ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

G_2 STRUCTURES WITH TORSION AND SOME APPLICATIONS IN STRING THEORY

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

JUNE 2016

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

BURULMALI G_2 YAPILARI VE BAZI SİCİM TEORİSİ UYGULAMALARI

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To my family,

FOREWORD

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ABBREVIATIONS

M	**:** Manifold
g	**:** Metric tensor
$T_p(M)$	**:** The vector space of all tangent vectors at p
TM	**:** The tangent bundle of M
$T_p^*(M)$	**:**The cotangent space at p
T^*M	**:** The cotangent bundle of M
$\wedge^k$	**:** The space of k-forms on M
T_0^r	**:** r covariant tensor
δ	**:** the coderivative
∇	**:** affine connection
Hol	**:** the holonomy group of the connection
$\mathbb{O}$	**:** Octonions
$\mathbb{H}$	**:** Quaternions
φ	**:** Three form
ψ	**:** Four form associated to the three form
$curl(X)$	**:** Curl of a vector field X
$grad$	**:** Gradient
div	**:** Divergent
Δ	**:** Laplacian operator

G_2 STRUCTURES WITH TORSION AND SOME APPLICATIONS IN STRING THEORY

SUMMARY

A G_2-structure can be defined on any seven dimensional smooth manifold M as a reduction of the structure group of the frame bundle of M to the compact, exceptional Lie group G_2. The Lie group G_2 can be described as the subgroup of the general linear group $GL(7,R)$ which preserves a positive 3-form φ, called the associative form. The Hodge dual, $\psi = *\varphi$ which is a 4-form, is called the coassociative form. ψ depends on φ nonlinearly, as the metric with respect to which the Hodge duality is defined, is also determined by the 3-form φ.

G_2 manifolds are manifolds with G_2 holonomy. This is a further differential geometric condition imposed on the 3-form φ. More precisely φ has to be parallel with respect to the Levi-Civita connection. It is known that if M is a G_2-manifold, then M is a Ricci-flat, orientable, spin manifold.

Manifolds with G_2 holonomy are important in physics, especially in string theory. Recently, manifolds with G_2-structure, rather than G_2-holonomy has found interesting applications in string theory. In this case, the associative 3-form φ and its Hodge dual ψ are not necessarily parallel. And the tool that measures how far they are from being parallel is given by the torsion classes of the G_2-structure.

The aim of this thesis is to study the differential geometric properties of manifolds with G_2-structure. We are particularly interested in the case when we have just G_2-structure rather than G_2 holonomy. Hence, we study in detail the description of the torsion classes of a given G_2-structure. We also study an application in string theory.

The outline of this thesis is as follows. We start by a preliminary chapter in differential geometry and algebra reviewing the topics which are essential for the rest of the thesis. In the following chapter, we start describing the G_2-structures. First, we construct the associative 3-form via the octonions. Then, we consider the properties of a G_2-structure and a G_2-manifold. After that, we see the relation between the metric, cross product and the associative 3-form. Then, we describe the decomposition of each space of k forms into irreducible G_2 representations. Afterwards, we decompose $d\phi$, $d*\phi$ into irreducible G_2 representations, which defines the torsion forms for us. By considering these torsion forms we see their relation to the concept of being torsion free ϕ. In the last chapter, we consider an application of G_2-structures in string theory. We start with a classical solution of ten dimensional supergravity of the form $R_{2,1} \times Y_7$ where Y^7 is a G_2 manifold. Then we ask if the metric of G_2 holonomy can be modified to compensate for quantum corrections, which are callled α' corrections. Equivalently we ask, if there is a small deformation $\phi' = \phi + \delta\phi$ of the associative 3-form ϕ such that the corresponding metric g' solves the α' corrected equations of the quantum theory. This amounts to analyzing the existence of a coupled system of partial differential

equations for ϕ', and its Hodge dual (with respect to g') where the source terms are determined by physics, and are related to the torsion forms of the G_2 structure ϕ'. We show that such a solution always exists.

BURULMALI G_2 YAPILARI VE BAZI SİCİM TEORİSİ UYGULAMALARI

ÖZET

G_2-yapısı düzgün bir manifold üzerinde tanımlanabilir. Eğer M düzgün 7 boyutlu bir manifold ise G_2-yapısı, çerçeve demetinin yapı grubunun kompakt, istisnai Lie grubu G_2'ye indirgenmesidir. G_2 grubu beş istisnai Lie grubundan biridir. Bununla birlikte, oktanyonların otomorfizm grubu olarak ya da genel lineer grup $GL(7,R)$'nin pozitif 3-formu koruyan bir alt grubu olarak da tanımlanabilir. Bu 3-formun duali $\psi = *\varphi$ şeklinde olup, φ'ye nonlineer bir biçimde bağlıdır. Bir M manifoldunun G_2 yapısına sahip olmasının iki denk koşulu vardır. Birinci ve ikinci Stiefel–Whitney sınıflarının sıfırlanması ya da buna denk olarak M manifoldunun yönlendirilebilir ve spin yapısına sahip olması gerekir.

G_2 manifoldları ise G_2 holonomisi olan manifoldlardır. Bu, pozitif 3-form φ üzerine diferansiyel geometrik bir koşuldur. Bu koşul, φ'nin Levi-Civita konneksiyonuna göre paralel olmasıdır. Bunun için, $\nabla\varphi = 0$ koşulu ancak ve ancak $d\varphi = d*\varphi = 0$ olması ile sağlanır. Metrik de bu G_2 yapısıyla tanımlanmaktadır. M manifoldunun G_2-manifoldu olabilmesi için Ricci düz, yönlendirilebilir ve spin bir manifold olması gerekir [1]. G_2 holonomisi olan manifoldlar ilk defa 1966 yılında Edmond Bonan tarafından bulunmuştur. Paralel 3-form ve paralel 4-formu inşaa etmiş ve bu manifoldların Ricci düz olduğunu göstermiştir [2]. G_2 holonomisi olan 7 boyutlu tam ancak kompakt olmayan manifoldlar ilk kez Robert Bryant ve Salamon tarafından 1989 yılında bulunmuştur [3, 4]. G_2 holonomisi olan 7 boyutlu kompakt manifoldlar ise ilk kez Dominic Joyce tarafından 1994 yılında bulunmuştur [5]. Özellikle fizik literatüründe kompakt G_2 manifoldları Joyce manifoldları olarak da anılır.

G_2 holonomisi olan manifoldlar fizikte özellikle sicim kuramında büyük bir öneme sahiptir. Son zamanlarda, G_2 holonomisinden ziyade G_2 yapısı olan manifoldlar, sicim kuramı uygulamalarında daha çok önem kazanmıştır. Bu durumda, pozitif 3-form φ ve onun Hodge duali olan ψ paralel olmak zorunda değildir. Ve bunların paralellikten ne kadar uzak olduklarını ölçen yapıya G_2 yapısının burulma sınıfları adı verilir. Biz bu burulma sınıflarının tanım ve özelliklerini inceleyecek ve sicim kuramındaki bir uygulamasını çalışacağız.

Bu tez çalışmasının temel amacı, G_2 yapısı olan manifoldların diferansiyel geometrik özelliklerini incelemektir. Özellikle G_2 holonomisinden ziyade G_2 yapısı olan manifoldları incelemektir. G_2 yapısının burulma sınıfları üzerinde detaylıca durmak ve bunların sicim kuramına uygulamalarını incelemeyi hedeflemekteyiz.

Bu tez 4 ayrı bölümden oluşmakta olup, birinci bölümde bu tez boyunca gerekli olacak bazı cebirsel ve diferansiyel geometrik kavramların tanımları incelenmektedir. İlk olarak, Diferansiyel Geometri alt bölümünde diferansiyel manifoldların genel tanımı verildikten sonra tanjant ve kotanjant uzaylarının tanımları verilmiştir. Tanjant

ve kotanjant demetinin tanımları ve r-kovaryant tensör vasıtasıyla dış çarpım cebiri tanımlanmış olup bunların elemanlarının ise diferansiyel formlar olduğu belirtilmiştir. Dış çarpımın bazı özellikleri verilmiştir. Diferansiyel formların lokal koordinatlarda gösterimi verilmiştir. Riemann metriği tanımlandıktan sonra lokal koordinatlarda yazılmıştır. Bu metrik yardımıyla M üzerindeki volüm formu ve Hodge yıldız operatörü $*$ lokal olarak tanımlanmıştır. Affin konneksiyonun tanımı verilerek bunun üzerinden paralel taşıma ve holonomi kavramları incelenmiştir. Kısıtlı holonomi grubunun tanımı verilmiştir. Metrik uyumlu ve burulmasız olan yegane affin konneksiyonu Levi-Civita konneksiyonu, Riemann eğriliği ile bu alt bölüm sonlandırılmıştır. Cebirsel temel kavramların incelendiği ikinci alt bölüm, normlu bölüm cebirlerinin ve vektör çarpımının tanımları ile başlamaktadır. Boyutları sırasıyla 1, 2, 4, 8 olan $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ dışında normlu bölüm cebirlerinin olmadığı vurgulanmıştır. Daha sonra $\mathbb{R}^3$'teki iki vektörün vektör çarpımı ile kuaterniyon çarpımı ilişkilendirilmiştir. Aynı şekilde $\mathbb{R}^7$'de bu durumun oktanyon çarpımı ile ilişkilendirildiği belirtilmiş olup bu çarpımların özellikleri incelenmiştir. Oktanyonlar yardımıyla tanımlanan yeni vektör çarpımının aynı $\mathbb{R}^3$'teki vektör çarpımı gibi

$$u \times v = -v \times u, \qquad \langle u \times v, u \rangle = 0, \qquad ||u \times v||^2 = ||u \wedge v||^2,$$

özelliklerine sahip olduğu görülmüştür. Ancak,

$$u \times (v \times w) + \langle u, v \rangle w - \langle u, w \rangle v$$

ifadesi $\mathbb{R}^3$'te olduğu gibi sıfırlanmamıştır. Bu, $\mathbb{R}^7$'de vektör çarpımının birleşme özelliğine sahip olmadığını gösterir.

İkinci bölümde, G_2 yapısı olan manifoldlar incelenmiştir. Öncelikle 3-form, volüm form ve 4-formun oktanyonlar üzerinden Cayley-Dickson prosesi ile nasıl tanımlandığı gösterilmiştir. Daha sonra G_2 yapısının ve G_2 grubunun tanımı verilmiştir. G_2 yapısının genel özellikleri verilmiştir. G_2 yapısı olan bir manifoldun metrik, vektör çarpım ve 3-form arasındaki

$$\varphi(u, v, w) = \langle u \times v, w \rangle.$$

ilişkiye sahip olduğu belirtilmiştir. Daha sonra (φ, g) G_2 yapısının burulması $\nabla\varphi$ olarak tanımlanmış olup, bu yapının burulmasız olması için $\nabla\varphi = 0$ olması gerektiği belirtilmiştir. 7 boyutlu bir manifoldun G_2 yapısına sahip olabilmesi için yönlendirilebilir ve spin olması gerekiği buna denk olarak da birinci ve ikinci Stiefel-Whitney sınıflarının sıfırlanması gerektiği vurgulanmıştır. Holonomi, G_2 yapısın burulmasız olması ve 3-form ile 4-formun paralel olmasıyla ilgili ilişkiler verilerek bu alt bölüm sonlandırılmıştır.

Sonraki alt bölümde ise metrik, 3-form ve vektör çarpım arasındaki ilişkiler incelenmiştir. Bu tez boyunca kullanılacak olan bazı temel özellikler ve ilişkiler verilmiş olup bunların yanısıra lokal koordinatlardaki gösterimleri de detaylıca incelenmiştir. Vektör alanlarının dış çarpımı, vektör çarpımı ve 4-form arasındaki ilişki verilmiştir. φ, g ve volüm form arasındaki genel ilişki incelenmiştir. Aynı ilişki lokal koordinatlarda detaylıca incelenmiştir. Bu kavramların birlikte kullanıldığı diğer eşitlikler ve ilişkiler incelenmiştir.

Bir sonraki alt bölümde 3-form φ ve onun duali olan 4-form ψ'nin birlikte bulunduğu eşitlikler incelenmiştir. 3-form φ ve onun duali olan 4-form ψ lokal koordinatlarda gösterilmiştir. Metrik, vektör çarpım ve 3-form arasındaki ilişkilerden ve kovaryant türevden yararlanılarak 3-form ve 4-form arasındaki ilişkiler ve özellikler belirlenmiştir. Sadece 3-form φ ve metrik g içeren eşitlikler incelenmiştir. Son olarak yalnızca 4-form ψ ve metrik g içeren eşitlikler üzerinde durulmuştur. Bunlar lokal koordinatlarda detaylıca incelenmiştir. Bir sonraki adım olarak, 3-form φ, 4-form ψ'nin kovaryant türevlerini içeren eşitlikler verilmiştir.

Sonraki alt bölümde k-formların uzayının indirgenemez ve ortogonal G_2 temsillemelerine ayrılması incelenmiştir. 2-formların uzayı 7 ve 14 boyutlu ortogonal alt modüllere ayrılmıştır. Bunun Hodge duali olan 5-formların uzayı da aynı şekilde 7 ve 14 boyutlu alt modüllere ayrılmıştır. 3-formların uzayı ve onun Hodge duali olan 4-formların uzayı ise 1, 7, 27 boyutlu alt modüllere ayrılmıştır. 3-form φ'nin özelliklerinden ve daha önce hesaplanan G_2 ilişkilerinden yararlanılarak her bir alt modül ayrı ayrı tanımlanmıştır. Daha fazla özelliklerinin incelenmesi için 3-formların uzayı kullanılarak simetrik bir tensör tanımlanmış ve bununla 3-formların uzayı arasındaki ilişkiler irdelenmiştir. Böylece 3-formların uzayının 27 boyutlu alt modülü için yeni bir tanım elde edilmiştir. Daha sonra, paralel olmayan 3-form ve onun Hodge duali olan paralel olmayan 4-form alınmıştır. $d\varphi$'nin 4 formların uzayına ve $d\psi$'nin 5 formların uzayına ait olduğu ve bu uzayların alt modüllere ayrılması gibi $d\varphi$ ve $d\psi$'nin de alt modüllere ayrıştığı elde eldilmiştir. Bu ayrışma sonucu burulma sınıfları elde edilmiştir.

$$\begin{aligned} d\varphi &= \tau_0\,\psi + 3\tau_1 \wedge \varphi + *\tau_3 \\ d\psi &= 4\tau_1 \wedge \psi + *\tau_2 \end{aligned}$$

Burada $d\varphi$ ve $d\psi$ ifadelerinde adı geçen τ_1'in aynı olduğu ispatlanmıştır. Burulma sınıflarının genel özellikleri incelendikten sonra kovaryant türev yardımıyla tam burulma tensörü tanımlanmıştır. Bu tensörün simetrik ve simetrik olmayan kısımlarının sırasıyla τ_0, τ_3 ve τ_1, τ_2 cinsinden yazılabildiğini elde edilmiştir. Ayrıca bu tensörün, burulma sınıfları cinsinden yazılması gibi her bir burulma sınıfının da bu tensörün simetrik ya da simetrik olmayan bölümleri cinsinden yazıldığını elde edilmiştir.

Bu bölümün son kısmında ise sicim teorisi uygulamalarında kullanılmak üzere *curl*, *div*, *grad* gibi bazı diferansiyel geometrik operatörler tanımlanmış olup G_2 yapılı manifoldlar üzerinde sağladığı bazı özellikleri verilmiştir.

Son bölümde ise G_2-yapılarının sicim kuramına uygulanması incelenmiştir. Bunun için, on boyutlu süper kütle çekimi denklemlerinin klasik bir çözümü ele alınmıştır. Daha sonra, G_2 holonomisinin metriğinin kuantum düzeltmelerini sağlamak üzere değiştirilebilme durumu araştırılmıştır. Buna denk olarak, 3-form φ'ye bir pertürbasyon uygulanarak elde edilen yeni metriğin, düzeltilmiş kuantum denklemlerini çözebilme durumu araştırılmıştır. Bu da bizi yeni oluşturulmuş $\phi^{'}$ ve onun Hodge duali $\psi^{'}$ tarafından oluşturulan kısmi diferansiyel denklem sisteminin varlığını araştırmaya yönlendirmiştir. Bu sistemin çözümünün, bazı şartlar altında her zaman var olduğu gösterilmiştir.

1. INTRODUCTION

A G_2-structure can be defined on any seven dimensional smooth manifold M as a reduction of the structure group of the frame bundle of M to the compact, exceptional Lie group G_2. The group G_2 is one of the five exceptional Lie groups. It can be described as the automorphism group of the octonions, or as the subgroup of the general linear group $GL(7,R)$ which preserves a positive 3-form φ, called the associative form. The Hodge dual $\psi = *\varphi$ which is a 4-form, is called the coassociative form. M admits a G_2-structure if and only if the first and second Stiefel-Whitney classes of M vanish. Equivalently, M is orientable and admits a spin structure [1].

G_2 manifolds are manifolds with G_2 holonomy. This is a further differential geometric condition imposed on the 3-form φ. More precisely φ has to be parallel with respect to the Levi-Civita connection. The metric is also given via the G_2-structure as will be described in the main body of the thesis. It can be shown that φ is parallel, namely $\nabla\varphi = 0$ if and only if $d\varphi = d*\varphi = 0$. It is known that if M is a G_2-manifold, then M is a Ricci-flat, orientable, spin manifold [1]. Manifolds with G_2 holonomy were first introduced by Edmond Bonan in 1966 [2]. He constructed the parallel 3-form, the parallel 4-form and showed that these manifolds are Ricci-flat. The first complete, but noncompact 7-dimensional manifolds with G_2 holonomy were constructed by Robert Bryant and Salamon in 1989. The first compact 7-dimensional manifolds with G_2 holonomy were constructed by Dominic Joyce in 1994 [5], and compact G_2 manifolds are sometimes known as "Joyce manifolds", especially in the physics literature.

Manifolds with G_2 holonomy are important in physics, especially in string theory. They break the original supersymmetry to $\frac{1}{8}$ of the original amount. Also they are models for the extra dimension in M-theory. They also play a role in particle physics, especially the standard model of particle physics. Recently, manifolds with G_2-structure, rather than G_2-holonomy has found interesting applications in string theory. In this case, the associative 3-form φ and its Hodge dual ψ are not necessarily parallel. And the tool that measures how far they are from being parallel is given by the torsion classes of the G_2-structure.

The aim of this thesis is to study the differential geometric properties of manifolds of G_2-structure. We are particularly interested in the case when we have just G_2-structure rather than G_2 holonomy. Hence, we study in detail the description of the torsion classes of a given G_2-structure. We also study an application in string theory.

The outline of this thesis is as follows. We start by a preliminary chapter in differential geometry and algebra reviewing briefly the topics which are essential for the rest of the thesis. Firstly, in the Differential Geometry section we review some basic concepts such as a differentiable manifolds, Riemannian metric and then we write differential forms, interior product, Hodge star operator and some useful concepts in local coordinates. Secondly, in the Algebra section we start by the definition of the normed division algebra. Using the quaternion multiplication we define the cross product of two vectors in $\mathbb{R}^3$. We use this concept to generate cross product on $\mathbb{R}^7$ via octonions. Then, we state the properties of cross products on $\mathbb{R}^3$ and $\mathbb{R}^7$. We also give the multiplication table of the octonion product. In chapter 2, we study G_2-structures. We construct the 3-form and its dual 4-form via the octonions. Then, we state the relations between a metric, cross product and 3-form. We use the wedge product, interior product and covariant derivate to get some useful relations. Secondly, we consider the equalities involving contractions of φ and ψ. Then we study the decomposition of the space of k-forms into irreducible G_2 representations. For each k, the space of k-forms decomposes as a direct sum of submodules, each of which is invariant under the action of G_2. Also, these submodules are orthogonal to each other with respect to the metric determined by φ. Next, using the decomposition of the spaces of differential forms we decompose $d\varphi$ and $d * \varphi$ into irreducible G_2 representations. Then, this defines the torsion forms. By considering these torsion forms we see their relation to the concept of being torsion free φ. After that, we define the full torsion tensor and we see some identities involving it. Lastly, in this section we define some useful operators like *curl*, *div*, *grad* on G_2-manifolds which are useful in the application in string theory. In the last chapter, we consider an application of G_2-structures in string theory. We start with a classical solution of ten dimensional supergravity of the form $R_{2,1} \times Y_7$ where Y^7 is a G_2 manifold. Then we ask if the metric of G_2 holonomy can be modified to compensate for quantum corrections, which are callled α' corrections. Equivalently we ask, if there is a small deformation $\phi' = \phi + \delta\phi$ of the associative 3-form ϕ such that the corresponding metric g' solves the α' corrected equations of the quantum theory. This amounts to analyzing the existence of a coupled system of partial differential equations for ϕ', and its Hodge dual (with respect to g') where the source terms are determined by physics, and are related to the torsion forms of the G_2 structure ϕ'. We show that such a solution always exists.

2. PRELIMINARIES

2.1 Preliminaries in Differential Geometry

In this section, we briefly review some basic concepts in differential geometry which we will need in this thesis.

Definition 2.1.1 *A differentiable manifold of dimension n is a set M and a family of injective mappings*
$\boldsymbol{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ *of open sets* U_α *of* $\mathbb{R}^n$ *into M such that:*

1. $\bigcup_\alpha \boldsymbol{x}_\alpha(U_\alpha) = M$
2. *for any pair* α, β *with* $\boldsymbol{x}_\alpha(U_\alpha) \cap \boldsymbol{x}_\beta(U_\beta) = W \neq \varnothing$, *the sets* $\boldsymbol{x}_\alpha^{-1}(W)$ *and* $\boldsymbol{x}_\beta^{-1}(W)$ *are open sets in* $\mathbb{R}^n$ *and the mappings* $\boldsymbol{x}_\beta^{-1} \circ \boldsymbol{x}_\alpha$ *are differentiable.*
3. *The family* $\{(U_\alpha, \boldsymbol{x}_\alpha)\}$ *is maximal relative to the conditions above.*

The pair $(U_\alpha, \boldsymbol{x}_\alpha)$ *with* $p \in \boldsymbol{x}_\alpha(U_\alpha)$ *is called a parametrization of M at p;* $\boldsymbol{x}_\alpha(U_\alpha)$ *is then called a coordinate neighborhood at p. A family* $\{(U_\alpha, \boldsymbol{x}_\alpha)\}$ *satisfying the conditions is called a differentiable structure on M [6].*

Definition 2.1.2 *The tangent space* $T_p(M)$ *at the point p is a vector space spanned by the basis* $e_i = \frac{\partial}{\partial x^i}$.
A tangent vector v can be written as $v = v^i e_i$. *A tangent vector at p is the tangent vector at* $t = 0$ *of some curve* $\alpha : (-\varepsilon, \varepsilon) \to M$ *with* $\alpha(0) = p$.

Definition 2.1.3 *The cotangent space* $T_p^*(M)$ *at the point p is a vector space of linear maps*

$$\alpha : T_p(M) \to \mathbb{R} \quad v \mapsto \langle \alpha, v \rangle$$

spanned by the basis $w^i = dx^i$. *This basis is dual to the basis* e_i *in the sense that* $\langle e_i, w^j \rangle = \delta_i^j$.

Definition 2.1.4 *The tangent bundle TM is the disjoint union of the tangent spaces $T_p(M)$, for all $p \in M$.*

$$TM = \{(p, v); p \in M, v \in T_p(M)\}$$

The cotangent bundle T^*M is the disjoint union of the cotangent spaces $T_p^*(M)$, for all $p \in M$.

$$T^*M = \{(p, w); p \in M, w \in T_p^*(M)\} \tag{2.1}$$

The tangent bundle TM and the cotangent bundle T^*M of a manifold M are also manifolds.

Definition 2.1.5 *Let $T_p^*(M)$ be the cotangent space and $\phi \in T_0^r(T_p^*(M))$, where $T_0^r(T_p^*(M))$ is a collection of all tensors of covariant order r, namely $\phi : T_p^*(M) \times T_p^*(M) \times \cdots \times T_p^*(M) \to \mathbb{R}$. $\phi \in T_0^r(T_p^*(M))$ is called r-covariant tensor. ϕ is symmetric if $\phi(v_1, \ldots, v_r) = \phi(v_\sigma(1), \ldots, v_\sigma(r)$ and is alternating if $\phi(v_1, \ldots, v_r) = sign(\sigma)\phi(v_\sigma(1), \ldots, v_\sigma(r))$ for every $v_1, \ldots, v_r$ and permutation σ. The alternating tensors in $T_0^r(T_p^*(M))$ form a subspace which we denote by $\wedge^r(T_p^*(M))$.*

Definition 2.1.6 *We define a linear transformation on the vector space $T_0^r(T_p^*(M))$; alternating mapping $\boldsymbol{A} : T_0^r(T_p^*(M)) \to \wedge^r(T_p^*(M))$ by*

$$(\boldsymbol{A}\phi)(v_1, \ldots, v_r) = \frac{1}{r!}\Sigma_\sigma sign(\sigma)\phi(v_\sigma(1), \ldots, v_\sigma(r))$$

Definition 2.1.7 *The mapping $\wedge^r(T_p^*(M)) \times \wedge^s(T_p^*(M)) \to \wedge^{r+s}(T_p^*(M))$ defined by*

$$(A, B) \to \frac{(r+s)!}{r!s!}\boldsymbol{A}(A \otimes B)$$

is called the exterior product of A and B and is denoted by $A \wedge B$. The exterior product is bilinear and associative. The space $\wedge^r(T_p^*(M))$ equipped with the exterior product is an algebra. Elements of this algebra are called differential forms. Differential one-forms are elements of T^*M.

Here are some properties of a k-form, from [7].

The space of k-forms on M will be denoted by $\wedge^k$. It is the space of sections of the bundle $\Lambda^k(T^*M)$. A differential k-form α on M can be written as

$$\alpha = \frac{1}{k!}\alpha_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

in local coordinates $(x^1, \ldots, x^7)$, where the sums are all from 1 to 7, and $\alpha_{i_1 i_2 \cdots i_k}$ is skew-symmetric in its indices. By this, α can also be written as

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

We take the interior product $(\frac{\partial}{\partial x^m}) \lrcorner \alpha$ of the k-form α with a vector field $(\frac{\partial}{\partial x^m})$, we obtain the $(k-1)$-form

$$(\frac{\partial}{\partial x^m}) \lrcorner \alpha = \frac{1}{(k-1)!}\alpha_{m i_1 i_2 \cdots i_{k-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}}$$

Definition 2.1.8 (Riemannian Metric) *A Riemannian metric on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle\, ,\rangle$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM which varies differentiably in the following sense:*

If $\boldsymbol{x} : U \subset \boldsymbol{R}_n \to M$ is a system of coordinates around p, with

$$\boldsymbol{x}(x_1, x_2, ..., x_n) = q \in \boldsymbol{x}(U)$$

and

$$\frac{\partial}{\partial x^i} = d\boldsymbol{x}_q(0, ..., 1, ..., 0)$$

then $\langle \frac{\partial}{\partial x^i}(q), \frac{\partial}{\partial x^j}(q)\rangle = g_{ij}(x_1, ..., x_n)$ is a differentiable function on U [6].

Definition 2.1.9 *Let M be an n-dimensional smooth manifold. For any open set $U \subseteq M$, an n-tuple of vector fields $(X_1, \ldots, X_n)$ over U is called a frame over U if and only if $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space $T_p(M)$, for every $p \in U$. Over every point p in M, the Riemannian metric determines the set of orthonormal frames, i.e., the possible choices for an orthonormal basis for the tangent space $T_p(M)$. The collection of orthonormal frames is the frame bundle.*

Here are the identities arise from the metric g, which follows from [7].

Remark 2.1.10 *Given a Riemannian metric g on M, it induces a metric on k-forms which is defined on decomposable elements to be*

$$\begin{aligned} g(dx^{i_1}\wedge\cdots\wedge dx^{i_k}, dx^{j_1}\wedge\cdots\wedge dx^{j_k}) &= \det_{a,b=1,\dots,k}(g(dx^{i_a}, dx^{j_b})) = \det(g^{i_a j_b}) \\ &= \sum_{\sigma\in S_7} \mathrm{sgn}(\sigma) g^{i_1 j_{\sigma(1)}} g^{i_2 j_{\sigma(2)}} \dots g^{i_k j_{\sigma(k)}} \end{aligned}$$

where $g^{ij} = g(dx^i, dx^j)$ is the induced metric on the cotangent bundle and g^{ij} is the inverse matrix of the matrix g_{ij}. By this, the inner product of two k-forms $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ and $\beta = \frac{1}{k!}\beta_{j_1\cdots j_k} dx^{j_1}\wedge\cdots\wedge dx^{j_k}$ is

$$g(\alpha,\beta) = \frac{1}{k!}\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1}\dots g^{i_k j_k} \tag{2.2}$$

The metric g determines a musical isomorphism between the tangent and cotangent bundles of M. If v is a vector field, then the metric dual 1*-form $v^\flat$ is defined by $v^\flat(w) = g(v,w)$ for all vector fields w. In local coordinates, $\left((\frac{\partial}{\partial x^i})\right)^\flat = g_{ik}dx^k$. Similarly a* 1*-form α has a metric dual vector field $\alpha^\sharp$ defined by $\beta(\alpha^\sharp) = g(\alpha,\beta)$ for all* 1*-forms β, and $\left(dx^i\right)^\sharp = g^{ik}(\frac{\partial}{\partial x^k})$.*

We denote the volume form on M associated to a metric g and an orientation by vol. In local coordinates the volume form can be written as

$$\mathrm{vol} = \sqrt{\det(g)}\, dx^1\wedge\cdots\wedge dx^n$$

where $\det(g)$ *is the determinant of the matrix $g_{ij} = g((\frac{\partial}{\partial x^i}),(\frac{\partial}{\partial x^j}))$.*

The metric and orientation together determine the Hodge star operator $*$ taking k-forms to $(n-k)$-forms, denoted by the relation

$$\alpha\wedge *\beta = g(\alpha,\beta)\ \mathrm{vol}$$

on two k-forms α and β. Then,

$$\begin{aligned} *\alpha &= \frac{\sqrt{\det(g)}}{k!(n-k)!}\alpha_{i_1\cdots i_k}\varepsilon^{i_1\cdots i_k}_{j_{k+1}\cdots j_n} dx^{j_{k+1}}\wedge\cdots\wedge dx^{j_n} \\ \varepsilon^{i_1\cdots i_k}_{j_{k+1}\cdots j_n} &= g^{i_1 j_1}\dots g^{i_k j_k}\varepsilon_{j_1\cdots j_k j_{k+1}\cdots j_n} \end{aligned}$$

where $\varepsilon_{j_k\cdots j_n} = sgn(\sigma)$. Also we have, $*1 = \frac{\sqrt{\det(g)}}{n!}\varepsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$

The operator δ can be written in terms of d and $*$ as

$$\delta = (-1)^{nk+n+1} * d * \tag{2.3}$$

The exterior derivative $d\alpha$ of a k-form α can be written in terms of the covariant derivative as

$$d\alpha = \frac{1}{k!}(\nabla_m \alpha_{i_1,i_2,\cdots,i_k})dx^m \wedge dx^{i_1} \wedge dx^{i_k}$$

The coderivative δ can be written in terms of the metric g and the covariant derivative ∇ as follows:

$$\begin{aligned} \delta\alpha &= \frac{1}{(k-1)!}(\delta\alpha)_{i_1 i_2 \cdots i_{k-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}} \\ \text{where} \quad (\delta\alpha)_{i_1 i_2 \cdots i_{k-1}} &= -g^{lm} \nabla_l \alpha_{m i_1 \cdots i_{k-1}} \end{aligned} \tag{2.4}$$

Definition 2.1.11 *Let (M,g) be a compact Riemannian manifold. The Laplacian Δ is a positive operator on M. It is defined by*

$$\Delta : \wedge^r(M) \to \wedge^r(M) \tag{2.5}$$

$$\Delta = dd^\dagger + d^\dagger d \tag{2.6}$$

where $d^\dagger = - * d *$.
An r-form ω is called harmonic if $\Delta\omega = 0$ and closed if $d\omega = 0$. It is coclosed if $d^\dagger \omega = 0$ [8].

The following theorem is a direct consequence.

Theorem 2.1.12 *An r-form ω is harmonic if and only if ω is closed and coclosed. We denote the set of harmonic r-forms on M by $Harm^r(M)$ [8].*

Theorem 2.1.13 *(Hodge Decomposition Theorem) Let (M,g) be a compact Riemannian manifold without a boundary. Then $\wedge^r(M)$ is uniquely decomposed as*

$$\wedge^r(M) = d \wedge^{r-1}(M) \oplus d^\dagger \wedge^{r+1}(M) \oplus Harm^r(M) \tag{2.7}$$

That is any r-form ω_r is written globally as

$$\omega_r = d\alpha_{r-1} + d^\dagger \beta_{r+1} + \gamma_r \tag{2.8}$$

where $\alpha_{r-1} \in \wedge^{r-1}(M)$, $\beta_{r+1} \in \wedge^{r+1}(M)$ and $\gamma_r \in Harm^r(M)$ [8].

Remark 2.1.14 *If $\omega \in \wedge^r(M)$ is such that $\omega = \Delta\beta$ for some $\beta \in \wedge^r(M)$, then we have that $(\omega,\gamma) = 0$ for all $\gamma \in Harm^r(M)$, with respect to the L^2 inner product on M which is given by*

$$(\alpha,\beta) = \int_M g(\alpha,\beta)dvol$$

The next definition follows from [6].

Definition 2.1.15 *Let M be a smooth manifold and $C^\infty(M,TM)$ be the space of vector fields on M, that is, the space of smooth sections of the tangent bundle TM. Then an affine connection on M is a bilinear map*
$\nabla : C^\infty(M,TM) \times C^\infty(M,TM) \to C^\infty(M,TM)$
such that for all smooth functions $f,g \in C^\infty(M,\mathbb{R})$ and all vector fields X,Y on M,

1. $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X(fY) = df(X)Y + f\nabla_X Y$ *that is ∇ satisfies the Leibniz rule in the second variable*

Definition 2.1.16 *Let ∇ be an affine connection on TM. Let $\gamma : [0,1] \to M$ be a closed curve at p in M, namely $\gamma(0) = \gamma(1) = p$. Parallel transport along γ is the map*

$$\begin{aligned} P_\gamma : T_pM &\to T_pM, \\ P_\gamma(v) &= \sigma(1) \end{aligned}$$

where σ is the (unique) parallel section of γ^*TM such that $\sigma(0) = v$.

Definition 2.1.17 *Let $p \in M$. The holonomy group of the connection ∇ is the group of transformations of T_pM given as parallel translations along piecewise smooth curves based at p. The group is denoted $Hol(\nabla, p)$.*

Definition 2.1.18 *The restricted holonomy group is*

$$Hol_p^0(\nabla) = \{P_\gamma : \gamma \text{ is null-homotopic loop based at } p\}$$

A loop γ based at p is null-homotopic if it can be deformed to the constant loop at p.

The next definitions follow from [6].

Definition 2.1.19 (Levi-Civita connection) *Given a Riemannian metric $\langle \cdot, \cdot \rangle$, there exists a unique affine connection ∇ on M. If M is endowed with a Riemannian metric g, then there is a unique connection ∇ that satisfies*

- $\nabla_X Y - \nabla_Y X = [X,Y]$. ∇ *is torsion free*
- $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad X,Y,Z \in X(M)$
  that is ∇ is metric compatible.

Definition 2.1.20 (Curvature) *The curvature R of a Riemannian manifold M is a correspondence that associates every pair $X,Y \in X(M)$ a mapping*

$$R(X,Y) : X(M) \to X(M)$$

is given by $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$, $Z \in X(M)$ where ∇ is the Riemannian connection of M.

2.2 Preliminaries in Algebra

In this section, we briefly review some basic concepts in algebra which will be needed in this thesis.
We start with defining the normed division algebra.

Definition 2.2.1 *Let $\mathbf{A}$ be a finite dimensional vector space with a norm $|.|$ we say $\mathbf{A}$ is a normed division algebra if it has the structure of an (not necessarily associative) algebra with identity such that $|ab| = |a|.|b|$ where $a,b \in \mathbf{A}$. Further, the norm is connected to the inner product by the relation $|a| = \langle a, a\rangle$ for all $a \in \mathbf{A}$.*

Remark 2.2.2 *Up to isomorphism, there are only four normed division algebras real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ of dimension 1, 2, 4 and 8, respectively. Also $\mathbb{H}$ is noncommutative and $\mathbb{O}$ is noncommutative and nonassociative. It is known that $\mathbb{O}$ octonions are called the exceptional normed division algebra.*

In 1898, Hurwitz proved that fact.

Definition 2.2.3 (Cross Product) *A bilinear map*

$$x : Im(\mathbf{A}) \times Im(\mathbf{A}) \to Im(\mathbf{A})$$

is defined by $a \times b = Im(ab) = -(b \times a)$*, where* $Im(\mathbf{A})$ *is the imaginary part of* $\mathbf{A}$.

Here are the properties of cross product on $\mathbb{R}^3$ and $\mathbb{R}^7$, which follows from [1,9].
The cross product of two vectors in $\mathbb{R}^3$ is the pure part of the quaternion product of two pure quaternions, that is, $a \times b = Im(ab)$ for $a, b \in \mathbb{R}^3 \subset \mathbb{H}$. The cross product of two vectors in $\mathbb{R}^7$ can be defined in terms of an orthonormal basis $e_1, \ldots, e_7$ by antisymmetry, $e_i \times e_j = -e_j \times e_i$ and in the form $e_i \times e_{i+1} = e_{i+3}$, where the indices are permuted cyclically and translated modulo 7.

The octonions come equipped with a positive definite inner product. The span of the identity element 1 is called the *real* octonions. Its orthogonal complement is called the *imaginary* octonions $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$. This is analogous to the quaternions $\mathbb{H}$, except the non-associativity. We define a *cross product* on $\mathbb{R}^7$ as follows. Let $u, v \in \mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$ and define $u \times v = \mathrm{Im}(uv)$, where uv denotes the octonion product. The real part of uv is equal to $-\langle u, v \rangle$, just as it is for quaternions, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. This cross product satisfies the following relations:

$$u \times v = -v \times u, \qquad \langle u \times v, u \rangle = 0, \qquad ||u \times v||^2 = ||u \wedge v||^2,$$

exactly like the cross product on $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$. However, there is a difference. Unlike the cross product in $\mathbb{R}^3$, the following expression is *not* zero:

$$u \times (v \times w) + \langle u, v \rangle w - \langle u, w \rangle v$$

but is a measure of being nonassociativity: $(uv)w - u(vw) \neq 0$.

Denote the product of $a, b \in \mathbb{O}$ by $a \circ b$. Let $1, e_1, e_2, \ldots, e_7$ be a basis of $\mathbb{O}$. Define the product in terms of the basis by $e_i \circ e_i = -1$, $e_i \circ e_j = -e_j \circ e_i$ for $i \neq j$ in the form $e_i \circ e_{i+1} = e_{i+3}$, where the indices are permuted cyclically and translated modulo 7. This can be shown by the multiplication table, taken from [9].

$e_1 \circ e_2 = e_4$	$e_2 \circ e_4 = e_1$	$e_4 \circ e_1 = e_2$
$e_2 \circ e_3 = e_5$	$e_3 \circ e_5 = e_2$	$e_5 \circ e_2 = e_3$

$e_3 \circ e_4 = e_6$	$e_4 \circ e_6 = e_3$	$e_6 \circ e_3 = e_4$
$e_4 \circ e_5 = e_7$	$e_5 \circ e_7 = e_4$	$e_7 \circ e_4 = e_5$
$e_5 \circ e_6 = e_1$	$e_6 \circ e_1 = e_5$	$e_1 \circ e_5 = e_6$
$e_6 \circ e_7 = e_2$	$e_7 \circ e_2 = e_6$	$e_2 \circ e_6 = e_7$
$e_7 \circ e_1 = e_3$	$e_1 \circ e_3 = e_7$	$e_3 \circ e_7 = e_1$

Similarly a seven dimensional cross product of the octonion product of two pure octonions, that is, $a \times b = \langle a \circ b \rangle_1$. The octonion algebra $\mathbb{O}$ is a normed division algebra with unity 1. The vector part $\mathbb{R}^7$ in $\mathbb{O} = \mathbb{R}^7 \oplus \mathbb{R}$ is also an algebra with cross product, that is, $a \times b = \frac{1}{2}(a \circ b - b \circ a)$ for $a, b \in \mathbb{R}^7 \subset \mathbb{O} = \mathbb{R}^7 \oplus \mathbb{R}$. The octonion product is given by $\mathbf{a} \circ \mathbf{b} = \alpha\beta + \alpha b + a\beta - a.b + a \times b$ for $\mathbf{a} = \alpha + a$ and $\mathbf{b} = \beta + b$ in $\mathbb{R}^7 \oplus \mathbb{R}$ [9].

3. MANIFOLDS WITH G_2 STRUCTURE

A G_2-manifold is a Riemannian manifold whose holonomy group is contained in the exceptional Lie group G_2. For the purposes of this thesis, the importance of the group G_2 does not arise from the fact that it is one of the five exceptional Lie groups, but rather than that it is the automorphism group of the octonions $\mathbb{O}$. A manifold has a G_2-structure if there is an isomorphism between its tangent spaces and the imaginary octonions $Im(\mathbb{O}) \cong \mathbb{R}^7$ [1].

3.1 G_2-Structures

In this section, we review the construction and properties of G_2-structures. We define the multiplication on the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{e} = \mathbb{R}^8$ via the Cayley-Dickson process [10], we have

$$(a + b\mathbf{e}) \cdot (c + d\mathbf{e}) = (ac - \bar{d}b) + (da + b\bar{c})\mathbf{e} \qquad a, b, c, d \in \mathbb{H}$$

in terms of quaternion multiplication, where $\bar{c}$ is the conjugate of c. Let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product on $\mathbb{R}^8$. Imaginary octonions can be considered as $\mathbb{R}^7$, $\mathrm{Im}(\mathbb{O}) = \mathbb{R}^7$ we define the 3-form φ by

$$\varphi(x, y, z) = \langle x, yz \rangle \qquad x, y, z \in \mathrm{Im}(\mathbb{O}),$$

and its dual 4-form ψ by

$$\psi(x, y, z, w) = \frac{1}{2}\langle x, [y, z, w] \rangle \qquad x, y, z, w \in \mathrm{Im}(\mathbb{O}),$$

where $[x, y, z] = (xy)z - x(yz)$ is the associator.

In terms of the standard basis for $\mathbb{R}^8 = \mathbb{O}$ we have the coordinates x^0, x^1, x^2, x^3, y^0, y^1, y^2, y^3 where the x^i's are coordinates on $\mathbb{H}$ and the y^j's are coordinates on $\mathbb{H}\mathbf{e}$ [10]. We take the orientation given by the volume form

$$\mathrm{vol}_8 = dx^{0123} \wedge dy^{0123}.$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. The induced orientation on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ is given by

$$\mathrm{vol}_7 = dx^{123} \wedge dy^{0123}.$$

With respect to this orientation, the 4-form ψ is the Hodge dual (on $\mathbb{R}^7$) of φ:

$$\psi = *_7 \varphi.$$

In these coordinates, from [10] the forms φ and ψ can be written as

$$\begin{aligned} \varphi &= dx^{123} - dx^1 \wedge dy^{23} - dy^1 \wedge dx^2 \wedge dy^3 - dy^{12} \wedge dx^3 \\ &\quad - dy^0 \wedge dx^1 \wedge dy^1 - dy^0 \wedge dx^2 \wedge dy^2 - dy^0 \wedge dx^3 \wedge dy^3, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \psi &= dy^{0123} - dy^{01} \wedge dx^{23} - dy^0 \wedge dx^1 \wedge dy^2 \wedge dx^3 - dy^0 \wedge dx^{12} \wedge dy^3 \\ &\quad - dx^2 \wedge dy^2 \wedge dx^3 \wedge dy^3 - dx^3 \wedge dy^3 \wedge dx^1 \wedge dy^1 - dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2. \end{aligned} \tag{3.2}$$

It can be seen that $\varphi \wedge \psi = 7\,\mathrm{vol}_7$.

We have the following equivalent definitions of G_2. The next definition follows from [11].

Definition 3.1.1 *We can define G_2 as the group of automorphisms of octonions as in*

$$G_2 = Aut(\mathbb{O}) = \{g \in \mathrm{GL}(\mathbb{O}); g(xy) = g(x)g(y) \text{ for all } x,y \in \mathbb{O}\} \tag{3.3}$$

Instead of this definition, we prefer to use the following definition of G_2, taken from [12, 13], which is more useful in differential geometric point of view. It can be shown that these definitions are equivalent.

Definition 3.1.2 *Let $x^1, \ldots, x^7$ be coordinates on $\mathbb{R}^7$. Write $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$ on $\mathbb{R}^7$. Define a 3-form φ_0 on $\mathbb{R}^7$ by*

$$\varphi_0 = dx^{123} - dx^{167} - dx^{527} - dx^{563} + dx^{415} + dx^{426} + dx^{437} \tag{3.4}$$

The subgroup of $\boldsymbol{GL}(7,\mathbb{R})$ preserving φ_0 is the exceptional Lie group G_2. That is given by

$$G_2 = \{A \in \mathrm{GL}(7,\mathbb{R}); A^*(\varphi_0) = \varphi_0\}$$

It is compact, connected, simply-connected, semisimple and 14*-dimensional, and it also fixes the* 4*-form, the Hodge star dual* $*\varphi_0$ *of* φ_0*.*

$$*\varphi_0 = dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526} \tag{3.5}$$

the standard Euclidean metric

$$g = \sum_{k=1}^{7} dx^k \otimes dx^k$$

and the orientation on $\mathbb{R}^7$*.*

The proof can be found in R.L. Bryant [3]. We have the following definition

Definition 3.1.3 *A smooth 7 dimensional manifold M has a* G_2*-structure if its tangent frame bundle reduces to a* G_2 *bundle. It is equivalent to saying that M has a* G_2*-structure if there exists a 3-form* $\varphi \in \wedge^3\mathbb{R}^7$ *such that at each* $p \in M$*,* $(T_p(M), \varphi_p)$ *is isomorphic to* $(T_0(\mathbb{R}^7), \varphi_0)$*, pointwise. We call* (M, φ) *a manifold with* G_2*-structure [13].*

A G_2-structure φ determines a metric g and a cross product $\times$ on the tangent bundle of M. The next relation follows from [13].

$$\begin{aligned} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi &= -6g(u,v)\,\mathrm{vol} \\ g(u \times v, w) &= \varphi(u,v,w) \end{aligned}$$

The proof of the first identity can be found in Lemma 3.2.2.

By using that relation one can derive the following expression, from [14], that shows the relation between the metric and the 3-form as follows:

Theorem 3.1.4 *Let v be a tangent vector at a point p and let* $e_1, e_2, \ldots, e_7$ *be any basis for* T_pM*. Then the length* $|v|$ *of v is given by*

$$|v|^2 = 6^{\frac{2}{9}} \frac{\left((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi\right)(e_1, e_2, \ldots, e_7)}{\left(\det\left(\left((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi\right)(e_1, e_2, \ldots, e_7)\right)\right)^{\frac{1}{9}}} \tag{3.6}$$

The detailed proof can be found in S. Karigiannis [14].
We have the next definition follows from [14].

Definition 3.1.5 *Let u and v be vector fields on M. The* cross product, *$u \times v$ is a vector field on M whose associated 1-form under the metric isomorphism satisfies the relations as follows:*

$$\begin{aligned} (u\times v)^{\flat} &= v \lrcorner u \lrcorner \varphi \\ g(u\times v, w) &= (u\times v)^{\flat}(w) = w \lrcorner v \lrcorner u \lrcorner \varphi = \varphi(u,v,w) \end{aligned}$$

The proof can be found in Appendix A.1., Lemma 1.0.3.

Because G_2, by definition, preserves φ, it also preserves the metric, the cross product and the volume form determined by the 3-form. Hence, an equivalent (but redundant) definition of G_2 is the following

$G_2 = \{A \in \mathrm{GL}(7,\mathbb{R}), \quad u,v \in Im(\mathbb{O}); \quad A^*\varphi_0 = \varphi_0, \quad A^*(g) = g, \quad detA = 1, A_*u \times A_*v = A_*(u\times v)\}$

For this reason G_2 is a subgroup of $SO(7)$.

Now, we consider some important facts, from [12], for defining a G_2-manifold.

Definition 3.1.6 *Let M be a 7-manifold, (φ, g) a G_2-structure on M, and ∇ the Levi-Civita connection of g. We call $\nabla\varphi$ the torsion of (φ, g). If $\nabla\varphi = 0$ then (φ, g) is called torsion-free.*

Definition 3.1.7 *We define a G_2-manifold to be a triple (M,φ,g) where M is a 7-manifold and (φ, g) a torsion-free G_2-structure on M.*

By [1], the existence of a G_2-structure is given by the following.

Remark 3.1.8 *Let us consider a 7-manifold with a G_2-structure φ. This structure exists if and only if M is orientable and spin, which is equivalent to the vanishing of the fist and second Stiefel-Whitney classes $w_1(M) = w_2(M) = 0$.*

The metric g and orientation determine a Hodge star operator $*$, and we have the associated dual 4-form $\psi = *\varphi$. The metric also determines the Levi-Civita connection ∇, and the manifold (M,φ) is called a G_2 manifold if $\nabla\varphi = 0$. We note that this is a nonlinear partial differential equation for φ. Such manifolds have Riemannian holonomy $Hol_g(M)$ contained in the exceptional Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$ [1]. It is given by the following proposition, taken from [12].

Proposition 3.1.9 *Let M be a 7-manifold and (φ, g) a G_2-structure on M. Then the following are equivalent:*

(i) $Hol(g) \subseteq G_2$*, and* φ *is the induced* 3*-form,*

(ii) $\nabla\varphi = 0$ *on M, where* ∇ *is the Levi-Civita connection of g, and*

(iii) $\mathrm{d}\varphi = \mathrm{d}*\varphi = 0$ *on M.*

Equivalence of the conditions (i) and (ii) was first proved by Fernandez and Gray [15]. We will study the proof of this theorem in section 3.5.
The following remark gives the definition of a positive 3-form that arises from G_2 structures [12, 14].

Remark 3.1.10 *Let M be an oriented 7-manifold. The 3-forms on M that arise from a G_2-structure are called positive 3-forms. For each $p \in M$, define $\mathrm{P}^3_p M$ to be the subset of 3-forms $\varphi \in \wedge^3 T^*_p(M)$ for which there exists an oriented isomorphism between $T_p(M)$ and $\mathbb{R}^7$ identifying φ and the associative 3-form φ_0. A form $\alpha \in \wedge^3\mathbb{R}^7$ is said to be positive if and only if $\alpha \in \mathrm{P}^3_p M$. A positive 3-form α is of the form $g^*(\varphi)$ for some $g \in \mathrm{GL}(7,\mathbb{R})$ and uniquely determines an associated positive definite inner product and orientation. Then $\mathrm{P}^3_p M$ is isomorphic to $\boldsymbol{GL}(7,\mathbb{R})/G_2$ which naturally imbeds in $\wedge^3\mathbb{R}^7$ as $\mathrm{P}^3_p M$, since φ_0 has symmetry group G_2. It can be seen that both $\boldsymbol{GL}(7,\mathbb{R})/G_2$ and $\wedge^3\mathbb{R}^7$ are 35 dimensional.*

Before we close this section, we would like to emphasize that we do not study the representation theory of G_2 as a Lie group. Information in this direction can be found in [2].

3.2 Metric, Cross Product and the 3-form Identities

In this section, we state the relations between the Riemannian metric g, cross product and the 3-form φ. Also we state the basic and useful properties which will be essential for the rest of the thesis. We start with the next corollary follows from [7].

Corollary 3.2.1 *Let a, b, c, d be vector fields. Then we have*

$$g(a\times b, c\times d) = g(a\wedge b, c\wedge d) - \psi(a,b,c,d) \tag{3.7}$$

Proof. According to [7], we compute

$$\begin{aligned} g(a\times b, c\times d) &= \varphi(a,b,c\times d) \\ &= -\varphi(a,c\times d,b) \\ &= -g(a\times(c\times d),b) \\ &= -g(-g(a,c)d+g(a,d)c-(a\lrcorner c\lrcorner d\lrcorner\psi)^{\sharp},b) \\ &= g(a,c)g(b,d)-g(a,d)g(b,c)+\psi(d,c,a,b) \end{aligned} \tag{3.8}$$

We have the relation involving the metric, 3-form and metric dual 1-forms, which follows from [7].

Lemma 3.2.2 *Let u, v, and w be vector fields on M. Let $u^\flat$, $v^\flat$, and $w^\flat$ denote their dual 1-forms with respect to the metric g. Then we have the following identity:*

$$*((u\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge(w\lrcorner\varphi)) = -2g(u,v)w^\flat - 2g(u,w)v^\flat - 2g(v,w)u^\flat$$

Proof. According to [7], we begin with the relation between φ, g, and the volume form

$$(u\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\varphi = -6g(u,v)\,\mathrm{vol}$$

First, we should prove

$$(v\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\varphi = 6|v|^2\,\mathrm{vol} \tag{3.9}$$

From Lemma 1.0.1 and Proposition 1.0.2 we have

$$v\lrcorner\varphi = *(v^\flat\wedge*\varphi)$$

and

$$(v\lrcorner\varphi)\wedge\varphi = 2(v^\flat\wedge*\varphi)$$

Thus we obtain

$$(v\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\varphi = 2|v^\flat\wedge*\varphi|^2\,\mathrm{vol} = 6|v|^2\,\mathrm{vol}$$

By polarizing (3.9) in v, we have the following the relation:

$$(v\lrcorner\varphi)\wedge(w\lrcorner\varphi)\wedge\varphi = 6\langle v,w\rangle\,\mathrm{vol}$$

Taking the interior product of this equation with w and using $w \lrcorner \mathrm{vol} = *w^\flat$, we obtain

$$\begin{aligned} -6g(u,v) * w^\flat &= (w \lrcorner u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi + (u \lrcorner \varphi) \wedge (w \lrcorner v \lrcorner \varphi) \wedge \varphi \\ &\quad + (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \\ &= -2(u \times w)^\flat \wedge *(v \lrcorner \varphi) - 2(v \times w)^\flat \wedge *(u \lrcorner \varphi) \\ &\quad + (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \end{aligned}$$

where we have used equation (**A.8**) and the relation $(v \lrcorner \varphi) \wedge \varphi = -2 * (v \lrcorner \varphi)$ from Proposition 1.0.2. We rearrange this equation and use $*(v \lrcorner \varphi) = v^\flat \wedge \psi$ to obtain

$$\begin{aligned} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) &= -6g(u,v) * w^\flat - 2v^\flat \wedge (u \times w)^\flat \wedge \psi \\ &\quad - 2u^\flat \wedge (v \times w)^\flat \wedge \psi \end{aligned}$$

We now use (**A.8**) and (**A.9**), and take $*$ of both sides to get

$$\begin{aligned} & * ((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi)) \\ &= -6g(u,v)w^\flat - 2(v \times (u \times w))^\flat - 2(u \times (v \times w))^\flat \\ &= -6g(u,v)w^\flat - 2\left(-g(u,v)w^\flat + g(v,w)u^\flat - (v \lrcorner u \lrcorner w \lrcorner \psi)^\sharp\right) \\ &\quad -2\left(-g(u,v)w^\flat + g(u,w)v^\flat - (u \lrcorner v \lrcorner w \lrcorner \psi)^\sharp\right) \\ &= -2g(u,v)w^\flat - 2g(u,w)v^\flat - 2g(v,w)u^\flat \end{aligned}$$

and the proof is complete.

Now, we denote the previous lemma in local coordinates, which follows from [7].

Remark 3.2.3 *Let $u = \frac{\partial}{\partial x^i}$, $v = \frac{\partial}{\partial x^j}$, and $w = \frac{\partial}{\partial x^l}$ be vector fields, then our identity becomes*

$$* \left((\frac{\partial}{\partial x^i} \lrcorner \varphi) \wedge (\frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge (\frac{\partial}{\partial x^l} \lrcorner \varphi) \right) = -2\left(g_{ij}g_{lm} + g_{il}g_{jm} + g_{jl}g_{im}\right) dx^m$$

If we take $*$ of both sides of this equation, and wedge both sides with an arbitrary 1-form $\alpha = \alpha_k dx^k$, we get

$$\begin{aligned} &\alpha \wedge \left((\frac{\partial}{\partial x^i} \lrcorner \varphi) \wedge (\frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge (\frac{\partial}{\partial x^l} \lrcorner \varphi) \right) = -2\left(g_{ij}g_{lm} + g_{il}g_{jm} + g_{jl}g_{im}\right) \alpha_k dx^k \wedge *dx^m \\ &\frac{1}{8}\alpha_{s_7}\varphi_{is_1s_2}\varphi_{js_3s_4}\varphi_{ls_5s_6} dx^{s_1} \wedge \ldots \wedge dx^{s_7} = -2\left(g_{ij}g_{lm} + g_{il}g_{jm} + g_{jl}g_{im}\right) \alpha_k g^{km} \mathrm{vol} \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{8} \sum_{\sigma \in S_7} \mathrm{sgn}(\sigma)\, \varphi_{i\sigma(1)\sigma(2)} \varphi_{j\sigma(3)\sigma(4)} \varphi_{l\sigma(5)\sigma(6)} \alpha_{\sigma(7)}\, dx^1 \wedge \ldots \wedge dx^7 \\ &= -2\left(g_{ij}\alpha_l + g_{il}\alpha_j + g_{jl}\alpha_i\right) \sqrt{\det(g)}\, dx^1 \wedge \ldots \wedge dx^7 \end{aligned}$$

Thus, we get the following relation.

$$\sum_{\sigma\in S_7} \mathrm{sgn}(\sigma)\varphi_{i\sigma(1)\sigma(2)}\varphi_{j\sigma(3)\sigma(4)}\varphi_{l\sigma(5)\sigma(6)}\alpha_{\sigma(7)} = -16\left(g_{ij}\alpha_l + g_{il}\alpha_j + g_{jl}\alpha_i\right) \sqrt{\det(g)} \quad (3.10)$$

Similarly, we have the following corollary that can be derived from the Lemma 3.2.2, which also follows from [7].

Corollary 3.2.4 *Let u, v, and w be vector fields on M. Then the following holds:*

$$*((v\lrcorner w\lrcorner\varphi)\wedge(u\lrcorner\varphi)\wedge\varphi) = 2g(u,v)w^\flat - 2g(u,w)v^\flat + 2*(u\lrcorner v\lrcorner w\lrcorner\psi)^\sharp$$

We can denote the previous corollary in local coordinates by [7].

Remark 3.2.5 *Let $v = \frac{\partial}{\partial x^l}$, $w = \frac{\partial}{\partial x^i}$, and $u = \frac{\partial}{\partial x^j}$ be vector fields, then our identity becomes*

$$*\left((\frac{\partial}{\partial x^l}\lrcorner\frac{\partial}{\partial x^i}\lrcorner\varphi)\wedge(\frac{\partial}{\partial x^j}\lrcorner\varphi)\wedge\varphi\right) = 2(g_{lj}g_{im} - g_{ji}g_{lm} + \psi_{iljm})dx^m$$

If we take $*$ of both sides of this equation, and wedge both sides with an arbitrary 1-form $\alpha = \alpha_k dx^k$, according to [7] we have

$$\alpha\wedge\left((\frac{\partial}{\partial x^l}\lrcorner\frac{\partial}{\partial x^i}\lrcorner\varphi)\wedge(\frac{\partial}{\partial x^j}\lrcorner\varphi)\wedge\varphi\right) = 2(g_{lj}g_{im} - g_{ji}g_{lm} + \psi_{iljm})\alpha_k dx^k\wedge *dx^m$$

$$\frac{1}{12}\alpha_{s_1}\varphi_{ils_2}\varphi_{js_3s_4}\varphi_{s_5s_6s_7}dx^{s_1}\wedge\ldots\wedge dx^{s_7} = 2(g_{lj}g_{im} - g_{ji}g_{lm} + \psi_{iljm})\alpha_k g^{km}\,\mathrm{vol}$$

and hence

$$\frac{1}{12}\sum_{\sigma\in S_7}\mathrm{sgn}(\sigma)\,\alpha_{\sigma(1)}\varphi_{il\sigma(2)}\varphi_{j\sigma(3)\sigma(4)}\varphi_{\sigma(5)\sigma(6)\sigma(7)}\,dx^1\wedge\ldots\wedge dx^7$$
$$= 2\left(g_{lj}\alpha_i - g_{ji}\alpha_l + \psi_{iljm}g^{km}\alpha_k\right)\sqrt{\det(g)}\,dx^1\wedge\ldots\wedge dx^7$$

Thus, we get the following useful relation.

$$\sum_{\sigma\in S_7}\mathrm{sgn}(\sigma)\alpha_{\sigma(1)}\varphi_{il\sigma(2)}\varphi_{j\sigma(3)\sigma(4)}\varphi_{\sigma(5)\sigma(6)\sigma(7)} = 24\left(g_{lj}\alpha_i - g_{ji}\alpha_l + \psi_{iljm}g^{km}\alpha_k\right)\sqrt{\det(g)} \quad (3.11)$$

According to [7], we have the following proposition, its proof and local coordinate representation.

Proposition 3.2.6 *Let u, v, and w be vector fields on M. Then, the following holds:*

$$(u \lrcorner v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge \varphi = (v \lrcorner \psi) \wedge (u \lrcorner w \lrcorner \varphi) \wedge \varphi$$

Proof. We begin with the 8-form which vanishes on a 7-manifold M.

$$(v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge \varphi = 0$$

and take the interior product with u. This gives

$$(u \lrcorner v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge \varphi = (v \lrcorner \psi) \wedge (u \lrcorner w \lrcorner \varphi) \wedge \varphi + (v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge (u \lrcorner \varphi)$$

We use the fact that $(v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge (u \lrcorner \varphi) = 0$ for any u, v, w which is proved in [14], Theorem 2.4.7, we have the desired result.

Remark 3.2.7 *If we put* $u = \frac{\partial}{\partial x^i}$, $w = \frac{\partial}{\partial x^j}$, *and* $v = \frac{\partial}{\partial x^l}$, *this identity becomes*

$$(\frac{\partial}{\partial x^i} \lrcorner \frac{\partial}{\partial x^l} \lrcorner \psi) \wedge (\frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge \varphi = (\frac{\partial}{\partial x^l} \lrcorner \psi) \wedge (\frac{\partial}{\partial x^i} \lrcorner \frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge \varphi$$

According to [7], in local coordinates this becomes

$$\frac{1}{24} \psi_{lis_1s_2} \varphi_{js_3s_4} \varphi_{s_5s_6s_7} dx^{s_1} \wedge \ldots \wedge dxs_7 = (\frac{\partial}{\partial x^l} \lrcorner \psi) \wedge (\frac{\partial}{\partial x^i} \lrcorner \frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge \varphi$$

and hence

$$\sum_{\sigma \in S_7} \operatorname{sgn}(\sigma)\, \psi_{li\sigma(1)\sigma(2)} \varphi_{j\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)\sigma(7)} \qquad \textit{is skew-symmetric in } i, j. \tag{3.12}$$

We also note that the identity $(v \lrcorner \psi) \wedge (w \lrcorner \varphi) \wedge (u \lrcorner \varphi) = 0$ *in local coordinates we can denote it as*

$$\sum_{\sigma \in S_7} \operatorname{sgn}(\sigma)\, \varphi_{i\sigma(1)\sigma(2)} \varphi_{j\sigma(3)\sigma(4)} \psi_{l\sigma(5)\sigma(6)\sigma(7)} = 0 \tag{3.13}$$

Now, we give the identities involving the 3-form, Hodge dual 4-form, metric and the metric dual 1-forms, which follows from [7].

Proposition 3.2.8 *Let* v, w, a, b, c, d *be vector fields on M. We have the followings:*

$$\begin{aligned}
a^\flat \wedge b^\flat \wedge c^\flat \wedge \psi &= \varphi(a,b,c)\,\mathrm{vol} \\
a^\flat \wedge b^\flat \wedge c^\flat \wedge d^\flat \wedge \varphi &= \psi(a,b,c,d)\,\mathrm{vol} \\
a^\flat \wedge b^\flat \wedge c^\flat \wedge w^\flat \wedge (v \lrcorner \psi) &= (g(v,w)\varphi(a,b,c) - g(a,v)\varphi(w,b,c) \\
&\quad - g(b,v)\varphi(a,w,c) - g(c,v)\varphi(a,b,w))\,\mathrm{vol} \\
a^\flat \wedge b^\flat \wedge c^\flat \wedge d^\flat \wedge w^\flat \wedge (v \lrcorner \varphi) &= (g(v,w)\psi(a,b,c,d) - g(a,v)\psi(w,b,c,d) \\
&\quad - g(b,v)\psi(a,w,c,d) - g(c,v)\psi(a,b,w,d) \\
&\quad - g(d,v)\psi(a,b,c,w))\,\mathrm{vol}
\end{aligned}$$

Proof. According to [7], the first two equations follow from Lemma 1.0.1.

$$\begin{aligned} a^\flat \wedge b^\flat \wedge c^\flat \wedge \psi &= *(a \lrcorner b \lrcorner c \lrcorner \varphi) \\ &= *\varphi(a,b,c) \\ &= \varphi(a,b,c)\sqrt{\det(g)}\, e^1 \wedge \ldots \wedge e^7 = \varphi(a,b,c)\,\mathrm{vol} \end{aligned}$$

$$a^\flat \wedge b^\flat \wedge c^\flat \wedge d^\flat \wedge \varphi = *(a \lrcorner b \lrcorner c \lrcorner d \lrcorner \psi) = \psi(a,b,c,d)\,\mathrm{vol}$$

To prove the third, we begin with the 8-form which vanishes

$$a^\flat \wedge b^\flat \wedge c^\flat \wedge w^\flat \wedge \psi = 0$$

and take the interior product with v, we get

$$\begin{aligned} (v \lrcorner a^\flat) \wedge b^\flat \wedge c^\flat \wedge w^\flat \wedge \psi - a^\flat \wedge (v \lrcorner b^\flat) \wedge c^\flat \wedge w^\flat \wedge \psi - a^\flat \wedge b^\flat \wedge (v \lrcorner c^\flat) \wedge w^\flat \wedge \psi \\ - a^\flat \wedge b^\flat \wedge c^\flat \wedge (v \lrcorner w^\flat) \wedge \psi - a^\flat \wedge b^\flat \wedge c^\flat \wedge w^\flat \wedge (v \lrcorner \psi) = 0 \end{aligned}$$

After rearranging terms, we have the following.

$$\begin{aligned} a^\flat \wedge b^\flat \wedge c^\flat \wedge w^\flat \wedge (v \lrcorner \psi) &= (g(v,w)\varphi(a,b,c) - g(a,v)\varphi(w,b,c) \\ &\quad - g(b,v)\varphi(a,w,c) - g(c,v)\varphi(a,b,w))\,\mathrm{vol} \end{aligned}$$

We have the following identity involving the 3-form, Hodge dual 4-form, metric, volume form and the metric dual 1-forms, which also follows from [7]

Proposition 3.2.9 *Let a,b,c,d be vector fields on M. The following relation holds:*

$$a^\flat \wedge b^\flat \wedge (c \lrcorner \varphi) \wedge (d \lrcorner \psi) = (2g(a \wedge b, c \wedge d) + \psi(a,b,c,d))\,\mathrm{vol}\,.$$

Proof. We begin with the relation $\psi \wedge (c^\flat \lrcorner \varphi) = 3 * c^\flat$ from Proposition 1.0.2. Then, we take the interior product with d, also use Lemma 1.0.3. According to [7], after rearranging we get the following

$$(c \lrcorner \varphi) \wedge (d \lrcorner \psi) = 3 * (c^\flat \wedge d^\flat) - (c \times d)^\flat \wedge \psi$$

We take the wedge product with $a^\flat \wedge b^\flat$,

$$\begin{aligned} a^\flat \wedge b^\flat \wedge (c \lrcorner \varphi) \wedge (d \lrcorner \psi) &= 3\,(a^\flat \wedge b^\flat) \wedge *(c^\flat \wedge d^\flat) - a^\flat \wedge b^\flat \wedge (c \times d)^\flat \wedge \psi \\ &= 3\,g(a \wedge b, c \wedge d)\,\mathrm{vol} - \varphi(a,b,c \times d)\,\mathrm{vol} \\ &= 3\,g(a \wedge b, c \wedge d)\,\mathrm{vol} - g(a \times b, c \times d)\,\mathrm{vol} \\ &= (2\,g(a \wedge b, c \wedge d) + \psi(a,b,c,d))\,\mathrm{vol}\,. \end{aligned}$$

In the second equality, we used $u^\flat \wedge v^\flat \wedge w^\flat \wedge \psi = \varphi(u,v,w)\,\mathrm{vol}$ from Proposition 3.2.8, in the third equality we used $\varphi(u,v,w) = g(u \times v, w)$, and in the final equality we used Corollary 3.2.1.

3.3 Equalities Involving Contractions of φ and ψ

We consider some identities consisting of φ, ψ, and their derivatives. We also denote them in local coordinates, which follows from [7].

In local coordinates $x^1, x^2, \ldots, x^7$, the 3-form φ and the dual 4-form ψ can be written as

$$\begin{aligned}\varphi &= \frac{1}{6}\varphi_{ijk}\, dx^i \wedge dx^j \wedge dx^k \\ \psi &= \frac{1}{24}\psi_{ijkl}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l\end{aligned}$$

φ_{ijk} and ψ_{ijkl} are skew-symmetric in their indices. The metric can be written as $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. The cross product is a $(2,1)$ tensor which can be written as

$$\frac{\partial}{\partial x^i} \times \frac{\partial}{\partial x^j} = P^k_{ij}\frac{\partial}{\partial x^k} \tag{3.14}$$

where $P^k_{ij} = -P^k_{ji}$. Therefore,

$$\varphi_{ijk} = g_{kl}P^l_{ij} \qquad\qquad P^l_{ij} = g^{kl}\varphi_{ijk} \tag{3.15}$$

If we set $u = \frac{\partial}{\partial x^i}$, $v = \frac{\partial}{\partial x^j}$ and $w = \frac{\partial}{\partial x^k}$ in

$$\frac{\partial}{\partial x^i} \times \left(\frac{\partial}{\partial x^j} \times \frac{\partial}{\partial x^k}\right) = -g_{ij}\frac{\partial}{\partial x^k} + g_{ik}\frac{\partial}{\partial x^j} + \psi_{ijkl}(\frac{\partial}{\partial x^l})^\sharp$$

$$P^m_{il}P^l_{jk}\frac{\partial}{\partial x^m} = -g_{ij}\frac{\partial}{\partial x^k} + g_{ik}\frac{\partial}{\partial x^j} + \psi_{ijkl}g^{lm}\frac{\partial}{\partial x^m} \tag{3.16}$$

We have the following identities and the proof by [7].

Lemma 3.3.1 *Let the tensors g, φ, ψ, and P be as given above. Then the following identities hold:*

$$\begin{aligned}P^k_{il}P^l_{jk} &= -6g_{ij} \\ \varphi_{ijk}\varphi_{abc}g^{ia}g^{jb}g^{kc} &= 42 \\ \varphi_{ijk}\varphi_{abc}g^{jb}g^{kc} &= 6g_{ia} \\ \varphi_{ijk}\varphi_{abc}g^{kc} &= g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}\end{aligned}$$

Proof. We first prove the last equation. The other identities follow by contraction with g^{ij} and using (3.15). To obtain the last equation, we take the inner product of (3.16) with $\frac{\partial}{\partial x}$:

$$\begin{aligned} P^m_{il} P^l_{jk} g_{mn} &= -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{ijkn} \\ \varphi_{ila} g^{am} \varphi_{jkb} g^{bl} g_{mn} &= -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{injk} \\ \varphi_{iln} \varphi_{jkb} g^{bl} &= -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{injk} \end{aligned}$$

since $\varphi_{iln} = -\varphi_{inl}$, we get the desired equation.

For the third equation, we contract the last equation by g^{jb},

$$\varphi_{ijk} \varphi_{abc} g^{kc} g^{jb} = g_{ia} g_{jb} g^{jb} - g_{ib} g_{ja} g^{jb} - \psi_{ijab} g^{jb} = 6 g_{ia} \tag{3.17}$$

For the second, we contract the third equation by g^{ia},

$$\varphi_{ijk} \varphi_{abc} g^{jb} g^{kc} g^{ia} = 6 g_{ia} g^{ia} = 6.7 = 42 \tag{3.18}$$

We also note that the second identity is just the pointwise norm $|\varphi|^2$ of φ is 7.
For the first equation, we use the equation (3) of Proposition 1.0.2 and (3.15).

The next identities, from [7], involves contractions of φ with ψ.

Lemma 3.3.2 *Let the tensors g, φ, and ψ be as given above. Then the following identities hold:*

$$\begin{aligned} \varphi_{ijk} \psi_{abcd} g^{ib} g^{jc} g^{kd} &= 0 \\ \varphi_{ijk} \psi_{abcd} g^{jc} g^{kd} &= -4 \varphi_{iab} \\ \varphi_{ijk} \psi_{abcd} g^{kd} &= g_{ia} \varphi_{jbc} + g_{ib} \varphi_{ajc} + g_{ic} \varphi_{abj} \\ &\quad - g_{aj} \varphi_{ibc} - g_{bj} \varphi_{aic} - g_{cj} \varphi_{abi} \end{aligned}$$

Proof. According to [7], the first two follow from the last. To prove the third, we take the inner product of (**A.9**) with (3.14):

$$g\left(\frac{\partial}{\partial x^a} \times \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^i} \times \left(\frac{\partial}{\partial x^j} \times \frac{\partial}{\partial x^k}\right)\right) = g_{ai} \varphi_{jkb} - g_{ib} \varphi_{jka} - \psi_{abil} P^l_{jk}$$

Here we used $g(a \times b, c \times d) = g(a,c)g(b,d) - g(a,d)g(b,c) + \psi(d,c,a,b)$. But this also equals

$$\begin{aligned}
&= g\left(P^l_{ab}\frac{\partial}{\partial x^l}, -g_{ij}\frac{\partial}{\partial x^k} + g_{ik}\frac{\partial}{\partial x^j} + \psi_{ijkn}g^{nm}\frac{\partial}{\partial x^m}\right) \\
&= -g_{ij}g_{lk}P^l_{ab} + g_{ik}g_{lj}P^l_{ab} + P^l_{ab}g^{nm}g_{lm}\psi_{ijkn} \\
&= -g_{ij}\varphi_{abk} + g_{ik}\varphi_{abj} + P^l_{ab}\psi_{ijkl}
\end{aligned}$$

We use these two equations and rearrange them. Then, we have the following

$$g_{ia}\varphi_{jkb} - g_{ib}\varphi_{jka} + g_{ij}\varphi_{abk} - g_{ik}\varphi_{abj} - \varphi_{jkc}\psi_{abil}g^{cl} - \varphi_{abc}\psi_{ijkl}g^{cl} = 0$$

We denote this expression by A_{ijkab}. Then it can be seen that

$$A_{ijkab} + A_{ajkbi} + A_{bijka} - A_{kijab} - A_{jkabi} = 0$$

By this, we get desired expression. For the second equation we start contracting the last equation with g^{jc}, we have

$$\varphi_{ijk}\psi_{abcd}g^{kd}g^{jc} = g_{ia}g^{jc}\varphi_{jbc} + g_{ib}g^{jc}\varphi_{ajc} + g_{ic}g^{jc}\varphi_{abj} - g_{aj}g^{jc}\varphi_{ibc} - g_{bj}g^{jc}\varphi_{aic} - g_{cj}g^{jc}\varphi_{abi}$$

becomes

$$\varphi_{ijk}\psi_{abcd}g^{kd}g^{jc} = \varphi_{iab} - \varphi_{iab} + \varphi_{iab} + \varphi_{iab} + \varphi_{iab} - 7\varphi_{iab} = -4\varphi_{iab}$$

by using the fact that $|\varphi|^2 = 7$. Also, for the first equation we contact the second with g^{ib}.

Now we contract ψ with itself, which follows from [7].

Lemma 3.3.3 *Let the tensors g, φ, and ψ be as given above. Then the following identities hold:*

$$\begin{aligned}
\psi_{ijkl}\psi_{abcd}g^{ia}g^{jb}g^{kc}g^{ld} &= 168 \\
\psi_{ijkl}\psi_{abcd}g^{jb}g^{kc}g^{ld} &= 24g_{ia} \\
\psi_{ijkl}\psi_{abcd}g^{kc}g^{ld} &= 4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab} \\
\psi_{ijkl}\psi_{abcd}g^{ld} &= -\varphi_{ajk}\varphi_{ibc} - \varphi_{iak}\varphi_{jbc} - \varphi_{ija}\varphi_{kbc} \\
&\quad + g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} \\
&\quad - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \\
&\quad - g_{ia}\psi_{jkbc} - g_{ja}\psi_{kibc} - g_{ka}\psi_{ijbc} \\
&\quad + g_{ab}\psi_{ijkc} - g_{ac}\psi_{ijkb}
\end{aligned}$$

Next, we consider identities involving the covariant derivatives of φ and ψ, which are given by [7].

Proposition 3.3.4 *Let the tensors g, φ, and ψ be as given above. Then the following identities hold:*

$$\begin{aligned}
(\nabla_m \varphi_{ijk})\varphi_{abc} g^{ia} g^{jb} g^{kc} &= 0 \\
(\nabla_m \psi_{ijkl})\psi_{abcd} g^{ia} g^{jb} g^{kc} g^{ld} &= 0 \\
(\nabla_m \varphi_{ijk})\psi_{abcd} g^{ib} g^{jc} g^{kd} &= -\varphi_{ijk}(\nabla_m \psi_{abcd}) g^{ib} g^{jc} g^{kd}
\end{aligned}$$

$$\begin{aligned}
(\nabla_m \varphi_{ijk})\varphi_{abc} g^{jb} g^{kc} &= -\varphi_{ijk}(\nabla_m \varphi_{abc}) g^{jb} g^{kc} \\
(\nabla_m \psi_{ijkl})\psi_{abcd} g^{jb} g^{kc} g^{ld} &= -\psi_{ijkl}(\nabla_m \psi_{abcd}) g^{jb} g^{kc} g^{ld} \\
(\nabla_m \varphi_{ijk})\psi_{abcd} g^{jc} g^{kd} &= -\varphi_{ijk}(\nabla_m \psi_{abcd}) g^{jc} g^{kd} - 4\nabla_m \varphi_{iab}
\end{aligned}$$

and also

$$\nabla_m \psi_{abcd} = -(\nabla_m \varphi_{abj})\varphi_{cdk} g^{jk} - \varphi_{abj}(\nabla_m \varphi_{cdk}) g^{jk} \tag{3.19}$$

Proposition 3.3.5 *The following relation holds between $\nabla\varphi$ and $\nabla\psi$:*

$$(\nabla_m \psi_{ijkl})\psi_{abcd} g^{jb} g^{kc} g^{ld} = 3(\nabla_m \varphi_{ijk})\varphi_{abc} g^{jb} g^{kc}$$

Proof *According to [7], we substitute* (3.19) *into the left hand side above, and use Lemma 3.3.2. Then,*

$$\begin{aligned}
&-\left((\nabla_m \varphi_{ijp})\varphi_{klq} g^{pq} + \varphi_{ijp}(\nabla_m \varphi_{klq}) g^{pq}\right)\psi_{abcd} g^{jb} g^{kc} g^{ld} \\
=\;& -(\nabla_m \varphi_{ijp}) g^{pq} g^{jb}(\varphi_{qkl}\psi_{abcd} g^{kc} g^{ld}) - (\nabla_m \varphi_{klq}) g^{pq} g^{kc} g^{ld}(\varphi_{pij}\psi_{cdab} g^{jb}) \\
=\;& -(\nabla_m \varphi_{ijp}) g^{pq} g^{jb}(-4\varphi_{qab}) \\
& -(\nabla_m \varphi_{klq}) g^{pq} g^{kc} g^{ld}(g_{pc}\varphi_{ida} + g_{pd}\varphi_{cia} + g_{pa}\varphi_{cdi} - g_{ic}\varphi_{pda} - g_{id}\varphi_{cpa} - g_{ia}\varphi_{cdp}) \\
=\;& 4(\nabla_m \varphi_{ijp})\varphi_{abq} g^{pq} g^{jb} + 0 + 0 \\
& -(\nabla_m \varphi_{klq})(\delta^q_a g^{kc} g^{ld}\varphi_{cdi} - \delta^k_i g^{pq} g^{ld}\varphi_{pda} - \delta^l_i g^{pq} g^{kc}\varphi_{cpa} - g_{ia} g^{pq} g^{kc} g^{ld}\varphi_{cdp}) \\
=\;& 4(\nabla_m \varphi_{ijk})\varphi_{abc} g^{jb} g^{kc} - (\nabla_m \varphi_{kla})\varphi_{cdi} g^{kc} g^{ld} + (\nabla_m \varphi_{ilq})\varphi_{pda} g^{pq} g^{ld} \\
& +(\nabla_m \varphi_{kiq})\varphi_{cpa} g^{pq} g^{kc} + (\nabla_a \varphi_{klq})\varphi_{cdp} g^{pq} g^{kc} g^{ld} g_{ia}
\end{aligned}$$

Using Proposition 3.3.4 on the second and final terms, the final term vanishes and the remaining terms become

$$3(\nabla_m \varphi_{ijk})\varphi_{abc} g^{jb} g^{kc}$$

and the proof is complete.

3.4 Decomposition of $\bigwedge^*(M)$ Into Irreducible G_2-Representations

The group G_2 acts on $\mathbb{R}^7$, and hence acts on the spaces $\wedge^*$ of differential forms on M. Let $A \in GL(7,\mathbb{R})$, and we have

$$A \cdot \varphi = \frac{1}{6}\varphi_{ijk}(A^* dx^i) \wedge (A^* dx^j) \wedge (A^* dx^k)$$

One can decompose each space $\wedge^k$ into irreducible G_2-representations. This is analogous to the decomposition of the space of k-forms to the space of differential forms of type (p,q) on a complex manifold. The results of this decomposition are presented below.

Proposition 3.4.1 *Let M be a 7-manifold and (φ, g) a G_2-structure on M. Then $\wedge^k T^*M$ splits orthogonally into components as follows, where $\wedge^k_l$ corresponds to an irreducible representation of G_2 of dimension l:*

1. $\wedge^1 T^*M = \wedge^1_7$
2. $\wedge^2 T^*M = \wedge^2_7 \oplus \wedge^2_{14}$
3. $\wedge^3 T^*M = \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27}$
4. $\wedge^4 T^*M = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}$
5. $\wedge^5 T^*M = \wedge^5_7 \oplus \wedge^5_{14}$
6. $\wedge^6 T^*M = \wedge^6_7$

The Hodge star $*$ gives an isometry between $\wedge^k_l$ and $\wedge^{7-k}_l$. Note also that $\wedge^3_1 = \langle\varphi\rangle$ and $\wedge^4_1 = \langle *\varphi\rangle$, and that the spaces $\wedge^k_7$ for $k = 1,2,\ldots,6$ are canonically isomorphic [12].

Proposition 3.4.2 *The map $\alpha \mapsto \varphi \wedge \alpha$ is an isomorphism between the following spaces:*

$$\wedge^0_1 \cong \wedge^3_1 \qquad \wedge^1_7 \cong \wedge^4_7$$
$$\wedge^2_7 \cong \wedge^5_7 \qquad \wedge^2_{14} \cong \wedge^5_{14}$$
$$\wedge^3_7 \cong \wedge^6_7 \qquad \wedge^4_1 \cong \wedge^7_1$$

The map $\alpha \mapsto *\varphi \wedge \alpha$ is an isomorphism between the following spaces:

$$\wedge^0_1 \cong \wedge^4_1 \qquad \wedge^1_7 \cong \wedge^5_7$$
$$\wedge^2_7 \cong \wedge^6_7 \qquad \wedge^3_1 \cong \wedge^7_1$$

In addition, if α is a 1-form, we have the following identities from [14] :

$$*(\varphi \wedge *(\varphi \wedge \alpha)) = -4\alpha \tag{3.20}$$
$$*\varphi \wedge *(\varphi \wedge \alpha) = 0$$
$$*(*\varphi \wedge *(*\varphi \wedge \alpha)) = 3\alpha \tag{3.21}$$
$$\varphi \wedge *(*\varphi \wedge \alpha) = 2(*\varphi \wedge \alpha) \tag{3.22}$$

We now describe the decomposition, from [14], starting with

The case $k = 2,5$

$$\begin{aligned} \wedge^2_7 &= \{w \lrcorner \varphi; w \in \Gamma(T(M))\} \\ &= \{\beta \in \wedge^2; *(\varphi \wedge \beta) = 2\beta\} \end{aligned} \tag{3.23}$$

Here for (3.23) we used the (11) equation of Proposition 1.0.2. Also using the (6) equation we can write
$\wedge^2_7 = \{\beta \in \wedge^2; *(*\varphi \wedge (*(*\varphi \wedge \beta))) = 3\beta\}$

$$\begin{aligned} \wedge^2_{14} &= \{\beta \in \wedge^2; *\varphi \wedge \beta = 0\} \\ &= \{\beta \in \wedge^2; *(\varphi \wedge \beta) = -\beta\} \end{aligned} \tag{3.24}$$

For (3.24), by Proposition 3.4.2 we know that there exists a linear map

$$L : \wedge^2 \to \wedge^6 = \wedge^6_7 \simeq \wedge^2_7$$
$$L : \alpha \mapsto \psi \wedge \alpha$$

Now, let $V \in \wedge^2$ then $V = Im(L) \oplus Ker(L)$.
Thus $\wedge^2_{14} = Ker(L) = \psi \wedge \alpha = 0$

The decomposition of $\wedge^5$

$\wedge^5 = \wedge^5_7 \oplus \wedge^5_{14}$ is obtained by taking the Hodge star of the decompositions of $\wedge^2$.

$$\begin{aligned}\wedge^5_7 &= \{\alpha\wedge *\varphi; \alpha\in\wedge^1_7\} \quad (3.25)\\ &= \{\gamma\in\wedge^5; \varphi\wedge *\gamma = 2\gamma\}\\ &= \{\gamma\in\wedge^5; *\varphi\wedge(*(*\varphi\wedge *\gamma)) = 3\gamma\}\end{aligned}$$

For (3.25) we used the Proposition 1.0.2, equations (8) and (6). Similarly, we get

$$\begin{aligned}\wedge^5_{14} &= \{\gamma\in\wedge^5; \varphi\wedge *\gamma = -\gamma\} \quad (3.26)\\ &= \{\gamma\in\wedge^5; *\varphi\wedge *\gamma = 0\}\end{aligned}$$

<u>The decompositions for $k = 3,4$</u>

$$\begin{aligned}\wedge^3_1 &= \{f\varphi; f\in C^\infty(M)\} \quad (3.27)\\ &= \{\eta\in\wedge^3; \varphi\wedge(*(*\varphi\wedge\eta)) = 7\eta\}\end{aligned}$$

For (3.27) we write $\eta = f\varphi$

$$\begin{aligned}\varphi\wedge(*(*\varphi\wedge f\varphi)) &= \varphi\wedge(*(f*\varphi\wedge\varphi))\\ &= 7\varphi f = 7\eta\end{aligned}$$

$$\begin{aligned}\wedge^3_7 &= \{*(\varphi\wedge\alpha); \alpha\in\wedge^1_7\} \quad (3.28)\\ &= \{w\lrcorner *\varphi; w\in\Gamma(T(M))\}\\ &= \{\eta\in\wedge^3; *(\varphi\wedge *(\varphi\wedge\eta)) = -4\eta\}\end{aligned}$$

For (3.28) we use the Proposition 1.0.2 in Appendix A.1. equation (5).

<u>Finally, for $\wedge^3_{27}$ we define</u>

$$L : \wedge^3 \to \wedge^6_7 \oplus \wedge^7_1 \simeq \wedge^3_7 \oplus \wedge^3_1$$
$$L : \alpha \mapsto \psi\wedge\alpha + \varphi\wedge\alpha$$

If $\psi\wedge\alpha = 0$ then $\alpha\in(\wedge^3_7)^\perp$.
If $\varphi\wedge\alpha = 0$ then $\alpha\in(\wedge^3_1)^\perp$. $Ker(L) = \psi\wedge\alpha + \varphi\wedge\alpha = 0$ if and only if $\psi\wedge\alpha = 0$ and $\varphi\wedge\alpha = 0$, since they belong to the orthogonal modules of the decomposition of 3-forms. We have

$$\wedge^3_{27} = \{\eta\in\wedge^3; \varphi\wedge\eta = 0 \text{ and } *\varphi\wedge\eta = 0\} \quad (3.29)$$

The decomposition of $\wedge^4$

Similarly we use the fact that the decomposition of $\wedge^4$ is obtained by taking the Hodge star of the decomposition of $\wedge^3$ to get the following expressions.

$$\begin{aligned}
\wedge^4_1 &= \{f*\varphi; f\in C^\infty(M)\} && (3.30)\\
&= \{\sigma\in\wedge^4; *\varphi\wedge(*(\varphi\wedge\sigma)) = 7\sigma\}\\
\wedge^4_7 &= \{\varphi\wedge\alpha; \alpha\in\wedge^1_7\} && (3.31)\\
&= \{\sigma\in\wedge^4; (\varphi\wedge*(\varphi\wedge*\sigma)) = -4\sigma\}\\
\wedge^4_{27} &= \{\sigma\in\wedge^4; \varphi\wedge\sigma = 0 \text{ and } \varphi\wedge*\sigma = 0\} && (3.32)
\end{aligned}$$

By using the definitions above, we can define the following expressions, from [14], for $\beta\in\wedge^2$

$$\begin{aligned}
*(\varphi\wedge\beta) &= *(\varphi\wedge(\pi_7(\beta)+\pi_{14}(\beta))\\
&= *(\varphi\wedge(\pi_7(\beta))+*(\varphi\wedge\pi_{14}(\beta))\\
&= 2\pi_7(\beta)-\pi_{14}(\beta)\\
\beta &= \pi_7(\beta)+\pi_{14}(\beta)\\
\pi_7(\beta) &= \frac{\beta+*(\varphi\wedge\beta)}{3} && (3.33)\\
\pi_{14}(\beta) &= \frac{2\beta-*(\varphi\wedge\beta)}{3} && (3.34)
\end{aligned}$$

and for $\gamma\in\wedge^5$ using (3.25) and (3.26)

$$\begin{aligned}
\varphi\wedge*\gamma &= (\varphi\wedge(\pi_7(*\gamma))+(\varphi\wedge\pi_{14}(*\gamma))\\
&= 2\pi_7(\gamma)-\pi_{14}(\gamma)\\
\gamma &= \pi_7(\gamma)+\pi_{14}(\gamma)\\
\pi_7(\gamma) &= \frac{\gamma+\varphi\wedge*\gamma}{3} && (3.35)\\
\pi_{14}(\gamma) &= \frac{2\gamma-\varphi\wedge*\gamma}{3} && (3.36)
\end{aligned}$$

where π_7,π_{14} are projections of $\wedge^2$ to $\wedge^2_7$ and $\wedge^2_{14}$, respectively.

$$\wedge^3_{27} = \{h_{ij}g^{jl}dx^i\wedge\left(\frac{\partial}{\partial x^l}\lrcorner\varphi\right); h_{ij}=h_{ji}, \mathrm{Tr}_g(h_{ij}) = g^{ij}h_{ij}=0\} \qquad (3.37)$$

For $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j$ is a 2-form, according to [14], we have

$$\pi_7(\beta) = \frac{1}{2}\left(\frac{1}{3}\beta_{ab} + \frac{1}{6}\beta_{ij}g^{il}g^{jm}\psi_{lmab}\right)dx^a \wedge dx^b \tag{3.38}$$

$$\pi_{14}(\beta) = \frac{1}{2}\left(\frac{2}{3}\beta_{ab} - \frac{1}{6}\beta_{ij}g^{il}g^{jm}\psi_{lmab}\right)dx^a \wedge dx^b \tag{3.39}$$

Now, we give a summary

$$\begin{aligned}
\wedge^2_7 &= \{w \lrcorner \varphi; w \in \Gamma(T(M))\} \\
&= \{\beta \in \wedge^2; *(\varphi \wedge \beta) = 2\beta\} \\
&= \{\beta \in \wedge^2; *(*\varphi \wedge (*(*\varphi \wedge \beta))) = 3\beta\} \\
\wedge^2_{14} &= \{\beta \in \wedge^2; *\varphi \wedge \beta = 0\} \\
&= \{\beta \in \wedge^2; *(\varphi \wedge \beta) = -\beta\} \\
&= \{\sum a_{ij}e^i \wedge e^j; (a_{ij}) \in \mathfrak{g}_2\}
\end{aligned}$$

$$\begin{aligned}
\wedge^5_7 &= \{\alpha \wedge *\varphi; \alpha \in \wedge^1_7\} \\
&= \{\gamma \in \wedge^5; \varphi \wedge *\gamma = 2\gamma\} \\
&= \{\gamma \in \wedge^5; *\varphi \wedge (*(*\varphi \wedge *\gamma)) = 3\gamma\} \\
\wedge^5_{14} &= \{\gamma \in \wedge^5; \varphi \wedge *\gamma = -\gamma\} \\
&= \{\gamma \in \wedge^5; *\varphi \wedge *\gamma = 0\}
\end{aligned}$$

$$\begin{aligned}
\wedge^3_1 &= \{f\varphi; f \in C^\infty(M)\} \\
&= \{\eta \in \wedge^3; \varphi \wedge (*(*\varphi \wedge \eta)) = 7\eta\} \\
\wedge^3_7 &= \{*(\varphi \wedge \alpha); \alpha \in \wedge^1_7\} \\
&= \{w \lrcorner *\varphi; w \in \Gamma(T(M))\} \\
&= \{\eta \in \wedge^3; *(\varphi \wedge *(\varphi \wedge \eta)) = -4\eta\} \\
\wedge^3_{27} &= \{\eta \in \wedge^3; \varphi \wedge \eta = 0 \text{ and } *\varphi \wedge \eta = 0\}
\end{aligned}$$

$$
\begin{aligned}
\wedge^4_1 &= \{f*\varphi; f\in C^\infty(M)\}\\
&= \{\sigma\in\wedge^4; *\varphi\wedge(*(\varphi\wedge\sigma)) = 7\sigma\}\\
\wedge^4_7 &= \{\varphi\wedge\alpha; \alpha\in\wedge^1_7\}\\
&= \{\sigma\in\wedge^4; (\varphi\wedge*(\varphi\wedge*\sigma)) = -4\sigma\}\\
\wedge^4_{27} &= \{\sigma\in\wedge^4; \varphi\wedge\sigma = 0 \text{ and } \varphi\wedge*\sigma = 0\}
\end{aligned}
$$

We derive some further properties of the representation modules $\wedge^k_l$, that we will need in this thesis, which follows from [7]. For this, we define maps $i : S^2(T) \to \wedge^3$ and $j : \wedge^3 \to S^2(T)$ as follows:

$$
i(h_{ij}) = h_{ij}g^{jl}\frac{\partial}{\partial x^i}\wedge\left(\frac{\partial}{\partial x^l}\lrcorner\varphi\right) = \frac{1}{2}h^l_i\varphi_{ljk}\,dx^i\wedge dx^j\wedge dx^k \tag{3.40}
$$

$$
(j(\eta))(v,w) = *((v\lrcorner\varphi)\wedge(w\lrcorner\varphi)\wedge\eta) \tag{3.41}
$$

Now, on a manifold with G_2-structure we have the relation between a symmetric 2-tensor, its trace, an arbitrary 3-form and its Hodge dual, which also follows from [7].

Proposition 3.4.3 *Suppose that h_{ij} is a symmetric tensor. It corresponds to the form $\eta = i(h_{ij})$ in $\wedge^3$, given by*

$$
\eta = h_{ij}g^{jl}\,dx^i\wedge\left(\frac{\partial}{\partial x^l}\lrcorner\varphi\right) = \frac{1}{2}h^l_i\varphi_{ljk}\,dx^i\wedge dx^j\wedge dx^k
$$

Then the Hodge star $*\eta$ of η is

$$
*\eta = \left(\frac{1}{4}\mathrm{Tr}_g(h)\,g_{ij} - h_{ij}\right)g^{jl}dx^i\wedge\left(\frac{\partial}{\partial x^l}\lrcorner\psi\right)
$$

where $\mathrm{Tr}_g(h) = g^{ij}h_{ij}$.

We can also see the relation which is given by [7].

Proposition 3.4.4 *The map $j : \wedge^3 \to S^2(T)$ is an isomorphism between $\wedge^3_1\oplus\wedge^3_{27}$ and $S^2(T)$. $\wedge^3_7$ is the kernel of j. Explicitly, we have*

$$
\begin{aligned}
&\textit{If}\quad \eta = h_{ij}g^{jl}\,dx^i\wedge\left(\frac{\partial}{\partial x^l}\lrcorner\varphi\right) + (X\lrcorner\psi) = i(h) + (X\lrcorner\psi)\\
&\textit{then}\quad j(\eta) = -2\,\mathrm{Tr}_g(h)g_{ij} - 4h_{ij}
\end{aligned}
$$

To summarize, we have seen that an arbitrary 3-form η on a manifold M with G_2-structure φ consists of a vector field X and a symmetric 2-tensor h [7]. We have

$$\begin{aligned}\eta &= h_{ij}g^{jl}\,dx^i \wedge \left(\frac{\partial}{\partial x^l} \lrcorner \varphi\right) + X^l \frac{\partial}{\partial x^l} \lrcorner \psi \\ &= \frac{1}{2} h_i^l \varphi_{ljk}\, dx^i \wedge dx^j \wedge dx^k + \frac{1}{6} X^l \psi_{lijk}\, dx^i \wedge dx^j \wedge dx^k\end{aligned}$$

Lastly, we have the useful relation between traceless, tracefull part of a symmetric 2-tensor and the submodules of space of 3-forms, which follows from [7]

Remark 3.4.5 *Note that the symmetric* 2*-tensor* h_{ij} *decomposes as* $h_{ij} = \frac{1}{7}\mathrm{Tr}_g(h)\, g_{ij} + h^0_{ij}$ *where* h^0_{ij} *is the traceless part of* h_{ij}*. Hence the first term in the above expression can be written as*

$$\frac{3}{7} h\varphi + \frac{1}{2}(h^0)^l_i \varphi_{ljk}\, dx^i \wedge dx^j \wedge dx^k$$

which is exactly the $\wedge^3_1$ *and* $\wedge^3_{27}$ *components.*

3.5 The Torsion Forms of a G_2-Structure

Using the decomposition of the spaces of differential forms on M determined by φ given in Section 3.4, we can decompose $d\varphi$ and $d\psi$ into irreducible G_2 representations. This defines the torsion forms of the G_2-structure [7]. According to [7], we have

Definition 3.5.1 *There are four independent torsion forms corresponding to a* G_2*-structure* φ*.*

$$\tau_0 \in \wedge^0_1 \qquad \tau_1 \in \wedge^1_7$$
$$\tau_2 \in \wedge^2_{14} \qquad \tau_3 \in \wedge^3_{27}$$

They are defined by the equations

$$\begin{aligned} d\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3 \\ d\psi &= 4\tau_1 \wedge \psi + *\tau_2 \end{aligned} \tag{3.42}$$

$d\varphi \in \wedge^4$ and $d\psi \in \wedge^5$.

Remark 3.5.2 *We call* τ_0 *the* scalar torsion*,* τ_1 *the* vector torsion*,* τ_2 *the* Lie algebra torsion*, and* τ_3 *the* symmetric traceless torsion *[7].*

Theorem 3.5.3 *In the expressions $d\varphi$ and $d\psi$, the same one-form τ_1 appears [7].*

Proof. We begin by not assuming that the two τ_1's are the same. Let $d\varphi = \tau_0\,\psi + 3\tau_1 \wedge \varphi + *\tau_3$ and $d\psi = 4\tilde{\tau}_1 \wedge \psi + *\tau_2$. We must show that $\tilde{\tau}_1 = \tau_1$. Then, we have

$$\begin{aligned} d\varphi &= \tau_0\,\psi + 3\tau_1 \wedge \varphi + *\tau_3 & \qquad d\psi &= 4\tilde{\tau}_1 \wedge \psi + *\tau_2 \\ *(d\varphi) &= \tau_0\,\varphi + 3*(\tau_1 \wedge \varphi) + \tau_3 & *(d\psi) &= 4*(\tilde{\tau}_1 \wedge \psi) + \tau_2 \\ \varphi \wedge *(d\varphi) &= 0 - 3\varphi \wedge *(\varphi \wedge \tau_1) + 0 & \psi \wedge *(d\psi) &= 4\psi \wedge *(\psi \wedge \tilde{\tau}_1) + 0 \\ \varphi \wedge *(d\varphi) &= 12 * \tau_1 & \psi \wedge *(d\psi) &= 12 * \tilde{\tau}_1 \end{aligned}$$

Therefore, we see that

$$\begin{aligned} \tau_1 = \tilde{\tau}_1 \quad &\Leftrightarrow \quad \varphi \wedge *(d\varphi) = \psi \wedge *(d\psi) \\ &\Leftrightarrow \quad dx^p \wedge \varphi \wedge *(d\varphi) = dx^p \wedge \psi \wedge *(d\psi) \qquad \text{for all } p \\ &\Leftrightarrow \quad g(d\varphi, dx^p \wedge \varphi) = g(d\psi, dx^p \wedge \psi) \qquad \text{for all } p \end{aligned}$$

Let $X = X_i dx^i$ be an arbitrary one-form. Then, we have

$$\begin{aligned} X \wedge \varphi &= \frac{1}{6} X_q \varphi_{ijk}\, dx^q \wedge dx^i \wedge dx^j \wedge dx^k \\ &= \frac{1}{24}\left(X_q \varphi_{ijk} - X_i \varphi_{qjk} - X_j \varphi_{iqk} - X_k \varphi_{ijq}\right) dx^q \wedge dx^i \wedge dx^j \wedge dx^k \\ &= \frac{1}{24} A_{qijk}\, dx^q \wedge dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

where we have skew-symmetrized the coefficients [7].

Similarly we have

$$\begin{aligned} d\varphi &= \frac{1}{6}\left(\nabla_m \varphi_{abc} - \nabla_a \varphi_{mbc} - \nabla_b \varphi_{amc} - \nabla_c \varphi_{abm}\right) dx^m \wedge dx^a \wedge dx^b \wedge dx^c \\ &= \frac{1}{24} B_{mabc}\, dx^m \wedge dx^a \wedge dx^b \wedge dx^c \end{aligned}$$

Now using (2.2), we have

$$\begin{aligned} g(X \wedge \varphi, d\varphi) &= \frac{1}{24} A_{qijk} B_{mabc} g^{qm} g^{ia} g^{jb} g^{kc} \\ &= \frac{1}{6}\left(X_q \varphi_{ijk} - X_i \varphi_{qjk} - X_j \varphi_{iqk} - X_k \varphi_{ijq}\right)\left(\nabla_m \varphi_{abc}\right) g^{qm} g^{ia} g^{jb} g^{kc} \end{aligned}$$

Let $X = dx^p$, so that $X_i = \delta_i^p$, and this expression becomes

$$\begin{aligned} g(dx^p \wedge \varphi, d\varphi) &= \frac{1}{6}\left(\delta_q^p \varphi_{ijk} - \delta_i^p \varphi_{qjk} - \delta_j^p \varphi_{iqk} - \delta_k^p \varphi_{ijq}\right)\left(\nabla_m \varphi_{abc}\right) g^{qm} g^{ia} g^{jb} g^{kc} \\ &= \frac{1}{6} \varphi_{ijk}\left(\nabla_m \varphi_{abc}\right) g^{pm} g^{ia} g^{jb} g^{kc} - \frac{1}{2} \varphi_{qjk}\left(\nabla_m \varphi_{abc}\right) g^{qm} g^{pa} g^{jb} g^{kc} \end{aligned}$$

By Proposition 3.3.4, the first term vanishes, and the second term becomes

$$g(dx^p \wedge \varphi, d\varphi) = \frac{1}{2}(\nabla_m \varphi_{ijk})\varphi_{abc} g^{im} g^{pa} g^{jb} g^{kc} \tag{3.43}$$

By an analogous calculation, we have the following expression

$$g(dx^p \wedge \psi, d\psi) = \frac{1}{6}(\nabla_m \psi_{ijkl})\psi_{abcd} g^{im} g^{pa} g^{jb} g^{kc} g^{ld} \tag{3.44}$$

$$\begin{aligned} X \wedge \psi &= \frac{1}{24} X_q \psi_{ijkl}\, dx^q \wedge dx^i \wedge dx^j \wedge dx^k \wedge dx^l \\ &= \frac{1}{120}\left(X_q \psi_{ijkl} - X_i \psi_{qjkl} - X_j \psi_{iqkl} - X_k \psi_{ijql} - X_l \psi_{ijkq}\right) \\ &\quad dx^q \wedge dx^i \wedge dx^j \wedge dx^k \wedge dx^l \\ &= \frac{1}{120} A_{qijkl}\, dx^q \wedge dx^i \wedge dx^j \wedge dx^k \wedge dx^l \end{aligned}$$

where we have skew-symmetrized the coefficients [7].

Similarly, we have

$$\begin{aligned} d\psi &= \frac{1}{24}(\nabla_m \psi_{abcd} - \nabla_a \psi_{mbcd} - \nabla_b \psi_{amcd} - \nabla_c \psi_{abmd} - \nabla_d \psi_{abcm}) \\ &\quad dx^m \wedge dx^a \wedge dx^b \wedge dx^c \wedge dx^d \\ &= \frac{1}{120} B_{mabcd}\, dx^m \wedge dx^a \wedge dx^b \wedge dx^c \wedge dx^d \end{aligned}$$

Now using (2.2), we have

$$\begin{aligned} g(X \wedge \psi, d\psi) &= \frac{1}{120} A_{qijkl} B_{mabcd} g^{qm} g^{ia} g^{jb} g^{kc} g^{ld} \\ &= \frac{1}{24}\left(X_q \psi_{ijkl} - X_i \psi_{qjkl} - X_j \psi_{iqkl} - X_k \psi_{ijql} - X_l \psi_{ijkq}\right) \\ &\quad (\nabla_m \psi_{abcd}) g^{qm} g^{ia} g^{jb} g^{kc} g^{ld} \end{aligned}$$

Let $X = dx^p$, so that $X_i = \delta_i^p$, and this expression becomes

$$\begin{aligned} g(dx^p \wedge \psi, d\psi) &= \frac{1}{24}\left(\delta_q^p \psi_{ijkl} - \delta_i^p \psi_{qjkl} - \delta_j^p \psi_{iqkl} - \delta_k^p \psi_{ijql} - \delta_l^p \psi_{ijkq}\right) \\ &\quad (\nabla_m \psi_{abcd}) g^{qm} g^{ia} g^{jb} g^{kc} g^{ld} \end{aligned}$$

By Proposition 3.3.4, the first term vanishes, and the remaining terms become

$$g(dx^p \wedge \psi, d\psi) = \frac{1}{6}(\nabla_m \psi_{ijkl})\psi_{abcd} g^{im} g^{pa} g^{jb} g^{kc} g^{ld} \tag{3.45}$$

Combining the two expressions, $g(dx^p \wedge \varphi, d\varphi) = g(dx^p \wedge \psi, d\psi)$ if and only if

$$(\nabla_m \psi_{ijkl})\psi_{abcd} g^{im} g^{jb} g^{kc} g^{ld} = 3\,(\nabla_m \varphi_{ijk})\varphi_{abc} g^{im} g^{jb} g^{kc}$$

But this is the Proposition 3.3.5, after contracting with g^{im}.

We state the reason of considering torsion forms, which follows from [7].

Remark 3.5.4 *We consider the torsion forms of a* G_2*-structure* φ *because* φ *is torsion-free if and only if all four torsion forms vanish, and these forms are independent. This is because the decomposition of* $\wedge^k$ *into* G_2*-representations is orthogonal, and because the maps* $\alpha \mapsto \varphi \wedge \alpha$ *from* $\wedge^1_7 \to \wedge^4_7$ *and* $\alpha \mapsto \psi \wedge \alpha$ *from* $\wedge^1_7 \to \wedge^5_7$ *are isomorphisms.*

Lemma 3.5.5 *For any vector field X, the* 3*-form* $\nabla_X \varphi$ *lies in the subspace* $\wedge^3_7$ *of* $\wedge^3$*. Therefore, the covariant derivative* $\nabla\varphi$ *lies in the space* $\wedge^1_7 \otimes \wedge^3_7$*, a* 49*-dimensional space (pointwise) [7].*

Proof. Let $X = \frac{\partial}{\partial x^l}$, and consider the 3-form $\nabla_l \varphi$. According to [7], η of $\wedge^3_1 \oplus \wedge^3_{27}$ can be written in terms of a symmetric tensor h_{ij} as follows:

$$\eta = \frac{1}{2} h_i^m \varphi_{mjk}\, dx^i \wedge dx^j \wedge dx^k = \frac{1}{6}\left(h_i^m \varphi_{mjk} + h_j^m \varphi_{imk} + h_k^m \varphi_{ijm}\right) dx^i \wedge dx^j \wedge dx^k$$

Using (2.2), the inner product of η with $\nabla_l \varphi = \frac{1}{6} \nabla_l \varphi_{abc}\, dx^a \wedge dx^b \wedge dx^c$ is

$$\begin{aligned} g(\nabla_l \varphi, \eta) &= \frac{1}{6}(\nabla_l \varphi_{abc})\left(h_i^m \varphi_{mjk} + h_j^m \varphi_{imk} + h_k^m \varphi_{ijm}\right) g^{ai} g^{bj} g^{ck} \\ &= \frac{1}{2}(\nabla_l \varphi_{abc}) h_i^m \varphi_{mjk} g^{ai} g^{bj} g^{ck} = \frac{1}{2}(\nabla_l \varphi_{abc}) h^{ma} \varphi_{mjk} g^{bj} g^{ck} \end{aligned}$$

which vanishes since the third equation of Proposition 3.3.4 says that $(\nabla_l \varphi_{abc})\varphi_{mjk} g^{bj} g^{ck}$ is skew-symmetric in a and m. Since $g(\nabla_l \varphi, \eta) = 0$ for all $\eta \in \wedge^3_1 \oplus \wedge^3_{27}$, we have that $\nabla_l \varphi \in \wedge^3_7$ for all $l = 1, \ldots, 7$.

We have the following theorems from [7].

Theorem 3.5.6 *The covariant derivative* $\nabla\varphi$ *of the* 3*-form* φ *can be written as*

$$\nabla_l \varphi_{abc} = T_{lm} g^{mn} \psi_{nabc}$$

where the full torsion tensor T_{lm} *is*

$$T_{lm} = \frac{\tau_0}{4} g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}$$

Proof. According to [7], we start to write the full torsion tensor as $T_{lm} = S_{lm} + C_{lm}$, where $S_{lm} = \frac{1}{2}(T_{lm} + T_{ml})$ and $C_{lm} = \frac{1}{2}(T_{lm} - T_{ml})$ are the symmetric and skew-symmetric parts of T_{lm}, from [7]. Thus, we have that

$$\nabla_l \varphi_{abc} = (S_{lm} + C_{lm}) g^{mn} \psi_{nabc} \tag{3.46}$$

Since $d\varphi = \tau_0 \psi + 3\,\tau_1 \wedge \varphi + *\tau_3$, the $\wedge^3_1 \oplus \wedge^3_{27}$ component of $*d\varphi$ is $\tau_0 \varphi + \tau_3$, we can write it as $\frac{3}{7}\left(\frac{7}{3}\tau_0\right)\varphi + \tau_3$. By Remark 3.4.5, this is $f_{ij} g^{jl} dx^i \wedge \left(\frac{\partial}{\partial x^l} \lrcorner \varphi\right)$, where $f_{ij} = \frac{1}{7}\left(\frac{7}{3}\tau_0\right) g_{ij} + (\tau_3)_{ij}$. Therefore, by Proposition 3.4.3, we have that the $\wedge^4_1 \oplus \wedge^4_7$ component of $d\varphi = *(*d\varphi)$ is $\left(\frac{1}{4}\mathrm{Tr}_g(f) g_{ij} - f_{ij}\right) g^{jl} dx^i \wedge \left(\frac{\partial}{\partial x^l} \lrcorner \psi\right)$. However, $\mathrm{Tr}_g(f) = \frac{7}{3}\tau_0$, so

$$\frac{1}{4}\mathrm{Tr}_g(f) g_{ij} - f_{ij} = \frac{7}{12}\tau_0 g_{ij} - \frac{4}{12}\tau_0 g_{ij} - (\tau_3)_{ij} = \frac{1}{4}\tau_0 g_{ij} - (\tau_3)_{ij} \tag{3.47}$$

Now, we can also write $d\varphi = \frac{1}{6}\nabla l \varphi_{abc}\, dx^l \wedge dx^a \wedge dx^b \wedge dx^c$, by (3.46) we have that

$$\begin{aligned} d\varphi &= \frac{1}{6}\left(S_{lm} + C_{lm}\right) g^{mn} \psi_{nabc}\, dx^l \wedge dx^a \wedge dx^b \wedge dx^c \\ &= S_{lm} g^{mn} dx^l \wedge \left(\frac{\partial}{\partial x^n} \lrcorner \psi\right) + C_{lm} g^{mn} dx^l \wedge \left(\frac{\partial}{\partial x^n} \lrcorner \psi\right) \end{aligned}$$

The second term belongs to $\wedge^4_7$. Therefore, if we compare the $\wedge^4_1 \oplus \wedge^4_{27}$ term of $d\varphi$ and by (3.47), we see that

$$S_{lm} = \frac{\tau_0}{4} g_{lm} - (\tau_3)_{lm}$$

Secondly, we write $\delta\varphi = -*d*\varphi$ in two different ways. First, since $d\psi = 4\,\tau_1 \wedge \psi + *\tau_2$, we have $\delta\varphi = -*d\psi = -4(\tau_1{}^{\sharp} \lrcorner \varphi) - \tau_2$, using Lemma 1.0.3. in Appendix A.1.Therefore, we have

$$\delta\varphi = -4\frac{1}{2}(\tau_1)_{ab} dx^a \wedge dx^b - \frac{1}{2}(\tau_2)_{ab} dx^a \wedge dx^b \tag{3.48}$$

From (2.4), we also have $\delta\varphi = -\frac{1}{2} g^{lk} \nabla_l \varphi_{kab}\, dx^a \wedge dx^b$. Using (3.46), this is

$$\delta\varphi = -\frac{1}{2} g^{lk} \left(S_{lm} + C_{lm}\right) g^{mn} \psi_{nkab}\, dx^a \wedge dx^b$$

The first term vanishes since S_{lm} is symmetric and ψ_{nkab} is skew-symmetric. Now we decompose $C_{lm} = (C_7)_{lm} + (C_{14})_{lm}$ into $\wedge^2_7 \oplus \wedge^2_{14}$ components and we interchange n and k. Thus,we see that

$$\begin{aligned} \delta\varphi &= \frac{1}{2} g^{lk} \left((C_7)_{lm} + (C_{14})_{lm}\right) g^{mn} \psi_{knab}\, dx^a \wedge dx^b \\ &= \frac{1}{2}\left(-4(C_7)_{ab} + 2(C_{14})_{ab}\right) dx^a \wedge dx^b \end{aligned}$$

By comparing this to (3.48), we see that $(C_7)_{ab} = (\tau_1)_{ab}$ and $(C_{14})_{ab} = -\frac{1}{2}(\tau_2)_{ab}$, hence

$$C_{lm} = (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}$$

and the proof is complete.

Now, we will prove the fact that we mention before in Section 3.1.

Corollary 3.5.7 *The* 3*-form* φ *is parallel if and only if it is both closed and co-closed [12].*

Proof. According to [7], a parallel form is always closed and co-closed, since the exterior derivative d and the coderivative δ can both be written in terms of the covariant derivative ∇, which is the Levi-Civita connection and remember that it is torsion-free. Conversely, from Theorem 3.5.6, $d\varphi = 0$ and $\delta\varphi = 0$ hold if and only if all four torsion forms vanish, thus $T_{lm} = 0$ and then $\nabla_l \varphi_{abc} = 0$.

We have another definition of full torsion tensor in terms torsion, metric and Hodge dual 4-form, which follows from [7].

Lemma 3.5.8 *The full torsion tensor* T_{lm} *is*

$$T_{lm} = \frac{1}{24}(\nabla_l \varphi_{abc}) \psi_{mijk} g^{ia} g^{jb} g^{kc} \tag{3.49}$$

Proof. [7] We begin with $\nabla_l \varphi_{abc} = T_{lk} g^{kn} \psi_{nabc}$ and use Lemma 3.3.3 to get the following:

$$\begin{aligned} \nabla_l \varphi_{abc} \psi_{nijk} g^{ia} g^{jb} g^{kc} &= T_{lk} g^{kn} \psi_{nabc} \psi_{mijk} g^{ia} g^{jb} g^{kc} \\ &= T_{lk} g^{kn} (24 g_{nm}) = 24 T_{lm} \end{aligned}$$

Proposition 3.5.9 *The four torsion forms can be written in terms of* $T_{pq} = S_{pq} + C_{pq}$ *as follows:*

$$\begin{aligned} \tau_0 &= \frac{4}{7} g^{pq} S_{pq} \\ (\tau_3)_{pq} &= \frac{1}{4} \tau_0 g_{pq} - S_{pq} \\ (\tau_1)_{pq} &= \frac{1}{3} C_{pq} - \frac{1}{6} C_{ij} g^{ia} g^{jb} \psi_{abpq} \\ (\tau_2)_{pq} &= -\frac{4}{3} C_{pq} - \frac{1}{3} C_{ij} g^{ia} g^{jb} \psi_{abpq} \end{aligned}$$

Hence, we can write all independent four torsion forms in terms of the symmetric or anti-symmetric part of the full torsion tensor by [7].

3.6 Some Differential Operators on G_2 Manifolds

In this section, we will consider some necessary definitions and properties that will be useful for the application in String Theory. The next definition follows from [16].

Definition 3.6.1 *We define the* curl *of a vector field X to be the vector field curlX, given by*

$$curlX = *(dX \wedge \psi) \tag{3.50}$$

Here we denote the vector field X and its metric dual 1-form by the same notation. In local coordinates we have

$$(curlX)_k = g^{pi} g^{qj} (\nabla_p X_q) \varphi_{ijk}. \tag{3.51}$$

$$divX = -d^{\dagger} X^{\flat} = *d * X^{\flat}, \tag{3.52}$$

where $d^{\dagger}$ is the adjoint to the exterior derivative d. The identity $d^{\dagger} = -*d*$ is true for 1-forms, since our manifolds is an odd-dimensional manifold [16].

We have the following definition

$$gradf = (df)^{\sharp}. \tag{3.53}$$

There are various relations between the operators *grad*, *div*, and *curl* on a manifold M with a torsion-free G_2 structure. First, we recall some identities that are satisfied for torsion-free G_2 structures. Let $X_k dx^k$ be a 1-form on M. The Ricci identities say that

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = -R_{ijkl} g^{lm} X_m, \tag{3.54}$$

where R_{ijkl} is the Riemann curvature tensor [16]. If we contract (3.54) with g^{jk}, we get

$$(graddivX)^{\flat} = \nabla_i (g^{jk} \nabla_j X_k) = g^{jk} \nabla_j \nabla_i X_k - R_{ijkl} g^{jk} g^{lm} X_m = g^{jk} \nabla_j \nabla_i X_k, \tag{3.55}$$

here we have used the fact that $R_{ijkl} g^{jk} = R_{il}$ is the Ricci tensor, which vanishes for a torsion-free G_2 structure. The Ricci-flatness of the metric also implies that the

Laplacian agrees with the Hodge Laplacian $\Delta_d = dd^\dagger + d^\dagger d$ on 1-forms. In particular by [16], if $X = X^k \frac{\partial}{\partial x^k}$ is a vector field on M, then

$$g^{ij}\nabla_i\nabla_j X_k = -\Delta_d X^\flat = -(dd^\dagger + d^\dagger d)X^\flat. \tag{3.56}$$

Because φ is torsion-free, the Riemann curvature tensor R_{ijkl} lies in $\mathrm{Sym}^2(\wedge^2_{14})$ says that

$$R_{ijkl}g^{ia}g^{jb}\psi_{abcd} = 2R_{cdkl}.$$

Contracting it with g^{kc} gives the following equation, which follows from [16].

$$R_{ijkl}g^{ia}g^{jb}g^{kc}\psi_{abcd} = 2R_{cdkl}g^{kc} = -2R_{dl} = 0. \tag{3.57}$$

Now according to [16], we have the following relations between *grad*, *div*, and *curl* on a G_2 manifold

Proposition 3.6.2 *Let f be any function and X be any vector field on a manifold M with a* torsion-free G_2 *structure. The following relations hold:*

$$curl(grad f) = 0, \tag{3.58}$$

$$div(curl X) = 0, \tag{3.59}$$

$$curl(curl X) = -dd^\dagger X + \Delta X = -d^\dagger dX \tag{3.60}$$

Proof. To establish (3.58), we note that, from [16], equations (3.51) and (3.53) show that

$$*(curl(grad f))^\flat = d(df)\wedge\psi,$$

which vanishes since $d(df) = 0$. To establish (3.59), we note that equations (3.52) and (3.51) show that

$$div(curl X) = *d*(*(dX^\flat\wedge\psi)) = *d(dX^\flat\wedge\psi) = 0,$$

using the facts that $*^2 = 1$, $d(dX^\flat) = 0$, and $d\psi = 0$. Note that (3.59) does not necessarily requires the G_2 structure to be parallel but $d\psi = 0$. Finally, we prove (3.60), using local coordinates, and the full torsion-free hypothesis. Using (3.51), we compute:

$$\begin{aligned}
(curl curl X))_k &= \nabla_p(curl X)_q g^{pa}g^{qb}\varphi_{abk} \\
&= \nabla_p\left(\nabla_\alpha X_\beta g^{\alpha i}g^{\beta j}\varphi_{ijq}\right)g^{pa}g^{qb}\varphi_{abk} \\
&= (\nabla_p\nabla_\alpha X_\beta)g^{\alpha i}g^{\beta j}g^{pa}(\varphi_{ijq}\varphi_{kab}g^{qb}) \\
&= (\nabla_p\nabla_\alpha X_\beta)g^{\alpha i}g^{\beta j}g^{pa}(g_{ik}g_{ja} - g_{ia}g_{jk} - \psi_{ijka}),
\end{aligned}$$

where we have used Lemma 3.3.1. This expression becomes

$$\begin{aligned}(curl(curlX))_k &= g^{\beta p}(\nabla_p\nabla_k X_\beta) - g^{\alpha p}(\nabla_p\nabla_\alpha X_k) - (\nabla_p\nabla_\alpha X_\beta)g^{\alpha i}g^{\beta j}g^{pa}\psi_{ijka} \\ &= (grad(divX))_k + (\Delta_d X^\flat)_k + (\nabla_p\nabla_\alpha X_\beta)g^{pa}g^{\alpha i}g^{\beta j}\psi_{aijk},\end{aligned}$$

using (3.55) and (3.56). Thus to prove (3.60) the only thing we have to show that the last term vanishes. By the skew-symmetry of ψ_{aijk}, we can write the last term as follows:

$$\begin{aligned}(\nabla_p\nabla_\alpha X_\beta)g^{pa}g^{\alpha i}g^{\beta j}\psi_{aijk} &= \tfrac{1}{2}(\nabla_p\nabla_\alpha X_\beta - \nabla_\alpha\nabla_p X_\beta)g^{pa}g^{\alpha i}g^{\beta j}\psi_{aijk} \\ &= -\tfrac{1}{2}R_{p\alpha\beta m}g^{mn}X_n g^{pa}g^{\alpha i}g^{\beta j}\psi_{aijk} = 0,\end{aligned}$$

using (3.54) and (3.57).

We have the next lemma that states the relation between the projections of space of 3-forms into submodules and curl, which follows from [16].

Lemma 3.6.3 *Let X be a vector field, and consider the form $X\lrcorner\varphi \in \wedge^2_7$. Then*

$$\pi_1\big(d(X\lrcorner\varphi)\big) = -\frac{3}{7}(d^*X)\varphi, \qquad \pi_7\big(d(X\lrcorner\varphi)\big) = \frac{1}{2}*\big((curlX)\wedge\varphi\big). \tag{3.61}$$

Proof. According to [16], we have

$$\pi_1(d(X\lrcorner\varphi)) = h\varphi \quad \text{for some } h \in \wedge^0_1.$$

Using the fact that $\wedge^3_7 \oplus \wedge^3_{27}$ lies in the kernel of wedge product with ψ, we compute

$$d((X\lrcorner\varphi)\wedge\psi) = d(X\lrcorner\varphi)\wedge\psi = \pi_1(d(X\lrcorner\varphi))\wedge\psi = h\varphi\wedge\psi = 7h\,\mathrm{vol}.$$

Hence, we find that

$$d(3*X) = d((X\lrcorner\varphi)\wedge\psi) = 7h\,\mathrm{vol},$$

and then $h = \frac{3}{7}*d(*X) = -\frac{3}{7}d^*X$. Similarly, we have

$$\pi_7(d(X\lrcorner\varphi)) = *(Y\wedge\varphi) \quad \text{for some } Y \in \wedge^1_7.$$

Using the fact that $\wedge^3_1 \oplus \wedge^3_{27}$ lies in the kernel of wedge product with φ, we compute

$$d((X\lrcorner\varphi)\wedge\varphi) = d(X\lrcorner\varphi)\wedge\varphi = \pi_7(d(X\lrcorner\varphi))\wedge\varphi = *(Y\wedge\varphi)\wedge\varphi = -4*Y.$$

Hence, we find that

$$-4*Y = d((X\lrcorner\varphi)\wedge\varphi) = d(-2*(X\lrcorner\varphi)) = -2d(X\wedge\psi) = -2(dX)\wedge\psi,$$

and thus $Y = \frac{1}{2} * ((dX) \wedge \psi) = \frac{1}{2} curlX$ [16].

We now consider the relation involving the 2-form dX, its projection onto the 7-dimensional submodule of the space of 2-forms and curl, which follows from [16].

Lemma 3.6.4 *Consider the vector field X as a 1-form using the metric. Then $dX \in \wedge^2 = \wedge^2_7 \oplus \wedge^2_{14}$. The $\wedge^2_7$ component of dX is given by*

$$\pi_7(dX) = \frac{1}{3}(curlX) \lrcorner \varphi = \frac{1}{3} * \big((curlX) \wedge \psi\big). \tag{3.62}$$

Proof. According to [16], we have that $\pi_7(dX) = W \lrcorner \varphi$ for some vector field W. Then by using the definition $\wedge^2_7$ and the curl, we have the following

$$\begin{aligned} curlX &= *(dX \wedge \psi) \\ &= *(\pi_7(dX) \wedge \psi) \\ &= *((W \lrcorner \varphi) \wedge \psi) \\ &= *(3 * W) = 3W \end{aligned}$$

4. AN APPLICATION IN STRING THEORY

String theory is a theory of quantum gravity, which unifies quantum field theory with Einstein's theory of general relativity. The low energy limit of string theory is ten dimensional supergravity theory, which is a supersymmetric theory of gravity. A variant of string theory is M-theory, whose low energy limit is eleven dimensional supergravity theory. As we live in a four dimensional space-time, one has to assume that the extra seven dimensions are small and belong to a compact internal manifold. Indeed, eleven dimensional supergravity has classical solutions of the form

$$M_4 \times Y_7$$

where M_4 is four dimensional Minkowski space-time that we observe and Y_7 is a compact manifold. Related to this (type II) string theory has solutions of the form $R_{2,1} \times Y_7$. To have only one unbroken supersymmetry in four dimension requires that the seven dimensional manifold Y_7 must be of G_2 holonomy. This follows from the requirement of the existence of a parallel spinor field η , which plays the role of generating supersymmetry transformations. Then one can construct a parallel 3-form field out of this spinor field, as a spinor bilinear $\varphi_{abc} = \eta^T \Gamma_{abc} \eta$ [1] and this 3-form field indeed defines a G_2 structure. From the fact that the 3-form field is parallel, it follows that the internal manifold should have G_2 holonomy.

These low energy, supergravity solutions acquire corrections from the quantum theory, which we call α' corrections, for reasons which are not relevant to this thesis. All we have to know is that these corrections imply that the spinor field, which generates the supersymmetry transformations is not covariantly constant, any more. This in turn means that the 3-form field constructed out of this spinor field is not parallel and hence the internal manifold is not of G_2 holonomy. Then it is natural to ask whether the classical metric of G_2 holonomy can be modified to compensate for these corrections. In other words, is there a small deformation of the G_2 metric, and hence of the G_2 3-form, such that the deformed metric solves the equations to all orders in α'? In this

[1] Γ_{abc} are products of Dirac matrices, which are the generators of an appropriate Clifford algebra.

part of the thesis we will only consider α' corrections to first order and we follow closely the paper [17].

Our goal is to find a globally-defined G_2 structure ϕ' which is close to ϕ,

$$\phi' = \phi + \delta\phi \tag{4.1}$$

where ϕ is the torsionless G_2-structure, corresponding to the Ricci-flat metric that solves the supergravity equations. The deformed metric g', associated to the 3-form ϕ' is to solve the α'-corrected equations of motion. The following theorem assures that such a small perturbation of the original 3-form ϕ defining the G_2 structure also yields a G_2-structure, which is given by Joyce [5]:

Theorem 4.1 Let $\varepsilon_1 > 0$ be a universal constant such that whenever $\{\phi, g\}$ is a G_2-structure on a 7 dimensional manifold.
If $\phi' \in C^\infty(\wedge^3 T^*M)$ and $\|\phi' - \phi\|_{C^0} \leq \varepsilon_1$ then $\phi' \in C^\infty(P^3M)$.
So, ϕ' defines a G_2-structure $\{\phi', g'\}$.

We assume $\delta\phi$ is such that

$$\|\phi' - \phi\|_{C^0} = \|\delta\phi\|_{C^0} \leq \varepsilon_1 \tag{4.2}$$

We define the norm $\|\cdot\|_{C^0}$ on $C^0(M)$ by $\|f\|_{C^0} = sup_M \mid f \mid$.
It can be shown that the equations for supersymmetry can be converted to the following equations for ϕ' and its Hodge dual ψ' with respect to the deformed metric:

$$d\phi' = \alpha \tag{4.3}$$

$$d\psi' = \beta \tag{4.4}$$

Here $\alpha = d\chi$ and $\beta = d\xi$ can be calculated from physics and it can be shown that they are exact forms, which is obviously a necessary condition for the existence of a solution the the above system of partial differential equations. The aim of this section is to show that this condition is sufficient and such a solution always exists.

We have the solution of the first equation

$$\phi' = \phi + \chi + db \tag{4.5}$$

where b is a two form.
For the second equation we have for the dual four form $\psi' = *'\phi'$

$$d\psi' = d * (\frac{4}{3}\pi_1 + \pi_7 - \pi_{27}(\chi + db) \tag{4.6}$$

Here π_1, π_7, π_{27} are the projections of three forms onto $\wedge^3_1, \wedge^3_7, \wedge^3_{27}$ We have that

$$*'\phi' \simeq *(\frac{4}{3}\pi_1 + \pi_7 - \pi_{27})(\chi + db)$$

Then we have

$$*(d*(\frac{4}{3}\pi_1 + \pi_7 - \pi_{27})(\chi + db)) = *d\xi \tag{4.7}$$

This is the linearization, from [12], of the $*'$ operator as can be seen from the following

Proposition 4.2 Let ε_1 be as in the Theorem 4.1. Then there exists constants $\varepsilon_2, \varepsilon_3 > 0$ such that whenever M is a 7-manifold and (φ, g) a G_2-structure on M with $d\varphi = 0$ then the following is true. Suppose $\chi \in C^\infty(\wedge^3 T^*M)$ and $|\chi| \leq \varepsilon_1$. Then $\varphi + \chi \in C^\infty(P^3M)$ and $\theta(\varphi, \chi)$ is given by

$$\begin{aligned}\theta(\varphi,\chi) &= *\varphi + \frac{4}{3}*\pi_1(\chi) + *\pi_7(\chi) - *\pi_{27}(\chi) - F(\chi) \\ &= *\varphi + \frac{7}{3}*\pi_1(\chi) + 2*\pi_7(\chi) - *\chi - F(\chi)\end{aligned}$$

where F is a smooth function from the closed ball of radius ε_1 in $\wedge^3 T^*M$ to $\wedge^4 T^*M$ with $F(0) = 0$.

Applying (4.7) we have

$$\begin{aligned}*(d*(\frac{4}{3}\pi_1 + \pi_7 - \pi_{27}))db &= *d\xi - *d*(\frac{4}{3}\pi_1 + \pi_7 - \pi_{27})\chi \\ -d^\dagger(\frac{4}{3}\pi_1 + \pi_7 - \pi_{27})db &= d^\dagger(-*\xi + (\frac{4}{3}\pi_1 + \pi_7 - \pi_{27})\chi)\end{aligned}$$

Thus, we have an equation for b

$$d^\dagger(\pi_{27} - \pi_7 - \frac{4}{3}\pi_1)db = d^\dagger\rho, \qquad \rho = -*\xi - (\pi_{27} - \pi_7 - \frac{4}{3}\pi_1)\chi \tag{4.8}$$

Here $d^\dagger = - * d *$.

We want to understand whether this equation has a solution for b or not. Our aim is to convert this equation to a Laplacian equation for b.

$\Delta b = d^\dagger\rho$ under the assumption that $d^\dagger b = 0$ where $\Delta = dd^\dagger + d^\dagger d$.

Putting the equation in that form ensures that we can always solve for b. A coexact form is orthogonal to all harmonic forms by Hodge decomposition theorem. Our claim is that

$$d^\dagger(\pi_{27} - \pi_7 - \frac{4}{3}\pi_1)db = d^\dagger(\pi_{27} + \pi_7 + \pi_1)db = d^\dagger db = \Delta b \tag{4.9}$$

In order to show it, first we prove

- $b \in \wedge^2_{14}$
- $\pi_7(db) = 0$
- $\pi_1(db) = 0$

First, let φ be a torsion-free G_2-structure and define $\eta = d(X \lrcorner \varphi)$, by using the definition of $\wedge^2_7$, for some vector field X. We claim that

$$d(*(\frac{4}{3}\pi_1\eta + \pi_7\eta - \pi_{27}\eta)) = 0$$

From Lemma 3.6.3 we find

$$\begin{aligned} \frac{4}{3} * \pi_1\eta + *\pi_7\eta - *\pi_{27}\eta &= \frac{7}{3} * \pi_1\eta + 2 * \pi_7\eta - *\eta & (4.10) \\ &= -(d^*X)\psi + (curlX) \wedge \varphi - *\eta. & (4.11) \end{aligned}$$

Hence we have

$$d(*(\frac{4}{3}\pi_1\eta + \pi_7\eta - \pi_{27}\eta)) = -(dd^*X) \wedge \psi + (dcurlX) \wedge \varphi - d * \eta. \quad (4.12)$$

and we use Definition 3.6.1, the third term above can be rewritten as

$$\begin{aligned} -d * \eta = -d * d(X \lrcorner \varphi) &= -d * d * (X \wedge \psi) \\ &= dd^*(X \wedge \psi) \\ &= (\Delta - d^*d)(X \wedge \psi) \\ &= (\Delta X) \wedge \psi - d^*\big((dX) \wedge \psi\big) \\ &= (\Delta X) \wedge \psi - d^*(*curlX) \\ &= (\Delta X) \wedge \psi - *(dcurlX). \end{aligned}$$

Substituting the above into (4.12) and using the definitions of $\wedge^2_7$ and $\wedge^2_{14}$, we obtain

$$\begin{aligned} d(*(\frac{4}{3}\pi_1\eta + \pi_7\eta - \pi_{27}\eta)) &= -(dd^*X) \wedge \psi + \big(-2 * \pi_7(dcurlX) + *\pi_{14}(dcurlX)\big) \\ &\quad + (\Delta X) \wedge \psi - \big(*\pi_7(dcurlX) + *\pi_{14}(dcurlX)\big) \\ &= (d^*dX) \wedge \psi - 3 * \pi_7(dcurlX). \end{aligned}$$

Applying Lemma 3.6.4 to the last equation (for the vector field $curlX$), we obtain

$$d(*(\frac{4}{3}\pi_1\eta + \pi_7\eta - \pi_{27}\eta)) = (d^*dX) \wedge \psi - (curlcurlX) \wedge \psi.$$

The right hand side above vanishes by $curl(curlX) = d^*dX$ for any vextor field X by Proposition 3.6.2. It means that $b \in \wedge^2_{14}$.

Now, we will show that $\pi_1(db) = 0$ if $b \in \wedge^2_{14}$.

Since $b \in \wedge^2_{14}$, then $*\varphi \wedge b = 0$. Taking derivative of boths sides we get

$d(*\varphi \wedge b) = d * \varphi \wedge b + *\varphi \wedge db = 0$

Since φ is a G_2 structure then $d * \varphi = 0$. Then $*\varphi \wedge db = 0$

Since db is a 3-form, we have

$$*\varphi \wedge (\pi_1(db) + \pi_7(db) + \pi_{27}(db)) = 0 \tag{4.13}$$

$$*\varphi \wedge \pi_1(db) + *\varphi \wedge \pi_7(db) + *\varphi \wedge \pi_{27}(db) = 0 \tag{4.14}$$

Now ,we know that $\pi_7(db) = *(\varphi \wedge \alpha)$ where $\alpha \in \wedge^1_7$. Also,

$$*\varphi \wedge \pi_7(db) = *\varphi \wedge *(\varphi \wedge \alpha) = 0 \tag{4.15}$$

by (3.21)

Also $*\varphi \wedge \pi_{27}(db) = 0$ by the definiton of $\wedge^3_{27}$. Thus,

$$*\varphi \wedge \pi_1(db) = 0 \tag{4.16}$$

by the definition of $\wedge^3_1$ we write $\pi_1(db) = f\varphi$ where $f \in C^\infty(M))$. $f * \varphi \wedge \varphi = 0$ then $fvol = 0 \Leftrightarrow f = 0$. Thus, $\pi_1(db) = 0$, as claimed.

Now, we will show that $\pi_7(db) = 0$ because $d^\dagger b = 0$.

Since $b \in \wedge^2_{14}$, then $*\varphi \wedge b = 0$.We have $*\varphi \wedge db = 0$ as above. Also, $b \in \wedge^2_{14}$, then $*(\varphi \wedge b) = -b$. Taking $d^\dagger$ of both sides we have

$d^\dagger(*(\varphi \wedge b)) = -d^\dagger b = 0$ by the assumption $d^\dagger b = 0$.

$$- * d *^2 (\varphi \wedge b) = *d(\varphi \wedge b) = *(d\varphi \wedge b) + *(\varphi \wedge db) = 0 \tag{4.17}$$

by using $d\varphi = 0$ we have $*(\varphi \wedge db) = 0$. Also $(\varphi \wedge db) = 0$.

Using the facts that $*\varphi \wedge db = 0$ and $(\varphi \wedge db) = 0$ we conclude that $db \in \wedge^3_{27}$. Then, $\pi_7(db) = 0$.

Then we have showed that the equation (4.9) for b takes the simple form $d^\dagger db = \Delta b = d^\dagger \rho$ which always has a solution by Hodge decomposition theorem.

The above analysis shows that the system of equations (4.3) and (4.4) always has a solution, for given exact forms α and β, determined by physics. This in turn proves our claim: the G_2 holonomy metric of the supergravity solution can be corrected to a metric of G_2 structure, which solves the $\alpha^{'}$ corrected equations of motion of the string theory.

5. CONCLUSION

The purpose of this thesis is to study the differential geometric properties of manifolds with G_2-structure, with a special emphasis on the torsion forms.

Firstly, we studied various equivalent definitions of G_2. We investigate conditions for the existence of a G_2-structure. Also, we consider the properties of a G_2-structure and a G_2-manifold. Then, we described the decomposition of each space of k forms into irreducible G_2 representations. Afterwards, using the decomposition of the spaces of differential forms we decompose $d\phi$, $d*\phi$ into irreducible G_2 representations. This led us to introducing the torsion forms for G_2 structures. As we studied in detail in the thesis, a G_2 structure is parallel if and only if all the torsion forms vanish. Hence, torsion forms are an important tool in studying manifolds with G_2 structure, but not of G_2 holonomy. Since our aim is to study such manifolds, we studied in detail some important properties satisfied by the torsion forms.

Recently, manifolds of G_2 structure, rather than G_2 holonomy, has found important applications in string theory. As one such application, we investigated if a metric of G_2 holonomy can be modified to compensate for α' corrections or equivalently if there is a perturbation of the associative 3-form ϕ such that the corresponding metric g' solves the α' corrected equations of the quantum theory. This led us to the investigation of the existence of a coupled system of partial differential equations for ϕ', and its Hodge dual (with respect to g') where the source terms are determined by physics, and are related to the torsion forms of the G_2 structure ϕ'. Then, by analyzing this system of equations, we showed that the G_2 holonomy metric of the supergravity solution can be corrected to a metric of G_2 structure, which solves the $\alpha^{'}$ corrected equations of motion of the string theory.

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APPENDICES

APPENDIX A.1 : Some Useful Identities

APPENDIX A.1

We have the following Lemma that gives the relations between the contractions and the wedge product.
By [14], we have that

$$*^2 = (-1)^{k(n-k)} \tag{A.1}$$

on k-forms. We also have $*^2 = 1$. Suppose v is a vector field and α is a k-form.

Lemma 1.0.1 According to [14], we have the following four identities :

$$\begin{aligned} *(w\lrcorner\alpha) &= (-1)^{k+1}(w^\flat \wedge *\alpha) && \textbf{(A.2)} \\ (w\lrcorner\alpha) &= (-1)^{nk+n} * (w^\flat \wedge *\alpha) && \textbf{(A.3)} \\ *(w\lrcorner*\alpha) &= (-1)^{nk+n+1}(w^\flat \wedge \alpha) && \textbf{(A.4)} \\ (w\lrcorner*\alpha) &= (-1)^{k} * (w^\flat \wedge \alpha) && \textbf{(A.5)} \end{aligned}$$

and when $\alpha = \text{vol}$, the special case

$$w\lrcorner\text{vol} = *w^\flat \tag{A.6}$$

Proof. According to [14], we have

$$\begin{aligned} \langle \beta, w\lrcorner\alpha\rangle \,\text{vol} &= \beta \wedge *(w\lrcorner\alpha) \\ &= (w\lrcorner\alpha)(\beta^\sharp)\,\text{vol} \\ &= \alpha(w \wedge \beta^\sharp)\,\text{vol} \\ &= \left\langle \alpha, w^\flat \wedge \beta \right\rangle \text{vol} \\ &= (w^\flat \wedge \beta) \wedge *\alpha \\ &= (-1)^{k-1}\beta \wedge (w^\flat \wedge \alpha) \end{aligned}$$

Since β is arbitrary, (**A.2**) follows. Substituting $*\alpha$ for α and using (**A.1**), we obtain (**A.4**). The other two are obtained by taking $*$ of both sides of the first two identities.

We have the following relations involving the 3-form, its Hodge dual 4-form, metric dual 1-forms and Hodge star, which follows from [7],

Proposition 1.0.2 Let α be a 1-form on M, let w be a vector field on M, and $w^\flat$ be the 1-form dual to w. Then the following relations hold:

1. $|\varphi|^2 = 7$
2. $|\psi|^2 = 7$
3. $|\varphi \wedge \alpha|^2 = 4|\alpha|^2$
4. $|\psi \wedge \alpha|^2 = 3|\alpha|^2$

5. $*(\varphi \wedge *(\varphi \wedge \alpha)) = -4\alpha$

6. $*(\psi \wedge *(\psi \wedge \alpha)) = 3\alpha$

7. $\psi \wedge *(\varphi \wedge \alpha) = 0$

8. $\varphi \wedge *(\psi \wedge \alpha) = -2\psi \wedge \alpha$

9. $*(\varphi \wedge w^\flat) = w \lrcorner \psi$

10. $*(\psi \wedge w^\flat) = w \lrcorner \varphi$

11. $\varphi \wedge (w \lrcorner \varphi) = -2 * (w \lrcorner \varphi)$

12. $\psi \wedge (w \lrcorner \varphi) = 3 * w^\flat$

13. $\varphi \wedge (w \lrcorner \psi) = -4 * w^\flat$

14. $\psi \wedge (w \lrcorner \psi) = 0$

Proof. According to [14], for (1), using (**A.4**) and (**A.5**) we start by necessarily zero eight form

$$\begin{aligned} 0 = w^\flat \wedge \alpha \wedge *\alpha &= w \lrcorner (w^\flat \wedge \alpha \wedge *\alpha) \\ &= |w|^2 \alpha \wedge *\alpha - w^\flat \wedge (w \lrcorner \alpha) \wedge *\alpha + (-1)^{k+1} w^\flat \wedge \alpha \wedge (w \lrcorner *\alpha) \\ &= |w|^2 |\alpha|^2 vol - (w \lrcorner \alpha) \wedge *(w \lrcorner \alpha) - (w \lrcorner *\alpha) \wedge *(w \lrcorner *\alpha) \\ &= |w|^2 |\alpha|^2 vol - |w \lrcorner \alpha|^2 vol - |w \lrcorner *\alpha|^2 vol \end{aligned}$$

We have $|w|^2|\alpha|^2 = |w \lrcorner \alpha|^2 + |w \lrcorner *\alpha|^2$.
Now, we write

$$\begin{aligned} |\varphi|^2|\alpha|^2 = |\psi \lrcorner \alpha|^2 + |\varphi \lrcorner \alpha|^2 &= 3|\alpha|^2 + 4|\alpha|^2 = 7|\alpha|^2 \\ |\varphi|^2 &= 7 \end{aligned}$$

Similarly, for (2) we write

$$\begin{aligned} |\psi|^2|\alpha|^2 = |\psi \lrcorner \alpha|^2 + |\varphi \lrcorner \alpha|^2 &= 4|\alpha|^2 + 3|\alpha|^2 = 7|\alpha|^2 \\ |\psi|^2 &= 7 \end{aligned}$$

For (5), let $\alpha = \alpha_1 dx^1$ be a one form.

$$\begin{aligned} \varphi \wedge \alpha &= \alpha_1(-dx^{5271} - dx^{5631} + dx^{4261} + dx^{4371}) \\ *(\varphi \wedge \alpha) &= \alpha_1(-dx^{364} - dx^{274} + dx^{375} + dx^{265}) \\ \varphi \wedge *(\varphi \wedge \alpha) &= \alpha_1(-dx^{527364} - dx^{563274} + dx^{426375} + dx^{437265}) \\ *(\varphi \wedge *(\varphi \wedge \alpha)) &= 4\alpha_1 dx^1 = 4\alpha \end{aligned}$$

For (7), $*\varphi \wedge *(\varphi \wedge \alpha) = \alpha_1 0 = 0$

For (8),

$$\begin{aligned} *\varphi \wedge \alpha &= \alpha_1(dx^{45671} - dx^{45231} + dx^{26371}) \\ *(*\varphi \wedge \alpha) &= -\alpha_1(dx^{23} - dx^{67} + dx^{45}) \\ \varphi \wedge *(*\varphi \wedge \alpha) &= -2\alpha_1(dx^{26371} - dx^{45231} + dx^{45671}) = -2(\psi \wedge \alpha) \end{aligned}$$

For (3), we have from (5) $(\varphi\wedge*(\varphi\wedge\alpha)) = -4*\alpha$

$$\begin{aligned}(\varphi\wedge\alpha)\wedge*(\varphi\wedge\alpha) &= -4\alpha\wedge*\alpha\\ &= -|\varphi\wedge\alpha|^2 vol\\ &= -4|\alpha|^2 vol\end{aligned}$$

Therefore, $|\varphi\wedge\alpha|^2 = 4|\alpha|^2$.

For (6),

$$\begin{aligned}*(\psi\wedge*(\psi\wedge\alpha)) &= *(\alpha_1(dx^{456723}+dx^{452367}+dx^{263745}))\\ &= \alpha_1(dx^1+dx^1+dx^1) = 3\alpha\end{aligned}$$

For (4), we have from (6) $\psi\wedge*(\psi\wedge\alpha) = 3*\alpha$

$$\begin{aligned}\alpha\wedge\psi\wedge*(\psi\wedge\alpha) &= 3\alpha\wedge*\alpha\\ |\psi\wedge\alpha|^2 vol &= 3|\alpha|^2 vol\end{aligned}$$

Therefore, $|\psi\wedge\alpha|^2 = 3|\alpha|^2$. For (9), we have from (**A.5**)

$$w\lrcorner\psi = -*(w^\flat\wedge\varphi) = *(\varphi\wedge w^\flat)$$

Thus, $w\lrcorner\psi = *(\varphi\wedge w^\flat)$.

For (10), by using (**A.3**), we have $*(\psi\wedge w^\flat) = w\lrcorner\varphi$.

For (12), from (6) we have $\psi\wedge*(\psi\wedge\alpha) = 3*\alpha$ If we set $\alpha = w^\flat$, then by (10) we have $*(\psi\wedge\alpha) = w\lrcorner\varphi$. Plugging into the first equation we have

$$\psi\wedge(w\lrcorner\varphi) = 3*w^\flat$$

For (14), from (7) we have $\psi\wedge*(\varphi\wedge\alpha) = 0$. But here
$*(\varphi\wedge\alpha) = *(\varphi\wedge w^\flat) = w\lrcorner\psi$ from (9). Thus $\psi\wedge(w\lrcorner\psi) = 0$.

For (11), from (8) we have $\varphi\wedge*(\psi\wedge w^\flat) = -2\psi\wedge w^\flat$.
Also by (10) we have,

$$*(\psi\wedge w^\flat) = w\lrcorner\varphi$$

So, $-2*(\psi\lrcorner\varphi) = \varphi\wedge(w\lrcorner\varphi)$.

Finally, for (13) we use (5) $(\varphi\wedge*(\varphi\wedge w^\flat)) = -4*w^\flat$ and (**A.5**)

$$\varphi\wedge(w\lrcorner\psi) = -4*w^\flat$$

We have the next lemma of the G_2-structure, which follows from [7].

Lemma 1.0.3 The metric g, cross product $\times$, and 3-form φ satisfy the following relations:

$$\begin{aligned}g(u\times v,w) &= \varphi(u,v,w) && \textbf{(A.7)}\\ (u\times v)^\flat &= v\lrcorner u\lrcorner\varphi \;=\; *(u^\flat\wedge v^\flat\wedge\psi) && \textbf{(A.8)}\\ u\times(v\times w) &= -g(u,v)w+g(u,w)v-(u\lrcorner v\lrcorner w\lrcorner\psi)^\sharp && \textbf{(A.9)}\end{aligned}$$

where u, v, w are vector fields and $v^\flat$ denotes the 1-form which is metric dual to v.

Proof. Let u and v be vector fields on M. According to [14], the cross product $u \times v$ is a vector field on M whose associated 1-form under the metric isomorphism satisfies the following:

$$(u \times v)^\flat = v \lrcorner u \lrcorner \varphi \qquad (\mathbf{A.10})$$

From this, we have the relation between $\times$, φ, and the metric g:

$$g(u \times v, w) = (u \times v)^\flat(w) = w \lrcorner v \lrcorner u \lrcorner \varphi = \varphi(u, v, w). \qquad (\mathbf{A.11})$$

Another characterization of the cross product can be obtained from this one using Lemma 1.0.1:

$$\begin{aligned} (u \times v)^\flat &= v \lrcorner u \lrcorner \varphi \qquad (\mathbf{A.12}) \\ &= -*(v^\flat \wedge *(u \lrcorner \varphi)) \\ &= -*(v^\flat \wedge u^\flat \wedge *\varphi) \\ &= *(u^\flat \wedge v^\flat \wedge *\varphi) \end{aligned}$$

For the last one, from (**A.12**), we have

$$(u \times (v \times w))^\flat = *\left(u^\flat \wedge *\left(v^\flat \wedge w^\flat \wedge *\varphi\right) \wedge *\varphi\right)$$

Now since $\beta \wedge *\varphi = 0$ for $\beta \in \wedge^2_{14}$, we can replace $v^\flat \wedge w^\flat$ by $\pi_7(v^\flat \wedge w^\flat) = \frac{1}{3}\left(v^\flat \wedge w^\flat - *\left(\varphi \wedge v^\flat \wedge w^\flat\right)\right)$. Then using (3.37), we have

$$\begin{aligned} *\left(\pi_7(v^\flat \wedge w^\flat) \wedge *\varphi\right) \wedge *\varphi &= 3 * \pi_7(v^\flat \wedge w^\flat) \\ &= *\left(v^\flat \wedge w^\flat - *\left(\varphi \wedge v^\flat \wedge w^\flat\right)\right) \\ &= *\left(v^\flat \wedge w^\flat + v \lrcorner w \lrcorner *\varphi\right) \end{aligned}$$

which we substitute back to obtain

$$\begin{aligned} (u \times (v \times w))^\flat &= *\left(u^\flat \wedge *\left(v^\flat \wedge w^\flat + v \lrcorner w \lrcorner *\varphi\right)\right) \\ &= -u \lrcorner \left(v^\flat \wedge w^\flat + v \lrcorner w \lrcorner *\varphi\right) \\ &= -g(u, v) w^\flat + g(u, w) v^\flat - u \lrcorner v \lrcorner w \lrcorner *\varphi \end{aligned}$$

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