UNIFORMLY CONVERGENT APPROXIMATION ON SPECIAL MESHES

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ABSTRACT

UNIFORMLY CONVERGENT APPROXIMATION ON SPECIAL MESHES

We consider finite difference methods for the approximation of one−dimensional convection−diffusion problem with a small parameter multiplying the diffusion term. An analysis of the centered difference and upwind difference schemes on equidistant meshes shows that these methods are not uniformly convergent in the discrete maximum norm. However, we show that the upwind method over a set of suitably distributed mesh points produce uniformly convergent approximations in the discrete maximum norm. We further investigate the upwind difference method for the approximation of the convection−diffusion problem with a point source. Theoretical findings are supported with the numerical results.

ÖZET

ÖZEL AĞLAR ÜZERİNDE DÜZGÜN YAKINSAYAN ÇÖZÜMLER

Difüzyon terimi küçük bir parametreyle çarpılmış olan konveksiyon−difüzyon probleminin bir boyutlu çözümleri için sonlu fark metodları ele alınmaktadır. Merkez ve geri fark metotlarının ayrık maksimum normda düzgün yakınsak olmadığı bir analizle gösterilmektedir. Geri fark metodunun yine de, ağ noktalarının özel bir seçimi ile ayrık maksimum normda düzgün yakınsak olduğu gösterilmiştir. Ayrıca noktasal bir kaynağa sahip olan konveksiyon−difüzyon denkleminin geri fark metodu ile yaklaşık sonuçları üzerinde çalışılmıştır. Teorik sonuçlar sayısal sonuçlarla desteklenmiştir.

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CHAPTER 1

INTRODUCTION

We study the numerical solution techniques on both equidistant and piecewise uniform meshes for the following convection$-$diffusion problem on the interval $\Omega = [0,1]$.

$$\left\{ \begin{array}{cc} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = u_0\ ,\ u(1) = u_1 \\ and\ for\ all\ x \in \Omega, \qquad Lu = -\epsilon u^{''}(x) + a(x)u^{'}(x) = f(x) \end{array} \right\} \tag{1.1}$$

where ϵ is a small parameter used to measure the relative amount of diffusion to convection. $a(x)$ and $f(x)$ are smooth functions and the function $a(x)$ satisfies the following strict inequality.

$$a(x) > \alpha > 0$$

The convection$-$diffusion problem (1.1) arises in diverse areas such as the moisture transport in desiccated soil, the potential function of fluid injection through one side of a long vertical channel, the potential for a semiconductor device modeling and steady flow of a viscous, incompressible fluid. Although the problem (1.1) may not be applied directly to real applications, it is important to find its solution, because it is an important stage in investigation of many practical applications.

The main difficulty is to obtain a numerical solution which converges $\epsilon - uniformly$ to the exact solution of the problem (1.1) since it is a singularly perturbed problem. When the standard finite difference operators are employed on a uniform mesh to solve this problem, for example the centered difference scheme, then the numerical solutions oscillate unless the mesh size h is chosen sufficiently small compared to ϵ. Although the upwind difference scheme gives more stable result, Kellog and Tsan (Malley, 1991) have analyzed the behavior of the error of the standard upwind scheme on a uniform mesh and they show that it is not $\epsilon - uniform$ in the discrete maximum norm in the layer. Therefore, we need more efficient methods in order to capture numerical solutions which has the feature of $\epsilon - uniform$ convergence. These methods can be given on a $uniform$ mesh or on a $non - uniform$ mesh. In this thesis, we investigate the numerical approximations of the convection$-$diffusion problem both on a $uniform$ and $non - uniform$ meshes. Thus, it is organized as follows:

In Chapter 2, we illustrate the behavior of the problem in one dimension using a simple problem and introduce some notations and definitions used in the subsequent chapters. In Chapter 3, we analyze the centered difference and upwind difference methods using the solutions of the associated difference equations. We present some numerical results to demonstrate the qualitative behavior of these methods for different configurations of ϵ relative to h. In Chapter 4, we derive a uniformly convergent method on an equidistant mesh, called Il'in$-$Allen$-$Southwell method and present some numerical results. In Chapter 5, a piecewise uniform mesh so called Shishkin mesh is introduced. We first consider a problem with regular data and whose convective term has a constant coefficient to obtain some results which are used in the convergence analysis of the problem (1.1) on this piecewise uniform mesh. We give an $\epsilon - uniform$ error estimate in section 5.2 and present two numerical experiments that verify the uniform convergence of the method under investigation. Further, we consider a different type of convection$-$diffusion problem with irregular data in section 5.3 and use again the upwind finite difference method on Shishkin mesh for discretization of this problem.

CHAPTER 2

OVERVIEW OF THE CONVECTION DIFFUSION PROBLEM

2.1. The Analytical Behavior of Convection-Diffusion Problem

We begin by explaining where the convection$-$diffusion phenomenon occurs and then introduce a convection diffusion equation in one dimension on the interval $[0, 1]$, together with the behavior of the exact solution.

Mathematical models that involve a combination of convective and diffusive processes are among the most widespread in all of science, engineering and other fields where mathematical modeling is important. Water quality problems, convective heat transfer problems, simulation of the semiconductor devices can be given as an example of these models. Also the linearization of the Navier-Stokes equation and drift-diffusion equation of semiconductor device modeling are important instances.

Very often the dimensionless parameter that measures the relative strength of the diffusion is quite small; so one often meets with situations where thin boundary and interior layers are present and singular perturbation problems arise. The following problem on the unit interval $\Omega = (0, 1)$ leads us to deduce the analytical behavior of the problem in one$-$dimension.

$$\left\{ \begin{array}{c} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = u_0 \quad u(1) = u_1 \\ and\ for\ all\ x \in \Omega, \qquad Lu = -\epsilon u''(x) + bu'(x) = 0 \end{array} \right\} \tag{2.1}$$

where b is a constant which satisfies $b > 0$ and $C^2(\Omega)$ denotes the space of two times differentiable functions on Ω. It can be solved exactly:

$$u(x) \quad = \quad u_0 + (u_1 - u_0)\frac{e^{-b(1-x)/\epsilon} - e^{-b/\epsilon}}{1 - e^{-b/\epsilon}} \tag{2.2}$$

Since the exponential function in the solution has the argument $(1 - x)/\epsilon$, the solution changes rapidly in the subinterval $(1 - \epsilon, 1)$. That is, there is a boundary layer around $x = 1$ as ϵ tends to zero and it is of width ϵ . The Figure 2.1 is plotted by setting $u_0 = 0$, $u_1 = 1$ and $b = 1$ and for the values of $\epsilon = 1, 10^{-1}, 10^{-2}, 10^{-3}$.

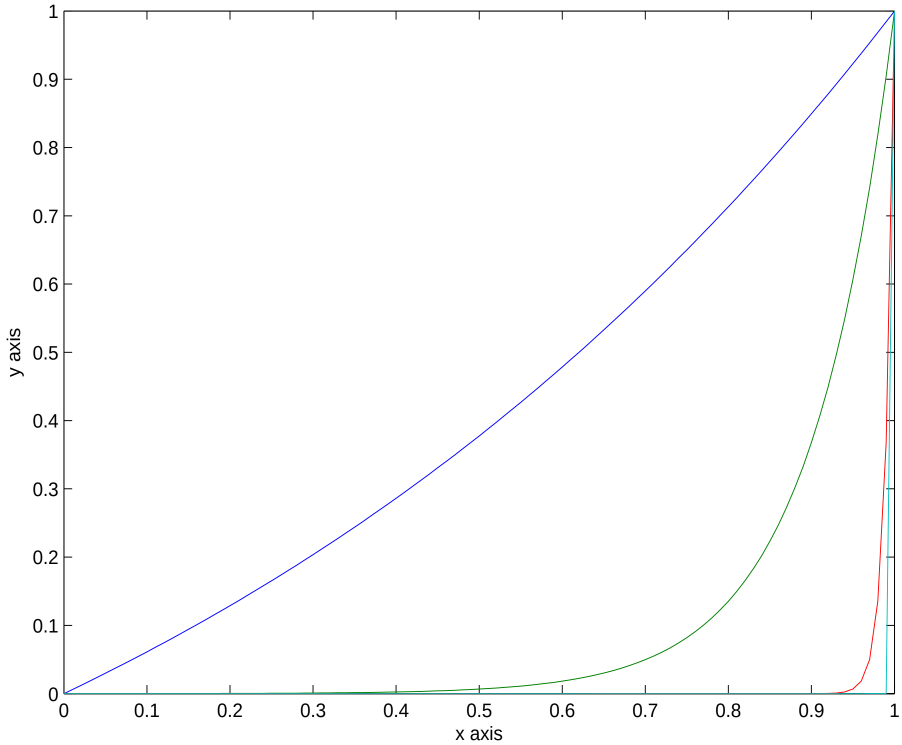

Figure 2.1. Exact solution of the Problem (2.1) for several values of ϵ

It shows that the thickness of the boundary layer narrows as ϵ gets smaller. However, it is difficult to find the numerical solution of the problem. Therefore, it is important to devise efficient algorithms for the approximation of the convection$-$diffusion problems.

2.2. Numerical Methods for The Singularly Perturbed Problems

In this section, we overview the numerical methods used to solve the singularly perturbed problems and introduce some notations, finite difference operators, function spaces, norms and seminorms which are used in the subsequent chapters.

Let D be a bounded domain in R. Typically $D = \Omega$ or $D = \overline{\Omega}$ where Ω is a bounded open interval. Let $C^0(D)$ denote the space of continuous functions on D with the norm of any $f \in C^0(D)$ defined by

$$\|f\|_D = \sup |f(x)| \quad \forall x \in D.$$

For each integer $k \geq 1$ let $C^k(D)$ denote the space of k$-$times differentiable functions on D, with continuous derivatives up to and including those of order k. The explicit reference to D is dropped whenever the domain in question is evident. For any mesh function V on an arbitrary mesh $\overline{\Omega}^N = \{x_i\}_0^N$, the discrete maximum norm is defined by

$$\|V\|_{\overline{\Omega}^N} = \max |V_i| \quad 0 \leq i \leq N.$$

The linear vector space of all mesh functions defined on $\overline{\Omega}^N$, and furnished with the norm $\|.\|_{\overline{\Omega}^N}$, is denoted by $V(\overline{\Omega}^N)$. When the mesh $\overline{\Omega}^N$ is evident it may be dropped from the notation.

In order to construct the numerical methods considered in the following chapters, we need the following mesh descriptions, finite difference operators and definition:

On the interval $\Omega = (0,1)$ for each integer $N \geq 2$, **the uniform mesh** $\overline{\Omega}^N = \{x_i\}_0^N$ is defined by taking the $N+1$ mesh points

$$x_i = i/N \quad for \quad 0 \leq i \leq N$$

that is they are separated by a uniform distance

$$h = x_i - x_{i-1} = 1/N \quad for \quad 0 \leq i \leq N.$$

An alternate way of arriving at the same result is to divide Ω into N mesh elements $\Omega_i = (x_{i-1}, x_i)$ which have the length $h = 1/N$.

First and second order finite difference operators are now defined on these uniform meshes as follows:

$$D^+V_i = \frac{V_{i+1} - V_i}{h} \qquad D^-V_i = \frac{V_i - V_{i-1}}{h}$$
$$D^oV_i = \frac{D^+ + D^-}{2} V_i = \frac{V_{i+1} - V_{i-1}}{2h} \tag{2.3}$$

D^+ and D^- give a first order approximation to the first derivative of any function while D^0 gives a second order. Second order difference operator D^2 is obtained by composing forward and backward difference operator and gives a second order approximation to the second derivative of any function:

$$D^2V_i = \frac{(D^+ - D^-)}{h} V_i = \frac{V_{i+1} - 2V_i + V_{i-1}}{h^2} \tag{2.4}$$

Since in the following chapters the meshes are no longer uniform, we need to extend the above definition from uniform to **non-uniform meshes**. If the mesh points in an arbitrary non$-$uniform mesh with N subintervals $\quad \Omega_i = (x_{i-1}, x_i) \;\; for \;\; 1 \leq i \leq N$ are denoted by $\overline{\Omega}^N = \{x_i\}_0^N$, then the mesh points are separated by a distance

$$h_i = x_i - x_{i-1} \quad for \;\; 1 \leq i \leq N.$$

First and second order finite difference operator for the non$-$uniform meshes are given by

$$D^+V_i = \frac{V_{i+1} - V_i}{h_{i+1}} \qquad D^-V_i = \frac{V_i - V_{i-1}}{h_i}$$
$$D^oV_i = \frac{h_{i+1}D^+ + h_iD^-}{2\overline{h_i}}V_i \qquad D^2V_i = \frac{(D^+ - D^-)}{\overline{h_i}}V_i \tag{2.5}$$

where

$$\overline{h}_i = \frac{h_{i+1} + h_i}{2}$$

for $1 \leq i \leq N - 1$.

Early numerical solutions of problems involving singularly perturbed differential equations were obtained by using a standard finite difference operator defined in (2.3) on a **uniform mesh** and then refining the mesh more and more in order to capture the boundary or interior layers as the singular perturbation parameter decreased in magnitude. Thus, even for problems in one dimension, the methods were inefficient, and accurate solutions could not be obtained for problems in higher dimensions. We deal with in the next chapter why these methods fail to capture the accurate solutions. A natural question then arises: Is it possible to construct numerical methods that behave uniformly well for all values of the singular perturbation parameter, no matter how small ?

We need the following definition in the subsequent chapters to say that a numerical method has $\epsilon -$ **uniform convergence**.

Definition 2.1 *Consider a family of mathematical problems parameterized by a singular perturbation parameter ϵ, where ϵ lies in the semi$-$open interval $0 < \epsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by u, and that each u is approximated by a sequence of numerical solutions $\{(U, \overline{\Omega}^N)\}_{N=1}^{\infty}$ where U is defined on* *the mesh $\overline{\Omega}^N$ and N is a discretization parameter. Then the numerical solution of u are said to* ***converge*** *$\epsilon -$* ***uniformly*** *to the exact solution u, if there exist a positive integer N_0, and positive numbers C and p, where N_0, C and p are all independent of N and ϵ, such that for all $N \geq N_0$*

$$sup\|U - u\|_{\overline{\Omega}^N} \leq CN^{-p}$$

Here p is called the $\epsilon -$ uniform rate of convergence and C is called the $\epsilon -$ uniform error constant.

A finite difference method has two major ingredients: the finite difference operator L^N that is used to approximate the differential operator L and the mesh Ω^N that replaces the continuous domain Ω. By standard finite difference methods is meant almost all of the finite difference methods that have been applied successfully to problems that are not singularly perturbed. Many of these methods are well known and are named after some of their inventors. Generally, these methods are stable and accurate, and hence their solutions converge to the exact solution as $N \to \infty$. It turns out however that none of these methods is $\epsilon-$ uniform, and some new attribute is required.

In the construction of $\epsilon-$ *uniform methods* two approaches have generally been taken to date. The first of these involves replacing the standard finite difference operator by a finite difference operator which reflects the singularly perturbed nature of the differential operator. Such finite difference operators are referred to in general as **fitted finite difference operators**. In some cases, for example for linear problems, they may be constructed by choosing their coefficients so that some or all of the exponential functions in the null space of the differential operator, or part of it, are also in the null space of the finite difference operator. In such cases the finite difference operator is referred to as an exponentially fitted finite difference operator. The corresponding numerical method is then obtained by applying the fitted finite difference operator to obtain a system of finite difference equations on a standard mesh, which in practice is often a uniform mesh. This system is then solved in the useful way to obtain approximate solutions. Other approaches to constructing fitted finite difference operators are illustrated in (Roos, 1994).

The second successful approach to the construction of $\epsilon-$ uniform numerical methods involves the use of a mesh that is adapted to the singular perturbation. Such methods are referred to here as **fitted mesh methods**. A standard finite difference operator is applied on the fitted mesh to obtain a system of finite difference equations, which is then solved in the usual way to obtain approximate solutions. It is often sufficient to construct a piecewise uniform mesh, that is a mesh which is a union of a finite number of uniform meshes having different mesh parameters. These piecewise uniform fitted meshes were first introduced by Shishkin (Shishkin, 1988) and corresponding numerical methods were further developed and shown to be $\epsilon - uniform$ in a series of papers culminating in (Shishkin,1992). The first numerical results using a fitted mesh method were presented in (Miller et al. 1991). Different approaches to adapting the mesh, involving complicated redistribution of the mesh points, have been taken by other authors, for example Bakhvalov, Gartland, Liseikin and Vulanovic (Bakhvalov 1969, Gartland 1988, Liseikin 1983, Vulanovic 1986) but none has the simplicity of the piecewise uniform fitted meshes.

The above considerations show that both fitted operators and fitted meshes need to be developed. In the chapters 4 and 5, examples of each technique are presented. In practice, methods using fitted meshes are recommended whenever possible because they are usually simpler to implement than methods using fitted operators. Moreover, they are easier to generalize to problems in more than one dimension and to nonlinear problems.

CHAPTER 3

STANDARD NUMERICAL METHODS

In this chapter, we use the standard numerical methods for the problem (2.1) on a uniform mesh and explain that why these methods fail to converge to the analytical solution of the problem. We approximate the first derivative by the centered difference operator D^0 and the upwind difference operator D^-, respectively. We take fix boundary conditions $u_0 = 0$ and $u_1 = 1$.

3.1. Centered Difference Method for the Convection Diffusion Problem

Consider discrete operator

$$L^N = -\epsilon D^2 + bD^0$$

for the uniform partition $\overline{\Omega}^N$ of the Ω. It approximates the first derivative in the problem (2.1) with the centered difference operator D^0 and the second derivative with the second order difference operator D^2

$$\left\{ \begin{array}{c} Find\ U \in V(\overline{\Omega}^N)\ such\ that\ U_0 = 0\ \ U_N = 1\ and\ for\ all\ x_i \in \Omega^N, \\ \\ L^N U_i = -\epsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b\frac{U_{i+1} - U_{i-1}}{2h} = 0, \quad 1 \le i \le N-1 \end{array} \right\} \tag{3.1}$$

where $U_i \approx u(x_i)$. Then, combining terms with the same indices leads to the following difference equation

$$(-1 + \rho)U_{i+1} + 2U_i + (-1 - \rho)U_{i-1} = 0 \tag{3.2}$$

where $\rho = bh/2\epsilon$. It gives a system of equations with $N - 1$ unknowns. We can obtain the approximate solution of the problem (2.1) by solving this system. Some numerical results are given together with the exact solution in the Figures from 3.1 to 3.4 for the different values of ϵ with $N = 50$ and $b = 1$. They shows that the numerical solution to be consistent with the exact solution for the large values of ϵ. But it oscillates for the $\epsilon = 10^{-3}$.

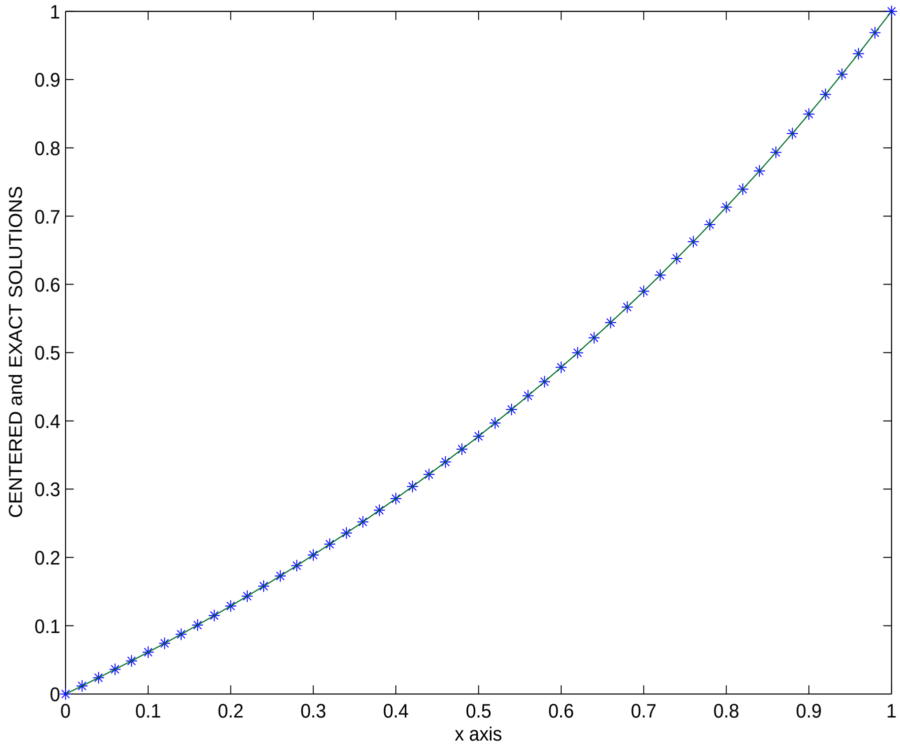

Figure 3.1. Exact($-$) and Centered Difference solution($*$) for $\epsilon = 1$

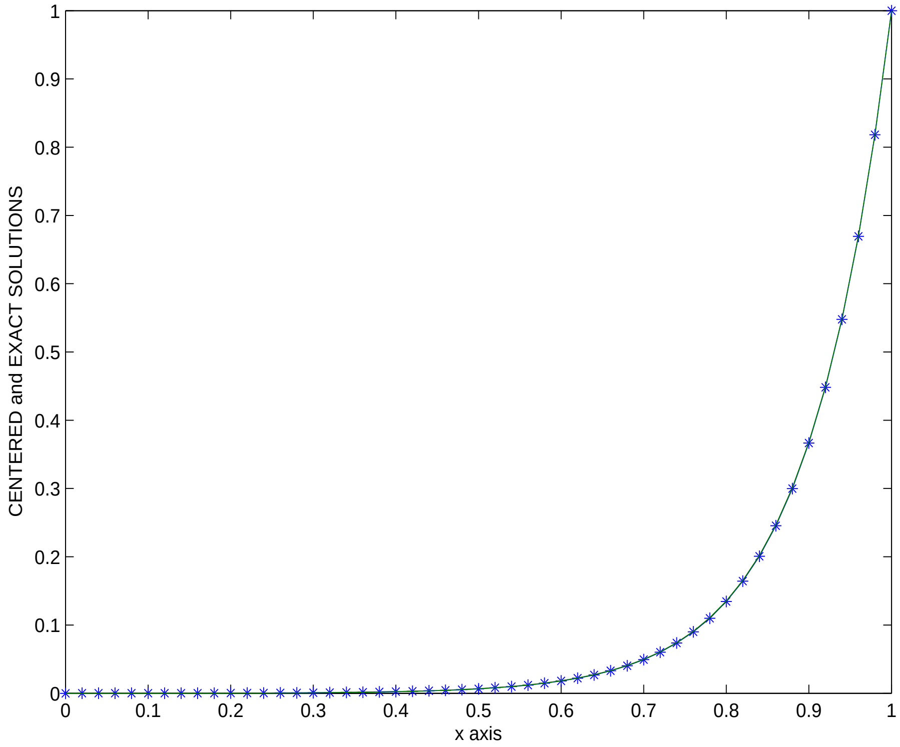

Figure 3.2. Exact($-$) and Centered Difference solution($*$) for $\epsilon = 0.1$

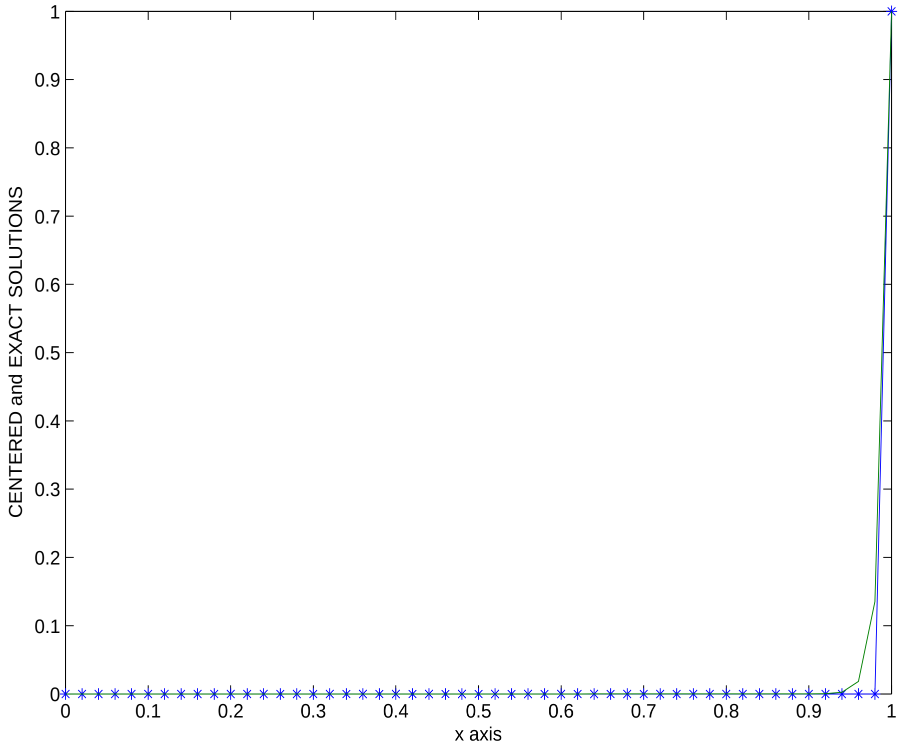

Figure 3.3. Exact($-$) and Centered Difference solution($*$) for $\epsilon = 0.01$

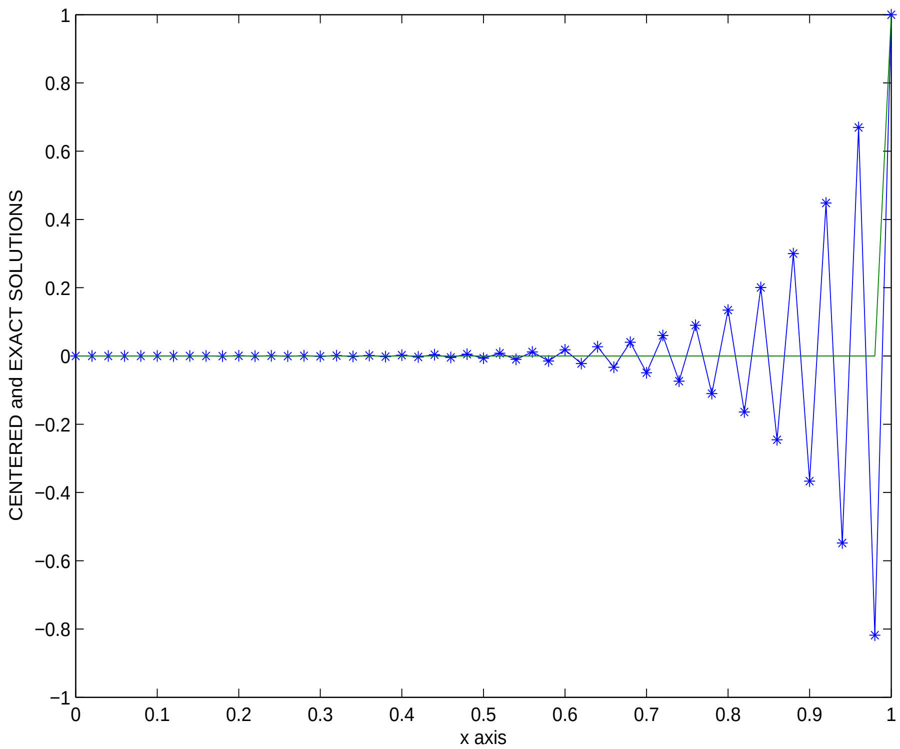

Figure 3.4. Exact($-$) and Centered Difference solution($*$) for $\epsilon = 0.001$

The situation can be explained by solving the difference equation (3.2) exactly. Setting $U_i = r^i$ in the difference equation and dividing the resulting expression with r^{i-1} then leads to the following characteristic equation

$$(-1+\rho)r^2 + 2r + (-1-\rho) = 0$$

and its roots are obtained as

$$r_1 = 1 \qquad\qquad r_2 = \frac{1+\rho}{1-\rho}.$$

Thus, the general solution to the difference equation (3.2) can be given by

$$U_i = a_1 r_1^i + a_2 r_2^i = a_1 + a_2 (\frac{1+\rho}{1-\rho})^i \tag{3.3}$$

Imposing the boundary conditions, i.e $U_0 = 0$ and $U_N = 1$, we obtain the unique solution of the difference equation (3.2) as follows

$$U_i = \frac{1 + (\frac{1+\rho}{1-\rho})^i}{1 - (\frac{1+\rho}{1-\rho})^N} \tag{3.4}$$

for all i, $0 \leq i \leq N$.

If $\rho < 1$, we see that numerical solution gives good results. However, the solution (3.4) clearly shows that if $\rho > 1$ $(\epsilon < 0.01)$, then the numerical solution oscillates since the second root r_2 would be negative in this case. Thus, we can conclude that the centered difference method is not robust for the problem (2.1).

3.2. Upwind Difference Method for the Convection Diffusion Problem

In this section, we approximate the first derivative with the backward difference operator D^- . The associated discrete operator is given by

$$L^N = -\epsilon D^2 + bD^-$$

and discrete problem is obtained as

$$\left\{ \begin{array}{c} Find\ U \in V(\overline{\Omega}^N)\ such\ that\ U_0 = 0\ \ U_N = 1\ \ and\ for\ all\ x_i \in \Omega^N, \\ \\ L^N U_i = -\epsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b\frac{U_i - U_{i-1}}{h} = 0, \quad 1 \leq i \leq N-1 \end{array} \right\} \tag{3.5}$$

Combining terms with the same indices we have the difference equation:

$$-U_{i+1} + (2+\rho)U_i + (-1-\rho)U_{i-1} = 0$$

where $\rho = bh/\epsilon$. Setting again $U_i = r^i$ and dividing resulting expression with the r^{i-1} results in

$$-r^2 + (2+\rho)r + (-1-\rho) = 0$$

and the roots of this characteristic equation are

$$r_1 = 1 \qquad\qquad r_2 = 1 + 2\rho.$$

Thus, the general solution can be expressed as follows

$$U_i = a_1 r_1^i + a_2 r_2^i = a_1 + a_2(1+2\rho)^i \tag{3.6}$$

The solution satisfying the boundary conditions can be immediately written as in the following form:

$$U_i = \frac{1-(1+2\rho)^i}{1-(1+2\rho)^N} \tag{3.7}$$

for all i, $0 \leq i \leq N$.

Since r_2 is always positive, we do not have oscillatory approximation in this case. Therefore, upwind difference method gives more stable result than the centered difference method as it can be seen in the Figures from 3.5 to 3.8. However, the error at the interior mesh point closest to the boundary $x = 1$ is

$$(U-u)(x_{N-1}) = \frac{1-3^{N-1}}{1-3^N} - \frac{e^{-1}-e^{-N}}{1-e^{-N}}$$

if $\rho = h/\epsilon = 1$. It follows that

$$\lim_{N\to\infty}(U-u)(x_{N-1}) = -\frac{1}{3} - \frac{1}{e} \neq 0.$$

and the upwind difference method is not convergent in the layer.

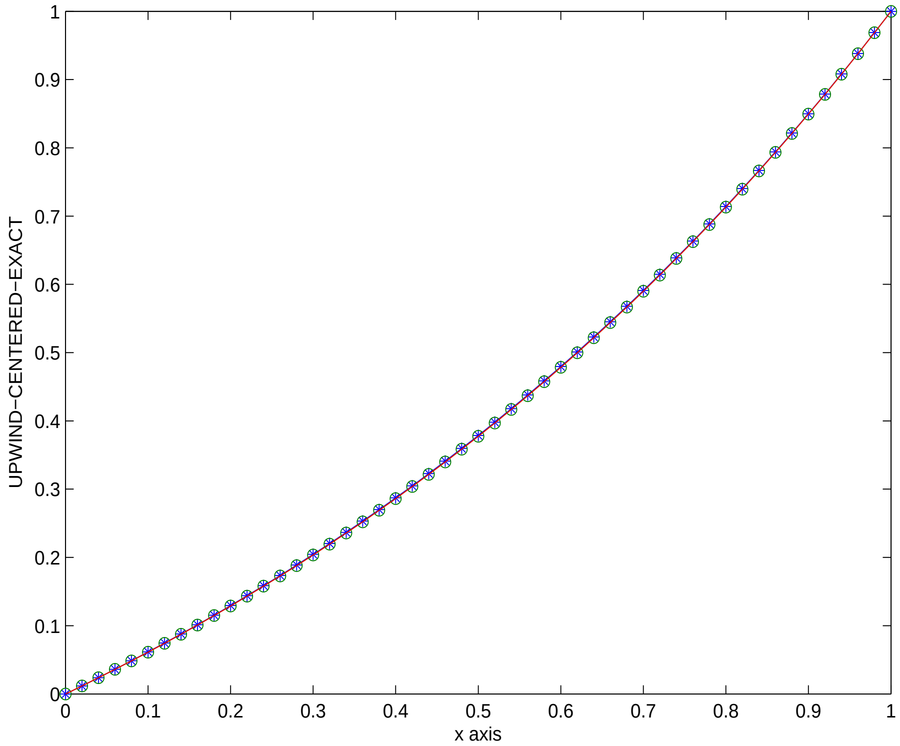

Figure 3.5. Centered(o) and Upwind($*$)Difference solution for $\epsilon = 1$

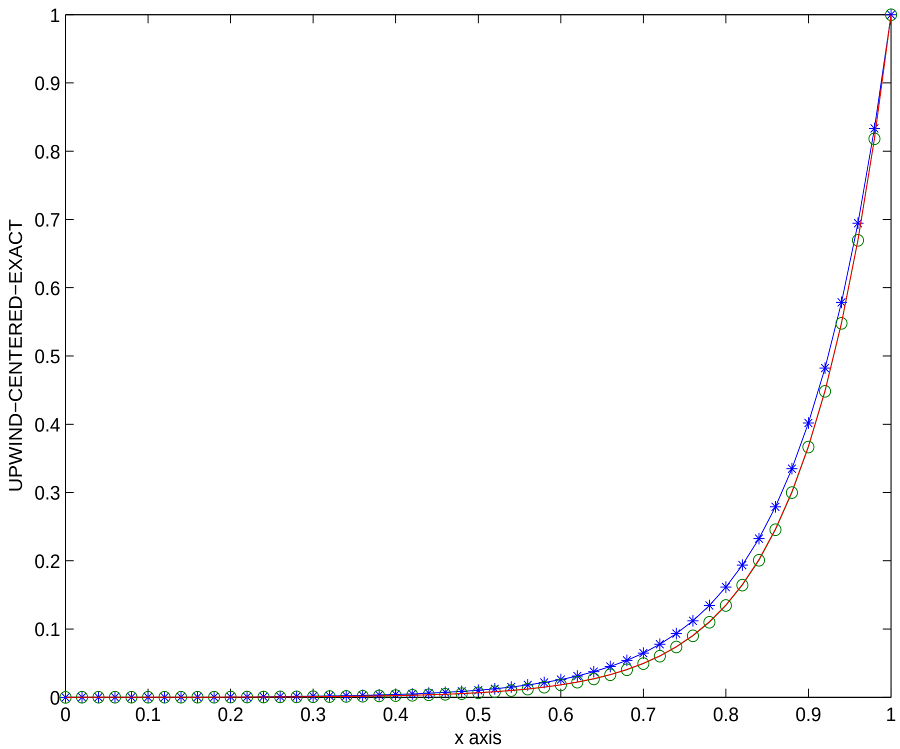

Figure 3.6. Centered(o) and Upwind($*$)Difference solution for $\epsilon = 0.1$

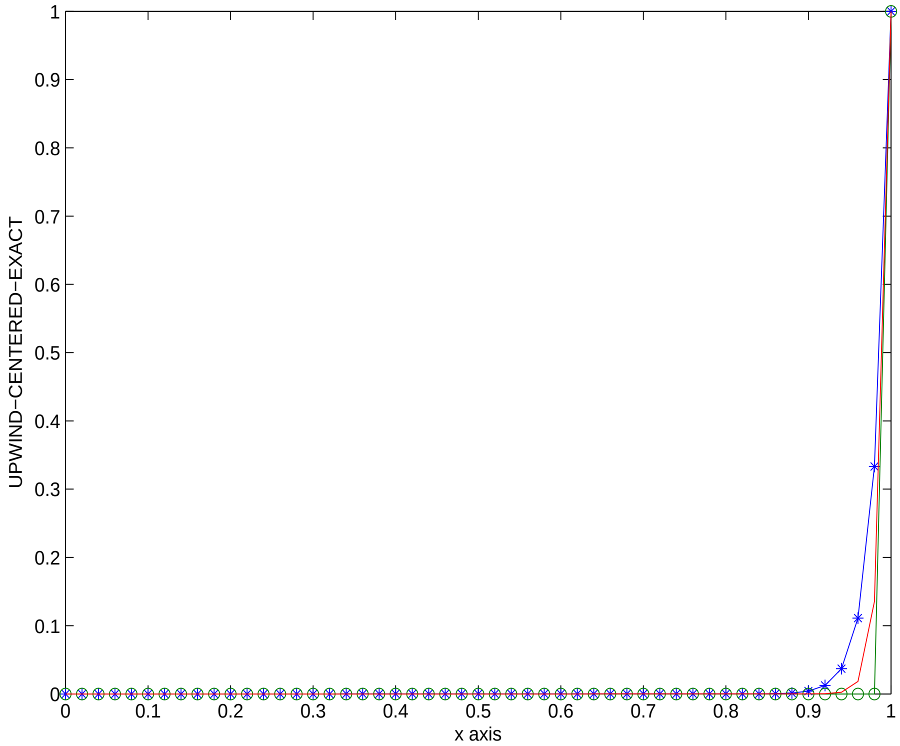

Figure 3.7. Centered(o) and Upwind$(*)$Difference solution for $\epsilon = 0.01$

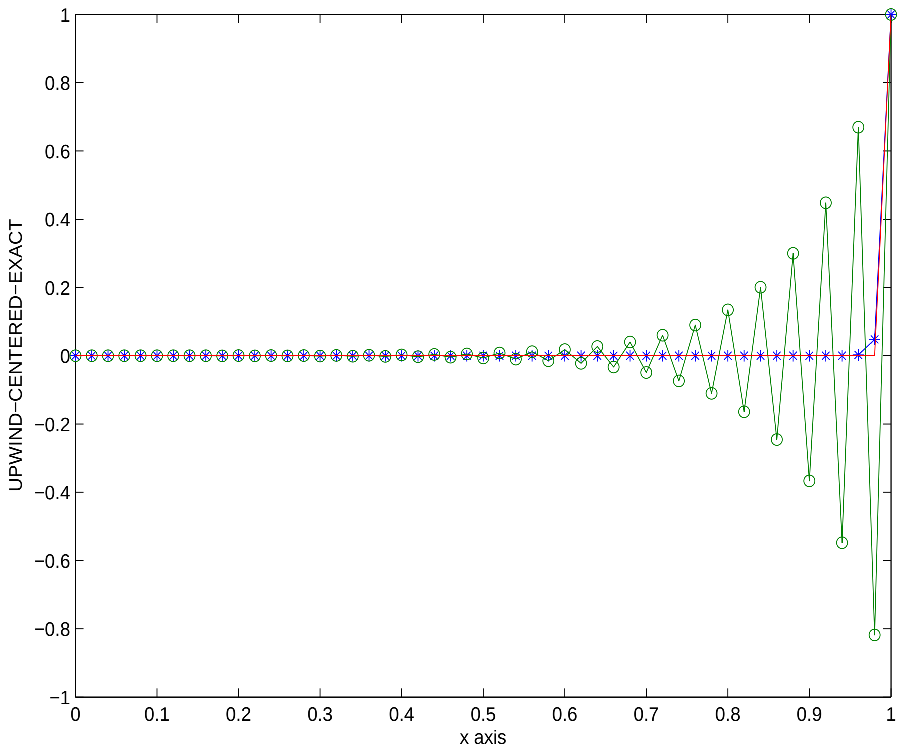

Figure 3.8. Centered(o) and Upwind$(*)$Difference solution for $\epsilon = 0.001$

CHAPTER 4

A UNIFORMLY CONVERGENT METHOD ON EQUIDISTANT MESHES

In this chapter, we deal with a uniformly convergent method, so called Il'in$-$Allen$-$Southwell method as an example of fitted numerical methods on a uniform mesh. We show that how to construct this method and present its convergence properties briefly.

Consider the following problem on the unit interval $\Omega = (0, 1)$

$$\left\{ \begin{array}{cc} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = 0\ ,\ u(1) = 0 \\ and\ for\ all\ x \in \Omega, \qquad Lu = -\epsilon u^{''}(x) + bu^{'}(x) = f(x) \end{array} \right\} \tag{4.1}$$

where b is a constant satisfying $b > 0$. The formal adjoint operator of L is defined by

$$L^* = -\epsilon \frac{d^2}{dx^2} - b\frac{d}{dx}$$

Let g_i be the local Green's function of L^* with respect to the discrete point x_i. Then the associated problem with the point x_i on the local domain $\overline{\Omega}_i \cup \overline{\Omega}_{i+1}$ can be given as follow:

$$\left\{ \begin{array}{c} Find\ g_i \in C(\overline{\Omega}_i \cup \overline{\Omega}_{i+1}) \cap C^2(\Omega_i \cup \Omega_{i+1})\ such\ that \\ g_i(x_{i-1}) = 0\ \ ,\ \ g_i(x_{i+1}) = 0 \\ and\ for\ all\ x^* \in \Omega_i \cup \Omega_{i+1}, \qquad L^* g_i = -\epsilon g_i^{''}(x) - b g_i^{'}(x) = 0 \end{array} \right\} \tag{4.2}$$

where $\Omega_i = (x_{i-1}, x_i)$ and with the additional condition

$$\epsilon(g_i^{'}(x_i^-) - g_i^{'}(x_i^+)) = 1. \tag{4.3}$$

Thus, multiplying the equation $Lu = f$ with g_i and integrating the resulting expression from x_{i-1} to x_{i+1} we obtain the following equality

$$\int_{x_{i-1}}^{x_{i+1}} (Lu) g_i dx = \int_{x_{i-1}}^{x_{i+1}} f g_i dx.$$

Then, using of integration by parts and the continuity of $u^{'}$ with the boundary conditions of the problem (4.2) and the condition (4.3) respectively we get the following equation

$$-\epsilon g_i'(x_{i-1})u_{i-1} + u_i + \epsilon g_i'(x_{i+1})u_{i+1} = f\int_{x_{i-1}}^{x_{i+1}} g_i dx \tag{4.4}$$

where $u_i \approx u(x_i)$. This gives a difference scheme since we are able to evaluate each $g_i^{'}$'s exactly, see $[5]$ for details. The solution of the equation (4.2) is given by

$$g_i(x^-) = c_1 + c_2\frac{-\epsilon}{b}e^{-bx/\epsilon} \qquad on\ \ (x_{i-1}, x_i) \tag{4.5}$$

$$g_i(x^+) = c_3 + c_4\frac{-\epsilon}{b}e^{-bx/\epsilon} \qquad on\ \ (x_i, x_{i+1}). \tag{4.6}$$

To determine the coefficients c_1, c_2, c_3 and c_4 we need four equations. These come from the conditions

$$g_i(x_{i-1}) = 0 \qquad g_i(x_{i+1}) = 0$$

$$\epsilon(g_i^{'}(x_i^-) - g_i^{'}(x_i^+)) = 1$$

and, from the continuity of g_i at $x = x_i$

$$g_i(x_i^-) = g_i(x_i^+).$$

Using of these conditions yields the following solution for the problem described by (4.2) and (4.3)

$$g_i(x^-) = \frac{1}{b}\frac{e^\rho - 1}{(e^\rho - e^{-\rho})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1 - e^{-\rho})}{(e^\rho - e^{-\rho})}(\frac{-\epsilon}{b})e^{\frac{-bx}{\epsilon}} \quad on\ \ [x_{i-1}, x_i] \tag{4.7}$$

$$g_i(x^+) = \frac{1}{b}\frac{e^{-\rho} - 1}{(e^\rho - e^{-\rho})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1 - e^{\rho})}{(e^\rho - e^{-\rho})}(\frac{-\epsilon}{b})e^{\frac{-bx}{\epsilon}} \qquad on\ \ [x_i, x_{i+1}] \tag{4.8}$$

where $\rho = \frac{bh}{\epsilon}$ and $\alpha_i = \frac{bx_i}{\epsilon}$. Thus, we obtain $g_i^{'}(x_{i-1}^-)$ and $g_i^{'}(x_{i+1}^+)$ by taking the derivatives of (4.7) and (4.8), respectively. They are given by

$$g_i^{'}(x_{i-1}^-) = \frac{1}{\epsilon}\frac{(e^\rho - 1)}{(e^\rho - e^{-\rho})} \qquad g_i^{'}(x_{i+1}^+) = \frac{1}{\epsilon}\frac{(e^{-\rho} - 1)}{(e^\rho - e^{-\rho})}. \tag{4.9}$$

Hence, using (4.7) and (4.8) we evaluate the integral in (4.4) and substituting the resulting expression together with (4.9) into the equation (4.4) leads to following fitted finite difference method

$$-\frac{e^\rho - 1}{e^\rho - e^{-\rho}}u_{i-1} + u_i - \frac{1 - e^{-\rho}}{e^\rho - e^{-\rho}}u_{i+1} = f\ \frac{h}{b}\ \frac{e^\rho - 1}{e^\rho + 1} \tag{4.10}$$

where $\rho = \frac{bh}{\epsilon}$. Its solution is given by

$$u_i = \frac{1 - e^{-i\rho}}{1 - e^{N\rho}}$$

and it satisfies the following error estimate.

Theorem 4.1 *The fitted finite difference method* (4.10) *with the uniform mesh* Ω^N*, is* $\epsilon-$*uniform for the problem* (4.1). *Moreover, the solution* u *of* (4.1) *and the solution* u_i *of* (4.10) *satisfy the following* $\epsilon-$*uniform error estimate*

$$\sup_{0<\epsilon\leq 1} \parallel u(x_i) - u_i \parallel_{\overline{\Omega}^N} \leq CN^{-1} \quad for \;\; 0 \leq i \leq N$$

where C is a constant independent of ϵ.

Proof:(Roos et al. 1994, Demirayak 2004).■

Notice that although the Il'in$-$Allen$-$Southwell method has first order convergence in the discrete maximum norm, it is based on the exact solution of the local problem (4.2). This is a disadvantage of the method.

Example: We take $f(x) = x$, $b = 1$ in the problem (4.1) and apply the Il'in$-$Allen$-$Southwell method. The numerical solution and exact solution are plotted on the same window for different values of ϵ and we also give the error at the boundary layer for a fixed value of ϵ when N increases. The Figures from 4.1 to 4.6 indicate that the error decreases in the boundary layer if we refine the uniform mesh $\overline{\Omega}^N$.

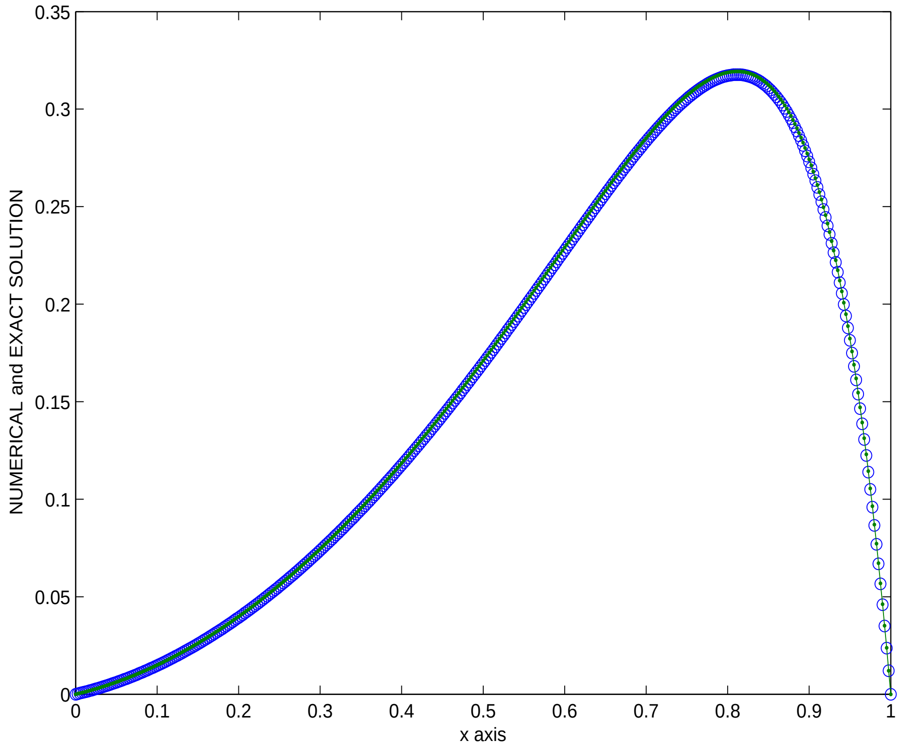

Figure 4.1. The Uniformly Convergent Method(o) for $N = 400$, $\epsilon = 0.1$

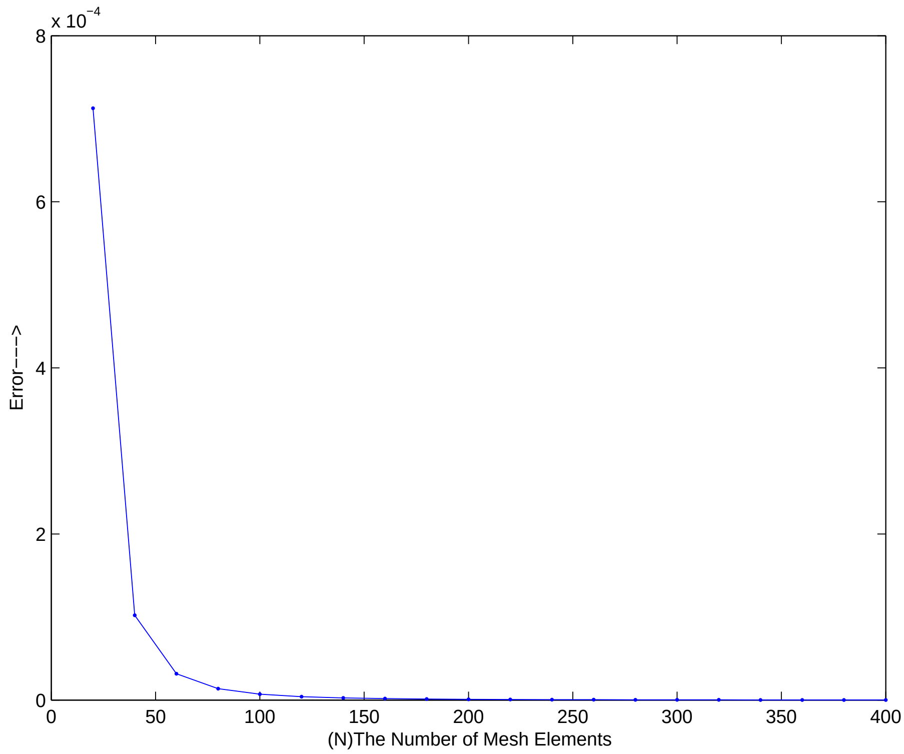

Figure 4.2. Error at the boundary for $\epsilon = 0.1$

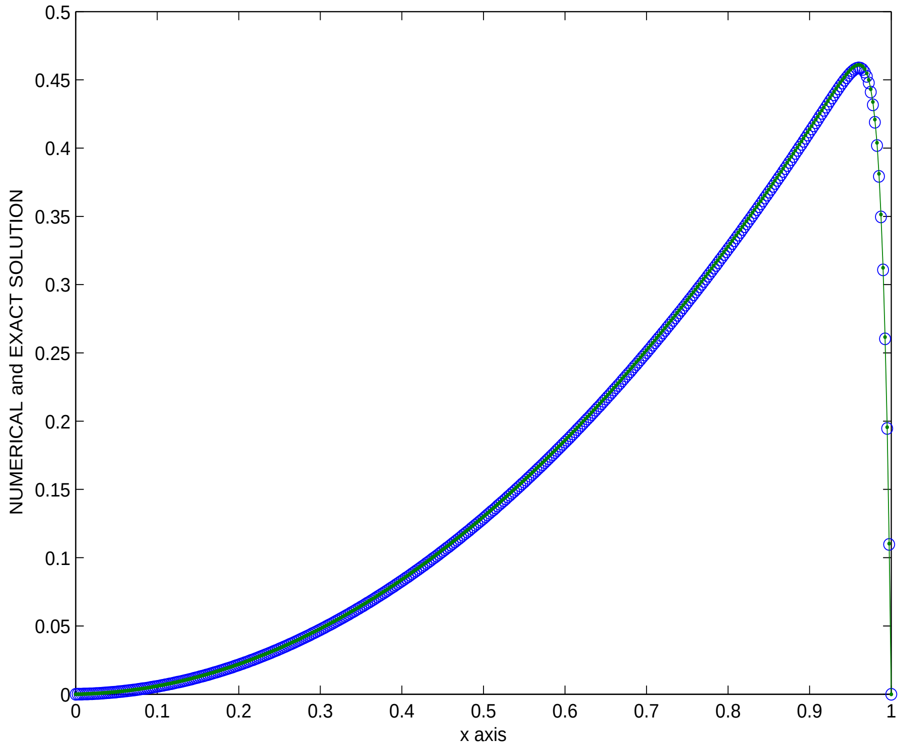

Figure 4.3. The Uniformly Convergent Method(o) for $N = 400$, $\epsilon = 0.01$

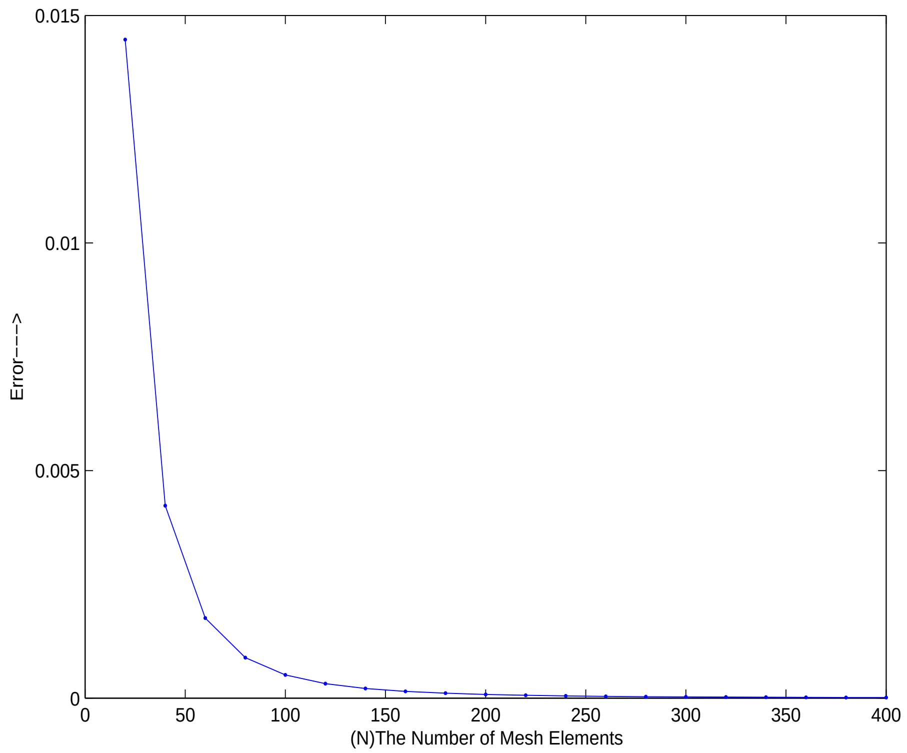

Figure 4.4. Error at the boundary for $\epsilon = 0.01$

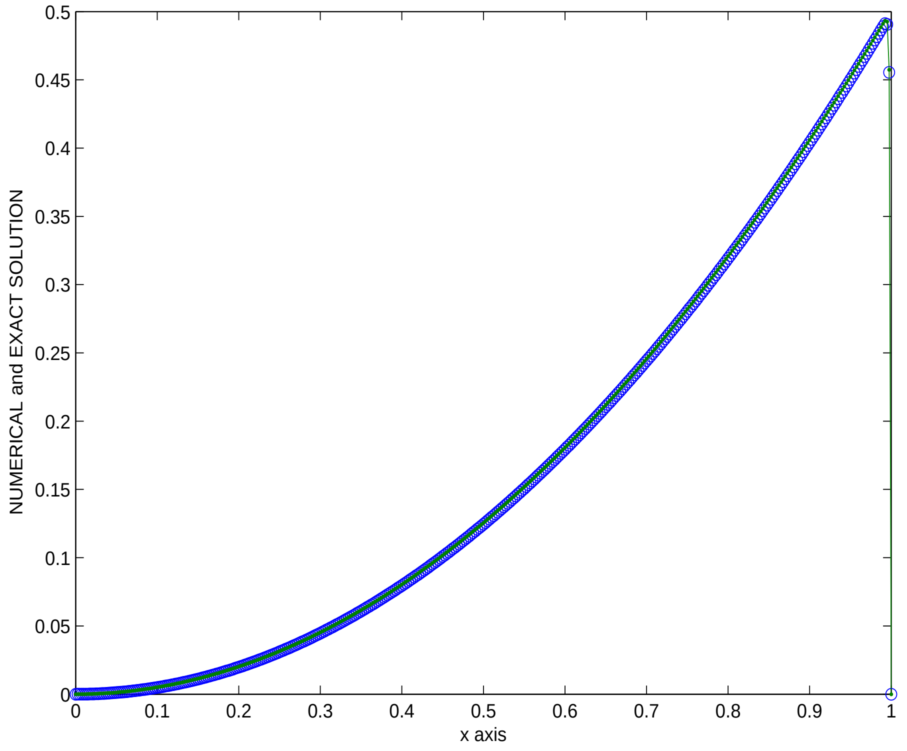

Figure 4.5. The Uniformly Convergent Method(o) for $N = 400$, $\epsilon = 0.001$

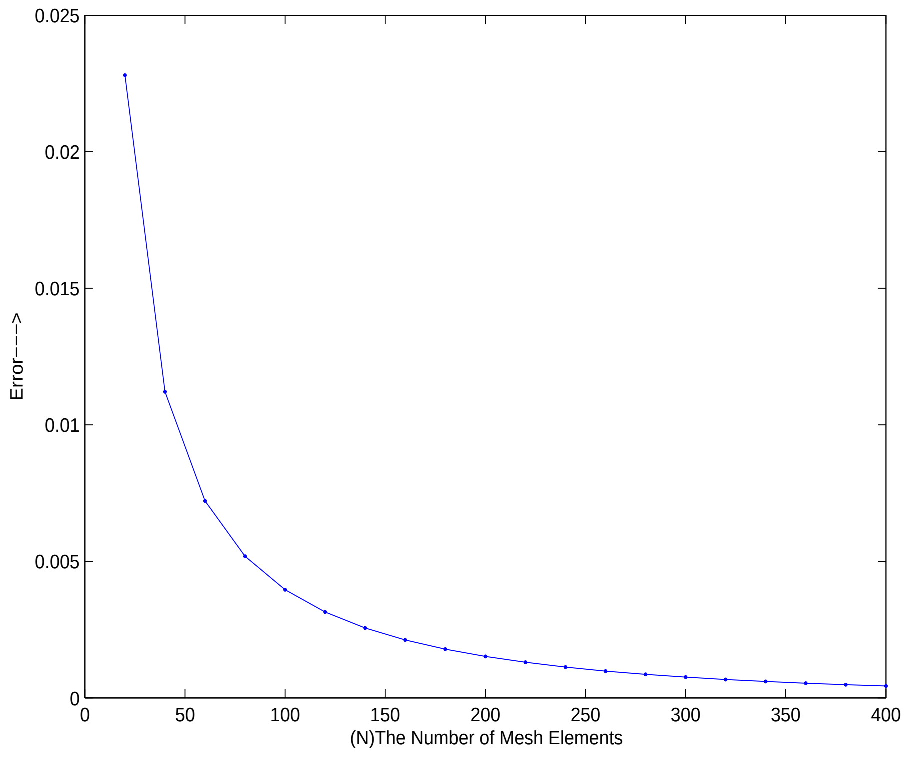

Figure 4.6. Error at the boundary for $\epsilon = 0.001$

CHAPTER 5

A UNIFORMLY CONVERGENT METHOD ON PIECEWISE UNIFORM MESHES

In this chapter, $\epsilon - uniform$ fitted mesh method is constructed for the convection diffusion problem. To introduce the idea of such method the problem (2.1) is considered here again. A piecewise uniform fitted mesh turns out to be sufficient for the construction of an $\epsilon{-}uniform$ method for a wide variety of problem configurations. Of course, more complicated fitted meshes may also be used. However, for simplicity, the piecewise uniform meshes is considered to be one of the most attractive choices.

A simple example of a piecewise uniform mesh is constructed on the interval $\Omega = (0, 1)$ as follows. Choose a point $1 - \tau$ satisfying $0 < \tau \leq 1/2$ and assume that $N = 2^r$, for some $r \geq 2$. The point $1 - \tau$ divides Ω into the two subintervals $(0, 1 - \tau)$ and $(1 - \tau, 1)$. The corresponding piecewise uniform mesh is constructed by dividing both $(0, 1 - \tau)$ and $(1 - \tau, 1)$ into $N/2$ equal subintervals denoted by Ω_τ^N. The figure 5.1 shows the piecewise uniform mesh Ω_τ^8

Figure 5.1. The Piecewise Uniform Mesh Ω_τ^8

where

$$\tau = \min\{\frac{1}{2}, \epsilon \ln N\}.$$

Notice that, as might be expected, τ depends on both ϵ and N. This means that locations of the mesh points change whenever ϵ or N changes. Note also that whenever N is sufficiently large τ takes the value $1/2$, and therefore the mesh Ω_τ^N becomes the uniform mesh with N subintervals. This happens when N satisfies

$$\epsilon \ln N \geq \frac{1}{2} \qquad or \qquad N \geq e^{\frac{1}{2\epsilon}}.$$

For all other permissible values of τ, $0 < \tau \leq 1/2$, the subinterval $(1-\tau, 1)$ is smaller than the subinterval $(0, 1-\tau)$. In these cases each of the $N/2$ uniform mesh elements of $(1-\tau, 1)$ is of length $2\tau/N$ which is shorter than the length $2(1-\tau)/N$ of the $N/2$ uniform mesh elements of $(0, 1-\tau)$. In such cases the global mesh is piecewise uniform rather than uniform and, because the subintervals in a neighborhood of 1 are small when τ is close to 0, the mesh is said to be condensing in a neighborhood of the boundary point $x = 1$, or more concisely, condensing at the point $x = 1$. Notice that, whatever the value of τ, all of the meshes consist of N mesh elements and consequently the mesh points are $\overline{\Omega}_\tau^N = \{x_i\}_0^N$ where the points x_i are the endpoints of these N mesh elements. It is not hard to see that the transition point $1-\tau$ coincides with the meshpoint $x_{N/2}$ and for the mesh $\overline{\Omega}_\tau^N = \{x_i\}_0^N$ the following inequalities hold

$$\begin{aligned}
h_i &\leq 2/N && for \quad 1 \leq i \leq N \\
h_i &\geq 1/N && for \quad 1 \leq i \leq N/2 \\
h_i &\leq 2\tau/N && for \quad N/2+1 \leq i \leq N \\
\overline{h}_i &\geq h_i/2 && for \quad 1 \leq i \leq N-1.
\end{aligned} \tag{5.1}$$

The $\epsilon-$uniform error analysis of many numerical methods on piecewise uniform fitted meshes depends on the following basic lemma.

Lemma 5.1 *For all integers* $N \geq 1$

$$(1 + \frac{2\ln N}{N})^{-N/2} \leq 2N^{-1}.$$

Proof: The inequality is trivial for $N = 1, 2, 3$. For $N \geq 4$ write the inequality in the form

$$(1 + \frac{2\ln N}{N})^{N/2} \geq \frac{N}{2}.$$

Letting $x = N/2$ this becomes the inequality

$$(1 + \frac{ln 2x}{x})^x \geq x, \qquad for \;\; all \;\; x \geq 2.$$

Taking the natural logarithm of both sides and dividing by x, this is equivalent to

$$\ln{(1 + \frac{ln 2x}{x})^x} \geq \frac{\ln x}{x}, \qquad for \;\; all \;\; x \geq 2.$$

Defining

$$g(x) = \ln{(1+\frac{ln2x}{x})^x} - \frac{\ln x}{x}$$

it is also equivalent to $g(x) \geq 0$, for all $x \geq 2$. Now, since

$$\lim_{x\to\infty} g(x) = \ln 1 = 0$$

and

$$\begin{aligned} g^{'}(x) &= \frac{1-\ln 2x}{x^2(1+\frac{\ln 2x}{x})} - \frac{1-\ln x}{x^2} \\ &= \frac{1}{x^2(1+\frac{\ln 2x}{x})}[1-\ln 2x-(1+\frac{\ln 2x}{x})(1-\ln x)] \\ &= \frac{-1}{x^3(1+\frac{\ln 2x}{x})}(x\ln 2+(1-\ln x)\ln 2x) \\ &= -\frac{h(x)}{x^3(1+\frac{\ln 2x}{x})} \end{aligned}$$

where

$$h(x) = x\ln 2 + (1-\ln x)\ln 2x$$

if we show that $g^{'}(x) < 0$, for all $x \geq 2$, we can say that $g(x)$ is a monotone decreasing function for $x \geq 2$ and it follows that $g(x) \geq 0$ for all $x \geq 2$. Thus, we need to determine the sign of the function $h(x)$ for $x \geq 2$. Since

$$h(2) = 2(2-\ln 2)\ln 2 > 0$$

and

$$h^{'}(x) = \frac{1}{x}(x\ln 2 + 1 - \ln x - \ln 2x) = \frac{k(x)}{x}$$

where

$$k(x) = x\ln 2 + 1 - \ln 2x^2,$$

it is seen that

$$k(2) = 1 - \ln 2 > 0$$

and

$$k^{'}(x) = \ln 2 - \frac{2}{x}, \qquad k^{''}(x) = \frac{2}{x^2}.$$

Thus, for $x \geq 2$, k has a minimum at $x = 2/\ln 2$. Its value there is

$$k(\frac{2}{\ln 2}) = 3(1 - \ln 2) + 2\ln\ln 2 > 0.$$

It follows that $k(x) > 0$ for all $x \geq 2$. This show also that $h^{'}(x) > 0$ and $h(x) > 0$ for all $x \geq 2$. Thus $g^{'}(x) < 0$ for all $x \geq 2$ and since $\lim_{x\to\infty} g(x) = 0$ it follows that $g(x) \geq 0$ for all $x \geq 2$ as required.■

In addition to this Lemma, the following contains two standard results for local truncation errors on general $non\!-\!uniform$ meshes.

Lemma 5.2 *Let $x_i \in \overline{\Omega}_\tau^N$. Then for any $\varphi \in C^2(\overline{\Omega})$,*

$$\mid (D^- - \frac{d}{dx})\varphi(x_i) \mid \leq \frac{1}{2}(x_i - x_{i-1}) \mid \varphi \mid_2$$

and, for any $\varphi \in C^3(\overline{\Omega})$,

$$\mid (D^2 - \frac{d^2}{dx^2})\varphi(x_i) \mid \leq \frac{1}{3}(x_{i+1} - x_{i-1}) \mid \varphi \mid_3$$

Proof: Using integration by parts we can show that

$$\begin{aligned}
\frac{1}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i}(x_{i-1} - s)\varphi^{''}(s)ds &= \frac{1}{x_i - x_{i-1}}\mathbf{[}(x_{i-1} - s)\varphi^{'}(s) \mid_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i}\varphi^{'}(s)ds\mathbf{]}\\
&= \frac{1}{x_i - x_{i-1}}\mathbf{[}(x_{i-1} - x_i)\varphi^{'}(x_i) + \varphi(x_i) - \varphi(x_{i-1})\mathbf{]}\\
&= \mathbf{[}\frac{\varphi(x_i) - \varphi(x_{i-1})}{x_i - x_{i-1}} - \varphi^{'}(x_i)\mathbf{]}\\
&= (D^- - \frac{d}{dx})\varphi(x_i).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mid (D^- - \frac{d}{dx})\varphi(x_i) \mid &\leq \frac{\mid \varphi \mid_2}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i}(s - x_{i-1})ds\\
&= \frac{\mid \varphi \mid_2}{x_i - x_{i-1}}\mathbf{[}\frac{s^2}{2} - x_{i-1}s\mathbf{]}_{x_{i-1}}^{x_i}\\
&= \frac{\mid \varphi \mid_2}{x_i - x_{i-1}}\mathbf{[}\frac{x_i^2 - x_{i-1}^2}{2} - x_{i-1}(x_i - x_{i-1})\mathbf{]}\\
&= \frac{\mid \varphi \mid_2}{x_i - x_{i-1}}.\frac{(x_i - x_{i-1})^2}{2}\\
&= \frac{1}{2}(x_i - x_{i-1}) \mid \varphi \mid_2
\end{aligned}$$

which is the first result. Similarly, using integration by parts twice we see that

$$\begin{aligned}
\frac{1}{x_{i+1} - x_{i-1}}\mathbf{[}\int_{x_i}^{x_{i+1}}\frac{(x_{i+1} - s)^2\varphi^{'''}(s)}{x_{i+1} - x_i}ds - \int_{x_{i-1}}^{x_i}\frac{(s - x_{i-1})^2\varphi^{'''}(s)}{x_i - x_{i-1}}ds\mathbf{]}\\
= (D^2 - \frac{d^2}{dx^2})\varphi(x_i)
\end{aligned}$$

and it follows that

$$| (D^2 - \frac{d^2}{dx^2})\varphi(x_i) | \leq \frac{| \varphi |_3}{x_{i+1} - x_{i-1}} \mathbf{[} \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - s)^2}{x_{i+1} - x_i} ds - \int_{x_{i-1}}^{x_i} \frac{(s - x_{i-1})^2}{x_i - x_{i-1}} ds \mathbf{]}$$
$$\leq \frac{1}{3}(x_{i+1} - x_{i-1}) \mid \varphi \mid_3 .$$

This completes the proof.■

5.1. Properties of Upwind Finite Difference Operator on Piecewise Uniform Fitted Meshes

Next, we overview properties of upwind finite difference operator on the piecewise uniform meshes to obtain some arguments perform on the convergence analysis of fitted mesh method related to the convection$-$diffusion problem with regular data. Consider the discrete operator

$$L^N = -\epsilon D^2 + bD^-$$

on the fitted piecewise uniform mesh $\overline{\Omega}_\tau^N$ defined at beginning of the chapter. Notice that the finite difference operators D^2 and D^- are used in the form introduced in (2.5). Thus, discrete problem related to the problem (2.1) is given by

$$\left\{ \begin{array}{l} Find\ U \in V(\overline{\Omega}_\tau^N)\ such\ that\ U_0\ \ and\ \ U_N\ \ given\ and\ for\ all\ x_i \in \Omega_\tau^N, \\ \\ L^N U_i = -\epsilon \frac{(D^+ - D^-)}{\bar{h}_i} U_i + bD^- U_i = 0, \qquad 1 \leq i \leq N-1 \end{array} \right\} \quad (5.2)$$

or equivalently

$$\left\{ \begin{array}{l} Find\ U \in V(\overline{\Omega}_\tau^N)\ such\ that\ U_0\ \ and\ \ U_N\ \ given\ and\ for\ all\ x_i \in \Omega_\tau^N, \\ \\ L^N U_i = -\epsilon (\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}) \frac{1}{\bar{h}_i} + b \frac{U_i - U_{i-1}}{h_i} = 0, \quad 1 \leq i \leq N-1 \end{array} \right\} \quad (5.3)$$

where b is a constant satisfying the strict inequality

$$b > \alpha > 0 \qquad (5.4)$$

for some constant α. The fitted piecewise uniform mesh $\overline{\Omega}_\tau^N = \{x_i\}_0^N$ is defined by

$$x_i - x_{i-1} = \left\{ \begin{array}{ll} \frac{2(1-\tau)}{N} & for \quad 0 < i \leq N/2 \\ \frac{2\tau}{N} & for \ \ N/2 < i \leq N \end{array} \right\}$$

where

$$\tau = \min\{\frac{1}{2}, \frac{\epsilon}{\alpha}\ln N\}$$

and it is assumed that $\tau \leq 1/2$. In the Figure 5.2 the solution of the discrete problem (5.3) is plotted for the special choices $U_0 = 0$ and $U_N = 1$.

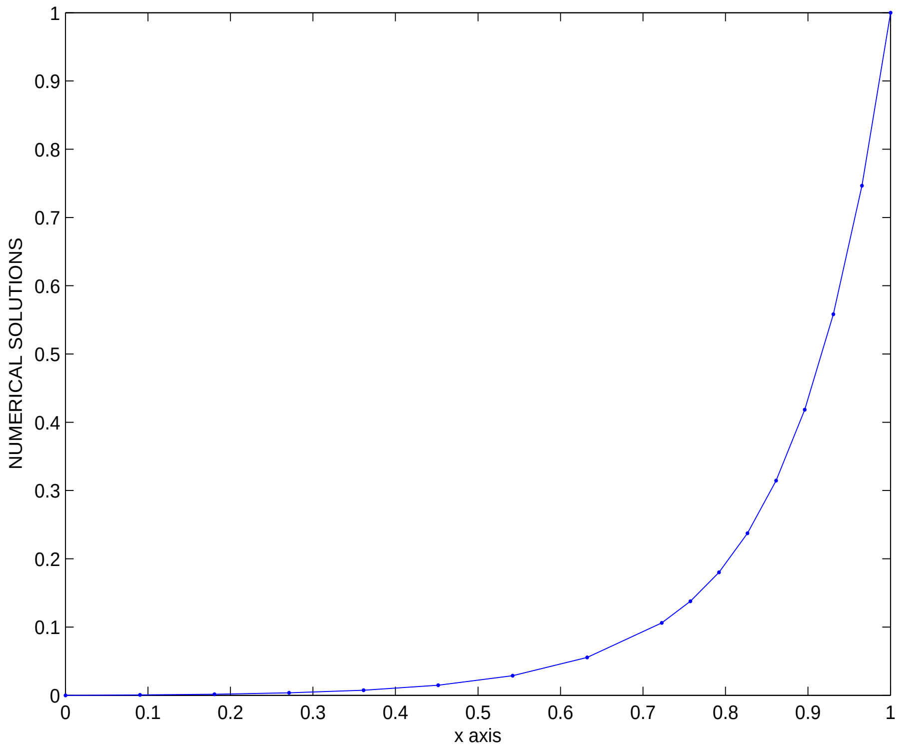

Figure 5.2. The mesh function U_i with $U_0 = 0$, $U_{16} = 1$ and $N = 16$

In the next section, we will show that the mesh function U_i is an $\epsilon - uniform$ approximation of the continuous boundary layer function $e^{-b(1-x)/\epsilon}$ which appears in the solution (2.2).

It is convenient to introduce the following notation

$$h_1 = \frac{2(1-\tau)}{N}, \qquad h_2 = \frac{2\tau}{N}, \qquad \overline{h} = \frac{h_1 + h_2}{2}$$

$$\lambda_1 = 1 + \frac{bh_1}{\epsilon}, \qquad \lambda_2 = 1 + \frac{bh_2}{\epsilon}, \qquad \overline{\lambda} = \frac{\lambda_1 + \lambda_2}{2}. \tag{5.5}$$

Then, it is clear that

$$\overline{h} = \frac{1}{N} \qquad \overline{\lambda} = 1 + \frac{b\overline{h}}{\epsilon}$$

and

$$1 < \lambda_1 \leq 2\overline{\lambda} \qquad 1 < \lambda_2 \leq 2\overline{\lambda}. \tag{5.6}$$

We turn back now to the difference equation in (5.3) and try to obtain its solution. It can be written separately in each of the subinterval $[0, 1-\tau]$ and $[1-\tau, 1]$ as follows:

$$(-1)U_{i+1} + (1+\lambda_1)U_i + (-\lambda_1)U_{i-1} = 0; \qquad if \qquad 1 \leq i < N/2$$

$$(-\frac{h_1}{h_2})U_{N/2+1} + (\frac{h_1}{h_2} + \overline{\lambda})U_{N/2} + (-\overline{\lambda})U_{N/2-1} = 0; \qquad if \qquad i = N/2 \tag{5.7}$$

$$(-1)U_{i+1} + (1+\lambda_2)U_i + (-\lambda_2)U_{i-1} = 0; \qquad if \qquad N/2 < i \leq N-1$$

Since the roots of the characteristic polynomial are

$$r_1 = 1 \qquad r_2 = \lambda_1 \quad for \;\; 1 \leq i < N/2$$
$$r_1 = 1 \qquad r_2 = \lambda_2 \quad for \;\; N/2 < i \leq N-1$$

we assume that the difference solution are of the form

$$U_i = \left\{ \begin{array}{ccc} a_1 + a_2\lambda_1^i & if & 0 \leq i \leq N/2 \\ & & \\ a_3 + a_4\lambda_2^i & if & N/2 \leq i \leq N \end{array} \right\}. \tag{5.8}$$

We have four unknown coefficient a_1, a_2, a_3, a_4 and need four equations. Two equations come from the boundary conditions U_0 and U_N. One of the other two equations is obtained by using the difference equation at the discrete node $x_{N/2}$ and the other can be obtained by using continuity condition at the same node. Thus, resulting system of equations can be given in the matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & \lambda_1^{N/2} & -1 & -\lambda_2^{N/2} \\ 1 & \kappa & -1 & -\lambda_2^{N/2+1} \\ 0 & 0 & 1 & \lambda_2^N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} U_0 \\ 0 \\ 0 \\ U_N \end{bmatrix}$$

where $\kappa = \lambda_1^{N/2-1}(\lambda_1 + \overline{\lambda}(\lambda_2 - 1))$. We solve this system and obtain the following results

$$a_1 = \frac{1}{\zeta}[\lambda_1\lambda_2^{-N/2}U_0 + (1-\lambda_2^{-N/2})\overline{\lambda}U_0 - \lambda_1^{-N/2+1}\lambda_2^{-N/2}U_N]$$
$$a_2 = \frac{1}{\zeta}[\lambda_1^{-N/2+1}\lambda_2^{-N/2}(U_N - U_0)]$$
$$a_3 = \frac{1}{\zeta}[\overline{\lambda}(U_0 - \lambda_2^{-N/2}U_N) - \lambda_1^{-N/2+1}\lambda_2^{-N/2}U_N + \lambda_1\lambda_2^{-N/2}U_N]$$
$$a_4 = -\frac{1}{\zeta}[\overline{\lambda}(U_0 - U_N)]$$

where

$$\zeta = \lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^{N/2} - 1) + (1-\lambda_2^{-N/2})\overline{\lambda}.$$

Substituting these coefficients into the form of the solution (5.8), we get

$$U_i = \left\{ \begin{array}{lll} U_0 + \frac{(U_N-U_0)\lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^i-1)}{\lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^{N/2}-1)+(1-\lambda_2^{-N/2})\overline{\lambda}} & for & 0 \le i \le N/2 \\ & & \\ U_N + \frac{(U_0-U_N)\overline{\lambda}(1-\lambda_2^{i-N})}{\lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^{N/2}-1)+(1-\lambda_2^{-N/2})\overline{\lambda}} & for & N/2 < i < N \end{array} \right\}$$

or in a more compact form

$$U_i = \left\{ \begin{array}{lll} U_0 + (U_N - U_0)\varphi_i^N & for & 0 \le i \le N/2 \\ U_N + (U_0 - U_N)\psi_i^N & for & N/2 \le i \le N \end{array} \right\} \tag{5.9}$$

where

$$\varphi_i^N = \frac{\mu_i^N}{d_N}, \qquad \psi_i^N = \frac{\nu_i^N}{d_N}$$

with

$$\mu_i^N = \lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^i - 1), \qquad \nu_i^N = \overline{\lambda}(1 - \lambda_2^{i-N})$$

and

$$d_N = \mu_{N/2} + \nu_{N/2}.$$

The following lemma shows that this solution is monotone increasing.

Lemma 5.3 *Assume that $U_N > U_0$, then*

$$U_0 < U_i < U_N \qquad for \qquad 1 \le i \le N-1$$

and

$$D^-U_i > 0 \qquad for \qquad 1 \le i \le N.$$

Proof: From the explicit expression (5.9) for U_i, it is clear that

$$D^-U_i = \left\{ \begin{array}{lll} (U_N - U_0)D^-\varphi_i^N & for & 1 \le i \le N/2 \\ (U_0 - U_N)D^-\psi_i^N & for & N/2+1 \le i \le N \end{array} \right\}.$$

Since

$$\begin{aligned} D^-\varphi_i^N &= \frac{\varphi_i^N - \varphi_{i-1}^N}{h_1} = \frac{\lambda_1(\lambda_1\lambda_2)^{-N/2}}{d_N}\frac{(\lambda_1^i - \lambda_1^{i-1})}{h_1} = \frac{(\lambda_1\lambda_2)^{-N/2}}{d_N}\frac{(\lambda_1^{i+1} - \lambda_1^i)}{h_1} \\ &= \frac{(\lambda_1\lambda_2)^{-N/2}}{d_N}\frac{\lambda_1^i(\lambda_1 - 1)}{h_1} = \frac{\lambda_1^{i-N/2}\lambda_2^{-N/2}}{d_N}\frac{\frac{bh_1}{\epsilon}}{h_1} = \frac{b}{\epsilon}\frac{\lambda_1^{i-N/2}\lambda_2^{-N/2}}{d_N} \end{aligned}$$

and

$$D^-\psi_i^N = \frac{\psi_i^N - \psi_{i-1}^N}{h_1} = \frac{\overline{\lambda}}{d_N}\frac{(-\lambda_2^{i-N} + \lambda_2^{i-N-1})}{h_2} = \frac{\overline{\lambda}}{d_N}\frac{\lambda_2^{i-N-1}(-\lambda_2 + 1)}{h_2} = -\frac{b}{\epsilon}\frac{\overline{\lambda}\lambda_2^{i-N-1}}{d_N}$$

it follows that

$$D^-\varphi_i^N > 0 \qquad D^-\psi_i^N < 0$$

since $d_N > 0$. Therefore, since $U_N > U_0$, for all i, $1 \le i \le N$

$$D^-U_i > 0$$

which is the second part of lemma. The first is an immediate consequence of the second.■

The next lemma shows that the solution is small outside a neighborhood of the boundary layer, if the boundary condition at the inflow boundary point is chosen appropriately.

Lemma 5.4 *Let $U_0 = e^{-b/\epsilon}U_N$. Then, for all i, $0 \le i \le N/2$,*

$$0 < U_i \le CN^{-1}U_N$$

fore some constant C independent of ϵ.

Proof: Since the hypothesis of the previous lemma are fulfilled for all i, $1 \le i \le N-1$, it follows that $e^{-b/\epsilon}U_N \le U_i \le U_N$ and that U_i is monotone increasing. To complete the proof it suffices therefore to show that for some constant C, independent of ϵ,

$$U_{N/2} \le CN^{-1}U_N. \tag{5.10}$$

From the explicit expression (5.9), it follows that for $0 \le i \le N/2$

$$U_{N/2} = U_N[e^{-b/\epsilon} + (1 - e^{-b/\epsilon})\varphi_{N/2}^N].$$

Since $\tau = \frac{\epsilon}{b}\ln N \leq \frac{1}{2}$, it is clear that

$$e^{-b/\epsilon} \leq e^{-\alpha/\epsilon} = e^{-\frac{1}{\tau}\ln N} = N^{-1/\tau} \leq N^{-2}$$

and so

$$| U_{N/2} | \leq | U_N | \, [N^{-2} + | \varphi^N_{N/2} | \,].$$

Thus, to establish (5.10), it suffices to prove that

$$| \varphi^N_{N/2} | \leq 8N^{-1}. \tag{5.11}$$

Using (5.6) in the explicit expression (5.9) leads to

$$| \varphi^N_{N/2} | = \frac{\lambda_1 \lambda_2^{-N/2}(1 - \lambda_1^{-N/2})}{d_N} \leq \frac{\lambda_1}{d_N}\lambda_2^{-N/2}. \tag{5.12}$$

But

$$\lambda_2 = 1 + \frac{\alpha h_2}{\epsilon} = 1 + \frac{2\alpha\tau}{\epsilon N} = 1 + \frac{2\ln N}{N}$$

and so, by Lemma 5.1 , it follows that

$$\lambda_2^{-N/2} \leq 2N^{-1}. \tag{5.13}$$

Then, from the explicit expression in (5.9)

$$\begin{aligned} d_N &= \lambda_1(\lambda_1\lambda_2)^{-N/2}(\lambda_1^{N/2} - 1) + (1 - \lambda_2^{-N/2})\overline{\lambda} \\ &\geq \overline{\lambda}(1 - \lambda_2^{-N/2}) \\ &\geq \overline{\lambda}(1 - 2N^{-1}) \\ &\geq \frac{\overline{\lambda}}{2}. \end{aligned}$$

Combining this with (5.6) gives

$$\frac{1}{d_N} < \frac{\lambda_1}{d_N} \leq 2\frac{\overline{\lambda}}{d_N} \leq 4. \tag{5.14}$$

Using (5.13) and (5.14) in (5.12) then leads to (5.11).■

The next lemma shows that the solution Z_i of a discrete problem whose upwind operator multiplied by a variable coefficient are less than U_i which produced by a discrete problem whose upwind operator multiplied by a constant coefficient b for all i, $0 \leq i \leq N$ provided that some conditions are satisfied.

Lemma 5.5 *Let U_i be the solution of (5.3) with $U_0 = e^{-b/\epsilon}U_N$, and let*

$$L^N = -\epsilon D^2 + a_i D^-$$

and Z_i be the solution of the problem

$$\left\{\begin{array}{c} Find\ Z \in V(\overline{\Omega}_\tau^N)\ such\ that\ Z_0 = e^{-a_0/\epsilon}Z_N\ \ and\ \ Z_N = U_N \\ and\ for\ all\ x_i \in \Omega_\tau^N \\ L^N U_i = -\epsilon(\frac{U_{i+1}-U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i})\frac{1}{\overline{h}_i} + a_i\frac{U_i-U_{i-1}}{h_i} = 0, \quad 1 \le i \le N-1 \end{array}\right\} \quad (5.15)$$

where it is assumed that for all i, $0 \le i \le N$, $a_i \ge b$. Then, for all i, $0 \le i \le N$,

$$Z_i \le U_i$$

Proof: Let $\Phi_i = U_i - Z_i$. Then, using the assumption, $a_i \ge b$, for all i, $0 \le i \le N$ and the condition $Z_N = U_N$ leads

$$\Phi_0 = (e^{-b/\epsilon} - e^{-a_0/\epsilon})U_N \ge 0$$
$$\Phi_N = 0.$$

Using Lemma 5.3, it follows that

$$\begin{aligned}(-\epsilon D^2 + a_i D^-)\Phi_i &= (-\epsilon D^2 + a_i D^-)(U_i - Z_i) \\ &= (-\epsilon D^2 + a_i D^-)U_i - (-\epsilon D^2 + a_i D^-)Z_i \\ &= -\epsilon D^2 U_i + a_i D^- U_i \\ &= -bD^- U_i + a_i D^- U_i \\ &= (a_i - b)D^- U_i \\ &> 0.\end{aligned}$$

By the discrete maximum principle for the finite difference operator

$$L^N = -\epsilon D^2 + a_i D^-$$

in the section 5.2, it follows that

$$\Phi_i \ge 0$$

as required.■

5.2. Convergence of Fitted Mesh Methods with Regular Data

In this section, the $\epsilon-$ uniform convergence of the numerical solutions obtained by a fitted mesh method for linear convection diffusion problem in one dimension with smooth data $f(x)$ is established. The problem considered is the following second order non self$-$adjoint problem with a variable coefficient.

$$\left\{\begin{array}{ll} \quad Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = u_0 & u(1) = u_1 \\ and\ for\ all\ x \in \Omega, & Lu = -\epsilon u^{''}(x) + a(x)u^{'}(x) = f(x) \end{array}\right\} \tag{5.16}$$

where u_0, u_1 are given constants, the functions a, $f \in C^3(\overline{\Omega})$ and $0 < \epsilon \leq 1$. It is assumed furthermore that the coefficient function satisfies the condition

$$a(x) > \alpha > 0 \qquad for\ \ all\ \ x \in \overline{\Omega}. \tag{5.17}$$

If the two boundary values u_0, and u_1 depend on ϵ, then it is assumed that $\mid u_0 \mid, \mid u_1 \mid$ are bounded above independently of ϵ.

The differential operator L defined in the problem (5.16) satisfies the following maximum principle on Ω, for all $\psi \in C^2(\overline{\Omega})$.

Maximum Principle: Assume that $\psi(0) \geq 0$ and $\psi(1) \geq 0$. Then, $L\psi(x) \geq 0$ for all $x \in \Omega$ implies that $\psi(x) \geq 0$ for all $x \in \overline{\Omega}$.

Proof: (Protter and Weienberger 1984) .■

The reduced problem corresponding to the problem (5.16) is the following first order problem

$$\left\{\begin{array}{ll} \ Find\ v_0 \in C^1(\overline{\Omega})\ such\ that\ v(0) = u_0 & \\ and\ for\ all\ x \in \Omega, & a(x)v_0^{'}(x) = f(x) \end{array}\right\}. \tag{5.18}$$

The unique solution of the problem (5.18) is

$$v_0(x) = u_0 + \int_0^x \frac{f(t)}{a(t)}dt$$

and it is clear, from the assumptions on a and f, that for $0 \leq k \leq 3$

$$\mid v_0^k(x) \mid \leq C \qquad for\ \ all\ x \in \overline{\Omega}.$$

The following lemma contains bounds on the function u and its derivatives up to $k-th$ order, $0 \leq k \leq 3$.

Lemma 5.6 *Let u be the solution of the problem* (5.16)*. Then, for* $\; 0 \leq k \leq 3$*,*

$$| \, u^{(k)}(x) \, | \leq C(1 + \epsilon^{-k} e^{-\alpha(1-x)/\epsilon}) \qquad for \;\; all \; x \in \overline{\Omega}.$$

Proof: The proof is by induction. A bound on the solution u of the problem (5.16) is obtained easily from the maximum principle as follows:

Consider the functions

$$\psi^{\pm}(x) = C(1+x) \pm u(x)$$

where C is a constant chosen sufficiently large that the following inequalities are fulfilled

$$\psi^{\pm}(0) \geq 0, \qquad \psi^{\pm}(1) \geq 0$$

and

$$\begin{aligned} L\psi^{\pm}(x) = -\epsilon(\pm u^{''}(x)) + Ca(x) \pm a(x)u^{'}(x) &= L(\pm u(x)) + Ca(x) \\ &= Ca(x) \pm f(x) \\ &\geq C\alpha \pm f(x) \\ &\geq 0 \end{aligned}$$

since $a(x) > \alpha$. Then the maximum principle for L gives $\psi^{\pm}(x) \geq 0$ and so

$$| \, u(x) \, | \leq C \qquad for \;\; all \; x \in \overline{\Omega}.$$

To obtain the required estimates of the derivatives of u is more difficult. The first step is to find the differential equation satisfied by these derivatives by differentiating k times the original equation $Lu = f$. This gives

$$Lu^{(k)} = f_k$$

where $f_0 = f$ and for $\; 1 \leq k \leq 3$

$$f_k = f^{(k)} - \sum_{s=0}^{k-1} \begin{pmatrix} k \\ s \end{pmatrix} a^{(k-s)} u^{(s+1)}.$$

Thus, the inhomogeneous term f_k of the equation satisfied by $u^{(k)}$ depends on the kth and lower order derivatives of u and the coefficient a, and on the kth order derivative of f. This observation suggests that the following induction step:

Assume that for all j, $\ 0 \leq j \leq k$, the following estimates hold

$$| \, u^{(j)}(x) \, | \leq C(1 + \epsilon^{-j} e^{-\alpha(1-x)/\epsilon}) \qquad for \quad all \ x \in \overline{\Omega}.$$

From the above assumptions it is clear that

$$Lu^{(k)} = f_k$$

where

$$| \, u^{(k)}(x) \, | \leq C(1 + \epsilon^{-k} e^{-\alpha(1-x)/\epsilon})$$

and

$$| \, f_k(x) \, | \leq C(1 + \epsilon^{-k} e^{-\alpha(1-x)/\epsilon}).$$

In particular then

$$| \, u^{(k)}(0) \, | \leq C(1 + \epsilon^{-k} e^{-\alpha/\epsilon}) \leq C(1 + \epsilon^{-(k-1)})$$
$$| \, u^{(k)}(1) \, | \leq C(1 + \epsilon^{-k})$$

since $\epsilon^{-1} e^{-\alpha/\epsilon} \leq C$. Define the functions

$$\theta_k(x) = \frac{1}{\epsilon} \int_x^1 f_k(t) e^{-(A(x)-A(t))} dt$$
$$u_p^{(k)}(x) = -\int_x^1 \theta_k(t) dt$$

where $A(x) = \int_x^1 a(s)ds$. Then, it follows that since $A^{'}(x) = -a(x)$

$$\begin{aligned}
Lu_p^{(k)}(x) &= -\epsilon u_p^{(k+2)}(x) + a(x) u_p^{(k+1)}(x) \\
&= -\epsilon \theta_k^{'}(x) + a(x)\theta_k(x) \\
&= -\epsilon \frac{d}{dx} \mathbf{[} \frac{1}{\epsilon} e^{-A(x)} \int_x^1 f_k(t) e^{A(t)} dt \mathbf{]} + a(x)\theta_k(x) \\
&= -\mathbf{[} - A^{'}(x) e^{-A(x)} \int_x^1 f_k(t) e^{A(t)} dt + e^{-A(x)} \frac{d}{dx} (\int_x^1 f_k(t) e^{A(t)} dt) \mathbf{]} + a(x)\theta_k(x) \\
&= -a(x) \int_x^1 f_k(t) e^{-(A(x)-A(t))} dt - e^{-A(x)}(-f_k(x) e^{A(x)}) + a(x)\theta_k(x) \\
&= -a(x)\theta_k(x) + f_k(x) + a(x)\theta_k(x) \\
&= f_k(x)
\end{aligned}$$

and so $u_p^{(k)}(x)$ is a particular solution of the equation

$$Lu^{(k)} = f_k$$

Its general solution can therefore be written in the form

$$u^{(k)} = u_p^{(k)} + u_h^{(k)}$$

where the homogeneous solution $u_h^{(k)}$ satisfies

$$Lu_h^{(k)} = 0, \qquad u_h^{(k)}(0) = u^{(k)}(0) - u_p^{(k)}(0), \quad u_h^{(k)}(1) = u^{(k)}(1).$$

Introducing the function

$$\varphi(x) = \frac{\int_x^1 e^{-A(t)/\epsilon} dt}{\int_0^1 e^{-A(t)/\epsilon} dt}$$

it is clear that $\varphi(0) = 1$, $\varphi(1) = 0$ and

$$\begin{aligned} L\varphi &= -\epsilon\varphi^{''}(x) + a(x)\varphi^{'}(x) \\ &= \frac{1}{\int_0^1 e^{-A(t)/\epsilon} dt}[-\epsilon\frac{d}{dx}(-e^{-A(x)/\epsilon}) + a(x)(-e^{-A(x)/\epsilon})] \\ &= \frac{1}{\int_0^1 e^{-A(t)/\epsilon} dt}[-\epsilon\frac{1}{\epsilon}A^{'}(x)e^{-A(x)/\epsilon} - a(x)e^{-A(x)/\epsilon}] \\ &= 0. \end{aligned}$$

Then $u_h^{(k)}$ is given by

$$u_h^{(k)}(x) = (u^{(k)}(0) - u_p^{(k)}(0))\varphi(x) + u^{(k)}(1)(1 - \varphi(x)).$$

The above leads to the following expression for $u^{(k+1)}$

$$u^{(k+1)} = u_p^{(k+1)} + u_h^{(k+1)} = \theta_k + (u^{(k)}(0) - u_p^{(k)}(0) - u^{(k)}(1))\varphi^{'}.$$

Since

$$\varphi^{'}(x) = \frac{-e^{-A(x)/\epsilon}}{\int_0^1 e^{-A(t)/\epsilon} dt}$$

the upper and lower bounds of $a(x)$ lead to the estimate

$$\mid \varphi^{'}(x) \mid \leq C\epsilon^{-1}e^{-\alpha(1-x)/\epsilon}.$$

Furthermore the lower bound on the coefficient a and the estimate for f_k lead to

$$\begin{aligned} \mid \theta_k(x) \mid &= \frac{1}{\epsilon}\int_x^1 \mid f_k(t) \mid\mid e^{-(A(x)-A(t))} \mid dt \\ &\leq C\epsilon^{-1}\int_x^1 (1 + \epsilon^{-k}e^{-\alpha(1-t)/\epsilon})e^{-(A(x)-A(t))} dt \\ &\leq C\epsilon^{-1}\int_x^1 (1 + \epsilon^{-k}e^{-\alpha(1-t)/\epsilon})e^{-\alpha(t-x)/\epsilon}. \end{aligned}$$

Evaluating the integral exactly and estimating the terms in the resulting expression then gives

$$| \theta_k(x) | \leq C(1 + \epsilon^{-(k+1)} e^{-\alpha(1-x)/\epsilon}).$$

Since

$$\begin{aligned}
| u_p^{(k)}(0) | = | -\int_0^1 \theta_k(t)dt | \leq \int_0^1 | \theta_k(t) | \, dt &\leq \int_0^1 C_1(1 + \epsilon^{-(k+1)} e^{-\alpha(1-t)/\epsilon})dt \\
&= \int_0^1 C_1 dt + C_1 \epsilon^{-(k+1)} e^{-\alpha/\epsilon} \int_0^1 e^{\alpha t/\epsilon} \\
&= C_1 + C_1 \epsilon^{-(k+1)} e^{-\alpha/\epsilon} \frac{\epsilon}{\alpha}(e^{\alpha/\epsilon} - 1) \\
&= C_1 + C_1 \epsilon^{-k} - C_1 \epsilon^{-k} e^{-\alpha/\epsilon} \\
&\leq C_1 + C_1 \epsilon^{-k} \\
&\leq C \epsilon^{-k}
\end{aligned}$$

the above estimates give

$$| u^{(k+1)} | \leq | \theta_k | + (| u^{(k)}(0) | + | u_p^{(k)}(0) | + | u^{(k)}(1) |) | \varphi^{'} | .$$

Thus,

$$| u^{(k+1)} | \leq C(1 + \epsilon^{-(k+1)} e^{-\alpha(1-x)/\epsilon})$$

as required.■

These bounds for the derivatives of u were first obtained in Kellog $[3]$. However, the stronger results of Shishkin $[8]$ are required to obtain the $\epsilon-$ uniform convergence result in this section. To find these solution u has to be decomposed into smooth and singular components as follows:

$$u = v_0 + \epsilon y_1 + w_0$$

where v_0 is the solution of reduced problem (5.18), y_1 satisfies

$$Ly_1 = v_0^{''}, \qquad y_1(0) = -\epsilon^{-1} w_0(0), \qquad y_1(1) = 0$$

and consequently w_0 is the solution of the homogeneous problem

$$Lw_0 = 0, \quad w_0(0) = w_0(1) e^{-\alpha/\epsilon}, \quad w_0(1) = u_1 - v_0(1).$$

Since $\epsilon^{-1}e^{-\alpha/\epsilon} \leq C$, and $\mid v_0^k(x) \mid \leq C$ it is clear that $\mid w_0(0) \mid$, $\mid w_0(1) \mid$, $\mid y_1(0) \mid$, $\mid v_0^{''} \mid$ are all bounded by a constant independent of ϵ. Therefore y_1 is the solution of a problem similar to (5.16). This implies that for $0 \leq k \leq 3$

$$\mid y_1^{(k)}(x) \mid \leq C(1 + \epsilon^{-k}e^{-\alpha(1-x)/\epsilon}) \qquad for \;\; all \; x \in \overline{\Omega}.$$

Bounds on the singular component of the solution, w_0, and on its derivatives are now obtained as follows. Define the two functions

$$\psi^{\pm}(x) = \mid w_0(1) \mid e^{-\alpha(1-x)/\epsilon} \pm w_0(x).$$

Then, since the inequalities

$$\psi^{\pm}(0) = \mid w_0(1) \mid e^{-\alpha/\epsilon} \pm w_0(0) = \mid u_1 - v_0(1) \mid e^{-\alpha/\epsilon} \pm (u_1 - v_0(1))e^{-\alpha/\epsilon} \geq 0$$
$$\psi^{\pm}(1) = \mid w_0(1) \mid \pm w_0(1) \geq 0$$

and

$$L\psi^{\pm} = \mid w_0(1) \mid [-\frac{\alpha^2 \epsilon}{e}^{-\alpha(1-x)/\epsilon} + a(x)\frac{\alpha}{\epsilon}e^{-\alpha(1-x)/\epsilon}] \geq 0$$

are fulfilled for all $x \in \Omega$, the maximum principle gives $\psi^{\pm}(x) \geq 0$ and so

$$\mid w_0(x) \mid \leq Ce^{-\alpha(1-x)/\epsilon} \qquad for \;\; all \; x \in \overline{\Omega}$$

w_0 can also be written in the form

$$w_0 = w_0(0)\varphi + w_0(1)(1 - \varphi)$$

where φ was defined above. Therefore

$$w_0^{'} = (w_0(0) - w_0(1))\varphi^{'}$$

and so

$$\mid w_0^{'}(x) \mid \leq C \mid \varphi^{'}(x) \mid \leq C\epsilon^{-1}e^{-\alpha(1-x)/\epsilon}.$$

Since $Lw_0 = 0$, the second and third derivatives of w_0 can be estimated immediately from the estimates of w_0 and $w_0^{'}$. Thus, for $0 \leq k \leq 3$,

$$\mid w_0^{(k)}(x) \mid \leq C\epsilon^{-k}e^{-\alpha(1-x)/\epsilon}.$$

Since

$$u^{(k)} = v_0^{(k)} + \epsilon y_1^{(k)} + w_0^{(k)}$$

the above estimates yield, for $0 \leq k \leq 3$, and for all $x \in \overline{\Omega}$,

$$| \, (v_0^{(k)} + \epsilon y_1^{(k)})(x) \, | \leq C(1 + \epsilon^{-(k-1)} e^{-\alpha(1-x)/\epsilon})$$
$$| \, w_0^{(k)}(x) \, | \leq C\epsilon^{-k} e^{-\alpha(1-x)/\epsilon}.$$

In particular, this shows that the smooth component $v_0 + \epsilon y_1$ and its first derivative are bounded for all values of ϵ. However, y_1 can now be decomposed in the same manner as was u, leading immediately to $y_1 = v_1 + \epsilon v_2 + w_1$ where, for $0 \leq k \leq 3$, and for all $x \in \overline{\Omega}$,

$$| \, v_1^{(k)}(x) \, | \leq C$$
$$| \, v_2^{(k)}(x) \, | \leq C(1 + \epsilon^{-k} e^{-\alpha(1-x)/\epsilon})$$
$$| \, w_1^{(k)}(x) \, | \leq C\epsilon^{-k} e^{-\alpha(1-x)/\epsilon}.$$

Combining these two decompositions gives

$$u = v + w$$

where

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2$$
$$w = w_0 + \epsilon w_1$$

and the above results are summarized in the following theorem.

Theorem 5.7 *The solution u of the problem (5.16) has the decomposition*

$$u = v + w$$

where, for all k, $0 \leq k \leq 3$, and all $x \in \overline{\Omega}$, the smooth component v satisfies

$$| \, v^{(k)}(x) \, | \leq C(1 + \epsilon^{-(k-2)} e^{-\alpha(1-x)/\epsilon})$$

and the singular component w satisfies

$$| \, w^{(k)}(x) \, | \leq C\epsilon^{-k} e^{-\alpha(1-x)/\epsilon}$$

for some constant C independent of ϵ.

Proof: Since $v = v_0 + \epsilon v_1 + \epsilon^2 v_2$ and $w = w_0 + \epsilon w_1$,

$$\begin{aligned}
\mid v^{(k)}(x) \mid \leq \mid v_0^{(k)}(x) \mid + \epsilon \mid v_1^{(k)}(x) \mid + \epsilon^2 \mid v_2^{(k)}(x) \mid &\leq C_1 + C_1\epsilon + C_1\epsilon^2(1 + \epsilon^{-k}e^{-\alpha(1-x)/\epsilon}) \\
&\leq C(1 + \epsilon^{-(k-2)}e^{-\alpha(1-x)/\epsilon})
\end{aligned}$$

and

$$\begin{aligned}
\mid w^{(k)}(x) \mid \leq \mid w_0^{(k)}(x) \mid + \epsilon \mid \leq \mid w_1^{(k)}(x) \mid &\leq C_1\epsilon^{-k}e^{-\alpha(1-x)/\epsilon} + C_1\epsilon\epsilon^{-k}e^{-\alpha(1-x)/\epsilon} \\
&\leq C\epsilon^{-k}e^{-\alpha(1-x)/\epsilon}
\end{aligned}$$

as required.■

This theorem shows that the smooth function v and both its first and second derivatives are bounded for all values of ϵ, while the singular component w satisfies the same estimate as the singular component in the first decomposition. Notice that v and w satisfy the following equations

$$\begin{aligned}
Lv = f, \qquad & v(0) = u_0 - w(0), \quad v(1) = u_1 - w(1) \\
Lw = 0, \qquad & w(0) = w(1)e^{-\alpha/\epsilon}
\end{aligned}$$

where $w(1)$ is chosen so that the first and the second derivatives of v are bounded uniformly in ϵ.

The numerical method used to solve (5.16) is the standard upwind finite difference operator on the piecewise uniform fitted mesh $\overline{\Omega}_\tau^N = \{x_i\}_0^N$ condensing at the boundary point $x_N = 1$. The transition parameter τ is chosen to satisfy

$$\tau = \min\{\frac{1}{2}, \frac{\epsilon}{\alpha}\ln N\} \tag{5.19}$$

and it is assumed that $N \geq 4$, which guarantees that there is at least one point in the boundary layer. The resulting fitted mesh finite difference method is

$$\left\{ \begin{array}{c}
Find\ U \in V(\overline{\Omega}^N)\ such\ that\ U_0 = u_0 \quad U_N = u_1 \\
and\ for\ all\ x_i \in \Omega_\tau^N \\
L^N U_i = -\epsilon D^2 U_i + a_i D^- U_i = f_i, \quad 1 \leq i \leq N-1
\end{array} \right\}. \tag{5.20}$$

The finite difference operator in (5.20) is defined by

$$L^N = -\epsilon D^2 + a_i D^-$$

and it satisfies the following discrete maximum principle on Ω_τ^N.

Discrete Maximum Principle: Assume that the mesh function Ψ_i satisfies $\Psi_0 \geq 0$ and $\Psi_N \geq 0$. Then $L^N\Psi_i \geq 0$ for $1 \leq i \leq N-1$ implies that $\Psi_i \geq 0$ for all $0 \leq i \leq N$.

Proof: Let k be such that $\Psi_k = \min \Psi_i$ and suppose that $\Psi_k < 0$. It is clear that $k \neq 0$ or $k \neq 1$, $\Psi_{k+1} - \Psi_k \geq 0$ and $\Psi_k - \Psi_{k-1} \leq 0$. Therefore

$$L^N\Psi_k = -\epsilon(\frac{\Psi_{k+1} - \Psi_k}{h_{k+1}} - \frac{\Psi_k - \Psi_{k-1}}{h_k})\frac{1}{\overline{h}_k} + a_k\frac{\Psi_k - \Psi_{k-1}}{h_k} \leq 0$$

with a strict inequality if $\Psi_k - \Psi_{k-1} < 0$. But this is false and so on, leads to

$$\Psi_0 = \Psi_1 = ... = \Psi_{k-1} = \Psi_k < 0$$

which is false. It follows that $\Psi_k \geq 0$ and thus that $\Psi_i \geq 0$ for all i, $0 \leq i \leq N$.■

An immediate consequence of this discrete maximum principle is the following $\epsilon-$uniform stability result for the operator L^N.

Lemma 5.8 *If Z_i is any mesh function such that $Z_0 = Z_N = 0$, then*

$$\mid Z_i \mid \leq \frac{1}{\alpha} \max_{1\leq j\leq N-1} \mid L^N Z_j \mid \qquad for \;\; 0 \leq i \leq N.$$

Proof: Introduce

$$M = \frac{1}{\alpha} \max_{1\leq j\leq N-1} \mid L^N Z_j \mid$$

and the two mesh functions

$$\Psi_i^{\pm} = Mx_i \pm Z_i.$$

Clearly $\Psi_0^{\pm} = 0$, $\Psi_N^{\pm} \geq 0$ and for $1 \leq i \leq N-1$

$$L^N\Psi_i^{\pm} = Ma_i \pm L^N Z_i \geq 0$$

because $a_i > \alpha$. The discrete maximum principle then implies that $\Psi_i^{\pm} \geq 0$ for $0 \leq i \leq N$ and the proof is complete.■

The main result of this chapter is contained in the following theorem.

Theorem 5.9 *The fitted mesh finite difference method* (5.20) *with the standard upwind finite difference operator and the piecewise uniform fitted mesh Ω_τ^N, condensing at the* *boundary point $x = 1$, is $\epsilon-$uniform for the problem (5.16) provided that τ is chosen to satisfy the condition (5.19) above. Moreover, the solution u of (5.16) and the solution U of (5.20) satisfy the following $\epsilon-$uniform error estimate*

$$\sup_{0<\epsilon\leq 1} \parallel U - u \parallel_{\overline{\Omega}_\tau^N} \leq CN^{-1}(\ln N)^2$$

where C is a constant independent of ϵ.

Proof: The solution U of the discrete problem is decomposed in an analogous manner to the above second decomposition of (5.16). Thus

$$U = V + W$$

where V is the solution of the inhomogeneous problem

$$L^N V = f, \qquad V(0) = v(0) \quad V(1) = v(1)$$

and W is the solution of the homogeneous problem

$$L^N W = 0, \qquad W(0) = w(0) \quad W(1) = w(1)$$

The error can then be written in the form

$$U - u = (V - v) + (W - w)$$

and so the errors in the smooth and singular components of the solution can be estimated separately.

The estimate of the smooth component is obtained by means of the following classical argument. From the differential and difference equations

$$L^N(V - v) = f - L^N v = (L - L^N)v = -\epsilon(\frac{d^2}{dx^2} - D^2)v + a(\frac{d}{dx} - D^-)v$$

Using the two estimates in Lemma 5.2 gives

$$\begin{aligned} \mid L^N(V - v)(x_i) \mid &\leq\mid \epsilon(\frac{d^2}{dx^2} - D^2)v \mid + a \mid (\frac{d}{dx} - D^-)v \mid \\ &\leq C_1\epsilon(x_{i+1} - x_{i-1}) \mid v \mid_3 + C_2(x_i - x_{i-1}) \mid v \mid_2 \\ &\leq C_3(x_{i+1} - x_{i-1})(\epsilon \mid v \mid_3 + \mid v \mid_2) \end{aligned}$$

Noting that $x_{i+1} - x_{i-1} \leq 2N^{-1}$ is always true, the estimates of $v^{''}$ and $v^{'''}$ obtained above then yield

$$\begin{aligned}
| L^N(V-v)(x_i) | &\leq C_4 N^{-1}(\epsilon \mid v \mid_3 + \mid v \mid_2) \\
&\leq C_4 N^{-1}[C_5\epsilon(1+\epsilon^{-1}e^{-\alpha(1-x)/\epsilon}) + C_5(1+e^{-\alpha(1-x)/\epsilon})] \\
&= C_4 N^{-1}[C_5\epsilon + C_5 e^{-\alpha(1-x)/\epsilon} + C_5 + C_5 e^{-\alpha(1-x)/\epsilon}] \\
&\leq CN^{-1}
\end{aligned}$$

An application of Lemma 5.8 to the mesh function $V - v$ yields the estimate

$$| (V-v)(x_i) | \leq CN^{-1} \tag{5.21}$$

To estimate the singular component of the local truncation error $L^N(W - w)$, the argument depends on whether $\tau = 1/2$ or $\tau = (\epsilon \ln N)/\alpha$.

In the first case the mesh is uniform and $(\epsilon \ln N)/\alpha \geq 1/2$. The classical argument, used above to estimate $V - v$, leads in this case to

$$| L^N(W-w)(x_i) | \leq C(x_{i+1} - x_{i-1})(\epsilon \mid w \mid_3 + \mid w \mid_2)$$

Since $x_{i+1} - x_{i-1} = 2N^{-1}$, the estimates for $w^{''}$ and $w^{'''}$ lead to

$$| L^N(W-w)(x_i) | \leq C\epsilon^{-2}N^{-1}$$

But, in this case $\epsilon^{-1} \leq (2 \ln N)/\alpha$ and so

$$| L^N(W-w)(x_i) | \leq CN^{-1}(\ln N)^2$$

An application of Lemma 5.8 to the mesh function $W - w$ then gives

$$| (W-w)(x_i) | \leq CN^{-1}(\ln N)^2$$

In the second case the mesh is piecewise uniform with the mesh spacing $2(1-\tau)/N$ in the subinterval $[0, 1-\tau]$ and $2\tau/N$ in the subinterval $[1-\tau, 1]$. A different argument is used to bound $\mid W - w \mid$ in each of the subintervals.

In the subinterval with no boundary layer $[0, 1-\tau]$ both W and w are small, and because $\mid W - w \mid \leq \mid W \mid + \mid w \mid$, it suffices to bound w and W separately. Note first that

$$\frac{w_0^{'}(x)}{w_0(1)} = \frac{(w_0(0) - w_0(1))}{w_0(1)}\varphi^{'} = -(1 - e^{-\alpha/\epsilon})\varphi^{'}(x) > 0 \quad \textit{and} \quad \frac{w_0(0)}{w_0(1)} = e^{-\alpha/\epsilon}$$

Thus, $\frac{w_0(x)}{w_0(1)}$ is positive and increasing in the interval Ω. It follows that for all x in $[0, 1-\tau]$

$$0 \leq \frac{w_0(x)}{w_0(1)} \leq \frac{w_0(1-\tau)}{w_0(1)}$$

and so

$$| w_0(x) | \leq | w_0(1-\tau) |$$

The same is true of $w_1(x)$ and since

$$w = w_0 + \epsilon w_1$$

it follows that for all $x \in [0, 1-\tau]$

$$| w(x) | \leq | w(1-\tau) |$$

Using the estimate given in the Theorem 5.7 for $| w |$ and the relation $\tau = (\epsilon \ln N)/\alpha$ it follows that for $x \in [0, 1-\tau]$

$$| w(x) | \leq Ce^{-\alpha(1-x)/\epsilon} \leq Ce^{-\alpha\tau/\epsilon} = CN^{-1}$$

To obtain a similar bound on W an auxiliary mesh function $\widetilde{W}$ is defined analogously to W except that the coefficient a in the difference operator L^N is replaced by its lower bound α. Then, by Lemma 5.5,

$$| W(x_i) | \leq | \widetilde{W}(x_i) | \quad for \;\; 0 \leq i \leq N$$

Furthermore Lemma 5.4 leads immediately to

$$| W(x_i) | \leq CN^{-1} \quad for \;\; 0 \leq i \leq N/2$$

The above estimates of $W(x_i)$ and $w(x_i)$, for $0 \leq i \leq N/2$, show that in the interval $[0, 1-\tau]$

$$| W(x_i) - w(x_i) | \leq CN^{-1}$$

On the other hand in the subinterval $[1-\tau, 1]$ the classical arguments leads as before to the following estimate of the local truncation error for $N/2+1 \leq i \leq N-1$

$$| L^N(W-w)(x_i) | \leq C_1(x_{i+1} - x_{i-1})(\epsilon \mid w \mid_3 + \mid w \mid_2)$$

Using $x_{i+1} - x_{i-1} = 4\tau/N$ and the estimate given in the Theorem 5.7 for w leads to

$$| L^N(W-w)(x_i) | \leq C\epsilon^{-2}\tau N^{-1}$$

Furthermore

$$| W(1) - w(1) | = 0$$

and

$$| W(x_{N/2}) - w(x_{N/2}) | \leq | W(x_{N/2}) | + | w(x_{N/2}) | \leq CN^{-1}$$

from the result just obtained in the other subinterval. Introducing the barrier function

$$\Phi_i = (x_i - (1-\tau))C_1\epsilon^{-2}\tau N^{-1} + C_2N^{-1}$$

it follows that for a suitable choice of C_1 and C_2 the mesh functions

$$\Psi_i^{\pm} = \Phi_i \pm (W-w)(x_i)$$

satisfy the inequalities

$$\Psi_{N/2}^{\pm} = \Phi_{N/2} \pm (W-w)(x_{N/2}) = C_2N^{-1} \pm (W-w)(x_{N/2}) \geq 0$$
$$\Psi_N^{\pm} = \Phi_N \pm (W-w)(x_N) = C_1\epsilon^{-2}\tau^2N^{-1} + C_2N^{-1} \geq 0$$

and for $N/2 + 1 \leq i \leq N-1$

$$L^N\Psi_i^{\pm} = L^N\Phi_i \pm L^N(W-w)(x_i) = C_1\epsilon^{-2}\tau N^{-1}a_i \pm L^N(W-w)(x_i) \geq 0$$

The discrete maximum principle on the interval $[1-\tau, 1]$ then gives

$$\Psi_i^{\pm} \geq 0, \qquad N/2 \leq i \leq N$$

and it follows that

$$| (W-w)(x_i) | \leq \Phi_i \leq C_1\epsilon^{-2}\tau^2N^{-1} + C_2N^{-1}$$

But since $\tau = (\epsilon \ln N)/\alpha$ this gives

$$| (W-w)(x_i) | \leq CN^{-1}(\ln N)^2$$

Combining the separate estimates in the two subintervals $[0, 1-\tau]$ and $[1-\tau, 1]$ then gives

$$| (W - w) | \leq CN^{-1}(\ln N)^2 \quad for \;\; 0 \leq i \leq N \tag{5.22}$$

Since

$$| U - u | \leq | V - v | + | W - w |$$

the inequalities (5.21) and (5.22) then give

$$| (U - u)(x_i) | \leq CN^{-1}(\ln N)^2$$

as required.■

Example 1: We consider the following test problem

$$\left\{ \begin{array}{ll} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = 0 \;\; u(1) = 1 \\ and\ for\ all\ x \in \Omega, \qquad Lu = -\epsilon u^{''}(x) + u^{'}(x) = 0 \end{array} \right\}$$

and estimate the $\epsilon - uniform$ error by

$$\eta^N = \max_{\epsilon=1,10^{-1},\ldots,10^{-9}} \| U^N - u \|$$

We compute the rate of convergence using the formula

$$r^N = \log_2(\frac{\eta^N}{\eta^{2N}})$$

and obtain following results.

N	16	32	64	128	256	512	1024	2048	4096
η^N	1.14e-1	6.5e-2	3.59e-2	1.96e-2	1.0e-3	5.823e-3	3.14e-3	1.69e-3	9.08e-4
r^N	0.81	0.85	0.86	0.87	0.88	0.88	0.88	0.89	0.89

Example 2: We show some numerical results confirm the Theorem 5.9 for the second test problem

$$\left\{ \begin{array}{ll} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = 0 \;\; u(1) = 0 \\ and\ for\ all\ x \in \Omega, \qquad Lu = -\epsilon u^{''}(x) + u^{'}(x) = x \end{array} \right\}$$

The following figures contains the exact and numerical solutions and also give the error at the discrete node x_{N-1} for each values of ϵ when the number of mesh elements N increases.

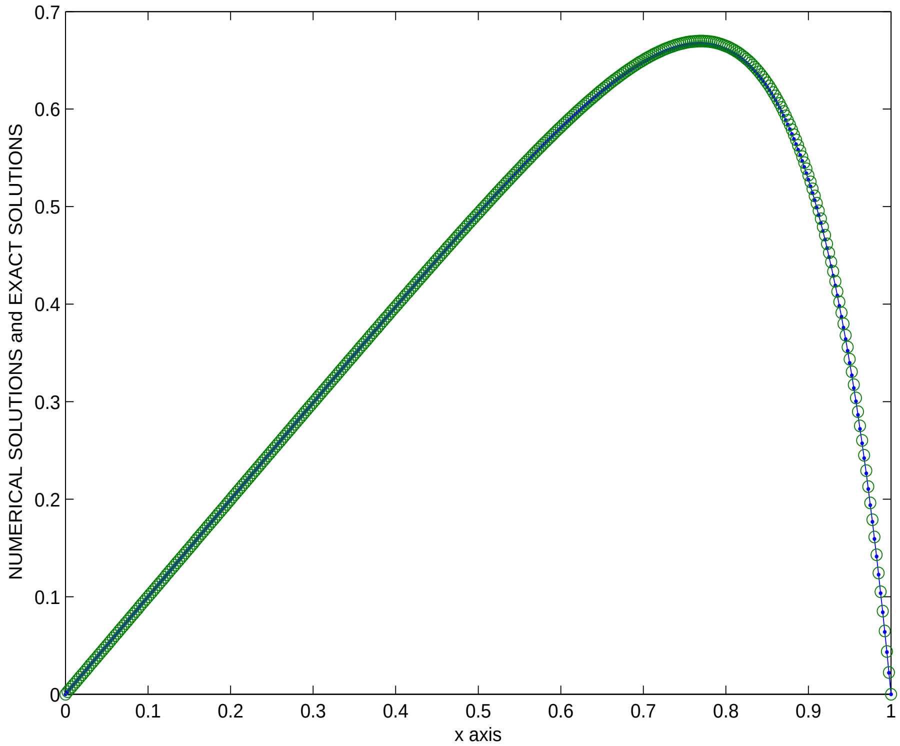

Figure 5.3. The Fitted Mesh Method(o) for $N = 400$, $\epsilon = 0.1$

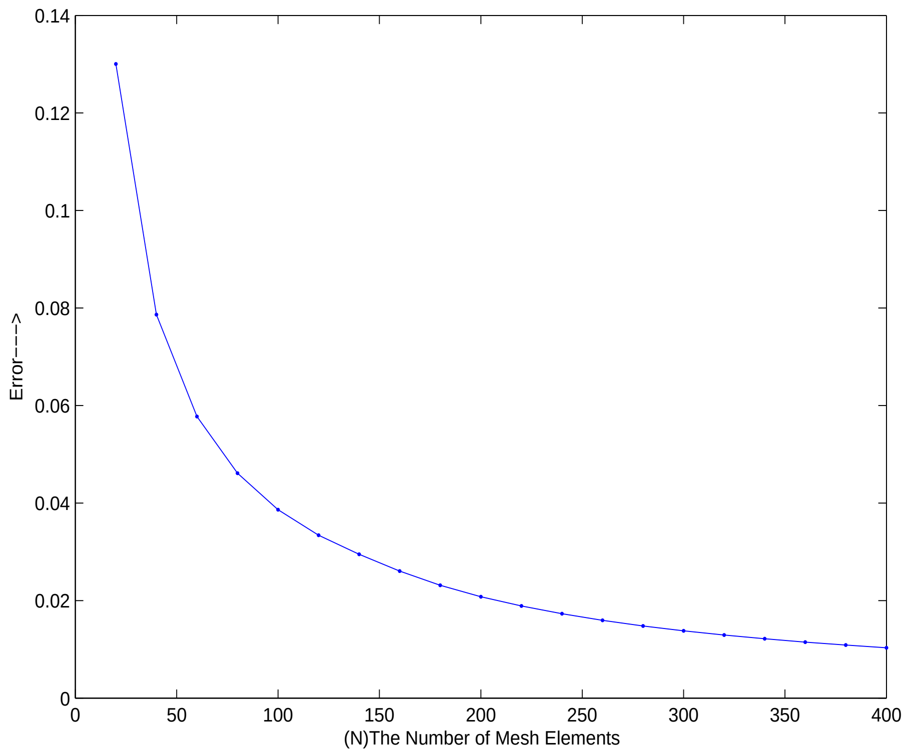

Figure 5.4. Error at the boundary for $\epsilon = 0.1$

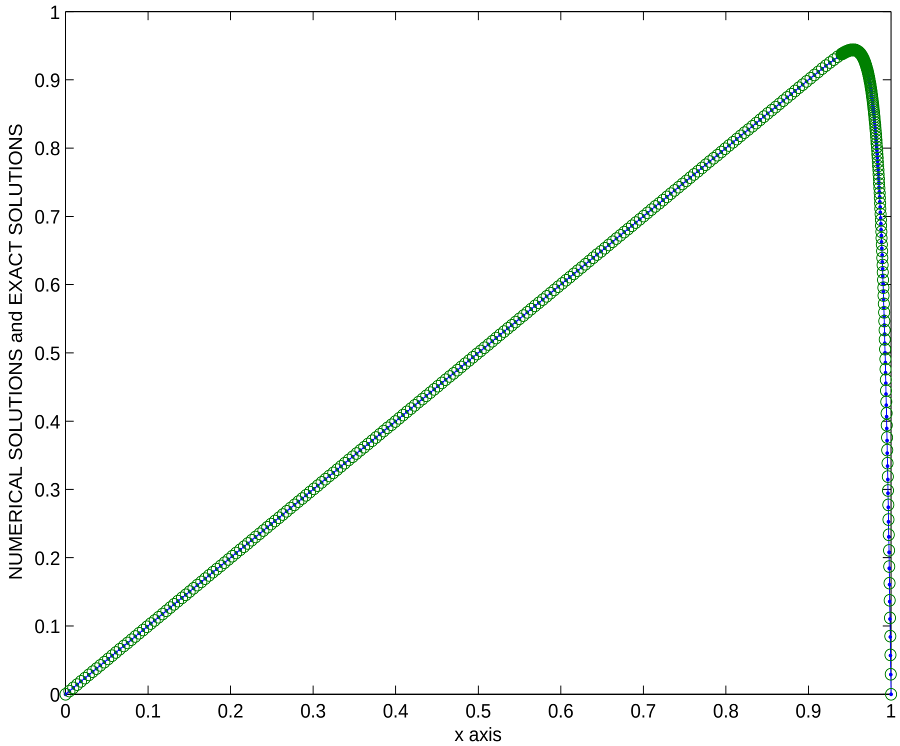

Figure 5.5. The Fitted Mesh Method(o) for $N = 400$, $\epsilon = 0.01$

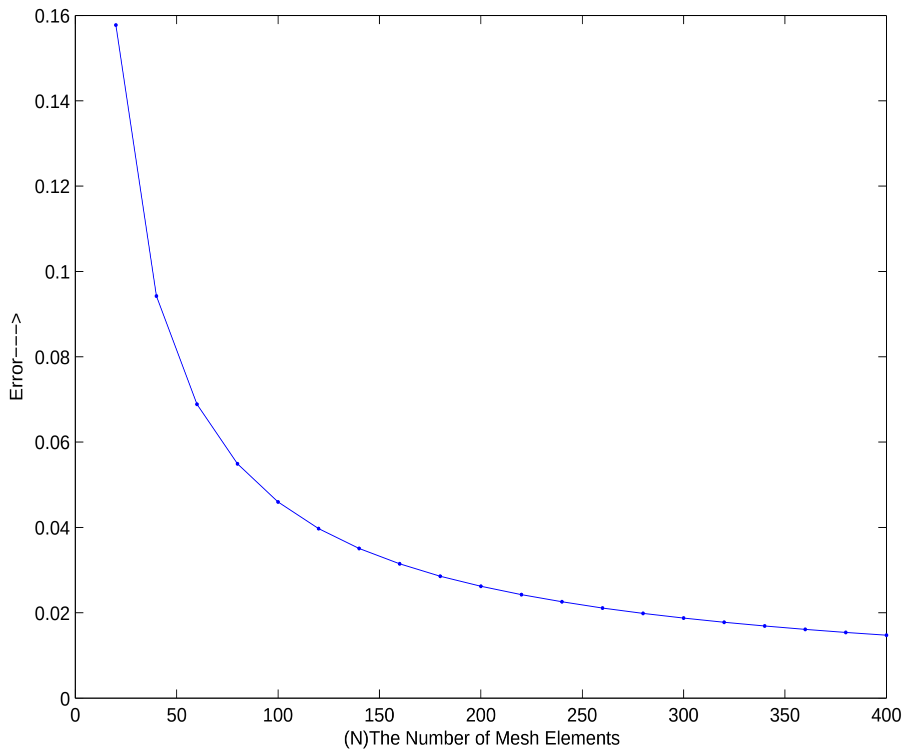

Figure 5.6. Error at the boundary for $\epsilon = 0.01$

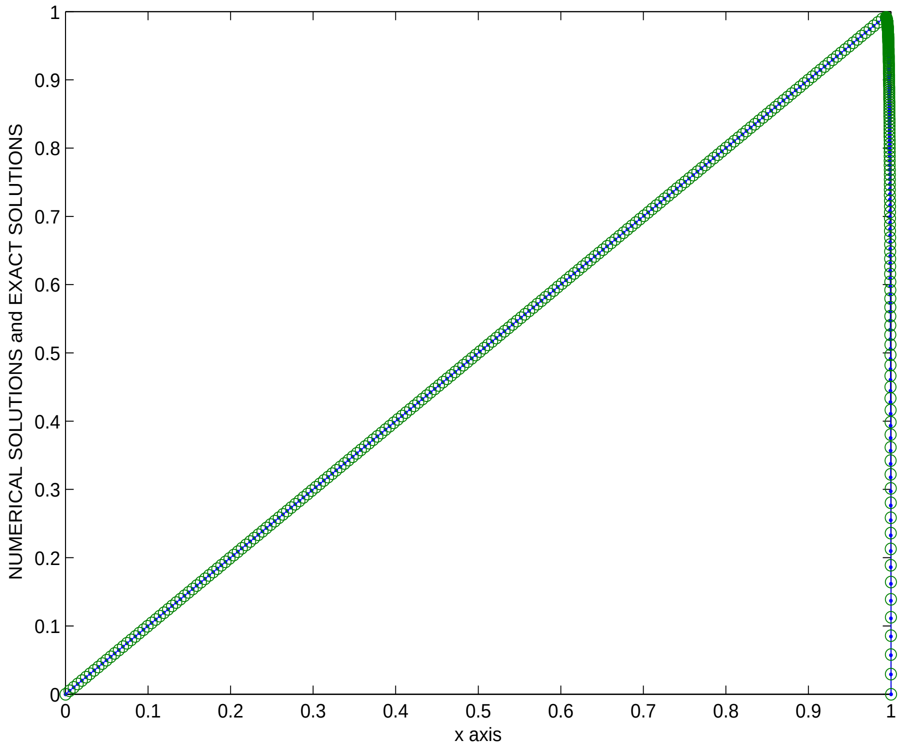

Figure 5.7. The Fitted Mesh Method(o) for $N = 400$, $\epsilon = 0.001$

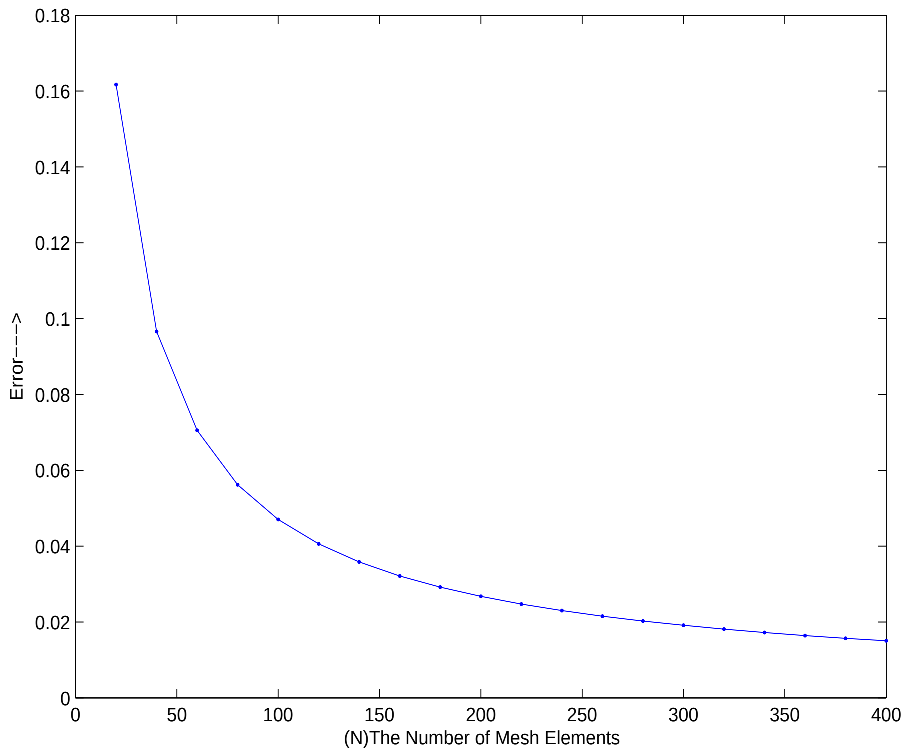

Figure 5.8. Error at the boundary for $\epsilon = 0.001$

5.3. Convergence of Fitted Mesh Methods with Irregular Data

In this section, we consider the following convection$-$diffusion problem with a concentrated source and show that $\epsilon-uniformly$ convergent methods can be constructed for problems with irregular data

$$\left\{ \begin{array}{c} Find\ u \in C^2(\overline{\Omega})\ such\ that\ u(0) = 0 \quad u(1) = 0 \\ and\ for\ all\ x \in \Omega, \quad Lu = -\epsilon u^{''}(x) - bu^{'}(x) = f(x) + \delta_d(x) \end{array} \right\} \tag{5.23}$$

where δ_d is the shifted $dirac-$delta function

$$\delta_d = \delta(x - d)$$

with $d \in (0, 1)$ and $0 < \epsilon \leq 1$. The convection coefficient b may also have discontinuity at $x = d$, but in this section we assume that b is a constant satisfying $b > \alpha > 0$. Alternatively to this problem, we may seek a solution which satisfies the problem

$$\left\{ \begin{array}{c} Find\ u \in C(\overline{\Omega}) \cap C^2((0, d) \cup (d, 1))\ such\ that\ u(0) = 0 \quad u(1) = 0 \\ and\ for\ all\ x \in (0, d) \cup (d, 1), \quad Lu = -\epsilon u^{''}(x) - bu^{'}(x) = f(x) \end{array} \right\} \tag{5.24}$$

with the additional condition

$$-\epsilon[u^{'}](d) - [b](d)u(d) = 1$$

where $[v](d) = v(d+0) - v(d-0)$ denotes the jump of v in $x = d$. Since b is a constant, this condition reduces

$$u^{'}(d^-) - u^{'}(d^+) = \frac{1}{\epsilon}. \tag{5.25}$$

The equivalence of these problems can be seen by integrating the differential equation in (5.23) from $d - \epsilon$ to $d + \epsilon$.

The solution u typically has an exponential boundary layer at the outflow boundary $x = 0$ and an internal layer at $x = d$ caused by the concentrated source or the discontinuity of the convective field. Figure 5.9 depicts a typical solution of the problem.

Next, we give a theorem which contains bounds on the solution u of (5.23) and its derivatives.

Theorem 5.10 *Let u be the solution of* (5.23)*, then*

$$u^k(x) \leq C[1 + \epsilon^{-k}(e^{-bx/\epsilon} + H_d e^{-b(x-d)/\epsilon})] \quad for\ \ x \in (0, d) \cup (d, 1) \tag{5.26}$$

where $k = 0, 1, ..., q$ *and* H_d *denotes the shifted Heaviside function,*

$$H_d(x) = \left\{ \begin{array}{ll} 0 & for \;\; x < d \\ 1 & for \;\; x > d \end{array} \right\}$$

and the maximal order q *depends on the smoothness of* b *and* f *on* $(0, d)$.

Proof: (Linß, 2002) .■

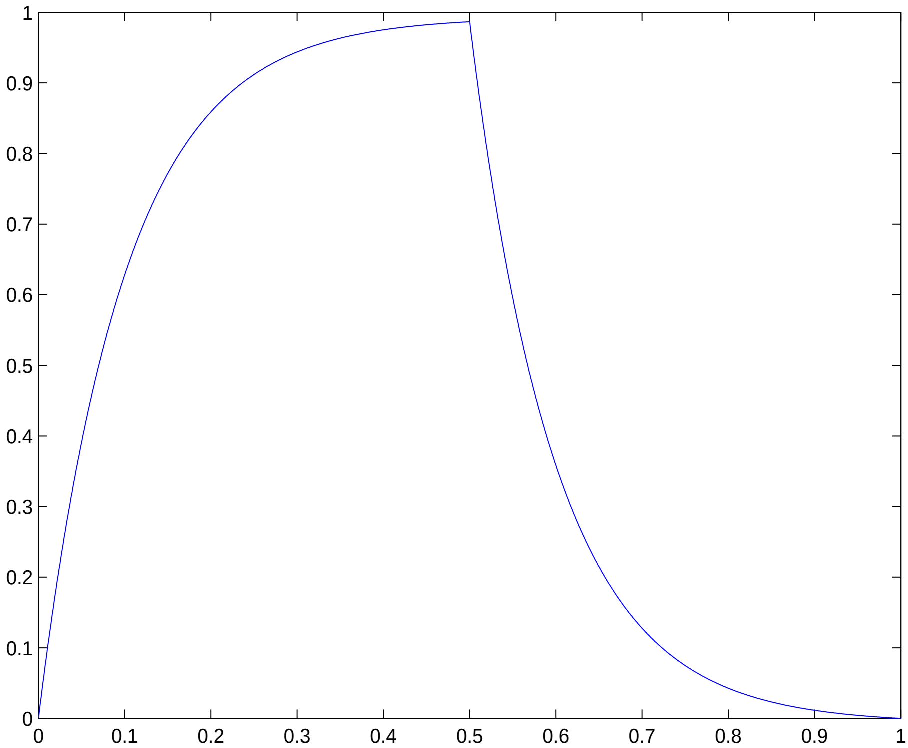

Figure 5.9. $Lu = \delta_{1/2}$ with $b = 1$ and $\epsilon = 0.1$

If we use again Shishkin mesh to solve the problem (5.23) approximately, then construction of the piecewise uniform mesh is different from the previous one because there are a boundary and an internal layer at $x = 0$ and $x = d$ respectively. To construct such mesh, choose three points τ, d and $d + \tau$, this divides the domain $\overline{\Omega}$ into the four subintervals

$$I_1 = [0, \tau] \qquad I_2 = [\tau, d] \qquad I_3 = [d, d + \tau] \qquad I_4 = [d + \tau, 1]$$

where τ satisfies the condition

$$\tau = \min\{\frac{1}{4}, \frac{\epsilon}{\alpha} \ln N\}. \tag{5.27}$$

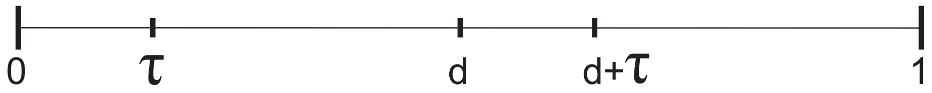

Figure 5.10. Subdomains for the discretization of the problem (5.23)

The corresponding piecewise uniform mesh is established by dividing each subintervals in the Figure 5.10 into $N/4$ equidistant subintervals. Thus, the resulting mesh $\overline{\Omega}_\tau^N$ can be described by

$$h_i = x_i - x_{i-1} = \left\{ \begin{array}{cc} \frac{4\tau}{N} & for\ \ 0 < i \leq N/4\ \ or\ \ N/2 < i \leq 3N/4 \\ \frac{4(d-\tau)}{N} & for\ \ \ N/4 < i \leq N/2\ \ or\ \ 3N/4 < i \leq N \end{array} \right\}.$$

It is convenient now to introduce the following notation before the discretization of the problem (5.23)

$$h_1 = \frac{4\tau}{N} \qquad\qquad h_2 = \frac{4}{N}(d-\tau)$$
$$\lambda_1 = 1 + \frac{bh_1}{\epsilon} \qquad \lambda_2 = 1 + \frac{bh_2}{\epsilon}.$$

Then, using the upwind operator

$$L^N = -\epsilon D^2 - bD^+$$

yields the following discretization

$$\left\{ \begin{array}{l} Find\ U \in V(\overline{\Omega}_\tau^N)\ such\ that\ \ U_0 = 0 \quad U_N = 0 \\ \\ L^N U_i = -\epsilon \frac{1}{h_i}(\frac{U_{i+1}-U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}) - b(\frac{U_{i+1}-U_i}{h_{i+1}}) = f_i + \Delta_{d,i} \end{array} \right\} \tag{5.28}$$

where $i = 1, 2, ..., N-1$ and

$$\Delta_{d,i} = \left\{ \begin{array}{cc} \frac{1}{h_{i+1}} & if \quad d \in [x_i, x_{i+1}) \\ 0 & otherwise \end{array} \right\}$$

is an approximation of the shifted $Dirac-$delta function. For simplicity, set $f = 0$ and $d = 1/2$, then combining the terms having the same indices leads to the following difference equation

$$(-\lambda_j)U_{i+1} + (\frac{h_{i+1}}{h_i} + \lambda_j)U_i + (-\frac{h_{i+1}}{h_i})U_{i-1} = \Delta_{1/2,i}; \qquad i = 1, 2, ..., N-1. \quad (5.29)$$

where $U_i \approx u(x_i)$ for $j = 1, 2$ and λ_j is defined by

$$\lambda_j = \left\{ \begin{array}{lllll} \lambda_1 & if & 1 \leq i \leq N/4 & or & N/2 < i \leq 3N/4 \\ \lambda_2 & if & N/4 < i \leq N/2 & or & 3N/4 < i \leq N-1. \end{array} \right\}.$$

The difference equation (5.29) can be written explicitly as

$$\begin{aligned} (-\lambda_1)U_{N/4+1} + (\frac{h_2}{h_1} + \lambda_1)U_{N/4} + (-\frac{h_2}{h_1})U_{N/4-1} &= 0; \qquad if \;\; i = N/4. \\ (-\lambda_2)U_{N/2+1} + (\frac{h_1}{h_2} + \lambda_2)U_{N/2} + (-\frac{h_1}{h_2})U_{N/2-1} &= \frac{h_2}{\epsilon}; \qquad if \;\; i = N/2. \qquad (5.30) \\ (-\lambda_1)U_{3N/4+1} + (\frac{h_2}{h_1} + \lambda_1)U_{3N/4} + (-\frac{h_2}{h_1})U_{3N/4-1} &= 0; \quad if \;\; i = 3N/4. \\ (-\lambda_j)U_{i+1} + (1 + \lambda_j)U_i + (-1)U_{i-1} &= 0; \qquad otherwise. \end{aligned}$$

Since the roots of the characteristic polynomial of the last difference equation are

$$r_1 = 1 \quad and \quad r_2 = \lambda_j^{-1},$$

we assume that the difference solution has the form

$$U_i = \left\{ \begin{array}{lll} a_1 + a_2\lambda_1^{-i} & if & 0 \leq i \leq N/4 \\ a_3 + a_4\lambda_2^{-i} & if & N/4 \leq i \leq N/2 \\ a_5 + a_6\lambda_1^{-i} & if & N/2 \leq i \leq 3N/4 \\ a_7 + a_8\lambda_2^{-i} & if & 3N/4 \leq i \leq N \end{array} \right\}. \qquad (5.31)$$

We have eight unknown coefficients and need to determine them in order to obtain the difference solution exactly. The boundary conditions $U_0 = U_N = 0$ give us two equations and also three equations comes from the difference equations related to the nodes $x_{N/4}$, $x_{N/2}$ and $x_{3N/4}$ as they can be seen in (5.9). Finally, the other three equations are obtained by using the continuity conditions below.

$$\begin{aligned} a_1 + a_2\lambda_1^{-N/4} &= a_3 + a_4\lambda_2^{-N/4} \\ a_3 + a_4\lambda_2^{-N/2} &= a_5 + a_6\lambda_1^{-N/2} \\ a_5 + a_6\lambda_1^{-3N/4} &= a_7 + a_8\lambda_2^{-3N/4}. \end{aligned}$$

The resulting system of equations is given in the matrix form as follows:

$$LA = B$$

The matrices in this system are given by

$$L = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \kappa_1^{-1} & -1 & -\kappa_2^{-1} & 0 & 0 & 0 & 0 \\ \lambda_1 & \kappa_1^{-1}(1-\lambda_2) & -\lambda_1 & -\kappa_2^{-1}\kappa_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \kappa_2^{-2} & -1 & -\kappa_1^{-2} & 0 & 0 \\ 0 & 0 & \lambda_2 & \kappa_2^{-2}(1-\lambda_1) & -\lambda_2 & -\kappa_1^{-2}\kappa_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \kappa_1^{-3} & -1 & -\kappa_2^{-3} \\ 0 & 0 & 0 & 0 & \lambda_1 & \kappa_1^{-3}(1-\lambda_2) & -\lambda_1 & -\kappa_2^{-3}\kappa_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \kappa_2^{-4} \end{bmatrix}$$

$$A = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8]^T \qquad B = [0 \quad 0 \quad 0 \quad 0 \quad \frac{h_2}{\epsilon} \quad 0 \quad 0 \quad 0]^T$$

where $\kappa_1 = \lambda_1^{N/4}$, $\kappa_2 = \lambda_2^{N/4}$ and $\kappa_3 = \lambda_1\lambda_2^{-1}$. We solve this system and obtain the following solutions

$$\begin{aligned} a_1 &= \frac{h_2}{\eta}\kappa_1\kappa_2\kappa_3 & a_2 &= -a_1 \\ a_3 &= \frac{h_2}{\eta}(-\kappa_2\kappa_3 + \kappa_1\kappa_2\kappa_3 + \kappa_2) & a_4 &= -\frac{h_2}{\eta}\kappa_2^2 \\ a_5 &= -\frac{h_2}{\eta}(1 - \kappa_2 + \kappa_2\kappa_3) & a_6 &= \frac{h_2}{\eta}\kappa_1^3\kappa_2\kappa_3 \\ a_7 &= -\frac{h_2}{\eta} & a_8 &= \frac{h_2}{\eta}\kappa_2^4 \end{aligned}$$

where $\eta = \epsilon(\lambda_1 - 1)(1 + \lambda_1^{N/4}\lambda_2^{N/4})$. To complete the solution we substitute these coefficients into the solution form (5.31). This results in

$$U_i = \left\{ \begin{array}{cc} \frac{h_2}{\eta}\kappa_1\kappa_2\kappa_3(1-\lambda_1^{-i}) & if \;\; 0 \le i \le N/4 \\ \frac{h_2}{\eta}(\kappa_1\kappa_2\kappa_3 + \kappa_2 - \kappa_2\kappa_3 - \lambda_2^{-i+N/2}) & if \;\; N/4 \le i \le N/2 \\ \frac{h_2}{\eta}(\kappa_2 - \kappa_2\kappa_3 - 1 + \kappa_2\kappa_3\lambda_1^{-i+3N/4}) & if \;\; N/2 \le i \le 3N/4 \\ \frac{h_2}{\eta}(-1 + \lambda_2^{-i+N}) & if \;\; 3N/4 \le i \le N \end{array} \right\} \qquad (5.32)$$

and the solutions u of (5.23) and U of (5.28) satisfies the following error estimate in the discrete maximum norm.

Theorem 5.11 *The finite difference method* (5.28) *with the piecewise uniform fitted mesh* Ω_τ^N *is* $\epsilon-$*uniform for the problem* (5.23) *provided that* τ *is chosen to satisfy the condition* (5.27) *above. Moreover, the solution* u *of* (5.23) *and the solution* U *of* (5.28) *satisfy the following* $\epsilon-$*uniform error estimate*

$$\sup_{0<\epsilon\leq 1} \parallel U - u \parallel_{\overline{\Omega}_\tau^N} \leq CN^{-1}\ln N$$

where C is a constant independent of ϵ*.*

Proof:(Linß, 2002) .■

CHAPTER 6

CONCLUSION

In this thesis, we mainly investigated the numerical methods which are $\epsilon - uniform$ on both equidistant and non$-$equidistant meshes for the convection$-$diffusion problem. We observed that the centered and upwind finite difference method are not $\epsilon - uniform$. That led us to obtain the numerical methods which are $\epsilon - uniformly$ convergent. In order to achieve it we use either a fitted operator method or a fitted mesh method. We started with derivation of Il'in$-$Allen$-$Southwell method as an example of fitted operator methods on equidistant meshes and saw that it is first$-$order uniformly convergent in the discrete maximum norm. But, since it is based on the exact solution of the problem, we tend to study on the other ideas to construct an $\epsilon - uniform$ method which is not based on the exact solution. Thus, we used Shishkin mesh, because of its simplicity, to construct a method as an example of fitted mesh methods. In the last chapter, we considered the problem in two cases to develop a more efficient method as a further aim. First, we considered a problem with regular data and proved that the resulting method using upwind operator on Shishkin mesh is $\epsilon - uniform$. Then, we studied on a problem which has an irregular data and used again Shishkin mesh. We gave an $\epsilon - uniform$ error estimate for this method. We have observed that theoretical findings are compatible with the numerical results for each methods.

We intend to study on a more efficient method by using the facts obtained in the last chapter as a future work. We will try to approximate the local Green's function and assemble the resulting solutions into the difference equation discussed in the Chapter 4.

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APPENDIX A

MATLAB CODES

EXACT SOLUTION FOR SEVERAL VALUES OF ϵ

```
% This program plot the figure in Chapter 2 FIGURE 2.1
function chp2(b,U0,U1) e1=1 ; e2=.1 ;
e3=.01 ; e4=.001; x=0:0.01:1;
Y1=U0+[(U1-U0)/(1-exp(-b/e1))]
*[exp((-b*(1-x))/e1)-exp(-b/e1)];
Y2=U0+[(U1-U0)/(1-exp(-b/e2))]
*[exp((-b*(1-x))/e2)-exp(-b/e2)];
Y3=U0+[(U1-U0)/(1-exp(-b/e3))]
*[exp((-b*(1-x))/e3)-exp(-b/e3)];
Y4=U0+[(U1-U0)/(1-exp(-b/e4))]
*[exp((-b*(1-x))/e4)-exp(-b/e4)];
plot(x,Y1,x,Y2,x,Y3,x,Y4) xlabel('x axis');
 ylabel('y axis');
```

CENTERED DIFFERENCE METHOD IN CHAPTER 3

```
% This program solves the convection diffusion problem below
% approximately using centered difference method on a
% uniform mesh.
% -eUxx + bUx = 0    on (0,1)
%  U(0)=U0    and   U(1)=UN
%  where e and a given constant
%  N: the number of mesh elements
function chp3CENT(N,e,b,U0,UN)
% h denotes the width of the mesh elements
h=1/N;
% k1,k2 and k3 denotes the coefficients of the
```

```
% algebraic equation
% produced by centered difference method
k1=-1-h/(2*e); k2=2; k3=-1+h/(2*e);
% x(i) denotes the grid points
x(1)=0; x(N+1)=1; for i=2:N
    x(i)=x(i-1)+h;
end
%%%% COMPUTATION OF THE EXACT SOLUTION %%%%%
for i=1:N+1
    U(i)=U0+[(UN-U0)/(1-exp(-b/e))]
    *[exp((-b*(1-x(i)))/e)-exp(-b/e)];
end

%%%% COMPUTATION OF THE NUMERICAL SOLUTION %%%%%
A=zeros(N-1); S=zeros(N-1,1); B=zeros(N-1,1); for i=1:N-1
    for j=1:N-1
        if i==j
            A(i,j)=k2;
        elseif i-1==j
            A(i,j)=k1;
        elseif i+1==j
            A(i,j)=k3;
        else A(i,j)=0;
        end
    end
end for i=1:N-1
    if i==N-1
        B(i)=-k3;
    else B(i)=0;
    end
end S=A\B; s(1)=U0; s(N+1)=UN; for i=2:N
    for j=1:1
```

```
        s(i)=S(i-1,j);
    end
end plot(x,s,'*-',x,U,'-') xlabel('x axis');
 ylabel('CENTERED and
EXACT SOLUTIONS');
```

UPWIND DIFFERENCE METHOD IN CHAPTER 3

```
% This program solves the convection diffusion problem
% below approximately using upwind difference method
% and centered difference method on a uniform mesh.
% It gives both the centered difference solutions and
% upwind difference solution on the same window with
% the exact solution.
% -eUxx + bUx = 0    on (0,1)
%  U(0)=U0    and   U(1)=UN
%  where e and a given constant
%  N: the number of mesh elements
function chp3UPW(N,e,b,U0,UN)
% h denotes the width of the mesh elements
h=1/N;
% k1,k2 and k3 denotes the coefficients of the
% algebraic equation
% produced by upwind difference method
k1=-1-h/e; k2=2+h/e; k3=-1; p1=-1-h/(2*e);
p2=2; p3=-1+h/(2*e);
% x(i) denotes the grid points
x(1)=0; x(N+1)=1; for i=2:N
    x(i)=x(i-1)+h;
end
%%%%% COMPUTATION OF THE EXACT SOLUTION %%%%%
for i=1:N+1
    U(i)=U0+[(UN-U0)/(1-exp(-b/e))]
    *[exp((-b*(1-x(i)))/e)-exp(-b/e)];
```

```
end

%%%% COMPUTATION OF THE NUMERICAL SOLUTION %%%%
A1=zeros(N-1); S1=zeros(N-1,1); B1=zeros(N-1,1);
A2=zeros(N-1);
S2=zeros(N-1,1); B2=zeros(N-1,1);

for i=1:N-1
    for j=1:N-1
        if i==j
            A1(i,j)=k2;
        elseif i-1==j
            A1(i,j)=k1;
        elseif i+1==j
            A1(i,j)=k3;
        else A1(i,j)=0;
        end
    end
end for i=1:N-1
    if i==N-1
        B1(i)=-k3;
    else B1(i)=0;
    end
end S1=A1\B1; s1(1)=U0; s1(N+1)=UN; for i=2:N
    for j=1:1
        s1(i)=S1(i-1,j);
    end
end



for i=1:N-1
    for j=1:N-1
```

```
        if i==j
            A2(i,j)=p2;
        elseif i-1==j
            A2(i,j)=p1;
        elseif i+1==j
            A2(i,j)=p3;
        else A2(i,j)=0;
        end
    end
end for i=1:N-1
    if i==N-1
        B2(i)=-p3;
    else B2(i)=0;
    end
end S2=A2\B2; s2(1)=U0; s2(N+1)=UN; for i=2:N
    for j=1:1
        s2(i)=S2(i-1,j);
    end
end

plot(x,s1,'*-',x,s2,'o-',x,U,'-') xlabel('x axis');
ylabel('UPWIND-CENTERED-EXACT');
```

Il'IN−ALLEN−SOUTHWELL METHOD IN CHAPTER 4

```
% PRODUCES FIGURE (4.1), (4.3) and (4.5)
% This program solves the convection diffusion
% problem defined below
% using The Il'in-Allen-Southwell Method.
% -epsU'' + bU' = x    on (0,1)
%  U(0)=0    and   U(1)=0
%  where eps and b given constant
%  N: the number of mesh elements
function chp4alt(N,e,b)
```

```
%SOME COEFFICIENTS and PARAMETERS TO CALCULATE
%THE DIFFERENCE SOLUTION
h=1/N; q=(b*h)/e; k1=-(1-exp(-q))/(1-exp(-2*q));
k2=-(exp(q)-1)/(exp(2*q)-1);
k3=[(h/b)*(1-exp(-q))]/(1+exp(-q));

% DEFINITION OF THE DISCRETE NODES
x(1)=0; x(N+1)=1; for i=2:N, x(i)=x(i-1)+h;end;

% COMPUTATION OF THE EXACT SOLUTION
for i=1:N+1, U(i)=x(i)^(2)/2+e*x(i)-(e+1/2)*
[(exp((-b*(1-x(i)))/e)-exp(-b/e))/(1-exp(-b/e))];
end;

%COMPUTATION OF THE NUMERICAL SOLUTION
for i=1:N-1
    for j=1:N-1
        if i==j
            A(i,j)=1;
        elseif i-1==j
            A(i,j)=k1;
        elseif i+1==j
            A(i,j)=k2;
        else A(i,j)=0;
        end
    end
end for i=1:N-1
  for j=1:1
    B(i,j)=x(i)*k3;
  end
end S=A\B; s(1)=0; s(N+1)=0; for i=2:N
    for j=1:1
```

```
        s(i)=S(i-1,j);
    end
end

% ERROR BETWEEN NUMERICAL AND EXACT SOLUTION
%E=abs(s-U);
plot(x,s,’o’,x,U,’.-’) xlabel(’x axis’);
ylabel(’NUMERICAL and EXACT
SOLUTION’);
%pause
%plot(x,E,’.-’)
%xlabel(’x axis’); ylabel(’Error’);
```

Il’IN−ALLEN−SOUTHWELL METHOD IN CHAPTER 4

```
% PRODUCES FIGURE (4.2), (4.4) and (4.6)
% This program solves the convection diffusion problem
% defined below
% using The Il’in-Allen-Southwell Method on a uniform mesh
% -epsU’’ + bU’ = x    on (0,1)
%  U(0)=0    and   U(1)=0
%  where eps and b given constant
%  N: the number of mesh elements
function chp4alt2(e,b)

N(1)=20; for k=1:20 N(k)=N(1)+(k-1)*20;
% SOME COEFFICIENTS and PARAMETERS TO CALCULATE
% THE DIFFERENCE SOLUTION
h=1/N(k); q=(b*h)/e k1=-(1-exp(-q))/(1-exp(-2*q))
k2=-(exp(q)-1)/(exp(2*q)-1);
k3=[(h/b)*(1-exp(-q))]/(1+exp(-q));

%DEFINITION OF THE DISCRETE NODES
x(1)=0; x(N(k)+1)=1; for i=2:N(k), x(i)=x(i-1)+h;end;
```

```
clear A clear S clear B clear s clear U

%COMPUTATION OF THE EXACT SOLUTION

for i=1:N(k)+1,
U(i)=x(i)^(2)/2+e*x(i)-(e+1/2)*
[(exp((-b*(1-x(i)))/e)-exp(-b/e))/(1-exp(-b/e))];
end;

%COMPUTATION OF THE NUMERICAL SOLUTION
clear A clear S clear B clear s

A=zeros(N(k)-1); S=zeros(N(k)-1,1); B=zeros(N(k)-1,1);

for i=1:N(k)-1
    for j=1:N(k)-1
        if i==j
            A(i,j)=1;
        elseif i-1==j
            A(i,j)=k1;
        elseif i+1==j
            A(i,j)=k2;
        else A(i,j)=0;
        end
    end
end for i=2:N(k)
  for j=1:1
    B(i-1,j)=x(i)*k3;
  end
end S=A\B; s(1)=0; s(N(k)+1)=0; for i=2:N(k)
    for j=1:1
        s(i)=S(i-1,j);
```

```
    end
end

%%%% ERROR AT THE LAYER %%%%%
 E(k)=abs(s(N(k))-U(N(k)))
end
plot(N,E,'.-')
xlabel('(N)The Number of Mesh Elements');
ylabel('Error--->');
```

FITTED MESH METHOD IN CHAPTER 5 $(Example 1)$

```
% This program solves the boundary value problems
% defined below
% using difference operators on a fitted mesh.
% -eps*u'' + c*u' = 0  on (0,1)
%  u(0)=0    and   u(1)=1
%
%  N denotes the number of  elements
function upws2(c,N) format long g for j=1:10
    eps(j)=10^(-j+1);

clear x ye mu fi yds

%Boundary conditions
u0=0; u1=1;

%  tau: the transition parameter
%  h1 and h2: the width of the fine and coarse mesh
%  elements
tau=min(1./2.,eps(j)*log(N)/c); h2=2*tau/N; h1=2*(1.-tau)/N;
if
(h1<1.e-14) | (h2<1.e-14)
    ('mesh is too small')
```

```
    stop
end L1=1+c*h1/eps(j); L2=1+c*h2/eps(j); LB=(L1+L2)/2;

% x(i) denotes the grid points
x(1)=0; x(N+1)=1.; for i=2:N/2+1,        x(i)=x(i-1)+h1;
end; for
i=N/2+2:N,        x(i)=x(i-1)+h2; end;
%plot(x,'-');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Solution of three-point difference equation
muN2=(L2)^(-N/2)*L1*(1-L1^(-N/2)) nuN2=LB*(1-L2^(-N/2)) dN
=(muN2+nuN2); (L1^(N/2)-1) for i=1:N/2+1
    mu(i)=(L2)^(-N/2)*L1*(L1^(i-1-N/2)-(L1)^(-N/2));
    fi(i)=mu(i)/dN;
    yds(i)=u0+(u1-u0)*fi(i);
end for i=N/2+2:N+1
    nu(i)=LB*(1-L2^(i-1-N));
    psi(i)=nu(i)/dN;
    yds(i)=u1+(u0-u1)*psi(i);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Compute the Exact solution at grid points
for i=1:N+1
        ye(i)=u0+[(u1-u0)/(1-exp(-c/eps(j)))]*
        [exp((-c*(1-x(i)))/eps(j))-exp(-c/eps(j))];
end
ER(j)=max(abs(ye-yds))
end
maxER=max(ER)
```

FITTED MESH METHOD IN CHAPTER 5 $(Example 2)$

```
% This program solves the convection diffusion problem
% using upwind difference operators on a piecewise
% uniform meshes.
% -eUxx + bUx = x    on (0,1)
%  U(0)=0    and   U(1)=0
%  where e and a given constant
%  N: the number of mesh elements

function chp5alt(N,e,b)
%  T denotes the transition parameter
%  h1 and h2 denote the width of the fine and coarse
%  mesh elements
%  L1 and L2 are parameters related to the discrete problem
N(1)=N; for k=1:20 N(k)=N(1)+(k-1)*20;



T1=[e*log(N(k))]/b; T=min(1/2,T1); h1=(2*(1-T))/N(k);
h2=(2*T)/N(k);
h3=(h1+h2)/2; L1=1+(b*h1)/e; L2=1+(b*h2)/e; L3=(L1+L2)/2;
% x(i) denotes the discrete nodes
x(1)=0; x(N(k)+1)=1;



for i=2:N(k)
     if i<=N(k)/2+1
        x(i)=x(i-1)+h1;
     else x(i)=x(i-1)+h2;
     end
end

clear U clear S clear B clear s clear Y
```

```
%%%% COMPUTATION OF THE EXACT SOLUTION %%%%

i=1:N(k)+1;
U(i)=x(i)-[exp((-b*(1-x(i)))/e)-exp(-b/e)]/(1-exp(-b/e));

%%%% COMPTATION OF THE NUMERICAL SOLUTION %%%%
p1=-L1;p2=(1+L1); p3=-1; p4=-L3; p5=(h1/h2+L3);
p6=-h1/h2; p7=-L2;
p8=1+L2; p9=p3; p10=h1^(2)/e; p11=h2^(2)/e; p12=h1*h3/e;

for i=1:N(k)-1
    for j=1:N(k)-1
        if i<N(k)/2 && i==j
            Y(i,j)=p2;
        elseif i<N(k)/2 && i-1==j
            Y(i,j)=p1;
        elseif i<N(k)/2 && i+1==j
            Y(i,j)=p3;
        elseif i==N(k)/2 && i==j
            Y(i,j)=p5;
        elseif i==N(k)/2 && i-1==j
            Y(i,j)=p4;
        elseif i==N(k)/2 && i+1==j
            Y(i,j)=p6;
        elseif i>N(k)/2 && i==j
            Y(i,j)=p8;
        elseif i>N(k)/2 && i-1==j
            Y(i,j)=p7;
        elseif i>N(k)/2 && i+1==j
            Y(i,j)=p9;
        else Y(i,j)=0;
        end
```

```
    end
end for i=1:N(k)-1
    for j=1:1
        if i<N(k)/2
           B(i,j)=x(i)*p10;
        elseif i==N(k)/2
           B(i,j)=x(i)*p12;
        elseif i>N(k)/2
            B(i,j)=x(i)*p11;
        end
    end
end S=Y\B; s(1)=0; s(N(k)+1)=0; for i=2:N(k)
    for j=1:1
        s(i)=S(i-1,j);
    end
end
%%%% ERROR AT THE BOUNDARY LAYER %%%%
E(k)=abs(s(N(k))-U(N(k))); end plot(N,E,'.-')
xlabel('(N)The Number
of Mesh Elements'); ylabel('Error--->');
```