ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

ROGUE WAVES IN THE GENERALIZED DAVEY-STEWARTSON SYSTEM

M.Sc. THESIS

Mervenur BELİN

Department of Mathematical Engineering

Mathematical Engineering Programme

JUNE 2017

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ISTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

GENELLEŞTİRİLMİŞ DAVEY-STEWARTSON SİSTEMİNDE DEV DALGALAR

YÜKSEK LİSANS TEZİ

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Mervenur BELİN, a M.Sc. student of ITU Graduate School of Science Engineering and Technology student ID 509131059, successfully defended the thesis entitled "ROGUE WAVES IN THE GENERALIZED DAVEY-STEWARTSON SYSTEM", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family,

FOREWORD

I would like to express my gratitude to my supervisor Assist. Prof. Dr. İrma Hacınlıyan for her endless and unconditional support, patience, encouragement and invaluable advice during this study. Without her support this study would never been completed.

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Mervenur BELİN
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ABBREVIATIONS

DS	: Davey-Stewartson
DSI	: Davey-Stewartson I
DSII	: Davey-Stewartson II
dNLS	: Discrete Nonlinear Schrödinger
GDS	: Generalized Davey-Stewartson
GDSI	: Generalized Davey-Stewartson I
GDSII	: Generalized Davey-Stewartson II
KdV	: Korteweg deVries
mKdV	: Modified Korteweg deVries
NLS	: Nonlinear Schrödinger
PDE	: Partial Differential Equation

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ROGUE WAVES
IN THE GENERALIZED
DAVEY-STEWARTSON SYSTEM

SUMMARY

In this study, we investigate the rogue waves in the generalized Davey-Stewartson system, derived in acoustics,

$$
\begin{aligned}
&iu_\zeta = u_{\xi\xi} + \gamma_1 u_{\eta\eta} + \gamma_2 u_{\xi\eta} + u(\beta_1\partial_\xi + \beta_2\partial_\eta)v + \chi|u|^2 u,\\
&\alpha_{11}v_{\xi\xi} + \alpha_{22}v_{\eta\eta} = -2(\beta_1\partial_\xi + \beta_2\partial_\eta)|u|^2,
\end{aligned}
$$

where u is the complex envelope function, v is the mean field, ζ is the time coordinate and (ξ, η) denotes for spatial coordinates. This system can be classified in two categories depending on the relationship between the coefficients: the generalized Davey-Stewartson I (GDSI) (or the elliptic-hyperbolic generalized Davey-Stewartson) system if $\gamma_2^2 - 4\gamma_1 < 0$ and $\alpha_{11}\alpha_{22} < 0$, and the generalized Davey-Stewartson II (GDSII) (or the hyperbolic-elliptic generalized Davey-Stewartson) system if $\gamma_2^2 - 4\gamma_1 > 0$ and $\alpha_{11}\alpha_{22} > 0$. On the other hand, if the terms $u_{\xi\eta}$, v_η and $(|u|^2)_\eta$ vanish in the equation, the generalized Davey-Stewartson system becomes the Davey-Stewartson equations whose rogue wave solutions are obtained in previous studies. In these studies, for the rogue wave solution, the Hirota direct method via determinants of matrices is used and, according to the solution function and graph, the rogue wave is sub-named as a fundamental rogue wave, a multi-rogue wave and a higher-order rogue wave.

In this thesis, we specifically focus on elliptic-hyperbolic generalized Davey-Stewartson system and derive their rogue wave solutions by the same method for the Davey-Stewartson equations. Specifically, the flow of the thesis is as follows:

In Chapter 1, we start with the nonlinear Schrödinger system, which is the origin of the Davey-Stewartson system, and describe some sub-groups of the nonlinear Schrödinger system. Then, we move to the generalized Davey-Stewartson system and share the equations derived in different media. After that we introduce a rogue wave, specially discussing its definition, the reasons for its occurrence and the classification of it.

In Chapter 2, we inform briefly Hirota direct method by giving the definition of bilinear operator and explaining the bilinearization techniques. Then soliton solutions of the nonlinear Schrödinger equation are established by this method.

In Chapter 3, we deal with the generalized Davey-Stewartson system and derive the rogue wave solution for elliptic-hyperbolic case using the direct method along with determinants. We first obtain rational solutions which form rogue waves. Then rogue wave solutions are shared as fundamental rogue waves, multi-rogue waves and higher-order rogue waves. The graphics of solutions are also interpreted in terms of the properties observed.

In chapter 4, conclusion part, we discuss and summarize the findings. Additionally, we observe that the blow up solution leading to the exploding rogue wave which can be derived from the multi-rogue wave or higher order rogue wave solution of the hyperbolic-elliptic system.

GENELLEŞTİRİLMİŞ
DAVEY-STEWARTSON SİSTEMİNDE
DEV DALGALAR

ÖZET

Bu çalışmada, akustikte türetilen genelleştirilmiş Davey-Stewartson (GDS) sistemindeki dev dalgaları araştırdık;

$$\begin{aligned} iu_\zeta &= u_{\xi\xi} + \gamma_1 u_{\eta\eta} + \gamma_2 u_{\xi\eta} + u(\beta_1\partial_\xi + \beta_2\partial_\eta)v + \chi|u|^2 u, \\ \alpha_{11}v_{\xi\xi} + \alpha_{22}v_{\eta\eta} &= -2(\beta_1\partial_\xi + \beta_2\partial_\eta)|u|^2, \end{aligned} \tag{1}$$

Burada u karmaşık zarf fonksiyonu, v reel ortalama alan, ζ zaman koordinatı ve $(\xi,\, \eta)$ uzaysal koordinatları belirtmektedir. Bu sistem katsayılar arası ilişkiye bağlı olarak iki gruba ayrılabilir: İlki $\gamma_2^2 - 4\gamma_1 < 0$ ve $\alpha_{11}\alpha_{22} < 0$ durumudur ki genelleştirilmiş Davey-Stewartson I (GDSI) veya eliptik-hiperbolik GDS sistemi olarak adlandırılır. Diğeri ise $\gamma_2^2 - 4\gamma_1 > 0$ ve $\alpha_{11}\alpha_{22} > 0$ eşitsizliklerinin sağlanması durumundaki sistemdir. Bu denklem hiperbolik-eliptik GDS olarak adlandırılır ve GDSII ile gösterilir. Öte yandan, (1) denkleminde $u_{\xi\eta}$, v_η ve $(|u|^2)_\eta$ terimleri yoksayıldığında GDS sistemi, Davey-Stewartson (DS) sistemine dönüşür.

Bu tezde, özellikle GDSI sistemindeki dev dalgalar incelendi. Dev dalgalar, eskiden beri üzerinde durulan bir araştırma konusudur. Bu dalgalar aniden maksimum genlikle oluşup aniden kaybolan dalgalar olarak tanımlanmaktadır. Okyanusta, plazmada, akustikte ve optikte dev dalgalara rastlanmış ve matematiksel olarak modellenmiştir. Dev dalgalar ilk olarak Peregrine tarafından doğrusal olmayan Schrödinger denkleminin çözümü sırasında bulunmuş ve çözüm Peregrine solitonu olarak adlandırılmıştır. Sonraki çalışmalarda dev dalgaların oluşumuna yönelik sebepler araştırılmış ve doğrusal olmama durumunun yanısıra, değişimsel kararsızlığın da etkisinin olduğu sonucuna ulaşılmıştır. Değişimsel kararsızlık güçlü taşıyıcı dalga ve kenar bantlarının etkileşimi sonucu oluşmaktadır. Dev dalgaların oluşum sebebinin yanısıra matematikçiler dev dalgaların sınıflandırılması ile ilgili de çalışmalar yürütmüşlerdir. Ohta ve Yang okyanusta oluşan dev dalgalar üzerinde çalışmış ve DS denklemlerine odaklanarak oluşan, dev dalgaları 3 sınıfa ayırmıştır: Temel dev dalga, çoklu dev dalga ve yüksek mertebe dev dalga. Temel dev dalga, doğru formunda oluşur ve (x, y) düzlemi boyunca ilerler. Ara zamanlarda genliği maksimuma ulaşıp, sonrasında dalga kaybolur ve durağan bir yapı ortaya çıkar. Çoklu dev dalga, birden fazla temel dev dalganın birleşiminden oluşur. Ara bir zamanda bu dalgalar birbiriyle etkileşir ve dalgaların kesişim alanı maksimum genliğe sahip olur. Sonrasında dalgalar birbirinden ayrılmaya başlar ve ayrı ayrı maksimum genliğe ulaşır. Zaman ilerledikçe de dalgalar kaybolur ve durağan bir yapı ortaya çıkar. Yüksek mertebe dev dalgalar, temel dev dalgalar ve çoklu dev dalgalardan farklı bir yapı gösterirler. Çünkü dalgalar tamamen ortadan kaybolmaz ve durağan bir yapı elde edilmez. Dalgaların ufak bir parçası hala belirgindir.

Mevcut çalışmada da Ohta ve Yang'ın yöntemine bağlı kalınarak dev dalga sınıflandırması kullanılmış ve oluşan dev dalgaların zamana bağlı değişimi açıklanmaya çalışılmıştır.

Dev dalga çözümü için öncelikle Hirota direkt yöntemi kullanılarak eliptik-hiperbolik GDS sisteminin

$$u = \frac{\sqrt{2\gamma_1}}{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}\frac{G}{F} \quad , v = \frac{2-\beta_2}{\beta_1}\xi + \eta + \frac{4\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}(\beta_1\partial_\xi + \beta_2\partial_\eta)\log F,$$

dönüşümü ile bilineer formu

$$\begin{aligned} &(-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta)GF = 0, \\ &(\alpha_{11}D_\xi^2 + \alpha_{22}D_\eta^2)(FF) = -2|G|^2 + 2F^2, \end{aligned} \tag{2}$$

elde edilmiştir. Burada G karmaşık değerli ve F reel değerli fonksiyonlardır. Ayrıca bilineer formdaki katsayılar arasındaki bağıntılar

$$\gamma_2 = \frac{4\beta_1\beta_2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1} \quad \alpha_{11} = \frac{\beta_2^2-\beta_1^2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1} \quad \alpha_{22} = -\gamma_1\left(\frac{\beta_2^2-\beta_1^2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}\right)$$

ile verilmektedir. Daha sonra, $N \times N$ boyutlu $\tau_n = \det\left(m_{ij}^{(n)}\right)$ determinant olmak üzere, GDSI denkleminin rasyonel çözümleri $F = \tau_0$ ve $G = \tau_1$ için bulunmuştur. p_i karmaşık bir sabit olmak üzere τ_n determinantının girdileri aşağıda verilen ifade yardımı ile hesaplanmaktadır.

$$\begin{aligned} m_{ij}^{(n)} &= \sum_{k=0}^{n_i} c_{ik}(p_i\partial_{p_i} + \mu_i^{'} + n)^{n_i-k} \times \sum_{l=0}^{n_j} c_{jl}^*(p_j^*\partial_{p_j^*} + (\mu_j^{'})^* - n)^{n_j-l}\frac{1}{p_i+p_j^*} \\ \mu_i^{'} &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2}\left(\frac{1}{p_i(\beta_2+\beta_1\sqrt{\gamma_1})} - \frac{p_i}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\xi \\ &+ \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2\sqrt{\gamma_1}}\left(\frac{1}{p_i(\beta_2+\beta_1\sqrt{\gamma_1})} + \frac{p_i}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\eta + \frac{p_i^2+p_i^{-2}}{\sqrt{-1}}\zeta. \end{aligned}$$

Bu rasyonel çözümlerin türetilmesinde

$$\begin{aligned} x_1 &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2(\beta_2-\beta_1\sqrt{\gamma_1})}\left(-\xi + \frac{1}{\sqrt{\gamma_1}}\eta\right) \\ x_{-1} &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2(\beta_2+\beta_1\sqrt{\gamma_1})}\left(-\xi - \frac{1}{\sqrt{\gamma_1}}\eta\right) \\ x_2 &= -\frac{i\zeta}{2} \qquad\qquad x_{-2} = \frac{i\zeta}{2}, \end{aligned}$$

dönüşümüyle (2) formuna indirgenen yardımcı bilineer form kullanılmıştır.

$$\begin{aligned} &(D_{x_1}D_{x_{-1}} - 2)FF = -2|G|^2 \\ &(D_{x_1}^2 - D_{x_2})GF = 0 \\ &(D_{x_{-1}}^2 + D_{x_{-2}})GF = 0. \end{aligned}$$

Bulunan rasyonel çözümler kullanılarak temel, çoklu ve yüksek mertebe dev dalga çözümleri oluşturularak zamana bağlı değişimleri grafiklerle ifade edilmiştir ve grafiklerde gözlemlenen özellikler yorumlanmıştır. Sonuç olarak DS sisteminde olduğu gibi GDSI sisteminde de aynı tipte dev dalgalar gözlemlenebilmiştir.

Bu tez, 4 ayrı bölümden oluşmuş olup, giriş bölümünde genelleştirilmiş Davey-Stewartson sisteminin içerdiği doğrusal olmayan Schrödinger denkleminden bahsedilmiştir. Ardından genelleştirilmiş Davey-Stewartson sistemi için elastik ve akustik ortamlarda türetilmiş denklem örnekleri paylaşılmıştır. Sonrasında dev dalgaların oluşumu ile ilgili araştırmalara yer verilmiştir ve dev dalgaların oluşum sebepleri ve oluşabilecek dev dalga çeşitleri üzerinde durulmuştur.

İkinci bölümde, dev dalga çözümlerinin bulunmasında kullanılacak olan Hirota direkt yöntemi üzerinde durulmuştur. Öncelikle Hirota bilineer operatörü tanımlanmış ve Hirota bilineer formu elde etmek için uygulanabilecek 3 (logaritmik, bi-logaritmik ve rasyonel dönüşümler) tip dönüşümden bahsedilmiştir. Sonrasında örnek olarak bu yöntemle doğrusal olmayan Schrödinger denkleminin 1-soliton ve 2-soliton çözümleri verilmiştir.

Üçüncü bölümde akustikte türetilmiş olan eliptik-hiperbolik genelleştirilmiş Davey-Stewartson sistemi verilmiş ve birimsiz büyüklükteki değişken dönümüşüyle boyutsuzlaştırılmıştır. Sonra Hirota bilineer formu bulunarak belli katsayı kısıtları altında GDSI sistemindeki dev dalga çözümleri elde edilmiştir. Çözümler matris determinantları üzerinden bulunmuştur. Daha sonra her bir dev dalga türü için (Temel dev dalga, çoklu dev dalga ve yüksek mertebe dev dalga) zamana bağlı çözüm grafiği çizilmiş ve dev dalgaların ne tür özelliklere sahip olduğundan bahsedilmiştir.

Sonuç bölümünde ise elde edilen sonuçlar özetlenmiştir. Ayrıca hiperbolik-eliptik genelleştirilmiş Davey-Stewartson sisteminde görülen ve çoklu dev dalga çözümleri veya yüksek mertebe dev dalga çözümlerinden türeyen özel durumlarda ortaya çıkan patlama yapan dev dalgalara da vurgu yapılmıştır. Bir örnek üzerinden patlama yapan dev dalga açıklanmıştır.

1. INTRODUCTION

Waves exist all around us and the most obvious examples being sound, light and water waves. They are generally modelled mathematically using partial differential equations (PDE), and constitute a time evolution phenomenon. Thus, they can occur in various scientific and engineering disciplines such as fluid mechanics, structural mechanics, optics, quantum mechanics, electromagnetism, solid mechanics, etc. Moreover, most people have a general notion of what a wave is, based on their everyday experience. Thus, the term 'wave' is not an easy concept to define. On the other hand, the mathematical model covers the fundamental theory of both linear and non-linear waves. One of the important differences between these is that in non-linear systems shock waves appear. The nonlinear Schröndinger (NLS) equation is a well-known example for non-linear waves.

NLS is a dispersive nonlinear PDE, so there is a balance between dispersive and nonlinear effects. Due to this, solutions of NLS are solitons. It was derived from another nonlinear differential equation called Korteweg deVries (KdV) [1]. Since many years, the NLS equation has been used to analyze physical phenomena in a variety of field of physics such as optics, acoustics, fluids and electromagnetism etc. The importance of NLS Equations has been emphasized in several fields such as superconductivity, superfluidity and electromagnetism since 1950 [2]. Apart form its essence in physics, mathematically, weakly nonlinear dispersive and energy-preserving systems may turn to the NLS equation. Therefore, Benney and Newell (1967) characterize the NLS equations as universially accepted [2].

Ablowitz and Prinari explain the NLS system in two types: Continuous or discrete. Discrete NLS (dNLS) system consists of Discrete (1+1) and scalar discrete (2+1) dimensional sytems [2]. For instance,

$$iA_{n,t} + \frac{1}{a^2}(A_{n+1} - 2A_n + A_{n-1}) + \gamma|A_n|^2 A_n = 0. \tag{1.1}$$

is an example of dNLS equation where A_n is complex wave amplitude, γ is an anharmonic parameter and $n = 0, \pm1, \pm2, \pm3, ..$ or $n = 1, 2, 3, ..., k$ for which $A_{n+k} = A_n$ [3, 4].

On the other hand, continuous NLS system includes, scalar (1+1) dimensional and multi-dimensional systems, vector (1+1) dimensional and multi-dimensional systems, and special equations, that is, Benny-Rockes and Davey-Stewartson (DS) equations.

The scalar multi-dimensional NLS equation is given by

$$iA_t + \Delta_\gamma A \pm 2|A|^2 A = 0, \tag{1.2}$$

where A is the complex amplitude of the wave propagation in $(\mathbb{R}^n \times \mathbb{R})$ and $\Delta_\gamma = \gamma_1 \partial^2_{x_1 x_1} + \gamma_2 \partial^2_{x_2 x_2} + \cdots + \gamma_n \partial^2_{x_n x_n}$, beside t is the time coordinate and $(x_1,\, x_2,\, \cdots,\, x_n)$ is the spatial coordinate. Similarly, the equation

$$\begin{aligned} &iA_{1,t} + A_{1,xx} + \gamma A_{1,yy} + \chi(|A_1|^2 + |A_2|^2)A_1 = 0, \\ &iA_{2,t} + A_{2,xx} + \gamma A_{2,yy} + \chi(|A_1|^2 + |A_2|^2)A_2 = 0 \end{aligned} \tag{1.3}$$

can be considered as a vector (2+1) dimensional system. System (1.3) is known as the coupled NLS system and occurs if two short transverse waves co-exist in a medium.

Another continuous model for NLS type equation is the DS equations. The simplest form of the DS equations is that

$$\begin{aligned} &iA_t + \gamma A_{xx} + A_{yy} = \chi|A|^2 A + \beta A Q_x, \\ &Q_{xx} + \alpha Q_{yy} = (|A|^2)_x, \end{aligned} \tag{1.4}$$

where A is complex wave amplitude and Q is real field. It can be divided into two type: DSI and DSII equations when the respective sign of the coefficients $(\gamma,\, \alpha)$ in (1.4) are $(+,\, -)$ and $(-,\, +)$.

In this study, we consider the expanded form of the DS system (1.4). In the recent studies, there are two expended forms corresponding to the derivation in different media. However both authors call the expanded system the generalized DS system.

In the first study, Generalized Davey-Stewartson (GDS) equation was derived in acoustics [5]. The equation was given as

$$
\begin{aligned}
&i\, A_t = \gamma_{11}A_{xx} + \gamma_{22}A_{yy} + \gamma_{12}A_{xy} + \chi\, |A|^2 A + A(\beta_1\partial_x + \beta_2\partial_y)Q, \\
&\alpha_{11}Q_{xx} + \alpha_{22}Q_{yy} = -2(\beta_1\partial_x + \beta_2\partial_y)|A|^2,
\end{aligned} \tag{1.5}
$$

where A is the complex envelope function and Q is the mean field. The GDS equation is classified in two types depending on the relationship between the coefficients. $(1.5)_1$ is **elliptic** if $\gamma_{12}^2 - 4\gamma_{11}\gamma_{22} < 0$, meanwhile $(1.5)_2$ is **hyperbolic** if $\alpha_{11}\alpha_{22} < 0$. Thus the GDS system for $\gamma_{12}^2 - 4\gamma_{11}\gamma_{22} < 0$ and $\alpha_{11}\alpha_{22} < 0$ may be called the elliptic-hyperbolic GDS (GDSI) system and is reduced to the DSI equations when $\gamma_{12} = \beta_2 = 0$. At the same time, $(1.5)_1$ is **hyperbolic** if $\gamma_{12}^2 - 4\gamma_{11}\gamma_{22} > 0$, and $(1.5)_2$ is **elliptic** if $\alpha_{11}\alpha_{22} > 0$. In this situation, the GDS equations for $\gamma_{12}^2 - 4\gamma_{11}\gamma_{22} > 0$ and $\alpha_{11}\alpha_{22} > 0$ may be called the hyperbolic-elliptic GDS (GDSII) system and is reduced to the DSII equations when $\gamma_{12} = \beta_2 = 0$.

In the other study, Babaoğlu and Erbay derived Generalized Davey-Stewartson Equaion in elastic medium with (2+1) dimensional waves propagation [6]:

$$
\begin{aligned}
&iA_t + pA_{xx} + rA_{yy} = q|A|^2A + \frac{k^2}{2w}(\gamma_3 Q_{1,x} + \gamma_1 Q_{2,y})A, \\
&(c_g^2 - c_1^2)Q_{1,xx} - c_2^2 Q_{1,yy} - (c_1^2 - c_2^2)Q_{2,xy} = \gamma_3 k^2(|A|^2)_x, \\
&(c_g^2 - c_2^2)Q_{2,xx} - c_1^2 Q_{2,yy} - (c_1^2 - c_2^2)Q_{1,xy} = \gamma_1 k^2(|A|^2)_y,
\end{aligned} \tag{1.6}
$$

where A is the complex amplitude of the free short transverse wave mode and Q_1 and Q_2 are free long longitudinal and long transverse wave modes. The coefficients denote for c_g as group velocity of transverse waves; c_1 as phase speed of longitudinal wave and c_2 as phase speed of transverse wave; k as the wave number and w as the frequency. System (1.6) is reduced to the DS system under the transformation $Q = Q_{1,x} + Q_{2,y} - \gamma_1|A|^2/(c_g^2 - c_1^2)$.

Exact solutions which have specific properties, such as shock wave, homoclinic structures, rouge waves, etc, is one of the biggest resource areas in the nonlinear wave theory. The solutions of NLS type equations can usually be found by two methods: Inverse-Scattering Transform and Direct Method. In this study, we deal with GDS equations (1.5) in acoustics and investigate the rogue waves by using direct method with determinants.

Before focusing on rogue waves, we need to explain what a soliton is since they are related. Soliton is defined as a solitary wave maintaining its shape after the collision with a wave of the same type. They were first observed by John Scott Russel in 1834 and investigated mathematically by Korteweg deVries in 1895 by modeling shallow water waves through KdV equation.

Rogue waves are called freak waves or giant waves which "appear from nowhere and disappear without a trace" ([7], p.1), [8]. The reason for obtaining this kind of waves could be due to the fact that initial conditions might grow up exponentially and reach very high amplitudes. Peregrine was the first one obtaining fundamental rogue wave in the NLS equation $iA_t + A_{xx} + 2|A|^2A = 0$ in 1983 [9, 10]. The solution is also called Peregrine soliton. The interesting issue was for Peregrine soliton was that it had both spatial and temporal localizations as follows [9]:

$$|A|^2 = \beta^2 + 2\frac{p(t) - \xi^2}{(p(t) + \xi^2)^2}, \tag{1.7}$$

where

$$p(t) = \frac{1}{4\beta^2} + (2\beta t)^2, \qquad \xi = x - 2\alpha t. \tag{1.8}$$

where β and α are real constants.

According to previous research, rogue waves may appear in oceans, in plasma [11], in acoustics [12] and also in the field of optics as found in 2007 [13]. In fact that the criteria for considering a wave as rogue wave is to determine the ratio of the amplitude of the wave height to significant wave height [8]. If the ratio is greater than 2 then the wave is called as rogue wave. However, there were the cases for which the ratio was not satisfied [8].

Many studies dealt with the reasons of obtaining rogue waves. It might depend on linear or nonlinear mechanisms but a study showed that due to the large amplitudes of rogue waves, the nonlinearity has an important contribution [8]. Peregrine and other researchers considered modulational instability to be the reason for the formation of the rogue waves [9, 14, 15]. Zakharov and Ostrovsky explained the occurrence of modulation instability as a result of interaction of two waves; one is strong carrier harmonic wave and the other is sidebands [16].

In a study about how rogue waves are formed in oceans, one dimensional NLS equation was considered and the reason of rogue waves in the ocean were considered as Akhmediev Breathers which is the name of modulational instability in water waves. [15]. Therefore, when the collision of two 'Akhmediev breathers' was investigated, the highest amplitude was obtained [7]. In another study [8] in hyrodynamics, researchers investigated the source of rogue waves and analyzed modulation instability, naming the modulation instability as Benjamin-Feir instability. When optical systems were analyzed in terms of forming rogue waves, it was investigated that modulational instabilities were convective rogue waves that occurred in optical systems and in order to define this term we need first to define absolute instability. "An instability is **absolute** if the dispersion relation has an unstable saddle point and the saddle point satisfies the pinching condition. An instability is convective if it is not absolute." ([8], p.7) Based on the definition given above, convective instability was considered as the instability which is not absolute [8].

Other than the source of rogue waves, recently, Ohta & Yang investigated rogue waves in the DSI and DSII equations and they found that rational solution of the equations are responsible for forming rogue waves under specific conditions [17, 18]. Also, they classified rogue waves in three forms: Fundamental Rogue Waves, Multi-rogue Waves and Higher-Order Rogues Waves. Fundamental Rogue waves are in the forms of lines and go in the direction of (x, y) plane and therefore, they are also called line rogue waves. In the intermediate times the amplitude of this wave reaches its maximum. Multirogue waves are the interaction of multiple fundamental rogue waves. In this type of waves, in the intermediate times fundamental rogue waves interact with each other and disappear into the background again. On the other hand, in the intermediate times higher-order rogue waves act as follows: different than multi rogue waves, these waves cannot approach to the constant background uniformly as $t \to \infty$. Only some parts of the waves approach to the constant background uniformly.

In the current study, the rogue wave solutions for the elliptic-hyperbolic GDS system (GDSI) in acoustics is obtained via direct method along with determinants. Specifically, in Chapter 2, we introduce the Hirota Direct method that is, the definition of bilinear operator, the bilinearization techniques and obtain a solution of Nonlinear Schrödinger Equation by direct method. In Chapter 3, we deal with the GDSI system (1.5) and summarize the rogue waves solution method by determinants along with direct method which was demonstrated by Ohta and Yang [17, 18]. After that the rogue wave solutions for the elliptic-hyperbolic GDS (GDSI) equation derived in acoustics, is shared in terms of the classification, fundamental rogue wave, multi rogue wave and higher order rogue wave. In Chapter 4, that is the conclusion part, we discuss and summarize the findings. Additionally, we discuss that the blow up solution of the hyperbolic-elliptic GDS (GDSII) system leads to exploding rogue wave which can be derived from the multi-rogue wave or higher order rogue wave solution.

2. HIROTA DIRECT METHOD

In this chapter, Hirota Direct Method will be explained with its properties and it wil be applied to (1+1) scalar NLS equation.

2.1 Hirota Direct Method

Hirota Direct Method was introduced by Ryogo Hirota in 1971 for finding soliton solutions to integrable nonlinear differential equations. This method operates by first linearizing the nonlinear differential equations and then solving the bilinear form via various methods such as via perturbation, via Wronskian determinants of block matrices [19, 20]. There are many methods for solving nonlinear differential equations. However, this method became superior to other methods by being algebraic rather than analytic and helping to get solutions in a quick way for instance for KdV, modified KdV (mKdV), sine-Gordon and NLS equations [21].

2.1.1 Hirota D-Operator and Its Properties

In order to bilinearize a nonlinear differential equation, a differential operator D was introduced:

$$D_x^a D_y^b[fg] = (\partial_x - \partial_{x'})^a(\partial_y - \partial_{y'})^b(f(x,y)g(x',y'))|_{x'=x,y'=y}. \tag{2.1}$$

We can write some of the Hirota derivatives based on the given expression as follows:

$$\begin{aligned} D_x(fg) &= (\partial_x - \partial_{x'})(f(x,y)g(x',y'))|_{x'=x,y'=y} = f_x g - f g_x, \\ D_x^2(fg) &= (\partial_x - \partial_{x'})^2(f(x,y)g(x',y'))|_{x'=x,y'=y} = f_{xx} g - 2f_x g_x + f g_{xx}, \end{aligned}$$

$$
\begin{aligned}
D_x^3(fg) &= (\partial_x - \partial_{x'})^3(f(x,y)g(x',y'))|_{x'=x,y'=y} \\
&= f_{xxx}g - 3f_{xx}g_x + 3f_xg_{xx} - fg_{xxx}, \\
D_xD_y(fg) &= (\partial_x - \partial_{x'})(\partial_y - \partial_{y'})(f(x,y)g(x',y'))|_{x'=x,y'=y} \\
&= f_{xy}g - f_xg_y - f_yg_x + fg_{xy}. \qquad (2.2)
\end{aligned}
$$

In addition, Hirota D-operator has the following properties derived from the definition:

$$
\begin{aligned}
& D_x(f1) = \partial_x f, \\
& D_x^a(ff) = 0, \\
& D_x^a(fg) = (-1)^a D_x^a(gf), \\
& \partial_x^2 log(F) = \frac{D_x^2(FF)}{2F^2}. \qquad (2.3)
\end{aligned}
$$

2.1.2 Bilinearization Methods

In order to obtain Hirota bilinear form from nonlinear differential equations, three kinds of transformations could be maintained:

2.1.2.1 Logarithmic Transformation

The logarithmic transformation of a nonlinear differential equation can be done in the following way: Let A be the solution of a nonlinear differential equation. Then the transformation will be

$$A = 2(\log F)_{xx} \qquad (2.4)$$

in this form. As an example, we can bilinearize the KdV equation with this method: This equation is given as

$$A_t + 6AA_x + A_{xxx} = 0. \qquad (2.5)$$

For the bilinearizition $A = 2(\log F)_{xx}$ can be used. First the the solution A was written in terms of a new variable w as follow: $A = w_x$ with $w = 2(\log F)_x$ Then the KdV equation integrated once with respect to x and

$$w_t + 3w_x^2 + w_{xxx} = c \qquad (2.6)$$

as c being a constant, is obtained. Then, by the substitution of w into the equation,

$$(D_x^4 + D_x D_t)FF = cF^2 \tag{2.7}$$

can be obtained as the Hirota Bilinear form of KdV Equation.

2.1.2.2 Bilogarithmic Transformation

Another transformation is bilogarithmic transformation which can be done by the following substitution for the solution A of a nonlinear differential equation:

$$A = \log\left(\frac{G}{F}\right). \tag{2.8}$$

As an example we can use this transformation for the mKdV equation which can be represented as

$$A_t + 6A^2 A_x + A_{xxx} = 0. \tag{2.9}$$

First A is written in terms of a new variable w as $A = iw_x$. Then the mKdV equation turns into the following equation:

$$w_{xt} - 6w_x^2 w_{xx} + w_{xxxx} = 0. \tag{2.10}$$

By integrating once with respect to x, we get

$$w_t - 2w_x^3 + w_{xxx} = c. \tag{2.11}$$

Now, we can use the bilogarithmic transformation as $w = \log(G/F)$ and obtain the following bilinear form for mKdV:

$$\begin{aligned}(D_t + D_x^3)GF &= 3\lambda D_x GF,\\ D_x^2 GF &= \lambda GF.\end{aligned} \tag{2.12}$$

2.1.2.3 Rational Transformation

The rational transformation of a nonlinear differential equation can be done in the following way: Let A be the solution of a nonlinear differential equation. Then the transformation will be

$$A = \frac{G}{F} \tag{2.13}$$

in this form. As an example, we can bilinearize the KdV equation with rational transformation. By the substitution of G/F into the equation (2.5),

$$\begin{aligned}(D_t + D_x^3)GF = 3\lambda D_x GF,\\ D_x^2 - 2GF = \lambda F^2.\end{aligned} \tag{2.14}$$

can be obtained as the Hirota Bilinear form of the KdV equation.

2.2 Solution of a NLS Equation by Hirota Direct Method via Perturbation

The NLS equation can be taken as follows:

$$iA_t + A_{xx} + 2c|A|^2 A = 0, \qquad\qquad c = \pm 1, \tag{2.15}$$

where A is a complex valued function. Via the transformation $A = G/F$ where G is complex valued and F is real valued functions, the NLS equation turns out to the following Hirota Bilinear form:

$$\begin{aligned}P_1(D)(GF) = (iD_t + D_x^2)(GF) = 0,\\ P_2(D)(FF) = D_x^2(FF) = 2c|G|^2.\end{aligned} \tag{2.16}$$

Now we will consider the following expansions for G and F

$$\begin{aligned}G &= g_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + \cdots,\\ F &= f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots\end{aligned} \tag{2.17}$$

and plug them in (2.16). This leads to a hierarchy of perturbation equations for like powers of ϵ:

$$\begin{aligned}\epsilon^0 \quad : \quad & P_1(D)(f_0 g_0) = 0,\\ & P_2(D)(f_0^2) = 2c|g_0|^2.\end{aligned} \tag{2.18}$$

Thus, for $\mathcal{O}(\epsilon^0)$, we get $g_0 = 0$ and $f_0 \in \mathbb{R}$. Then, we have

$$
\begin{aligned}
\epsilon \quad &: \quad P_1(D)(f_0 g_1) = 0, \\
&\quad\;\; 2P_2(D)(f_0 f_1) = 0. && (2.19) \\
\epsilon^2 \quad &: \quad P_1(D)(f_1 g_1 + f_0 g_2) = 0, \\
&\quad\;\; P_2(D)(f_1^2 + 2f_0 f_2) = 2c|g_1|^2. && (2.20) \\
\epsilon^3 \quad &: \quad P_1(D)(f_2 g_1 + f_1 g_2 + f_0 g_3) = 0, \\
&\quad\;\; P_2(D)(2f_1 f_2 + 2f_0 f_3) = 2c(g_1^* g_2 + g_1 g_2^*). && (2.21) \\
\epsilon^4 \quad &: \quad P_1(D)(f_3 g_1 + f_2 g_2 + f_1 g_3 + f_0 g_4) = 0, \\
&\quad\;\; P_2(D)(f_2^2 + 2f_1 f_3 + 2f_0 f_4) = 2c(|g_2|^2 + g_1^* g_3 + g_1 g_3^*). && (2.22)
\end{aligned}
$$

Note that $*$ denotes the complex conjugate of the related term. For the higher order perturbation equations, we need to determine the form of g_1 whether represents one-soliton or a combination of solitons.

2.2.1 One-Soliton Solution of the NLS Equation

In the previous section, it is obtained that the NLS equation has the trivial solution (Vacuum soliton) for $G = 0$ and $F = 1$ (also $f_0 = 1$). In order to find one-soliton solution, let us first try $g_1 = G_1 \exp(\mu)$ where $\mu = \mu_1 x + \mu_2 t + \mu_0$; and G_1, μ and μ_i $(i = 0, 1, 2)$ are complex constants. Using $(2.19)_1$, we get

$$
(i\mu_2 + \mu_1^2) G_1 e^{\mu} = 0, \tag{2.23}
$$

which implies that $\mu_2 = i\mu_1^2$. On the other hand, thanks to $(2.19)_2$, we find $f_{1,xx} = 0$, so $f_1 = 0$. Now let us use $(2.20)_3$: The equations

$$
\begin{aligned}
& i g_{2,t} + g_{2,xx} = 0, \\
& f_{2,xx} = c|G_1|^2 \, \mathrm{e}^{\mu + \mu^*}
\end{aligned} \tag{2.24}
$$

imply that $g_2 = 0$ and $f_2 = \dfrac{c|G_1|^2}{(\mu_1 + \mu_1^*)^2} \exp(\mu + \mu^*)$. Now set $G_1 = 1$ and $\epsilon = 1$ to get the one-soliton solution for the NLS equation,

$$
A = \frac{\mathrm{e}^{\mu}}{1 + \dfrac{c}{\mu_1 + \mu_1^*} \, \mathrm{e}^{\mu + \mu^*}}. \tag{2.25}
$$

2.2.2 Two-Soliton Solution of the NLS Equation

For the two-soliton solution, let us try $g_1 = e^{\mu} + e^{\eta}$ where $\mu = \mu_1 x + \mu_2 t + \mu_0$ and $\eta = \eta_1 x + \eta_2 t + \eta_0$. When we plug g_1 into (2.19), we get

$$i(\mu_2 e^{\mu} + \eta_2 e^{\eta}) + \mu_1^2 e^{\mu} + \eta_1^2 e^{\eta} = 0. \tag{2.26}$$

Therefore, $\mu_2 = i\mu_1^2$ and $\eta_2 = i\eta_1^2$. The next order perturbation equations $(2.20)_3$ and $(2.20)_4$ imply

$$\begin{aligned}
& ig_{2,t} + g_{2,xx} = 0, \\
& f_{2,xx} = c(\mathrm{e}^{\mu} + \mathrm{e}^{\eta})(\mathrm{e}^{\mu^*} + \mathrm{e}^{\eta^*}). && (2.27)\\
\Rightarrow\quad & g_2 = 0, \\
& \frac{f_2}{c} = \frac{1}{\mu_1 + \mu_1^*}\mathrm{e}^{\mu+\mu^*} + \frac{1}{\mu_1 + \eta_1^*}\mathrm{e}^{\mu+\eta^*} + \frac{1}{\eta_1 + \mu_1^*}\mathrm{e}^{\eta+\mu^*} + \frac{1}{\eta_1 + \eta_1^*}\mathrm{e}^{\eta+\eta^*}. && (2.28)
\end{aligned}$$

By using $(2.21)_5$ and $(2.21)_6$, we have

$$\begin{aligned}
& ig_{3,t} + g_{3,xx} + c|g_1|^2 g_1 - ig_1 f_{2,t} - 2f_{2,x} g_{1,x} = 0, \\
& f_{3,xx} = 0,
\end{aligned} \tag{2.29}$$

which lead to $f_3 = 0$ and

$$ig_{3,t} + g_{3,xx} = \frac{2c(\mu_1 - \eta_1)^2}{(\mu_1 + \eta_1^*)(\eta_1 + \eta_1^*)}\mathrm{e}^{\mu+\eta+\eta^*} + \frac{2c(\mu_1 - \eta_1)^2}{(\mu_1 + \mu_1^*)(\eta_1 + \mu_1^*)}\mathrm{e}^{\mu+\mu^*+\eta}. \tag{2.30}$$

Then, setting $g_3 = B_1 \exp(\mu + \eta + \eta^*) + B_2 \exp(\mu + \mu^* + \eta)$ gives

$$B_1 = \frac{c(\mu_1 - \eta_1)^2}{(\mu_1 + \eta_1^*)^2(\eta_1 + \eta_1^*)^2} \quad \text{and} \quad B_2 = \frac{c(\mu_1 - \eta_1)^2}{(\mu_1 + \mu_1^*)^2(\eta_1 + \mu_1^*)^2}. \tag{2.31}$$

For $\mathcal{O}(\epsilon^4)$, simplifying equation (2.22), we obtain

$$\begin{aligned}
& g_{4,t} + g_{4,xx} = 0, \\
& f_{4,xx} + cf_2|g_1|^2 - c(g_1^* g_3 + g_1 g_3^*) - f_{2,x}^2 = 0, . && (2.32)\\
\Rightarrow\quad & g_4 = 0, \\
& f_4 = \frac{c^2(\mu_1 - \eta_1)^2(\mu_1^* - \eta_1^*)^2}{(\mu_1 + \mu_1^*)^2(\eta_1 + \mu_1^*)^2(\mu_1 + \eta_1^*)^2(\eta_1 + \eta_1^*)^2}\,\mathrm{e}^{\mu+\mu^*} + \eta + \eta^*. && (2.33)
\end{aligned}$$

Therefore, the two-soliton solution for the NLS equation is $A = G/F$, where

$$
\begin{aligned}
G &= e^{\mu} + e^{\eta} + \frac{c(\mu_1 - \eta_1)^2\, \mathrm{e}^{\mu+\eta+\eta^*}}{(\mu_1 + \eta_1^*)^2(\eta_1 + \eta_1^*)^2} + \frac{c(\mu_1 - \eta_1)^2\, \mathrm{e}^{\mu+\mu^*+\eta}}{(\mu_1 + \mu_1^*)^2(\eta_1 + \mu_1^*)^2} \\
F &= 1 + \frac{c\, \mathrm{e}^{\mu+\mu^*}}{\mu_1 + \mu_1^*} + \frac{c\, \mathrm{e}^{\mu+\eta^*}}{\mu_1 + \eta_1^*} + \frac{c\, \mathrm{e}^{\eta+\mu^*}}{\eta_1 + \mu_1^*} + \frac{c\, \mathrm{e}^{\eta+\eta^*}}{\eta_1 + \eta_1^*} \\
&+ \frac{c^2(\mu_1 - \eta_1)^2(\mu_1^* - \eta_1^*)^2\, \mathrm{e}^{\mu+\mu^*+\eta+\eta^*}}{(\mu_1 + \mu_1^*)^2(\eta_1 + \mu_1^*)^2(\mu_1 + \eta_1^*)^2(\eta_1 + \eta_1^*)^2}.
\end{aligned} \tag{2.34}
$$

Remark: In this section, soliton solutions for NLS are presented by perturbating the bilinear form. However, to find rogue wave solutions, we will consider a different approach which is used for the DS system in [17, 18]. In fact, rogue wave solutions for NLS is also derived by this approach in [22]. We will use the direct method along with the determinant of matrices which will be explained in the next chapter. Therefore, the solution of the GDS equation will consist of determinants of matrices.

3. ROGUE WAVE SOLUTIONS FOR THE GDS EQUATIONS

In this section, we will consider the GDS system, which is given by

$$\begin{aligned} &i\ A_t = \gamma_{11}A_{xx} + \gamma_{22}A_{yy} + \gamma_{12}A_{xy} + \chi\ |A|^2 A + A(\beta_1\partial_x + \beta_2\partial_y)Q, \\ &\alpha_{11}Q_{xx} + \alpha_{22}Q_{yy} = -2(\beta_1\partial_x + \beta_2\partial_y)|A|^2, \end{aligned} \tag{3.1}$$

to find its rogue wave solutions by using the Hirota bilinear form.

3.1 Bilinear Forms

Before converting the GDS system (3.1) into the Hirota Bilinear form, it is appropriate to express this system in a dimensionless form. To do this, we first introduce the dimensionless variables:

$$\xi = x, \quad \eta = y, \quad \zeta = \gamma_{11}t, \quad A = \frac{1}{\sqrt{\gamma_{11}}}u, \quad Q = v. \tag{3.2}$$

Then, substituting the new variables into the equations (3.1) gives the dimensionless form of the GDS system

$$\begin{aligned} &iu_\zeta = u_{\xi\xi} + \gamma_1 u_{\eta\eta} + \gamma_2 u_{\xi\eta} + u(\overline{\beta_1}\partial_\xi + \overline{\beta_2}\partial_\eta)v + \overline{\chi}|u|^2 u, \\ &\alpha_{11}v_{\xi\xi} + \alpha_{22}v_{\eta\eta} = -2(\overline{\beta_1}\partial_\xi + \overline{\beta_2}\partial_\eta)|u|^2, \end{aligned} \tag{3.3}$$

where $\gamma_1 = \gamma_{22}/\gamma_{11}$, $\gamma_2 = \gamma_{12}/\gamma_{11}$, $\overline{\beta_1} = \beta_1/\gamma_{11}$, $\overline{\beta_2} = \beta_2/\gamma_{11}$ and $\overline{\chi} = \chi/\gamma_{11}^2$. Also, the DSI equation is obtained in the case that $\gamma_2 = \beta_2 = 0$, $\gamma_1 = 1$ and $\alpha_{11}\alpha_{22} < 0$, whereas DSII is obtained in the case $\gamma_2 = \beta_2 = 0$, $\gamma_1 = -1$ and $\alpha_{11}\alpha_{22} > 0$. Therefore, system (3.3) for $\gamma_2^2 - 4\gamma_1 < 0$ and $\alpha_{11}\alpha_{22} < 0$, the elliptic-hyperbolic GDS, may be called the GDSI system. Similarly, the hyperbolic-elliptic GDS system, where $\gamma_2^2 - 4\gamma_1 > 0$ and $\alpha_{11}\alpha_{22} > 0$, may be called the GDSII system. From now on, for the ease of flow, we eliminate the bar notation in $\overline{\beta_1}$, $\overline{\beta_2}$ and $\overline{\chi}$; and use them as β_1, β_2 and χ.

Now, in order to obtain Hirota bilinear form for (3.3), the variable transformations are defined by

$$u = a_1 \frac{G}{F} \quad \text{and} \quad v = a_2\xi + a_3\eta + a_4(\beta_1\partial_\xi + \beta_2\partial_\eta)\log F, \tag{3.4}$$

where G is complex valued and F is real-valued functions; and a_i, $(i = 1, \cdots, 4)$, are undetermined coefficients. Hence, equation (3.3) becomes

$$\begin{aligned}
&GF\left\{\left[\left(1 - \frac{a_4\beta_1^2}{2}\right) D_\xi^2 + \left(\gamma_1 - \frac{a_4\beta_2^2}{2}\right) D_\eta^2 + (\gamma_2 - a_4\beta_1\beta_2)D_\xi D_\eta\right](FF)\right. \\
&\qquad \left. -\chi a_1^2|G|^2\right\} = FF\left[-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta + a_2\beta_1 + a_3\beta_2\right](GF), \\
&(\beta_1\partial_\xi + \beta_2\partial_\eta)\left(\alpha_{11}\frac{D_\xi^2(FF)}{F^2} + \alpha_{22}\frac{D_\eta^2(FF)}{F^2}(FF) + \frac{4a_1^2}{a_4}\frac{|G|^2}{F^2}\right) = 0.
\end{aligned} \tag{3.5}$$

Then, equation $(3.5)_1$ turns to the simpler form when we set

$$\left[-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta + a_2\beta_1 + a_3\beta_2\right](GF) = (a_2\beta_1 + a_3\beta_2)GF,$$

which helps to separate $(3.5)_1$ according to GF and FF:

$$\begin{aligned}
&\left[-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta\right](GF) = 0, \\
&\left[\left(1 - \frac{a_4\beta_1^2}{2}\right) D_\xi^2 + \left(\gamma_1 - \frac{a_4\beta_2^2}{2}\right) D_\eta^2 + (\gamma_2 - a_4\beta_1\beta_2)D_\xi D_\eta\right](FF) \\
&\quad = \chi a_1^2|G|^2 + (a_2\beta_1 + \alpha_3\beta_2)(FF), \\
&(\beta_1\partial_\xi + \beta_2\partial_\eta)\left(\alpha_{11}\frac{D_\xi^2(FF)}{F^2} + \alpha_{22}\frac{D_\eta^2(FF)}{F^2}(FF) + \frac{4a_1^2}{a_4}\frac{|G|^2}{F^2}\right) = 0.
\end{aligned} \tag{3.6}$$

At this stage, we need to assume the following equations

$$\begin{aligned}
\alpha_{11} &= 1 - \frac{a_4\beta_1^2}{2}, \\
\alpha_{22} &= \gamma_1 - \frac{a_4\beta_2^2}{2}, \\
\chi &= -\frac{4}{a_4}, \\
a_4 &= \frac{\gamma_2}{\beta_1\beta_2},
\end{aligned} \tag{3.7}$$

One can observe that equation $(3.6)_3$ is directly satisfied under the assumptions (3.7). In this way, we obtain the bilinear form for equation (3.3)

$$\begin{aligned}
&(-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta)GF = 0, \\
&(\alpha_{11}D_\xi^2 + \alpha_{22}D_\eta^2)(FF) = \chi a_1^2|G|^2 + (a_2\beta_1 + a_3\beta_2)F^2.
\end{aligned} \tag{3.8}$$

In the next section, we will adapt the theorems in which is presented the rogue wave solutions of DSI [17] for the GDSI system. In fact, we first need to find the rational solutions for GDSI, (3.3) for $\gamma_1^2 - 4\gamma_1 < 0$ and $\alpha_{11}\alpha_{22} < 0$, since the rogue wave is also a rational wave. However, we still have the free coefficients a_1, a_2 and a_3. In addition, the restrictions (3.7) do not guarantee that equation (3.3) is in the elliptic-hyperbolic case. Therefore, we write more assumptions

$$\begin{aligned} \gamma_2 &= \frac{4\beta_1\beta_2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}, \quad \gamma_1 > 0, \quad \beta_2 - \beta_1\sqrt{\gamma_1} < 0, \\ a_1 &= \frac{\sqrt{2\gamma_1}}{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}, \quad a_2 = \frac{2-\beta_2}{\beta_1}, \quad a_3 = 1. \end{aligned} \tag{3.9}$$

Now, using the all restrictions (3.7) and (3.9) in (3.4) and (3.8) gives the transformation

$$u = \frac{\sqrt{2\gamma_1}}{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}\frac{G}{F}, \quad v = \frac{2-\beta_2}{\beta_1}\xi + \eta + \frac{4\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}(\beta_1\partial_\xi + \beta_2\partial_\eta)\log F, \tag{3.10}$$

and the final bilinear form

$$\begin{aligned} &(-iD_\zeta + D_\xi^2 + \gamma_1 D_\eta^2 + \gamma_2 D_\xi D_\eta)GF = 0, \\ &(\alpha_{11}D_\xi^2 + \alpha_{22}D_\eta^2)(FF) = \frac{2\chi\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}|G|^2 + 2F^2, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \gamma_2 &= \frac{4\beta_1\beta_2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}, \quad \alpha_{11} = \frac{\beta_2^2-\beta_1^2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}, \\ \alpha_{22} &= -\gamma_1\left(\frac{\beta_2^2-\beta_1^2\gamma_1}{\beta_2^2+\beta_1^2\gamma_1}\right), \quad \chi = -\frac{\beta_2^2+\beta_1^2\gamma_1}{\gamma_1}, \end{aligned} \tag{3.12}$$

respectively. Note that the conditions $\gamma_1 > 0$ and $\beta_2 - \beta_1\sqrt{\gamma_1} < 0$ gives the case of the GDSI equation (3.3).

On the other hand, there is another bilinear form,

$$\begin{aligned} &(D_{x_1}D_{x_{-1}} - 2)FF = -2|G|^2, \\ &(D_{x_1}^2 - D_{x_2})GF = 0, \\ &(D_{x_{-1}}^2 + D_{x_{-2}})GF = 0, \end{aligned} \tag{3.13}$$

which is reduced to (3.11) by the independent variables transformation

$$\begin{aligned} x_1 &= \tfrac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2(\beta_2-\beta_1\sqrt{\gamma_1})}\left(-\xi+\tfrac{1}{\sqrt{\gamma_1}}\eta\right), \qquad & x_2 &= -\frac{i\zeta}{2} \\ x_{-1} &= \tfrac{-\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2(\beta_2+\beta_1\sqrt{\gamma_1})}\left(\xi+\tfrac{1}{\sqrt{\gamma_1}}\eta\right), \qquad & x_{-2} &= \frac{i\zeta}{2}. \end{aligned} \tag{3.14}$$

Indeed, to the rogue wave solutions for (3.3), the bilinear form (3.13) is considered as a special case of the bilinear form

$$\begin{aligned} &(D_{x_1}D_{x_{-1}}-2)\tau_n\tau_n = -2\tau_{n+1}\tau_{n-1}, \\ &(D_{x_1}^2-D_{x_2})\tau_{n+1}\tau_n = 0, \\ &(D_{x_{-1}}^2+D_{x_{-2}})\tau_{n+1}\tau_n = 0, \end{aligned} \tag{3.15}$$

where the determinant

$$\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right). \tag{3.16}$$

It is easy to see that equation (3.15) for $n=0$ becomes (3.13) when defined $\tau_0 = F$, $\tau_1 = G$ and $\tau_{-1} = G^*$. Before passing the discussion of the solutions, we prove the following lemma about the bilinear form (3.15):

Lemma 3.1.1 ([17]) *If the functions $m_{ij}^{(n)}$, $\varphi_i^{(n)}$, and $\psi_j^{(n)}$ of x_1, x_2, x_{-1} and x_{-2} satisfy the differential equations*

$$\begin{aligned} \partial_{x_1}m_{ij}^{(n)} &= \varphi_i^{(n)}\psi_j^{(n)}, \\ \partial_{x_2}m_{ij}^{(n)} &= \varphi_i^{(n+1)}\psi_j^{(n)}+\varphi_i^{(n)}\psi_j^{(n-1)}, \\ \partial_{x_{-1}}m_{ij}^{(n)} &= -\varphi_i^{(n-1)}\psi_j^{(n+1)}, \\ \partial_{x_{-2}}m_{ij}^{(n)} &= -\varphi_i^{(n-2)}\psi_j^{(n+1)}-\varphi_i^{(n-1)}\psi_j^{(n+2)}, \\ \partial_{x_k}\varphi_i^{(n)} &= \varphi_i^{(n+k)}, \\ \partial_{x_k}\psi_j^{(n)} &= -\psi_j^{(n-k)} \qquad\qquad (k=-2,-1,1,2) \end{aligned} \tag{3.17}$$

and the difference relation

$$m_{ij}^{(n+1)} = m_{ij}^{(n)}+\varphi_i^{(n)}\psi_j^{(n+1)}, \tag{3.18}$$

then the bilinear equations (3.15) is satisfied by the determinant τ_n in (3.16).

Proof: To compute derivatives of τ_n, we apply the differential formula for a determinant

$$\partial_x \det_{1\le i,j\le N}(a_{ij}) = \sum_{i,j=1}^{N} A_{i,j}\partial_x a_{ij}, \tag{3.19}$$

where $A_{i,j}$ represents the $(i,\, j)$ cofactor of the matrix (a_{ij}), to τ_n with the differential equations (3.17) and then these derivatives are expressed in terms of the expansion of a bordered determinant

$$\begin{vmatrix} a_{ij} & b_i \\ c_j & d \end{vmatrix} = -\sum_{i.j=1} A_{i,j} b_i c_j + d\det(a_{ij}). \tag{3.20}$$

The first order derivatives and the second order derivatives of τ_n are given in Table 3.1 and in Table 3.2, respectively:

Table 3.1 : First order derivatives of τ_n.

$\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n)} & 0 \end{vmatrix}$	$\partial_{x_2}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix}$
$\partial_{x_{-1}}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n+1)} & 0 \end{vmatrix}$	$\partial_{x_{-2}}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-2)} \\ \psi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n+2)} & 0 \end{vmatrix}$

Table 3.2 : Second order derivatives of τ_n.

$\partial_{x_1}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix}$
$\partial_{x_{-1}}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-2)} \\ \psi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n+2)} & 0 \end{vmatrix}$
$(\partial_{x_1}\partial_{x_{-1}} - 1)\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} & \varphi_i^{(n)} \\ \psi_j^{(n+1)} & 0 & -1 \\ -\psi_j^{(n)} & -1 & 0 \end{vmatrix}$

By using $\partial^2_{x_1}\tau_n$, $\partial_{x_2}\tau_n$, $\partial^2_{x_{-1}}\tau_n$ and $\partial_{x_{-2}}\tau_n$ in Table 3.1 and in Table 3.2, we deduce the following determinants:

$$\begin{aligned}\frac{1}{2}(\partial^2_{x_1}+\partial_{x_2})\tau_n &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix}, \\ \frac{1}{2}(\partial^2_{x_{-1}}-\partial_{x_{-2}})\tau_n &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n+2)} & 0 \end{vmatrix}.\end{aligned} \tag{3.21}$$

At this step, the difference equation (3.18) is taken into account for the bordered determinant representations of τ_{n+1} and τ_{n-1}:

$$\begin{aligned}\tau_{n+1} &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & 1 \end{vmatrix}, \\ \tau_{n-1} &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n)} & 1 \end{vmatrix}.\end{aligned} \tag{3.22}$$

Then, the procedure done to τ_n is applied to τ_{n+1} and find the first and second order derivatives given in Table 3.3 and Table 3.4, respectively.

Table 3.3 : First order derivatives of τ_{n+1}.

$\partial_{x_1}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix}$	$\partial_{x_{-1}}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n+2)} & 0 \end{vmatrix}$
$\partial_{x_2}\tau_{n+1} = -\begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix}$	
$\partial_{x_{-2}}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n+2)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n+3)} & 0 \end{vmatrix}$	

Similarly, we deduce the following determinants involving the derivatives of τ_{n+1}:

$$\begin{aligned}\frac{1}{2}(\partial^2_{x_1}-\partial_{x_2})\tau_{n+1} &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \\ \frac{1}{2}(\partial^2_{x_{-1}}+\partial_{x_{-2}})\tau_{n+1} &= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n+2)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}.\end{aligned} \tag{3.23}$$

Table 3.4 : Second order derivatives of τ_{n+1}.

$\partial_{x_1}^2 \tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix}$
$\partial_{x_{-1}}^2 \tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n+2)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n+3)} & 0 \end{vmatrix}$

In the last step, using equations (3.21), (3.23) and (3.22), and the following formula which is called Jacobi Formula for determinants

$$\begin{vmatrix} a_{ij} & b_i & c_i \\ d_j & e & f \\ g_j & h & k \end{vmatrix} \times \begin{vmatrix} a_{ij} \end{vmatrix} = \begin{vmatrix} a_{ij} & c_i \\ g_j & k \end{vmatrix} \times \begin{vmatrix} a_{ij} & b_i \\ d_j & e \end{vmatrix} - \begin{vmatrix} a_{ij} & b_i \\ g_j & h \end{vmatrix} \times \begin{vmatrix} a_{ij} & c_i \\ d_j & f \end{vmatrix}, \tag{3.24}$$

we get the following equalities:

$$\begin{aligned} &(\partial_{x_1}\partial_{x_{-1}} - 1)\tau_n \times \tau_n = \partial_{x_1}\tau_n \times \partial_{x_{-1}}\tau_n - (-\tau_{n+1})(-\tau_{n-1}), \\ &\frac{1}{2}(\partial_{x_1}^2 - \partial_{x_2})\tau_{n+1} \times \tau_n = \partial_{x_1}\tau_{n+1} \times \partial_{x_1}\tau_n - (\tau_{n+1})\frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2})\tau_n, \\ &\frac{1}{2}(\partial_{x_{-1}}^2 + \partial_{x_{-2}})\tau_{n+1} \times \tau_n = \partial_{x_{-1}}\tau_{n+1} \times \partial_{x_{-1}}\tau_n - (\tau_{n+1})\frac{1}{2}(\partial_{x_{-1}}^2 - \partial_{x_{-2}})\tau_n, \end{aligned} \tag{3.25}$$

which are together equivalent to (3.15). □

3.2 Rational Solutions

In this section, inspired by Theorem 1 in [17], we prove the following main theorem which presents the rational solutions of the GDSI system.

Theorem 3.2.1 *The non-singular rational solutions of the GDSI equation (3.3) with the restrictions (3.7) and (3.9) are given by*

$$u = \frac{\sqrt{2\gamma_1}}{\sqrt{\beta_2^2 + \beta_1^2\gamma_1}} \frac{\tau_1}{\tau_0}, \tag{3.26}$$

where τ_n is a determinant of the $N \times N$ matrix $\left(m_{ij}^{(n)}\right)$ whose entires are

$$m_{ij}^{(n)} = \sum_{k=0}^{n_i} c_{ik} \left(n + \mu_i' + p_i \partial_{p_i}\right)^{n_i - k} \times \sum_{l=0}^{n_j} c_{jl}^* \left(-n + \left(\mu_j'\right)^* + p_j^* \partial_{p_j^*}\right)^{n_j - l} \frac{1}{p_i + p_j^*}, \qquad (3.27)$$

with

$$\mu_i' = \frac{\sqrt{\beta_2^2 + \beta_1^2 \gamma_1}}{2} \left(\frac{1}{p_i(\beta_2 + \beta_1 \sqrt{\gamma_1})} - \frac{p_i}{\beta_2 - \beta_1 \sqrt{\gamma_1}} \right) \xi + \frac{\sqrt{\beta_2^2 + \beta_1^2 \gamma_1}}{2\sqrt{\gamma_1}} \left(\frac{1}{p_i(\beta_2 + \beta_1 \sqrt{\gamma_1})} + \frac{p_i}{\beta_2 - \beta_1 \sqrt{\gamma_1}} \right) \eta + \frac{p_i^2 + p_i^{-2}}{\sqrt{-1}} \zeta. \qquad (3.28)$$

Proof: We need to verify that the function u in (3.26) is a solution of $GDSI$. To do this, we recall that the auxiliary bilinear form (3.15) is reduced to the bilinear form of the GDSI equation (3.3) in (3.11) under the transformations (3.14) with $\tau_1 = G$, $\tau_{-1} = G^*$ and $\tau_0 = F$. Thus, we have two complex conjugate conditions

$$\tau_n^* = \tau_{-n} \quad \text{and} \quad x_2^* = x_{-2}. \qquad (3.29)$$

Moreover, it is established that the determinant τ_n in (3.16) satisfies the auxiliary form (3.15) when its elements $m_{ij}^{(n)}$ hold the differential and difference relations (3.17) and (3.18) (see Lemma 3.1.1). Under these circumstances, it is sufficient to choose the suitable $m_{ij}^{(n)}$, so we assume that

$$\varphi_i^{(n)} = p_i^n \mathrm{e}^{\mu_i} \quad \text{and} \quad \psi_j^{(n)} = (-q_j)^{-n} \mathrm{e}^{\lambda_j}, \qquad (3.30)$$

where

$$\mu_i = \frac{1}{p_i^2} x_{-2} + \frac{1}{p_i} x_{-1} + p_i x_1 + p_i^2 x_2 \ \text{ and } \ \lambda_j = \frac{-1}{q_j^2} x_{-2} + \frac{1}{q_j} x_{-1} + q_j x_1 - q_j^2 x_2 \qquad (3.31)$$

with p_i and q_j being complex numbers. By substituting (3.30) into equation (3.17), we get $m_{ij}^{(n)} = \int^{x_1} \varphi_i^{(n)} \psi_j^{(n)} \, dx_1 = \left(-p_i q_j^{-1}\right)^n \mathrm{e}^{\mu_i + \lambda_j} / (p_i + q_j)$. It is easy to verify that $m_{ij}^{(n)}$ also satisfy the other relations, but this choice does not lead to rational solution. Therefore, we must improve the assumptions and take into account the derivatives of the functions $\varphi_i^{(n)}$, $\psi_j^{(n)}$ and $m_{ij}^{(n)}$. The new assumptions are given by

$$\varphi_i^{(n)} = A_i p_i^n \mathrm{e}^{\mu_i}, \quad \psi_j^{(n)} = B_j (-q_j)^{-n} \mathrm{e}^{\lambda_j},$$
$$m_{ij}^{(n)} = A_i B_j \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j}\right)^n \mathrm{e}^{\mu_i + \lambda_j} \qquad (3.32)$$

where A_i and B_j denote differential operators of order n_i and n_j with respect to p_i and q_j as

$$A_i = \sum_{k=0}^{n_i} c_{ik}(p_i\partial_{p_i})^{n_i-k} \text{ and } B_j = \sum_{l=0}^{n_j} d_{lj}(q_j\partial_{q_j})^{n_j-l} \tag{3.33}$$

for which c_{ik}, d_{lj} are complex numbers and n_i and n_j are positive integers.

At this stage, we find the solution expression in Theorem 3.2.1 leading the rational solutions since it is also easy to show the functions $\varphi_i^{(n)}$, $\psi_j^{(n)}$ and $m_{ij}^{(n)}$ are solutions of (3.17) and (3.18). Now, we first expand the term $(p_i\partial_{p_i})\,p_i^n e^{\mu_i}$:

$$\begin{aligned}(p_i\partial_{p_i})\,p_i^n e^{\mu_i} &= p_i\left(\partial_{p_i}p_i^n\right)\mathrm{e}^{\mu_i} + p_i p_i^n\left(\partial_{p_i}\mathrm{e}^{\mu_i}\right) + p_i^{n+1}\mathrm{e}^{\mu_i}\partial_{p_i} \\ &= np_i^n\mathrm{e}^{\mu_i} + p_i^{n+1}\left(-\frac{2}{p_i^3}x_{-2} - \frac{1}{p_i^2}x_{-1} + x_1 + 2p_i x_2\right)\mathrm{e}^{\mu_i} + p_i^{n+1}\mathrm{e}^{\mu_i}\partial_{p_i} \\ &= p_i^n\mathrm{e}^{\mu_i}\left(n + \left(-\frac{2}{p_i^2}x_{-2} - \frac{1}{p_i}x_{-1} + p_1x_1 + 2p_1^2x_2\right) + p_i\partial_{p_i}\right), \end{aligned} \tag{3.34}$$

which implies the operator relation

$$(p_i\partial_{p_i})\,p_i^n\mathrm{e}^{\mu_i} = p_i^n\mathrm{e}^{\mu_i}\left(n + \mu_i' + p_i\partial_{p_i}\right), \tag{3.35}$$

if we set $\mu_i' = -2x_{-2}/p_i^2 - x_{-1}/p_i + p_ix_1 + 2p_1^2x_2$.

Next, we do same procedure for $\left(q_j\partial_{q_j}\right)(-q_j)^{-n}\,\mathrm{e}^{\lambda_j}$:

$$\begin{aligned}\left(q_j\partial_{q_j}\right)(-q_j)^{-n}\,\mathrm{e}^{\lambda_j} &= q_j\left(\partial_{q_j}(-q_j)^{-n}\right)\mathrm{e}^{\lambda_j} - q_j^{-n+1}\left(\partial_{q_j}\mathrm{e}^{\lambda_j}\right) - q_j^{-n+1}\mathrm{e}^{\lambda_j}\partial_{q_j} \\ &= nq_j^{-n}\mathrm{e}^{\lambda_j} - q_j^{-n+1}\left(\frac{2}{q_j^3}x_{-2} - \frac{1}{q_j^2}x_{-1} + x_1 - 2q_jx_2\right)\mathrm{e}^{\lambda_j} - q_j^{-n+1}\mathrm{e}^{\lambda_j}\partial_{q_j} \\ &= (-q_j)^{-n}\,\mathrm{e}^{\lambda_j}\left(-n + \left(\frac{2}{q_j^2}x_{-2} - \frac{1}{q_j}x_{-1} + q_jx_1 - 2q_j^2x_2\right) + q_j\partial_{q_j}\right) \end{aligned} \tag{3.36}$$

If we define $\lambda_j' = 2x_{-2}/q_j^2 - x_{-1}/q_j + q_jx_1 - 2q_j^2x_2$ then the expression (3.36) becomes

$$\left(q_j\partial_{q_j}\right)(-q_j)^{-n}\,\mathrm{e}^{\lambda_j} = (-q_j)^{-n}\,\mathrm{e}^{\lambda_j}\left(-n + \lambda_j' + q_j\partial_{q_j}\right). \tag{3.37}$$

Therefore, applying the operator relations (3.35) and (3.37) to (3.32), the matrix entries $m_{ij}^{(n)}$ can be represented as follows:

$$\begin{aligned} m_{ij}^{(n)} &= \left(-\frac{p_i}{q_j}\right)^n \mathrm{e}^{\mu_i+\lambda_j}\sum_{k=0}^{n_i} c_{ik}\left(n + \mu_i' + p_i\partial_{p_i}\right)^{n_i-k} \\ &\qquad \times \sum_{l=0}^{n_j} d_{jl}\left(-n + \lambda_j' + q_j\partial_{q_j}\right)^{n_j-l}\frac{1}{p_i+q_j}. \end{aligned} \tag{3.38}$$

Due to the complex conjugate condition $\tau_n^* = \tau_{-n}$, we take the restrictions $d_{jl} = c_{ik}^*$ and $q_j = p_j^*$. Then, we substitute these restrictions and the variables transformation (3.14) into the expressions μ_i' and λ_i':

$$\begin{aligned}
\mu_i' &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2}\left(\frac{1}{p_i(\beta_2+\beta_1\sqrt{\gamma_1})}-\frac{p_i}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\xi+\frac{p_i^2+p_i^{-2}}{\sqrt{-1}}\zeta \\
&+ \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2\sqrt{\gamma_1}}\left(\frac{1}{p_i(\beta_2+\beta_1\sqrt{\gamma_1})}+\frac{p_i}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\eta, \\
\lambda_j' &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2}\left(\frac{1}{p_i^*(\beta_2+\beta_1\sqrt{\gamma_1})}-\frac{p_i^*}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\xi-\frac{(p_i^*)^2+(p_i^*)^{-2}}{\sqrt{-1}}\zeta \\
&+ \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2\sqrt{\gamma_1}}\left(\frac{1}{p_i^*(\beta_2+\beta_1\sqrt{\gamma_1})}+\frac{p_i^*}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\eta, \qquad (3.39)
\end{aligned}$$

which implies that $\lambda_j' = \mu_j'^*$. As a result, we obtain

$$\begin{aligned}
m_{ij}^{(n)} &= \left(-\frac{p_i}{p_j^*}\right)^n \mathrm{e}^{\mu_i+\mu_j^*}\sum_{k=0}^{n_i} c_{ik}\left(n+\mu_i'+p_i\partial_{p_i}\right)^{n_i-k} \\
&\qquad \times\sum_{l=0}^{n_j} c_{jl}^*\left(-n+\left(\mu_j'\right)^* p_j^*\partial_{p_j^*}\right)^{n_j-l}\frac{1}{p_i+p_j^*}, \qquad (3.40)
\end{aligned}$$

so $m_{ij}^{(n)*} = m_{ji}^{(-n)}$ and $\tau_n^* = \tau_{-n}$. Finally, we use the gauge freedom of τ_n to get equation (3.27).

Moreover, the nonsingularity of rational solutions are satisfied depending on the fact that the real part of p_i is positive. As mentioned previously F is defined as $F = det_{1\le i,j\le N}(m_{ij}^{(0)})$. Specifically the entries of the $N\times N$ matrix are

$$m_{ij}^{(0)} = \int_{-\infty}^{x_1} A_iB_je^{\mu_i+\mu_j^*}dx_1$$

where B_j is obtained as the complex conjugate of A_j with the expression

$$B_j = \sum_{l=0}^{n_j} c_{lj}^*(p_j^*\partial_{p_j^*})^{n_j-l}$$

When the real part of p_i is positive, then in the integral, the part $e^{\mu_i+\mu_j^*}$ vanishes at $x_1 = -\infty$.

Now, assume that $v = (v_1, v_2, v_3, ..., v_N)$ is a nonzero vector and $(\overline{v})^T$ is the conjugate transpose of v. Then,

$$v\left(m_{ij}^{(0)}\right)_{i,j=1}^{N}(\overline{v})^T = \int_{-\infty}^{x_1}\left|\sum_{i=1}^{N} v_iA_ie^{\mu_i}\right|^2 dx_1 > 0$$

is obtained. this shows that the matrix $\left(m_{ij}^{(0)}\right)_{i,j=1}^{N}$ is positive definite which means that the determinant of the matrix is positive, $f > 0$. Hence, the solution is nonsingular. The similar procedure can be followed for the case that the real parts of p_i's are negative. Also, in this case, the nonsingularity of the rational solutions is obtained. □

In the next sections, we present rogue waves as fundamental rogue wave, multi-rogue wave and higher order rogue wave in the GDSI system (3.3) under the constraints $\gamma_1 > 0$ and $\beta_2 - \beta_1\sqrt{\gamma_1} < 0$.

3.3 Fundamental Rogue Waves

The 1- dimensionol rational solution of first order ($N = 1$ and $n_1 = 1$) with $p_1 \in \mathbb{R}$ forms a fundamental rouge wave. To obtain a fundamental rogue wave, we first impose $N = 1$, $n_1 = 1$ to the rational solution given in Theorem 3.2.1:

$$\begin{aligned} G &= \tau_1 = m_{11}^{(1)} \\ &= \sum_{k=0}^{1} c_{1k}\left(1+\mu_1' + p_1\partial_{p_1}\right)^{1-k} \times \sum_{l=0}^{1} c_{1l}^*\left(-1+(\mu_j')^* + p_1^*\partial_{p_1^*}\right)^{1-l}\frac{1}{p_1+p_1^*} \\ &= \frac{1}{p_1+p_1^*}\left(\left(\mu_1'+1+c_{11}-\frac{p_1}{p_1+p_1^*}\right)\right. \\ &\qquad \left.\times\left((\mu_1')^* - 1 + c_{11}^* - \frac{p_1^*}{p_1+p_1^*}\right) + \frac{|p_1|^2}{(p_1+p_1^*)^2}\right), \end{aligned}$$

$$\begin{aligned} F &= \tau_0 = m_{11}^{(0)} \\ &= \sum_{k=0}^{1} c_{1k}\left(p_1\partial_{p_1}+\mu_1'\right)^{1-k} \times \sum_{l=0}^{1} c_{1l}^*\left(p_1^*\partial_{p_1^*}+(\mu_1')^*\right)^{1-l}\frac{1}{p_1+p_1^*} \\ &= \frac{1}{p_1+p_1^*}\left(\left(\mu_1'+c_{11}-\frac{p_1}{p_1+p_1^*}\right)\right. \\ &\qquad \left.\times\left((\mu_1')^* + c_{11}^* - \frac{p_1^*}{p_1+p_1^*}\right) + \frac{|p_1|^2}{(p_1+p_1^*)^2}\right), \end{aligned} \tag{3.41}$$

where p_1 and c_{11} are free complex constants and

$$\begin{aligned} \mu_1' &= \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2}\left(\frac{1}{p_1(\beta_2+\beta_1\sqrt{\gamma_1})} - \frac{p_1}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\xi \\ &+ \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2\sqrt{\gamma_1}}\left(\frac{1}{p_1(\beta_2+\beta_1\sqrt{\gamma_1})} + \frac{p_1}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\eta + \frac{p_1^2+p_1^{-2}}{\sqrt{-1}}\zeta. \end{aligned} \tag{3.42}$$

Then putting μ_1' in (3.42) into this solutions, we have

$$\begin{aligned} G &= \frac{1}{p_1+p_1^*}\left(|\mu|^2+\mu^*-\mu-1+\frac{|p_1|^2}{(p_1+p_1^*)^2}\right), \\ F &= \frac{1}{p_1+p_1^*}\left(|\mu|^2+\frac{|p_1|^2}{(p_1+p_1^*)^2}\right), \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} \mu &= c_{11}-\frac{p_1}{p_1+p_1^*}+\frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2}\left(\frac{1}{p_1(\beta_2+\beta_1\sqrt{\gamma_1})}-\frac{p_1}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\xi \\ &+ \frac{\sqrt{\beta_2^2+\beta_1^2\gamma_1}}{2\sqrt{\gamma_1}}\left(\frac{1}{p_1(\beta_2+\beta_1\sqrt{\gamma_1})}+\frac{p_1}{\beta_2-\beta_1\sqrt{\gamma_1}}\right)\eta+\frac{p_1^2+p_1^{-2}}{\sqrt{-1}}\zeta. \end{aligned} \tag{3.44}$$

Now, we observe that the coefficients of ξ and η in (3.44) are real and the coefficient of ζ in (3.44) is pure imaginary, if $p_1 \in \mathbb{R}$. In this case, the solution $u = G/F = 1 + (\mu^* - \mu - 1)/(|\mu|^2 + 1/4)$ is in the form of line, so it is called a line wave. However, it does not move in the direction of (ξ, η)-plane and is approaching to uniformly constant background as $\zeta \to \pm\infty$. Moreover, its highest amplitude occur in the intermediate times. Thus, the wave solution (3.43) appears from nowhere and disappears with no trace. In other words, this wave is a line rogue wave. By reducing GDSI system to DSI system considering the condition that $\gamma_2 = \beta_2 = 0$ and $\gamma_1 = 1$ and then reducing the DSI system to nLS equation, the fundamental rogue wave turns to the Peregrine soliton.

At this step, we simulate the line rogue wave solutions of the GDSI system for two different values of $(\gamma_1, \beta_1, \beta_2)$:

(a) The case $(\gamma_1, \beta_1, \beta_2) = (1, 1, 0.5)$:

For this values, The GDSI equation (3.3) becomes

$$\begin{aligned} iu_\zeta &= u_{\xi\xi}+u_{\eta\eta}+\frac{8}{5}u_{\xi\eta}+u\left(\partial_\xi+\frac{1}{2}\partial_\eta\right)v-\frac{5}{4}|u|^2u, \\ v_{\xi\xi}-v_{\eta\eta} &= \frac{10}{3}\left(\partial_\xi+\frac{1}{2}\partial_\eta\right)|u|^2. \end{aligned} \tag{3.45}$$

Assuming $c_{11} = 0.5$ and $p_1 = 1$, the solutions u and v are obtained as

$$u = 2\sqrt{\frac{2}{5}}\left(1 + \frac{144i\zeta - 36}{9 + 80(\xi - \eta/2)^2 + 144\zeta^2}\right),$$
$$\left(\partial_\xi + \frac{1}{2}\partial_\eta\right) v = 2 + 288\frac{9 - 80(\xi - \eta/2)^2 + 144\zeta^2}{\left(9 + 80(\xi - \eta/2)^2 + 144\zeta^2\right)^2}. \tag{3.46}$$

As seen in Figure 3.1, the solution describes a line rogue wave. Thus, fundamental rogue wave can be considered as line rogue wave. In the intermediate times |u| reaches the maximum amplitude which is close to 4. However, as $\zeta \to \pm\infty$, the solution |u| approaches to constant background which is 1 in the (ξ, η) plane. Therefore, the line rogue wave appears and disappears suddenly.

(b) The case $(\gamma_1,\ \beta_1,\ \beta_2) = (0.25,\ 3,\ 0.5)$

This time, we consider the following system:

$$iu_\zeta = u_{\xi\xi} + \frac{1}{4}u_{\eta\eta} + \frac{3}{5}u_{\xi\eta} + u\left(3\partial_\xi + \frac{1}{2}\partial_\eta\right) v - 10|u|^2 u,$$
$$v_{\xi\xi} - \frac{1}{4}v_{\eta\eta} = \frac{2}{5}\left(3\partial_\xi + \frac{1}{2}\partial_\eta\right)|u|^2 \tag{3.47}$$

Similarly, assuming $c_{11} = 0.5$ and $p_1 = 1$, the solutions u and v are found as

$$u = \frac{\sqrt{5}}{5}\left(1 + \frac{128i\zeta - 32}{8 + 5(3\xi - 2\eta)^2 + 128\zeta^2}\right),$$
$$\left(3\partial_\xi + \frac{1}{2}\partial_\eta\right) v = 2 + 256\frac{8 - 5(3\xi - 2\eta)^2 + 128\zeta^2}{\left(8 + 5(3\xi - 2\eta)^2 + 128\zeta^2\right)^2}. \tag{3.48}$$

In Figure 3.2, at time $\zeta = 0$ the wave reaches its maximum amplitude and $|u|$ approaches to 1.35. In this case, the fundamental rogue wave has smaller amplitude than one in the first case. As $\zeta \to \pm\infty$, it shows a constant background. For $\zeta = -4$ and $\zeta = 4$ illustrate the constant background. Just obtaining maximum amplitude at $\zeta = 0$ shows that rogue waves suddenly appears and disappears again.

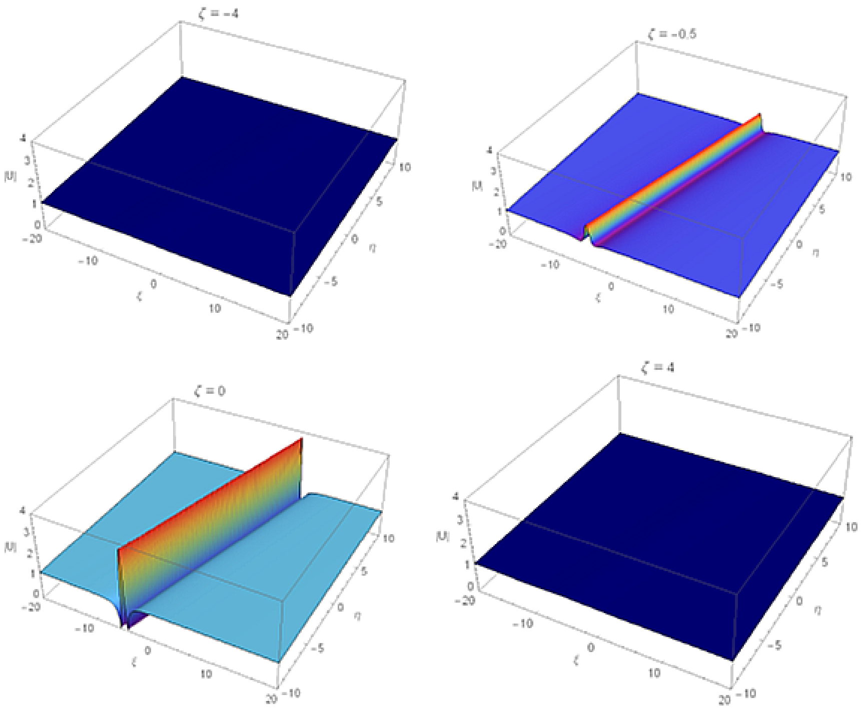

Figure 3.1 : The fundamental rogue wave solution for the case $(\gamma_1,\ \beta_1,\ \beta_2) = (1,\ 1,\ 0.5)$ with $p_1 = 1$.

3.4 Multi-Rogue Waves

Multi-rogue waves are the interaction of multiple fundamental rogue waves. Multi-rogue waves behave as follows: in the intermediate times they interact with each other and at the intersection region the amplitude reaches the maximum value. After the intersection region fades, they separately reach their highest amplitude. Finally, they disappear into the constant backround again. Since we dealt with combination of multiple line rogue waves, in order to obtain multi-rogue wave, the condition should be that $N > 1$, $n_1 = n_2 = ... = n_N = 1$ and $(p_1, p_2, ..., p_N)$ are real-valued.

Let us first take $N = 2$ and assume that $c_{i0} = 1$. Now, the solution for F and G are the following determinants:

$$G = \begin{vmatrix} m_{11}^{(1)} & m_{12}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} \end{vmatrix}, \qquad F = \begin{vmatrix} m_{11}^{(0)} & m_{12}^{(0)} \\ m_{21}^{(0)} & m_{22}^{(0)} \end{vmatrix}, \tag{3.49}$$

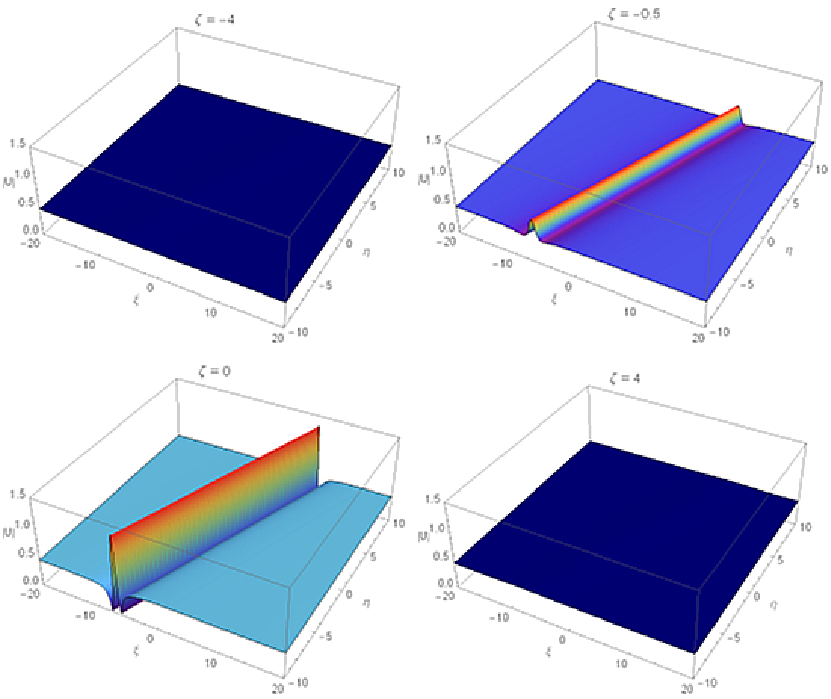

Figure 3.2 : The fundamental rogue wave solution for the case $(\gamma_1,\, \beta_1,\, \beta_2) = (0.25,\, 3,\, 0.5)$ with $p_1 = 1$.

where

$$
\begin{aligned}
m_{ij}^{(0)} &= \sum_{k=0}^{1} c_{ik} \left(\mu_i' + p_i \partial_{p_i}\right)^{1-k} \times \sum_{l=0}^{1} c_{jl}^{*} \left(\left(\mu_j'\right)^{*} + p_j^{*} \partial_{p_j^{*}}\right)^{1-l} \frac{1}{p_i + p_j^{*}} \\
&= \frac{1}{p_i + p_j^{*}} \left(\left(\mu_i' + c_{i1} - \frac{p_i}{p_i + p_j^{*}} \right) \left(\left(\mu_j'\right)^{*} + c_{j1}^{*} - \frac{p_j^{*}}{p_i + p_j^{*}} \right) + \frac{p_i p_j^{*}}{\left(p_i + p_j^{*}\right)^{2}} \right) \\
&= \frac{1}{p_i + p_j} \left(\left(\mu_i' + c_{i1} - \frac{p_i}{p_i + p_j} \right) \left(\left(\mu_j'\right)^{*} + c_{j1}^{*} - \frac{p_j}{p_i + p_j} \right) + \frac{p_i p_j}{\left(p_i + p_j\right)^{2}} \right)
\end{aligned}
$$

$$
\begin{aligned}
m_{ij}^{(1)} &= \sum_{k=0}^{1} c_{ik} \left(1 + \mu_i' + p_i \partial_{p_i}\right)^{1-k} \times \sum_{l=0}^{1} c_{jl}^{*} \left(-1 + \left(\mu_j'\right)^{*} + p_j^{*} \partial_{p_j^{*}}\right)^{1-l} \frac{1}{p_i + p_j^{*}} \\
&= \frac{1}{p_i + p_j^{*}} \left(\left(\mu_i' + 1 + c_{i1} - \frac{p_i}{p_i + p_j^{*}} \right) \right. \\
&\qquad \left. \times \left(\left(\mu_j'\right)^{*} - 1 + c_{j1}^{*} - \frac{p_j^{*}}{p_i + p_j^{*}} \right) + \frac{p_i p_j^{*}}{(p_i + p_j^{*})^{2}} \right) \\
&= \frac{1}{p_i + p_j} \left(\left(\mu_i' + 1 + c_{i1} - \frac{p_i}{p_i + p_j} \right) \right. \\
&\qquad \left. \times \left(\left(\mu_j'\right)^{*} - 1 + c_{j1}^{*} - \frac{p_j}{p_i + p_j} \right) + \frac{p_i p_j}{(p_i + p_j)^{2}} \right) \qquad (3.50)
\end{aligned}
$$

with p_1 and p_2 are arbitrary real parameters, and c_1 and c_{21} are arbitrary complex parameters, beside μ_j' is given in (3.28).

(a) The case $(\gamma_1,\ \beta_1,\ \beta_2) = (1,\ 1,\ 0.5)$

By taking $p_1 = 1$, $p_2 = 1.5$, $c_{11} = 0$ and $c_{21} = 0$ the two-rogue wave solution is given in Figure 3.3. As seen in the figure the two fundamental rogue waves arise from the constant background. Then the interaction region reaches the highest amplitude which is close to 4 at $\zeta = -1$. After the higher amplitude fade, the two fundamental rogue waves in the far field appear to be with their highest amplitude at $\zeta = 0$. The wave fronts at this stage is well separated. These curvy wave fronts are caused by the interaction of the two fundamental rogue waves. When ζ becomes larger, the waves go back to the constant background as seen at $\zeta = -10$ and $\zeta = 10$. In all cases, the amplitude did not pass through 4. Thus, this means that the interaction does not guarantee very high peaks.

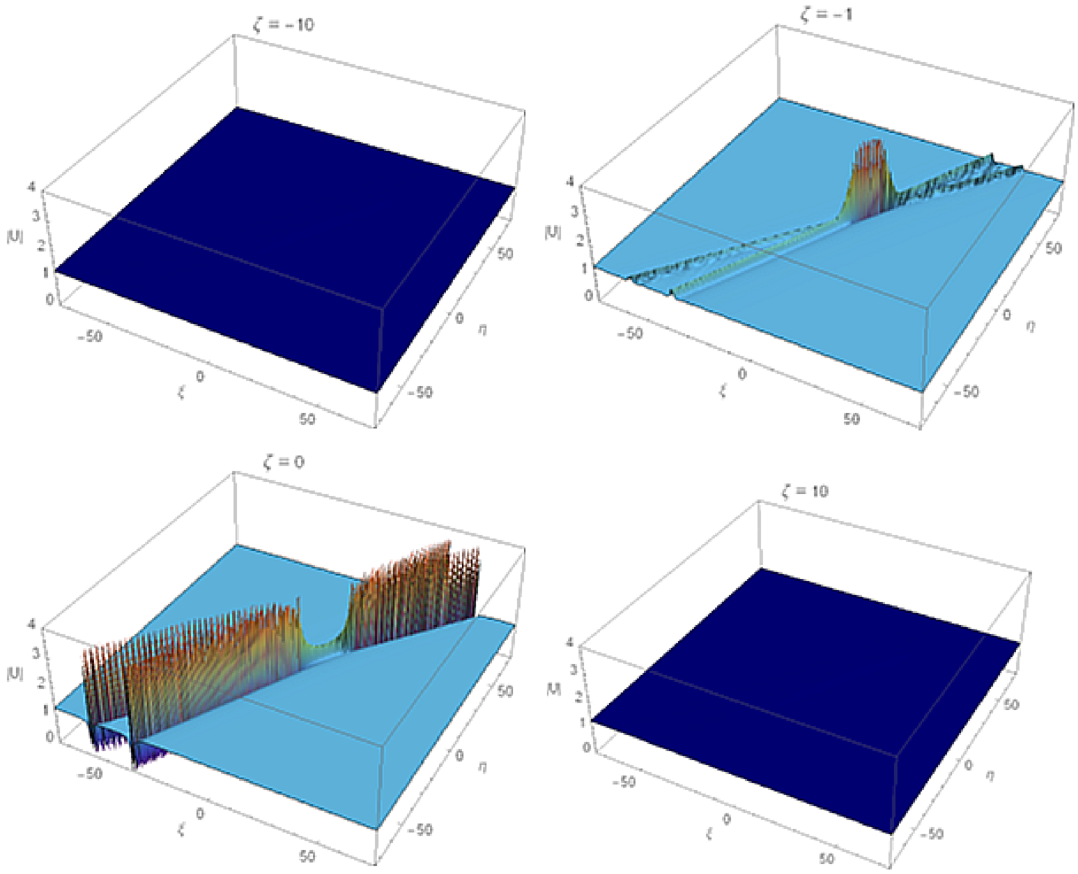

Figure 3.3 : The two-rogue wave solution for the case $(\gamma_1,\ \beta_1,\ \beta_2) = (1,\ 1,\ 0.5)$ with $p_1 = 1$ and $p_2 = 1.5$.

(b) The case $(\gamma_1, \beta_1, \beta_2) = (0.25, 3, 0.5)$

By taking $p_1 = 1$, $p_2 = 1.5$, $c_{11} = 0$ and $c_{21} = 0$ the two-rogue wave solution is given in Figure 3.4. As seen in the figure the two fundamental rogue waves arise from the constant backround. Then the interaction region reaches the highest amplitude which is close to 1.5 at $\zeta = -1.1$. After the higher amplitude fade, the two fundamental rogue waves in the far field appear to be with their highest amplitude at $\zeta = 0$. The wave fronts at this stage is well separated. These curvy wave fronts are caused by the interaction of the two fundamental rogue waves. When ζ becomes larger, the waves go back to the constant background as seen at $\zeta = -5$ and $\zeta = 5$. In all cases, the amplitude did not pass through 1.5. Thus, this means that the interaction does not guarantee very high peaks.

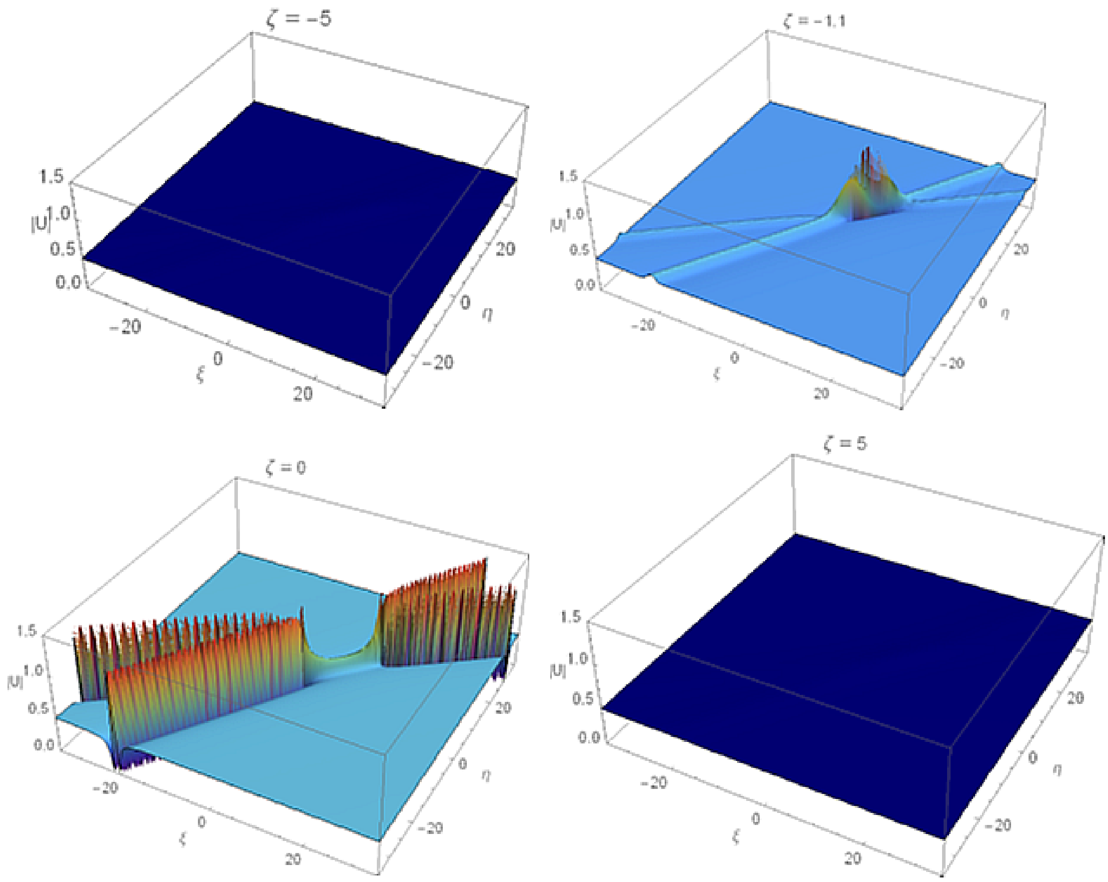

Figure 3.4 : The two-rogue wave solution for the case $(\gamma_1, \beta_1, \beta_2) = (0.25, 3, 0.5)$ with $p_1 = 1$ and $p_2 = 1.5$.

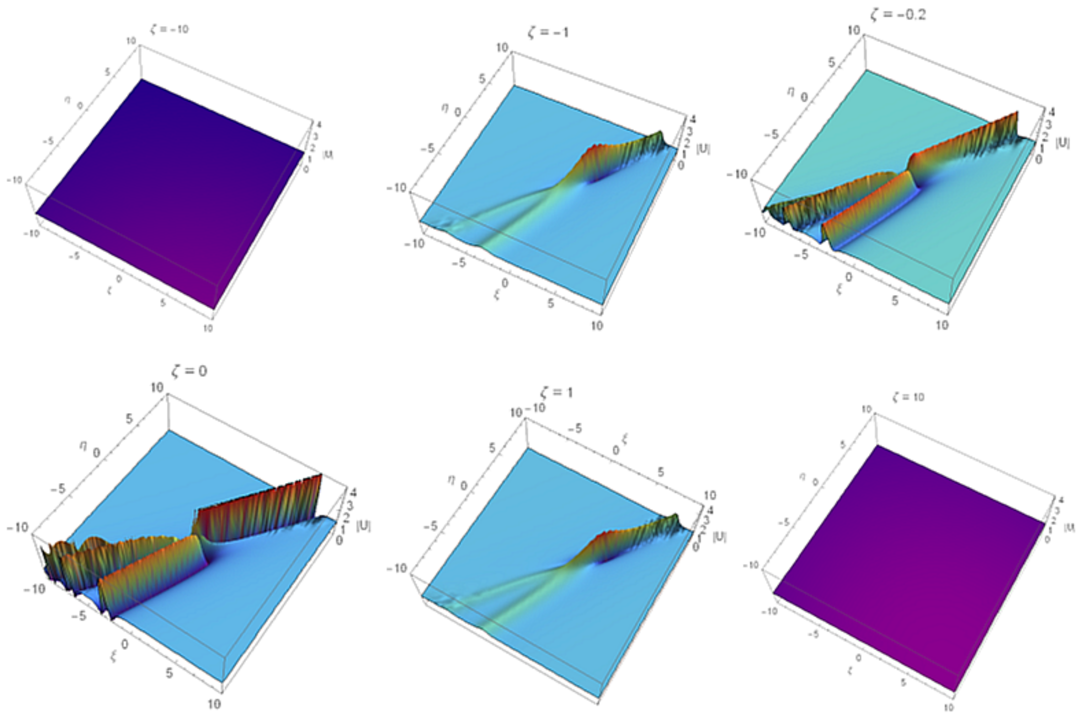

Figure 3.5 : The three-rogue wave solution for the case $(\gamma_1,\, \beta_1,\, \beta_2) = (1,\, 1,\, 0.5)$ with $p_1 = 1$, $p_2 = 1.5$ and $p_3 = 2$.

Let us now take $N = 3$. Then, F and G are

$$G = \begin{vmatrix} m_{11}^{(1)} & m_{12}^{(1)} & m_{13}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} & m_{23}^{(1)} \\ m_{31}^{(1)} & m_{32}^{(1)} & m_{33}^{(1)} \end{vmatrix}, \qquad F = \begin{vmatrix} m_{11}^{(0)} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{21}^{(0)} & m_{22}^{(0)} & m_{23}^{(0)} \\ m_{31}^{(0)} & m_{32}^{(0)} & m_{33}^{(0)} \end{vmatrix}, \tag{3.51}$$

where $m_{ij}^{(0)}$ and $m_{ij}^{(1)}$, $1 \leq i, j \leq 3$ are obtained in (3.50).

(a) The case $(\gamma_1,\, \beta_1,\, \beta_2) = (1,\, 1,\, 0.5)$

Setting $p_1 = 1$, $p_2 = 1.5$, $p_3 = 2$ and $c_{i1} = 0$ for $i = 1, 2, 3$, the three-rogue waves are obtained in Figure 3.5. Similar to the case of two-rogue wave, the three rogue waves interact with each other. However, different than the previous case the wave fronts are more complicated due to the number of interacting waves. The graphics show that at $\zeta = 0$ the three waves separately have higher amplitudes which is close to 4. However, the intersection of three waves reaches the higher amplitude at $\zeta = -1$ and $\zeta = 1$ which is approximately the same as in the case of separate amplitude. After a while the rogue wave disappears without any trace. Since in all cases the amplitude did not pass through 4, this implies that the interaction does not create very high peaks.

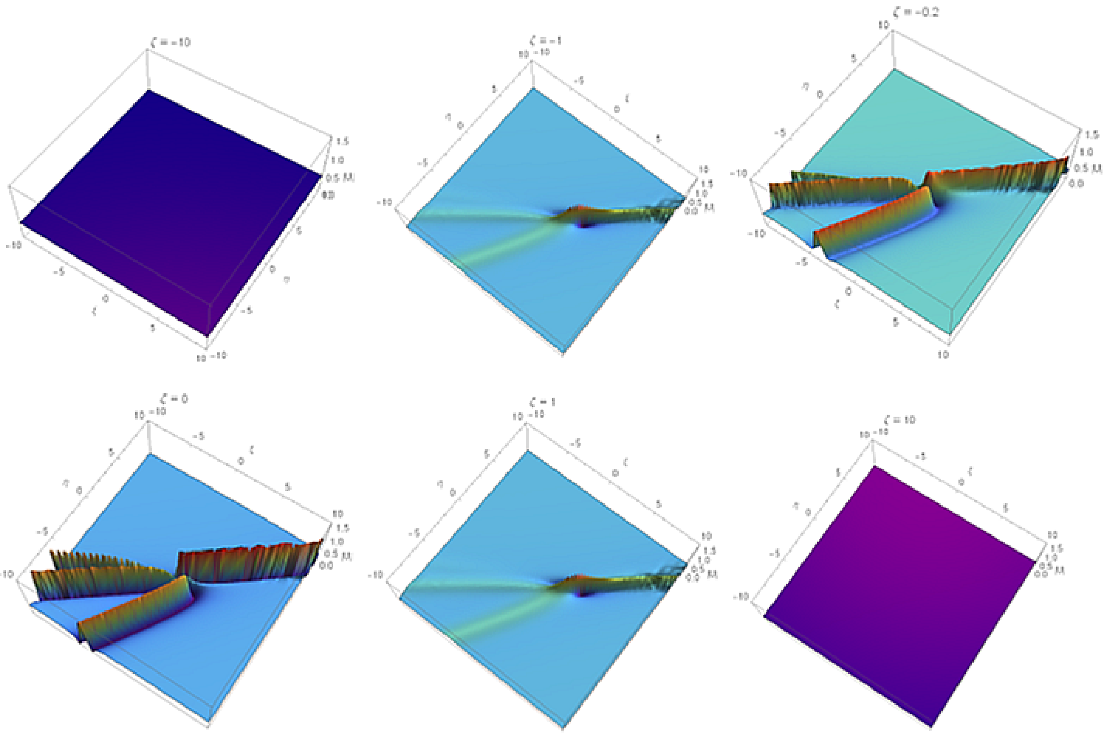

Figure 3.6 : The three-rogue wave solution for the case $(\gamma_1,\, \beta_1,\, \beta_2) = (0.25,\, 3,\, 0.5)$ with $p_1 = 1$, $p_2 = 1.5$ and $p_3 = 2$.

(b) the case $(\gamma_1,\, \beta_1,\, \beta_2) = (0.25,\, 3,\, 0.5)$

For the values $p_1 = 1$, $p_2 = 1.5$, $p_3 = 2$ and $c_{i1} = 0$ for $i = 1, 2, 3$, the three-rogue waves are obtained in Figure 3.6. The graphics show that at $\zeta = 0$ the three waves separately have higher amplitudes which is close to 1.5. However, the intersection of three waves reaches the highest amplitude at $\zeta = -1$ and $\zeta = 1$ which is close to 1.5. After a while the rogue wave disappears without any trace. Since in all cases the amplitude did not pass through 1.5, this implies that the interaction does not create very high peaks.

3.5 Higher Order Rogue Waves

Higher-order rogue waves are different than multi-rogue waves. Therefore, they act differently. These waves cannot approach to the constant background uniformly as $\zeta \to \infty$. Only some parts of the waves approach to the constant background uniformly. In order to find higher order rogue waves the necessary condition is that $N = 1$ and $n_1 > 1$ with p_1 being real-valued. Now let us take

$n_1 = 2$. Then, F and G are

$$\begin{aligned} G &= m_{11}^{(0)} \\ &= \sum_{k=0}^{2} c_{1k}\left(1+\mu_1'+p_1\partial_{p_1}\right)^{1-k} \times \sum_{l=0}^{2} c_{1l}^{*}\left(-1+\left(\mu_j'\right)^{*}+p_1^{*}\partial_{p_1^{*}}\right)^{1-l}\frac{1}{p_1+p_1^{*}}, \\ F &= m_{11}^{(0)} \\ &= \sum_{k=0}^{2} c_{1k}\left(p_1\partial_{p_1}+\mu_1'\right)^{1-k} \times \sum_{l=0}^{2} c_{1l}^{*}\left(p_1^{*}\partial_{p_1^{*}}+(\mu_1')^{*}\right)^{1-l}\frac{1}{p_1+p_1^{*}}. \end{aligned} \tag{3.52}$$

By taking $c_{10} = 1$, $c_{11} = 0$, $c_{12} = 0$ and $c_{21} = 0$, the solution F and G in (3.52) are reduced to the following form:

$$\begin{aligned} G &= \left(1+\mu_1'+p_1\partial_{p_1}\right)^2\left(-1+(\mu_1')^{*}+p_1^{*}\partial_{p_1^{*}}\right)^2\frac{1}{p_1+p_1^{*}} \\ F &= \left(\mu_1'+p_1\partial_{p_1}\right)^2\left((\mu_1')^{*}+p_1^{*}\partial_{p_1^{*}}\right)^2\frac{1}{p_1+p_1^{*}}, \end{aligned} \tag{3.53}$$

where μ_1' is given in (3.42).

(a) The case $(\gamma_1,\ \beta_1,\ \beta_2) = (1,\ 1,\ 0.5)$

By taking $p_1 = 1$, we have the solution $u = 2\sqrt{2}G/(\sqrt{5}F)$ with

$$\begin{aligned} F &= 8\left(t^2+\frac{5}{9}\left(x-\frac{y}{2}\right)^2+\frac{1}{4}\sqrt{5}(y-x)\right)^2+4t^2+\frac{1}{4} \\ &+\left(\frac{1}{27}\left(90y-9\sqrt{5}\right)\left(x-\frac{y}{2}\right)-\frac{10}{27}\left(x-\frac{y}{2}\right)^2\left(3-4\sqrt{5}(x-2y)\right)\right), \\ G &= F+it\left(16t^2+\frac{80}{9}\left(x-\frac{y}{2}\right)^2+4\sqrt{5}(y-x)\right) \\ &-12t^2-\frac{20}{9}\left(x-\frac{y}{2}\right)-\frac{1}{2} \end{aligned} \tag{3.54}$$

Then the higher order rogue wave is formed as in Figure 3.7. Unlike fundamental rogue wave and multi rogue wave, in this case the rogue wave does not disappear without any trace. For instance, at $\zeta = -5$ and $\zeta = 10$ some part of the wave still exists. When the graphs are examined, it can be said that in the case that $\zeta \gg 1$, the solution is localized lump sitting on the constant background. As ζ increases, the lump accelarates. When $\zeta = 1$ the lump disappears and turns to a parabola shaped rogue wave that rises from the background.

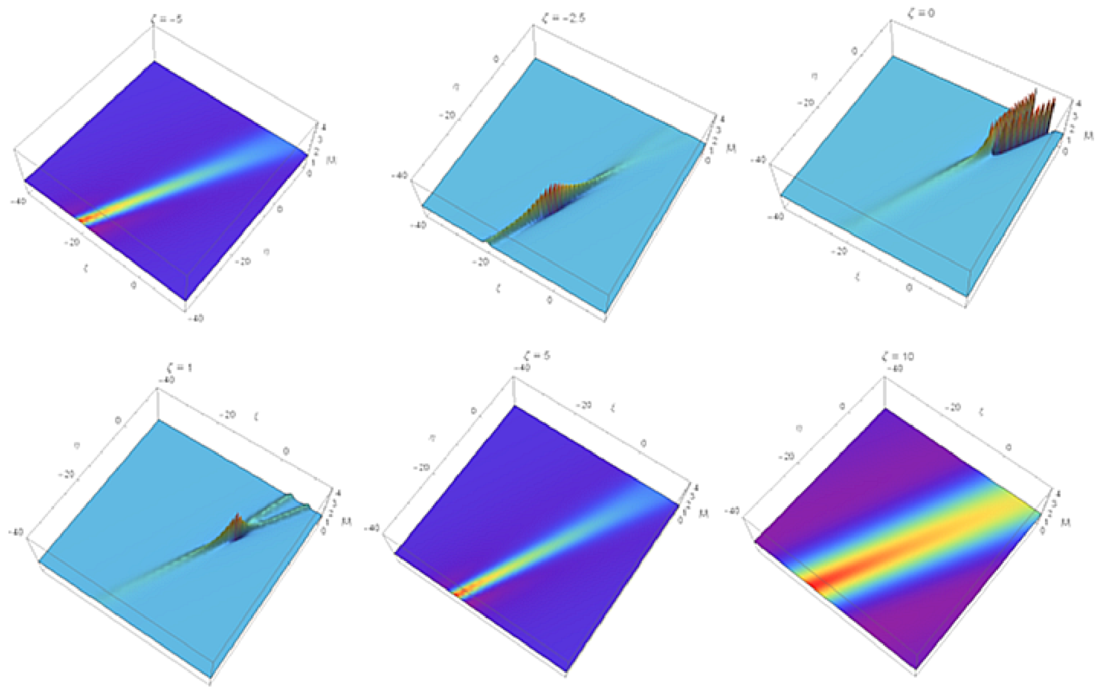

Figure 3.7 : The higher order rogue wave solution for the case $(\gamma_1,\ \beta_1,\ \beta_2) = (1,\ 1,\ 0.5)$ with $p_1 = 1$.

(b) The case $(\gamma_1,\ \beta_1,\ \beta_2) = (0.25,\ 3,\ 0.5)$:

By taking $p_1 = 1$, we have the solution $u = G/(\sqrt{5}F)$ with

$$F = 8\left(t^2 + \frac{45}{128}\left(x - \frac{2y}{3}\right)^2 + \frac{1}{8}\sqrt{10}(2y - x)\right)^2 + 4t^2 + \frac{1}{4}$$
$$+\frac{3x-2y}{128}\left(15\sqrt{10}\left(x - \frac{10y}{3}\right)^2 - \frac{2}{3}\left(15x + 160\sqrt{10}y^2 - 330y + 12\sqrt{10}\right)\right),$$

$$G = F + \frac{1}{8}it\left(16\left(8t^2 - \sqrt{10}(x-2y)\right) + 45\left(x - \frac{2y}{3}\right)^2\right) - 12t^2 - \frac{1}{2}$$
$$-\frac{5}{32}(3x-2y)^2 + \sqrt{\frac{5}{2}}(x-2y). \tag{3.55}$$

Then the higher order rogue wave is formed as in Figure 3.8. Unlike fundamental rogue wave and multi rogue wave, in this case the rogue wave does not disappear without any trace. For instance, at $\zeta = 2.5$ and $\zeta = 5$ some part of the wave still exists. When the graphs are examined, it can be said that in the case that $\zeta \gg 1$ the solution is localized lump sitting on the constant backround. As ζ increases,

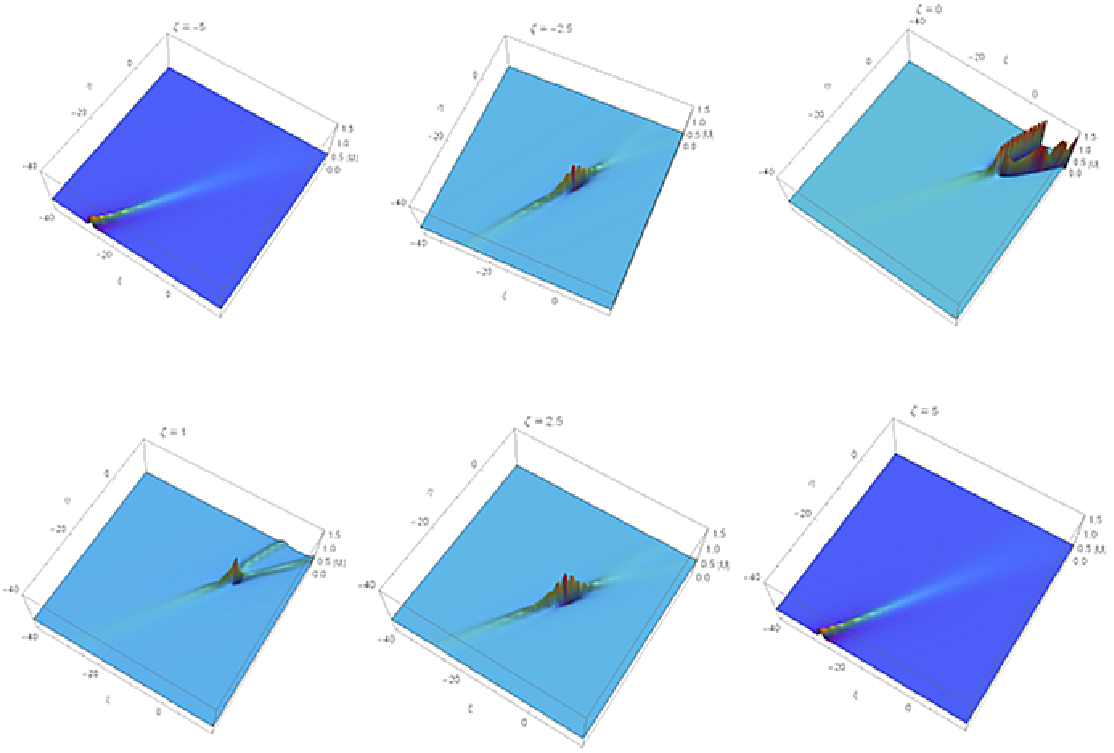

Figure 3.8 : The higher order rogue wave solution for the case $(\gamma_1,\, \beta_1,\, \beta_2) = (0.25,\, 3,\, 0.5)$ with $p_1 = 1$.

the lump accelarates. When $\zeta = 1$ the lump disappears and turns to a parabola shaped rogue wave that rises from the backround.

4. CONCLUSION

The aim of this study is to investigate rogue waves in elliptic-hyperbolic GDS system which is derived in acoustics. Therefore, we focus on NLS type equations since it is the origin of DS system. After giving the classification of NLS, we elaborate on the generalized DS systems in different media such as elastic medium and acoustics. Then, we focus on rogue waves and explain what is rogue wave, origin of the rogue waves and classification of rogue waves. In order to find the rogue wave solutions for elliptic-hyperbolic generalized Davey-Stewartson equation, we mention the method we use which was also suggested in previous studies [17, 18, 22]. The method is finding solutions by determinants of matrices using Hirota Direct method. Therefore, we mention the properties of the direct method and share the proof of lemma and theorem which are related to the rogue wave solution by determinants of matrices. Then, we solve the elliptic-hyperbolic generalized DS system and determine the rogue wave solutions. The results show that the properties of the waves obtained in the solution are compatible with Ohta and Yang's classification of rogue waves as fundamental rogue wave, multi-rogue wave and higher order rogue wave. We search for high amplitude and low amplitude. Starting with the case of fundamental rogue wave, as time approaches to $-\infty$ we observed a constant background and in the intermediate times we investigated fundamental rogue wave with a line profile by appearing suddenly and disappearing into the constant background again as time approaches to ∞. In the case of multi-rogue wave, we searched for $N = 2$ and $N = 3$, which gives two-rogue/three-rogue wave solution. Again starting with constant background as time approaches to $-\infty$, in the intermediate times, the two/three line rogue waves interact with each other and at the intersection region, the highest amplitude is reached. After the intersection region fade, the two/three line rogue waves separately reach their highest amplitudes. As time approaches to ∞, they disappear into the constant background again. In the case of higher-order rogue wave, we investigated a different sitaution compared to the fundamental rogue wave and multi-rogue wave but is compatible with the findings of Ohta and Yang. These waves cannot approach to the constant background uniformly as time goes to $-\infty$ and ∞. Only some parts of the waves approach to the constant background uniformly. There are still some parts remaining in the graph of the wave.

Now, we want to mention as a further study to focus on hyperbolic-elliptic generalized DS equation. Ohta and Yang [18] investigated the rogue waves in DSII equation and emphasized a different type of rogue wave stem from multi rogue wave or higher order rogue wave in DS II equation and named it as exploding rogue wave. Different than the rogue wave solution for DSI equation as using $N \times N$ determinanats, in DSII equation, considering the appropriate variable transformation x_{-2}, x_{-1}, x_1, x_2 from

$$\begin{aligned}
&(D_{x_1} D_{x_{-1}} - 2)FF = -2|G|^2,\\
&(D_{x_1}^2 - D_{x_2})GF = 0,\\
&(D_{x_{-1}}^2 + D_{x_{-2}})GF = 0
\end{aligned} \tag{4.1}$$

into the GDSII equation, the size of the matrices are considered as $2N \times 2N$ due to complex conjugate condition $x_{-1} = -x_1^*$ and $x_{-2} = x_2^*$ We investigate the exploding rogue wave for the following hyperbolic-elliptic generalized DS system through multi-rogue wave solution:

$$\begin{aligned}
&iu_\zeta = u_{\xi\xi} - u_{\eta\eta} + \frac{8}{3}u_{\xi\eta} + u(-2\partial_\xi - \partial_\eta)v - 3|u|^2 u\\
&-\frac{5}{3}v_{\xi\xi} - \frac{5}{3}v_{\eta\eta} = -2(-2\partial_\xi - \partial_\eta)|u|^2
\end{aligned} \tag{4.2}$$

The solution is of the form $u = \sqrt{\frac{2}{3}}\frac{G}{F}$ and $v = -\frac{3}{2}\xi + \eta + \frac{4}{3}(-2\partial_\xi - \partial_\eta)\log F$. For finding $\tau_0 = F$ and $\tau_1 = G$ we can use the τ_n expression given below [18]:

$$\begin{aligned}
\tau_n = \quad & \left((\mu_1 + n)(\mu_2 + n) - \frac{1}{|p_1 - p_2|^2}\right)\left((\mu_1^* - n)(\mu_2^* - n) - \frac{1}{|p_1 - p_2|^2}\right)\\
&+\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{|p_{3-i} - p_{3-j}|}(\mu_i + n)(\mu_j^* - n) + \frac{1}{2}\frac{1}{|p_1 + p_2|^2}\\
&+\frac{1}{16}\left|\frac{p_1 - p_2}{p_1 + p_2}\right|^4,
\end{aligned} \tag{4.3}$$

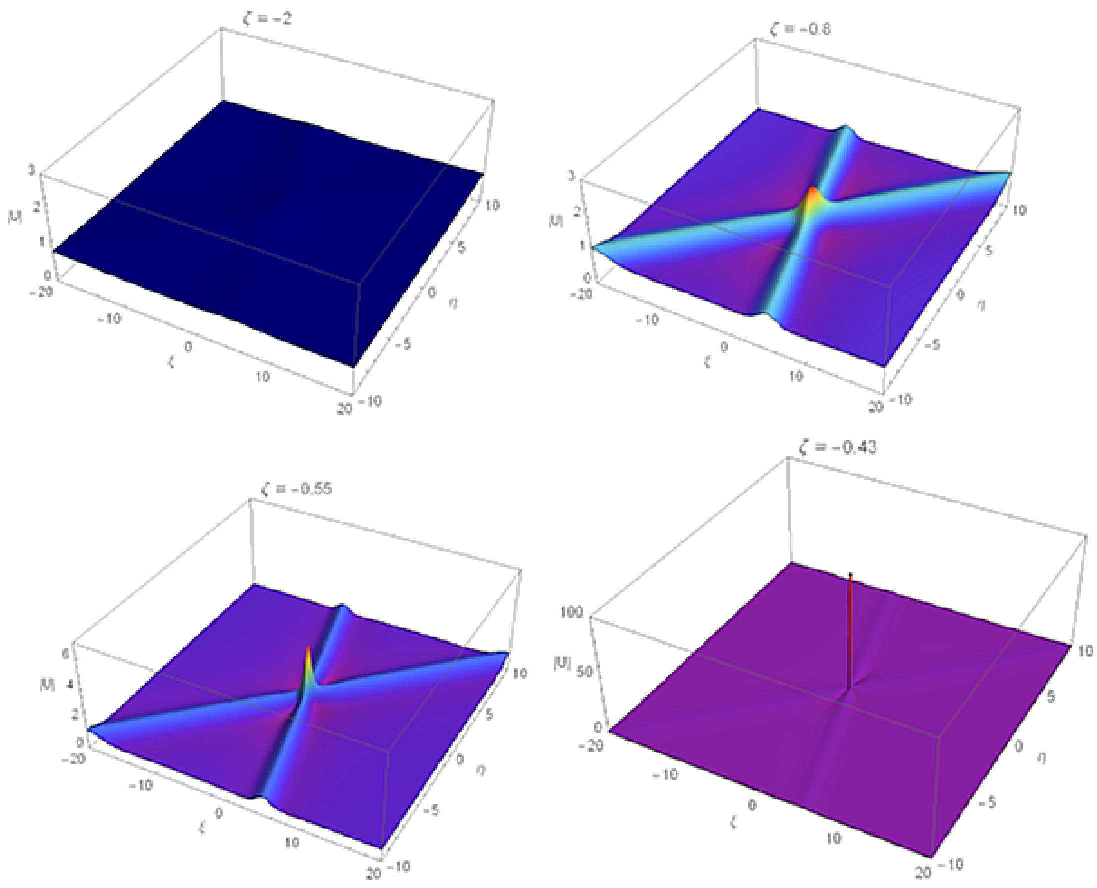

Figure 4.1 : Exploding rogue wave in GDSII.

where for the solution of (4.2), $p_1 = 1$, $p_2 = i$, $\mu_1 = \frac{\sqrt{3}}{5}\xi - \frac{2\sqrt{3}}{5}\eta + 2i\zeta$ and $\mu_2 = -\frac{2\sqrt{3}}{5}\xi - \frac{\sqrt{3}}{5}\eta - 2i\zeta$.

When the change in the amplitude with respect to the time is investigated, Figure 4.1 is obtained. Unlike the two-rogue wave solution in elliptic-hyperbolic GDS system, in this case we obtain that the wave initially has a constant background but as $\zeta \to -0.43$ the amplitude of the wave approaches to 100 as we can say the wave explodes.

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