NONLINEAR EULER POISSON DARBOUX EQUATIONS EXACTLY SOLVABLE IN MULTIDIMENSIONS

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Barış ATEŞ

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We approve the thesis of **Barış ATEŞ**

Prof. Dr. Oktay K. PASHAEV
Supervisor

Assist. Prof. Dr. Şirin ATILGAN
Committee Member

Assist. Prof. Dr. Seçil ARTEM
Committee Member

12 June 2008
Date

Prof. Dr. Oğuz YILMAZ
Head of the Mathematics Department

Prof. Dr. Hasan BÖKE
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

NONLINEAR EULER POISSON DARBOUX EQUATIONS EXACTLY SOLVABLE IN MULTIDIMENSIONS

The method of spherical means is the well known and elegant method of solving initial value problems for multidimensional PDE. By this method the problem reduced to the 1+1 dimensional one, which can be solved easily. But this method is restricted by only linear PDE and can not be applied to the non-linear PDE. In the present thesis we study properties of the spherical means and nonlinear PDE for them. First we briefly review the main definitions and applications of the spherical means for the linear heat and the wave equations. Then we study operator representation for the spherical means, especially in two and three dimensional spaces. We find that the spherical means in complex space are determined by modified exponential function. We study properties of these functions and several applications to the heat equation with variable diffusion coefficient. Then nonlinear wave equations in the form of the Liouville equation, the Sine-Gordon equation and the hyperbolic Sinh-Gordon equations in odd space dimensions are introduced. By some combinations of functions we show that models are reducible to the 1+1 dimensional one on the half line. The Backlund transformations and exact particular solutions in the form of progressive waves are constructed. Then the initial value problem for the nonlinear Burgers equation and the Liouville equations are solved. Application of our solutions to spherical symmetric multidimensional problems is discussed.

ÖZET

YÜKSEK BOYUTLARDA TAM ÇÖZÜMLENEBİLEN DOĞRUSAL OLMAYAN EULER POISSON DARBOUX DENKLEMLERİ

Küresel ortalama metodu iyi bilinen ve yüksek boyutlu kısmi türevli diferansiyel denklemler için başlangıç değer problemlerini çözmekte oldukça kullanışlı bir metottur. Bu metodla yüksek boyutlu problem kolaylıkla çözülebilen bir boyutlu probleme indirgenir. Fakat bu metot doğrusal kısmi türevli diferansiyel denklemlerle sınırlıdır ve doğrusal olmayan kısmi türevli diferansiyel denklemlere uygulanamaz. Biz bu tezde küresel ortalamanın özelliklerini ve küresel ortalamayla ilişkilendirilebilen doğrusal olmayan kısmi türevli diferansiyel denklemleri çalıştık. İlk olarak küresel ortalamanın temel tanımlarını, doğrusal ısı ve dalga denklemlerine uygulamalarını yeniden inceledik. Daha sonra küresel ortalamanın operatör temsilini iki ve üç boyutlu uzaylarda çalıştık. Küresel ortalamanın karmaşık uzayda modifiye üstel fonksiyon tarafından belirlendiğini bulduk. Bu fonksiyonların özelliklerini ve değişken katsayılı ısı denklemine bir çok uygulamalarını çalıştık. Daha sonra Lioville, Sinüs Gordon ve Hiperbolik tipte Sinüs hiperbolik Gordon formunda doğrusal olmayan dalga denklemleri tek boyutlu uzaylarda verildi. Bazı fonksiyonel kombinasyonlarla bu denklemlerin 1+1 boyutlu yarı doğru üzerine indirgenebilecekleri gösterildi. Bäcklund transformasyonu ve progresif dalga tarzındaki kesin çözümler oluşturuldu. Daha sonra Liouville ve doğrusal olmayan Burgers denklemi için başlangıç değer problemleri çözüldü. Çözümlerimizin yüksek boyutlu küresel simetrik problemlere uygulamarı tartışıldı.

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CHAPTER 1

INTRODUCTION

The method of spherical means is the well known and elegant method of solving initial value problems for multidimensional PDE (Courant and Hilbert 1962 John 1981). By this method the problem reduced to the 1+1 dimensional one, which can be solved easily (Courant and Hilbert 1962 John 1981). Then by taking the limit $r \to 0$ it is shown that the spherical means are reducible to the original function and this way solution of the wave equation in the D'Alembert form was given. But since the spherical means are averages of given function around arbitrary spheres, the method can't be applied to the nonlinear PDE. This is why all studies in this field are restricted by the linear PDE. In the present thesis we show that idea of spherical means and way how they solve the problem could be helpful in study of some nonlinear PDE in multi-dimensions. The main idea is motivated by the Darboux equation reducing the action of multidimensional Laplacian on the spherical means to the one dimensional linear operator of the second order. Then a multi-dimensional problem for spherical means is reducible to the 1+1 dimensional one. The last problem in many cases can be solved exactly.

From the theory of integrable models we know that some class of integrable models called the C integrable, by some transformation of unknown function can reduce the nonlinear model to the linear one. If we consider this linear equation as the equation for the spherical means, then the nonlinear counterpart gives 1+1 dimensional PDE. These equations can be studied in a full capacity of integrable systems. This allows us to describe some multidimensional nonlinear PDE, with the set of particular solutions and the initial value problems.

This thesis is arranged as follows;

In Chapter II, we briefly review the method of spherical means and consider its operator representation in one and two dimensional space. In Section 2.2, we consider operator representation for spherical means in the complex plane. For this case in Section 2.3, we give explicit formula for the spherical means operator in terms of modified exponential functions. We study some properties of this function and related functions such as the modified sine-cosine and hyperbolic sine-cosine functions. Then we introduce the differential equations satisfied by these modified functions. As an application of the modified exponential function, we relate it with a Heat equation whose diffusion coefficient is linear function of x. Using the Cole-Hopf transformation we construct the corresponding nonlinear Burgers equation such that its solution can be expressed in terms of the modified exponential function. We also describe the Hierarchy of the Burgers equations.

Chapter III starts from brief review of solution for the initial value problem for the wave equation in 1+1 dimensional space. In Section 3.2, we present solutions of I.V.P for the wave equation by the method of spherical means in $3 + 1$ dimensional space. In Section 3.3, by the method of spherical means I.V.P for the wave equation in $5 + 1$ dimensional space is solved. In Section 3.4, we solve I.V.P $7 + 1$ dimensional spaces by the same method. In Section 3.5, solution of the I.V.P for the wave equation in arbitrary odd dimensional space is given by the method of spherical means. In Section section 3.6, we review the Hadamard Method of Descent which is useful to study i.v.p in even dimensional spaces.

In Chapter IV, we start with review of the relation between Liouville equation and surface theory. In Section 4.2, we give the expression of the spherical Liouville equation which is defined in $3 + 1$ dimensional space. We relate it with the spherically symmetric wave equation by Backlund transformation which allows us to write the general solution of spherical Liouville equation. In Section 4.3, we solve initial value problem in a particular form for the spherical Liouville equation where initial velocity is zero. The progressing wave solution of Spherical Liouville equation and expression for the Lax pair is given in Section 4.4. For arbitrary dimensional space, we give the spherical Liouville equation with some potential and write its general solution. We solve initial value problem for this case and give progressing wave solutions in Section 4.5. In Section 4.6, we write spherical Liouville equation for arbitrary odd dimensional space.

In Chapter V, we introduce the spherical Sine-Gordon and spherical Sinh-Gordon equations. In Section 5.1, we study Sine-Gordon equation in 1+1 dimensional space. In Section 5.2, we give the expression of the spherical Sine Gordon equation. Using Backlund transformation we write Bianchi permutability formula for this equations and find its kink and anti-kink like solutions. After giving its

Lax pair we relate this equation with Riccati equation. In Section 5.3, we write the expression for spherical Sinh-Gordon equation. Writing Backlund transformation for this equation allows us to write the soliton like solutions. To construct new solutions we write the Bianchy Permutability formula for this equation. Then we consider two soliton like solution of this equation and progressing wave solution. Finally, we relate this equation with Riccati equation.

In the last Chapter we consider the application of the method of spherical means to the heat equation. In Section 6.1, we solve initial value problem for the heat equation in 2 + 1 and 3 + 1 dimensional space by the method of spherical means. In addition to these we write he solution of the initial value problem for arbitrary odd dimensional spaces. These applications allows us to write the solutions of I.V.P for the corresponding nonlinear Burgers equations. In Section 6.2, we introduce Cylindrical Burgers equation and write its general solution. In Section 6.3, we introduce the spherical Burgers equation in 3+1 dimensional space and write its general solution. In Section 6.4, we consider the heat and Burgers hierarchies and write the spherical and cylindrical Burgers Hierarchies.

CHAPTER 2

THE METHOD OF SPHERICAL MEANS

In the present chapter, we briefly review the method of spherical means (Courant and Hilbert 1962 John 1981), its operator representation in two and three dimensional spaces and in the complex plane. In the complex plane, we give explicit representation of the spherical means operator in terms of generalized exponential functions. Then we study some properties of the generalized exponential function and the related functions such as the generalized sine-cosine and the hyperbolic sine-cosine functions. Some applications of these functions to partial differential equations are discussed.

2.1. Main Definitions and Properties

The method of spherical means, averaging functions on the sphere, date back to the studies of Fritz John (John 1955). The method of spherical means plays very important role in the theory of partial differential equations. It is very powerful method to study partial differential equations in the higher dimensional spaces. It appears in different areas of mathematics like integral geometry, inversion of the Fourier transform and in the study of Radon transforms,etc.

Definition 2.1 *Spherical mean of a continuous function $u(x) = u(x_1, x_2, \ldots, x_n)$ in R^n is the average of u on $(n-1)$ sphere with given radius and center. Spherical means is denoted by $M_u(x, r)$ and given by the following formula (John 1981)*

$$M_u(x, r) = \frac{1}{\omega_n r^{n-1}} \int\limits_{|y-x|=r} u(y) dS_y. \tag{2.1}$$

In this formula ω_n is the surface area of the unit sphere, $x = (x_1, \ldots, x_n)$ is the center of the sphere with radius r, $\omega_n r^{n-1}$ is the surface area and dS_y is the area element of this sphere. By setting $y = x + r\xi$, with $|\xi| = 1$, we can find another representation for the spherical means;

$$M_u(x,r) = \frac{1}{\omega_n} \int\limits_{|\xi|=1} u(x+r\xi)dS_\xi. \tag{2.2}$$

In this representation ω_n and dS_ξ are the surface area and surface element of the unit sphere respectively. In limiting case when r approaches zero, the spherical means gives exactly the original function u.

$$\lim_{r\to o} M_u(x,r) = u(x). \tag{2.3}$$

Proposition 2.1 *(John 1981) Spherical means of a function satisfies the Darboux equation.*

$$\Delta_x M_u(x,r) = (\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r). \tag{2.4}$$

Proof Differentiating both sides of equation (2.2) with respect to r gives ;

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{\omega_n}\int\limits_{|\xi|=1}\sum_{i=1}^{n} u_{\eta_i}(x+r\xi)\xi_i dS_\xi, \tag{2.5}$$

where $\eta_i = x_i + r\xi_i$. The Divergence Theorem is read as

$$\int\limits_{\Omega} D_k u(x)dx = \int\limits_{\partial\Omega} u(x)\xi_k dS_x \tag{2.6}$$

where $D_k = \frac{\partial}{\partial x_k}$, $\xi = \xi(\xi_1,\ldots,\xi_n)$ outward unit normal, $dx = dx_1\ldots dx_n$ and dS_x is the volume and the surface element correspondingly. Applying Divergence Theorem (2.6) to the equation (2.5) gives,

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{r^{1-n}}{\omega_n}\Delta_x \int\limits_{|y-x|<r} u(y)dy \tag{2.7}$$

$$= \frac{r^{1-n}}{\omega_n}\Delta_x \int\limits_{0}^{r} d\rho \int\limits_{|y-x|=\rho} u(y)dS_y \tag{2.8}$$

$$= r^{1-n}\Delta_x \int_0^r \rho^{n-1}M_u(x,\rho)d\rho. \tag{2.9}$$

Multiplying equation (2.9) by r^{n-1} and differentiating with respect to r yields,

$$\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}M_u(x,r)\right) = \Delta_x r^{n-1}M_u(x,r). \tag{2.10}$$

Thus the spherical mean $M_u(x,r)$ of function $u(x)$ satisfies the partial differential equation

$$(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r) = \Delta_x M_u(x,r) \tag{2.11}$$

which is known as Darboux's Equation. From this equation we can see that $M_u(x,-r)$ is also satisfies the Darboux Equation. □

Proposition 2.2 *(John 1981) If a function $u(x,t)$, which depends on n space variables $x_1, x_2, \dots, x_n$ and time t, satisfies the wave equation $u_{tt} - c^2 \triangle u = 0$, then its spherical means satisfy the Euler-Poisson-Darboux equation.*

$$(M_u)_{tt} - c^2\left((M_u)_{rr} + \frac{n-1}{r}(M_u)_r\right) = 0. \tag{2.12}$$

Proof If function $u(x)$ is also depends on time t, then its spherical means is found by the following formula

$$M_u(x,t;r) = \frac{1}{\omega_n}\int\limits_{|\xi|=1} u(x + r\xi, t)dS_\xi. \tag{2.13}$$

If the Laplacian operator acts on the Spherical means we find that

$$\Delta_{\mathrm{x}} M_u = \frac{1}{\omega_{\mathrm{n}}}\int\limits_{|\xi|=1} \Delta_x u(x + \mathrm{r}\xi, \mathrm{t})dS_\xi. \tag{2.14}$$

Since $u(x,t)$ satisfies the wave equation we can write

$$\Delta_{\mathrm{x}} M_u = \frac{1}{\omega_{\mathrm{n}}}\int\limits_{|\xi|=1} \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x + \mathrm{r}\xi, \mathrm{t})dS_\xi. \tag{2.15}$$

Using that t is independent from ξ, we can interchange differentiation with integration. After using the definition of spherical means we have

$$\Delta_{\mathrm{x}} M_u = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} M_u. \tag{2.16}$$

This equation means that the spherical means of any solution of the wave equation is solution of the same equation. By using Darboux equation (2.4) we find that spherical means of function $u(x,t)$ satisfies

$$(M_u)_{tt} = c^2\left((M_u)_{rr} + \frac{n-1}{r}(M_u)_r\right). \tag{2.17}$$

□

2.1.1. Operator Representation of Spherical Means

Let function $u(x)$ is real analytic function in the disk $|\xi| \leq 1$. Then expanding it into the Taylor series and using the definition of spherical means we have

$$M_u(x,r) \;\;=\;\; \frac{1}{\omega_n}\int\limits_{|\xi|=1} dS_\xi e^{r\xi.\nabla_x} u(x). \tag{2.18}$$

Since x is independent from ξ, we can threat the last expression as an operator acting on function $u(x)$. On the other hand, the spherical means is an even function of r, $M_u(x,r) = M_u(x,-r)$. Using this property and representation $e^{r\xi\nabla_x} = \cosh r\xi\nabla_x + \sinh r\xi\nabla_x$, we deduce that spherical means operator M has the form

$$M = \frac{1}{\omega_n} \int\limits_{|\xi|=1} dS_\xi \, \cosh\left(r\xi.\nabla_x\right). \tag{2.19}$$

2.1.2. Spherical Means in One Dimensional Space

Spherical means of a function u in one dimensional space is equal to the standard mean value of the function u;

$$M_u(x;r) = \frac{u(x-r) + u(x+r)}{2}. \tag{2.20}$$

Expanding u into the Taylor series, we find

$$M_u(x;r) = \cosh\left(r\frac{d}{dx}\right)u(x). \tag{2.21}$$

This representation is even in r and gives the original function when r approaches to zero.

2.1.3. Spherical Means in Two Dimensional Space

In two dimensional space, expanding function $u(x,t)$ in the Taylor series allows us to write the spherical means as

$$M_u(x,y;r) = \frac{1}{2\pi} \int_o^{2\pi} d\theta e^{(r\cos\theta\partial_x + r\sin\theta\partial_y)} u(x,y). \tag{2.22}$$

If we split the integrand in hyperbolic cosine function and hyperbolic sine function, we find the operator representation of the spherical means in two dimensional space as

$$M_u(x,y;r) = \frac{1}{2\pi} \int\limits_o^{2\pi} d\theta \cosh\left(r\cos\theta\partial_x + r\sin\theta\partial_y\right)u(x,y). \tag{2.23}$$

Using equation (2.23), it is instructive to derive the Darboux (2.4) equation by alternative method. If we differentiate $M_u(x,r)$ with respect to r we find

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \sinh(r\cos\theta\partial_x + r\sin\theta\partial_y)(\cos\theta\partial_x + \sin\theta\partial_y)u(x,y),$$

$$\frac{\partial^2}{\partial r^2}M_u(x,r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cosh(r\cos\theta\partial_x + r\sin\theta\partial_y)(\cos\theta\partial_x + \sin\theta\partial_y)^2 u(x,y).$$

In order to combine these two equations, let us write the $\frac{\partial}{\partial r}M_u(x,r)$ in terms of the hyperbolic cosine function.

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{2\pi}\int_0^{2\pi} r\cosh(r\cos\theta\partial_x + r\sin\theta\partial_y)(-\sin\theta\partial_x + \cos\theta\partial_y)^2 u(x,y)d\theta. \quad (2.24)$$

If we write the obtained results in the Darboux equation (2.4)

$$\Delta_x M_u(x,r) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})M_u(x,r) \quad (2.25)$$

we find,

$$\Delta_x M_u(x,r) = \frac{1}{2\pi}\int_0^{2\pi} \cosh(r\cos\theta\partial_x + r\sin\theta\partial_y)(\partial x^2 + \partial y^2)u(x,y)d\theta. \quad (2.26)$$

2.2. Spherical Means Operator in Complex Plane

Spherical means operator can be expressed in terms of complex variables which is useful to study analytic and harmonic functions.

Proposition 2.3 *Spherical means operator in complex domain is given by the following formula*

$$M_f(z,r) = \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{d\xi}{\xi}\cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}})f(z). \quad (2.27)$$

Proof In two dimensional space, operator representation of the spherical means (2.23) is

$$M_u(x,r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cosh(r\cos\theta\partial_x + r\sin\theta\partial_y)u(x,y). \quad (2.28)$$

Using the definition of sine and cosine functions we find that;

$$M_u(x,y,r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cosh\left[r\left(\frac{e^{i\theta}}{2}(\partial_x - i\partial_y) + \frac{e^{-i\theta}}{2}(\partial_x + i\partial_y)\right)\right]u(x,y). \quad (2.29)$$

Defining $z = x + iy$ and $\xi = e^{i\theta}$ we find

$$M_f(z,r) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{d\xi}{\xi}\cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}})f(z). \quad (2.30)$$

□

Theorem 2.1 *(Cauchy Integral Formula) Let F be analytic in domain D within simple closed curve $C \subset D$. For a point z_0 interior to the C, following relation holds:*

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz. \tag{2.31}$$

The meaning of this formula in terms of spherical means is given by the next theorem.

Theorem 2.2 *Spherical means of an analytic function $f(z)$ is independent of r and identical to the function's itself in the region of analyticity.*

$$M_f(z, r) = f(z). \tag{2.32}$$

Proof Let us consider spherical means operator (2.27) in complex domain, in the region of analyticity of function $f(z)$,

$$M_f(z, r) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{d\xi}{\xi} \cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}}) f(z). \tag{2.33}$$

Then expanding cosine hyperbolic function in the Taylor series we find

$$M_f(z, r) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \binom{2n}{k} \frac{r^{2n} \xi^{2n-k} \bar{\xi}^k}{(2n)!} \partial_z^{2n-k} \partial_{\bar{z}}^k f(z). \tag{2.34}$$

Since $f(z)$ is analytic then its derivative with respect to $\bar{z}$ gives zero. According to this, only terms with $k = 0$ will survive

$$M_f(z, r) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \sum_{n=0}^{\infty} \frac{r^{2n} \xi^{2n}}{(2n)!} \partial_z^{2n} f(z). \tag{2.35}$$

According to the Cauchy integral formula, in this integral only the term with ξ^{-1} will survive. Hence n must be zero.

$$M_f(z, r) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} f(z) \tag{2.36}$$

$$M_f(z, r) = f(z) \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{1}{\xi} d\xi \tag{2.37}$$

$$M_f(z, r) = f(z). \tag{2.38}$$

Another way to prove this theorem is based on the Cauchy Integral formula (2.31)

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz. \tag{2.39}$$

□

Let us consider contour C as the circle: $|z - z_0| = r$. Then $z - z_0 = re^{i\theta}$, $0 \leq \theta < 2\pi$, $dz = ire^{i\theta}d\theta$, so

$$f(z_0) = \frac{1}{2\pi}\oint f(z_0 + re^{i\theta})d\theta = M_f(z_0). \tag{2.40}$$

Proposition 2.4 *The relation between differential operator and integro-differential operator is given as*

$$\left(\frac{d}{dz}\right)^k = \frac{k!}{2\pi i}\oint \frac{d\xi}{\xi^{k+1}}e^{\xi\frac{\partial}{\partial z}}. \tag{2.41}$$

Proof Expanding exponential function, we observe that only term of order k gives nontrivial contribution. □

Theorem 2.3 *(generalized Cauchy Formula) If a function is not analytic but continuous in a region Ω bounded by a closed curve C, then at any point z_0 in the ω following formula holds;*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z_0}d\xi - \frac{1}{\pi}\int\int_\Omega \frac{\partial f/\partial\bar{\xi}}{\xi - z_0}dA \tag{2.42}$$

where dA is the surface element.

Theorem 2.4 *Spherical means of an arbitrary complex function is given by the following formula*

$$M_f(z,\bar{z},r) = \left(\sum_{n=0}^{\infty}\frac{r^{2n}}{(n!)^2}\partial_z^n\partial_{\bar{z}}^n\right)f(z,\bar{z}). \tag{2.43}$$

Proof The proof can be done by two ways.

1) spherical means operator in complex notation is given by

$$M_f(z,\bar{z},r) = \frac{1}{2\pi i}\oint_{|\xi|=1}\frac{d\xi}{\xi}\cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}})f(z,\bar{z}). \tag{2.44}$$

From the Generalized Cauchy integral formula (2.42), we find

$$M_f(z,\bar{z},r) = \cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}})f(z,\bar{z})|_{\xi=0} + \frac{1}{\pi}\int\int_\Omega \frac{dA}{\xi}\frac{\partial\left(\cosh(r\xi\partial_z + r\bar{\xi}\partial_{\bar{z}})\right)}{\partial\bar{\xi}}f(z,\bar{z}) \tag{2.45}$$

where dA is the surface element. Thus spherical means of an arbitrary function is equal to the

$$M_f(z,\bar{z},r) = f(z,\bar{z}) + \frac{1}{\pi}\int\int_\Omega dA\sum_{n=0}^{\infty}\sum_{k=1}^{2n}\binom{2n}{k}\frac{kr^{2n}\xi^{2n-k-1}\bar{\xi}^{k-1}}{(2n)!}\partial_z^{2n-k}\partial_{\bar{z}}^k f(z,\bar{z}). \tag{2.46}$$

Letting $\xi = Re^{i\theta}$, allows us to write

$$M_f(z,\bar{z},r) = f + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=1}^{2n}\int_{\theta=0}^{2\pi}\int_0^1 dRd\theta\frac{\binom{2n}{k}kr^{2n}e^{i\theta(2n-2k)}R^{2n-1}}{(2n)!}\partial_z^{2n-k}\partial_{\bar{z}}^k f(z,\bar{z})$$

$$M_f(z,\bar{z},r) = f + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=1}^{2n}\int_{\theta=0}^{2\pi}d\theta\frac{\binom{2n}{k}kr^{2n}e^{i\theta(2n-2k)}}{2n(2n)!}\partial_z^{2n-k}\partial_{\bar{z}}^k f(z,\bar{z}). \tag{2.47}$$

If $n \neq k$ then the integral gives zero. Hence the expression reduces to the

$$M_f(z,\bar{z},r) = f(z) + \sum_{n=1}^{\infty} \frac{r^{2n}}{(2n)!}\partial_z^n \partial_{\bar{z}}^n f(z,\bar{z}). \tag{2.48}$$

Thus spherical means of a function which depend on z and $\bar{z}$ is given by

$$M_f(z,\bar{z},r) = \left(\sum_{n=0}^{\infty} \frac{r^{2n}}{(n!)^2}\partial_z^n \partial_{\bar{z}}^n \right) f(z,\bar{z}). \tag{2.49}$$

Since $\triangle = 4\partial_z \partial_{\bar{z}}$ then spherical means of function $f(x,y)$ satisfies

$$M_f(x,y,r) = \left(\sum_{n=0}^{\infty} \frac{r^{2n}\ \triangle^n}{(n!)^2 2^{2n}} \right) f(x,y). \tag{2.50}$$

2) The second way to prove is to calculate the integral directly. Spherical means operator in complex plane is given by

$$M_f(z,\bar{z},r) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \binom{2n}{k} \frac{r^{2n}\xi^{2n-k}\bar{\xi}^k}{(2n)!} \partial_z^{2n-k} \partial_{\bar{z}}^k f(z,\bar{z}). \tag{2.51}$$

Since $|\xi| = 1$ we get, $\xi\bar{\xi} = 1$. Using this relation we can write

$$M_f(z,\bar{z},r) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \binom{2n}{k} \frac{r^{2n}\xi^{2n-2k}}{(2n)!} \partial_z^{2n-k} \partial_{\bar{z}}^k f(z,\bar{z}). \tag{2.52}$$

Now, according to the Cauchy integral formula (2.31) only terms will survive when $n = k$, so we obtain,

$$M_f(z,\bar{z},r) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{r^{2n}}{(2n)!} (\partial_z \partial_{\bar{z}})^n f(z,\bar{z}) \tag{2.53}$$

$$M_f(z,\bar{z},r) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(n!)^2} (\partial_z \partial_{\bar{z}})^n f(z,\bar{z}). \tag{2.54}$$

□

Let us define new function

$$e(z;2) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}, \tag{2.55}$$

by ratio test it can be shown that the series converges for all z in the whole complex plane, then the function $e(z;2)$ is an entire function. Using this function, we can write the spherical mean operator M as

$$M = e\left(\frac{r^2}{4} \triangle ; 2 \right). \tag{2.56}$$

Theorem 2.5 *The value of a harmonic function $u = u(x,y)$, at the center of a disk, is equal to the average value of the function u on the boundary of the disk.*

Proof If function $u(x,y)$ is harmonic then it satisfies Laplace equation

$$\triangle\, u(x,y) = 0. \tag{2.57}$$

Hence from the spherical means operator

$$M_u(x,y;r) = e\left(\frac{r^2}{4}\triangle;2\right)u$$

and by the definition of $e(x;2)$ (2.55), it is found that spherical means of a harmonic function is

$$M_u(x,y;r) = u(x,y). \tag{2.58}$$

This equality tells us also that, average of a harmonic function is independent from the radius of the circle. □

If function $u(x,y)$ is not harmonic then for the spherical means we have the next formula

$$M_u(x,y;r) = u(x,y) + \frac{r^2}{4}\triangle\, u + \frac{r^4}{16}\triangle^2\, u + \ldots + \frac{r^{2n}}{4^n}\triangle^n\, u + \ldots \tag{2.59}$$

Explicitly dependent of the radius r.

Now, as an application of spherical means operator (2.56), let us evaluate the spherical means of some special functions which depend only radial part.

1) In two dimensional space let us consider consider function

$$u(x,y) = \frac{1}{\sqrt{x^2+y^2}} \tag{2.60}$$

or

$$u(R) = \frac{1}{R},\ R = \sqrt{x^2+y^2}. \tag{2.61}$$

Since the function depends only on radial part then Laplace operator has the form

$$\triangle = \partial_R^2 + \frac{1}{R}\partial_R.$$

If we evaluate the Laplacian of function $\frac{1}{R}$ we find

$$\triangle \frac{1}{R} = \frac{1}{R^3} \quad (2.62)$$

$$\triangle^2 \frac{1}{R} = \frac{3^2}{R^5} \quad (2.63)$$

$$\triangle^3 \frac{1}{R} = \frac{3^2 5^2}{R^7} \quad (2.64)$$

$$\triangle^4 \frac{1}{R} = \frac{3^2 5^2 7^2}{R^9} \quad (2.65)$$

$$\triangle^5 \frac{1}{R} = \frac{3^2 5^2 7^2 9^2}{R^{11}} \quad (2.66)$$

$$\vdots$$

$$\triangle^n \frac{1}{R} = \frac{3^2 5^2 7^2 \ldots (2n-1)^2}{R^{2n+1}}. \quad (2.67)$$

Writing the the operator form of spherical means we find spherical means of function $\frac{1}{R}$ as

$$M_{\frac{1}{R}}(R;r) = \sum_{n=0}^{\infty} \frac{(\frac{r^2}{4} \triangle)^n}{(n!)^2} \frac{1}{R} \quad (2.68)$$

$$M_{\frac{1}{R}}(R;r) = \sum_{n=0}^{\infty} \frac{(\frac{r^2}{4})^n}{(n!)^2} \frac{[(2n-1)!!]^2}{R^{2n+1}} \quad (2.69)$$

$$M_{\frac{1}{R}}(R;r) = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{r^2}{4R^2}\right)^n [\frac{(2n-1)!!}{(n!)}]^2. \quad (2.70)$$

If $R = r$, then we find

$$M_{\frac{1}{R}} = \sum_{n=0}^{\infty} (\frac{1}{4})^n [\frac{(2n-1)!!}{n!}]^2 \quad (2.71)$$

Let $\frac{r}{R} = \rho$, then equation (2.70) takes the form

$$M_{\frac{1}{R}} = \frac{1}{R} \sum_{n=0}^{\infty} (\frac{\rho^2}{4})^n [\frac{(2n-1)!!}{n!}]^2. \quad (2.72)$$

2) Now let us consider function in two dimensional space as

$$u(x,y) = (\sqrt{x^2+y^2})^N, \quad u(R) = R^N. \quad (2.73)$$

Power of Laplacians are found as

$$\triangle R^N = N^2 R^{N-2} \quad (2.74)$$

$$\triangle^2 R^N = N^2 (N-2)^2 R^{N-4} \quad (2.75)$$

$$\triangle^3 R^N = N^2 (N-2)^2 (N-4)^2 R^{N-6} \quad (2.76)$$

$$\vdots$$

$$\triangle^n R^N = N^2 (N-2)^2 (N-4)^2 .... ((N-(2n-2)))^2 R^{N-2n}. \quad (2.77)$$

Then spherical means of function R^N is found as

$$M_{R^N} = R^N \sum_{n=0}^{\infty} \left(\frac{r^2}{4R^2} \right)^n \frac{[N(N-2)(N-4)...(N-(2n-1))]^2}{(n!)^2}. \tag{2.78}$$

3) Let $u(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, this function is harmonic in three dimensional space so spherical means of $u(\frac{1}{R}) = \frac{1}{R}$ is equal to itself

$$M_{\frac{1}{R}} = \frac{1}{R}. \tag{2.79}$$

This also tells us that for the harmonic functions we do not need to take the limit $r \to 0$ to recover the original function from the spherical means of the function.

4) If we consider the function

$$u(x, y, z) = R^N, R = \sqrt{x^2 + y^2 + z^2}. \tag{2.80}$$

Laplacians of function are found as

$$\triangle R^N = \frac{(N+1)!}{(N-1)!} R^{N-2} \tag{2.81}$$

$$\triangle^2 R^N = \frac{(N+1)!}{(N-3)!} R^{N-4} \tag{2.82}$$

$$\triangle^3 R^N = \frac{(N+1)!}{(N-5)!} R^{N-6} \tag{2.83}$$

$$\vdots$$

$$\triangle^n R^N = \frac{(N+1)!}{(N-(2n-1))!} R^{N-2n} \tag{2.84}$$

$$M_{R^N} = R^N \sum_{n=0}^{\infty} \left(\frac{r^2}{4R^2} \right)^n \frac{[N(N-2)(N-4)...(N-(2n-1))]^2}{(n!)^2}. \tag{2.85}$$

In three dimensional space spherical means operator is given by the following formula (Sabelfeld and Shalimova 1997)

$$M_u(x, y, z; r) = \frac{\sinh(r\sqrt{\triangle})}{r\sqrt{\triangle}} u(x, y, z). \tag{2.86}$$

Then using the definition of sine hyperbolic function

$$M_u(x, y, z; r) = \sum_{n=0}^{\infty} \frac{r^{2n} \triangle^n}{(2n+1)!} u. \tag{2.87}$$

Hence

$$M_{R^N} = \sum_{n=0}^{\infty} \frac{r^{2n}(N+1)!}{(2n+1)!\,(N-(2n-1))!} R^{N-2n} \tag{2.88}$$

$$M_{R^N} = R^N \sum_{n=0}^{\infty} (\frac{r^2}{R^2})^n \frac{(N+1)!}{(2n+1)!\,(N-(2n-1))!}. \tag{2.89}$$

In addition to above examples, spherical means of an arbitrary function is evaluated in appendix (A).

2.3. Properties of Modified Exponential Function

In previous section we introduced modified exponential function derived by the next formula ,

$$e(x;2) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}. \tag{2.90}$$

This section is devoted for the properties of modified exponential function.

Proposition 2.5 *Function $e(x;2)$ satisfies the second order differential equation*

$$\frac{d}{dx}\left(x\frac{d}{dx}y(x)\right) - y(x) = 0. \tag{2.91}$$

Proof Let us take derivative of $e(x;2)$ with respect to x

$$\frac{d}{dx}e(x;2)(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n!)^2}. \tag{2.92}$$

Multiplying equation (2.92) with x gives,

$$x\frac{d}{dx}e(x;2) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n!)^2}. \tag{2.93}$$

Now, in order to eliminate $(n+1)$ in denominator, let us take one more derivative

$$\frac{d}{dx}(x\frac{d}{dx}e(x;2)) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = e(x;2) \tag{2.94}$$

thus $e(x;2)$ is a solution of the following differential equation

$$\frac{d}{dx}(x\frac{d}{dx}e(x;2)) = e(x;2). \tag{2.95}$$

□

Equation (2.95) can be transformed to the Schrödinger equation with exponential potential. To do this, let us multiply equation (2.95) with x

$$x\frac{d}{x}x\frac{d}{x}e(x;2) = xe(x;2). \tag{2.96}$$

We see that this equation is Euler type. So by substitution

$$x = e^y, \quad y = \ln x, \quad \frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx} = x\frac{d}{dx} \tag{2.97}$$

it becomes

$$\frac{d^2}{dy^2}e(e^y;2) = e^y e(e^y;2) \tag{2.98}$$

or

$$\left(-\frac{d^2}{dy^2} + e^y\right)\varphi(y) = 0. \tag{2.99}$$

So, solution of this equation is

$$\varphi(y) = e(e^y; 2) \tag{2.100}$$

or

$$\varphi(y) = \sum_{n=0}^{\infty} \frac{(e^y)^n}{(n!)^2} = \sum_{n=0}^{\infty} \frac{e^{yn}}{(n!)^2}. \tag{2.101}$$

Equation (2.99) can be extended to the complex domain. In particular, for pure imaginary $y = i\xi$, we have following equation

$$\left(\frac{d^2}{d\xi^2} + e^{i\xi}\right)\varphi(i\xi) = \left(\frac{d^2}{d\xi^2} + e^{i\xi}\right)\Psi(\xi). \tag{2.102}$$

The meaning of this equation is

$$-\frac{d^2}{d\xi^2}\Psi(\xi) = e^{i\xi}\Psi(\xi) = (\cos\xi + \sin\xi)\Psi(\xi). \tag{2.103}$$

Solution of equation (2.102) is

$$\Psi(\xi) = \sum_{n=0}^{\infty} \frac{e^{i\xi n}}{(n!)^2} = \sum_{n=0}^{\infty} \frac{\cos n\xi + i\sin n\xi}{(n!)^2} \tag{2.104}$$

Proposition 2.6 *(Abramowich 1983) Function $e(x; 2)$ satisfies the differential equation*

$$\frac{d^k}{dx^k}(x^k \frac{d^k}{dx^k} e(x; 2)) = e(x; 2). \tag{2.105}$$

Proposition 2.7 *Function $e(\lambda x; 2)$ where λ is a constant, satisfies the differential equation*

$$\frac{d^k}{dx^k} x^k \frac{d^k}{dx^k} e(\lambda x; 2) = \lambda e(\lambda x; 2). \tag{2.106}$$

Proposition 2.8 *Modified exponential function can be expressed in terms of modified Bessel function*

$$e(x; 2) = I_0(2\sqrt{x}). \tag{2.107}$$

In addition to above properties modified exponential function $e(x; 2)$ has infinite number of zeros located on the negative axis x.

$$e(x_n; 2) = 0, \quad x_n < 0, \quad n = 1, 2, \ldots$$

Proposition 2.9 *If λ_k $(k = 1, 2, 3, \ldots)$ are zeros of modified exponential function $e(x; 2)$, then functions $e(\lambda_k x; 2)$ satisfies the following relations*

$$\int_0^1 e(\lambda_m x; 2) e(\lambda_n x; 2) dx = \begin{cases} 0, & n \neq m \\ -\frac{1}{\lambda}(e'(\lambda; 2))^2 & n = m \end{cases} \tag{2.108}$$

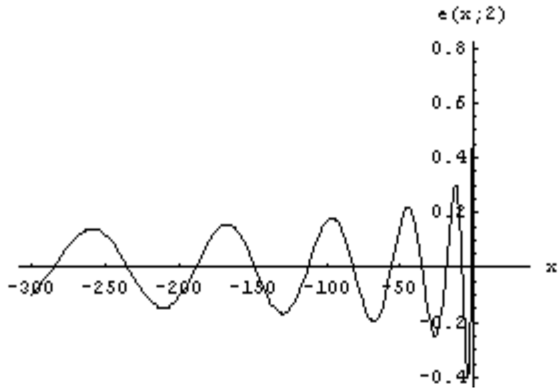

Figure 2.1. Zeros of $e(x;2)$.

Proposition 2.10 *(Abramowich 1983) Let* $e^{(0)}(x;2) = e(x;2)$ *and*

$$e^{(k)}(x;2) = \frac{d^k}{dx^k}e(x;2), \quad e^{(-k)}(x,2) = \int_0^x e^{(-k+1)}(s;2)ds, \quad k \geq 1 \tag{2.109}$$

then functions $e^{(k)}(x;2)$ *possess the generating function*

$$\exp(t + \frac{x}{t}) = \sum_{k=-\infty}^{\infty} t^k e^{(k)}(x;2). \tag{2.110}$$

Proof Expanding left hand side of (2.110) in Taylor series and using the following identities

$$e^{(k)}(x;2) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+k)!}, \quad e^{(-k)}(x;2) = \sum_{n=k}^{\infty} \frac{x^n}{n!(n-k)!}, \tag{2.111}$$

we find that

$$\exp(t + \frac{x}{t}) = \sum_{k=-\infty}^{\infty} t^k e^{(k)}(x;2). \tag{2.112}$$

□

As in the standard case of exponential function we can find related analogs of trigonometric and hyperbolic functions for the modified exponential function. If we write $e(ix;2)$ and use the definition of modified exponential function (2.55), we obtain analog of Euler formula

$$e(x;2) = c(x;2) + is(x;2), \quad i^2 = -1 \tag{2.113}$$

where $c(x;2)$ and $s(x;2)$ modified cosine and sine functions given by

$$c(x;2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k!)^2}, \quad s(x;2) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{((2k-1)!)^2}. \tag{2.114}$$

Functions $c(x;2)$ and $s(x;2)$ obey the rules

$$\frac{d}{dx}\left(x\frac{d}{dx}c(x;2)\right) = -s(x;2), \tag{2.115}$$

$$\frac{d}{dx}\left(x\frac{d}{dx}s(x;2)\right) = c(x;2). \tag{2.116}$$

Applying the operator $\frac{d}{dx}x\frac{d}{dx}$ to the equations (2.115) and (2.116) one more time, leads to the following proposition.

Proposition 2.11 *Functions $c(x;2)$ and $s(x;2)$ are solutions of the differential equation*

$$\left(\frac{d}{dx}x\frac{d}{dx}\right)^2 u + u = 0. \tag{2.117}$$

Furthermore, as in the standard case, we can find modified hyperbolic type functions

$$ch(x;2) = \frac{e(x;2) + e(-x;2)}{2}, \quad sh(x;2) = \frac{e(x;2) - e(-x;2)}{2}$$

where $ch(x;2)$ and $sh(x;2)$ are given by the following formulas

$$ch(x;2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k!)^2}, \quad sh(x;2) = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{((2k-1)!)^2}. \tag{2.118}$$

From the definition of the modified hyperbolic functions (2.118) we find that modified cosine and sine functions satisfy following equations

$$\frac{d}{dx}(x\frac{d}{dx}ch(x;2)) = sh(x;2), \tag{2.119}$$

$$\frac{d}{dx}(x\frac{d}{dx}sh(x;2)) = ch(x;2). \tag{2.120}$$

Hence, following proposition holds.

Proposition 2.12 *Functions $ch(x;2)$ and $sh(x;2)$ are the solutions of the following differential equation*

$$\left(\frac{d}{dx}x\frac{d}{dx}\right)^2 u - u = 0. \tag{2.121}$$

Proposition 2.13 *(Abramowich 1983) Addition rule for the $e(x;2)$ is given by the following formula*

$$e(x+y;2) = \sum_{n=-\infty}^{\infty} e^{(n)}(\frac{x}{2};2)e^{(-n)}(\frac{y}{2};2). \tag{2.122}$$

Proof Using the generating function relation, given in proposition (2.10), we can write

$$e^{t+\frac{x}{t}} = \sum_{n=-\infty}^{\infty} t^n e^n(x;2) \tag{2.123}$$

$$e^{t+\frac{y}{t}} = \sum_{m=-\infty}^{\infty} t^m e^m(y;2). \tag{2.124}$$

Multiplying equations (2.123) and (2.124) then equating the same power of t, we find

$$e(2(x+y);2) = \sum_{n=-\infty}^{\infty} e^n(x;2)e^{-n}(y;2) \tag{2.125}$$

writing $x \to \frac{x}{2}$ and $y \to \frac{y}{2}$, we find

$$e(x+y;2) = \sum_{n=-\infty}^{\infty} e^{(n)}(\frac{x}{2};2)e^{(-n)}(\frac{y}{2};2). \tag{2.126}$$

□

Proposition 2.14 *(Abramowich 1983) The solution of the integral equation*

$$u(x_1, x_2) = 1 + \int\limits_0^{x_1}\int\limits_0^{x_2} u(t_1, t_2)dt_1dt_2 \tag{2.127}$$

with $x_2 = 1$, is given by the modified exponential function $e(x; 2)$.

Proof Let us consider the integral equation

$$u(x_1, x_2) = 1 + \int\limits_0^{x_1}\int\limits_0^{x_2} u(t_1, t_2)dt_1dt_2. \tag{2.128}$$

As an first approximation to the solution, let us take $u(x) = 1$ and substitute into the integral equation and then we find after n iterations

$$\begin{aligned}
u_1(x_1, x_2) &= 1 + x_1x_2 && (2.129)\\
u_2(x_1, x_2) &= 1 + (x_1x_2) + \frac{(x_1x_2)^2}{2^2} && (2.130)\\
u_3(x_1, x_2) &= 1 + (x_1x_2) + \frac{(x_1x_2)^2}{2^2} + \frac{(x_1x_2)^3}{(3!)^2} && (2.131)\\
&\vdots\\
u(x_1, x_2) &= \sum_{k=0}^{\infty} \frac{(x_1x_2)^k}{(k!)^2}. && (2.132)
\end{aligned}$$

Taking $x_2 = 1$, gives the desired result

$$u(x_1, 1) = \sum_{k=0}^{\infty} \frac{(x_1)^k}{(k!)^2}. \tag{2.133}$$

□

2.3.1. Application of Modified Exponential Function to the PDEs

Let us consider a heat equation where the diffusion coefficient is linear function of x

$$u_t = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u, \quad u = u(x, t). \tag{2.134}$$

Separation of variables $u(x, t) = X(x)T(t)$ gives us;

$$\frac{T'}{T} = \frac{(\partial_x x \partial_x)X}{X} = \lambda. \tag{2.135}$$

Thus we find that functions T and X are given by,

$$T(t) = e^{\lambda t}, \quad X(x) = e(\lambda x; 2). \tag{2.136}$$

Since this heat equation is linear, we can write the general solution in the form,

$$u(x,t) = \sum_{i=0}^{\infty} a_{\lambda_i} e^{\lambda_i t} e(\lambda_i x; 2). \tag{2.137}$$

It is well known that one-dimensional heat equation is related with nonlinear PDE called Burgers equation (Burgers 1948) by the Cole-Hopf (Cole 1951 Hopf 1950) transformation. This relation allows not only to find shock soliton solutions of Burgers equation, but also to solve IVP for the last one. Now for our modified version of the heat equation (2.134), we can construct corresponding PDEs. To do this let us define $\phi = \ln u$. Then from the heat equation (2.134), we get the analog of the potential Burgers equation.

$$\phi_t = x\phi_{xx} + \phi_x + x\phi_x^2. \tag{2.138}$$

Taking derivative with respect to x and defining $\phi_x = v$, we get the analog of the Burgers equation with space dependent coefficients given by

$$v_t = xv_{xx} + 2v_x + v^2 + 2xvv_x. \tag{2.139}$$

Then a solution of this Burgers equation is given by the Cole-Hopf transformation

$$v(x,t) = \frac{u_x(x,t)}{u(x,t)}. \tag{2.140}$$

By using solution of heat equation (2.137), we find solution of (2.138) and (2.139) depending on only $e(x,2)$ and its derivatives. Moreover zeros of function $u(x,t)$ becomes the poles of equation (2.139).

$$v(x,t) = \frac{a_{\lambda_i} e^{\lambda_i t} \frac{d}{dx} e(\lambda_i x; 2)}{a_{\lambda_i} e^{\lambda_i t} e(\lambda_i x; 2)}. \tag{2.141}$$

2.3.2. Burgers Hierarchy

We can generalize previous result to the hierarchy of the heat equations (Pashaev and Gürkan 2007). Let us consider the hierarchy of heat equations given for the different times $(t_1, t_2, \ldots)$ as

$$\partial_{t_n} u(x,t) = \partial_x^n x^n \partial_x^n u(x,t), \quad n = 1, 2, \ldots \tag{2.142}$$

Then by the Cole-Hopf transformation (2.140), we find the Burgers equations hierarchy for different times $(t_1, t_2, \ldots)$ as

$$\partial_{t_n} v = \partial_x \left[(\partial_x + v)^n x^n (\partial_x + v)^n .1 \right], \quad n = 1, 2, \ldots \tag{2.143}$$

2.3.3. Spherical Means in Three Dimensional Space

In three dimensional dimensional space, we can write the expression of spherical means as

$$M_u(x,r) = \frac{1}{4\pi}\int_o^{2\pi}\int_0^{\pi} \cosh\left(r\sin\theta\cos\phi\partial_x + r\sin\theta\sin\phi\partial_y + r\cos\theta\partial z\right)u(x,y)\sin\theta d\theta d\phi. \tag{2.144}$$

Derivative of $M_u(x,r)$ is given by

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{4\pi}\int_o^{2\pi}\int_0^{\pi}(\sinh\alpha)\Big(\sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial z\Big)u(x,y)\sin\theta d\theta d\phi \tag{2.145}$$

where $\alpha = (r\sin\theta\cos\phi\partial_x + r\sin\theta\sin\phi\partial_y + r\cos\theta\partial z)$. Differentiating this equation for two times with respect to θ gives us the relation

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{4\pi}\int_o^{2\pi}\int_0^{\pi} r(\cosh\alpha)\Big(\cos\theta\cos\phi\partial_x + \cos\theta\sin\phi\partial_y - \sin\theta\partial z\Big)^2 u(x,y)\sin\theta d\theta d\phi \tag{2.146}$$

Second derivative of $M_u(x,r)$ is

$$\frac{\partial^2}{\partial r^2}M_u(x,r) = \frac{1}{4\pi}\int_o^{2\pi}\int_0^{\pi}(\cosh\alpha)\Big(\sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial z\Big)^2 u(x,y)\sin\theta d\theta d\phi \tag{2.147}$$

writing expressions (2.146) and (2.147), for the first and second derivatives of $M_u(x,r)$, into the Darboux equation (3.8) in three dimensional space we find

$$\Delta_x M_u(x,r) = \frac{1}{4\pi}\int_o^{2\pi}\int_0^{\pi}(\cosh\alpha)\ (\chi_1(\theta,\phi,x,y,z) + \chi_2(\theta,\phi,x,y,z))u(x,y,z)\sin\theta d\theta d\phi \tag{2.148}$$

where the functions α, χ_1, χ_2 are given with

$$\alpha(\theta,\phi,x,y,z) = (r\sin\theta\cos\phi\partial_x + r\sin\theta\sin\phi\partial_y + r\cos\theta\partial z) \tag{2.149}$$

$$\chi_1(\theta,\phi,x,y,z) = 2\Big(\cos\theta\cos\phi\partial_x + \cos\theta\sin\phi\partial_y - \sin\theta\partial z\Big)^2 \tag{2.150}$$

$$\chi_2(\theta,\phi,x,y,z) = \Big(\sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial z\Big)^2. \tag{2.151}$$

2.4. Power of Laplacians and Spherical Means

In n dimensional space the Darboux equation reads

$$\Delta_x M_u(x,r) = (\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r). \tag{2.152}$$

The first and the second derivatives of the expression are given by

$$\Delta_x \frac{\partial}{\partial r} M_u(x,r) = \frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r), \tag{2.153}$$

$$\Delta_x \frac{\partial^2}{\partial r^2} M_u(x,r) = \frac{\partial^2}{\partial r^2}(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r). \tag{2.154}$$

Multiplying equation (2.153) with $\frac{n-1}{r}$ and adding with equation (2.154) together we find that

$$\Delta_x \Delta_x M_u(x,r,t) = \Delta_x^2 M_u(x,r,t) = (\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})M_u(x,r). \tag{2.155}$$

If we do this calculation for arbitrary number k, we find that

$$\Delta_x^k M_u(x,r,t) = (\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r})^k M_u(x,r). \tag{2.156}$$

In the special case, for three dimensional space, we have following relations

$$\Delta_x M_u(x,r,t) = \frac{1}{r}(rM_u)_{rr} \tag{2.157}$$

$$\Delta_x(rM_u(x,r,t)) = (rM_u)_{rr} \tag{2.158}$$

$$\Delta_x^2(rM_u(x,r,t)) = (rM_u)_{rrrr} \tag{2.159}$$

$$\vdots$$

$$\Delta_x^k(rM_u(x,r,t)) = \frac{\partial^{2k}}{\partial r^{2k}}(rM_u). \tag{2.160}$$

Then the differential operator in the form of powers of Laplacian

$$L = \sum_{k=1}^{n} a_k \, \Delta_x^k, \tag{2.161}$$

acts on the function $u(x)$ as

$$L(u(x)) = \sum_{k=1}^{n} a_k \, \Delta_x^k \, u(x) = \frac{1}{r}\sum_{k=1}^{n} a_k \frac{\partial^{2k}}{\partial r^{2k}} \left(rM_u(x,r)\right). \tag{2.162}$$

This allows us to reduce the problem of solution for PDE in three dimensional space in the form $\partial_t u = Lu$, $\partial_{tt}^2 u = Lu$ or $Lu = 0$, to one-dimensional problem for the spherical means.

CHAPTER 3

SOLUTION OF IVP FOR THE WAVE EQUATION BY SPHERICAL MEANS

In the present chapter, we study solution of initial value problem for the wave equation by the method of spherical means. First we consider solution of I.V.P in $3, 5$ and 7 dimensional spaces and then for arbitrary odd dimensional spaces (Evans 1949).

3.1. IVP For The One-Dimensional Wave Equation

The wave equation is a hyperbolic type partial differential equation which arises in the study of many important physical problems involving wave propagation phenomena (Davis 2000 Young 1972), such as the transverse vibrations of an elastic string and the longitudinal vibrations or the torsional oscillations of a rod. It is given with the operator

$$L = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right), \tag{3.1}$$

where u is a function of two independent variables x, t. The function u physically represent the normal displacement of particles of the vibrating string. Initial value problem is given by

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && (3.2) \\ \mathrm{u(x,0)} = f(x), \quad u_t(x,0) &= g(x). && (3.3) \end{aligned}$$

Characteristics are given by,

$$x \pm ct = constant. \tag{3.4}$$

It is natural to take these characteristics as coordinates, say

$$\xi = x + ct, \qquad \eta = x - ct. \tag{3.5}$$

Then equation (3.2) takes the form,

$$u_{\xi\eta} = 0. \tag{3.6}$$

This equation tells us that u must have the form,

$$u = F(\xi) + G(\eta) \tag{3.7}$$

where F and G are arbitrary functions of their arguments. In original variables we find the general solution in the form,

$$u(x,t) = F(x+ct) + G(x-ct). \tag{3.8}$$

Graph of u in xt plane consists of two waves propagating without change of shape with velocity c in opposite directions along the x axis. If we consider wave equation (3.2) with the initial conditions (3.3), by using the solution (3.8) it is found that,

$$F'(x) = \frac{cf'(x) + g(x)}{2c}, \quad G'(x) = \frac{cf'(x) - g(x)}{2c}$$

or equivalently,

$$F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\xi)d\xi \tag{3.9}$$

$$G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\xi)d\xi. \tag{3.10}$$

So solution of the initial value problem for the wave equation is given by,

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi. \tag{3.11}$$

This solution shows that $u(x,t)$ is determined uniquely by values of the initial functions f, g in the interval [x-ct,x+ct] of the x axis whose end points are cut out by the characteristics through the point (x,t). This interval represents the domain of dependence for the solution at the point (x,t).

3.2. Solution of IVP for the Wave Equation in Three Dimensional Space

The initial value problem for the linear wave equation can be solved by the method of spherical means [(John 1981), (Courant and Hilbert 1962)]. Let us consider the function $u(x,t) = u(x_1, x_2, x_3, t)$ which depends on three space variables x_1, x_2, x_3 and time t.

Proposition 3.1 *(John 1981) Solution of the initial value problem for the wave equation*

$$\Box u = u_{tt} - c^2 \triangle u = 0 \tag{3.12}$$

$$\text{u}(\text{x}, 0) = f(x), \quad u_t(x,0) = g(x) \tag{3.13}$$

in $3+1$ *dimensional space is given by*

$$u(x,t) = \frac{1}{4\pi c^2 t} \int\limits_{|y-x|=ct} g(y) dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \int\limits_{|y-x|=ct} f(y) dS_y \right) \tag{3.14}$$

where dS_y *is the element of surface area of the sphere with radius ct and centered at* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., y_n)$ *is a point on the sphere.*

Proof In the method of spherical means by using the Darboux equation (3.8), the initial-value problem (3.12),(3.13) can be transformed into the one for $M_u(x,t;r)$,

$$\frac{\partial^2}{\partial t^2} M_u = c^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) M_u, \tag{3.15}$$

$$M_u(x,0;r) = M_f(x,r), \quad \frac{\partial}{\partial t} M_u(x,0;r) = M_g(x,r), \tag{3.16}$$

where M_f and M_g are spherical means of initial functions f and g correspondingly. But equation (3.15) implies;

$$\frac{\partial^2}{\partial t^2}(rM_u) = c^2 \frac{\partial^2}{\partial r^2}(rM_u), \tag{3.17}$$

thus $rM_u(x,t;r)$ as a function of r, t, can be threaded as a solution of one dimensional wave equation with initial values;

$$rM_u(x,t;r) = rM_f(x;r), \quad \frac{\partial}{\partial t}(rM_u(x,t;r)) = rM_g(x;r), \quad t = 0. \tag{3.18}$$

In terms of a new function $N(x,t;r) = rM_u(x,t;r)$, equation (3.17) takes the form

$$N_{tt} - c^2 N_{rr} = 0 \tag{3.19}$$

which has the general solution,

$$N(x,t;r) = \tilde{f}(x, r-ct) + \tilde{g}(x, r+ct). \tag{3.20}$$

From initial value problem (3.17) and (3.18) it can be found that;

$$\begin{aligned} M_u(x,t;r) &= \frac{1}{2r}[(ct+r)M_f(x, ct+r) - (ct-r)M_f(x, ct-r)] \\ &+ \frac{1}{2cr}\int_{ct-r}^{ct+r} \xi M_g(x,\xi) d\xi. \end{aligned} \tag{3.21}$$

Letting r goes to zero, solution of the initial value problem (3.12),(3.13) is found in the form;

$$u(x,t) = tM_g(x,ct) + \frac{\partial}{\partial t}\left(tM_f(x,ct)\right)$$

or

$$u(x,t) = \frac{1}{4\pi c^2 t} \int\limits_{|y-x|=ct} g(y) dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \int\limits_{|y-x|=ct} f(y) dS_y \right). \tag{3.22}$$

□

3.3. Solution of IVP for Wave Equation in Five Dimensional Space

Similar to the previous case, the method of spherical means allows us to find the solution of initial value problem for the wave equation in terms of spherical means of initial functions in five dimensional space.

Proposition 3.2 *Solution of the initial value problem for the wave equation in* $5+1$ *dimensional space*

$$\Box u = u_{tt} - c^2 \triangle u = 0 \tag{3.23}$$

$$\mathrm{u(x,0)} = f(x), \quad u_t(x,0) = g(x) \tag{3.24}$$

is given by

$$u(x,t) = (\frac{1}{3}t^2\frac{\partial}{\partial t} + t)M_g(x,ct) + \frac{\partial}{\partial t}(\frac{1}{3}t^2 + t)M_f(x,ct). \tag{3.25}$$

Proof When the space dimension is five the substitution $\tilde{\psi}(x,t;r) = rM_u(x,t;r)$ does not reduce the Euler-Poison-Darboux Equation (2.2) into the canonical wave equation (3.2) anymore. But the function given by,

$$N(x,t;r) = r^2\frac{\partial}{\partial r}M_u(x,t;r) + 3rM_u(x,t,;r) \tag{3.26}$$

reduces equation $\frac{\partial^2}{\partial t^2}M_u = c^2(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r})M_u$, into the canonical form of wave equation

$$\frac{\partial^2}{\partial t^2}N = c^2\frac{\partial^2}{\partial r^2}N.$$

When r approaches the zero, $M_u(x,t;r)$ approaches $u(x,t)$. Then by using equation (3.26), it is found that

$$u(x,t) = \lim_{r\to 0}\frac{N(x,r,t)}{3r}. \tag{3.27}$$

Finally, solution of the initial value problem (3.23) and (3.24) is given in terms of the spherical means of the initial functions as;

$$u(x,t) = (\frac{1}{3}t^2\frac{\partial}{\partial t} + t)M_g(x,ct) + \frac{\partial}{\partial t}(\frac{1}{3}t^2 + t)M_f(x,ct). \tag{3.28}$$

□

3.4. Solution of IVP for Wave Equation in Seven Dimensional Space

Solution of initial value problem in seven dimensional space is given by the following proposition.

Proposition 3.3 *Solution of the initial value problem for the wave equation in* $7+1$ *dimensional space*

$$\square u = u_{tt} - c^2 \triangle u = 0 \tag{3.29}$$

$$\mathrm{u(x,0)} = f(x), \quad u_t(x,0) = g(x) \tag{3.30}$$

is given by

$$u(x,t) = \left(\frac{t^3}{15}\frac{\partial^2}{\partial t^2} + \frac{9t^2}{15}\frac{\partial}{\partial t} + t\right) M_g(x,ct) + \frac{\partial}{\partial t}\left(\left(\frac{t^3}{15}\frac{\partial^2}{\partial t^2} + \frac{9t^2}{15}\frac{\partial}{\partial t} + t\right) M_f(x,ct)\right). \tag{3.31}$$

Proof Similar to the previous case, we can convert the Euler-Poisson-Darboux equation $\frac{\partial^2}{\partial t^2} M_u = c^2 (\frac{\partial^2}{\partial r^2} + \frac{6}{r}\frac{\partial}{\partial r}) M_u$, into the canonical wave form by transformation,

$$N(x,t;r) = r^3 \frac{\partial^2}{\partial r^2} M_u(x,t;r) + 9r^2 \frac{\partial}{\partial r} M_u(x,t;r) + 15 r M_u(x,t;r).$$

The function $u(x,t)$ can be recovered from $N(x,t;r)$ by the limit,

$$u(x,t) = \lim_{r\to 0} \frac{N(x,r,t)}{15r}.$$

Following the same procedure as before, we find solution of initial value problem for the wave equation in seven dimensional space as,

$$u(x,t) = \frac{t^3}{15}\frac{\partial^3}{\partial t^3} M_f + \frac{12t^2}{15}\frac{\partial^2}{\partial t^2} M_f + \frac{33t}{15}\frac{\partial}{\partial t} M_f + M_f \tag{3.32}$$

$$+ \frac{t^3}{15}\frac{\partial^2}{\partial t^2} M_g + \frac{9t^2}{15}\frac{\partial}{\partial t} M_g + t M_g. \tag{3.33}$$

After some rearrangements it can be written as;

$$u(x,t) = \left(\frac{t^3}{15}\frac{\partial^2}{\partial t^2} + \frac{9t^2}{15}\frac{\partial}{\partial t} + t\right) M_g(x,ct) + \frac{\partial}{\partial t}\left(\left(\frac{t^3}{15}\frac{\partial^2}{\partial t^2} + \frac{9t^2}{15}\frac{\partial}{\partial t} + t\right) M_f(x,ct)\right). \tag{3.34}$$

□

3.5. Solution of IVP for the Wave Equation in Arbitrary Odd Dimensional Space

We have solved the initial value problem for the wave equation by using spherical means in three, five and seven dimensional spaces and found the expressions for the function $u(x,t)$ in terms of spherical means of initial functions, and their derivatives.

Generalization of this idea immediately tells us that in principle, we can find a transformation which reduces the Euler-Poisson-Darboux equation (2.2) into the canonical wave equation (3.2). To do this let us search a function in the form;

$$N(x,t;r) = \sum_{k=0}^{\frac{n-3}{2}} a_k r^{k+1} M_u^{(k)}(x,t;r) \tag{3.35}$$

where n is the number of space variables, $M_u^{(k)}(x,t;r)$ denotes $\frac{\partial^k}{\partial r^k}M_u$ and a_k are constants especially $a_{\frac{n-3}{2}} = 1$. We require that function $N(x,t;r)$ obtained from the linear combination of the spherical means of u and its derivatives, satisfies the canonical wave equation

$$N_{tt}(x,t,;r) - c^2 N_{rr}(x,t;r) = 0. \tag{3.36}$$

By using the Euler- Poisson-Darboux equation (2.2), it is found that coefficients a_k must satisfy the following algebraic relation,

$$\sum_{k=0}^{\frac{n-3}{2}} a_k r^{k+1}\left(\frac{n-1}{r}M^{(1)}\right)^{(k)} - \sum_{k=0}^{\frac{n-3}{2}} a_k k(k+1) r^{k-1} M_u^{(k)} - 2\sum_{k=0}^{\frac{n-3}{2}} a_k (k+1) r^k M_u^{(k+1)} = 0. \tag{3.37}$$

Solving this equation allows us to find $N(x,t;r)$. We give explicit form of functions $N(x,t;r)$ for some different space variables n;

for $n = 3$, $N(x,t;r) = rM_u(x,t;r)$

for $n = 5$, $N(x,t;r) = r^2(M_u)_r(x,t;r) + 3rM_u(x,t;r)$

for $n = 7$, $N(x,t;r) = r^3(M_u)_{rr}(x,t;r) + 9r^2(M_u)_r(x,t;r) + 15rM_u(x,t;r)$

for $n = 9$, $N(x,t;r) = r^4(M_u)_{rrr}(x,t;r) + 18r^3(M_u)_{rr}(x,t;r) + 87r^2(M_u)_r(x,t;r) + 105rM_u(x,t;r)$.

Function $N(x,t;r)$ given with equation (3.35), allows us to solve initial value problem for the wave equation in higher dimensional spaces.

Proposition 3.4 *(Evans 1949) Solution of the initial value problem*

$$\Box u = u_{tt} - c^2 \triangle u = 0 \tag{3.38}$$

$$\mathrm{u(x,0)} = f(x), \quad u_t(x,0) = g(x) \tag{3.39}$$

for the wave equation in $n+1$ *dimensions, where* $n = 2k+1$, $(k = 1,2,3,...)$ *is given by*

$$u(x,t) = \frac{1}{a_0}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}M_f\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}M_g\right)\right] \tag{3.40}$$

where $a_0 = 1.3.5.7....(2k-1) = (2k-1)!!$.

Proof Spherical means M_u of function u satisfies the initial value problem

$$\frac{\partial^2}{\partial t^2}M_u = c^2\left(\frac{\partial^2}{\partial r^2} + \frac{\mathrm{n}-1}{r}\frac{\partial}{\partial r}\right)M_u, \tag{3.41}$$

$$M_u(x,0;r) = M_f(x,r), \quad \frac{\partial}{\partial t}M_u(x,0;r) = M_{\mathrm{g}}(x,r). \tag{3.42}$$

Then for function $N(x,t;r) = (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}M_u(x,t;r))$ initial value problem is given by

$$N_{tt} - N_{rr} = 0, \tag{3.43}$$

$$N(x,0,r) = F(x;r), \quad N_t(x,0;r) = G(x;r). \tag{3.44}$$

Initial value problem (3.43) and (3.44) has solution

$$N(x,t;r) = \frac{1}{2}[F(x,r+t) - F(x,r-t)] + \frac{1}{2}\int_{t-r}^{t+r} G(x,\rho)d\rho. \tag{3.45}$$

Original function $u(x,t)$ can be recovered by the following limit

$$\lim_{r\to 0}\frac{N(x,t;r)}{a_0 r} = u(x,t). \tag{3.46}$$

Then solution of the ivp (3.38) and (3.39) is given by

$$u(x,t) = \frac{1}{a_0}\lim_{r\to 0}\left[\frac{F(x,r+t) - F(x,t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} G(x,\rho)d\rho\right]. \tag{3.47}$$

Finally, we find solution of initial value problem $u(x,t)$, in terms of spherical means of initial functions,

$$\begin{aligned} u(x,t) &= \frac{1}{a_0}\left[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} f dS\right) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} g dS\right)\right] \\ u(x,t) &= \frac{1}{a_0}\left[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}M_f\right) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}M_g\right)\right]. \end{aligned} \tag{3.48}$$

Where $a_0 = 1.3.5.7....(2k-1) = (2k-1)!!$. □

3.6. Hadamard's Method of Descent

Above we have solved the initial value problem for wave equation in odd dimensional spaces by reducing it to i.v.p for the one-dimensional wave equation for new function $N(x,t;r)$ (3.35). But situation is different for the even dimensional spaces. In general one can not find a transformation which reduces Euler-Poisson-Darboux equation to the canonical wave form. But every function in even ($2n$) dimensional space can be considered as ($2n+1$) odd dimensional function with one component fixed. For example, two dimensional function $u = u(x_1,x_2,t)$ can be considered as a three dimensional function $u = u(x_1,x_2,x_3,t)$ with $x_3 = constant$. This way, solution for even dimensional spaces can be found. This method is Hadamard Method of Descent (John 1981). Let us consider, for the simplicity, $n = 2$ dimensional space and initial value problem

$$\Box u = u_{tt} - c^2\triangle u = 0 \tag{3.49}$$

$$\text{u(x,0)} = f(x), \quad u_t(x,0) = g(x), \quad x \in R^2. \tag{3.50}$$

To solve this problem, we start from $n = 3$ dimensional solution (3.14) and take limit $x_3 = 0$. We average function $u(x,t)$ on the sphere with radius ct,

$$|y-x| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + y_3^2} = ct. \tag{3.51}$$

The surface element for this sphere is

$$dS_y = \frac{ct}{|y_3|} d_{y_1} d_{y_2}. \tag{3.52}$$

By using the solution of initial value problem in three dimensional space

$$u(x,t) = \frac{1}{4\pi c^2 t} \int\limits_{|\mathrm{y}-\mathrm{x}|=\mathrm{ct}} g(\mathrm{y}) dS_{\mathrm{y}} + \frac{\partial}{\partial t}\Big(\frac{1}{4\pi c^2 t} \int\limits_{|\mathrm{y}-\mathrm{x}|=\mathrm{ct}} f(\mathrm{y}) dS_{\mathrm{y}}\Big) \tag{3.53}$$

we find the solution in two dimensional space as

$$u(x,t) = \frac{1}{2\pi c} \int\int\limits_{\mathrm{r<ct}} \frac{g(y_1,y_2)}{\sqrt{c^2t^2 - r^2}} d_{y_1} d_{y_2} + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \int\int\limits_{\mathrm{r<ct}} \frac{f(y_1,y_2)}{\sqrt{c^2t^2 - r^2}} d_{y_1} d_{y_2}\right). \tag{3.54}$$

where $r = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$.

CHAPTER 4

LIOUVILLE EQUATION

In the present chapter, we consider nonlinear wave equation with exponential nonlinearity called Liouville equation. First, we shortly review the relation between $1+1$ dimensional Liouville equation and the surface theory. Then we introduce the spherical Liouville equation which is defined in $3+1$ dimensional space and relate it with the spherically symmetric linear wave equation by the Bäcklund transformation. It allows us to write the general solution of spherical Liouville equation. After solving the initial value problem in a particular form for the spherical Liouville equation, we give the progressing wave solutions. We also give expression for the Lax pair of spherical Liouville equation. For arbitrary odd dimensional space, we consider the spherical Liouville equation with some potential. We find its general solution and solve initial value problem. In addition to these, progressing wave solution is constructed.

4.1. Liouville Equation and Surface Geometry

When the line element of a surface is given in terms of conformal coordinates, we have the following theorem.

Theorem 4.1 *(Dubrovin, Fomenko and Novikov 1984) If u and v are conformal coordinates on a surface in Euclidean 3-space, in terms of which the induced metric has the form*

$$dl^2 = g(u,v)\left(du^2 + dv^2\right), \tag{4.1}$$

then the Gaussian curvature of the surface is given by

$$K = -\frac{1}{2g(u,v)} \triangle \ln g(u,v) \tag{4.2}$$

where $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ *is the Laplace operator.*

Proof Suppose that in terms of conformal coordinate u, v the surface is given (locally) by $r = r(u,v)$; $r = (x,y,z)$ (where x,y,z are Euclidean coordinates for the R^3). Since the metric on the surface is given by $dl^2 = g(u,v)\left(du^2 + dv^2\right)$, we have

$$\langle r_u, r_u \rangle = \langle r_v, r_v \rangle = g(u,v), \quad \langle r_u, r_v \rangle = 0.$$

By differentiating these equations with respect to u, v, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial g(u,v)}{\partial u} &= \langle r_{uu}, r_u \rangle = \langle r_{uv}, r_v \rangle, & (4.3)\\ \frac{1}{2}\frac{\partial g(u,v)}{\partial v} &= \langle r_{vv}, r_v \rangle = \langle r_{uv}, r_u \rangle, & (4.4)\\ \langle r_{uu}, r_v \rangle + \langle r_u, r_{uv} \rangle &= 0, & (4.5)\\ \langle r_{uv}, r_v \rangle + \langle r_u, r_{vv} \rangle &= 0. & (4.6) \end{aligned}$$

We define unit vectors as

$$e_1 = \frac{r_u}{\sqrt{g(u,v)}}, e_2 = \frac{r_v}{\sqrt{g(u,v)}}, n = [e_1, e_2].$$

By the form of the metric and properties of the vector product, the frame e_1, e_2, n is orthonormal at each point of the surface. In addition, vectors e_1, e_2 are tangent to the surface and vector n is normal to it. Coefficients of the second fundamental form of the surface are

$$\begin{aligned} b_{11} &= L = \langle r_{uu}, n \rangle, \\ b_{12} &= M = \langle r_{uv}, n \rangle, & (4.7)\\ b_{22} &= N = \langle r_{vv}, n \rangle. \end{aligned}$$

It follows from (4.3),(4.4) and (4.7) that relative to the basis e_1, e_2, n the components of the vectors r_{uu}, r_{uv}, r_{vv} are as follows;

$$\begin{aligned} r_{uu} &= \left(\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial u}, -\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial v}, L \right) & (4.8)\\ r_{uv} &= \left(\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial v}, \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial u}, M \right) & (4.9)\\ r_{vv} &= \left(-\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial u}, \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial v}, N \right). & (4.10) \end{aligned}$$

Hence following relation holds

$$\langle r_{uu}, r_{vv} \rangle - \langle r_{uv}, r_{uv} \rangle = LN - M^2 - \frac{1}{2g}\left[(\frac{\partial g}{\partial u})^2 + (\frac{\partial g}{\partial v})^2 \right]. \tag{4.11}$$

From the results given by equations (), (4.3) and equation (4.4), it is found that

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 g}{\partial u^2} &= \langle r_{uuv}, r_v \rangle + \langle r_{uv}, r_{uv} \rangle & (4.12)\\ \frac{1}{2}\frac{\partial^2 g}{\partial u^2} &= \frac{\partial}{\partial v}\langle r_{uu}, r_v \rangle - \langle r_{uu}, r_{vv} \rangle + \langle r_{uv}, r_{uv} \rangle & (4.13)\\ \frac{1}{2}\frac{\partial^2 g}{\partial u^2} &= -\frac{1}{2}\frac{\partial^2 g}{\partial v^2} - (LN - M^2) + \frac{1}{2g}\left[(\frac{\partial g}{\partial u})^2 + (\frac{\partial g}{\partial v})^2 \right]. & (4.14) \end{aligned}$$

Then Gaussian curvature of the surface is given by,

$$K = \frac{det(b_{ij})}{det(g_{ij})} = \frac{LN - M^2}{(g(u,v))^2} = -\frac{1}{2g(u,v)} \vartriangle \ln g(u,v). \tag{4.15}$$

For a given surface, if the gaussian curvature is non-zero then the metric $g(u,v)$ satisfies the Liouville equation

$$\vartriangle \ln g = -2Kg. \tag{4.16}$$

Introducing new field by $g = e^{\varphi}$, we find the elliptic Liouville equation

$$\vartriangle \varphi = -2Ke^{\varphi}. \tag{4.17}$$

When the surface is pseudo-Euclidean, the metric in conformal coordinates is given by

$$dl^2 = g(t,x)(dt^2 - dx^2) \tag{4.18}$$

then for the constant gaussian curvature, it satisfies the hyperbolic Liouvile equation

$$(\partial_t^2 - \partial_x^2)\varphi = -2Ke^{\varphi} \tag{4.19}$$

where $g = e^{\varphi}$. In characteristic coordinates $\xi = x + t$, $\eta = x - t$ the equation has the form

$$\varphi_{\xi\eta} = -2Ke^{\varphi}. \tag{4.20}$$

□

4.1.1. Bäcklund Transformation for Liouville Equation

The Liouville equation is nonlinear but can be transformed directly to the linear wave equation, which allows to solve it exactly. This transformation is called the Bäcklund transformation (Clairin 1902), (Forsyth 1959).

Theorem 4.2 *Bäcklund transformation relating the Liouville equation $u_{\xi\eta} = e^u$ with the wave equation $\tilde{u}_{\xi\eta} = 0$ is given by*

$$\tilde{u}_{\xi} = u_{\xi} + ke^{\frac{\tilde{u}+u}{2}} \tag{4.21}$$

$$\tilde{u}_{\eta} = -u_{\eta} - \frac{2}{k}e^{-\frac{\tilde{u}-u}{2}}. \tag{4.22}$$

Proof Let us consider two functions u and $\tilde{u}$ satisfying

$$u_{\xi\eta} = e^u, \;\; \tilde{u}_{\xi\eta} = 0 \tag{4.23}$$

Now let us derive Bäcklund transformation relating these two equations. To do this let us consider following system

$$\tilde{u}_\xi = u_\xi + f(\tilde{u}, u) \tag{4.24}$$

$$\tilde{u}_\eta = -u_\eta + g(\tilde{u}, u) \tag{4.25}$$

where functions f, g are arbitrary and have first derivatives according to the arguments. Compatibility condition of this system by equation (4.23) gives us

$$\Omega = f_{\tilde{u}}(-u_\eta + g) + f_u u_\eta - g_{\tilde{u}}(u_\xi + f) - g_u u_\xi + 2e^u = 0. \tag{4.26}$$

Since the Ω is identically zero, its derivatives with respect to $u, \tilde{u}$ are also zero

$$\Omega_{(u_\xi)} = -g_u - g_{\tilde{u}} = 0 \tag{4.27}$$

$$\Omega_{(u_\eta)} = f_u - f_{\tilde{u}} = 0. \tag{4.28}$$

These two relation simplify the expression for the Ω as

$$\Omega = g f_{\tilde{u}} - f g_{\tilde{u}} + 2e^u. \tag{4.29}$$

Taking one more derivative with respect to u gives

$$g f_{\tilde{u}\tilde{u}} - f g_{\tilde{u}\tilde{u}} = 0. \tag{4.30}$$

According to the equation (4.27) and (4.28), we find that functions f, g must have the form

$$g(\tilde{u}, u) = g(\tilde{u} - u) \tag{4.31}$$

$$f(\tilde{u}, u) = f(\tilde{u} + u). \tag{4.32}$$

Let us define two new variables as

$$\xi = \tilde{u} - u \tag{4.33}$$

$$\eta = \tilde{u} + u. \tag{4.34}$$

Then equation (4.30) takes the form

$$g(\xi)\frac{d^2 f}{d\eta^2} - f(\eta)\frac{d^2 g}{d\xi^2} = 0. \tag{4.35}$$

Separating this equation and equating to the k^2, allows us to write

$$f(\eta) = ae^{k\eta} + be^{-k\eta} \tag{4.36}$$

$$g(\xi) = ce^{k\xi} + de^{-k\xi}. \tag{4.37}$$

Substituting the values of f and g in the equation (4.24), gives

$$\tilde{u}_\xi = u_\xi + ae^{k\eta} + be^{-k\eta} \tag{4.38}$$
$$\tilde{u}_\eta = -u_\eta + ce^{k\xi} + de^{-k\xi}. \tag{4.39}$$

Applying the compatibility condition and require that $\tilde{u}_{\xi\eta} = 0$, in the following equation

$$\tilde{u}_{\xi\eta} = u_{\xi\eta} + k(ace^{2k\tilde{u}} + ade^{2ku} - bce^{-2ku} - bde^{-2k\tilde{u}}) \tag{4.40}$$

Choosing $k = \frac{1}{2}, a = d = 0, c = \frac{2}{b}$, gives the Bäcklund transformation for the Liouville equation (Lamb 1975)

$$\tilde{u}_\xi = u_\xi + be^{\frac{\tilde{u}+u}{2}} \tag{4.41}$$
$$\tilde{u}_\eta = -u_\eta - \frac{2}{b}e^{-\frac{\tilde{u}-u}{2}}. \tag{4.42}$$

□

Once the relation between Liouville equation and linear wave equation is established, then general solution of Liouville equation can be found by using this relation.

Theorem 4.3 *General solution of the Liouville equation is given by (Liouville 1853)*

$$u(x,t) = \ln\left(\frac{A'(x)B'(t)}{(A(x)+B(x))^2}\right). \tag{4.43}$$

Proof The general solution of the Liouville equation can be found by using the Bäcklund transformation which relates the Liouville equation with the linear wave equation. Let us consider the Bäcklund transformation

$$u_x - \tilde{u}_x = -2ke^{\frac{1}{2}(u+\tilde{u})} \tag{4.44}$$
$$u_t + \tilde{u}_t = \frac{-1}{k}e^{\frac{1}{2}(u-\tilde{u})} \tag{4.45}$$

compatibility of equations (4.44) and (4.45) gives

$$u_{xt} - e^u = \tilde{u}_{xt}. \tag{4.46}$$

Solution of linear wave equation is of the form $\tilde{u} = f(x) + g(t)$. Substituting this solution into the Bäcklund transformation (4.44) and (4.45), we get

$$u_x - f' = -2ke^{\frac{1}{2}(u+f+g)} \tag{4.47}$$
$$u_t + g' = \frac{-1}{k}e^{\frac{1}{2}(u-f-g)}. \tag{4.48}$$

To handle with the exponential term, let us introduce

$$\phi(x,t) = e^{\frac{1}{2}u(x,t)}. \tag{4.49}$$

This new function satisfies the following equation

$$\begin{aligned} \phi_x - \frac{f'}{2}\phi &= -k\phi^2 e^{\frac{1}{2}(f+g)} && (4.50) \\ \phi_t - \frac{g'}{2}\phi &= \frac{-1}{2k}\phi^2 e^{\frac{-1}{2}(f+g)}. && (4.51) \end{aligned}$$

Multiplying equations (4.50) and (4.51) with $e^{\frac{-f}{2}}$ and $e^{\frac{g}{2}}$ respectively, gives

$$\begin{aligned} \frac{d}{dx}(e^{\frac{-f}{2}}\phi) &= -k\phi^2 e^{\frac{g}{2}} && (4.52) \\ \frac{d}{dt}(e^{\frac{g}{2}}\phi) &= \frac{-1}{2k}\phi^2 e^{\frac{-f}{2}}. && (4.53) \end{aligned}$$

If we multiply first and second equation $e^{\frac{g}{2}}$ and $e^{\frac{-f}{2}}$ respectively we can write

$$\begin{aligned} \frac{d}{dx}(e^{\frac{-f}{2}+\frac{g}{2}}\phi) &= -k\phi^2 e^{g} && (4.54) \\ \frac{d}{dt}(e^{\frac{-f}{2}+\frac{g}{2}}\phi) &= \frac{-1}{2k}\phi^2 e^{-f}. && (4.55) \end{aligned}$$

Introducing function $\chi(x,t)$ as

$$\chi(x,t) = \phi(x,t)e^{\frac{-f}{2}+\frac{g}{2}} \tag{4.56}$$

we find

$$\begin{aligned} \frac{d\chi}{dx} &= -k\chi^2 e^{f} && (4.57) \\ \frac{d\chi}{dt} &= \frac{-1}{2k}\chi^2 e^{-g}. && (4.58) \end{aligned}$$

Solution of these equations gives us

$$\chi(x,t) = \frac{1}{k\int^x e^{f(\rho)d\rho} + \frac{1}{2k}\int^t e^{-g(\rho)d\rho}}. \tag{4.59}$$

Returning the original function, we find the general solution the Liouville Equation as

$$u(x,t) = 2\ln\left(\frac{e^{\frac{1}{2}f}e^{-\frac{1}{2}g}}{k\int^x e^{f(\rho)d\rho} + \frac{1}{2k}\int^t e^{-g(\rho)d\rho}}\right). \tag{4.60}$$

Introducing two functions $A(x) = k\int^x e^{f(\rho)d\rho}$ and $B(t) = \frac{1}{2k}\int^t e^{-g(\rho)d\rho}$, we can write the general solution in a compact form as

$$u(x,t) = \ln\left(2\frac{A'(x)B'(t)}{(A(x)+B(x))^2}\right). \tag{4.61}$$

□

4.2. Spherical Liouville Equation

Described above the Liouville equation admits the general solution in one space dimension. In higher dimensions it does not work anymore. If one considers the Liouville equation in 3 + 1 dimensions,

$$v_{tt} - c^2 \triangle v = e^v \tag{4.62}$$

for the spherical symmetric solutions $v(x,t;r) = M(r,t)$; $r = \sqrt{x^2+y^2+z^2}$ it is possible to reduce the equation to 1 + 1 dimensional model

$$M_{tt} - c^2 M_{rr} - \frac{2c^2}{r} M_r = e^M. \tag{4.63}$$

However, this equation does not seems to be solvable in general. In the present section, we introduce modified form of this equation which could be studied at the same level of completeness as in one-dimension.

Definition 4.1 *The Spherical Liouville equation (SLE) for function $M_v(r,t)$ is defined by the equation*

$$(M_v)_{tt} - c^2 (M_v)_{rr} - \frac{2c^2}{r}(M_v)_r = -\frac{4c^2}{r} e^{r(M_v)}. \tag{4.64}$$

The form of the equation is motivated by the method of spherical means for the linear wave equation in 3 + 1 dimensions,

$$v_{tt} - c^2 \triangle v = 0 \tag{4.65}$$

which becomes formally in the form of 1 + 1 dimensional case

$$(rM_v)_{tt} - c^2 (rM_v)_{rr} = 0 \tag{4.66}$$

where M_v is the spherical means for the function v. It has solution

$$M_v(x,t;r) = \frac{f(x, r+ct) + g(x, r-ct)}{r}, \tag{4.67}$$

equation (4.66) can be threaded as an 1 + 1 dimensional wave equation with the general solution (4.67). But we know that the 1 + 1 dimensional wave equation can be related with the Liouville equation by the Bäcklund transformation (4.44),(4.45). In our case the similar Bäcklund transformation

$$(\partial_t + c\partial_r)(r(M_v - M_u)) = -4kce^{\frac{r}{2}(M_v+M_u)} \tag{4.68}$$

$$(\partial_t - c\partial_r)(r(M_v + M_u)) = \frac{2c}{k} e^{\frac{r}{2}(M_v-M_u)} \tag{4.69}$$

relates the spherical means $M_u(x,t;r)$ with function $M_v(x,t;r)$. From this Bäcklund Transformation it is found that,

$$(\partial_t^2 - c^2\partial_r^2)(rM_v) + 4c^2 e^{rM_v} = (\partial_t^2 - c^2\partial_r^2)(rM_u). \tag{4.70}$$

Hence if $M_u(x,t;r)$ is a spherical mean satisfying the D'Alambert equation

$$(\partial_t^2 - c^2\partial_r^2)(rM_u) = 0 \tag{4.71}$$

then $M_v(x,t;r)$ is the solution of following equation

$$(M_v)_{tt} - c^2(M_v)_{rr} - \frac{2c^2}{r}(M_v)_r = -\frac{4c^2}{r}e^{r(M_v)}. \tag{4.72}$$

It should be noted here that the function $M_v(x,t;r)$ can not be considered as the spherical means of equation (4.62) since equation is nonlinear. If we average the Liouville equation (4.62), we find

$$\begin{aligned} (M_v)_{tt} - c^2 \bigtriangleup M_v &= M_{e^v} && (4.73)\\ (M_v)_{tt} - c^2(M_v)_{rr} - \frac{2c^2}{r}(M_v)_r &= M_{e^v}. && (4.74) \end{aligned}$$

Spherical Liouville equation (4.64) can be considered as the spherical symmetric reduction of 3 + 1 dimensional model. Let function $u = u(x,y,z)$ satisfies the nonlinear equation

$$\partial_t^2 u - c^2 \bigtriangleup u = -\frac{4}{r}e^{ru}. \tag{4.75}$$

Then for the spherical symmetric solutions $u(x,y,z;t) = R(r;t)$, $r = \sqrt{x^2+y^2+z^2}$ we have the spherical Liouville equation (4.64)

$$\partial_t^2 R - c^2\partial_r^2 R - \frac{2c^2}{r}\partial_r R = -\frac{4c^2}{r}e^{rR}. \tag{4.76}$$

This type of equation also occurs when we consider Yang- Mills Equations with external sources(German G., 1984).

4.2.1. General Solution of Spherical Liouville Equation

Theorem 4.4 *General solution of spherical Liouville Equation (4.64) is given by*

$$M_v(x,t;r) = \frac{1}{r}\ln\frac{2A'(x,r+ct)B'(x,r-ct)}{(A(x,r+ct)+B(x,r-ct))^2} \tag{4.77}$$

where the primes denote derivative according to the second argument.

Proof Substituting solution of spherically symmetric wave equation (4.67) into the Bäcklund transformations (4.68),(4.69) and integrating gives,

$$M_v(x,t;r) = \frac{1}{r}\ln\frac{e^{\frac{1}{2}\mathrm{f(x,r+ct)}}e^{-\frac{1}{2}\mathrm{g(x,r-ct)}}}{(k\int^{\mathrm{r+ct}}e^{\mathrm{f(x,\rho)}}d\rho + \frac{1}{2k}\int^{\mathrm{r-ct}}e^{-\mathrm{g(x,\rho)}}d\rho)^2}. \tag{4.78}$$

If we introduce two new functions A and B instead of arbitrary functions f and g

$$A(x, r+ct) = k\int^{\mathrm{r+ct}} e^{\mathrm{f(x,\rho)}}d\rho, \quad B(x, r-ct) = \frac{1}{2k}\int^{\mathrm{r-ct}} e^{-\mathrm{g(x,\rho)}}d\rho,$$

then we obtain,

$$M_v(x,t;r) = \frac{1}{r}\ln\frac{2A'(x,r+ct)B'(x,r-ct)}{(A(x,r+ct)+B(x,r-ct))^2}. \tag{4.79}$$

□

4.3. Initial Value Problem for Spherical Liouville Equation

The general solution (4.79) can be applied to solve IVP for the SLE (4.64). Similar to the 1+1 dimensional case (Jorjadze, Pogrebkov and Polivanov 1978), we consider the following special case of initial value problem for SLE (4.64) for function $M_v(x,t;r)$, $r > 0$:

Proposition 4.1 *The solution of the initial value problem for the spherical Liouville equation*

$$(M_v)_{tt} - c^2(M_v)_{rr} - \frac{2c^2}{r}(M_v)_r = -\frac{4c^2}{r}e^{rM_v}, \tag{4.80}$$

$$M_v(x,0;r) = M_h(x,r), \quad (M_v)_t(x,0;r) = 0. \tag{4.81}$$

is given by

$$\begin{aligned} M_v(x,t;r) &= \frac{1}{2r}\left[(r+ct)M_h(x,r+ct) + (r-ct)M_h(x,r-ct)\right] \\ &- \frac{1}{r}\ln\cosh^2\frac{1}{\sqrt{2}}\int_{\mathrm{r-ct}}^{\mathrm{r+ct}} e^{\frac{\rho \mathrm{M_h(x,\rho)}}{2}}d\rho \end{aligned} \tag{4.82}$$

where $M_h(x,r)$ denotes the spherical means of function $h(x)$.

Proof To solve initial value problem, we rewrite the spherical Liouville equation (4.64) as;

$$(rM_v)_{tt} - c^2(rM_v)_{rr} = -4c^2e^{rM_v}. \tag{4.83}$$

If we introduce $\Psi(x,t;r) = rM_v(x,t;r)$, then initial value problem (4.80), (4.81) takes the form of the one for the $1+1$ dimensional case

$$\Psi_{tt} - c^2\Psi_{rr} = -4c^2e^{\Psi}, \tag{4.84}$$

$$\Psi(x,t;r) = rM_h(x,r) = \phi(x,r), \quad \Psi_t(x,t;r) = 0, \; for \;\; t = 0. \tag{4.85}$$

Equation (4.84) has the general solution given with equation (4.79).

$$\Psi(x,t;r) = \ln \frac{2A'(x,r+ct)B'(x,r-ct)}{(A(x,r+ct)+B(x,r-ct))^2}. \tag{4.86}$$

If we substitute this general solution into the initial conditions (4.85) , we can integrate the obtained system in terms of

$$z(x,r) = \sqrt{\frac{A'(x,r)}{B'(x,r)}} \tag{4.87}$$

as

$$\sqrt{2}\int^{r} e^{\frac{\phi(y)}{2}} dy = \ln\left|\frac{z-1}{z+1}\right|. \tag{4.88}$$

Defining

$$w(x,r) \equiv \int^{r} e^{\frac{\phi(x,\rho)}{2}} d\rho \tag{4.89}$$

we find

$$\ln A' = \frac{\phi}{2} - 2\ln\sinh(\frac{w}{\sqrt{2}}), \quad \ln B' = \frac{\phi}{2} - 2\ln\cosh(\frac{w}{\sqrt{2}}). \tag{4.90}$$

Integrating once more we get

$$A = -\sqrt{2}\coth(\frac{w}{\sqrt{2}}), \qquad B = \sqrt{2}\tanh(\frac{w}{\sqrt{2}}). \tag{4.91}$$

Collecting all these results, we get solution of initial value problem (4.84),(4.85) as,

$$\Psi(x,t;r) = \frac{\phi(x,r+ct)+\phi(x,r-ct)}{2} - \ln\cosh^2 \int_{\mathrm{r-ct}}^{\mathrm{r+ct}} \frac{e^{\frac{\phi(x,\rho)}{2}}}{\sqrt{2}} d\rho. \tag{4.92}$$

Finally the solution of initial value problem (4.80) (4.81) for SLE (4.64) is given by

$$\begin{aligned} M_v(x,t;r) &= \frac{1}{2r}\left[(r+ct)M_h(x,r+ct)+(r-ct)M_h(x,r-ct)\right] \\ &- \frac{1}{r}\ln\cosh^2 \frac{1}{\sqrt{2}} \int_{\mathrm{r-ct}}^{\mathrm{r+ct}} e^{\frac{\rho M_h(x,\rho)}{2}} d\rho. \end{aligned} \tag{4.93}$$

If the initial function $M_h(x,r)$ is a constant, say $M_h(x,r) = h$, then (4.93) gives,

$$M_v(x,t;r) = h - \frac{1}{r}\ln\cosh^2\left(\frac{2\sqrt{2}}{h} e^{r\frac{h}{2}} \sinh(\frac{h}{2}ct)\right). \tag{4.94}$$

Similarly if $h = 0$ it can be found that

$$M_v(x,t;r) = -\frac{1}{r}\ln\cosh^2(\sqrt{2}ct). \tag{4.95}$$

Two particular solutions given by (4.94) and (4.95) have singularities at the origin. These solutions also illustrate that spherical means does not work for SLE (4.64) at the origin. □

4.4. Progressive Wave Solution for Spherical Liouville Equation

Here we find particular solutions of SLE (4.64). If we write progressive wave form $\Psi(x,t;r) = f(x, r - vt)$, in arranged form of SLE (4.84) and define $\xi = r - vt$, we get

$$v^2 f_{\xi\xi} - c^2 f_{\xi\xi} + 4c^2 e^f = 0. \tag{4.96}$$

Defining $\kappa = 4c^2/(v^2 - c^2)$ and multiplying this second order differential equation with f_ξ we find

$$\frac{(f_\xi)^2}{2} + \kappa e^f = E(x) \tag{4.97}$$

where $E = E(x)$ is an arbitrary function of x. We have three possibilities:

1) $\kappa > 0$ and $E > 0$. We can rewrite differential equation (4.97) as;

$$\frac{\partial f}{\partial \xi} = \mp \sqrt{2E - 2\kappa e^f}. \tag{4.98}$$

By substitution

$$\frac{\kappa}{E} e^f = \sin^2 \varphi \tag{4.99}$$

we can integrate this equation as;

$$e^f = \frac{E}{\kappa} \frac{1}{\cosh^2[\sqrt{\frac{E}{2}}(\xi - \xi_0)]}. \tag{4.100}$$

Finally we find

$$M_v(x,t;r) = \frac{1}{r} \ln\left[\frac{E}{\kappa} \frac{1}{\cosh^2[\sqrt{\frac{E}{2}}(r - vt - r_0)]}\right], \tag{4.101}$$

where r_0 is a constant. For small r, it can be found that this solution takes the form;

$$M_v(x,t;r) = \frac{1}{r} \ln\frac{E}{\kappa} - \frac{2}{r} \ln\cosh \sqrt{\frac{E}{2}} vt. \tag{4.102}$$

2) $\kappa < 0$ and $E > 0$. Let κ be $\kappa = -a_0^2$ (a_0 is a constant), then equation (4.97) can be written as

$$\frac{\partial f}{\partial \xi} = \mp \sqrt{2E} \sqrt{1 + \frac{a_0^2}{E} e^f} \tag{4.103}$$

By substitution

$$\frac{a_0^2}{E} e^f = \sinh^2 \varphi \tag{4.104}$$

we can integrate the this equation as

$$e^f = \frac{E}{a_0^2} \frac{1}{\sinh^2[\sqrt{\frac{E}{2}}(\xi - \xi_0)]} \tag{4.105}$$

and

$$M_v(x,t;r) = \frac{1}{r}\ln\left[\frac{E}{a_0^2}\frac{1}{\sinh^2[\sqrt{\frac{E}{2}}(r-vt-r_0)]}\right], \tag{4.106}$$

where r_0 is a constant.

3) $\kappa < 0$ and $E < 0$ $\;(E = -|E|)$.

Equation (4.97) can be written as;

$$\frac{\partial f}{\partial \xi} = \mp\sqrt{2}\sqrt{|E|}\sqrt{\frac{a_0^2}{|E|}e^f - 1}$$

By substitution,

$$\frac{a_0^2}{|E|}e^f = \cosh^2\varphi \tag{4.107}$$

we have

$$e^f = \frac{|E|}{a_0^2}\frac{1}{\cos^2[\sqrt{\frac{|E|}{2}}(\xi-\xi_0)]}, \tag{4.108}$$

and

$$M_v(x,t;r) = \frac{1}{r}\ln\left[\frac{|E|}{a_0^2}\frac{1}{\cos^2[\sqrt{\frac{|E|}{2}}(r-vt-r_0)]}\right], \tag{4.109}$$

where r_0 is a constant.

4.4.1. Lax Pair for Spherical Liouville Equation.

In previous section, we constructed one soliton type solution of SLE (4.64). This equation admits also multisoliton solutions. They can be obtained by standard technique from the Lax Pair (Lax 1968).

$$\begin{pmatrix}\eta_1\\ \\ \eta_2\end{pmatrix}_r = \begin{pmatrix}\frac{r(M_v)_t}{4c} & \frac{e^{\frac{rM_v}{2}}}{\sqrt{2}\lambda}\\ \\ \frac{\lambda}{\sqrt{2}}e^{\frac{rM_v}{2}} & -\frac{r(M_v)_t}{4c}\end{pmatrix}\begin{pmatrix}\eta_1\\ \\ \eta_2\end{pmatrix} \tag{4.110}$$

$$\begin{pmatrix}\eta_1\\ \\ \eta_2\end{pmatrix}_t = \begin{pmatrix}\frac{1}{4}c(M_v + r(M_v)_r) & \frac{-ce^{\frac{rM_v}{2}}}{\sqrt{2}\lambda}\\ \\ \frac{\lambda c}{\sqrt{2}}e^{\frac{rM_v}{2}} & -\frac{1}{4}c(M_v + r(M_v)_r)\end{pmatrix}\begin{pmatrix}\eta_1\\ \\ \eta_2\end{pmatrix}. \tag{4.111}$$

Compatibility of (4.110) and (4.111) gives us the SLE (4.64).

4.5. Spherical Liouville Equation in Arbitrary n Dimensional Space

If spherical means of function $u(x,t)$ satisfies the following equation

$$(M_u)_{tt} - c^2\left((M_u)_{rr} + \frac{n-1}{r}(M_u)_r + \frac{n-1}{2}\frac{n-3}{2}\frac{1}{r^2}M_u\right) = 0, \tag{4.112}$$

defining new function

$$\Psi(x,t;r) = r^{\frac{n-1}{2}} M_u(x,r;t) \tag{4.113}$$

allows us to write equation (4.112) as

$$\Psi_{tt} - c^2\Psi_{rr} = 0. \tag{4.114}$$

By the Bäcklund Transformation

$$(\partial_t + c\partial_r)(r^{\frac{n-1}{2}}(M_v - M_u)) = -4kce^{r^{\frac{n-1}{2}}\frac{(M_v+M_u)}{2}} \tag{4.115}$$

$$(\partial_t - c\partial_r)(r^{\frac{n-1}{2}}(M_v + M_u)) = \frac{2c}{k}e^{r^{\frac{n-1}{2}}\frac{(M_v-M_u)}{2}} \tag{4.116}$$

we can relate it with the Liouville Equation

$$(M_v)_{tt} - c^2\left((M_v)_{rr} + \frac{n-1}{r}(M_v)_r + \frac{n-1}{2}\frac{n-3}{2}\frac{1}{r^2}M_v\right) = -4c^2 r^{\frac{1-n}{2}} e^{r^{\frac{n-1}{2}}M_v}, \tag{4.117}$$

which has general solution

$$M_v(x,t;r) = \frac{1}{r^{\frac{n-1}{2}}} \ln \frac{2A'(x,r+ct)B'(x,r-ct)}{(A(x,r+ct)+B(x,r-ct))^2}. \tag{4.118}$$

4.5.1. Initial Value Problem

Initial value problem for the Liouville equation (4.117) in arbitrary n dimensional space is given by

$$(M_v)_{tt} - c^2\left((M_v)_{rr} + \frac{n-1}{r}(M_v)_r + \frac{n-1}{2}\frac{n-3}{2}\frac{1}{r^2}M_v\right) = -4c^2 r^{\frac{1-n}{2}} e^{r^{\frac{n-1}{2}}M_v} \tag{4.119}$$

$$M_v(x,r,0) = M_h(x,r), \quad \frac{\partial}{\partial t}(M_v(x,r,t))|_{t=0} = 0. \tag{4.120}$$

This initial value problem can be written in a more compact form. By defining

$$\Psi(x,r,t) = r^{\frac{n-1}{2}} M_v \tag{4.121}$$

initial value problem (4.119),(4.120) reduces to the one

$$\Psi_{tt} - c^2\Psi_{rr} = -4c^2 e^{\Psi} \tag{4.122}$$

$$\Psi(x,r,0) = r^{\frac{n-1}{2}} M_u(x,r,0) = r^{\frac{n-1}{2}} h(x,r) = \phi(x,r) \tag{4.123}$$

$$\Psi_t(x,r,0) = r^{\frac{n-1}{2}}.0 = 0. \tag{4.124}$$

Using the general solution of Liouville Equation (4.118) and initial conditions (4.120), we find solution of initial value problem (4.119), (4.120) as

$$
\begin{aligned}
M_v(x,r,t) &= \frac{(r+ct)^{(\frac{n-1}{2})}h(x,r+ct)+(r-ct)^{(\frac{n-1}{2})}h(x,r-ct)}{2r^{\frac{n-1}{2}}} \\
&+ \frac{1}{r^{\frac{n-1}{2}}}\ln\cosh^2\frac{1}{\sqrt{2}}\int_{r-ct}^{r+ct} e^{\rho^{\frac{n-1}{2}}h(x,\rho)}d\rho. \qquad (4.125)
\end{aligned}
$$

4.5.2. Progressing Wave Solution

For the Liouville Equation (4.117), we can also find progressive wave solutions. To find this solutions, let us define $\Psi(x,t;r) = r^{\frac{n-1}{2}}M_v(x,t;r)$ and $\Psi(x,t;r) = f(x, r-vt)$ in the equation (4.117). Then we get

$$
v^2 f_{\xi\xi} - c^2 f_{\xi\xi} + 4c^2 e^f = 0 \qquad (4.126)
$$

where $\xi = r - vt$. Defining $\kappa = 4c^2/(v^2-c^2)$ and multiplying this second order differential equation with f_ξ, we find

$$
\frac{(f_\xi)^2}{2} + \kappa e^f = E(x) \qquad (4.127)
$$

where $E = E(x)$ is an arbitrary function of x. As before in section 3.4, we have three possibilities:

1) $\kappa > 0$ and $E > 0$.

For this case M_v is found as

$$
M_v(x,t;r) = \frac{1}{r^{\frac{n-1}{2}}}\ln\left[\frac{E}{\kappa}\frac{1}{\cosh^2[\sqrt{\frac{E}{2}}(r-vt-r_0)]}\right], \qquad (4.128)
$$

where r_0 is a constant.

2) $\kappa < 0$ and $E > 0$.

Let κ be $\kappa = -a_0^2$ (a_0 is a constant), then from the equation (4.97) by using the transformation

$$
\frac{a_0^2}{E}e^f = \sinh^2\varphi \qquad (4.129)
$$

we find

$$
M_v(x,t;r) = \frac{1}{r^{\frac{n-1}{2}}}\ln\left[\frac{E}{a_0^2}\frac{1}{\sinh^2[\sqrt{\frac{E}{2}}(r-vt-r_0)]}\right], \qquad (4.130)
$$

where r_0 is a constant.

3) $\kappa < 0$ and $E < 0$ $(E = -|E|)$.

Equation (4.97) can be written as;

$$\frac{\partial f}{\partial \xi} = \mp \sqrt{2}\sqrt{|E|}\sqrt{\frac{a_0^2}{|E|}e^f - 1}$$

By substitution

$$\frac{a_0^2}{|E|}e^f = \cosh^2\varphi \tag{4.131}$$

we have

$$M_v(x,t;r) = \frac{1}{r^{\frac{n-1}{2}}}\ln\left[\frac{|E|}{a_0^2}\frac{1}{\cos^2[\sqrt{\frac{|E|}{2}}(r - vt - r_0)]}\right], \tag{4.132}$$

where r_0 is a constant.

4.6. Spherical Liouville Equation in Odd Dimensional Spaces

When the space dimension is odd, in the wave equation

$$u_{tt} - c^2 \triangle u = 0. \tag{4.133}$$

After averaging function u, and defining function $N(x,t;r)$ given by equation (3.35), we can write

$$\frac{\partial^2}{\partial t^2}N = c^2\frac{\partial^2}{\partial r^2}N. \tag{4.134}$$

By Bäcklund transformation (4.21) and (4.22), we can relate this wave equation with Liouville equation

$$\tilde{N}_{\xi\eta} = e^{\tilde{N}} \tag{4.135}$$

where $\xi = r + ct, \eta = r - ct$ and $\tilde{N}(x,t;r)$ belongs to the same dimensional space with $N(x,t;r)$.

CHAPTER 5

SPHERICAL SINE-GORDON EQUATION

In this chapter, we introduce the spherical Sine-Gordon and spherical Sinh-Gordon equations. By using Bäcklund transformation, we write Bianchy permutability formula for these equations and find their kink and anti-kink like solutions. After giving their Lax pair, we relate these equations with the Riccati equation.

5.1. Sine-Gordon Equation

In general, the nonlinear wave equation is given by

$$u_{tt} - c^2\nabla^2 u + F(u) = 0 \tag{5.1}$$

where $F(u)$ is a nonlinear function of u. The form of $F(u)$ determines the character of the equation (5.1). When $F(u) = \sin u$, equation (5.1) is called the Sine-Gordon equation (Ablowitz, Kaup, Newell and Segur 1973). The Sine-Gordon equation has a lot of physical applications. It governs propagation of ultra short plane wave optical pulses in certain resonant media and it also governs propagation of quantized flux in Josephson junctions (Lamb 1980). The Sine-Gordon equation written in characteristic coordinates

$$u_{xt} = \sin u \tag{5.2}$$

admits Auto-Bäcklund transformation given by

$$u_x - \tilde{u}_x = 2\lambda \sin \frac{u + \tilde{u}}{2} \tag{5.3}$$

$$u_t + \tilde{u}_t = \frac{2}{\lambda} \sin \frac{u - \tilde{u}}{2}. \tag{5.4}$$

The Auto-Bäcklund transformation relates two different solutions of the same equation. The trivial solution $\tilde{u} = 0$ is a solution for the Sine-Gordon equation. Hence by using the Auto-Bäcklund transformation (5.3) and (5.4), one can find the kink solution

$$u(x,t) = 4 \arctan e^{\xi}, \;\; \xi = k(x + k^{-2}t) + constant, \; k \in R. \tag{5.5}$$

Furthermore, four solutions u_0, u_1, u_2, u_{12} of the Sine-Gordon equation satisfies algebraic relation known as the Bianchi permutability formula or the nonlinear superposition formula

$$\tan \frac{u_{12} - u_0}{4} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \tan \frac{u_1 - u_2}{4}. \tag{5.6}$$

Using this formula, one can construct two soliton solution from the vacuum solution $u_0 = 0$ and the kink solitons (5.5) u_1 with λ_1 and u_2 with λ_2.

5.2. Spherical Sine-Gordon Equation and its Bäcklund Transformation

As we discuss in chapter III, the spherical Liouville equation has singularities at the origin. Now let us consider the nonlinear wave equation whose limiting case when r approaches to zero is nonsingular.

Definition 5.1 *Spherical Sine-Gordon equation is defined by the equation*

$$\frac{\partial^2}{\partial t^2}M_v - \frac{2}{r}\frac{\partial}{\partial r}M_v - \frac{\partial^2}{\partial r^2}M_v + \frac{\sin(rM_v)}{r} = 0 \tag{5.7}$$

where $M_v(x,t;r)$ denotes the Spherical means of a function $v(x,t)$ and $x = (x_1, x_2, x_3)$.

When r approaches to the zero spherical Sine-Gordon equation (5.7) is equivalent to the linear wave equation. In fact

$$\frac{\partial^2 M_v}{\partial_t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)M_v + \frac{\sin rM_v}{r} = 0 \tag{5.8}$$

or

$$\frac{\partial_2 M_v}{\partial_t^2} - \triangle M_v + \frac{\sin rM_v}{r} = 0. \tag{5.9}$$

Since $\lim_{r\to 0} M_v = v$,

$$\frac{\partial^2 v}{\partial t^2} - \triangle v + v = 0. \tag{5.10}$$

So nonlinear equation (5.7) for the spherical means M_v of a function v in the limit r approaches zero, is equivalent to the linear wave equation for function v.

Proposition 5.1 *Spherical Sine-Gordon equation admits Auto-Bäcklund transformation.*

Proof Spherical Sine-Gordon equation could be arranged as

$$\Psi_{tt} - \Psi_{rr} + \sin\Psi = 0 \tag{5.11}$$

where $\Psi(x,r,t) = rM_v(x,t;r)$. After changing coordinates under the rules

$$r' = \frac{1}{2}(r-t), \tag{5.12}$$

$$t' = \frac{1}{2}(r+t) \tag{5.13}$$

and dropping the primes we obtain the equation

$$\Psi_{rt} = \sin\Psi \tag{5.14}$$

for which the Auto-Bäcklund transformation is given by;

$$(rM_v)_r - (r\tilde{M}_v)_r = 2\lambda \sin\frac{rM_v + r\tilde{M}_v}{2}, \tag{5.15}$$

$$(rM_v)_t + (r\tilde{M}_v)_t = \frac{2}{\lambda}\sin\frac{rM_v - r\tilde{M}_v}{2} \tag{5.16}$$

where (M_v) and $(\tilde{M}_v)$ are solutions of Sine-Gordon equation (5.7). □

5.2.1. Solitonlike Solution

Since $\tilde{M}_v = 0$ is an evident solution of the spherical Sine-Gordon equation (5.7), the Bäcklund transformation (5.15), (5.16) gives

$$(rM_v)_r = 2\lambda \sin\frac{rM_v}{2}, \tag{5.17}$$

$$(rM_v)_t = \frac{2}{\lambda}\sin\frac{rM_v}{2}. \tag{5.18}$$

From which we find

$$\frac{dr}{dt} = \frac{1}{\lambda^2}. \tag{5.19}$$

In order to convert the partial differentiation to ordinary differentiation we introduce new variable

$$\xi = r - r_0 - \frac{1}{\lambda^2}t. \tag{5.20}$$

From the first partial differential equation we find,

$$\frac{d(rM_v)}{2\sin\frac{rM_v}{2}} = \lambda d\xi. \tag{5.21}$$

From this equation, it is found that

$$M_v(x,t;r) = \frac{4}{r}\arctan\left(E(x)e^{\lambda(r-r_0)-\frac{1}{\lambda}t}\right) \tag{5.22}$$

where $E(x)$ is an arbitrary function of x. From the partial differential equation (5.18) we obtain the solution of the spherical Sine-Gordon equation as

$$M_v(x,t;r) = \frac{4}{r}\arctan\left(E(x)e^{-\lambda(r-r_0)+\frac{1}{\lambda}t}\right) \tag{5.23}$$

where $E(x)$ is an arbitrary function of x.

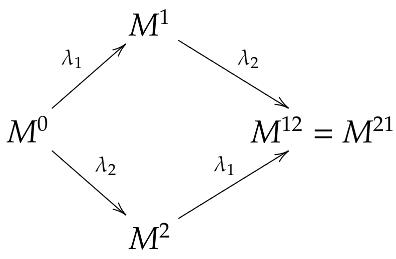

Figure 5.1. Schematic form of transformations occurring in the theorem of permutability

5.2.2. Bianchi Permutability Theorem

Since we know the Auto-Bäcklund transformation for the spherical Sine- Gordon equation, by using it we can find new solution. If we take the first equation of the Bäcklund transformation (5.15) and rewrite it for the different solutions corresponding to the different values of the λ, we find

$$\frac{\partial}{\partial r}((rM_v^1) - (rM_v^0)) = \lambda_1 \sin\frac{(rM_v^1) + (rM_v^0)}{2}, \tag{5.24}$$

$$\frac{\partial}{\partial r}((rM_v^2) - (rM_v^0)) = \lambda_2 \sin\frac{(rM_v^2) + (rM_v^0)}{2}, \tag{5.25}$$

$$\frac{\partial}{\partial r}((rM_v^{12}) - (rM_v^1)) = \lambda_2 \sin\frac{(rM_v^{12}) + (rM_v^1)}{2}, \tag{5.26}$$

$$\frac{\partial}{\partial r}((rM_v^{21}) - (rM_v^2)) = \lambda_1 \sin\frac{(rM_v^{21}) + (rM_v^2)}{2}. \tag{5.27}$$

Combining the first and the third equation, and the second and the fourth equation we obtain

$$\frac{\partial}{\partial r}((rM_v^{12}) - (rM_v^0)) = \lambda_1 \sin\frac{(rM_v^1) + (rM_v^0)}{2} + \lambda_2 \sin\frac{(rM_v^{12}) + (rM_v^1)}{2}, \tag{5.28}$$

$$\frac{\partial}{\partial r}((rM_v^{21}) - (rM_v^0)) = \lambda_1 \sin\frac{(rM_v^{12}) + (rM_v^2)}{2} + \lambda_2 \sin\frac{(rM_v^2) + (rM_v^0)}{2}. \tag{5.29}$$

According to the Bianchi permutability theorem $M_v^{12} = M_v^{21}$. Now if we open trigonometric expressions and use some trigonometric identities we find that

$$\tan\frac{(rM_v^{12}) - (rM_v^0)}{4} = (\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}) \tan\frac{(rM_v^1) - (rM_v^2)}{4}. \tag{5.30}$$

Finally we find pure algebraic nonlinear superposition relation between four solutions of spherical Sine-Gordon equation (5.7)

$$\tan\frac{r(M_v^{12} - M_v^0)}{4} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \tan\frac{r(M_v^1 - M_v^2)}{4}. \tag{5.31}$$

5.2.3. Two Solitonlike Solution of Spherical Sine-Gordon Equation

Nonlinear superposition formula (5.31), allows us to construct new solitonlike solutions from known three solutions of the Spherical Sine-Gordon Equation without quadratures. Evident solution $M_v^0 = 0$ is one solution for the Spherical Sine-Gordon equation, so when two solutions are given by

$$\tan\frac{rM_v^1}{4} = e^{\eta_1}, \eta_1 = \lambda_1(r - r_0) - \frac{1}{\lambda_1}\eta, \tag{5.32}$$

$$\tan\frac{rM_v^2}{4} = e^{\eta_2}, \eta_2 = \lambda_2(r - r_0) - \frac{1}{\lambda_2}\eta, \tag{5.33}$$

a new solution is found in the form,

$$M_v^{12} = \frac{4}{r}\arctan\left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\frac{e^{\eta_1} - e^{\eta_2}}{1 + e^{\eta_1+\eta_2}}\right). \tag{5.34}$$

This solution is analogues of two-soliton solution in standard case.

5.2.4. Lax Pair for Spherical Sine-Gordon Equation

Lax pair for spherical sine-Gordon equation is given by,

$$\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}_{\mathrm{r}} = \begin{pmatrix} -\lambda + \frac{1}{\lambda}\cos(rM_v) & \frac{(rM_v)_t+(rM_v)_r}{2} + \frac{1}{\lambda}\sin(rM_v) \\ \\ \frac{-(rM_v)_t-(rM_v)_r}{2} - \frac{1}{\lambda}\sin(rM_v) & \lambda - \frac{1}{\lambda}\cos(rM_v) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}, \tag{5.35}$$

$$\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}_{\mathrm{t}} = \begin{pmatrix} -\lambda - \frac{1}{\lambda}\cos(rM_v) & \frac{1}{\lambda}\sin(rM_v) + \frac{(rM_v)_r+(rM_v)_t}{2} \\ \\ \frac{1}{\lambda}\sin(rM_v) - \frac{(rM_v)_r+(rM_v)_t}{2} & \lambda + \frac{1}{\lambda}\cos(rM_v) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}. \tag{5.36}$$

5.2.5. Progressing Wave Solution of Spherical Sine-Gordon Equation

After rearranging spherical Sine-Gordon Equation

$$r\frac{\partial^2}{\partial t^2}M_v - 2\frac{\partial}{\partial r}M_v - r\frac{\partial^2}{\partial r^2}M_v + \sin(rM_v) = 0 \tag{5.37}$$

as

$$\Psi_{tt} - \Psi_{rr} + \sin\Psi = 0, \tag{5.38}$$

we can introduce a new function

$$f(x, r - vt) = \Psi(x, t; r) \tag{5.39}$$

which reduces nonlinear partial differential equation (5.38) to the nonlinear ODE

$$(v^2 - 1)f'' + \sin f = 0 \tag{5.40}$$

where primes denotes derivative according to the $\xi = r - vt$. Multiplying with f' and integrating once we get

$$(v^2 - 1)\frac{f'^2}{2} - \cos f = c_1(x). \tag{5.41}$$

If we consider the previous equation for $|v| < 1$, it gives

$$\frac{df}{\sqrt{1 - \cos f}} = \sqrt{\frac{2}{1 - v^2}}d\xi. \tag{5.42}$$

Using some trigonometric identities we can integrate it as

$$\ln(\tan\frac{f}{4}) = \frac{1}{\sqrt{1 - v^2}}\xi + \ln\alpha(x) \tag{5.43}$$

which gives us

$$M_v = \frac{4}{r}\arctan\left(\alpha(x)e^{\frac{r-vt}{\sqrt{1-v^2}}}\right) \tag{5.44}$$

where $\alpha(x)$ is an arbitrary function of x.

5.3. Sine-Gordon Equation and Riccati Equation

The Bäcklund transformation for the Sine-Gordon equation is given by

$$(rM_v)_r - (r\tilde{M}_v)_t = 2\lambda\sin\frac{(rM_v) + (r\tilde{M}_v)}{2}, \tag{5.45}$$

$$(rM_v)_r + (r\tilde{M}_v)_t = \frac{2}{\lambda}\sin\frac{(rM_v) - (r\tilde{M}_v)}{2}. \tag{5.46}$$

Let $\phi^{\pm} = \frac{1}{2}((rM_v) \pm (r\tilde{M}_v))$ (Pashaev 1996), then we find

$$\phi_r^- = \lambda\sin\phi^+, \tag{5.47}$$

$$\phi_t^+ = \frac{1}{\lambda}\sin\phi^-. \tag{5.48}$$

$$\tag{5.49}$$

Now let us introduce

$$\tan\frac{(rM_v) + (r\tilde{M}_v)}{4} = \tan\frac{\phi^+}{2} = \gamma. \tag{5.50}$$

By using some trigonometric identities, we obtain

$$\sin\phi^+ = \frac{2\tan\frac{\phi^+}{2}}{1 + \tan^2\frac{\phi^+}{2}} = \frac{2\gamma}{1 + \gamma^2}. \tag{5.51}$$

Differentiating expression (5.50) we find

$$\gamma_r = \frac{1}{2}\phi_r^2(1+\gamma^2). \tag{5.52}$$

Thus we have for the first Bäcklund transformation

$$\phi_r^+ = \frac{2\gamma_r}{1+\gamma^2}, \tag{5.53}$$

$$\phi_r^- = \frac{2\lambda\gamma}{1+\gamma^2}. \tag{5.54}$$

Adding these two equation we have

$$(rM_v)_r = 2\frac{\gamma_r + \lambda\gamma}{1+\gamma^2}. \tag{5.55}$$

If we arrange this equation we find

$$\gamma_r + \lambda\gamma - \frac{1}{2}(rM_v)_r(1+\gamma^2) = 0. \tag{5.56}$$

We know that transformation $\gamma = \frac{v_1}{v_2}$ linearizes the Riccati differential equation (Ince 1956), so we get the linear form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_r = \frac{1}{2}\begin{pmatrix} -\lambda & (rM_v)_r \\ -(rM_v)_r & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5.57}$$

Following the same strategy we find

$$\tan\frac{\phi^-}{2} = \tan\frac{u-\phi^+}{2} = \frac{\delta-\gamma}{1+\delta\gamma} \tag{5.58}$$

where $\delta = \tan\frac{(rM_v)}{2}$. Using trigonometric properties we can write

$$\sin\phi^- = 2\frac{(\delta-\gamma)(1+\delta\gamma)}{(1+\gamma^2)(1+\delta^2)}. \tag{5.59}$$

Differentiating γ with respect to t gives

$$\gamma_t = \frac{1}{2}\phi_t^+(1+\gamma^2) = \frac{1}{\lambda}\frac{(\delta-\gamma)(1+\delta\gamma)}{1+\delta^2}. \tag{5.60}$$

Finally we find

$$\gamma_t + 2\left(\frac{1}{2\lambda}\cos(rM_v)\right)\gamma + \left(\frac{1}{2\lambda}\sin(rM_v)\right)\gamma^2 - \frac{1}{2\lambda}\sin(rM_v) = 0. \tag{5.61}$$

For this Riccati equation the linear problem in the matrix form is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \frac{1}{2\lambda}\begin{pmatrix} -\cos(rM_v) & \sin(rM_v) \\ \sin(rM_v) & \cos(rM_v) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5.62}$$

Finally we obtain the matrix first-order linear problem for the Sine-Gordon equation (5.7)

$$v_r = Uv, \;\; v_t = Vv \tag{5.63}$$

with linear operators

$$U = \begin{pmatrix} -\lambda & (rM_v)_r \\ -(rM_v)_r & \lambda \end{pmatrix} \tag{5.64}$$

$$V = \begin{pmatrix} -\cos(rM_v) & \sin(r\tilde{M}_v) \\ \sin(r\tilde{M}_v) & \cos(r\tilde{M}_v) \end{pmatrix}. \tag{5.65}$$

Then consistency condition for this system

$$U_t - V_r + UV - VU = 0 \tag{5.66}$$

is equivalent to Spherical Sine-Gordon equation (5.7).

5.4. Spherical Sinh-Gordon Equation

In addition to the spherical Sine-Gordon equation, another nonlinear Klein-Gordon equation so called spherical Sinh-Gordon equation, whose limit when r approaches zero does not contains singularity, can be considered as a nonlinear Euler-Poisson-Darboux Equation.

Definition 5.2 *Spherical Sinh-Gordon equation is defined as*

$$r\frac{\partial^2}{\partial t^2}M_v - 2\frac{\partial}{\partial r}M_v - r\frac{\partial^2}{\partial r^2}M_v + \sinh(rM_v) = 0. \tag{5.67}$$

Proposition 5.2 *Spherical Sinh-Gordon equation admits the Auto-Bäcklund transformation*

$$\begin{aligned} (rM_v)_r - (r\tilde{M}_v)_r &= 2\lambda \sinh \frac{rM_v + r\tilde{M}_v}{2}, \\ (rM_v)_t + (r\tilde{M}_v)_t &= \frac{2}{\lambda} \sinh \frac{rM_v - r\tilde{M}_v}{2}. \end{aligned} \tag{5.68}$$

5.4.1. Solitonlike Solution of Spherical Sinh-Gordon Equation

Since $M_v = 0$ is a solution of spherical Sinh-Gordon equation substituting it into the Bäcklund transformation we get

$$(rM_v)_r = 2\lambda \sinh \frac{rM_v}{2}, \tag{5.69}$$

$$(rM_v)_t = \frac{2}{\lambda} \sinh \frac{rM_v}{2}. \tag{5.70}$$

From this Bäcklund transformation, we find

$$\frac{dr}{dt} = \frac{1}{\lambda^2}. \tag{5.71}$$

In order to convert the partial differentiation to ordinary differentiation, we introduce new variable as

$$\xi = r - r_0 - \frac{1}{\lambda^2}t \tag{5.72}$$

from the partial differential equation (5.69) we find

$$\frac{d(rM_v)}{2\sinh\frac{rM_v}{2}} = \lambda dr. \tag{5.73}$$

Integration of this equation gives us

$$M_v(x,t;r) = \frac{4}{r}\text{arctanh}\Big(e^{\lambda(r-r_0)-\frac{1}{\lambda}t}\Big). \tag{5.74}$$

From the second partial differential equation (5.70) we obtain

$$\frac{d(rM_v)}{2\sinh\frac{rM_v}{2}} = -\lambda d\xi. \tag{5.75}$$

Integration of which gives

$$M_v(x,t;r) = \frac{4}{r}\text{arctanh}\Big(e^{-\lambda(r-r_0)+\frac{1}{\lambda}t}\Big). \tag{5.76}$$

5.4.2. Bianchi Permutability Theorem

From the Bäcklund transformation we can write the following relations

$$\frac{\partial}{\partial r}((rM_v^1) - (rM_v^0)) = \lambda_1 \sinh\frac{(rM_v^1) + (rM_v^0)}{2}, \tag{5.77}$$

$$\frac{\partial}{\partial r}((rM_v^2) - (rM_v^0)) = \lambda_2 \sinh\frac{(rM_v^2) + (rM_v^0)}{2}, \tag{5.78}$$

$$\frac{\partial}{\partial r}((rM_v^{12}) - (rM_v^1)) = \lambda_2 \sinh\frac{(rM_v^{12}) + (rM_v^1)}{2}, \tag{5.79}$$

$$\frac{\partial}{\partial r}((rM_v^{21}) - (rM_v^2)) = \lambda_1 \sinh\frac{(rM_v^{21}) + (rM_v^2)}{2}. \tag{5.80}$$

Combining I,III and II,IV we get

$$\frac{\partial}{\partial r}((rM_v^{12}) - (rM_v^0)) = \lambda_1 \sinh\frac{(rM_v^1) + (rM_v^0)}{2} + \lambda_2 \sinh\frac{(rM_v^{12}) + (rM_v^1)}{2}, \tag{5.81}$$

$$\frac{\partial}{\partial r}((rM_v^{21}) - (rM_v^0)) = \lambda_1 \sinh\frac{(rM_v^{12}) + (rM_v^2)}{2} + \lambda_2 \sinh\frac{(rM_v^2) + (rM_v^0)}{2}. \tag{5.82}$$

(5.83)

If we open the expression and use some hyperbolic identities we find that

$$\tanh\frac{(rM_v^{12}) - (rM_v^0)}{4} = (\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2})\tanh\frac{(rM_v^1) - (rM_v^2)}{4}. \tag{5.84}$$

5.4.3. Two Solitonlike Solutions of Spherical Sinh-Gordon Equation

Bianchi permutability allows us two construct new soliton solutions when we know three solutions of the Spherical Sinh-Gordon Equation. Evidently $\Psi = 0$ is one solution of the Spherical Sinh-Gordon equation so when two solution given by

$$\tanh\frac{\Psi_1}{4} = e^{\eta_1}, \eta_1 = \lambda_1(r - r_0) - \frac{1}{\lambda_1}\eta, \tag{5.85}$$

$$\tanh\frac{\Psi_2}{4} = e^{\eta_2}, \eta_2 = \lambda_2(r - r_0) - \frac{1}{\lambda_2}\eta, \tag{5.86}$$

a new solution is found in the form,

$$(M_v)_{12} = \frac{4}{r}\text{arctanh}\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\frac{e^{\eta_1} - e^{\eta_2}}{1 + e^{\eta_1+\eta_2}}. \tag{5.87}$$

5.4.4. Lax Pair for Spherical Sinh-Gordon Equation

Lax pair for spherical sinh-Gordon equation is given by,

$$\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}_r = \begin{pmatrix} -\lambda + \frac{i}{\lambda}\cosh(rM_v) & \frac{i(rM_v)_t + i(rM_v)_r}{2} + \frac{i}{\lambda}\sinh(rM_v) \\ \\ \frac{-i(rM_v)_t - i(rM_v)_r}{2} - \frac{i}{\lambda}\sinh(rM_v) & \lambda - \frac{1}{\lambda}\cosh(rM_v) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}$$

$$\begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}_t = \begin{pmatrix} -\lambda - \frac{1}{\lambda}\cosh(rM_v) & \frac{i}{\lambda}\sinh(rM_v) + \frac{i(rM_v)_r + i(rM_v)_t}{2} \\ \\ \frac{i}{\lambda}\sinh(rM_v) - \frac{i(rM_v)_r + i(rM_v)_t}{2} & \lambda + \frac{1}{\lambda}\cosh(rM_v) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix} \tag{5.88}$$

5.4.5. Progressing wave solution of Spherical Sinh-Gordon Equation

Spherical Sinh-Gordon Equation is given by the following formula

$$r\frac{\partial^2}{\partial t^2}M_v - 2\frac{\partial}{\partial r}M_v - r\frac{\partial^2}{\partial r^2}M_v + \sinh(rM_v) = 0 \tag{5.89}$$

This equation could be arranged as

$$\Psi_{tt} - \Psi_{rr} + \sinh\Psi = 0 \tag{5.90}$$

where $\Psi(x, t; r) = rM_v(x, t; r)$, if we introduce a new function

$$f(x, r - vt) = \Psi(x, t; r) \tag{5.91}$$

Equation (5.90) reduces to the

$$(v^2 - 1)f'' + \sinh f = 0 \tag{5.92}$$

where primes denotes the derivative according to the $\xi = r - vt$. Multiplication equation (5.92) with f' allows us to integrate as

$$(v^2 - 1)\frac{f'^2}{2} + \cosh f = c_1 \tag{5.93}$$

If we consider the previous equation for $c_1 = 1, |v| < 1$, its integration gives

$$\ln(\tanh\frac{f}{4}) = \frac{1}{\sqrt{1 - v^2}}\xi + \ln\alpha(x) \tag{5.94}$$

which finally gives us

$$M_v(x, t; r) = \frac{4}{r}\text{arctanh}\left(\alpha(x)e^{\frac{1}{\sqrt{1-v^2}}(r-vt)}\right) \tag{5.95}$$

where $\alpha(x)$ is function of x.

5.4.6. Sinh-Gordon Equation and Riccati Equation

The Bäcklund transformation for Spherical Sinh-Gordon equation is given by

$$(rM_v)_r - (r\tilde{M}_v)_t = 2\lambda \sinh\frac{(rM_v) + (r\tilde{M}_v)}{2} \tag{5.96}$$

$$(rM_v)_r + (r\tilde{M}_v)_t = \frac{2}{\lambda}\sinh\frac{(rM_v) - (r\tilde{M}_v)}{2} \tag{5.97}$$

Let

$$\phi^{\pm} = \frac{1}{2}((rM_v) \pm (r\tilde{M}_v)) \tag{5.98}$$

then

$$\phi_r^- = \lambda\sinh\phi^+ \tag{5.99}$$

$$\phi_t^+ = \frac{1}{\lambda}\sinh\phi^- \tag{5.100}$$

Now let us introduce

$$\tanh\frac{(rM_v) + (r\tilde{M}_v)}{4} = \tanh\frac{\phi^+}{2} = \gamma \tag{5.101}$$

By using trigonometric identity

$$\sinh\phi^+ = \frac{2\tanh\frac{\phi^+}{2}}{1 - \tanh^2\frac{\phi^+}{2}} = \frac{2\gamma}{1 - \gamma^2} \tag{5.102}$$

Differentiating the expression (5.101) we find

$$\gamma_r = \frac{1}{2}\phi_r^2(1 - \gamma^2). \tag{5.103}$$

Thus we have for the first Bäcklund transformation

$$\phi_r^+ = \frac{2\gamma_r}{1-\gamma^2} \tag{5.104}$$

$$\phi_r^- = \frac{2\lambda\gamma}{1-\gamma^2} \tag{5.105}$$

Adding these two equation we have

$$(rM_v)_r = 2\frac{\gamma_r + \lambda\gamma}{1+\gamma^2} \tag{5.106}$$

If we arrange this equation we find

$$\gamma_r + \lambda\gamma - \frac{1}{2}(rM_v)_r(1-\gamma^2) = 0 \tag{5.107}$$

We know that the transformation

$$\gamma = \frac{v_1}{v_2}$$

linearize the Riccati differential equation and gives the linear form

$$(v_1)_r + \frac{\lambda}{2}v_1 = \frac{1}{2}(rM_v)_r v_2 \tag{5.108}$$

$$(v_2)_{(rM_v)} - \frac{\lambda}{2}v_2 = -\frac{1}{2}(rM_v)_r v_1 \tag{5.109}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_r = \frac{1}{2}\begin{pmatrix} -\lambda & (rM_v)_r \\ -(rM_v)_r & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.110}$$

Following the same strategy

$$\tanh\frac{\phi^-}{2} = \tanh\frac{(rM_v)-\phi^+}{2} = \frac{\delta-\gamma}{1-\delta\gamma} \tag{5.111}$$

where $\delta = \tanh\frac{(rM_v)}{2}$. Using the trigonometric properties we can write

$$\sinh\phi^- = 2\frac{(\delta-\gamma)(1-\delta\gamma)}{(1-\gamma^2)(1-\delta^2)} \tag{5.112}$$

Differentiating γ with respect to t gives

$$\gamma_t = \frac{1}{2}\phi_t^+(1-\gamma^2) = \frac{1}{\lambda}\frac{(\delta-\gamma)(1-\delta\gamma)}{1-\delta^2} \tag{5.113}$$

Finally we find

$$\gamma_t + 2(\frac{1}{2\lambda}\cosh(rM_v))\gamma + (\frac{1}{2\lambda}\sinh(rM_v))\gamma^2 - \frac{1}{2\lambda}\sinh(rM_v) = 0 \tag{5.114}$$

For this Riccati equation linear problem is given by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \frac{1}{2\lambda}\begin{pmatrix} -\cosh(rM_v) & \sinh(rM_v) \\ -\sinh(rM_v) & \cosh(rM_v) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.115}$$

Finally we obtain the first-order linear problem for Sinh-Gordon equation

$$v_r = Uv, \;\; v_t = Vv \tag{5.116}$$

with linear operators

$$U = \begin{pmatrix} -\lambda & (rM_v)_r \\ -(rM_v)_r & \lambda \end{pmatrix} \tag{5.117}$$

$$V = \begin{pmatrix} -\cosh(rM_v) & \sinh(rM_v) \\ -\sinh(rM_v) & \cosh(rM_v) \end{pmatrix} \tag{5.118}$$

Then consistency condition

$$U_t - V_r + UV - VU = 0 \tag{5.119}$$

is equivalent to Sinh-Gordon equation.

CHAPTER 6

HEAT EQUATION AND SPHERICAL MEANS

This chapter is devoted to the application of the method of spherical means to the linear and related nonlinear heat equations. Solution of initial value problem for the linear heat equation and related nonlinear heat equation, in higher dimensional spaces are given. By the method of spherical means, cylindrical and Spherical Burgers equations are constructed in two and three dimensional spaces respectively. Their general solutions are given. Hierarchy of Cylindrical and Spherical Burgers equations are given.

6.1. Solution of IVP for the Heat Equation by Spherical Means

Spherical means, similar to the wave equation, can also be applied to the heat equation to solve the initial value problem. Let us consider $n+1$ dimensional function $u = u(x_1, x_2, \ldots, x_n, t)$ satisfying the heat equation,

$$u_t - \nu \vartriangle u = 0. \tag{6.1}$$

The spherical means M_u of function u

$$M_{\mathrm{u}}(x, \mathrm{t}; \mathrm{r}) = \frac{1}{\omega_{\mathrm{n}}} \int\limits_{|\xi|=1} u(x + \mathrm{r}\xi, \mathrm{t}) dS_\xi, \tag{6.2}$$

satisfies the Darboux Equation;

$$\Delta_{\mathrm{x}} M_{\mathrm{u}}(x; \mathrm{r}) = \left(\frac{\partial^2}{\partial \mathrm{r}^2} + \frac{\mathrm{n}-1}{\mathrm{r}} \frac{\partial}{\partial \mathrm{r}} \right) M_{\mathrm{u}}(x; \mathrm{r}). \tag{6.3}$$

From another side, by the definition of spherical means following equality holds

$$\Delta_{\mathrm{x}} M_u = \frac{1}{\omega_{\mathrm{n}}} \int\limits_{|\xi|=1} \Delta_x u(x + \mathrm{r}\xi, \mathrm{t}) dS_\xi \tag{6.4}$$

$$\Delta_{\mathrm{x}} M_u = \frac{1}{\nu} \frac{\partial}{\partial t} \frac{1}{\omega_{\mathrm{n}}} \int\limits_{|\xi|=1} u(x + \mathrm{r}\xi, \mathrm{t}) dS_\xi \tag{6.5}$$

$$\Delta_{\mathrm{x}} M_u = \frac{1}{\nu} \frac{\partial}{\partial t} M_u. \tag{6.6}$$

Hence $M_u(x, t; r)$ also satisfies the heat equation

$$(M_u)_t - \nu \vartriangle_x M_u = 0. \tag{6.7}$$

6.1.1. Initial Value Problem for the Heat Equation in Two Dimensional Space

The initial value problem for the Heat equation in 2 dimensional space is given for the function $u = u(x_1, x_2, t)$ as

$$u_t - \nu \bigtriangleup u = 0, \quad u(x_1, x_2, 0) = f(x_1, x_2). \tag{6.8}$$

Then the corresponding initial value problem for the spherical means of function u is;

$$\frac{\partial}{\partial t} M_{\mathrm{u}} = \nu\left(\frac{\partial^2}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\right) M_{\mathrm{u}}, \quad M_{\mathrm{u}}(x, 0; \mathrm{r}) = M_{\mathrm{f}}(x, \mathrm{r}). \tag{6.9}$$

If we use zero order Hankel transformation;

$$M(x, t; r) = \int_0^\infty \rho F(x, t, \rho) J_0(r\rho) d\rho \tag{6.10}$$

and substitute it into the equation (6.9) gives us

$$F_t(x, t, \rho) = -\rho^2 F(x, t, \rho). \tag{6.11}$$

Hence $M(x, t; r)$ can be written as

$$M(x, t; r) = \int_0^\infty \rho F(x, 0, \rho) e^{-\rho^2 t} J_0(r\rho) d\rho. \tag{6.12}$$

Writing the value of $F(x, 0; \rho)$ from the inverse Hankel transformation allows us to write

$$M(x, t; r) = \int_0^\infty d\rho \, \rho \, e^{-\rho^2 t} J_0(r\rho) \int_0^\infty ds M(x, 0, s) s J_0(s\rho). \tag{6.13}$$

Changing order of integration and using identity

$$\int_0^\infty \rho e^{-\rho^2 t} J_0(s\rho) J_0(r\rho) d\rho = \frac{1}{2t} e^{-\frac{r^2+s^2}{4t}} I_0\left(\frac{rs}{2t}\right), \tag{6.14}$$

we find

$$M(x, t; r) = \int_0^\infty s M(x, 0, s) \frac{1}{2t} e^{-\frac{r^2+s^2}{4t}} I_0\left(\frac{rs}{2t}\right) ds \tag{6.15}$$

where I_0 is the modified Bessel function. Letting r go to zero, allows us to write the solution of initial value problem (6.8) as

$$u(x, t) = \frac{1}{2\nu t} \int_0^\infty s \, M_f(x, s) \, e^{-\frac{s^2}{4t}} ds. \tag{6.16}$$

6.1.2. Initial Value Problem for the Heat Equation in Three Dimensional Space

The initial value problem for the Heat equation in 3 dimensional space is given by

$$u_t - \nu \triangle u = 0, \quad u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3). \tag{6.17}$$

Spherical means $M_u(x, t; r)$ of function $u(x, t)$ satisfies following initial value problem

$$\frac{\partial}{\partial t} M_{\mathrm{u}} = \nu\left(\frac{\partial^2}{\partial \mathrm{r}^2} + \frac{2}{\mathrm{r}} \frac{\partial}{\partial \mathrm{r}}\right) M_{\mathrm{u}}, \quad M_{\mathrm{u}}(x, 0; \mathrm{r}) = M_{\mathrm{f}}(x, \mathrm{r}). \tag{6.18}$$

Transformation

$$N(x, t; r) = r M_u(x, t; r) \tag{6.19}$$

allows us to write initial value problem (6.18) in the form

$$N_t = \nu N_{rr}, \quad N(x, 0; r) = r M_u(x, 0; r). \tag{6.20}$$

The initial value problem for the heat equation on semi-infinite domain can be solved by the Fourier-Sine transformation,

$$N(x, t; r) = \int_0^\infty F(x, t; s) \sin(sr) ds. \tag{6.21}$$

Substituting this expression into heat equation (6.20) gives

$$\int_0^\infty (F_t(x, t; s) + \nu s^2 F(x, t; s)) \sin(sr) ds = 0. \tag{6.22}$$

Thus F must satisfy the following relation

$$\Big(F_t(x, t; s) + \nu s^2 F(x, t; s)\Big) = 0. \tag{6.23}$$

Solution of equation (6.23) gives

$$F(x, t; s) = F(x, 0; s) e^{-\nu s^2 t}. \tag{6.24}$$

By inverse Fourier Sine-transformation, $F(x, 0; s)$ is found as

$$F(x, 0; s) = \frac{2}{\pi} \int_0^\infty N(x, 0; \rho) \sin(s\rho) d\rho. \tag{6.25}$$

Substituting equations (6.24) and (6.25) into the Fourier transform (6.21) we find

$$N(x, t; r) = \frac{2}{\pi} \int_0^\infty \int_0^\infty N(x, 0; \rho) e^{-\nu s^2 t} \sin(s\rho) \sin(sr) d\rho ds. \tag{6.26}$$

Changing order of integration, and using some trigonometric identities we find

$$N(x,t;r) = \frac{1}{\pi}\int_0^\infty N(x,0;\rho)\int_0^\infty e^{-\nu s^2 t}(\cos(s(\rho-r)) - \cos(s(\rho+r)))ds d\rho. \tag{6.27}$$

After evaluating interior integral, we find

$$N(x,t;r) = \frac{1}{\sqrt{4\pi\nu t}}\int_0^\infty N(x,0;\rho)(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}})d\rho. \tag{6.28}$$

Taking the limit for r going to zero gives us the solution of the initial value problem (6.17)

$$u(x,t) = \frac{4\pi}{(4\pi\nu t)^{\frac{3}{2}}}\int_0^\infty \rho^2 M_f(x;\rho)e^{-\frac{\rho^2}{4\nu t}}d\rho. \tag{6.29}$$

It is shown in appendix(B) that at $t = 0$, solution (6.29) satisfies the initial condition (6.17). As known, nonlinear Burgers equation is related with the linear heat equation. Thus the initial value problem for the Burger's equation on half line

$$\phi_t(x,r,t) + \phi\phi_r = \nu\phi_{rr} \tag{6.30}$$

$$\phi_t(x,r,0) = h(x,r) \tag{6.31}$$

obtained from the heat equation by the transformation

$$\phi(x,t;r) = -2\nu\frac{N_r(x,t;r)}{N(x,t;r)} \tag{6.32}$$

has solution given by

$$\phi(x,t;r) = -2\nu\frac{\int_0^\infty N(x,0;\rho)[\frac{2(\rho-r)}{4\nu t}e^{-\frac{(\rho-r)^2}{4\nu t}} + \frac{2(\rho+r)}{4\nu t}e^{-\frac{(\rho+r)^2}{4\nu t}}]d\rho}{\int_0^\infty N(x,0;\rho)(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}})d\rho} \tag{6.33}$$

where $N(x,0;r)$ satisfy the relation

$$\phi(x,0;r) = -2\nu\frac{N_r(x,0;r)}{N(x,0;r)}. \tag{6.34}$$

Substituting the $N(x,0;r)$ in to the equation (6.33) allows us to write the solution of i.v.p for the Spherical Burgers equation is found as

$$\phi(x,t;r) = -2\nu\frac{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho h(x;k)dk}[\frac{2(\rho-r)}{4\nu t}e^{-\frac{(\rho-r)^2}{4\nu t}} + \frac{2(\rho+r)}{4\nu t}e^{-\frac{(\rho+r)^2}{4\nu t}}]d\rho}{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho h(x;k)dk}(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}})d\rho}. \tag{6.35}$$

6.1.3. I.V.P for the Heat Equation in Five Dimensional Spaces

Let us consider initial value problem for the heat equation in 5 dimensional space

$$u_t - \nu \vartriangle u = 0, \quad u(x,0) = f(x) \tag{6.36}$$

where $x = (x_1, x_2, \ldots, x_5)$. Then spherical means of u satisfies the initial value problem

$$\frac{\partial}{\partial t}M_{\mathrm{u}} = \nu\left(\frac{\partial^2}{\partial \mathrm{r}^2} + \frac{4}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\right)M_{\mathrm{u}}, \quad M_{\mathrm{u}}(x,0;\mathrm{r}) = M_{\mathrm{f}}(x,\mathrm{r}). \tag{6.37}$$

Transformation $N(x,t;r) = r^2(M_u)_r + 3rM_u$ allows us to write the initial value problem (6.37) into the canonical form

$$N_t = \nu N_{rr}, \quad N(x,0;r) = r^2(M_u)_r(x,0;r) + 3rM_u(x,0;r). \tag{6.38}$$

Fourier transformation gives us

$$N(x,t;r) = \frac{1}{\sqrt{4\pi\nu t}}\int_0^\infty N(x,0;\rho)\left(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}}\right)d\rho. \tag{6.39}$$

Writing the value of the N and taking the limit for r approaches to zero gives us

$$u(x,t) = \frac{4\pi}{3}\frac{1}{(4\pi\nu t)^{\frac{3}{2}}}\int_0^\infty \Big(\rho^3(M_f(x;\rho))_\rho + 3\rho^2 M_f(x;\rho)\Big)e^{-\frac{(\rho)^2}{4\nu t}}d\rho \tag{6.40}$$

Thus the initial value problem for the spherical Burger's equation

$$\phi_t(x,r,t) + \phi\phi_r = \nu\phi_{rr} \tag{6.41}$$

$$\phi_t(x,r,0) = g(x,r) \tag{6.42}$$

obtained from the heat equation by the transformation

$$\phi(x,r,t) = -2\nu\frac{N_r(x,r,t)}{N(x,r,t)} \tag{6.43}$$

has solution

$$\phi(x,t;r) = -2\nu\frac{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho g(x;k)dk}\left[\frac{2(\rho-r)}{4\nu t}e^{-\frac{(\rho-r)^2}{4\nu t}} + \frac{2(\rho+r)}{4\nu t}e^{-\frac{(\rho+r)^2}{4\nu t}}\right]d\rho}{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho g(x;k)dk}\left(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}}\right)d\rho}. \tag{6.44}$$

6.1.4. IVP for the Heat Equation in Odd Dimensional Spaces

Let us consider initial value problem for the heat equation in $n = 2k+1$ dimensional space

$$u_t - \nu \vartriangle u = 0, \quad u(x,0) = f(x). \tag{6.45}$$

Then Spherical means of u satisfies the equation

$$\frac{\partial}{\partial t}M_{\mathrm{u}} = \nu(\frac{\partial^2}{\partial \mathrm{r}^2} + \frac{n-1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}})M_{\mathrm{u}}, \quad M_{\mathrm{u}}(x,0;\mathrm{r}) = M_{\mathrm{f}}(x,\mathrm{r}) \tag{6.46}$$

Transformation

$$N(x,t;r) = (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}M_u(x,t;r)) \tag{6.47}$$

allows us to write equation (6.46) in to the canonical form

$$N_t = \nu N_{rr}, \quad N(x,0;r) = (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}M_u(x,0;r)). \tag{6.48}$$

Then Fourier transformation gives

$$N(x,t;r) = \frac{1}{\sqrt{4\pi\nu t}}\int_0^\infty N(x,0;\rho)(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}})d\rho. \tag{6.49}$$

Writing the value of the N and taking the limit for r approaches to zero gives us the equation

$$u(x,t) = \frac{4\pi}{a_k}\frac{1}{(4\pi\nu t)^{\frac{3}{2}}}\int_0^\infty d\rho\,\rho\,e^{-\frac{\rho^2}{4\nu t}}(\frac{1}{\rho}\frac{d}{d\rho})^{k-1}\rho^{2k-1}M_f(x;\rho) \tag{6.50}$$

where $a_k = 1.3.5....(2k-1)$.

The initial value problem for the Burger's equation

$$\phi_t(x,r,t) + \phi\phi_r = \nu\phi_{rr} \tag{6.51}$$

$$\phi_t(x,r,0) = h(x,r) \tag{6.52}$$

obtained from the heat equation by the transformation

$$\phi(x,r,t) = -2\nu\frac{N_r(x,r,t)}{N(x,r,t)} \tag{6.53}$$

has solution given by

$$\phi(x,t;r) = -2\nu\frac{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho h(x;k)dk}[\frac{2(\rho-r)}{4\nu t}e^{-\frac{(\rho-r)^2}{4\nu t}} + \frac{2(\rho+r)}{4\nu t}e^{-\frac{(\rho+r)^2}{4\nu t}}]d\rho}{\int_0^\infty e^{\frac{-1}{2\nu}\int^\rho h(x,0;k)dk}(e^{-\frac{(\rho-r)^2}{4\nu t}} - e^{-\frac{(\rho+r)^2}{4\nu t}})d\rho}. \tag{6.54}$$

6.2. Cylindrical Burgers Equation

Heat equation given in two dimensional space

$$u_t = \Delta u \tag{6.55}$$

can be converted to the following equation which is satisfied by the spherical means of function u.

$$(M_u)_t = (M_u)_{rr} + \frac{1}{r}(M_u)_r \tag{6.56}$$

Defining a new function $\Phi(x,t;r)$ as

$$M_u(x,t;r) = e^{\Phi(x,t;r)} \tag{6.57}$$

gives us

$$\Phi_t = \frac{2}{r}\Phi_r + \Phi_{rr} + \Phi_r^2 \tag{6.58}$$

If we change the unknown function again according to the following rule

$$\Psi(x,r,t) = \Phi_r(x,r,t)$$

we get the following equation celebrated as Cylindrical Burgers Equation.

$$\Psi_t = -\frac{1}{r^2}\Psi + \frac{1}{r}\Psi_r + 2\Psi\Psi_r + \Psi_{rr} \tag{6.59}$$

if we return back to the equation (6.56) and write

$$M_u(x,t;r) = X(x)R(r)T(t), \tag{6.60}$$

then we found

$$M_u(x,t;r) = \alpha(x)e^{-k^2t}(c_1J_0(kr) + c_2N_0(kr)) \tag{6.61}$$

Where the functions J and N are the Bessel Functions of the first and second kind respectively. Since spherical means of a function is even in r then general solution of Cylindrical Burgers Equation is given by

$$\Psi(x,t;r) = \frac{\left(e^{-k_1^2t}c_1J_0(k_1r) + e^{-k_2^2t}c_3J_0(k_2r)\right)_r}{e^{-k_1^2t}c_1J_0(k_1r) + e^{-k_2^2t}c_3J_0(k_2r)}.$$

6.3. Spherical Burgers Equation

Heat equation given in three dimensional space

$$U_t = \Delta U \tag{6.62}$$

can be converted to the following equation which is satisfied by the spherical means of function u

$$(M_u)_t = (M_u)_{rr} + \frac{2}{r}(M_u)_r. \tag{6.63}$$

Defining a new function $\Phi(x,t;r)$ as

$$\Phi(x,t;r) = \frac{(M_u)_r}{M_u}, \tag{6.64}$$

gives us so called spherical Burgers Equation

$$\Phi_t = \Phi_{rr} + 2\Phi\Phi_r + \frac{2}{r}\Phi_r - \frac{2}{r^2}\Phi. \tag{6.65}$$

If we return back to the equation (6.63) and write

$$M_u(x,t;r) = X(x)R(r)T(t) \tag{6.66}$$

we obtain

$$\frac{T'}{T} = \frac{R'' + \frac{2}{r}R'}{R} \tag{6.67}$$

If we equate obtained system to the constant k^2, then we found

$$M_u(x,t;r) = c(x)e^{k^2 t}\left(\alpha\frac{\sin(kr)}{kr} + \beta\frac{\cos(kr)}{kr}\right). \tag{6.68}$$

Since spherical means of any function must be even in r and must give the original function when r approaches the zero, we conclude that β must be identically zero. Then general solution of Spherical Burgers Equation obtained by

$$\Phi(x,t;r) = \frac{(M_u)_r}{M_u} \tag{6.69}$$

is given as

$$\Phi(x,t;r) = \frac{\left(c_1(x)e^{k_1^2 t}\frac{\sin(k_1 r)}{k_1 r} + c_2(x)e^{k_2^2 t}\frac{\sin(k_2 r)}{k_2 r}\right)_r}{\left(c_1(x)e^{k_1^2 t}\frac{\sin(k_1 r)}{k_1 r} + c_2(x)e^{k_2^2 t}\frac{\sin(k_2 r)}{k_2 r}\right)}. \tag{6.70}$$

6.4. Heat and Burgers Hierarchy

Linear heat equation

$$\Psi_t = \Psi_{xx} \tag{6.71}$$

can be related with the non-linear Burgers equation by the Cole-Hopf transformation

$$\phi = \frac{\Psi_x}{\Psi}. \tag{6.72}$$

According to the previous formula solution of the linear heat equation Ψ can be written in terms of ϕ

$$\Psi(x,t) = e^{\int^x \phi(y,t)dy}. \tag{6.73}$$

satisfying the Burgers equation. Substituting eqn (6.73) into the heat equation we get

$$(e^{\int^x \phi(y,t)dy})_t = (e^{\int^x \phi(y,t)dy})_{xx} \tag{6.74}$$

$$\int^x \phi_t(y,t)dy = \phi_x(x,t) + \phi_x^2(x,t) \tag{6.75}$$

Differentiating with respect to x, gives Burgers equation

$$\phi_t(x,t) = \phi_{xx}(x,t) + 2\phi\phi_x(x,t). \tag{6.76}$$

Now if we consider the equation

$$\Psi_t = \Psi_{xxx} \tag{6.77}$$

Cole-Hopf transformation (6.72) leads to the following equation

$$\phi_t(x,t) = \phi_{xxx}(x,t) + 3\phi(x,t)\phi_{xx}(x,t) + 3\phi_x^2 + 3\phi^2\phi_x \tag{6.78}$$

Similar calculations for the equation

$$\Psi_t = \Psi_{xxxx} \tag{6.79}$$

By the non-linear transformation (6.73) gives

$$\begin{aligned} \phi_t(x,t) &= \phi_{xxxx}(x,t) + 10\phi_x(x,t)\phi_{xx}(x,t) + 4\phi(x,t)\phi_{xxx}(x,t) \\ &+ 12\phi(x,t)\phi_x^2(x,t) + 6\phi^2(x,t)\phi_{xx}(x,t) + 4\phi^3(x,t)\phi_x(x,t). \end{aligned} \tag{6.80}$$

Generalization of these results says that for a equation given with

$$\Psi_{t_n}(x,t) = \frac{\partial^n}{\partial x^n}\Psi(x,t), \quad n = 2,3,4... \tag{6.81}$$

the Cole-Hopf transformation gives equations for different values of n obeying the rule given by (Pashaev and Gürkan 2007)

$$\partial_{t_n}\phi = \partial_x((\partial_x + \phi)^n.1) \tag{6.82}$$

Writing function $\phi(x,t)$ in the traveling wave form

$$\phi(x,t) = f(x - vt), \quad \xi = x - vt \tag{6.83}$$

we get the hierarchy of ordinary differentiation as

$$-v.\partial_\xi f = \partial_\xi((\partial_\xi + f)^n.1) \tag{6.84}$$

Some of the members of this hierarchy are given explicitly as

$$-vf = f_\xi + f^2, \ \text{for} \quad n = 2 \tag{6.85}$$

$$-vf = f_{\xi\xi} + 3ff_\xi + f^3, \ \text{for} \quad n = 3 \tag{6.86}$$

$$-vf = f_{\xi\xi\xi} + 3f_\xi^2 + 4ff_{\xi\xi} + 6f^2f_\xi + f^4, \ \text{for} \quad n = 4 \tag{6.87}$$

The equation obtained for the $n = 2$ is the well known Riccati Equation.Writing in the Riccati equation

$$f(\xi) = \frac{\phi_\xi}{\phi} \tag{6.88}$$

we reduce it to the linear second order differential equation as

$$-v\phi_\xi = \phi_{\xi\xi}. \tag{6.89}$$

solution of which enables us to write

$$\phi(x,t) = f(x-vt) = \frac{-vc_2 e^{-v(x-vt)}}{c_1 + c_2 e^{-v(x-vt)}}. \tag{6.90}$$

6.4.1. Heat Equation with Potential

Linear heat equation with potential

$$\Psi_t = \Psi_{xx} + V(x,t)\Psi \tag{6.91}$$

can be related with the non-linear Heat equation. By Cole-Hopf transformation

$$\phi = \frac{\Psi_x}{\Psi}. \tag{6.92}$$

Solution of the linear heat equation Ψ, can be written in terms of ϕ satisfying the nonlinear heat equation . Cole-Hopf transformation can be written as

$$\Psi(x,t) = e^{\int^x \phi(y,t)dy} \tag{6.93}$$

Substituting eqn (6.93) into the heat equation we get

$$(e^{\int^x \phi(y,t)dy})_t = (e^{\int^x \phi(y,t)dy})_{xx} + V(x,t)e^{\int^x \phi(y,t)dy} \tag{6.94}$$

After differentiation with respect to x, eqn(6.94) reduces to the

$$\phi_t(x,t) = \phi_{xx}(x,t) + 2\phi\phi_x(x,t) + V_x(x,t). \tag{6.95}$$

Now if we consider the equation

$$\Psi_t = \Psi_{xxx} + (V\Psi)_x \tag{6.96}$$

Cole-Hopf transformation (6.92) leads to the following equation

$$\begin{aligned}\phi_t(x,t) &= \phi_{xxx}(x,t) + 3\phi(x,t)\phi_{xx}(x,t) + 3\phi_x^2 + 3\phi^2\phi_x \\ &+ V_{xx} + V_x\phi + V\phi_x\end{aligned} \tag{6.97}$$

Similar calculations for the equation

$$\Psi_t = \Psi_{xxxx} + (V\Psi)_{xx} \tag{6.98}$$

By non-Linear transformation (6.92), we find

$$\begin{aligned}\phi_t(x,t) &= \phi_{xxxx}(x,t) + 10\phi_x(x,t)\phi_{xx}(x,t) + 4\phi(x,t)\phi_{xxx}(x,t)\\ &+ 12\phi(x,t)\phi_x^2(x,t) + 6\phi^2(x,t)\phi_{xx}(x,t) + 4\phi^3(x,t)\phi_x(x,t)\\ &+ V_{xxx} + 2V_{xx}\phi + 3V_x\phi_x + V\phi_{xx} + V_x\phi^2 + 2V\phi\phi_x. \end{aligned} \tag{6.99}$$

From the above calculations we can formulate the general case. For the linear wave equation given by

$$\partial_{t_n}\Psi(x,t) = \partial_x^n\Psi(x,t) + \partial_x^{n-2}(V(x)\Psi(x,t)), \quad n = 2,3,4... \tag{6.100}$$

the Cole-Hopf transformation gives the equations for different values of n obeying the rule given by

$$\partial_{t_n}\phi = \partial_x((\partial_x + \phi)^n.1) + \partial_x((\partial_x + \phi)^{n-2}.V), \;\; n = 2,3,4... \tag{6.101}$$

Writing functions $\phi(x,t), V(x,t)$ in the traveling wave form

$$\phi(x,t) = f(x - vt), \;\; V(x,t) = g(x,t), \quad \xi = x - vt \tag{6.102}$$

we get

$$-v.\partial_\xi f = \partial_\xi((\partial_\xi + f)^n.1 + (\partial_\xi + f)^{n-2}.g). \tag{6.103}$$

Taking one integration with respect to ξ gives

$$-vf = (\partial_\xi + f)^n.1 + (\partial_\xi + f)^{n-2}.g \tag{6.104}$$

First three member of the hierarchy is given by the

$$-vf = f_\xi + f^2 + g, \;\; \text{for} \quad n = 2 \tag{6.105}$$

$$-vf = f_{\xi\xi} + 3ff_\xi + f^3 + g_\xi + gf, \;\; \text{for} \quad n = 3 \tag{6.106}$$

$$\begin{aligned}-vf &= f_{\xi\xi\xi} + 3f_\xi^2 + 4ff_{\xi\xi} + 6f^2f_\xi + f^4\\ &+ g_{\xi\xi} + 2g_\xi f + gf_\xi + gf^2, \;\; \text{for} \quad n = 4\end{aligned} \tag{6.107}$$

The equation obtained for the $n = 2$ is well known Riccati Equation.

6.4.2. Spherical Burgers Hierarchy

When the Heat equation is given in three dimensional space for the function $u = u(x,t)$ as

$$u_t(x,t) = \triangle\, u(x,t) \tag{6.108}$$

then the spherical means M of function $u(x,t)$ satisfies

$$rM_t = (rM)_{rr} = 2M_r + rM_{rr} \tag{6.109}$$

For this heat equation we can write time evolution as

$$\begin{aligned}
(rM)_{t_2} &= (rM)_{rr} = 2M_r + rM_{rr} && (6.110)\\
(rM)_{t_3} &= (rM)_{rrr} = 3M_r + rM_{rrr} && (6.111)\\
&\vdots && (6.112)\\
(rM)_{t_n} &= (rM)_{rrr...r} = n\frac{d^{n-1}}{dr^{n-1}}M + r\frac{d^n}{dr^n}M && (6.113)\\
& && (6.114)
\end{aligned}$$

The Cole-Hopf transformation

$$\Phi(r,t) = \frac{M_r}{M} \tag{6.115}$$

allows us to write for $t = t_2$ spherical Burgers Equation as

6.4.3. Cylindrical Burgers Hierarchy

When the function $u = u(x_1, x_2, t)$ satisfies two dimensional Heat equation

$$u_t(x,t) = \Delta u(x,t) \tag{6.116}$$

spherical means $M(x,t;r)$ of function $u(x,t)$ satisfies.

$$(M_u)_t = (M_u)_{rr} + \frac{1}{r}(M_u)_r \tag{6.117}$$

Defining a new function $\Phi(x,t;r)$ as

$$\phi(x,t;r) = \frac{M_r(x,t;r)}{M(x,t;r)} \tag{6.118}$$

gives us following equation celebrated as Cylindrical Burgers Equation.

$$\phi_t = -\frac{1}{r^2}\phi + \frac{1}{r}\phi_r + 2\phi\phi_r + \phi_{rr} \tag{6.119}$$

we can write time evolution for this equation For $t = t_2$ we can write the equation (6.117) as

$$(rM)_{t_2} = (rM)_{rr} - M_r \tag{6.120}$$

Then for different times we can write equations corresponding to the function $\chi(x,t;r) = rM(x,t;r)$ as

$$(rM)_{t_2} = \frac{\partial^2}{\partial r^2}(rM) - \frac{\partial}{\partial r}(rM) \tag{6.121}$$

$$(rM)_{t_3} = \frac{\partial^3}{\partial r^3}(rM) - \frac{\partial^3}{\partial r^3}(rM) \tag{6.122}$$

$$\vdots$$

$$(rM)_{t_n} = \frac{\partial^n}{\partial r^n}(rM) - \frac{\partial^{n-1}}{\partial r^{n-1}}(rM) \tag{6.123}$$

Applying Cole-Hopf transformation gives us Cylindrical burgers Hierarchy

$$\partial_{t_n}\phi = \partial_r\left\{\frac{n-1}{r}(\partial_r + \phi)^{n-1}.1 + (\partial_r + \phi)^n.1\right\}, \quad n = 2,3,4\ldots \tag{6.124}$$

CHAPTER 7

CONCLUSION

In the present thesis we studied the method of spherical means, its properties and its relation between PDEs.

1)We found spherical means operator representation in 2-dimensional space could be expressible in terms of modified exponential functions. After studying properties of modified exponential function, we give related linear and nonlinear Heat equations and their general solutions.

2) By Backlund transformation we relate Euler-Poisson-Darboux Equation in arbitrary odd dimensional spaces with spherical Liouville Equation which exact spherically symmetric solutions. We showed that solutions of this equation are singular at origin and can not be spherical means of a function.

3)Nonlinear hyperbolic wave equations in the form of Sine-Gordon and Sinh-Gordon equations have been considered. We found solutions for the spherical Sine-Gordon and Spherical Sinh-Gordon equations which are not singular at the origin.

4) Nonlinear Heat equations in arbitrary odd dimensional spaces were considered. By Cole-Hopf transformation we introduced the Spherical and Cylindrical Burgers equations and their general solutions.

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APPENDIX A

APPLICATION OF SPHERICAL MEANS

In n dimensional space relation between cartesian coordinates $(x_1, x_2, x_3, \ldots, x_n)$ and hyperspherical coordinates $(r, \phi_1, \phi_2, \phi_3, ..., \phi_{n-1})$ is given by the following equations

$$x_1 = r\cos(\phi_1) \tag{A.1}$$

$$x_2 = r\sin(\phi_1)\cos(\phi_2) \tag{A.2}$$

$$x_3 = r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \tag{A.3}$$

$$\vdots$$

$$x_{n-1} = r\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1}) \tag{A.4}$$

$$x_n = r\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}) \tag{A.5}$$

and surface element is given by

$$d_{S^{n-1}}V = \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \ldots \sin\phi_{n-2}\ d_{\phi_1}d_{\phi_2}\ldots d_{\phi_{n-1}} \tag{A.6}$$

Surface are of sphere in n dimensions is given by the following formula (John 1955)

$$\omega_n = \frac{2\sqrt{\pi^n}}{\Gamma(\frac{n}{2})} \tag{A.7}$$

. Example: As an application of the method of spherical means let us evaluate the spherical means of following function at an arbitrary point $P = P(x, y)$ in two ways

$$u(x, y) = x^2 y. \tag{A.8}$$

1) By definition of spherical means:

Spherical means of a function is given by

$$M_u(x, r) = \frac{1}{\omega_n}\int\limits_{|\xi|=1} u(x + r\xi)dS_\xi. \tag{A.9}$$

Using equation (A.7) we find

$$M_u(x, r) = \frac{1}{2\pi}\int\limits_{|\xi|=1} u(x + r\xi)dS_\xi. \tag{A.10}$$

Equation (A.6) and (A.8) allows us to write

$$M_u(x, y, r) = \frac{1}{2\pi}\int\limits_{\theta=0}^{2\pi} (x + r\cos\theta)^2(y + r\sin\theta)d_\theta. \tag{A.11}$$

If we open the parenthesis we get

$$M_u(x,y,r) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}(x^2y+x^2r\sin\theta+2rxy\cos\theta+2xr^2\cos\theta\sin\theta+r^2y\sin^2\theta+r^3\cos^2\theta\sin\theta)d_\theta \tag{A.12}$$

From this equation we find the spherical means of function u as

$$M_u(x,y;r) = x^2y+\frac{r^2y}{2}. \tag{A.13}$$

2) By using operator representation:

Let us evaluate the Laplacian of function $u(x,y)$ (A.8)

$$\triangle\, u = 2y,\;\; \triangle^2\, u = 0,\;\; \ldots\, \triangle^n\, u = 0 \tag{A.14}$$

If we use the operator representation of spherical means given by

$$M = e\left(\frac{r^2}{4}\,\triangle;2\right) \tag{A.15}$$

then the spherical means of function (A.8), by using the equation (A.14), is found as

$$M_u(x,y;r) = u(x,y)+\frac{r^2}{4}\,\triangle\, u+\frac{r^4}{16}\,\triangle^2\, u+\ldots+\frac{r^{2n}}{4^n}\,\triangle^n\, u+\ldots \tag{A.16}$$

$$M_u(x,y;r) = x^2y+\frac{r^2y}{2}. \tag{A.17}$$

APPENDIX B

APPLICATION OF SPHERICAL MEANS TO THE HEAT EQUATION

Let us show that solution of initial value problem (6.17)

$$u(x,t) = \frac{4\pi}{(4\pi\nu t)^{\frac{3}{2}}} \int\limits_0^\infty M_f(x;\rho)\rho^2 e^{-\frac{\rho^2}{4\nu t}} d\rho \tag{B.1}$$

satisfies the initial condition, $u(x,o) = f(x)$.
To do this let us define

$$n = \frac{1}{\sqrt{4\nu t}} \tag{B.2}$$

$$F(\rho) = M_f(x,\rho) \tag{B.3}$$

In this definition if t approaches zero then n approaches to infinity. Thus by the definition (B.2)and (B.3), equation (B.1) takes the form

$$I = \frac{4n^3}{\sqrt{\pi}} \int_0^\infty F(\rho)\rho^2 e^{(-\rho^2 n^2)} d\rho. \tag{B.4}$$

From the property of spherical means $F(\rho)$ is an even function. If we expand the function $F(\rho)$ in Taylor series we obtain following equation

$$I = \frac{4n^3}{\sqrt{\Pi}} \int_{-\infty}^\infty \sum_{m=0}^\infty \frac{F^{(2m)}(0)}{(2m!)} \rho^2 e^{(-\rho^2 n^2)} d\rho \tag{B.5}$$

which can be arranged as follows

$$I = \frac{2n^3}{\sqrt{\pi}} \sum_{m=0}^\infty \frac{F^{(2m)}(0)}{(2m!)} \int_{-\infty}^\infty \frac{d^{m+1}}{d(n^2)^{m+1}} (e^{(-\rho^2 n^2)}) d\rho. \tag{B.6}$$

Using the error function we find

$$I = \frac{2n^3}{\sqrt{\pi}} \sum_{m=0}^\infty \frac{F^{(2m)}(0)}{(2m!)} \frac{d^{m+1}}{d(n^2)^{m+1}} (\sqrt{\frac{\pi}{n^2}}). \tag{B.7}$$

After evaluating the differentiation we find

$$I = 2\sum_{m=0}^\infty \frac{F^{(2m)}(0)}{(2m!)} \frac{1.3.5...(2m+1)}{2^{m+1}} .n^{-(2m+3)}. \tag{B.8}$$

It can be shown by the ratio test that the series is convergent and can be integrated term by term.

$$I = 2(\frac{F^0(0)}{2} + \frac{F^1(0)}{2}\frac{3}{n^2} + \frac{F^2(0)}{2.4}\frac{1.3.5}{n^4} + \dots. \tag{B.9}$$

When $n \to \infty$ only the first gives nonzero contribution.

$$I = F(0) = M_f(x, 0)$$

Since spherical means M_f approaches the function f when r approaches zero, then we find

$$I = F(0) = M_f(x, 0) = f(x). \tag{B.10}$$