PROPER CLASS GENERATED BY SUBMODULES THAT HAVE SUPPLEMENTS

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ABSTRACT

PROPER CLASS GENERATED BY SUBMODULES THAT HAVE SUPPLEMENTS

In this thesis, we study the class $\mathcal{S}$ of all short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\operatorname{Im}\alpha$ has a supplement in B, i.e. a minimal element in the set $\{V \subseteq B \mid V + \operatorname{Im}\alpha = B\}$. The corresponding elements of $\operatorname{Ext}_R(C, A)$ are called κ-elements. In general κ-elements need not form a subgroup in $\operatorname{Ext}_R(C, A)$, but in the category $\mathcal{T}_R$ of torsion R-modules over a Dedekind domain R, $\mathcal{S}$ is a proper class; there are no nonzero $\mathcal{S}$-projective modules and the only $\mathcal{S}$-injective modules are injective R-modules in $\mathcal{T}_R$. In this thesis we also give the structure of $\mathcal{S}$-coinjective R-modules in $\mathcal{T}_R$. Moreover, we define the class $\mathcal{SB}$ of all short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\operatorname{Im}\alpha$ has a supplement V in B and $V \cap \operatorname{Im}\alpha$ is bounded. The corresponding elements of $\operatorname{Ext}_R(C, A)$ are called β-elements. Over a noetherian integral domain of Krull dimension 1, β-elements form a proper class. In the category $\mathcal{T}_R$ over a Dedekind domain R, $\mathcal{SB}$ is a proper class; there are no nonzero $\mathcal{SB}$-projective R-modules and $\mathcal{SB}$-injective R-modules are only the injective R-modules. In the category $\mathcal{T}_R$, reduced $\mathcal{SB}$-coinjective R-modules are bounded R-modules.

ÖZET

TÜMLEYENİ OLAN ALTMODÜLLERİN ÜRETTİĞİ ÖZ SINIF

Bu tezde, $\operatorname{Im}\alpha$' nın B'de bir tümleyeni, yani $\{V \subseteq B | V + \operatorname{Im}\alpha = B\}$ kümesinin minimum elemanı bulunacak şekilde tüm $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ kısa tam dizilerinin $\mathcal{S}$ sınıfını inceliyoruz. $\operatorname{Ext}_R(C, A)$' nın bu dizilere karşılık gelen elemanlarına κ-elemanlar denir. Genelde κ-elemanlar bir öz sınıf oluşturmayabilir, fakat R Dedekind bölgesi üzerindeki burulma modüllerinin $\mathcal{T}_R$ kategorisinde $\mathcal{S}$ bir öz sınıftır; sıfırdan farklı $\mathcal{S}$-projektif modüller bulunmaz, $\mathcal{S}$-injektif modüller sadece injektif modüllerdir. Tezde $\mathcal{T}_R$ kategorisinde $\mathcal{S}$-eşinjektif modüllerin yapısını da verdik. Ayrıca $\operatorname{Im}\alpha$'nın B'de V diye bir tümleyeninin bulunduğu ve $V \cap \operatorname{Im}\alpha$' nın sınırlı olduğu $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ kısa tam dizilerinin $\mathcal{SB}$ sınıfını tanımladık. $\operatorname{Ext}_R(C, A)$' nın bu dizilere karşılık gelen elemanlarına β-elemanlar denir. Krull boyutu 1 olan Noether tamlık bölgesi üzerinde $\mathcal{SB}$' nin bir öz sınıf oluşturduğunu gösterdik. R Dedekind bölgesi üzerinde burulma modüllerinin $\mathcal{T}_R$ kategorisinde $\mathcal{SB}$ bir öz sınıftır; sıfırdan farklı $\mathcal{SB}$-projektif modüller bulunmaz, $\mathcal{SB}$-injektif modüller sadece injektif modüllerdir. $\mathcal{T}_R$ kategorisinde indirgenmiş $\mathcal{SB}$-eşinjektif modüller tam olarak sınırlı modüllerdir.

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NOTATION

R	an associative ring with unit unless otherwise stated
$R_{\mathfrak{p}}$	the localization of a ring R at a prime ideal $\mathfrak{p}$ of R
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$G[n]$	for a group G and integer n, $G[n] = \{g \in G \mid ng = 0\}$
G^1	the first Ulm subgroup of abelian group G: $G^1 = \bigcap_{n=1}^{\infty} nG$
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p (the p-primary part of the torsion group $\mathbb{Q}/\mathbb{Z}$)
R-module	*left* R-module
R-$\mathcal{M}od$	the category of *left* R-modules
$\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$	the category of abelian groups ($\mathbb{Z}$-modules)
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$T(M)$	the torsion submodule of the module M: $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathcal{T}_R$	the category of torsion R-modules
$\mathcal{B}$	the class of bounded R-modules
$\langle \mathcal{E} \rangle$	the smallest proper class containing the class $\mathcal{E}$ of short exact sequences
$\mathcal{P}$	a proper class of R-modules
$\hat{\mathcal{P}}$	the set $\{\mathbb{E} \mid r\mathbb{E} \in \mathcal{P} \text{ for some } 0 \neq r \in R\}$ for a proper class $\mathcal{P}$
$\pi(\mathcal{P})$	all $\mathcal{P}$-projective modules
$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules

$\iota(\mathcal{P})$	all $\mathcal{P}$-injective modules
$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\tau(\mathcal{P})$	all $\mathcal{P}$-flat *right* R-modules
$\tau^{-1}(\mathcal{M})$	the proper class of R-modules flatly generated by a class $\mathcal{M}$ of *right* R-modules
$\overline{k}(\mathcal{M})$	the proper class coprojectively generated by a class $\mathcal{M}$ of R-modules
$\underline{k}(\mathcal{M})$	the proper class coinjectively generated by a class $\mathcal{M}$ of R-modules
$\text{Ext}_R(C, A) = \text{Ext}^1_R(C, A)$	the set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\text{Text}_R(C, A)$	the set $\{\mathbb{E} \in \text{Ext}(C, A) \mid r\mathbb{E} \equiv 0 \text{ for some } 0 \neq r \in R\}$ of equivalence classes of short exact sequences of R-modules
$\text{Pext}(C, A)$	the set of all equivalence classes of pure-exact sequences starting with the group A and ending with the group C
$\text{Next}(C, A)$	the set of all equivalence classes of neat-exact sequences starting with the group A and ending with the group C
$\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of pure-exact sequences of abelian groups
$\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of neat-exact sequences of abelian groups
$\mathcal{A}$	an abelian category (like R-$\mathcal{M}od$ or $\mathbb{Z}$-$\mathcal{M}od = \mathcal{A}b$)
	For a suitable abelian category $\mathcal{A}$ like R-$\mathcal{M}od$ or $\mathbb{Z}$-$\mathcal{M}od$, the following classes are defined:

$\mathcal{S}plit_{\mathcal{A}}$	the smallest proper class consisting of *only splitting* short exact sequences in the abelian category $\mathcal{A}$
$\mathcal{A}bs_{\mathcal{A}}$	the largest proper class consisting of *all* short exact sequences in the abelian category $\mathcal{A}$
$Compl_{\mathcal{A}}$	the proper class of complements in the abelian category $\mathcal{A}$
$\mathcal{S}uppl_{\mathcal{A}}$	the proper class of supplements in the abelian category $\mathcal{A}$
$\mathcal{N}eat_{\mathcal{A}}$	the proper class of neats in the abelian category $\mathcal{A}$
$Co\text{-}\mathcal{N}eat_{\mathcal{A}}$	the proper class of coneats in the abelian category $\mathcal{A}$
$\mathcal{S}_{\mathcal{A}}$	the class of κ-exact sequences in the abelian category $\mathcal{A}$
$\mathcal{SB}_{\mathcal{A}}$	the class of β-exact sequences in the abelian category $\mathcal{A}$
$\cong$	isomorphic
$\leq$	submodule
$\ll$	small (=superfluous) submodule
$\subset^{\beta}$	$\mathcal{SB}$-submodule

CHAPTER 1

INTRODUCTION

Throughout R is an associative ring with identity and all modules are unital left R-modules unless otherwise stated. We will denote the category of torsion R-modules by $\mathcal{T}_R$ and bounded R-modules by $\mathcal{B}$. Definitions not given here can be found in (Anderson and Fuller 1992), (Wisbauer 1991), (Hungerford 1974), (Mac Lane 1995) and (Fuchs 1970).

In this thesis, we study the class $\mathcal{S}$ of κ-exact sequences where an element $\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ of $\operatorname{Ext}_R(C, A)$ is called κ-exact if $\operatorname{Im}\alpha$ has a supplement in B, i.e. a minimal element in the set $\{V \subseteq B | V + \operatorname{Im}\alpha = B\}$. We show that $\mathcal{S}$ is not a proper class in general. The class $\mathcal{W}supp$ consists of the short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ of R-modules such that $\operatorname{Im}\alpha$ has a weak supplement in B. We denote the class consisting of the short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\operatorname{Im}\alpha \ll B$ by $\mathcal{S}mall$. For a class $\mathcal{E}$, we denote by $\langle\mathcal{E}\rangle$ the smallest proper class containing $\mathcal{E}$ which is called the proper class generated by $\mathcal{E}$. Over a Dedekind domain R, the smallest proper class $\langle\mathcal{S}\rangle$ containing $\mathcal{S}$ coincides with the smallest proper class $\langle\mathcal{S}mall\rangle$ containing $\mathcal{S}mall$ and the smallest proper class $\langle\mathcal{W}supp\rangle$ containing $\mathcal{W}supp$. The class $\mathcal{SB}$ of short exact sequences is introduced as the class of short exact sequences $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, where $\operatorname{Im} f$ has a supplement V in B with $V \cap \operatorname{Im} f$ is bounded. The short exact sequences contained in $\mathcal{SB}$ form a proper class over a noetherian ring of Krull dimension 1 and $\mathcal{SB}$ coincides with the proper class $\underline{k}(\mathcal{B})$ generated by the class $\mathcal{B}$ of bounded R-modules in this case. In the category $\mathcal{T}_R$ of torsion R-modules over a Dedekind domain R, $\mathcal{S}$ and $\mathcal{SB}$ form proper classes. There are no nonzero $\mathcal{S}$-projective and nonzero $\mathcal{SB}$-projective R-modules , the only $\mathcal{S}$-injective and $\mathcal{SB}$-injective R-modules are injective modules in the category $\mathcal{T}_R$. The characterization of $\mathcal{S}$-coinjective and $\mathcal{SB}$-coinjective modules in the category $\mathcal{T}_R$ are given in Propositions 4.5 and 4.7, respectively.

In Chapter 2, the notions related to our work will be given, which includes the properties of the functor $\mathrm{Ext}_R(C, A)$ in terms of short exact sequences, supplements, supplemented modules and Dedekind domains.

The definition and the properties of a proper class will be given in Chapter 3. The class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ of pure-exact sequences of abelian groups is an important example of a proper class in the category of abelian groups. It is shown here that, if $\mathcal{M}$ is a given class of R-$\mathcal{M}od$ for an additive functor $T(M, \cdot) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$, the class of exact triples $\mathbb{E}$ such that $T(M, \mathbb{E})$ is exact form a proper class. This result is helpful in the definition of projectively, injectively or flatly generated proper classes.

In Chapter 4, the proper classes related to complements and supplements are studied. It is shown that κ-elements of $\mathrm{Ext}_R(C, A)$ need not form a proper class in general. Results due to Zöschinger show that when A and C are torsion abelian groups, the κ-elements of $\mathrm{Ext}(C, A)$ over the ring $\mathbb{Z}$ of integers form a proper class, which we denote by $\mathcal{S}$. For a Dedekind domain R, over the category $\mathcal{T}_R$ of torsion R-modules, there are no nonzero $\mathcal{S}$-projective R-modules and the $\mathcal{S}$-injectives are exactly injective modules in $\mathcal{T}_R$. We give the characterization of $\mathcal{S}$-coinjective R-modules in Proposition 4.5. The subgroup $\mathcal{SB}$ of $\mathrm{Ext}_R(C, A)$ is introduced as the set of elements $[0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0]$ such that $\mathrm{Im}\,\alpha$ has a supplement V in B and $V \cap \mathrm{Im}\,\alpha$ is bounded. For a noetherian integral domain of Krull dimension 1, in the category $\mathcal{T}_R$, we show that there are no nonzero $\mathcal{SB}$-projective modules and $\mathcal{SB}$-injective modules are only the injective modules in $\mathcal{T}_R$.

CHAPTER 2

PRELIMINARIES

This Chapter will consist of a short summary of Chapter IX from (Fuchs 1970) and Chapter 3 from (Mac Lane 1995), some preliminary information about supplements in module theory and Dedekind domains. One can find further information and missing proofs in (Fuchs 1970), (Vermani 2003) and (Mac Lane 1995) about group of extensions, in (Wisbauer 1991) about supplements, supplemented modules and in (Cohn 2002) about Dedekind domains.

2.1. Extensions As Short Exact Sequences

Given the R-modules A and C, the extension B of A by C can be visualized as a short exact sequence

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0,$$

where μ is a monomorphism and ν is an epimorphism with kernel $\mu(A)$. Then one can build up a category in which the objects are the short exact sequences and a morphism between two short exact sequences $\mathbb{E}$ and $\mathbb{E}'$ is defined as a triple (α, β, γ) of module homomorphisms such that the diagram

$$\begin{tikzcd} \mathbb{E}: 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ \mathbb{E}': 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\nu'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

has commutative squares. It is straightforward to show that in this way a category $\mathscr{E}$ arises.

The extensions $\mathbb{E}$ and $\mathbb{E}'$ with $A = A'$, $C = C'$ are said to be *equivalent*, denoted by $\mathbb{E} \equiv \mathbb{E}'$, if there is a morphism $(1_A, \beta, 1_C)$ with $\beta : B \to B'$ is an isomorphism. Indeed, the condition β being an isomorphism can be omitted, since this follows from the Five Lemma.

If A is a fixed R-module, for a homomorphism $\gamma : C' \to C$, to the extension $\mathbb{E}$ in 2.1, there is a pullback square

$$\begin{tikzcd} & & B' \arrow[r, "\nu'"] \arrow[d, "\beta"'] & C' \arrow[d, "\gamma"] & \\ 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

for some B', β and ν'. ν' is epic (since ν is epic), and $\operatorname{Ker}\nu' \cong \operatorname{Ker}\nu \cong A$, hence there is a monomorphism $\mu' : A \to B'$ (i.e. $\mu' a = (\mu a, 0) \in B'$ if B' is defined to be a submodule of $B \oplus C'$) such that the diagram

$$\begin{tikzcd} \mathbb{E}\gamma : & 0 \arrow[r] & A \arrow[r, "\mu'"] \arrow[d, equal] & B' \arrow[r, "\nu'"] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ \mathbb{E} : & 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows and pullback right square commutes. The top row is an extension of A by C' which we have denoted by $\mathbb{E}\gamma$ to indicate its origin from $\mathbb{E}$ and γ. Notice that $\gamma^* = (1_A, \beta, \gamma)$ is a morphism $\mathbb{E}\gamma \to \mathbb{E}$ in $\mathscr{E}$.

If the diagram

$$\begin{tikzcd} \mathbb{E}^\circ : & 0 \arrow[r] & A \arrow[r, "\mu^\circ"] \arrow[d, equal] & B^\circ \arrow[r, "\nu^\circ"] \arrow[d, "\beta^\circ"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ \mathbb{E} : & 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

has exact rows and commutes, then there is unique $\phi : B^\circ \to B'$ such that $\nu'\phi = \nu^\circ$ and $\beta\phi = \beta^\circ$. Since the maps $\phi\mu^\circ, \mu' : A \to B'$ are such that $\beta(\phi\mu^\circ) = \beta^\circ\mu^\circ = \mu = \beta\mu'$ and $\nu'(\phi\mu^\circ) = \nu^\circ\mu^\circ = 0 = \nu'\mu'$, with the uniqueness assertion in (Vermani 2003, 1.7.3), we have $\phi\mu^\circ = \mu'$. This shows that $\mathbb{E}_\gamma$ is unique up to equivalence and this yields the equivalences

$$\mathbb{E}1_C \equiv \mathbb{E} \qquad \text{and} \qquad \mathbb{E}(\gamma\gamma') \equiv (\mathbb{E}\gamma)\gamma'$$

for $C'' \xrightarrow{\gamma'} C' \xrightarrow{\gamma} C$. Now the contravariance of $\operatorname{Ext}(C, A)$ on C is evident.

Next let C be fixed and for a given $\alpha : A \to A'$, let B' be defined by the pushout square

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] & 0\, . \\ & A' \arrow[r, "\mu'"] & B' & & \end{tikzcd}$$

Here μ' is a monomorphism, and if B' is defined as the quotient module $(A' \oplus B)/H$ where H is the submodule of $A \oplus B$ consisting of elements of the form $(\mu(a), -\alpha(a))$

for $a \in A$, then $\nu' : B' \longrightarrow C$ defined by $\nu'((a', b) + H) = \nu(b)$ for $(a', b) \in A' \oplus B$, makes the diagram

$$\begin{array}{lc}
\mathbb{E}: & 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0 \\
& \quad\ \ \Big\downarrow{\scriptstyle\alpha} \qquad \Big\downarrow{\scriptstyle\beta} \qquad \Big\| \\
\alpha\mathbb{E}: & 0 \longrightarrow A' \xrightarrow{\mu'} B' \xrightarrow{\nu'} C \longrightarrow 0
\end{array}$$

with exact rows commutative. The bottom row of this diagram is an extension of A' by C which we denote by $\alpha\mathbb{E}$. Here $\alpha_* = (\alpha, \beta, 1_C)$ is a morphism $\mathbb{E} \to \alpha\mathbb{E}$ in $\mathscr{E}$.

If we have the commutative diagram

$$\begin{array}{lc}
\mathbb{E}: & 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0 \\
& \quad\ \ \Big\downarrow{\scriptstyle\alpha} \qquad \Big\downarrow{\scriptstyle\beta_\circ} \qquad \Big\| \\
\mathbb{E}_\circ: & 0 \longrightarrow A' \xrightarrow{\mu_\circ} B_\circ \xrightarrow{\nu_\circ} C \longrightarrow 0
\end{array}$$

with exact rows, then in view of (Vermani 2003, 1.7.6) there exists a unique $\phi : B' \to B_\circ$ such that $\phi\beta = \beta_\circ$ and $\phi\mu' = \mu_\circ$. From $(\nu_\circ\phi)\beta = \nu_\circ\beta_\circ = \nu = \nu'\beta$, $(\nu_\circ\phi)\mu' = 0 = \nu'\mu'$ we infer that $\nu_\circ\phi = \nu'$, thus $(1_{A'}, \phi, 1_C)$ is a morphism $\alpha\mathbb{E} \to \mathbb{E}_\circ$. Consequently, $\alpha\mathbb{E} \equiv \mathbb{E}_\circ$, i.e. $\alpha\mathbb{E}$ is unique up to equivalence. So, we obtain

$$1_A\mathbb{E} \equiv \mathbb{E} \qquad \text{and} \qquad (\alpha\alpha')\mathbb{E} \equiv \alpha(\alpha'\mathbb{E})$$

for $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$, which establishes the covariant dependence of $\mathrm{Ext}(C, A)$ on A.

With $\alpha : A \to A'$ and $\gamma : C' \to C$, we have the important associative law

$$\alpha(\mathbb{E}\gamma) \equiv (\alpha\mathbb{E})\gamma.$$

By making use of the pullback property of $(\alpha\mathbb{E})\gamma$, it is easy to prove the existence of a morphism $(\alpha, \beta', 1) : \mathbb{E}\gamma \to (\alpha\mathbb{E})\gamma$ and to show the commutativity of the square

$$\begin{array}{ccc}
\mathbb{E}_\gamma & \xrightarrow{(1,\beta_1,\gamma)} & \mathbb{E} \\
{\scriptstyle(\alpha,\beta',1)}\Big\downarrow & & \Big\downarrow{\scriptstyle(\alpha,\beta_2,1)} \\
(\alpha\mathbb{E})\gamma & \xrightarrow{(1,\beta,\gamma)} & \alpha\mathbb{E}.
\end{array}$$

The equivalence classes of extensions of A by C form a group.

In order to describe the group operation in the language of short exact sequences, we make use of diagonal map $\Delta_G : g \mapsto (g, g)$ and the codiagonal map

$\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$ of a module G. If we understand by the *direct sum* of two extensions

$$\mathbb{E}_i : \qquad 0 \longrightarrow A_i \xrightarrow{\mu_i} B_i \xrightarrow{\nu_i} C_i \longrightarrow 0 \qquad (i = 1, 2)$$

the extension

$$\mathbb{E}_1 \oplus \mathbb{E}_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\mu_1 \oplus \mu_2} B_1 \oplus B_2 \xrightarrow{\nu_1 \oplus \nu_2} C_1 \oplus C_2 \longrightarrow 0\,,$$

then we have :

Proposition 2.1 ((Mac Lane 1995), Theorem 2.1) *For given R-modules A and C, the set* $\mathrm{Ext}_R(C, A)$ *of all congruence classes of extensions of A by C is an abelian group under the binary operation which assigns to the congruence classes of extensions* $\mathbb{E}_1$ *and* $\mathbb{E}_2$*, the congruence class of the extension*

$$\mathbb{E}_1 + \mathbb{E}_2 = \nabla_A(\mathbb{E}_1 \oplus \mathbb{E}_2)\Delta_C.$$

The class of the split extension $0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$ *is the zero element of this group, while the inverse of any* $\mathbb{E}$ *is the extension* $(-1_A)\mathbb{E}$*. For homomorphisms* $\alpha : A \longrightarrow A'$ *and* $\gamma : C' \longrightarrow C$ *one has*

$$\alpha(\mathbb{E}_1 + \mathbb{E}_2) \equiv \alpha\mathbb{E}_1 + \alpha\mathbb{E}_2, \qquad (\mathbb{E}_1 + \mathbb{E}_2)\gamma \equiv \mathbb{E}_1\gamma + \mathbb{E}_2\gamma, \tag{2.2}$$

$$(\alpha_1 + \alpha_2)\mathbb{E} \equiv \alpha_1\mathbb{E} + \alpha_2\mathbb{E}, \qquad \mathbb{E}(\gamma_1 + \gamma_2) \equiv \mathbb{E}\gamma_1 + \mathbb{E}\gamma_2. \tag{2.3}$$

The equivalences in 2.2 and 2.3 express the fact that $\alpha_* : \mathbb{E} \mapsto \alpha\mathbb{E}$ and $\gamma^* : \mathbb{E} \mapsto \mathbb{E}\gamma$ are group homomorphisms

$$\alpha_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A'), \qquad \gamma^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A),$$

and that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ for $\alpha_1, \alpha_2 : A \longrightarrow A'$, $\gamma_1, \gamma_2 : C' \longrightarrow C$.

Theorem 2.1 ((Mac Lane 1995), Lemma 1.6) Ext_R *is an additive bifunctor on* $R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od$ *to* $\mathcal{A}b$ *which is contravariant in the first and covariant in the second variable.* □

In order to be consistent with the functorial notation for homomorphisms, we shall use the notation

$$\mathrm{Ext}_R(\gamma,\alpha) : \mathrm{Ext}_R(C,A) \to \mathrm{Ext}_R(C',A')$$

instead of $\gamma^*\alpha_* = \alpha_*\gamma^*$; that is, $\mathrm{Ext}_R(\gamma,\alpha)$ acts as shown by

$$\mathrm{Ext}_R(\gamma,\alpha) : \mathbb{E} \mapsto \alpha\mathbb{E}\gamma.$$

Given an extension

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{2.4}$$

representing an element of $\mathrm{Ext}_R(C,A)$, and homomorphisms $\eta : A \to G$ and $\xi : G \to C$, we know that $\eta\mathbb{E}$ is an extension of G by C and $\mathbb{E}\xi$ is an extension of A by G, i.e., $\eta\mathbb{E}$ represents an element of $\mathrm{Ext}_R(C,G)$ and $\mathbb{E}\xi$ represents an element of $\mathrm{Ext}_R(G,A)$. In this way we obtain the maps

$$E^* : \mathrm{Hom}(A,G) \to \mathrm{Ext}_R(C,G)$$

$$E_* : \mathrm{Hom}(G,C) \to \mathrm{Ext}_R(G,A)$$

defined as

$$E^* : \eta \mapsto \eta\mathbb{E} \text{ and } E_* : \xi \mapsto \mathbb{E}\xi.$$

From 2.3 we can show that E^* and E_* are homomorphisms. If $\phi : G \to H$ is any homomorphism, as we have $(\phi\eta)\mathbb{E} \equiv \phi(\eta\mathbb{E})$ and $\mathbb{E}(\xi\phi) \equiv (\mathbb{E}\xi)\phi$, the diagrams

$$\begin{tikzcd} \mathrm{Hom}(A,G) \arrow[r] \arrow[d] & \mathrm{Ext}_R(C,G) \arrow[d] \\ \mathrm{Hom}(A,H) \arrow[r] & \mathrm{Ext}_R(C,H) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(H,C) \arrow[r] \arrow[d] & \mathrm{Ext}_R(H,A) \arrow[d] \\ \mathrm{Hom}(G,C) \arrow[r] & \mathrm{Ext}_R(G,A) \end{tikzcd}$$

with the obvious maps commute. E^* and E_* are called *connecting homomorphisms* for the short exact sequence 2.4. This terminology is justified in the light of Theorem 2.2.

Lemma 2.1 ((Mac Lane 1995), Proposition 1.7) *Given a diagram*

$$\begin{tikzcd} \mathbb{E} : & 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\xi"] & C \arrow[r] & 0 \\ & & G & & & \end{tikzcd}$$

with exact row, there exists a $\xi : B \to G$ making the triangle commute if and only if $\eta\mathbb{E}$ splits.

Lemma 2.2 ((Mac Lane 1995), Proposition 1.7) *If the diagram*

$$\begin{tikzcd} & & & & G \arrow[dl, dashed, "\xi"'] \arrow[d, "\eta"] & \\ \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

has exact row, then there is a $\xi : G \to B$ such that $\beta\xi = \eta$ if and only if $\mathbb{E}\eta$ splits.

With the aid of these lemmas, we have the following theorem which establishes a close connection between Hom and Ext_R.

Theorem 2.2 ((Mac Lane 1995), Theorem 3.4) *If 2.4 is an exact sequence, then the sequences*

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \longrightarrow$$

$$\xrightarrow{E^*} \mathrm{Ext}_R(C,G) \xrightarrow{\beta^*} \mathrm{Ext}_R(B,G) \xrightarrow{\alpha^*} \mathrm{Ext}_R(A,G) \longrightarrow \cdots,$$

and

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C) \longrightarrow$$

$$\xrightarrow{E_*} \mathrm{Ext}_R(G,A) \xrightarrow{\beta_*} \mathrm{Ext}_R(G,B) \xrightarrow{\alpha_*} \mathrm{Ext}_R(G,C) \longrightarrow \cdots$$

are exact for every module G.

If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0$ is an extension of A by C, and if $\alpha : A \to A$, $\gamma : C \to C$ are endomorphisms of A and C, respectively, then $\alpha\mathbb{E}$ and $\mathbb{E}\gamma$ will be extensions of A by C. The correspondences

$$\alpha_* : \mathbb{E} \mapsto \alpha\mathbb{E} \qquad \text{and} \qquad \gamma^* : \mathbb{E} \mapsto \mathbb{E}\gamma$$

are endomorphisms of $\mathrm{Ext}_R(C,A)$, which are called *induced endomorphisms* of Ext_R. The formulas $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ show that the endomorphism ring of A acts on $\mathrm{Ext}_R(C,A)$ and similarly the dual of the endomorphism ring C operates on $\mathrm{Ext}_R(C,A)$. These commute as is shown by $\alpha_*\gamma^* = \gamma^*\alpha_*$; hence $\mathrm{Ext}_R(C,A)$ is a (unital) bimodule over endomorphism rings of A and C, acting from the left and right, respectively.

2.2. Supplements and Supplemented Modules

This section includes definitions and some results about supplements and supplemented modules. See (Wisbauer 1991) for more information about supplements and supplemented modules.

Let U be a submodule of an R-module M. If there exists a submodule V of M minimal with respect to the property $M = U + V$ then V is called a *supplement* of U in M.

A submodule K of an R-module M is called *superfluous* or *small* in M, written $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$. The following lemma is used frequently while studying supplements.

Lemma 2.3 *V is a supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$.*

The properties of supplements are given in the next proposition.

Proposition 2.2 ((Wisbauer 1991), 41.1) *Let U, $V \subseteq M$ and V be a supplement of U in M.*

1. If $W + V = M$ for some $W \subseteq U$, then V is a supplement of W.

2. If M is finitely generated, then V is also finitely generated.

3. If U is a maximal submodule of M, then V is cyclic and $U \cap V = \operatorname{Rad} V$ is a (the unique) maximal submodule of V.

4. If $K \ll M$, then V is a supplement of $U + K$.

5. If $K \ll M$, then $V \cap K \ll V$ and $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.

6. If $\operatorname{Rad} M \ll M$, then U is contained in a maximal submodule of M.

7. If $L \subseteq U$, $V + L/L$ is a supplement of U/L in M/L.

8. If $\operatorname{Rad} M \ll M$ or $\operatorname{Rad} M \subseteq U$ and $p : M \longrightarrow M/\operatorname{Rad} M$ is the canonical epimorphism, then $M/\operatorname{Rad} M = p(U) \oplus p(V)$.

Let M be a module. If every submodule of M has a supplement in M, then M is called a *supplemented module*. Artinian modules and semisimple modules are examples of supplemented modules. As an example to show that every module need not be supplemented, we can consider the ring $\mathbb{Z}$ of integers as a module over itself.

For the properties of supplemented modules, we have the following proposition.

Proposition 2.3 ((Wisbauer 1991), 41.2) *Let M be an R-module.*

1. Let U and V be submodules of M such that U is supplemented and $U + V$ have a supplement in M, then V has a supplement in M.

2. If $M = M_1 + M_2$ with M_1 and M_2 supplemented, then M is also supplemented.

3. If M is supplemented, then $M/\operatorname{Rad} M$ is semisimple.

2.3. Dedekind Domains

Let R be an integral domain, i.e. a commutative ring without zero divisors, and M be an R-module. The *torsion submodule* of M is defined as the set $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$. If $T(M) = M$, then M is called *torsion*, and if $T(M) = 0$, then M is called *torsion-free*. For a prime ideal $\mathfrak{p}$ of R, the submodule $\{m \in M \mid \mathfrak{p}^n m = 0 \text{ for some } n \geq 1\}$ is called the $\mathfrak{p}$-*primary part* of M. This submodule is indicated by $T_{\mathfrak{p}}(M)$. An R-module M is said to be *bounded* if there exists $0 \neq r \in R$ such that $rM = 0$.

A commutative ring R which is not a field is a *valuation ring*, if its ideals are totally ordered by inclusion. Additionally, if R is an integral domain, it is called a *valuation domain*. A Noetherian valuation domain with unique maximal ideal is said to be a *discrete valuation ring* (DVR for short). If R is a DVR then all its non-zero ideals are: $R > Rp > \cdots > Rp^n > \cdots$ for some $n \in \mathbb{N}$ where Rp is the unique maximal ideal of R.

Let R be an integral domain and K be its field of fractions. An element of K is said to be *integral* over R if it is a root of a monic polynomial in $R[X]$. A commutative domain R is *integrally closed* if the elements of K which are integral over R are exactly the elements of R.

An integral domain R is a *Dedekind domain* if the following conditions hold:

1. R is a Noetherian ring,
2. R is integrally closed in its field of fractions K, and
3. all non-zero prime ideals of R are maximal.

The following lemma is well-known, we include it for completeness.

Lemma 2.4 *Let R be a commutative ring and Ω be the set of all maximal ideals of R. Then for an R-module M,* $\operatorname{Rad} M = \bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}M$.

Proof For a maximal ideal $\mathfrak{p}$, we can consider $M/\mathfrak{p}M$ as a module over $R/\mathfrak{p}$, so $M/\mathfrak{p}M$ is semisimple and therefore $\operatorname{Rad} M \subseteq \mathfrak{p}M$. Then we obtain $\operatorname{Rad} M \subseteq \bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M$. Conversely, let $x \in M$ be such that $x \notin \operatorname{Rad} M$. Then there is a maximal submodule K in M such that $x \notin K$. M/K is a simple module, so $\mathfrak{q}M \subseteq K$ for some $\mathfrak{q} \in \Omega$. then we obtain $x \notin \mathfrak{q}M$, hence $x \notin \bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M$. Contradiction. □

Theorem 2.3 ((Cohn 2002), Propositions 10.5.1, 4, 6) *For a commutative domain R, the following are equivalent.*

(i) R is a Dedekind domain.

(ii) Every ideal of R is projective.

(iii) R is Noetherian and the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$ is a DVR for all maximal ideals $\mathfrak{p}$ of R.

(iv) Every ideal of R can be expressed uniquely as a finite product of prime ideals.

Proposition 2.4 ((Sharpe and Vamos 1972), Proposition 2.10) *Every divisible module over a Dedekind domain is injective.*

Over a Dedekind domain R, by the use of Proposition 2.4 together with Lemma 2.4 we have that the conditions for an R-module M being divisible, injective and radical, i.e. $\operatorname{Rad} M = M$, are equivalent. For torsion R-modules, we have the following important result.

Proposition 2.5 ((Cohn 2002), Proposition 10.6.9) *Any torsion module M over a Dedekind domain is a direct sum of its primary parts, in a unique way:*

$$M = \oplus T_{\mathfrak{p}}(M)$$

and when M is finitely generated, only finitely many terms on the right are different from zero.

For more information about Dedekind domains and modules over a Dedekind domain see (Hazewinkel, Gubareni and Kirichenko 2004) and (Sharpe and Vamos 1972).

CHAPTER 3

PROPER CLASSES

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3.1}$$

belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is said to be a *$\mathcal{P}$-epimorphism* (both are said to be *$\mathcal{P}$-proper* and the short exact sequence is said to be a *$\mathcal{P}$-proper* short exact sequence.). The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions ((Buchsbaum 1959), (Mac Lane 1995), (Sklyarenko 1978)):

P-1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

P-3′) The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

P-4′) If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

An important example for proper classes in abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$: The proper class of all short exact sequences (3.1) of abelian groups and abelian group homomorphisms such that $\mathrm{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see (Fuchs 1970) for the important notion of purity in abelian groups). The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ are called *pure-exact sequences* of abelian groups. The corresponding subgroup of $\mathrm{Ext}(C, A)$ is denoted by $\mathrm{Pext}(C, A)$. The following Theorem gives the structure of $\mathrm{Pext}(C, A)$ in terms of subgroups of $\mathrm{Ext}(C, A)$.

Theorem 3.1 ((Fuchs 1970), Theorem 53.3) *For every abelian groups A, C, $\mathrm{Pext}(C, A)$ coincides with the first Ulm subgroup of $\mathrm{Ext}(C, A)$, i.e.*

$$\mathrm{Pext}(C, A) = Ext(C, A)^1 = \bigcap_{n \in \mathbb{Z}^+} n\,\mathrm{Ext}(C, A).$$

The smallest proper class of R-modules consists of only *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit_{R\text{-}\mathcal{M}od}$. The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$ (*absolute purity*).

Another example is constructed by using the change of rings: Let $f : R \longrightarrow S$ be a homomorphism of rings. Then every S-module M can be made an R-module by $rm = f(r)m$, $\forall m \in M, r \in R$. Let $\mathcal{F} = \{\, \mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \mid \mathbb{E}$ is splitting as a sequence of R-modules $\}$. Then $\mathcal{F}$ is a proper class.

A subfunctor $\mathcal{F}$ of Ext^1_R such that $\mathcal{F}(C, A)$ is a subgroup of $\mathrm{Ext}^1_R(C, A)$ is called an *e-functor* (see (Butler and Horrocks 1961)). By (Nunke 1963, Theorem 1.1), an *e-functor* $\mathcal{F}$ of Ext^1_R gives a proper class if it satisfies one of the properties *P-3)* and *P-3')*. This result enables us to define a proper class in terms of subfunctors of Ext^1_R.

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism.

Let $T(M, \cdot) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ be an additive functor (covariant or contravariant), left or right exact and depending on an R-module M from $R\text{-}\mathcal{M}od$. If $\mathcal{M}$ is a given class of modules of this category, we denote by $t^{-1}(\mathcal{M})$ the class $\mathcal{P}$ of short exact sequences $\mathbb{E}$ such that $T(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$.

Lemma 3.1 *$\mathcal{P} = t^{-1}(\mathcal{M})$ is a proper class.*

Proof For example, suppose that T is covariant and right exact. Let $\mathbb{E} : \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ and $\mathbb{E}' : \quad 0 \longrightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \longrightarrow 0$ be isomorphic triples, i.e. there is an isomorphism $\alpha : B \longrightarrow B'$. Since T is right exact and $T(M, \mathbb{E})$ is exact we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & T(M,A) \arrow[r, "T(M,f)"] \arrow[d, equal] & T(M,B) \arrow[r, "T(M,g)"] \arrow[d, "T(M,\alpha)"] & T(M,C) \arrow[r] \arrow[d, equal] & 0 \\
 & T(M,A) \arrow[r, "T(M,f')"] & T(M,B') \arrow[r, "T(M,g')"] & T(M,C) \arrow[r] & 0
\end{tikzcd}$$

$T(M, f') = T(M, \alpha \circ f) = T(M, \alpha) \circ T(M, f)$ and $T(M, \alpha)$ is an isomorphism, as α is an isomorphism. Then $T(M, f')$ is a monomorphism, i.e. the second row is exact. Hence $\mathbb{E}' \in \mathcal{P}$.

If $\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0$ is a splitting short exact sequence, then there exist $\mu' : B \longrightarrow A$ and $\nu' : B \longrightarrow C$ such that $\mu' \circ \mu = 1_A$, $\nu \circ \nu' = 1_C$. Then we have $T(M, \mu') \circ T(M, \mu) = T(M, \mu' \circ \mu) = T(M, 1_A) = 1_{T(M,A)}$ and $T(M, \nu) \circ T(M, \nu') = T(M, \nu \circ \nu') = T(M, 1_C) = 1_{T(M,C)}$, i.e. $T(M, \mathbb{E})$ is exact.

Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be $\mathcal{P}$-monomorphisms. Then $T(M, \alpha)$ and $T(M, \beta)$ are monomorphisms and $T(M, \beta \circ \alpha) = T(M, \beta) \circ T(M, \alpha)$ is a monomorphism. So $\beta \circ \alpha$ is a $\mathcal{P}$-monomorphism.

Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be monomorphisms and $\beta \circ \alpha$ be a $\mathcal{P}$-monomorphism. Then we have the diagram

$$\begin{tikzcd}
 & T(M,A) \arrow[r, "T(M,\alpha)"] \arrow[d, equal] & T(M,B) \arrow[d, "T(M,\beta)"] \\
0 \arrow[r] & T(M,A) \arrow[r, "T(M,\beta\circ\alpha)"] & T(M,C)
\end{tikzcd}$$

If $x \in \operatorname{Ker} T(M, \alpha)$, then $T(M, \beta \circ \alpha)(x) = T(M, \beta) \circ T(M, \alpha)(x) = 0$, so $x \in \operatorname{Ker} T(M, \beta \circ \alpha) = 0$, i.e. α is a $\mathcal{P}$-monomorphism.

If $h : B \longrightarrow C$ and $g : C \longrightarrow D$ are epimorphisms and $A' = \operatorname{Ker} g \circ h$, then the mapping of derived functors $T_1(M, B) \longrightarrow T_1(M, C) \longrightarrow T_1(M, D)$ is epimorphic, therefore, $T(M, A') \longrightarrow T(M, B)$ is a monomorphism and $g \circ h \in \mathcal{P}$.

Let $\mu : B \longrightarrow C$ and $\nu : C \longrightarrow D$ be epimorphisms and $\nu \circ \mu$ be a $\mathcal{P}$-epimorphism. Then we have the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "h"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A \arrow[r, "w"] & B \arrow[r, "\mu"] \arrow[d, "\nu\circ\mu"] & C \arrow[r] \arrow[d, "\nu"] & 0 \\
 & & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where h, u, f and w are R-module homomorphisms. Applying the functor $T(M,.)$ to this diagram, we see that the second column of the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
T(M,A) \arrow[r, "T(M,h)"] \arrow[d, equal] & T(M,X) \arrow[r, "T(M,u)"] \arrow[d, "T(M,g)"] & T(M,N) \arrow[r] \arrow[d, "T(M,f)"] & 0 \\
T(M,A) \arrow[r, "T(M,w)"] & T(M,B) \arrow[r, "T(M,\mu)"] \arrow[d, "T(M,\nu\circ\mu)"] & T(M,C) \arrow[r] \arrow[d, "T(M,\nu)"] & 0 \\
 & T(M,D) \arrow[r, equal] \arrow[d] & T(M,D) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

is exact, since $\nu\circ\mu$ is a $\mathcal{P}$-epimorphism. In order to show that ν is a $\mathcal{P}$-epimorphism, we have to show that $T(M,f)$ is a monomorphism. Let $n \in \operatorname{Ker} T(M,f)$. $n = T(M,u)(x)$ for some $x \in T(M,X)$ since $T(M,u)$ is an epimorphism. $(T(M,\mu)\circ T(M,g))(x) = (T(M,f)\circ T(M,u)(x) = 0$. Then $T(M,g)(x) \in \operatorname{Ker} T(M,\mu) = \operatorname{Im} T(M,w)$, i.e. $T(M,g)(x) = T(M,w)(a)$ for some $a \in T(M,A)$. $T(M,g)(x) = T(M,w)(a) = (T(M,g)\circ T(M,h))(a) \Rightarrow x - T(M,h)(a) \in \operatorname{Ker} T(M,g) = 0$. Then $n = T(M,u)(x) = T(M,u)(T(M,h)(a)) = (T(M,u)\circ T(M,h))(a) = 0$. So $\operatorname{Ker} T(M,f) = 0$ and ν is a $\mathcal{P}$-epimorphism. □

Let $t(\mathcal{P})$ be the class of all modules M for which the triples $T(M,\mathbb{E})$, $\mathbb{E}\in\mathcal{P}$, are exact. As we can take the functors Hom or $\otimes$ for T, $t(\mathcal{P})$ and $t^{-1}(\mathcal{P})$ leads us to projectively, injectively or flatly generated proper classes.

For a proper class $\mathcal{P}$ over an integral domain R, we denote by $\hat{\mathcal{P}}$ the class of the short exact sequences $\mathbb{E}: \quad 0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ of R-modules such that $r\mathbb{E} \in \mathcal{P}$ for some $0 \neq r \in R$ where r also denotes the multiplication homomorphism by $r \in R$. Thus

$$\hat{\mathcal{P}} = \{\mathbb{E} \mid r\mathbb{E} \in \mathcal{P} \text{ for some } 0 \neq r \in R\}.$$

In case of abelian groups the class $\hat{\mathcal{P}}$ is studied in (Walker 1964), (Alizade 1986) and (Alizade, Pancar and Sezen 2004) for $\mathcal{P} = \mathcal{S}plit$ where it was denoted by *Text* since $\mathrm{Ext}^1_{\hat{\mathcal{S}plit}}(C, A) = T(\mathrm{Ext}(C, A))$ the torsion part of $\mathrm{Ext}(C, A)$.

Let $\mathcal{E}$ be a class of short exact sequences. The smallest proper class containing $\mathcal{E}$ is said to be *generated by* $\mathcal{E}$ and denoted by $<\mathcal{E}>$ (see (Pancar 1997)).

Since the intersection of any family of proper classes is proper, for a class $\mathcal{E}$ of short exact sequences

$$<\mathcal{E}> = \bigcap \{\mathcal{P} : \mathcal{E} \subseteq \mathcal{P}\,; \mathcal{P} \text{ is a proper class }\}.$$

For more information about proper classes generated by a class of short exact sequences see (Pancar 1997). We will give two results from this paper in the next section.

3.1. Projectives, Injectives, Coprojectives and Coinjectives with respect to a Proper Class

Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the epimorphism* g if any of the following equivalent conditions holds:

1. every diagram

$$\begin{tikzcd} \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ & & & & M \arrow[u, "\gamma"'] \arrow[ul, dashed, "\tilde{\gamma}"] & \end{tikzcd}$$

where the first row is $\mathbb{E}$ and $\gamma : M \longrightarrow C$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

2. The sequence

$$\mathrm{Hom}(M, \mathbb{E}): \quad 0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* f if any of the following equivalent conditions holds:

1. every diagram

$$\begin{tikzcd}
\mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[dl, dashed, "\tilde{\alpha}"] & C \arrow[r] & 0 \\
 & & M & & &
\end{tikzcd}$$

where the first row is $\mathbb{E}$ and $\alpha : A \longrightarrow M$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

2. The sequence

$$\mathrm{Hom}(\mathbb{E}, M): \quad 0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \longrightarrow 0$$

is exact.

An R-module M is said to be *$\mathcal{P}$-projective* [*$\mathcal{P}$-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$. The relative projectiveness [injectiveness] of M is equivalent to the requirement that $\mathrm{Ext}^1_{\mathcal{P}}(M, B) = 0$, for any B [$\mathrm{Ext}^1_{\mathcal{P}}(A, M) = 0$, for any A]. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$].

The Functor $\mathrm{Ext}^1_{\mathcal{P}}$: In a proper class $\mathcal{P}$ in R-$\mathcal{M}od$, there need not be a $\mathcal{P}$-epimorphism from some $\mathcal{P}$-projective module to a given R-module A. So in general, it is not possible to define the functor $\mathrm{Ext}^1_{\mathcal{P}}$ by using the derived functor of the functor Hom. However, the alternative definition of $\mathrm{Ext}^1_{\mathcal{P}}$ may be used in this case.

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ or shortly by $\mathrm{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences in $\mathcal{P}$ which start with A and end with C. This turns out to be a subgroup of $\mathrm{Ext}_R(C, A)$ and a bifunctor $\mathrm{Ext}^1_{\mathcal{P}} : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext^1_R.

A class $\mathcal{P}$ of R-modules is said to have *enough projectives* if for every module A we can find a $\mathcal{P}$-epimorhism from some $\mathcal{P}$-projective module P to A. A class $\mathcal{P}$ of R-modules is said to have *enough injectives* if for every module B we can find a $\mathcal{P}$-monomorphism from B to some $\mathcal{P}$-injective module J. A proper class $\mathcal{P}$ of R-modules with enough projectives [enough injectives] is also said to be a *projective proper class* [resp. *injective proper class*].

The following propositions give the relation between projective (resp. injective) modules with respect to a class $\mathcal{E}$ of short exact sequences and with respect to the proper class $< \mathcal{E} >$ generated by $\mathcal{E}$.

Proposition 3.1 ((Pancar 1997), Propositions 2.3 and 2.4)

(a) $\pi(\mathcal{E}) = \pi(< \mathcal{E} >)$.

(b) $\iota(\mathcal{E}) = \iota(< \mathcal{E} >)$.

An R -module C is said to be *$\mathcal{P}$-coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

ending with C is in the proper class $\mathcal{P}$. An R-module A is said to be *$\mathcal{P}$-coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

starting with A is in the proper class $\mathcal{P}$.

Using the functor $\mathrm{Ext}_{\mathcal{P}}$, the $\mathcal{P}$-projectives, $\mathcal{P}$-injectives, $\mathcal{P}$-coprojectives, $\mathcal{P}$-coinjectives are simply described in terms of the subgroup $\mathrm{Ext}_{\mathcal{P}}(C, A) \leq \mathrm{Ext}_R(C, A)$ being 0 or the whole of $\mathrm{Ext}_R(C, A)$:

1. An R-module C is *$\mathcal{P}$-projective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } A.$$

2. An R-module C is *$\mathcal{P}$-coprojective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } A.$$

3. An R-module A is *$\mathcal{P}$-injective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } C.$$

4. An R-module A is *$\mathcal{P}$-coinjective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } C.$$

Proposition 3.2 ((Misina and Skornjakov 1960), Propositions 1.9 and 1.14)
If in the short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$, modules M and K are $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective), then N is $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective).

Proof Let A be an R-module. Suppose that M and K are $\mathcal{P}$-coprojective. Then $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0 \in \mathcal{P}$. We have the following exact sequences

$$0 \longrightarrow \mathrm{Hom}(K, A) \longrightarrow \mathrm{Hom}(N, A) \longrightarrow \mathrm{Hom}(M, A) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(K, A) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(N, A) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(M, A) \longrightarrow \cdots$$

$$0 \longrightarrow \mathrm{Hom}(K, A) \longrightarrow \mathrm{Hom}(N, A) \longrightarrow \mathrm{Hom}(M, A) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{R}(K, A) \longrightarrow \mathrm{Ext}^1_{R}(N, A) \longrightarrow \mathrm{Ext}^1_{R}(M, A) \longrightarrow \cdots$$

Since M and K are $\mathcal{P}$-coprojective, we have the equalities in the following diagram.

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{P}}(K, A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{P}}(N, A) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{P}}(M, A) \arrow[d, equal] \\
\mathrm{Ext}^1_{R}(K, A) \arrow[r] & \mathrm{Ext}^1_{R}(N, A) \arrow[r] & \mathrm{Ext}^1_{R}(M, A)
\end{tikzcd}$$

Then $\mathrm{Ext}^1_{\mathcal{P}}(N, A) = \mathrm{Ext}^1_R(N, A)$ for every R-module A, which shows that N is $\mathcal{P}$-coprojective.
For the case of $\mathcal{P}$-coinjectives, the proof can be done by using the functor $\mathrm{Hom}(B, \cdot)$ for an R-module B. □

Proposition 3.3 ((Misina and Skornjakov 1960), Proposition 1.12) *An R-module M is $\mathcal{P}$-coprojective if and only if there is a $\mathcal{P}$-epimorphism from a projective R-module P to M.*

Proof ($\Rightarrow$) Take any epimorphism $\gamma : P \longrightarrow M$ from a projective R-module P to M. Since M is $\mathcal{P}$-coprojective, γ is a $\mathcal{P}$-epimorphism.
($\Leftarrow$) Let $\gamma : P \longrightarrow M$ be a $\mathcal{P}$-epimorphism and $K = \operatorname{Ker}\gamma$. Then the short exact sequence $0 \longrightarrow K \longrightarrow P \xrightarrow{\gamma} M \longrightarrow 0$ is in $\mathcal{P}$. For every R-module A, we have the following exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Hom}(K,A) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{P}}(M,A) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{P}}(P,A) & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \| & & \\
\cdots & \longrightarrow & \mathrm{Hom}(K,A) & \longrightarrow & \mathrm{Ext}^1_{R}(M,A) & \longrightarrow & \mathrm{Ext}^1_{R}(P,A) & \longrightarrow & \cdots
\end{array}$$

where the equality $\mathrm{Ext}^1_{\mathcal{P}}(P,A) = \mathrm{Ext}^1_R(P,A)=0$ holds, since P is projective. Then $\mathrm{Ext}^1_{\mathcal{P}}(M,A) = \mathrm{Ext}^1_R(M,A)$, hence M is $\mathcal{P}$-coprojective. □

Corollary 3.1 ((Misina and Skornjakov 1960), Proposition 1.13)
If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in a proper class $\mathcal{P}$ and B is $\mathcal{P}$-coprojective, then C is also $\mathcal{P}$-coprojective.

Dually, for $\mathcal{P}$-coinjective modules we have the following proposition:

Proposition 3.4 ((Misina and Skornjakov 1960), Proposition 1.7)
An R-module N is $\mathcal{P}$-coinjective if and only if there is $\mathcal{P}$-monomorphism from N to an injective module I.

Proof ($\Rightarrow$) Take any monomorphism $\alpha : N \longrightarrow I$ from N to an injective R-module I. Since N is $\mathcal{P}$-coinjective, α is a $\mathcal{P}$-monomorphism.
($\Leftarrow$) Let $\alpha : N \longrightarrow I$ be a $\mathcal{P}$-monomorphism and $L = I/\operatorname{Im}\alpha$. Then the short exact sequence $0 \longrightarrow N \xrightarrow{\alpha} I \longrightarrow L \longrightarrow 0$ is in $\mathcal{P}$. For every R-module B, we have the following exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Hom}(B,L) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{P}}(B,N) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{P}}(B,I) & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \| & & \\
\cdots & \longrightarrow & \mathrm{Hom}(B,L) & \longrightarrow & \mathrm{Ext}^1_{R}(B,N) & \longrightarrow & \mathrm{Ext}^1_{R}(B,I) & \longrightarrow & \cdots
\end{array}$$

where the equality $\mathrm{Ext}^1_{\mathcal{P}}(B,I) = \mathrm{Ext}^1_R(B,I)=0$ holds, since I is injective. Then $\mathrm{Ext}^1_{\mathcal{P}}(B,N) = \mathrm{Ext}^1_R(B,N)$, i.e. N is $\mathcal{P}$-coinjective. □

Corollary 3.2 ((Misina and Skornjakov 1960), Proposition 1.8) *If* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is a short exact sequence in a proper class* $\mathcal{P}$ *and* B *is* $\mathcal{P}$*-coinjective, then* A *is also* $\mathcal{P}$*-coinjective.*

3.2. Projectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \, \mathrm{Hom}(M, \mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective and it is called the proper class *projectively generated* by $\mathcal{M}$.

Proof This is a consequence of Lemma 3.1. Take $T(M, \cdot) = \mathrm{Hom}(M, \cdot)$. □

Proposition 3.5 *Let* $\mathcal{P}$ *be a proper class and* $\mathcal{M}$ *be a class of modules. Then we have*

1. $\mathcal{P} \subseteq \pi^{-1}(\pi(\mathcal{P}))$,
2. $\mathcal{M} \subseteq \pi(\pi^{-1}(\mathcal{M}))$,
3. $\pi(\mathcal{P}) = \pi(\pi^{-1}(\pi(\mathcal{P})))$,
4. $\pi^{-1}(\mathcal{M}) = \pi^{-1}(\pi(\pi^{-1}(\mathcal{M})))$.

For a proper class $\mathcal{P}$, $\pi^{-1}(\pi(\mathcal{P}))$ is called the *projective closure* of $\mathcal{P}$ and it always contains $\mathcal{P}$.

3.3. Injectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \, \mathrm{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective which is called the proper class *injectively generated* by $\mathcal{M}$.

Proof This is a consequence of Lemma 3.1. Take $T(M, \cdot) = \mathrm{Hom}(\cdot, M)$. □

3.4. Flatly Generated Proper Classes

When the ring R is *not* commutative, we must be careful with the sides for the tensor product analogues of projectives and injectives with respect to a proper class. Recall that by an R-module, we mean a *left* R-module.

Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms. We say that a *right* R-module M is *flat with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* g if

$$M \otimes \mathbb{E}: \quad 0 \longrightarrow M \otimes A \xrightarrow{1_M \otimes f} M \otimes B \xrightarrow{1_M \otimes g} M \otimes C \longrightarrow 0$$

is exact.

A *right* R-module M is said to be *$\mathcal{P}$-flat* if M is flat with respect to every short exact sequence $\mathbb{E} \in \mathcal{P}$, that is, $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{P}$.

For a given class $\mathcal{M}$ of *right* R-modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$:

$$\tau^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | M \otimes \mathbb{E} \text{ is exact for all } M \in \mathcal{M}\}.$$

$\tau^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{P}$-flat. It is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules.

3.5. Coprojectively and Coinjectively Generated Proper Classes

Let $\mathcal{M}$ and $\mathcal{J}$ be classes of modules over some ring R. The smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are coprojective (resp. coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. J).

Theorem 3.2 ((Alizade 1985), Theorem 2) *Let L be a class of modules closed under extensions. Consider the class $\mathcal{L}$ of exact triples, defined as:*

$$\mathrm{Ext}_{\mathcal{L}}(C, A) = \bigcup_{I,\alpha} \mathrm{Im}\{\mathrm{Ext}(C, I) \xrightarrow{\alpha_*} \mathrm{Ext}(C, A)\}$$

over all $I \in L$ and all homomorphisms $\alpha : I \longrightarrow A$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\mathrm{Ext}_{\mathcal{L}}(C, A)$, form a proper class and $\mathcal{L}$ coincides with $\underline{k}(L)$.

For more information about coprojectively and coinjectively generated proper classes see (Alizade 1985) and (Alizade 1986).

CHAPTER 4

THE PROPER CLASSES RELATED TO COMPLEMENTS AND SUPPLEMENTS

The proper classes $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ are defined in (Mermut 2004). One can find the properties of these classes and their relationship in the same work and (Clark, et al. 2006). Here we will give definitions and some results that will be useful for our work.

4.1. $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$

The class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.1}$$

in $R\text{-}\mathcal{M}od$ such that $\operatorname{Im} f$ is a complement of some submodule K of B, that is $\operatorname{Im} f \cap K = 0$ and K is maximal with respect to this property.

The class $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ consists of all short exact sequences 4.1 such that every simple R-module is relative projective for it, denoted by

$$\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \pi^{-1}_{R\text{-}\mathcal{M}od}\{S \in R\text{-}\mathcal{M}od \mid S \text{ is simple}\}.$$

The corresponding subset of $\operatorname{Ext}(C, A)$ is denoted by $\operatorname{Next}(C, A)$. Over the ring $\mathbb{Z}$ of integers, we have the following result that gives the structure of $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ in terms of the subgroups of $\operatorname{Ext}(C, A)$.

Corollary 4.1 ((Alizade, Pancar and Sezen 2004), Corollary 4.3)
For every abelian groups A, C, we have $\operatorname{Next}(C, A) = \bigcap_p p \operatorname{Ext}(C, A) = F(\operatorname{Ext}(C, A))$ *where p ranges over the prime numbers and $F(\operatorname{Ext}(C, A))$ is the Frattini subgroup of* $\operatorname{Ext}(C, A)$.

The class $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, consisting of all short exact sequences 4.1 such that $\operatorname{Im} f$ is a supplement of some submodule K of B, is a proper class (see (Erdoğan 2004) or (Clark, et al. 2006) for a proof). The properties of $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$*-coinjective* and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$*-coprojective* modules are investigated in (Erdoğan 2004).

Dual to the notion of $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$, $Co\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is defined as

$$Co\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}_{R\text{-}\mathcal{M}od}\{M \in R\text{-}\mathcal{M}od \mid \operatorname{Rad} M = 0\}.$$

We have the relations, $Compl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq Co\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ for arbitrary ring R.

4.2. The κ-Elements of $\operatorname{Ext}(C, A)$

For the rest of this chapter, we will write Ext instead of Ext_R. A short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.2}$$

is called *κ-exact* if $\operatorname{Im} f$ has a supplement in B, i.e. a minimal element in the set $\{V \subset B \mid V + \operatorname{Im} f = B\}$. In this case we say that $\mathbb{E} \in \operatorname{Ext}(C, A)$ is a *κ-element* and the set of all κ-elements of $\operatorname{Ext}(C, A)$ will be denoted by $\mathcal{S}$.

We denote by $\mathcal{W}supp$ the class of short exact sequences 4.2., where $\operatorname{Im} f$ has (is) a weak supplement in B, i.e. there is a submodule K of B such that $\operatorname{Im} f + K = B$ and $\operatorname{Im} f \cap K \ll B$. We denote by $\mathcal{S}mall$ the class of short exact sequences 4.2. where $\operatorname{Im} f \ll B$.

The κ-elements need not form a proper class in general. For instance, let $R = \mathbb{Z}$ and consider the composition $\beta \circ \alpha$ of the monomorphisms $\alpha : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ and $\beta : \mathbb{Z} \longrightarrow \mathbb{Q}$ where α and β are the corresponding inclusions. Then we have $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\beta\circ\alpha} \mathbb{Q} \longrightarrow \mathbb{Q}/2\mathbb{Z} \longrightarrow 0$ is a κ-element, but $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is not a κ-element as $2\mathbb{Z}$ does not have a supplement in $\mathbb{Z}$.

If X is a $\mathcal{S}mall$-submodule of an R-module Y, then Y is a supplement of X in Y, so X is $\mathcal{S}$-submodule of Y. If U is a $\mathcal{S}$-submodule of an R-module Z, then a supplement V of U in Z is also a weak supplement, therefore U is a $\mathcal{W}supp$-submodule of Z. These arguments give us the relation $\mathcal{S}mall \subseteq \mathcal{S} \subseteq \mathcal{W}supp$ for any ring R.

For the following proposition, recall that for a class $\mathcal{E}$ of short exact sequences $< \mathcal{E} >$ denotes the proper class generated by $\mathcal{E}$.

Proposition 4.1 $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{W}supp \rangle$.

Proof We have the relation $\mathcal{S}mall \subseteq \mathcal{S} \subseteq \mathcal{W}supp$ which implies $\langle \mathcal{S}mall \rangle \subseteq \langle \mathcal{S} \rangle \subseteq \langle \mathcal{W}supp \rangle$. Conversely, let $\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \in \mathcal{W}supp$. We can assume that A is a submodule of B. Let D be a weak supplement of A in B, i.e. $A + D = B$ and $A \cap D \ll B$. Then we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \cap D \arrow[r, equal] \arrow[d, "x"'] & A \cap D \arrow[d, "y"'] & & \\
\mathbb{E}: 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"'] & C \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}': 0 \arrow[r] & A/A \cap D \arrow[r, "h"] \arrow[d] & B/A \cap D \arrow[r, "j"] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where x, y are the corresponding inclusion homomorphisms and u, v are canonical epimorphisms. We have $A/A \cap D \oplus D/A \cap D = B/A \cap D$, therefore $\mathbb{E}' \in \mathcal{S}plit \subseteq \langle \mathcal{S}mall \rangle$. Since $A \cap D \ll B$, v and j are $\langle \mathcal{S}mall \rangle$ -epimorphisms. Then $g = j \circ v$ is a $\langle \mathcal{S}mall \rangle$ -epimorphism and $\mathbb{E} \in \langle \mathcal{S}mall \rangle$. Since $\langle \mathcal{S}mall \rangle$ is a proper class, we have that $\langle \mathcal{W}supp \rangle \subseteq \langle \mathcal{S}mall \rangle$. □

Proposition 4.2 *Let R be a domain. Then every bounded R-module is $\langle \mathcal{S}mall \rangle$-coinjective.*

Proof Let B be a bounded R-module and I be an injective hull of B such that $B \subset I$. We will show that $B \ll I$. Let $B + X = I$ for some $X \subset I$. Since B is bounded, there exists $0 \neq r \in R$ such that $rB = 0$. Then $I = rI = rB + rX = rX$, since I is divisible. Therefore $X = I$ and $B \ll I$. I is $\langle \mathcal{S}mall \rangle$-coinjective, since it is injective. Then B is $\langle \mathcal{S}mall \rangle$-coinjective by Corollary 3.2. □

Corollary 4.2 *If R is a domain, then $\underline{k}(\mathcal{B}) \subseteq \langle \mathcal{S}mall \rangle$.*

The main problem with the investigation of the κ-elements in $\mathrm{Ext}(C, A)$ is that they need not form a subgroup. The reason for this is the fact that, in general, for a homomorphism $g : C' \longrightarrow C$, the induced map $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ need not preserve κ-elements.

Let us consider the short exact sequences $\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ in which $V + \operatorname{Im}\alpha = B$ for some $V \subset B$, where $V \cap \operatorname{Im}\alpha \ll V$ and $V \cap \operatorname{Im}\alpha$ is bounded, i.e. V is a supplement of $\operatorname{Im}\alpha$ in B with $V \cap \operatorname{Im}\alpha$ is bounded. Following Zöschinger we will call such sequences β-exact and denote $\operatorname{Im}\alpha \subset^{\beta} B$. In this case we say that $\mathbb{E} \in \operatorname{Ext}(C, A)$ is a *β-element*. Over a Dedekind domain, any β-element of $\operatorname{Ext}_R(C, A)$ is a κ-element as well as a torsion element. Let us denote the β-elements of $\operatorname{Ext}_R(C, A)$ by $\mathcal{SB}$. In order to show that every κ-element need not be a β-element, we give an example over $R = \mathbb{Z}$.

Example 4.1 *Consider the inclusion homomorphism $f : \bigoplus_p \mathbb{Z}_p \longrightarrow \bigoplus_p \mathbb{Z}_{p^\infty}$ where p ranges over all prime numbers in $\mathbb{Z}$.* $\operatorname{Im} f = \bigoplus_p \mathbb{Z}_p$ *is small in $\bigoplus_p \mathbb{Z}_{p^\infty}$, so f is a $\mathcal{S}$-monomorphism. $\bigoplus_p \mathbb{Z}_{p^\infty}$ itself is the only supplement of* $\operatorname{Im} f$ *in $\bigoplus_p \mathbb{Z}_{p^\infty}$.* $\operatorname{Im} f = \bigoplus_p \mathbb{Z}_p$ *is not bounded, hence f is not an $\mathcal{SB}$-monomorphism.*

The following proposition holds for a noetherian integral domain of Krull dimension 1. Recall that $\mathcal{B}$ denotes the class of bounded R-modules.

Proposition 4.3 *Let R be a noetherian integral domain of Krull dimension 1. Then $\mathcal{SB} = \underline{k}(\mathcal{B})$. Hence $\mathcal{SB}$ is a proper class in this case.*

Proof ($\subseteq$) Let $\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be a short exact sequence in $\mathcal{SB}$. We can assume that α is the inclusion, i.e. $A \subseteq B$. Then there is a supplement V of A in B such that $V \cap A$ is bounded. We have the following commutative diagram

$$\begin{array}{cccccccccc}
\mathbb{E}: & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| & & \\
\pi\mathbb{E}: & 0 & \longrightarrow & A/(A \cap V) & \longrightarrow & B/(A \cap V) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

where the second row splits, since it is equivalent with the splitting short exact sequence $0 \longrightarrow A/(A \cap V) \longrightarrow A/(A \cap V) \oplus V/(A \cap V) \longrightarrow V/(A \cap V) \longrightarrow 0$. If we apply the functor $\operatorname{Hom}_R(C, \cdot)$ to the short exact sequence $0 \longrightarrow A \cap V \xrightarrow{\iota} A \xrightarrow{\pi} (A/A \cap V) \longrightarrow 0$ where ι is the inclusion homomorphism and π is the canonical epimorphism, we get the sequence

$$\cdots \longrightarrow \operatorname{Ext}_R(C, A \cap V) \xrightarrow{\iota^*} \operatorname{Ext}_R(C, A) \xrightarrow{\pi^*} \operatorname{Ext}_R(C, A/A \cap V) \longrightarrow \cdots$$

and $\pi^*(\mathbb{E}) = 0$ by the previous argument. Then $\mathbb{E} \in \operatorname{Ker} \pi^* = \operatorname{Im} \iota^*$, so there is an $\mathbb{E}' \in \operatorname{Ext}(C, A \cap V)$ such that $\iota^*(\mathbb{E}') = \mathbb{E}$. Since $A \cap V$ is bounded and $\underline{k}(\mathcal{B})$ is a proper class, $\mathbb{E} = \iota^*(\mathbb{E}')$ is an element of $\underline{k}(\mathcal{B})$. Hence $\mathcal{SB} \subseteq \underline{k}(\mathcal{B})$.

($\supseteq$) By (Zöschinger 1974b, Folgerung after Lemma 1.4) over a noetherian integral domain of Krull dimension 1, every bounded R-module M is $\mathcal{S}$-coinjective. As M is bounded, it is $\mathcal{SB}$-coinjective.

Let $\mathbb{E}: \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \underline{k}(\mathcal{B})$. Using Theorem 3.2, there exist $I \in \mathcal{B}$ and a homomorphism $\alpha : I \longrightarrow A$ such that $\alpha^*(\mathbb{E}') = \mathbb{E}$ for some $\mathbb{E}': \quad 0 \longrightarrow I \longrightarrow X \longrightarrow C \longrightarrow 0 \in \operatorname{Ext}(C, I)$. Then the diagram

$$\begin{tikzcd}
\mathbb{E}' : 0 \arrow[r] & I \arrow[r] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E} : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative. I is $\mathcal{SB}$-coinjective, since $I \in \mathcal{B}$. Therefore $\mathbb{E}' \in \mathcal{SB}$. By (Zöschinger 1978, Folgerung (b) after Lemma 1.3), α_* preserves β-elements. Then $\mathbb{E} = \alpha_*(\mathbb{E}') \in \mathcal{SB}$. □

Corollary 4.3 *Over a Dedekind domain R, we have $\mathcal{SB} = \underline{k}(\mathcal{B})$, therefore $\mathcal{SB}$ is a proper class.*

Let R denote the ring $\mathbb{Z}$ of integers till the end of this section.

A homomorphism $g : C' \longrightarrow C$ is called *coneat* if for every decomposition $g = \beta \circ \alpha$ where β is a small epimorphism, β is an isomorphism.

The following results establish a connection between coneat homomorphisms and the κ-elements of $\operatorname{Ext}(C, A)$.

Lemma 4.1 ((Zöschinger 1978), Lemma 2.2)

(a) An epimorphism $g : C' \longrightarrow C$ is coneat if and only if $\operatorname{Ker} g$ *is coclosed in C', i.e. for any submodule X of* $\operatorname{Ker} g$, $\operatorname{Ker} g/X \ll C'/X$ *implies* $X = \operatorname{Ker} g$.

(b) A splitting monomorphism $g : C' \longrightarrow C$ is coneat if and only if $\operatorname{Coker} g$ *has no small cover.*

(c) If $g = g_2 \circ g_1$ is coneat, then g_2 is also coneat.

Theorem 4.1 ((Zöschinger 1978), Satz 2.3) *For a homomorphism $g : C' \longrightarrow C$, the following are equivalent:*

(i) g *is coneat.*

(ii) $\operatorname{Ker} g$ *is coclosed in* C' *and* $\operatorname{Im} g \supset \operatorname{Soc} C$*.*

(iii) $g(C'[p]) = C[p]$ *for all prime numbers* p*.*

(iv) *If the diagram below is a pullback diagram and* β *is a small epimorphism, then* β' *is also a small epimorphism.*

$$\begin{tikzcd} B' \arrow[r, "\beta'"] \arrow[d, "g'"'] & C' \arrow[d, "g"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

Corollary 4.4 ((Zöschinger 1978), Folgerung 1 after Satz 2.3) *If* $g : C' \longrightarrow C$ *is coneat, then* $g^* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C', A)$ *preserves* κ*-elements.*

Corollary 4.5 ((Zöschinger 1978), Folgerung 2 after Satz 2.3) $g^* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C', A)$ *preserves* κ*-elements if* g *satisfies the following two conditions:*

(a) $\operatorname{Im} g \supset \operatorname{Soc}(C)$.

(b) $\operatorname{Ker} g$ *is supplemented and has a supplement in every extension.*

We can find an answer to the question if κ-elements of $\operatorname{Ext}(C, A)$ form a subgroup of $\operatorname{Ext}(C, A)$, in terms of C and A. The following results give an answer under some conditions on C and A. Note that the following two results for abelian groups can be generalized for modules over Dedekind domains.

Lemma 4.2 ((Zöschinger 1978), Lemma 2.1) *Let* A, A', C *and* C' *be* R*-modules.*

(I) *If* $f : A \longrightarrow A'$*, then* $f_* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C, A')$ *preserves* κ*-elements.*

(II) *Let* $g : C' \longrightarrow C$ *and* C' *be torsion. If either a primary component of* C *is zero or* A *is torsion, then* $g^* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C', A)$ *preserves* κ*-elements.*

Corollary 4.6 ((Zöschinger 1978), Folgerung 3 after Lemma 2.1) *If* C *is torsion, and either a primary component of* C *is zero or* A *is torsion, then the* κ*-elements of* $\operatorname{Ext}(C, A)$ *form a subgroup.*

4.3. The κ-Elements of $\mathrm{Ext}_R(C, A)$ over the Category $\mathcal{T}_R$ for a Dedekind Domain R

In this section, R denotes a Dedekind domain which is not a field and K denotes its field of fractions, we will denote the set of maximal ideals of R by Ω. Let $\mathcal{T}_R$ be the category of torsion R-modules. Consider the short exact sequences $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R-modules A, B, C and R-module homomorphisms f and g where $A, B, C \in \mathcal{T}_R$. From now on, we will consider the short exact sequences in the form given above.

By Corollary 4.6, κ-elements of $\mathrm{Ext}(C, A)$ in $\mathcal{T}_R$ form a subgroup. It is also a subfunctor of Ext by Lemma 4.2, so it is an e-functor.

In order to show that κ-elements form a proper class, we need the transitivity of the relation κ. The following lemma is proved when $R = \mathbb{Z}$ in (Zöschinger 1978), note that it holds for all R-modules, but we will use it only for modules in $\mathcal{T}_R$.

Lemma 4.3 ((Zöschinger 1978), Lemma 6.6) *Let $X \subset Y \subset Z$ be R-modules, V be a supplement of X in Y, and W be a supplement of Y in Z. Then we have:*

(a) If $\mathrm{Rad}(Y/X) = Y/X \cap \mathrm{Rad}(Z/X)$, then $V + W$ is a supplement of X in Z.

(b) X has a supplement in Z.

Proof (a) The condition on the radical implies that $(X + (W \cap Y)/X) = ((W + X) \cap Y)/X$ is small in Y/X. Then the canonical map $V \longrightarrow Y/X \longrightarrow Z/(W + X)$ is a small epimorphism. Therefore V is a supplement of $W + X$ in Z. We also have that W is a supplement of $V + X$ in Z. Hence $V + W$ is a supplement of X in Z.
(b) The R-module $((W + X) \cap Y)/X$ is small in Z/X, so it is coatomic. It has a supplement Y'/X in the torsion module Y/X such that $(W + X)/X$ and Y'/X are mutual supplements in Z/X. Then we have that Y' has a supplement in Z, $\mathrm{Rad}(Y'/X) = Y'/X \cap \mathrm{Rad}(Z/X)$ and $(V \cap Y') + X = Y'$. Therefore X has a supplement in Y'. By using the same argument in part (a), X has a supplement in Z. □

We can see that $\mathrm{Ext}_{\mathcal{S}}$ is an e-functor by Lemma 4.2 and Corollary 4.6 in the category $\mathcal{T}_R$. Lemma 4.3 also holds for modules in $\mathcal{T}_R$. Then $\mathcal{S}$ gives an *e-functor* and P-3 in the definition of a proper class is satisfied in the category $\mathcal{T}_R$, we have by (Nunke 1963, Theorem 1.1), that $\mathcal{S}$ form a proper class in the category $\mathcal{T}_R$.

Our next aim is to find the $\mathcal{S}$-injective and $\mathcal{S}$-projective R-modules in $\mathcal{T}_R$.

Proposition 4.4 *In the category $\mathcal{T}_R$, we have:*

(a) κ-elements of $\mathrm{Ext}(C, A)$ *form a proper class.*

(b) $\pi(\mathcal{W}supp) = \pi(\mathcal{S}) = \pi(\mathcal{S}mall) = \{0\}$.

(c) $\mathcal{S}$-injective R-modules are only the injective R-modules in $\mathcal{T}_R$.

Proof (a) Follows from the previous arguments.
(b) We always have the relation $\mathcal{W}supp \supseteq \mathcal{S} \supseteq \mathcal{S}mall$ which implies $\pi(\mathcal{W}supp) \subseteq \pi(\mathcal{S}) \subseteq \pi(\mathcal{S}mall)$.
Assume that there is a nonzero element P in $\pi(\mathcal{S}mall)$. $P = \bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(P)$ where $T_{\mathfrak{q}}(P) \neq 0$ for some $\mathfrak{q} \in \Omega$, since $P \neq 0$. Then there is a simple submodule $M \leq T_{\mathfrak{q}}(P)$, clearly $M \cong R/\mathfrak{q} \cong \mathfrak{q}^{-1}/R$ (see (Nunke 1959, Lemma 4.4) for the last isomorphism), and there is a nonzero homomorphism $\alpha : M \longrightarrow T_{\mathfrak{q}}(K/R)$. Since $T_{\mathfrak{q}}(K/R)$ is injective, there is a homomorphism $h : P \longrightarrow T_{\mathfrak{q}}(K/R)$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\beta"] \arrow[d, "\alpha"'] & P \arrow[dl, dashed, "h"] \\ & T_{\mathfrak{q}}(K/R) & \end{tikzcd}$$

commutative. Since $\alpha \neq 0$, we have $h \neq 0$.
The short exact sequence $0 \longrightarrow \mathfrak{q}^{-1}/R \xrightarrow{j} T_{\mathfrak{q}}(K/R) \xrightarrow{f} T_{\mathfrak{q}}(K/R) \longrightarrow 0$ where j and f are canonical homomorphisms, is in $\mathcal{S}mall$ (see (Wisbauer 1991, Ch. 8, §40.3, (4))). Since P is an element of $\pi(\mathcal{S}mall)$, there is a homomorphism $g : P \longrightarrow T_{\mathfrak{q}}(K/R)$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{q}^{-1}/R \arrow[r, "j"] & T_{\mathfrak{q}}(K/R) \arrow[r, "f"] & T_{\mathfrak{q}}(K/R) \arrow[r] & 0 \\ & & & P \arrow[u, "h"'] \arrow[ul, dashed, "g"] & \end{tikzcd}$$

commute, i.e. $h = f \circ g$, which implies $h|_M = (f \circ g)|_M$. Since $g(M)$ is simple in $T_{\mathfrak{q}}(K/R)$, $g(M) \cong \mathfrak{q}^{-1}/R$ or $g(M) = 0$. In both cases, we have $0 \neq \alpha(M) = h|_M = (f \circ g)|_M = 0$. This contradicts with $h = f \circ g$.

(c) We always have $\iota(\mathcal{W}supp) \subseteq \iota(\mathcal{S}) \subseteq \iota(\mathcal{S}mall)$, and we know that all these classes include injective R-modules. We will show that the $\mathcal{S}mall$-injective R-modules in $\mathcal{T}_R$ are only the injective ones.

Let I be a $\mathcal{S}mall$-injective R-module. Then $I = D(I) \oplus I'$, where $D(I)$ is the divisible part of I and I' is reduced. Since I' is a direct summand of a $\mathcal{S}mall$-injective R-module, I' is $\mathcal{S}mall$-injective.

Suppose that $I' \neq 0$. With similar arguments we used in part (b), there is a nonzero monomorphism $\gamma : \mathfrak{q}^{-1}/R \longrightarrow I'$ for some $\mathfrak{q} \in \Omega$. If we take the same short exact sequence we used in part (b), we get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{q}^{-1}/R \arrow[r, "j"] \arrow[d, "\gamma"'] & T_{\mathfrak{q}}(K/R) \arrow[r, "f"] \arrow[dl, dashed, "h"] & T_{\mathfrak{q}}(K/R) \arrow[r] & 0 \\ & I' & & & \end{tikzcd}$$

where the existence of h is guaranteed by the assumption of I being $\mathcal{S}mall$-injective.Then we have $\gamma = h \circ j$ and $0 \neq \gamma(\mathfrak{q}^{-1}/R) = (h \circ j)(\mathfrak{q}^{-1}/R) \subseteq h(T_{\mathfrak{q}}(K/R)) \subseteq D(I') = 0$, where $D(I') = 0$ as I' is reduced. This is a contradiction.

So $\mathcal{S}mall$-injective R-modules in $\mathcal{T}_R$ are only the injective modules in $\mathcal{T}_R$.

□

Corollary 4.7 *In the category $\mathcal{T}_{\mathbb{Z}}$ of torsion abelian groups we have:*

(a) *κ-elements of* $\mathrm{Ext}(C, A)$ *form a proper class.*

(b) $\pi(\mathcal{W}supp) = \pi(\mathcal{S}) = \pi(\mathcal{S}mall) = \{0\}$.

(c) *$\mathcal{S}$-injective abelian groups are only the injective abelian groups.*

In order to find the form of κ-coinjective R-modules in the category $\mathcal{T}_R$, we need the following lemmas.

Lemma 4.4 *Let A, B be R-modules and $A \subseteq B$. Then $A \ll B$ if and only if A is coatomic and $A \subseteq \mathrm{Rad}\, B$.*

Proof ($\Rightarrow$) Suppose that $\mathrm{Rad}(A/X) = A/X$ for some $X \subseteq A$. Then A/X is divisible, so A/X is a direct summand of B/X. Since A/X is an epimorphic image of A in B, $A/X \ll B/X$ which implies $A/X = 0$.

($\Leftarrow$) Suppose that $A + Y = B$ for some $Y \subsetneq B$. Then $A/A \cap Y \cong A + Y/Y = B/Y$ is also coatomic, so there is a maximal submodule Z of B containing Y and we have $\mathrm{Rad}\, B + Z = B$ which is a contradiction. □

Lemma 4.5 *Let S be a DVR, B be a reduced torsion S-module and A be a bounded submodule of B. If B/A is divisible, then B is also bounded.*

Proof If p is the generator of the maximal ideal of S, then $p(B/A) = B/A$, since B/A is divisible. Then $pB + A/A = B/A$ and $pB + A = B$. As A is bounded, $p^n A = 0$ for some $n \in \mathbb{Z}^+$. We have $p^n B = p^{n+1}B + p^n A = p^{n+1}B$, i.e. $p^n B$ is divisible. Then $p^n B = 0$, since B is reduced. □

Lemma 4.6 *Let S be a DVR and $A \subseteq B$ be torsion S-modules. If $A \ll B$, then A is bounded.*

Proof Let A and B be torsion S-modules and $A \ll B$. Then A is reduced and by Lemma 4.4, A is coatomic. By (Zöschinger 1974a, Lemma 2.1) A is bounded. □

Proposition 4.5 *In the category $\mathcal{T}_R$ of torsion R-modules, an R-module X is $\mathcal{S}$-coinjective if and only if every primary part of the reduced part of X is bounded.*

Proof ($\Rightarrow$) Let $X \in \mathcal{T}_R$ be $\mathcal{S}$-coinjective. Let D be the divisible part of X. Then $X = D \oplus T$ where T is reduced. By Corollary 3.2, T is $\mathcal{S}$-coinjective. Let $\mathfrak{p}$ be a maximal ideal of R and $Y = T_\mathfrak{p}(T)$. Again by Corollary 3.2 Y is also $\mathcal{S}$-coinjective. We can consider Y as an $R_\mathfrak{p}$-module,i.e. a module over a DVR. If I is the injective hull of Y, Y has a supplement A in I. As $Y \cap A$ is small in A, $Y \cap A$ is coatomic by Lemma 4.4 and bounded by (Zöschinger 1974a, Lemma 2.1). We have $Y/Y \cap A \cong Y + A/A = I/A$ is divisible. Then by Lemma 4.5, Y is bounded.
($\Leftarrow$) Let $X = D \oplus T$ where D is the divisible part of X. D is injective, hence D is $\mathcal{S}$-coinjective. Let $\mathfrak{p}$ be maximal ideal of R, $T_\mathfrak{p}(T)$ is $\mathcal{S}$-coinjective by (Zöschinger 1974b, Folgerung after Lemma 1.4). Let Y be an R-module containing T. We have $T_\mathfrak{p}(T) \subseteq T_\mathfrak{p}(Y)$ and $T_\mathfrak{p}(T)$ has a supplement $V_\mathfrak{p}$ in $T_\mathfrak{p}(Y)$. Then $\bigoplus_\mathfrak{p} V_\mathfrak{p}$ is a supplement of $\bigoplus_\mathfrak{p} T_\mathfrak{p}(T) = T$ in Y. Therefore T is $\mathcal{S}$-coinjective. Considering the splitting short exact sequence $0 \longrightarrow D \longrightarrow X \longrightarrow T \longrightarrow 0$, by Proposition 3.2, X is $\mathcal{S}$-coinjective. □

The following result holds when $R = \mathbb{Z}$ and it can be generalized for modules over a Dedekind domain. Recall that we denoted κ-elements of $\text{Ext}(C, A)$ by $\mathcal{S}$ and β-elements of $\text{Ext}(C, A)$ by $\mathcal{SB}$.

Lemma 4.7 ((Zöschinger 1978), Lemma 1.2) *If A, $C \in \mathcal{T}_R$, then*

$$\text{Ext}_{\mathcal{SB}}(C, A) = \text{Ext}_{\mathcal{S}}(C, A) \cap T(\text{Ext}(C, A)).$$

We have a similar result to Lemma 4.3 for β-elements.

Lemma 4.8 *Let $X \subset^\beta Y \subset^\beta Z$. If Z is torsion, then $X \subset^\beta Z$.*

Proof Following the proof of Lemma 4.3, there exists $Y' \subseteq Z$ such that X has a supplement V' in Y' and Y' has a supplement W' in Z. We know that $V' + W'$ is a supplement of X in Z. What we need to show is that $X \cap (V' + W')$ is bounded. We have $X \cap (V' + W') \subseteq (V' \cap (X + W')) + (W' \cap (X + V')) = (V' \cap (X + W')) + (W' \cap Y')$. We know that $W' \cap Y'$ is bounded. Let $v' = x + w' \in V' \cap (X + W')$, then $w' = v' - x \in W' \cap (V' + X) = W' \cap Y'$. $W' \cap Y'$ is bounded, so $r(v' - x) = 0$ for some $r \in R$. $rv' = rx \in V' \cap X$. $V' \cap X$ is also bounded, therefore $srv' = 0$ for some $s \in R$. Hence $V' \cap (X + W')$ is also bounded. □

By using (Nunke 1963, Theorem 1.1), we have that β-elements of $\mathrm{Ext}_R(C, A)$ form a proper class. With similar arguments in Proposition 4.4, we have the following proposition.

Proposition 4.6 *Let $\mathcal{T}_R$ be the category of torsion R-modules and $A, C \in \mathcal{T}_R$. Then we have:*

(i) *β-elements of* $\mathrm{Ext}_R(C, A)$ *form a proper class.*

(ii) $\pi(\mathcal{SB}) = \{0\}$.

(iii) *$\mathcal{SB}$-injective R-modules are only the injective R-modules in $\mathcal{T}_R$.*

Corollary 4.8 *In the category $\mathcal{T}_{\mathbb{Z}}$ of torsion abelian groups we have:*

(i) *β-elements of* $\mathrm{Ext}(C, A)$ *form a proper class.*

(ii) $\pi(\mathcal{SB}) = \{0\}$.

(iii) *$\mathcal{SB}$-injective abelian groups are only the injective abelian groups in $\mathcal{T}_{\mathbb{Z}}$.*

The following proposition characterize $\mathcal{SB}$-coinjective R-modules in the category $\mathcal{T}_R$.

Proposition 4.7 *In the category $\mathcal{T}_R$ of torsion R-modules, an R-module X is $\mathcal{SB}$-coinjective if and only if reduced part of X is bounded.*

Proof ($\Rightarrow$) Let $X \in \mathcal{T}_R$ be $\mathcal{SB}$-coinjective. Let D be the divisible part of X. Then $X = D \oplus Y$ where Y is reduced. By Corollary 3.2, Y is $\mathcal{SB}$-coinjective, so it is $\mathcal{S}$-coinjective. By Proposition 4.5, $T_{\mathfrak{p}}(Y)$ is bounded for every maximal ideal $\mathfrak{p}$ of R. Suppose that $T_{\mathfrak{p}}(Y) \neq 0$ for infinitely many maximal ideals $\mathfrak{p}$ of R. We can write $Y = \bigoplus_{\mathfrak{p} \in G} T_{\mathfrak{p}}(Y)$ where $G \subseteq \Omega$ and $T_{\mathfrak{p}}(Y) \neq 0$ for all $\mathfrak{p} \in G$. Let A be the supplement of Y in an injective hull I of Y. We have $(\bigoplus_{\mathfrak{p} \in G} T_{\mathfrak{p}}(Y)) + (\bigoplus_{\mathfrak{p} \in G} T_{\mathfrak{p}}A) = \bigoplus_{\mathfrak{p} \in G} T_{\mathfrak{p}}(I)$. Since $Y \cap A$ is bounded, there is $\mathfrak{q} \in G$ such that $T_{\mathfrak{q}}(Y \cap A) = T_{\mathfrak{q}}(Y) \cap T_{\mathfrak{q}}(A) = 0$. Then $T_{\mathfrak{q}}(Y) \oplus T_{\mathfrak{q}}(A) = T_{\mathfrak{q}}(I)$, so $T_{\mathfrak{q}}(Y) = 0$, since Y is reduced. This contradicts with our assumption that $T_{\mathfrak{q}}(Y) \neq 0$ for every $\mathfrak{q} \in G$. Therefore Y is bounded.

($\Leftarrow$) Let $X = D \oplus Y$ where D is the divisible part of X. D is injective, hence D is $\mathcal{SB}$-coinjective. Y is $\mathcal{S}$-coinjective by (Zöschinger 1974b, Folgerung after Lemma 1.4). Since Y is also bounded, it is $\mathcal{SB}$-coinjective. By Proposition 3.2, X is $\mathcal{SB}$-coinjective. □

Corollary 4.9 *In the category $\mathcal{T}_{\mathbb{Z}}$ of torsion abelian groups, an abelian group X is $\mathcal{SB}$-coinjective if and only if reduced part of X is bounded.*

CHAPTER 5

CONCLUSIONS

In this thesis we applied homological methods for description of the submodules of modules that have supplements. The corresponding elements in the module of extensions are called κ-elements. These elements for the case of abelian groups were studied in (Zöschinger 1978). We showed that when R is a Dedekind domain, the proper class $\langle \mathcal{S} \rangle$ generated by the class $\mathcal{S}$ consisting of κ-elements coincides with the classes $\langle \mathcal{S}mall \rangle$ and $\langle \mathcal{W}supp \rangle$. We have also investigated β-elements and showed that over a noetherian integral domain of Krull dimension 1, β-elements form a proper class and this proper class coincides with the proper class coinjectively generated by the class of bounded R-modules. We restricted our attention to the category $\mathcal{T}_R$ of torsion R-modules for a Dedekind domain R and characterized $\mathcal{S}$-projective, $\mathcal{S}$-injective, $\mathcal{SB}$-projective and $\mathcal{SB}$-injective R-modules. We have also given the characterization of $\mathcal{S}$-coinjective and $\mathcal{SB}$-coinjective R-modules in the category $\mathcal{T}_R$.

REFERENCES

Alizade, R. G. 1985. Proper Kepka Classes. *Math. Notes* 37(2):268-273 (English Translation, 153-156).

Alizade, R. G. 1986. Proper Classes of Short Exact Sequences in the Category of Abelian Groups. *Math. Notes* 40(1):3-15 (English Translation).

Alizade, R. G., A. Pancar and B. Sezen. 2004. Scalar Multiplication of Proper Classes of Short Exact Sequences. *Soochow J. Math.* 30(2):127-138.

Anderson, F. W. and K. R. Fuller, eds. 1992. *Rings and Categories of Modules*. New York: Springer.

Buchsbaum, D. 1959. A Note on Homology in Categories. *Ann. of Math.* 69(1):66-74.

Butler, M. C. R. and G. Horrocks. 1961. Classes of Extensions and Resolutions. *Philos. Trans. R. Soc. London* 254(A):155-222.

Clark, J., C. Lomp, N. Vanaja, and R. Wisbauer, eds. 2006. *Lifting Modules: Supplements and Projectivity in Module Theory*. Basel: Birkhäuser.

Cohn, P. M. 2002. *Basic Algebra: Groups, Rings and Fields*. Springer-Verlag.

Erdoğan, E. 2004. Absolutely Supplement and Absolutely Complement Modules. MSc Thesis, İzmir Institute of Technology.

Fuchs, L. 1970. *Infinite Abelian Groups, Vol. I*. New York: Academic Press.

Fuchs, L. 1973. *Infinite Abelian Groups, Vol. II*. New York: Academic Press.

Harrison, D. K. 1959. Infinite Abelian Groups and Homological Methods. *Ann. Math.* 69:366-391.

Hazewinkel, M., N. Gubareni, and V. V. Kirichenko, eds. 2004. *Algebras, Rings and Modules, Vol I*. Springer-Verlag.

Hungerford, T. W., 1974. *Algebra*. Springer-Verlag.

Kaplansky, I. 1969. *Infinite Abelian Groups*. Michigan: University of Michigan Press.

Mac Lane, S. 1995. *Homology*. Springer-Verlag.

Mermut, E. 2004. Homological Approach to Complements and Supplements. PhD diss., Dokuz Eylül University.

Misina, A. P. and L. A. Skornjakov, eds. 1960. *Abelian Groups and Modules*. Amer Mathematical Society.

Nunke, R. J. 1959. Modules of Extensions over Dedekind Rings. *Illinois J. Math.* 3:222-241.

Nunke, R. J. 1963. Purity and Subfunctor of the Identity. *Topics in Abelian groups* (Proc. Sympos., New Mexico State Univ., 1962) 121-171.

Pancar, A. 1997. Generation of Proper Classes of Short Exact Sequences. *Internat. J. Math. and Math. Sci.* 20(3):465-474.

Sharpe, D.W. and P. Vamos, eds. 1972. *Injective Modules*. Chambridge University Press.

Sklyarenko, E. G. 1978. Relative Homological Algebra in Categories of Modules. *Russian Math. Surveys* 33(3):97-137. Translated from Russian from *Uspehi Mat. Nauk* 33(201):85-120.

Vermani, L. R. 2003. *An Elementary Approach to Homological Algebra*. Chapman and Hall/CRC.

Walker, C. P. 1964. Properties of Ext and Quasisplitting of Abelian Groups. *Acta Math. Acad. Sci. Hung.* 15:157-160.

Wisbauer, R. 1991. *Foundations of Module and Ring Theory*. Gordon and Breach.

Zöschinger, H. 1974. Komplementierte Moduln über Dedekindringen. *J. Algebra* 29:42-56.

Zöschinger, H. 1974. Moduln die in jeder Erweiterung ein Komplement haben. *Math. Scand.* 35:267-287.

Zöschinger, H. 1978. Über Torsions- und κ-Elemente von Ext(C,A). *Journal of Algebra* 50:299-336.

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