motion which are the independent and normally distributed increments of Brownian motion.

After this brief historical explanation of Brownian motion in the subsequent sections derivation and properties of Brownian motion have been explained.

2.1. Derivations of the Brownian Motion

There are two kind of approaches for deriving the Brownian motion, random walk approach and Einstein's approach.First, the random walk approach, which uses the CLT(Central Limit Theorem), will be introduced and then the second derivation based on the Einstein's approach related with diffusion will be introduced.

Theorem 2.1 (Central Limit Theorem (CLT)) *Let $X_1, \ldots, X_n, \ldots$ identically and independent distributed random variables with*

$$E(X_i) = \mu, \quad V(X_i) = g^2 > 0 \quad i = 1, \ldots.$$

and

$$S_n \triangleq X_1 + \cdots + X_n. \tag{2.6}$$

Then the central limit theorem states that for all a, b such that $-\infty < a < b < +\infty$

$$\lim_{n\to\infty} P(a \leq \frac{S_n - n\mu}{\sqrt{n}\sigma} \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx \tag{2.7}$$

(Skorokhod 2005).

2.1.1. Random Walk Approach

Let $x_i,\ i = 1, 2, 3, \ldots$ be a random variable with

$$P(x_i = k) = p \quad and \quad P(x_i = -k) = q = 1 - p \tag{2.8}$$

where k denotes the size of the i^{th} step, with probability p that the walk is toward the positive direction and with probability q toward the negative direction. Then it can be shown that,

$$E(x_i) = (p - q)k \qquad and \qquad var(x_i) = 4pqk^2 \tag{2.9}$$

Definition 2.1 (Random Walk Process) *If X_n denotes the position of the random walk (i.e. $n = 1, 2, \ldots$, Let $X_n = x_1 + x_2 + \ldots + x_n$) after n steps on a line in time t, then stochastic process $\{X_n,\ \ n \geq 0\}$ is called a random walk process.*

Then from equation (2.9) it can be seen that,

$$E(X_n) = (p - q)nk \quad and \quad var(X_n) = 4pqk^2n \tag{2.10}$$

Let there be r random walks in unit time, then $\lambda = 1/r$, is the time interval between two random walks. In limit case,

$$\lim_{r\to\infty} \lambda = 0 \tag{2.11}$$

Let μ be the mean displacement and σ^2 be the variance in time t. Then, $n\lambda = t$ and from (2.10) ,

$$\mu = \frac{E(X_n)}{n\lambda} = (p - k)\frac{k}{\lambda} \tag{2.12}$$

$$\sigma^2 = \frac{var(X_n)}{n\lambda} = 4pq\frac{k^2}{\lambda} \tag{2.13}$$

Since $p+q=1$

$$p = \frac{1}{2}(1+\mu\frac{\lambda}{k}) \quad and \quad q = \frac{1}{2}(1-\mu\frac{\lambda}{k}) \tag{2.14}$$

from (2.13) and (2.14),

$$\sigma^2 = \frac{k^2}{\lambda} - \mu^2\lambda \approx \frac{k^2}{\lambda} \tag{2.15}$$

where λ has been assumed to be small.

Let $u(t,x)$ denote the probability that the particle takes the position x at time t,

$$u(x,t) = P(X_n = x) \quad at \quad t = n\lambda \tag{2.16}$$

and the probability function satisfies the recurrence relation

$$u(x, t+\lambda) = pu(x-k, t) + qu(x+k, t) \tag{2.17}$$

The Taylor expansion of both sides of the equation (2.17) is

$$\begin{aligned} u(x,t) + \lambda\frac{\partial u(x,t)}{\partial t} + O(\lambda^2) &= u(x,t) + k(q-p)\frac{\partial u(x,t)}{\partial x} + \frac{k^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + O(k^3) \\ \Rightarrow \frac{\partial u(x,t)}{\partial t} &= \left[(q-p)\frac{k}{\lambda}\right]\frac{\partial u(x,t)}{\partial x} + \frac{k^2}{2\lambda}\frac{\partial^2 u(x,t)}{\partial x^2} \end{aligned} \tag{2.18}$$

under the assumption $\lambda, k \to 0$. By plugging (2.15) into (2.18)

$$\frac{\partial u(x,t)}{\partial t} = -\mu\frac{\partial u(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{2.19}$$

which is called the Forward Kolmogorov Equation (Fokker Planck Equation) where μ is the drift rate and σ^2 is the diffusion rate. The solution of equation (2.19) is a Gaussian function and given as

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} \; exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) \tag{2.20}$$

This solution has a peak at $x = \mu t$ and its width is $\sigma^2 t$. This is again the probability density of the particle being at position x at time t.

2.1.2. Einstein's Approach

Consider a tube is filled with clear water and a unit amount of ink is injected at time $t = 0$ at location $x = 0$. Let $\rho(x,t)$ be the physical density of the ink particles at position x and at time t and let

$$\rho(x,t) = \delta_0 \tag{2.21}$$

Then assume that the probability density function of the event that an ink particle moves amount of y in small time τ , is $f(\tau, y)$. So ,

$$\begin{aligned} \rho(x, t+\tau) &= \int_{-\infty}^{\infty} \rho(x-y, t) f(\tau, y) dy \qquad (2.22) \\ &= \int_{-\infty}^{\infty} (\rho - \rho_x y + \frac{1}{2}\rho_x x y^2 + \ldots) f(\tau, y) dy. \end{aligned}$$

It is known that,

$$\int_{-\infty}^{\infty} f(\tau, y) dy = 1, \qquad (2.23)$$

and by symmetry

$$f(\tau, -y) = f(\tau, y) \qquad (2.24)$$

and

$$\int_{-\infty}^{\infty} y f dy = 0. \qquad (2.25)$$

The variance of the random variable y is linear in τ

$$\int_{-\infty}^{\infty} y^2 f dy = D\tau, \quad D > 0 \qquad (2.26)$$

plugging these identities into equation (2.22), one obtains

$$\frac{\rho(x, t+\tau) - \rho(x, t)}{\tau} = \frac{D\, \rho_{xx}(x, t)}{2} + (\text{lower} \quad \text{order} \quad \text{terms}) \qquad (2.27)$$

then

$$\lim_{\tau \to 0} \frac{\rho(x, t+\tau) - \rho(x, t)}{\tau} = \rho_t \qquad (2.28)$$

finally

$$\rho_t = \frac{D}{2}\rho_{xx} \quad \text{with initial condition} \quad \rho(t=0, x) = \delta(x) \qquad (2.29)$$

the form of the equation (2.29) is the form of *heat equation* which is also called *diffusion equation* with diffusion coefficient D. With initial condition $\rho(0, x) = \delta_0$ it has a solution, as

$$\rho(x, t) = \frac{1}{(2\pi D t)^{1/2}}\ e^{\frac{x^2}{2Dt}} \qquad (2.30)$$

which says that the *probability density* of a particle being at position x at time t is normally distributed with zero mean and Dt variance $\mathcal{N}(0, Dt)$, for some constant D.

In the next section solution of the heat equation will be derived.

2.2. Solution of Heat/Diffusion Equation

Let the Heat/Diffusion Equation with $D = 1$ be

$$\frac{\partial \rho(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho(t,x)}{\partial x^2}, \quad \rho(t=0,x) = \delta(x). \tag{2.31}$$

The fundamental solution of (2.31) is

$$\rho(t,x) = \frac{1}{\sqrt{2\pi t}} exp\left(-\frac{x^2}{2t}\right) \tag{2.32}$$

To obtain this solution separation of variables has been used as (WEB_4 2005),

$$\rho(t,x) = f(t)h(x), \quad \text{for} \quad t > 0, \quad \text{and} \quad -\infty < x < \infty \tag{2.33}$$

then substituting this into (2.31), it is obtained that

$$\begin{aligned} h(x)\frac{df(t)}{dt} &= \frac{1}{2}f(t)\frac{d^2h(x)}{dx^2} \\ \frac{2}{f(t)}\frac{df(t)}{dt} &= \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} \end{aligned} \tag{2.34}$$

These terms are equal if and only if both side of the equation equals to a constant λ i.e.,

$$\frac{2}{f(t)}\frac{df(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = \lambda. \tag{2.35}$$

Three cases of λ have been considered :

CASE I : If $\lambda > 0$ then $\lambda = k^2$

$$\frac{2}{f(t)}\frac{df(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = k^2 \tag{2.36}$$

solving for $f(t)$ and $h(x)$ it is obtained that

$$\begin{aligned} \frac{1}{f(t)}\frac{df(t)}{dt} &= \frac{1}{2}k^2 \\ \frac{df(t)}{f(t)} &= \frac{1}{2}k^2 dt \\ \int_0^t \frac{df(t)}{f(t)} &= \frac{1}{2}k^2 \int_0^t dt \\ f(t) &= Ce^{\frac{1}{2}k^2 t} \end{aligned} \tag{2.37}$$

$$\begin{aligned} \frac{d^2h(x)}{dx^2} &= k^2 h(x) \\ h(x) &= Ae^{kx} + Be^{-kx} \end{aligned} \tag{2.38}$$

Then the solution is

$$\begin{aligned} \rho(t,x) &= f(t)h(x) \\ &= Ce^{\frac{1}{2}k^2t}(Ae^{kx} + Be^{-kx}) \\ &= CAe^{\frac{1}{2}k^2t+kx} + CBe^{\frac{1}{2}k^2t-kx}. \end{aligned} \tag{2.39}$$

But the solution diverges as $x \to \pm\infty$

$$\begin{aligned} \lim_{x\to\ \infty} CAe^{\frac{1}{2}k^2t+kx} &= \infty \\ \lim_{x\to\ -\infty} CBe^{\frac{1}{2}k^2t-kx} &= \infty. \end{aligned}$$

Since the initial condition is not satisfied the from of (2.39) can not be a solution.

<u>**CASE II:**</u> $\lambda = 0$

$$\frac{2}{f(t)}\frac{df(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = 0 \tag{2.40}$$

Again solving for $f(t)$ and $h(t)$

$$\begin{aligned} \frac{1}{f(t)}\frac{df(t)}{dt} &= 0 \\ df(t) &= 0 \\ f(t) &= C \end{aligned} \tag{2.41}$$

$$\begin{aligned} \frac{d^2h(x)}{dx^2} &= 0 \\ h(x) &= A + Bx \end{aligned} \tag{2.42}$$

$$\begin{aligned} \rho(t,x) &= f(t)h(x) \\ &= C(A+Bx) \\ &= CA + CBx \end{aligned} \tag{2.43}$$

Solution (2.43) is not divergent if and only if $B = 0$. But on the other hand if $B = 0$ then (2.43) is independent of time t and position x. Therefore such a form of solution (2.43) does not satisfy the initial condition.

<u>**CASE III :**</u> $\lambda < 0$ so let $\lambda = -k^2$

$$\frac{2}{f(t)}\frac{df(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = -k^2 \tag{2.44}$$

solving for $f(t)$ and $h(x)$ it is obtained

$$\begin{aligned}
\frac{1}{f(t)}\frac{df(t)}{dt} &= -\frac{1}{2}k^2 \\
\frac{df(t)}{f(t)} &= -\frac{1}{2}k^2 dt \\
\int_0^t \frac{df(t)}{f(t)} &= -\frac{1}{2}k^2 \int_0^t dt \\
f(t) &= Ce^{-\frac{1}{2}k^2 t} \qquad (2.45)
\end{aligned}$$

$$\begin{aligned}
\frac{d^2h(x)}{dx^2} &= -k^2 h(x) \\
h(x) &= Ae^{ikx} + Be^{-ikx} \qquad (2.46)
\end{aligned}$$

Then the solution is

$$\begin{aligned}
\rho(t,x) &= f(t)h(x) \\
&= Ce^{-\frac{1}{2}k^2 t}(Ae^{ikx} + Be^{-ikx}) \\
&= CAe^{-\frac{1}{2}k^2 t + ikx} + CBe^{-\frac{1}{2}k^2 t - ikx} \qquad (2.47)
\end{aligned}$$

From Euler Formula

$$e^{ix} = \cos x + i \sin x \qquad (2.48)$$

(2.47) is oscillatory as $x \to \pm\infty \quad \forall t \geq 0$

$$\begin{aligned}
\lim_{x\to\pm\infty} CAe^{-\frac{1}{2}k^2 t + ikx} &= \lim_{x\to\pm\infty} CAe^{-\frac{1}{2}k^2 t}(\cos(kx) + i\sin(kx)) = \text{oscillatory} \\
\lim_{x\to\pm\infty} CBe^{-\frac{1}{2}k^2 t - ikx} &= \lim_{x\to\pm\infty} CBe^{-\frac{1}{2}k^2 t}(\cos(kx) + i\sin(kx)) = \text{oscillatory}
\end{aligned}$$

On the other hand solution (2.47) is bounded as $x \to \pm\infty \quad \forall t \geq 0$

$$\begin{aligned}
\lim_{x\to\pm\infty} |CAe^{-\frac{1}{2}k^2 t + ikx}| &= |CA|e^{-\frac{1}{2}k^2 t} \\
\lim_{x\to\pm\infty} |CBe^{-\frac{1}{2}k^2 t - ikx}| &= |CB|e^{-\frac{1}{2}k^2 t}
\end{aligned}$$

Therefore (2.47) is a candidate of the solution. Then the general solution of the equation (2.29) is an integral on k

$$\rho(t,x) = \int_0^\infty C(k)A(k)e^{-\frac{1}{2}k^2 t + ikx}dk + \int_0^\infty C(k)B(k)e^{-\frac{1}{2}k^2 t - ikx}dk$$

$$\rho(t,x) = \int_0^\infty C(k)A(k)e^{-\frac{1}{2}k^2 t + ikx}dk + \int_{-\infty}^0 C(k)B(k)e^{-\frac{1}{2}k^2 t - ikx}dk. \qquad (2.49)$$

The integrals in (2.49) can be written in a compact form as

$$\rho(t,x) = \int_{-\infty}^{\infty} Q(k)e^{-\frac{1}{2}k^2t+ikx}\ \frac{dk}{2\pi} \tag{2.50}$$

where

$$Q(k) = \begin{cases} 2\pi C(k)A(k) & \text{if } k > 0, \\ 2\pi C(-k)B(-k) & \text{if } k < 0. \end{cases} \tag{2.51}$$

The function $Q(k)$ is found by substituting the initial condition into equation (2.29)

$$\rho(t=0,x) = \delta(x) = \int_{-\infty}^{\infty} Q(k)e^{ikx}\frac{dk}{2\pi} \tag{2.52}$$

On the other hand, Fourier transform of $g(x)$ is

$$\tilde{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx$$

and inverse Fourier Transform of $\tilde{g}(k)$ is

$$g(x) = \int_{-\infty}^{\infty} \tilde{g}(k)e^{ikx}\frac{dk}{2\pi}.$$

Letting $g(x) = \delta(x)$

$$\int_{-\infty}^{\infty} 1\, e^{ikx}\frac{dx}{2\pi} = \delta(x) \tag{2.53}$$

From equation (2.53) and (2.52) it has found that

$$Q(k) = 1 \tag{2.54}$$

Then the below integral can be evaluated as :

$$\rho(t,x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2t+ikx}\ \frac{dk}{2\pi}. \tag{2.55}$$

Completing the square in the exponent in equation (2.55) it can be written as

$$\begin{aligned} k^2t - 2ikx &= t\left(k^2 - 2k\frac{ix}{t} + \left(\frac{ix}{t}\right)^2 - \left(\frac{ix}{t}\right)^2\right) \\ &= t\left(k - \frac{ix}{t}\right) + \frac{x^2}{t} \end{aligned} \tag{2.56}$$

Then substituting this into integral will result

$$\rho(t,x) = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}t\left(k - \frac{ix}{t}\right) - \frac{x^2}{2t}\right)\frac{dk}{2\pi} \tag{2.57}$$

$$= exp\left(-\frac{x^2}{2t}\right)\int_{-\infty}^{\infty} exp\left(-\frac{1}{2}t\left(k - \frac{ix}{t}\right)\right)\frac{dk}{2\pi} \tag{2.58}$$

using the following change of variables

$$u = \sqrt{t}\left(k - \frac{ix}{t}\right) \Rightarrow \quad du = \sqrt{t}dk \tag{2.59}$$

it can be found

$$\rho(t,x) = \frac{1}{\sqrt{2\pi t}}\, exp\left(-\frac{x^2}{2t}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du. \tag{2.60}$$

Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = 1 \tag{2.61}$$

which is a Gaussian integral then finally the solution ***heat / diffusion equation*** is

$$\rho(t,x) = \frac{1}{\sqrt{2\pi t}}\, exp\left(-\frac{x^2}{2t}\right). \tag{2.62}$$

2.3. Properties of Brownian Motion

After the physical interpretations of Brownian motion in Section 2.1. definition of Brownian motion and its properties have been given in this section.

Definition 2.2 (Brownian Motion/Wiener Process) *A Brownian motion, is the stochastic process describing the position of the pollen particles in a viscous fluid $B(t,\omega)$ or alternatively $W(t,\omega)$ Wiener Process after Norbert Wiener's works on Brownian motion (Wiener 1920, 1921, 1923), satisfying*

1. $B(0) = 0$ *;*
2. *For any* $0 \leq t_0 < t_1 < \ldots < t_n$*, the random variables* $B(t_k) - B(t_{k-1})$*, where* $1 \leq k \leq n$ *are independent ;*
3. *If* $0 \leq s < t$ *then* $B(t) - B(s) \sim \mathcal{N}(0, t-s)$*.*

The independence property is defined as below :

Definition 2.3 (Independence of random variables) *Let $X(t)$ be a random process then two random variables $X(t_1)$ and $X(t_2)$ are independent if the events $\{X(t_1) \leq x_1\}$ and $\{X(t_2) \leq x_2\}$ are independent for every combination of x_1 and x_2.*

Definition 2.4 (Independence of two events) *Two events A and B are independent if*

$$P(AB) = P(A)P(B) \tag{2.63}$$

where event AB is the intersection of events A and B.

Letting

$$\begin{aligned} A &\triangleq \{X(t_1) \leq x_1\} \\ B &\triangleq \{X(t_2) \leq x_2\} \\ AB &\triangleq \{X(t_1) \leq x_1\} \cap \{X(t_2) \leq x_2\} \end{aligned} \tag{2.64}$$

and the probability distributions are

$$\begin{aligned} F_{X(t_1)}(x_1) &\triangleq P(X(t_1) \leq x_1) \\ F_{X(t_2)}(x_2) &\triangleq P(X(t_2) \leq x_2) \\ F_{X(t_1)X(t_2)}(x_1, x_2) &\triangleq P(X(t_1) \leq x_1, X(t_2) \leq x_2) \end{aligned}$$

then it can be written as

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(t_1)}(x_1)F_{X(t_2)}(x_2) \ \ \forall x_1\, x_2 \tag{2.65}$$

if and only if $X(t_1)$ and $X(t_2)$ are independent. Furthermore it can be written for the probability density functions of the random variables.

$$\begin{aligned} f_{X(t_1)X(t_2)}(x_1, x_2) &= \frac{\partial^2 F_{X(t_1)X(t_2)}(x_1, x_2)}{\partial x_1 \partial x_2} \\ &= \frac{\partial F_{X(t_1)}(x_1)}{\partial x_1} \frac{\partial F_{X(t_2)}(x_2)}{\partial x_2} \\ &= f_{X(t_1)}(x_1) f_{X(t_2)}(x_2) \end{aligned}$$

This independent increment property (2) in the definition of Brownian motion has also been used to define the Markovian property.

To visualize Brownian motion, in Figure 2.1 the sample paths of one dimensional, two dimensional and three dimensional Brownian motion have been plotted using Matlab. One dimensional path can be considered as motion on a straight line has just forwards and backwards movements. In two dimensional case it has any direction movement on the xy-plane which R.Brown had observed. Three dimensional Brownian motion can be considered as a random flight because altitude of the motion also changes randomly.

In this section properties of Brownian motion have been discussed. These properties of the Brownian motion give the ability to evaluate the stochastic integrals which plays an important role in the solution of stochastic differential equation.

Although Brownian motion is continuous in t, it is not differentiable for all t. It is a Normal (Gaussian) process also its independent increments are normally distributed. It

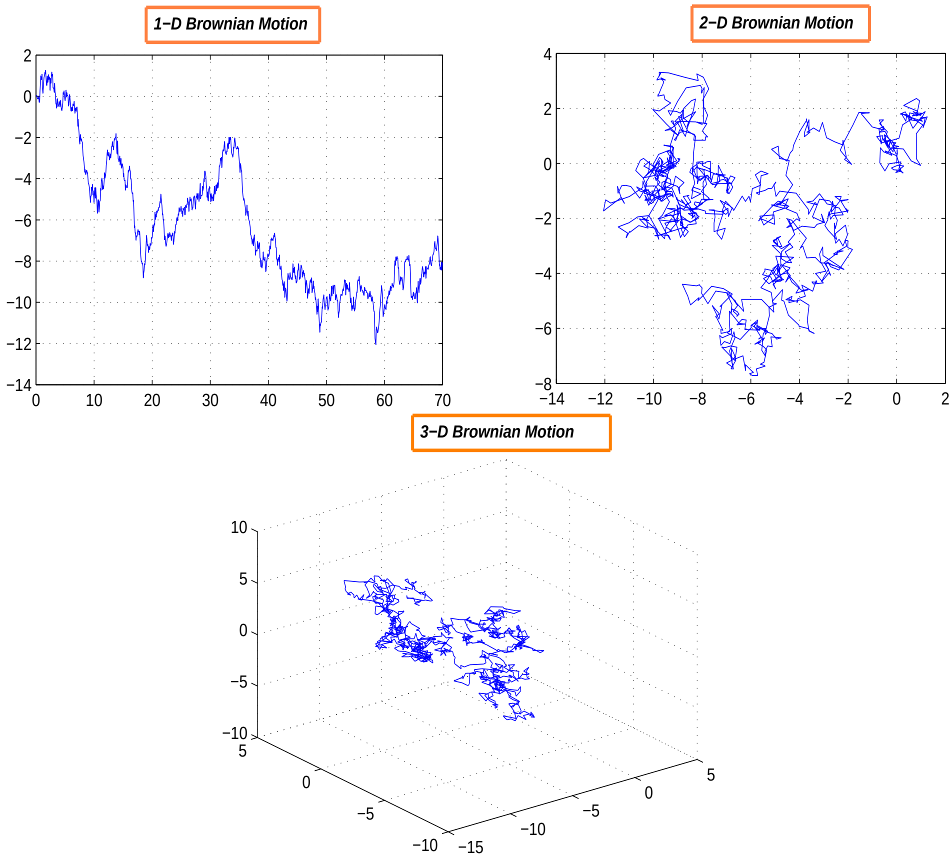

Figure 2.1. Brownian motion in 1-D, 2-D and 3-D.

has been mentioned that Brownian Motion has an infinite total variation, this property has a close relation with the differentiability since each finite variational function of t should be everywhere differentiable for all t (Situ 2005, Kannan 1979).

2.3.1. Infinite Total(First) Variation Property

In this section first and second variations have been defined and variations of Brownian motion have been evaluated. Variations of Brownian motion have been plotted in Figure 2.2 up to fourth order.

Definition 2.5 (Total Variation) *Let the interval $[0, T]$ partition into n pieces and $\triangle t(n) = \frac{T}{n}$ represents the stepsize of the partition and*

$$0 = t_0 < t_1 < \ldots < t_n = T \tag{2.66}$$

be the limits of the partition. Then the total variation of the Brownian path $B(\omega)$ is

$$\mathrm{V}_T(B(\omega)) = \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|. \tag{2.67}$$

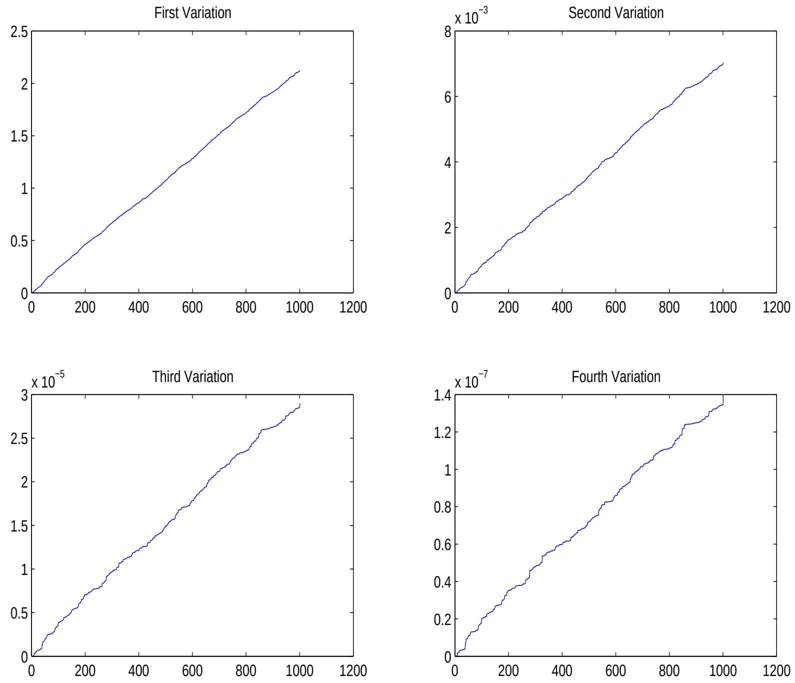

Figure 2.2. Variations of the Brownian path up to fourth order.

Theorem 2.2 *Almost surely no path of a Brownian motion has bounded first variation for every $T \geq 0$*

$$P(\omega : V_T(B(\omega)) < \infty) = 0. \tag{2.68}$$

Therefore it is necessary to define a higher variation to make Brownian motion has a bounded variation (Protter 2004). It has been shown that quadratic variation, second variation, of the Brownian motion has a finite variation t in (WEB_1 2005). The definition of quadratic variation has been given below.

Definition 2.6 (Quadratic Variation) *Again let the interval $[0, T]$ partition into n pieces and $\triangle\, t(n) = \frac{T}{n}$ represents the stepsize of the partition and*

$$0 = t_0 < t_1 < \ldots < t_n = T \tag{2.69}$$

$$\mathrm{QV}_T(B(\omega)) = \sup_{n \in \mathbb{N}} \lim_{n \to \infty} \sum_{k=0}^{n-1} (B(t_{k+1}) - B(t_k))^2. \tag{2.70}$$

Brownian motion has a quadratic variation equals to t that is

$$\sup_{n \in \mathbb{N}} \lim_{n \to \infty} \sum_{k=0}^{n-1} (B(t_{k+1}) - B(t_k))^2 = t. \tag{2.71}$$

Proof Let

$$E[B_{t_{j+1}} - B_{t_j}]^2 = [t_{j+1} - t_j] \tag{2.72}$$

then,

$$E\sum_{j=0}^{n-1}[B_{t_{j+1}} \;- B_{t_j}]^2 = t. \tag{2.73}$$

Therefore,

$$\lim_{n} E(\sum_{j=0}^{n-1}[B_{t_{j+1}} - B_{t_j}]^2 - t)^2 = 0 \tag{2.74}$$

If (2.74) holds then it will be proved that the Brownian motion has a quadratic variation t. Inside of the expectation operator can be written as,

$$\sum_{j=0}^{n-1}[B_{t_{j+1}} - B_{t_j}]^2 - t = \sum_{j=0}^{n-1}([B_{t_{j+1}} - B_{t_j}]^2 - [t_{j+1} - t_j])$$

$$\begin{aligned}
\lim_{n} E(\sum_{j=0}^{n-1}[B_{t_{j+1}} - B_{t_j}]^2 - t)^2 &= \sum_{j=0}^{n-1} E([B_{t_{j+1}} - B_{t_j}]^2 - [t_{j+1} - t_j])^2 \\
&= \sum_{j=0}^{n-1}(E[B_{t_{j+1}} - B_{t_j}]^4 + [t_{j+1} - t_j]^2 \\
&\quad -2[t_{j+1} - t_j]E[B_{t_{j+1}} - B_{t_j}]^2) \\
&= \sum_{j=0}^{n-1}(\{3E[B_{t_{j+1}} - B_{t_j}]^2\}^2 - [t_{j+1} - t_j]^2) \\
&= \sum_{j=0}^{n-1}(3[t_{j+1} - t_j]^2 - [t_{j+1} - t_j]^2))^2 \\
&= \sum_{j=0}^{n-1} 2[t_{j+1} - t_j]^2 \\
&\leq 2\max_j[t_{j+1} - t_j]\sum_{j=0}^{n-1}[t_{j+1} - t_j] \\
&= 2t\max_j[t_{j+1} - t_j]
\end{aligned}$$

as $n \to \infty$ last term goes to 0, this completes the proof. ♦

This result has been used in the derivation of Itô formula in chapter IV.

2.3.2. Gaussian Process Property

In the previous section it has been shown that the solution of the heat equation is a probability density function of a Gaussian random variable. And also the continuum limit of random walk shows that Brownian motion exhibits Gaussian process property.But on the other hand for a rigorous proof of Gaussian property the following theorems should be stated and the proofs should be made for the Brownian Motion.

Definition 2.7 *A stochastic process $\{X(t),\ t \in T\}$ is called a* ***normal*** *or* ***Gaussian process*** *if for any integer $n \geq 1$ and any infinite sequence $t_1 < t_2 < \ldots < t_n$ from T the random variables $X(t_1)\ \ldots,\ X(t_n)$ are jointly normally distributed.*

Theorem 2.3 *Let $X_1, \ldots, X_n$ be random variables jointly distributed as an n dimensional normal random vector. If $Y_1, \ldots, Y_k$ are the linear combinations*

$$Y_k = \sum_{i=1}^{n} a_{ki} X_i \quad k = 1 \ldots k \tag{2.75}$$

of $X_1, \ldots, X_n$, then $Y_1, \ldots, Y_k$ are jointly normally distributed.

Definition 2.8 *A stochastic process $\{X(t),\ t \in T\}$ is called a Gaussian process if every finite linear combination of the RVs $\{X(t),\ t \in T\}$ is normally distributed*

Theorem 2.4 *Brownian motion is a Gaussian process*

Proof To show that Brownian motion is a Gaussian process an arbitrary linear combination has been considered, when $a_i \in R$ and $0 \leq t_1 < \ldots < t_n$ then

$$\sum_{i=1}^{n} a_i B(t_i) = (\sum_{k=1}^{n} a_k)[B(t_1) - B(0)] + (\sum_{k=2}^{n} a_k)[B(t_2) - B(t_1)] + \cdots$$

$$+ \cdots + (\sum_{k=n-1}^{n} a_k)[B(t_{n-1}) - B(t_{n-2})]$$

$$+ a_n[B(t_n) - B(t_{n-1})]$$

It can be seen from the second statement of the definition of the Brownian motion that $\sum_{i=1}^{n} a_i B(t_i)$ is expressed as a linear combination of independent normally random variables, and hence the Brownian motion itself is a Gaussian process. ♦

2.3.3. Markovian Property

The Markovian property of Brownian motion is due to the independent increment property of the Brownian motion. Definition of independent increment is given as :

Definition 2.9 *A stochastic process $X(t)$, $t \geq 0$, is said to have* ***independent increments*** *if $\forall n \geq 1$ and time instants $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ the increments $X(t_0) - X(0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are stochastically independent random variables. A process $X(t)$ is said to have* ***stationary increments*** *if the distributions of the increments $X(t) - X(s)$, $s < t$, depend only on the length $(t - s)$ of the time interval $[s, t]$ over the individual increment.*

Theorem 2.5 *Every stochastic process $X(t)$, $t \geq 0$ with independent increments has the* ***Markov property*** *i.e. $\forall\ x \in R$ and $0 \leq t_0 < t_1 \ldots < t_n < \infty$,*

$$P\{X(t_n) < x | X(t_i),\, 0 \leq i < n\} = P\{X(t_n) < x | X(t_{n-1})\} \tag{2.76}$$

(Kannan 1979). Since by definition of Brownian motion, Brownian motion has Markov property.

2.3.4. Martingale Property

This property is based on the conditional expectation.In order to evaluate the Itô integral it has to be a martingale. Itô integral is a stochastic integral that has to be evaluated to find the solution of the SDE.

Definition 2.10 (Discrete Martingale) *Let $X_1, \ldots, X_n, \ldots$ be a sequence of real-valued random variables, with $E(|X_i|) < \infty$ $(i = 1, 2, \ldots)$. If*

$$E(X_j | X_k, \ldots, X_1) = X_k \ \ \forall j \geq k\ \ . \tag{2.77}$$

then $\{X_i\}_{i=1}^{\infty}$ is called a discrete martingale.

In order to give the definition of continuous version of martingale it has been necessary to define $\sigma - Algebra$ that has been used in defining the history of a process.

Definition 2.11 (σ-Algebra(Field)) *If Ω is a given set, then a σ -algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

1. $\varnothing \in \mathcal{F}$,

2. $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, *here* $F^C = \Omega \backslash F$ *is the complement of* F *in* Ω,

3. $A_1,\ A_2,\ \ldots \in \mathcal{F} \Rightarrow A \triangleq \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(Oksendal 2000). A σ*-field is closed under any countable set of unions, intersections and combinations (Stark and Woods 2002).*

Definition 2.12 *Let* $X(\cdot)$ *be a real valued stochastic process. Then the history of the process is,*

$$U(t) = U(X(s)|0 \le s \le t), \tag{2.78}$$

the σ *- algebra generated by the random variables* $X(s)$ *for* $0 \le s \le t$.

Definition 2.13 *Let* $X(\cdot)$ *be a stochastic process, such that* $E(|X(t)|) < \infty,\ \forall t \ge 0$, *if*

$$E(X(t)|U(s)) = X(s)\ \ \forall t \ge s \ge 0, \tag{2.79}$$

then $X(\cdot)$ *is called a martingale*

Theorem 2.6 *Brownian motion has martingale property*

Proof Let

$$\mathcal{B}(t) = U(B(s)|\, 0 \le s \le t) \tag{2.80}$$

$$\begin{aligned} E(B(t)|\mathcal{B}(s)) &= \underbrace{E(B(t) - B(s)|\mathcal{B}(s))}_{\text{from independent increment property}} + E(B(s)|\mathcal{B}(s)) \\ &= \underbrace{E(B(t) - B(s))}_{\text{from linearity property of the expectation}} + B(s) \\ &= E(B(t)) - E(B(s)) + B(s) \\ &= 0 - 0 + B(s) \end{aligned} \tag{2.81}$$

this completes the proof that Brownian Motion has the martingale property. ♦

2.3.5. Covariance of Brownian Motion

Covariance is a function that measures how two random variables vary together. Let two random variables are $B(t_1)$, $B(t_2)$ and their means are respectively $E(B(t_1)) = \mu(t_1)$, $E(B(t_2)) = \mu(t_2)$ then the covariance of $B(t_1)$, $B(t_2)$ is

$$K(t_1, t_2) = cov(B(t_1), B(t_2)) \tag{2.82}$$

$$= E(B(t_1)B(t_2)) - \underbrace{\mu(t_1)\mu(t_2)}_{0} \tag{2.83}$$

Then by using independent increment property,

$$E\{(B(t_1) - B(t_2))B(t_2)\} = E(B(t_1) - B(t_2)) \,.\, E(B(t_2)) = 0 \tag{2.84}$$

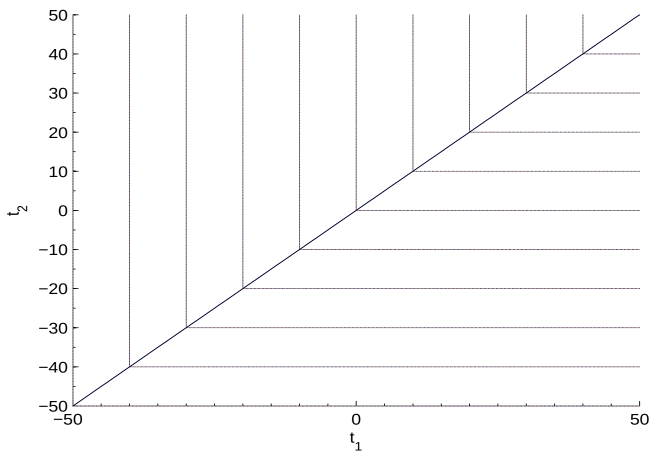

Figure 2.3. Contour Plot of $min(t_1, t_2)$ which shows that the minimum is a constant line lying in the $y = x$ axis.

Therefore,

$$E(B(t_1)B(t_2)) = \begin{cases} E(B^2(t_2)) = \sigma^2 t_2 & \text{if } t_1 \geq t_2, \\ E(B^2(t_1)) = \sigma^2 t_1 & \text{if } t_2 \geq t_1. \end{cases} \tag{2.85}$$

Finally the covariance can be written in a compact form as,

$$K(t_1, t_2) = \sigma^2 min(t_1, t_2) \tag{2.86}$$

$min(t_1, t_2)$ is plotted in Figure (2.3).

2.3.6. Continuity of Stochastic Process

Definition 2.14 *A stochastic process $X : T\times\Omega \to R$ is called* ***stochastically continuous*** *at a point $t_0 \in T$ if for any $\epsilon > 0$*

$$P\{|X(t) - X(t_0)| > \epsilon\} \to 0 \;\; as|t - t_0| \to 0 \tag{2.87}$$

If $X(t)$ is stochastically continuous at every point in T then it is called ***stochastically continuous on*** T. *A process $X(t)$, $t \in T$, is called* ***stochastically uniformly continuous on*** T. *if for arbitrary constants $\epsilon > 0$ and $\eta > 0$ there exists a $\delta > 0$ such that*

$$P\{|X(t) - X(t_0)| > \epsilon\} < \eta \;\; as\ long\ as\ |s - t| < \delta \tag{2.88}$$

Theorem 2.7 *If the process $X(t)$ is stochastically continuous on a compact T then $X(t)$ is stochastically uniformly continuous*

Definition 2.15 *A process $X(t)$, $t \in T$, is called a continuous process if almost all of its sample functions are continuous on T.*

Theorem 2.8 *Brownian motion $B(t)$ is a continuous process; that is almost all sample paths $X_w(\cdot)$ are continuous functions (Kannan 1979).*

2.3.7. Continuity in Mean Square Sense

Definition 2.16 *A second-order stochastic process $X(t)$, $t \in T$, is said to be* ***mean-square continuous*** *at $s \in T$ if*

$$\lim_{t\to s} E\{|X(t) - X(s)|^2\} = 0 \tag{2.89}$$

Theorem 2.9 *A second-order process $X(t)$, $t \in I$, is mean-square continuous at $t = \tau$ if and only if its covariance function $K(s,t)$ is continuous at $s = t = \tau$.*

Proof To show that this is sufficient condition :

$$\lim_{h\to 0} E\{|X(\tau + h) - X(\tau)|^2\}$$

$$\begin{aligned} &= \lim_{h\to 0}\{K(\tau + h, \tau + h) - K(\tau + h, \tau) - K(\tau, \tau + h) + K(\tau, \tau)\} \qquad (2.90)\\ &= 0 \;\; by\ the\ continuity\ of\ K\ at\ (\tau, \tau) \end{aligned}$$

To show that this is a necessary condition using the Schwartz inequality.

$$|K(\tau+h,\tau+h') - K(\tau,\tau)| \tag{2.91}$$

$$\begin{aligned}
&\leq E\{|X(\tau+h)-X(\tau)||X(\tau+h)-X(\tau)|\} \\
&\quad +E\{|X(\tau+h)-X(\tau)||X(\tau)|\} + E\{|X(\tau+h)-X(\tau)||X(\tau)|\} \\
&\leq \{E\{|X(\tau+h)-X(\tau)|^2\}E\{|X(\tau+h)-X(\tau)|^2\}\}^{\frac{1}{2}} \\
&\quad +\{E\{|X(\tau+h)-X(\tau)|^2\}E\{|X(\tau)|^2\}\}^{\frac{1}{2}} \\
&\quad +\{E\{|X(\tau+h)-X(\tau)|^2\}E\{|X(\tau)|^2\}\}^{\frac{1}{2}}
\end{aligned} \tag{2.92}$$

$\rightarrow 0$ as $h \rightarrow 0$ by the mean-square continuity of $X(t)$. This completes the proof ♦

Now to show that whether the Brownian motion(Wiener process) is a m.s. continuous process or not. Theorem 2.9 is applied by substituting $\sigma = 1$ in (2.86) ,

$$K(s,t) = min(s,t) \tag{2.93}$$

it is obtained that Brownian motion has a continuous covariance function. Therefore Brownian motion is a m.s. continuous process.

2.3.8. Differentiability

In stochastic processes the differentiability is defined in the sense of sample paths.

Definition 2.17 *A stochastic process $X(t)$, $t \in I$, is said to be sample-path differentiable if almost all sample paths possess continuous derivatives in I.*

Theorem 2.10 *Almost all sample paths of a Brownian motion $B(t)$ are nowhere differentiable (Kannan 1979).*

A detailed proof can be found in (Kannan 1979). But simply it has been explained that sample paths of Brownian motion have infinite first variation therefore their paths are nowhere differentiable.

2.3.9. Differentiability in Mean Square Sense

It has been shown that Brownian motion is nowhere differentiable in ordinary calculus sense, now we define the differentiability in mean square sense.

Definition 2.18 *A second order process $X(t)$, $t \in T$, is said to be mean-square differentiable at $t \in T$ if there exists a second-order process $Y(s)$, $s \in T$, such that*

$$\lim_{\varepsilon \to 0} E|\varepsilon^{-1}[X(t+h) - X(t)] - Y(t)|^2 = 0 \tag{2.94}$$

where

$$Y(t) \triangleq X'(t)$$

Theorem 2.11 *The process $X(t)$, $t \in I$, is mean-square-differentiable at t if and only if the second generalized derivative $\frac{\partial^2 K(t,u)}{\partial t \partial u}$ exists at (t,t).*

Brownian motion is **not** a mean-square-differentiable process.

Proof

$$\lim_{\varepsilon \to\ 0} E|\frac{[X(t+\varepsilon) - X(t)]}{\varepsilon} - Y(t)|^2 = \lim_{\varepsilon \to\ 0} E|\frac{\sigma^2 \varepsilon}{\varepsilon} - Y(t)|^2 \neq 0 \tag{2.95}$$

Therefore Brownian motion is not a m.s. differentiable process. ♦

From the first and the last two properties of Brownian motion it has been shown that Brownian motion is neither in ordinary sense nor in mean-square sense differentiable. So we can not use ordinary calculus to evaluate integrals which includes stochastic increment element. These integrals are encountered in the solution of stochastic differential equations. Therefore in the next chapter evaluation of such kind of integrals have been explained and hence the solution methods of SDE have been investigated.

CHAPTER 3

STOCHASTIC DIFFERENTIAL EQUATIONS(SDE)

After examining the properties of the Brownian motion, in this chapter it has been explained how the Brownian motion is an important process to construct a stochastic differential equation. On the other hand, comparisons have been made between ODEs and SDEs in the sense of uniqueness and existence of solutions. Then the Langevin's Equation is introduced. Finally, application fields of SDEs have been investigated. Let the differential form of SDE be as below

$$dX(t) = b(t, X(t))dt + g(t, X(t))dB(t) \quad X(0) = x_0 \in R^d \tag{3.1}$$

where

$$b(t, X(t)) = \begin{pmatrix} b_1(X(t)) \\ \vdots \\ b_d(X(t)) \end{pmatrix} \qquad g(t, X(t)) = \begin{pmatrix} g_{11}(X(t)) & \dots & g_{1d}(X(t)) \\ \vdots & & \vdots \\ g_{d1}(X(t)) & \dots & g_{dd}(X(t)) \end{pmatrix}$$

and

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_d(t) \end{pmatrix} \qquad dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_d(t) \end{pmatrix}$$

where $b(t, X(t))$ is the drift matrix and $g(t, X(t))$ is the diffusion matrix and the term $dB(t)$ is the stochastic increment. As mentioned before this is the increment of Brownian motion (Wiener process). At least with this random term SDEs are different from the deterministic ODE or PDE. Such representation of a stochastic dynamical system started with Langevin's work (Langevin 1908). He found the same result for mean square displacement of the particle in a fluid that Einstein had found in 1905. Langevin found this result by not solving a PDE, as Einstein did, but using Newton's second law of motion $F = ma$. In his paper, he first wrote the equation of motion by using theorem of equipartition of the kinetic energy that is

$$m\overline{\xi^2} = \frac{RT}{N} \tag{3.2}$$

where $\xi = \dfrac{dx}{dt}$ is the speed of the particle. Then using the Stoke's law he wrote

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X \tag{3.3}$$

Then he defined X that it is a force that moves the particle in a random manner and its magnitude is such that the motion never ceases due to the viscous resistant of the fluid. Equation (3.3) was the first stochastic differential equation in the history. But there had been no mathematical description about this force by Langevin. When he solved his equation he canceled out this force by multiplying x and finally found the result that Einstein found for the mean square displacement

$$\overline{\triangle_x^2} = \frac{RT}{N}\frac{1}{3\pi\mu a}\tau \tag{3.4}$$

The Details of the calculation can be found in (Langevin 1908).

Therefore he wrote a stochastic differential equation but did not solved a stochastic differential equation. The differential form given in (3.1) as first written by J.L. Doob in (Doob 1942) , after Wiener's description of Brownian motion as a probability measure in (Wiener 1923). Doob worked on solution of this stochastic differential equation but he wrote that "usual methods of solving differential equations are still acceptable to equation 3.25 (this equation number belonged to an equation that is in the form (3.1), in his paper) and again distribution of the solution turns out to be Gaussian". Then K.Itô in (Itô 1944) explained stochastic integration. Then with series of his papers (Itô 1950, 1951a,b) he built the stochastic calculus about the solution of stochastic differential equations that is sometimes called Itô calculus. Stratonovich gave an alternative explanation to Itô's formula of stochastic integration in (Stratonovich 1966) . These two solution methods have been discussed in the next section.

On the other hand solution of a stochastic differential equation is a stochastic process. Therefore stochastic differential equations can be used to generate a stochastic process. Usage of SDEs for this purpose, are listed in (Primak et al. 2004) and also a related example from this book, has been borrowed in the next chapter.

3.1. SOLUTION TECHNIQUES OF SDEs

In this chapter before starting to solutions techniques it has to be known that the solution exists and unique, so related definitions and theorems has been given in first part of this chapter. Let the SDE be defined as

$$dX(t) = b(t, X(t))dt + g(t, X(t))dB(t) \quad \text{with} \quad X(0) = x_0. \tag{3.5}$$

Then integrating the equation (3.5) over the interval $[0, t]$, results

$$X(t) = \int_0^t b(X(s))ds + \int_0^t g(X(s))dB(s) + x_0. \tag{3.6}$$

Definition 3.1 *A continuous stochastic process $\{X(t)\}_{t\geq 0}$ exists and is called the solution of (3.5) if :*

(a) $\{X(t)\}$ is non-anticipative,

(b) For every $t \geq 0$ (3.6) is satisfied with probability one

(Lasota and Mackey 1985).

Theorem 3.1 (Uniqueness of Solution) *If $b(t, X(t))$ and $g(t, X(t))$ satisfy the Lipschitz conditions*

$$|b(t, X(t_1)) - b(t, X(t_2))| \leq L|X(t_1) - X(t_2)| \tag{3.7}$$

and

$$|g(t, X(t_1)) - g(t, X(t_2))| \leq L|X(t_1) - X(t_2)| \tag{3.8}$$

with some constant L, then the initial value problem, in (3.5) has a unique solution $\{X(t)\}_{t\geq 0}$ as in (3.6) (Lasota and Mackey 1985).

3.2. Analytic Solution Methods

First two methods, Itô and Stratonovich, are based on how to evaluate the stochastic integral which appears in the solution of stochastic differential equation (1.3) . But by Fokker-Planck Equation method the solution of the stochastic differential equation is expressed by another solution of a partial differential equation called FPE which is not a stochastic, but a deterministic parabolic type of partial differential equations. Therefore solution of SDE depends on the solvability of FPE.

Two of the analytic methods Itô and Stratonovich are named according to the location of the chosen point τ_i in the following approximation sum of the integrals (Oksendal 2000, Protter 2004, Primak et al. 2004).

$$\int_{t_0}^{t} g(x(\tau), \tau)\mathrm{d}B(\tau) = \lim_{\triangle \to 0} \sum_{i=0}^{m-1} g(x(\tau_i), \tau_i)[B(t_{i+1}) - B(t_i)] \tag{3.9}$$

This gives different results because it changes according to the chosen point τ_i. However in ordinary calculus it does not depend on this decision.

3.2.1. ITÔ Solution

This solution method, developed by K. Itô , is closely related with the quadratic variation of the stochastic process that is the random part of stochastic differential equation (Itô 1944, 1950, 1951a,b). This method depends on the martingale property of the random term in the stochastic differential equation.

3.2.1.1. Definition of ITO Formula(Lemma) and Integral

Definition 3.2 *For $G \in \mathcal{L}^2(0,T)$, set*

$$I(t) = \int_0^t GdB(t) \quad 0 \leq t \leq T \tag{3.10}$$

is the indefinite integral of $G(\cdot)$, where I(0)=0 (WEB_1 2005).

Theorem 3.2 *If $G \in \mathcal{L}^2(0,T)$, then the indefinite integral $I(\cdot)$ is a martingale.*

Definition 3.3 *Suppose that $X(\cdot)$ is a real valued stochastic process satisfying*

$$X(r) = X(s) + \int_s^r Fdt + \int_s^t GdB(t) \tag{3.11}$$

for some $F \in \mathcal{L}^1(0,T), G \in \mathcal{L}^2(0,T)$, $\forall\, 0 \leq s \leq r \leq T$, it is said that $X(\cdot)$ has the stochastic differential equation

$$dX(t) = F(t)dt + G(t)dB(t) \tag{3.12}$$

Theorem 3.3 (Ito's Formula) *Suppose that$X(\cdot)$ has a stochastic differential*

$$dX(t) = F(t)dt + G(t)dB(t) \tag{3.13}$$

for $F \in \mathcal{L}^1(0,T), G \in \mathcal{L}^2(0,T)$. Assume $u : R \times [0,T] \to R$ is a continuous and that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ exists and are continuous. Let $Y(t) = u(X(t),t)$, then Y has the stochastic differential

$$\begin{aligned} dY(t) &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX(t) + \frac{\partial^2 u}{\partial x^2}G(t)^2 dt \\ &= (\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}F(t) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G(t)^2)dt + \frac{\partial u}{\partial x}G(t)dB(t). \end{aligned} \tag{3.14}$$

Equation (3.14) is called the **Ito's Formula** or **Ito's Chain Rule**.

Theorem 3.4 (Ito version of Integration by parts) *Let $f(s,w) = f(s)$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f(s)dB(s) = f(t)B(t) - \int_0^t df(s) \tag{3.15}$$

(WEB_1 2005).

Theorem 3.5 (Multi-dimensional Ito Formula) *Let $B(t) = (B_1(t), ..., B_m(t))$ denote m-dimensional Wiener Process (Brownian Motion) and $u_i(t)$ and v_{ij} satisfies the properties of Brownian motion then the following system of equations can be formed,*

$$\begin{cases} dX_1 = & u_1dt + v_{11}dB_1 + ... + v_{1m}dB_m \\ dX_2 = & u_2dt + v_{21}dB_1 + ... + v_{2m}dB_m \\ \vdots & \vdots \qquad\qquad\qquad\qquad \vdots \\ dX_n = & u_ndt + v_{n1}dB_1 + ... + v_{nm}dB_m \end{cases} \tag{3.16}$$

or, in matrix notation it can be written that,

$$dX(t) = udt + vdB(t) \tag{3.17}$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix}, \quad dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_n(t) \end{pmatrix}$$

and let,

$$\begin{aligned} g(t,x) &\triangleq (g_1(t_x), \dots, g_p(t,x)) \\ Y(t) &= g(t, X(t)) \end{aligned} \tag{3.18}$$

Then

$$dY_k = \frac{\partial g_k}{\partial t}(t, X(t))dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X(t))dX_i + \frac{1}{2}\sum_{ij} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X(t))dX_idX_j \tag{3.19}$$

where

$$dB_idB_j = \delta_{ij}dt, \; dB_idt = dtdB_i = 0 \tag{3.20}$$

(WEB_1 2005).

The derivation of the Itô integral and the usage of quadratic variation has been given in next section, on a simple example.

3.2.1.2. Derivation of ITO Integral

Let the stochastic differential equation be in the form as,

$$dX(t) = B(t)dB(t) \quad X(0) = 0 \tag{3.21}$$

and

$$\begin{aligned} \delta t &= T/N \\ t_j &= j\delta t \\ B_j &= B_{j-1} + dB_j \\ dB_j &= \sqrt{\delta t}\, \mathcal{N}(0,1) \end{aligned} \tag{3.22}$$

then integrating (3.21) has given

$$X(t) = \int_0^t B(s)\, dB(s). \tag{3.23}$$

If we consider the discrete case of the integral in (3.23)

$$X(t) = \sum_{j=0}^{N-1} B_j(B_{j+1} - B_j), \;\; t = N\triangle t, \;\; B_N = B(t), \;\; B_0 = B(0) \tag{3.24}$$

$$X(t) = \frac{1}{2}\sum_{j=0}^{N-1} B_{j+1}^2 - B_j^2 - (B_{j+1}^2 + 2B_jB_{j+1} + B_j^2) \tag{3.25}$$

$$\begin{aligned} &= \frac{1}{2}\sum_{j=0}^{N-1} B_{j+1}^2 - B_j^2 - (B_{j+1} - B_j)^2 \\ &= \frac{1}{2}[(B_1^2 - B_0^2) + (B_2^2 - B_1^2) + ... + (B_{N-1}^2 - B_{N-2}^2) \\ &\quad +(B_N^2 - B_{N-1}^2) - \frac{1}{2}\sum_{j=0}^{N-1}(B_{j+1} - B_j)^2] \\ &= \frac{1}{2}(B_N^2 - B_0^2) - \frac{1}{2}\sum_{j=0}^{N-1}(B_{j+1} - B_j)^2 \\ &= \frac{1}{2}(B_N^2 - B_0^2) - \frac{1}{2}\sum_{j=0}^{N-1}\triangle B_j^2 \end{aligned} \tag{3.26}$$

There are two approaches to continue from here to reach the result that is found by using the Itô formula. First is related with quadratic variation of Brownian motion, an the other is based on the fact that $\triangle B_j = B_{j+1} - B_j \sim \mathcal{N}(0, \triangle t)$ that is normally distributed with mean zero and variance $\triangle t$

First Approach: The last term in (3.25), $\sum_{j=0}^{N-1} \triangle B_j^2$, is equal to t. Therefore after plugging this value and the initial condition and the notation back into 3.25 as

$$B_0 = B(0) = 0\,, \;\; B_N = B(t) \tag{3.27}$$

then the result has been obtained as

$$X(t) = \frac{1}{2}B(t)^2 - \frac{1}{2}t \tag{3.28}$$

Here it has been explained that why the extra $\frac{t}{2}$ term comes into solution. Because Brownian motion has a quadratic variation which equals to t.

Second Approach:

$$\sum_{j=0}^{N-1} \triangle B_j^2 = N(\frac{1}{N}\sum_{j=0}^{N-1} \triangle B_j^2),\;\; var(\triangle W) = E(\triangle W^2) = \frac{1}{N}\sum_{j=0}^{N-1} \triangle B_j^2 \tag{3.29}$$

It is known that

$$\triangle B_j = B_{j+1} - B_j \sim N(0, \triangle t) \tag{3.30}$$

then,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} \triangle B_j^2 = \triangle t, \tag{3.31}$$

so that the approximation to the integral (3.23) becomes

$$X(t) = \frac{1}{2}(B_N^2 - B_0^2) - \frac{1}{2}N\triangle t, \tag{3.32}$$

with

$$B_0 = B(0) = 0,\;\; B_N = B(t),\;\; N\triangle t = t \tag{3.33}$$

Finally, the solution of SDE in (3.21) is obtained as by Itô

$$X(t) = \frac{1}{2}B(t)^2 - \frac{1}{2}t \tag{3.34}$$

(Higham 2001).

3.2.2. Stratonovich Solution

This solution is different from the Itô solution without the additional term $\frac{t}{2}$ and by choosing the point mentioned in Section 3.2.

3.2.3. Derivation of Stratonovich Integral

If the chosen τ_i is the midpoint in (3.9) then approximation of the integral 3.23 is,

$$X(t) = \sum_{j=0}^{N-1} B\left(\frac{t_j + t_{j+1}}{2}\right)(B_{j+1} - B_j) \tag{3.35}$$

Let

$$B(t_{j+1/2}) = \frac{1}{2}(B_j + B_{j+1}) + C_j \tag{3.36}$$

where C_j should be determined in order to satisfy the properties of Brownian motion given in Chapter 3. For convenience in the notation let,

$$Z(t_j) = B(t_{j+\frac{1}{2}}) \tag{3.37}$$

then the expectation of the increment is

$$\begin{aligned} E(\triangle Z_j) &= E(Z(t_j + \triangle t) - Z(t_j)) \\ &= E(Z(t_j + \triangle t)) - E(Z(t_j)) \\ &= 0. \end{aligned} \tag{3.38}$$

In order to satisfy the Markovian property of the Brownian motion

$$\begin{aligned} \triangle Z_j &= \frac{1}{2}[(B_{j+1} + B_{j+2}) + 2C_{j+1} - (B_j + B_{j+1}) - 2C_j] \\ &= \frac{1}{2}(B_{j+2} - B_j) + (C_{j+1} - C_j). \end{aligned} \tag{3.39}$$

(3.39) can be written as

$$\triangle Z_j = \frac{1}{2}(\triangle B_j + \triangle B_{j+1}) + \triangle C_j \tag{3.40}$$

where

$$\begin{aligned} B_{j+1} &= B_j + \triangle B_j \\ B_{j+2} &= B_{j+1} + \triangle B_{j+1} \\ \triangle C_j &= C_{j+1} - C_j. \end{aligned} \tag{3.41}$$

Then,

$$\begin{aligned} E(\triangle Z_j) &= E(\frac{1}{2}(\triangle B_j + \triangle B_{j+1}) + \triangle C_j) \\ &= \frac{1}{2}[E(\triangle B_j) + E(\triangle B_{j+1})] + E(\triangle C_j). \end{aligned} \tag{3.42}$$

On the other hand

$$\begin{aligned} E(\triangle B_j) &= 0 \\ E(\triangle B_{j+1}) &= 0 \\ E(\triangle Z_j) &= 0 \end{aligned} \tag{3.43}$$

and therefore,

$$E(\triangle C_j) = E(C_{j+1}) - E(C_j) = 0. \tag{3.44}$$

(3.44) can be satisfied by

$$E(C_j) = 0 \tag{3.45}$$

Then ,

$$\begin{aligned} \text{var}(\triangle Z_j) &= E(\triangle Z_j^2) - [E(\triangle Z_j)]^2 \\ &= E(\triangle Z_j^2) \end{aligned} \tag{3.46}$$

Substituting (3.40) into (3.46) variance can be obtained as,

$$\begin{aligned} \text{var}(\triangle Z_j) &= E(\frac{1}{4}(\triangle B_j + \triangle B_{j+1} + 2\triangle C_j)^2) \\ &= \frac{1}{4}[E((\triangle B_j + \triangle B_{j+1})^2) + 4E(\triangle C_j \triangle B_j) + 4E(\triangle C_j \triangle B_{j+1}) \\ &\quad + E(\triangle C_j^2)] \\ &= \frac{1}{4}[E(\triangle B_j^2) + 2E(\triangle B_j \triangle B_{j+1}) + 4E(\triangle C_j \triangle B_j) + 4E(\triangle C_j \triangle B_{j+1}) \\ &\quad + 4E(\triangle C_j^2)] \end{aligned} \tag{3.47}$$

and using the independent increment property of the Brownian motion,

$$\begin{aligned} E(\triangle B_j \triangle B_{j+1}) &= E(\triangle B_j)E(\triangle B_{j+1}) = 0 \\ E(\triangle C_j \triangle B_{j+1}) &= E(\triangle C_j)E(\triangle B_{j+1}) = 0 \\ E(\triangle C_j \triangle B_j) &= E(\triangle C_j)E(\triangle B_j) = 0. \end{aligned} \tag{3.48}$$

Then,

$$\text{var}(\triangle Z_j) = \frac{1}{4}[E(\triangle B_j^2) + E(\triangle B_{j+1}^2)] + E(\triangle C_j^2) \tag{3.49}$$

to satisfy the properties of Brownian motion,

$$\begin{aligned} var(\triangle Z_j) &= \triangle t \\ E(\triangle B_j^2) &= \triangle t \\ E(\triangle B_{j+1}^2) &= \triangle t \\ \Rightarrow E(\triangle C_j^2) &= \frac{\triangle t}{2} \end{aligned} \tag{3.50}$$

$$\Rightarrow E(C_{j+1}^2) - 2E(C_{j+1}C_j) + E(C_j^2) = \frac{\triangle t}{2} \tag{3.51}$$

Since

$$E(C_{j+1}C_j) = E(C_{j+1})E(C_j) = 0. \tag{3.52}$$

Therefore,

$$E(C_{j+1}^2) + E(C_j^2) = \frac{\triangle t}{2} \tag{3.53}$$

where equation (3.53) can be satisfied by

$$E(C_{j+1}^2) = E(C_j^2) = \frac{\triangle t}{4} \tag{3.54}$$

so that,

$$C_j \sim \mathcal{N}\left(0, \frac{\triangle t}{4}\right) \tag{3.55}$$

Substituting into (3.35), it has obtained that

$$\begin{aligned} X(t) &= \frac{1}{2}\sum_{j=0}^{N-1}(B_j + B_{j+1} - 2C_j)(B_{j+1} - B_j) \qquad (3.56)\\ &= \frac{1}{2}\sum_{j=0}^{N-1}(B_{j+1}^2 - B_j^2) - \sum_{j=0}^{N-1} C_j(B_{j+1} - B_j)\\ &= \frac{1}{2}[(B_1^2 - B_0^2) + (B_2^2 - B_1^2) + ... + (B_{N-1}^2 - B_{N-2}^2) + (B_N^2 - B_{N-1}^2)]\\ &\quad - \sum_{j=0}^{N-1} C_j(B_{j+1} - B_j)\\ &= \frac{1}{2}(B_N^2 - B_0^2) - N\left(\frac{1}{N}\sum_{j=0}^{N-1} C_j \triangle B_j\right) \end{aligned}$$

where

$$\lim_{N\to\infty} N(\frac{1}{N}\sum_{j=0}^{N-1}(C_j \triangle B_j) = NE(C_j \triangle B_j) = 0. \tag{3.57}$$

Therefore, with $B_0 = B(0) = 0$ and $B_N = B(t)$ approximation to the integral in (3.23) is

$$X(t) = \frac{1}{2}B(t)^2 \tag{3.58}$$

in Stratonovich sense.

3.3. Numerical Solution Methods

Various numerical methods to approximate the solution of a SDE are mentioned in the references (Kloeden and Platen 1992) and Chapter V of (Kannan and Lakshmikantham 2002). Definition of two of them have been given below that has been also used in the developed Simulink block.

3.3.1. Euler-Maruyama Method

Euler-Maruyama method is an approximate numerical solution to the linear stochastic differential equation given in the form

$$dX(t) = aX(t)dt + bX(t)dB(t), \quad X(0) = x_0, \quad a\,, b \in R \tag{3.59}$$

then the numerical approximation Y to the solution X in the interval $[0, T]$ is the recursive relation

$$Y_{n+1} = Y_n + a(Y_n)\delta + b(Y_n)\triangle B_n \tag{3.60}$$

where

$$\begin{aligned} 0 &= \tau_0 < \tau_1 < \ldots < \tau_N = T \\ \delta &= \frac{T}{N} \\ \triangle B_n &= B_{\tau_{n+1}} - B_{\tau_n}. \end{aligned}$$

3.3.2. Milstein Method

Milstein method, with the same settings of δ and $\triangle B_n$, is an approximate numerical solution to the stochastic differential equation given in the form

$$dX(t) = aX(t)dt + bX(t)dB(t), \quad X(0) = x_0. \tag{3.61}$$

Then the numerical approximation Y to the solution X in the interval $[0, T]$ is the recursive relation

$$Y_{n+1} = Y_n + a(Y_n)\delta + b(Y_n)\triangle B_n + \frac{1}{2}b(Y_n)b^{'}(Y_n)((\triangle B_n)^2 - \delta). \tag{3.62}$$

The convergence of both methods have been proved in (Kannan and Lakshmikantham 2002).

3.4. Fokker Planck Equation(FPE) Method

Fokker-Planck equation is a parabolic type partial differential equation describing the probability density of fluctuative microscopic variables with certain drift and diffusion coefficient in a system. This is a deterministic method to find the pdf of the solutions of SDE instead of finding a solution corresponding to a single initial condition. But sometimes FPE can be difficult to solve, too. FPE is an equation constructed with the coefficients of SDE that has been explained in sequel. This equation also called *forward Kolmogorov equation* (Risken 1989). After a brief introduction the usage of FPE is given below.

Let the stochastic differential equation be

$$dX(t) = b(X(t))dt + g(X(t))dB(t), \quad X(0) = x_0 \tag{3.63}$$

where

$$b(X(t)) = \begin{pmatrix} b_1(X(t)) \\ \vdots \\ b_d(X(t)) \end{pmatrix} \qquad g(X(t)) = \begin{pmatrix} g_{11}(X(t)) & \dots & g_{1d}(X(t)) \\ \vdots & & \vdots \\ g_{d1}(X(t)) & \dots & g_{dd}(X(t)) \end{pmatrix}$$

and

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_d(t) \end{pmatrix} \qquad dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_d(t) \end{pmatrix}$$

Let the probability density of the solution $X(t)$, be $u(t,x)$ that is

$$P\{X(t) \in B\} = \int_B u(t,z)dz \tag{3.64}$$

Further the drift and diffusion coefficients are satisfying the Lipschitz conditions given at the beginning of this chapter. Then in order to construct the FPE, let

$$a_{ij} = \sum_{k}^{d} g_{ik}(x)g_{jk}(x) \tag{3.65}$$

then the following theorem states the Fokker Planck Equation(FPE).

Theorem 3.6 *If the functions* g_{ij} , $\frac{\partial g_{ij}}{\partial x_k}$, $\frac{\partial^2 g_{ij}}{\partial x_k \partial x_i}$, b_i , $\frac{\partial b_i}{\partial x_j}$, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, *and* $\frac{\partial^2 u}{\partial x_i \partial x_j}$ *are continuous for* $t > 0$ *and* $x \in R^d$ *and if* b_i, g_{ij} *and their first derivatives are bounded, then* $u(t,x)$

*satisfies the equation (3.66) which is called **Fokker-Planck Equation** or **Kolmogorov Forward Equation***

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}u) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i u), \quad t > 0 \;\; x \in R^d \tag{3.66}$$

(Lasota and Mackey 1985).

3.5. Derivation of FPE

Derivation of FPE has been given using the Indicator (Index) function

Definition 3.4 (Indicator (Index or Characteristic) Function) *Indicator function is defined as,*

$$1_{\{x\in A\}}(x) = \begin{cases} 1 & \textit{if } x \in A, \\ 0 & \textit{if } x \notin A. \end{cases} \tag{3.67}$$

Indicator function, is a useful function when it is necessary to pass from probability of a random variable to the expectation of the random variable, this property has been used in the derivation of FPE.

Let SDE be written in the form below

$$dX(t) = \mu(t, X(t))dt + g(t, X(t))dW(t). \tag{3.68}$$

and Borel set(The minimal σ-algebra containing the open sets.) is denoted by B, then

$$P(X(t) \in B) = E(\mathbf{1}_{\{X(t)\in B\}}(X(t))) = \int_B p(t, x)dx \tag{3.69}$$

Suppose the indicator function **1** is approximated by some smooth function, it can be the Gaussian CDF $\Phi(x; \mu, \sigma)$ i.e.

$$\Phi(x; \mu, \sigma) = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) \tag{3.70}$$

where $erf(\cdot)$ is the Gaussian error function $\mu = 0$ and $\sigma << 1$. Taking derivative at both sides of (3.69) according to the Itô formula.

$$\begin{aligned} dP(X_t \in B) &= E(\mathbf{1}_{\{X(t)\in B\}}) \\ &= E(\mathbf{1}'_{\{X(t)\in B\}}dX(t)) + \frac{1}{2}E(\mathbf{1}''_{\{X(t)\in B\}}(dX_t)^2) \\ &= E(\mathbf{1}'_{\{X(t)\in B\}}\mu(t, X(t))]dt + \frac{1}{2}E(\mathbf{1}''_{\{X(t)\in B\}}\sigma^2(t, X(t))]dt \end{aligned} \tag{3.71}$$

where $1'$ denotes the derivative of indicator function. (3.71) can be written as,

$$\frac{d}{dt}P(X(t) \in B) = \int \mathbf{1}'_{\{X(t)\in B\}}\mu(t,x)p(t,x)dx + \frac{1}{2}\int \mathbf{1}''_{\{X(t)\in B\}}\sigma^2(t,x)p(t,x)dx. \tag{3.72}$$

By applying integrating-by-part technique,

$$\begin{aligned}
\int \mathbf{I}'_{\{X(t)\in B\}}\mu(t,X(t))p(t,X(t))dx &= \int \frac{\partial}{\partial X(t)}[\mathbf{1}_{\{X(t)\in B\}}\mu(t,X(t))p(t,X(t))] \\
&\quad -\mathbf{1}_{\{X(t)\in B\}}\frac{\partial}{\partial X(t)}[\mu(t,X(t))p(t,X(t))]\}dX(t) \\
&= -\int_B \frac{\partial}{\partial X(t)}[\mu(t,X(t))p(t,X(t))]dX(t)
\end{aligned}$$

because $\mathbf{1}_{\{X(t)\in B\}}\mu(t,x)p(t,x)$ disappears at the boundary. Similarly

$\int \mathbf{1}''_{\{X(t)\in B\}}\sigma^2(t,X(t))p(t,X(t))dX(t)$

$$\begin{aligned}
&= \int\{\frac{\partial}{\partial X(t)}[\frac{\partial}{\partial X(t)}[\mathbf{1}_{\{X(t)\in B\}}]\sigma^2(t,X(t))p(t,X(t))] \\
&\quad -\frac{\partial}{\partial X(t)}[\mathbf{1}_{\{X(t)\in B\}}]\frac{\partial}{\partial X(t)}[\sigma^2(t,X(t))p(t,X(t))]\}dX(t) \\
&= -\int \frac{\partial}{\partial x}[\mathbf{1}_{\{X(t)\in B\}}]\frac{\partial}{\partial X(t)}\sigma^2(t,X(t))p(t,X(t))dX(t) \\
&= -\int\{\frac{\partial}{\partial X(t)}[\mathbf{1}_{\{X(t)\in B\}}\frac{\partial}{\partial X(t)}\sigma^2(t,X(t))p(t,X(t))] \\
&\quad -\mathbf{1}_{\{X(t)\in B\}}\frac{\partial^2}{\partial X(t)^2}[\sigma^2(t,X(t))p(t,X(t))]\}dX(t) \\
&= \int_B \frac{\partial^2}{\partial x^2}[\sigma^2(t,X(t))p(t,X(t))]dX(t)
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}P(X_t \in B) &= \int_B \frac{\partial}{\partial t}p(t,x) \\
&= \int_B\{-\frac{\partial}{\partial x}[\mu(t,x)p(t,x)] + \frac{\partial^2}{\partial x^2}[\sigma^2(t,x)p(t,x)]\}dx
\end{aligned}$$

Since B is arbitrary, the integrands are equal. Hence the results is the ***Fokker-Planck equation*** (WEB_3 2005).

CHAPTER 4

APPLICATIONS

This chapter includes examples about analytic solution methods and application of SDE to the communication systems.

In the following example analytic solution techniques of Itô and Stratonovich have been demonstrated. Example has been borrowed from (Oksendal 2000).

Example 4.1 (Application of Itô Formula to Linear SDE)

$$dX(t) = \mu X(t)dt + \lambda X(t)dB(t) \quad X(0) = x_0 \tag{4.1}$$

Let $Y(t) = log(X(t))$ then according to the Itô formula

$$\begin{aligned} Y(t) &= u(X(t), t) \\ &= u(0, X_o) + \int_0^t \frac{\partial u}{\partial s} ds + \int_0^t \frac{\partial u}{\partial x} dX(s) \\ &+ \frac{1}{2} \int_0^t \lambda^2 X(t)^2 \frac{\partial^2 u}{\partial x^2} dX(s) \end{aligned} \tag{4.2}$$

Then plugging the derivatives and the differential $dX(s)$ from (4.1) into (4.2)

$$\begin{aligned} Y(t) &= log(X(t)) = log(X_o) + \int_0^t \mu ds \\ &+ \lambda \int_0^t dB(t) + \frac{1}{2}\lambda^2 \int_0^t dt \end{aligned} \tag{4.3}$$

The integral, with the stochastic differential $dB(t)$ in (4.3), intuitively equals to $B(t)$ then

$$X(t) = X_0 e^{(\mu - \frac{\lambda^2}{2})t + \lambda B(t)} \tag{4.4}$$

But according to the Stratonovich (4.1) has a solution as,

$$X(t) = X_0 e^{(\mu t + \lambda B(t)})} \tag{4.5}$$

The following example has demonstrated, how a stochastic differential equation can be used to model a part of the communication systems.

Example 4.2 (Error flow in a channel with Nakagami fading) *Let a communication channel experienced Nakagami fading, described by PDF*

$$p_A(A) = \frac{2}{\Gamma(m)}(\frac{m}{\Omega})^m A^{2m-1}\exp[-\frac{mA^2}{\Omega}] \tag{4.6}$$

The communication channel experiencing fading can be represented as

$$\hat{s}(t) = \mu(t)s_k(t-\tau) + K\xi(t) \tag{4.7}$$

where $\xi(t)$ is the white Gaussian noise(WGN), K^2 is the intensity of WGN, $\hat{s}(t)$ is the received signal, $s_k(t)$ is the transmitted signal. On the other hand SNR (Signal to Noise Raito) is

$$SNR \triangleq h(t) = \frac{\mu^2(t)E(s_k)}{K^2} = \gamma(t)$$

and the probability of error is

$$p_{err}(t) \triangleq F(h(t))$$

where $F(\cdot)$ function is defined by demodulation methods. Thus, $\mu(t)$ in (4.7) can be considered as a random process generated by the following SDE

$$\dot{\mu} = \frac{\Omega}{2m\tau_{\text{corr}}}(\frac{2m-1}{\mu} - \frac{2m\mu}{\Omega}) + \sqrt{\frac{\Omega}{m\tau_{\text{corr}}}}\xi(t) \tag{4.8}$$

Being proportional to the quantity $z(t) = \mu^2(t)$, the SNR $\gamma(t)$ can be generated by SDE

$$\dot{\gamma} = \frac{\overline{\Omega}}{m\tau_{\text{corr}}}(2m-1-\frac{2m\gamma}{\overline{\Omega}}) + 2\sqrt{\frac{\overline{\Omega}\gamma}{m\tau_{\text{corr}}}}\xi(t) \tag{4.9}$$

where $\overline{\Omega}/m$ is the instantaneous SNR. The solution of this SDE $\gamma(t)$ can be used to drive the intensity of a Poisson flow of errors (Primak et al. 2004).

4.1. Developed Block

Numerical and analytic solution methods of stochastic differential equations have been considered in previous chapter. After investigating the following references (Talay 1990, WEB_2 2005, Cyganowski et al. 2002, Higham 2001, Gilsing and Shardlow 2007) about simulation of stochastic differential equations on Fortran, Maple and Matlab environments, it has been seen that all proposed simulation methods are source-code based so they are first compiled then run on the platform that they are built. During the research of this thesis a new block for Matlab Simulink have been developed which outputs

Itô solution and the numerical solution methods of Euler-Maruyama and Milstein Scheme for the linear type of stochastic differential equation.

$$dX(t) = \mu X(t) + \lambda X(t) dB(t) \;\; X(0) = x_0. \tag{4.10}$$

According to the Itô calculus (Oksendal 2000) (4.10) has an explicit solution

$$X(t) = X_0 e^{(\mu - \frac{\lambda^2}{2})t + \lambda B(t)}. \tag{4.11}$$

The newly proposed block includes numerical integrations schemes inside, the random number input and outputs for Itô solution, Euler-Maruyama and Milstein methods for the linear stochastic differential equation. It also outputs the random increment and the Wiener process. To simulate a random variable, which is essential for simulating stochas-

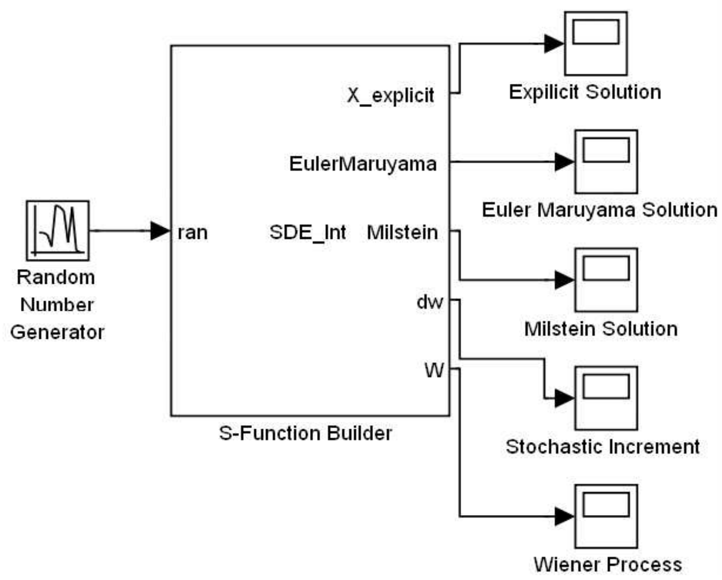

Figure 4.1. Stochastic Integrator uses Euler-Maruyama, Milstein and Itô solution methods for linear stochastic differential

tic differential equations, in a computer environment, one should consider the random number generation algorithms carefully. Briefly random numbers can be produced using the following algorithms, given the uniformly distributed sequences $U[0,1]$-distributed independent numbers (U_n, V_n).

Definition 4.1 (The Inverse Transform) *This method helps to convert a uniformly distributed random variable into a random variable with a desired distribution function* $F_x(x)$. *An invertible distribution* $F_x = F_x(x)$ *of random number variable* X *can be* *generated from uniformly distributed random numbers U by the following equation*

$$X(U) = inf\{x : U \leq F_x(x)\} \tag{4.12}$$

so

$$X(U) = F_x^{-1}(U)\ if\ F_x^{-1}\ exists \tag{4.13}$$

(Kannan and Lakshmikantham 2002).But on other hand the inverse of the function F_x can be complicated or has no compact form then this method does not work well or consumes a lot of computational resources such as cpu and ram. Therefore the following methods can be to generate a random variable.

Definition 4.2 (Box-Muller Algorithm) *This method uses the below transforms to obtain two independent Gaussian distributed numbers.*

$$G_n^{(1)} = \sqrt{-2ln(U_n)}\cos(2\pi V_n) \tag{4.14}$$

$$G_n^{(2)} = \sqrt{-2ln(U_n)}sin(2\pi V_n) \tag{4.15}$$

$(G_n^{(1)}, G_n^{(2)})$, correlated random variables can be generated from those independent pairs by algebraic multiplication with corresponding matrices arising from Cholesky factorization of given correlation matrix e.g. factorization of correlation matrix (Kannan and Lakshmikantham 2002).

$$\begin{aligned} CC^T &= \begin{pmatrix} \sqrt{\triangle} & 0 \\ \frac{\triangle^{\frac{3}{2}}}{2} & \frac{\triangle^{\frac{3}{2}}}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{\triangle} & 0 \\ \frac{\triangle^{\frac{3}{2}}}{2} & \frac{\triangle^{\frac{3}{2}}}{2\sqrt{3}} \end{pmatrix}^T \\ &= \begin{pmatrix} \triangle & \frac{\triangle^2}{2} \\ \frac{\triangle^2}{2} & \frac{\triangle^3}{3}. \end{pmatrix} \end{aligned} \tag{4.16}$$

Definition 4.3 (Polar Marsaglia Algorithm) *The Polar Marsaglia method also generates independent, standard Gaussian distribution pseudorandom numbers, which exhibits a slightly more computationally efficient generator than that of Box-Muller. This method avoids the time consuming generation of trigonometric functions by the following transformations, and algorithm.*

1. Let

$$\begin{aligned} \hat{U}_n &= 2U_n - 1 \\ \hat{V}_n &= 2V_n - 1 \end{aligned}$$

2. *Check whether*

$$B_n = \hat{U}_n^2 + \hat{V}_n^2 \leq 1 \tag{4.17}$$

or repeat until acceptance of pair $(\hat{U}_n, \hat{V}_n)$

3. *Using the transform* $B_n \leq 1$ *obtain*

$$\begin{aligned} G_n^{(1)} &= \hat{U}_n \sqrt{\frac{-2ln(B_n)}{B_n}} \\ G_n^{(2)} &= \hat{V}_n \sqrt{\frac{-2ln(B_n)}{B_n}} \end{aligned}$$

(Kannan and Lakshmikantham 2002).

Another method is to use the readily available Random Number Block that comes within the Simulink software.

For the random number generation (RNG) Simulink random-number block has been used which generates Gaussian random variables according to the given parameters. To implement the numerical integrator algorithms, Simulink S-function builder block has been used and therefore C-programming language has been chosen. S-function builder treats input and output ports of the block as pointers that can be dynamically allocated while the simulator is running. The results of block simulation has been shown in Figure.4.5. The differences between each solution can be seen in Figure.(4.6)

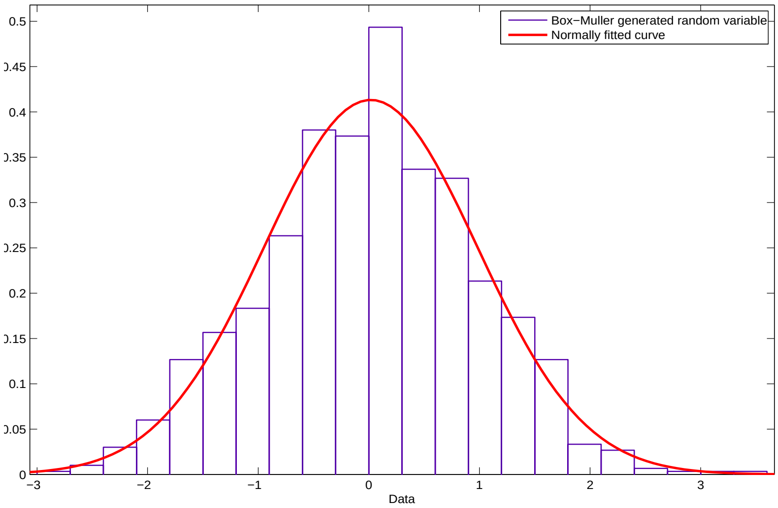

Figure 4.2. Box-Muller generated random numbers plotted with $hist()$ function of Matlab and fitted normally with the Distribution Fitting Tool. Length of the Gaussian sequence is 1000 samples.

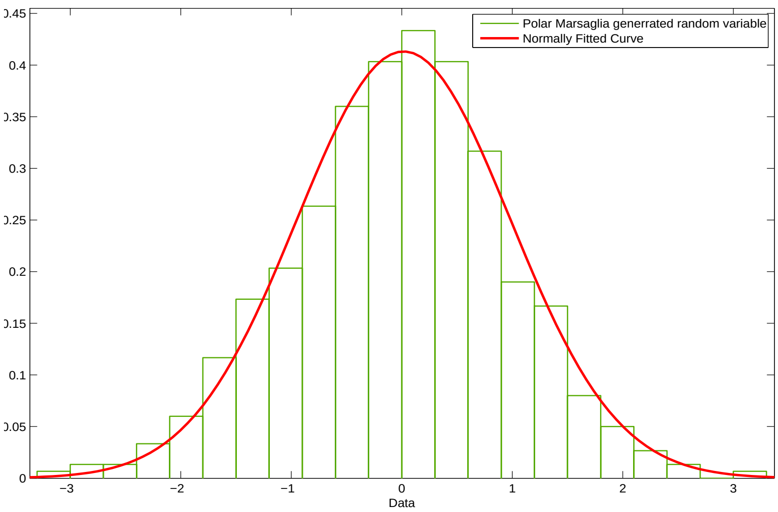

Figure 4.3. Polar-Marsaglia generated random numbers plotted with $hist()$ function of Matlab and fitted normally with the Distribution Fitting Tool. Length of the Gaussian sequence is 1000 samples.

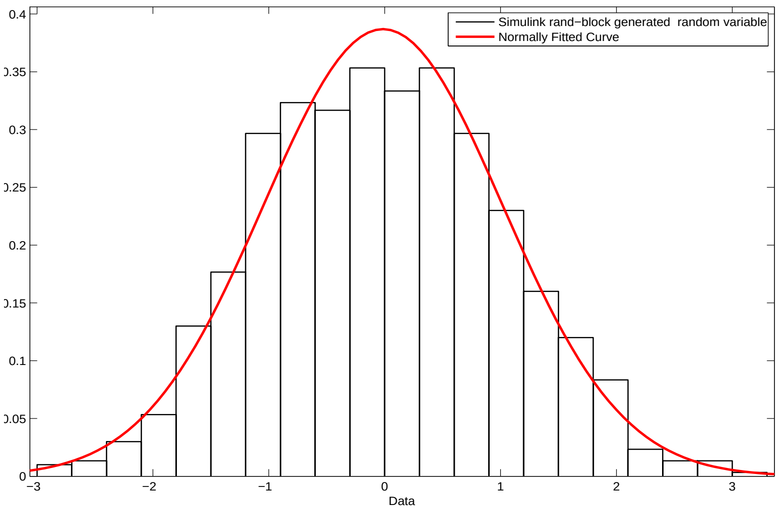

Figure 4.4. Simulink Random Number Block generated random numbers plotted with $hist()$ function of Matlab and fitted normally with the Distribution Fitting Tool. Length of the Gaussian sequence is 1000 samples.

Table 4.1. Comparisons of the parameters found by random number generation algorithms with the reference distribution $\mathcal{N}(0, 1)$ parameters

	Mean	Variance	Difference in Mean	Difference in Variance
Box-Muller Method	0.017029	0.932262	0.017029	0.067738
Polar Marsaglia	-0.02771	0.960039	0.027714	0.039961
Random Number Block	-0.01378	1.06318	0.013776	0.06318
Original Parameters	0	1		

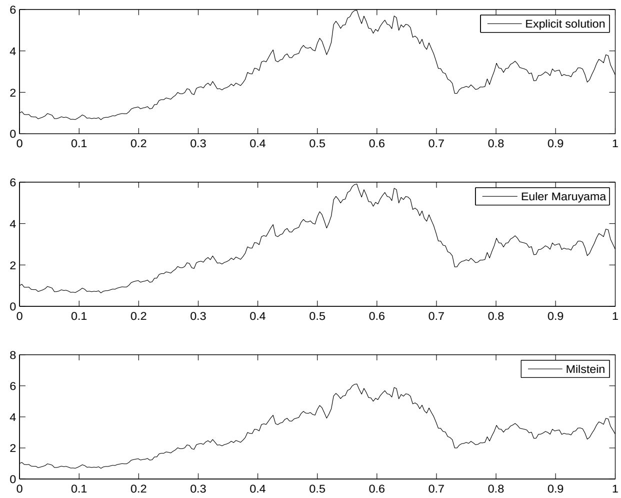

Figure 4.5. Solution with parameters $\lambda = 1$, $\mu = 1$ and the initial condition of SDE $x_0 = 1$

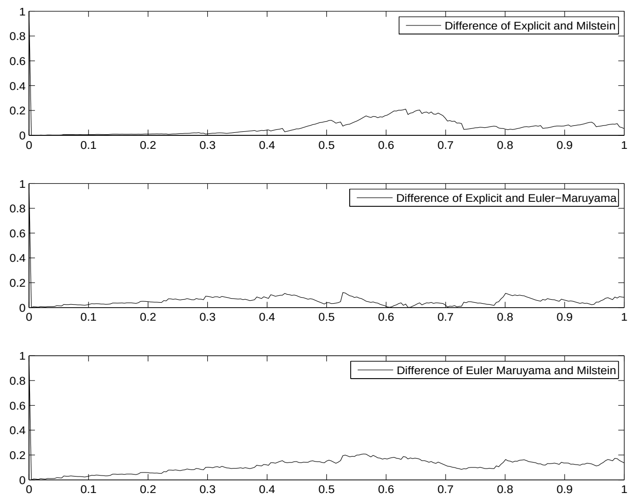

Figure 4.6. Differences between each two solutions Explicit and Milstein, Explicit and Euler-Maruyama and Euler Maruyama and Milstein

CHAPTER 5

CONCLUSION

In this thesis, the wide area of stochastic analysis have been studied in the sense of stochastic differential equations. Itô solution, Stratonovich solution and Fokker Planck equation method have been explained, then numerical methods such as Euler-Maruyama and Milstein method have been investigated. These methods also have been used in the developed numerical integrator Simulink block. But unfortunately convergence and stability problems of other numerical methods which are different than Euler Maruyama and Milstein, have not been studied in this thesis. On the other hand, a new Matlab Simulink Block has been developed that uses these numerical methods to solve the linear stochastic differential equation and the analytic method of Itô has been used to measure the performance of these numerical methods. This will be a useful practical tool when it is integrated with the abilities to evaluate nonlinear kind of stochastic differential equations in a Simulink environment, which is one of the popular and practical ways of simulating dynamical systems encountered in communication systems, mechanical systems, etc.. On a computer environment.to generate a random variable random number generation algorithms are tested using Matlab and results are compared with reference distribution and seen that readily available Simulink random number block is a good choice while working with s-function builder block because it generates random numbers dynamically which are synchronized with the simulation time. Numbers generated with the other explained algorithms, that has been explained, has to be input offline to the block. Offline method has drawbacks such as synchronization with the Simulink time steps. As a future work, an integrator will be developed to run on a hardware environment, by using a FPGA (Field Programmable Gate Array). Then with such a hardware implementation it would be possible to find a solution of a noisy system modeled by a SDE. It would also be possible to generate a desired random process by using the solution of a stochastic differential equation.

REFERENCES

Brown R., 1828. “Additional remarks on active molecules”, not published

Brown R., 1827. “A brief account of microscopical observations made in the months of June, July and August, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies”, not published

Cohen L., 2005. “History of Noise(on the 100th anniversary of its birth)”, *IEEE Signal Processing Magazine*. Vol. 22, No. 6, pp. 20-45.

Cyganowski, S. Grüne, L. and Kloeden, P.E., 2002. “MAPLE for Jump-Diffusion Stochastic Differential Equations in Finance”, *Programming Languages and Systems in Computational Economics and Finance*. pp.441-460.

Doob J.L., 1942. “The Brownian Movement and Stochastic Equations”, *The Annals of Mathematics*. Vol.43 pp.351-369.

Einstein A., 1905. “Investigations on the theory of the brownian movement”, *Annalen der Physik*. Vol.332 No. 8, pp. 549-560.

Friedman A., 1975. *Stochastic differential equations and applications Vol 2* (Academic Press).

Gilsing, H. Shardlow T., 2007. “SDELab:A package for solving stochastic differential equation in Matlab”, *Journal of Computational and Applied Mathematics* Vol.205, pp.1002-1018. (Paper was obtained and investigated in July 2006 by me)

Hänggi, P. Marchesoni, F. 2005. ”100 Years of Brownian Motion”, *Chaos : An Interdisciplinary Journal of Nonlinear Science*. Vol. 15, p.026101.

Higham D.J., 2001. “An algorithmic introduction to numerical simulation of stochastic differential equations”, *Society for Industrial and Applied Mathematics Philadelphia*. Vol.33, pp.525-546.

Itô K., 1950. “Stochastic differential equations in a differentiable manifold ”, *Nagoya Math. Joum.* Vol. 1, pp. 35-47.

Itô K., 1951. “On stochastic differential equations”, *Mem. Amer. Math. Soc.* Vol. 4, pp. 1-51.

Itô K., 1951. “On a formula concerning stochastic differentials”, *Nagoya Math. Joum.* Vol. 3, pp. 55-65.

Itô K., 1944. “Stochastic integral”. Proc. Imp. Acad. Tokyo Vol. 20, pp. 519-524.

Kannan, D., 1979. *Introduction to stochastic processes* (Elsevier North Holland, New York).

Kannan, D. and Lakshmikantham, V., 2002. *Handbook of Stochastic Anlaysis and Applications*, (Marcel Dekker Inc., New York).

Kloeden, P.E. and Platen, E., 1992. *Numerical Solution of Stochastic Differential Equations*, (Springer-Verlag, New York).

Langevin M.P., 1908. “Sur la théorie du mouvement brownien(On the theory of Brownian motion)”, *C. R. Acad. Sci. (Paris)*. Vol.146,pp.530-533.

Lasota, A. Mackey and M.C., 2004. *Probabilistic properties of deterministic systems* (Cambridge University Press)

McKean H.P., 1969. *McKean H.P. Stochastic integrals*, (Academic Press, New York and London).

Nelson E., 1967. *Dynamical Theory Of Brownian Motion*, (Princeton University Press, New Jersey).

Oksendal B., 2000. *Stochastic Differential Equations*, (Springer-Verlag, New York).

Perrin J.B., 1909. "Mouvement brownien et realite moleculaire, *Ann. de Chimie et de Physique*. Vol.18, pp. 5û114, translated "Brownian motion and molecular reality".

Primak S., Kontorovich V. and Lyanders V. 2004. *Stochastic Methods and Their Applications to Communications: Stochastic Differential Equations Approach*, John Wiley and Sons,England.

Protter P.E., 2004. *Stochastic integration and differential equations* (Springer).

Risken H., 1989. *The Fokker-Planck equation: methods of solution and applications* (Springer).

Ross S.M., 1997. *Introduction to probability models*, (Academic Press,USA).

Situ R., 2005. *Theory of Stochastic Differential Equations with Jumps and Applications* , (Springer,US).

Skorokhod, V.A., 2005. *Basic Principles and Applications of Probability Theory*, (Springer, USA).

Smoluchowski, M., 1906. "Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen", *Annalen der Physik*. Vol.326, No.14, pp. 756-780.

Stark, H. and Woods, J.W., 2002. *Probability and Random Processes with Applications to Signal Processing* (Prentice Hall, New Jersey).

Stratonovich R., 1966. "A New Representation for Stochastic Integrals and Equations", *SIAM Journal on Control*. Vol 4, pp.362-371.

Talay D., 1990. "Simulation and numerical analysis of stochastic differential systems: A review", *INRIA*. Report 1313,Rocquencourt.

WEB_1, 2005. Lawrence C. Evans's Home Page, 25/03/2005.
http://math.berkeley.edu/ eva ns/SDE.course.pdf

WEB_2, 2005. abstract, 27/03/2005.
http://www.wias-berlin.de/publications/preprints/453/

WEB_3, 2005. Quick Derivation of Fokker-Planck equation, 26/03/2005.
www.ssc.wisc.edu/ manuelli/DiTella/FokkerPlanckTanaka.pdf

WEB_4, 2005. Greens Function.pdf, 28/03/2005.
http://faculty.chicagogsb.edu/akash.bandyopadhyay/more/files/GreensFunction.pdf

Wiener N., 1921. "The Average of an Analytic Functional and the Brownian Movement", Proc. of the National Academy of Sciences of the United States of America, Vol. 7, pp. 294-298.

Wiener N., 1920. "The mean of a functional of arbitrary elements, *Ann. Math*. Vol. 22, pp.66-72.

Wiener N., 1923. "Differential Space, *J. Math. Phys.* Vol. 2, pp.105-124.