THE SIGNATURE OF COHERENT SYSTEMS AND ITS APPLICATION TO CONSECUTIVE k-WITHIN-m-OUT-OF-n SYSTEMS

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THE SIGNATURE OF COHERENT SYSTEMS AND ITS APPLICATION TO CONSECUTIVE k-WITHIN-m-OUT-OF-n SYSTEMS

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We have read the thesis entitled **"THE SIGNATURE OF COHERENT SYSTEMS AND ITS APPLICATION TO CONSECUTIVE k-WITHIN-m-OUT-OF-n SYSTEMS"** completed by **CİHANGİR KAN** under supervision of **Assoc. Prof. Dr. Serkan Eryılmaz** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

THE SIGNATURE OF COHERENT SYSTEMS AND ITS APPLICATION TO CONSECUTIVE k-WITHIN-m-OUT-OF-n SYSTEMS

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The signature of a coherent system has been found to be a useful tool in various applications including the evaluation of reliability characteristics of systems, and the comparison of the performance of complex systems. As a generalization of k-out-of-n:F and consecutive k-out-of-n:F systems, the consecutive k-within-m-out-of-n:F system consists of n linearly ordered components such that the system fails iff there are m consecutive components which include among them at least k failed components. In this study, the reliability properties of consecutive k-within-m-out-of-n:F systems with exchangeable components are studied. Bounds and approximations for the survival function are provided. Monte Carlo estimator of system signature is obtained and used to approximate the survival function. The results are illustrated and numerics are provided for exchangeable multivariate Pareto distribution.

Keywords: Exchangeable lifetimes; Mean time to failure; Monte Carlo simulation; Moving order statistics; Multivariate Pareto distribution; System signature.

ÖZ

UYUMLU SİSTEMLERİN İMZASI VE ARDIL n'DEN m İÇİNDE k'LI SİSTEMLERE UYGULAMASI

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Uyumlu sistemlerin imzası; sistemlerin güvenilirlik karakteristiklerinin incelenmesi, rekabet eden sistemlerin performanslarının karşılaştırılması gibi birçok alanda kullanılabilen önemli bir kavram olmuştur. n'den k'lı:F ve ardıl n'den k'lı:F sistemlerin genellemesi olan ve ardıl n'den m içinde k'lı:F sistem olarak adlandırılan sistem n bileşenden oluşmakta ve ardışık m bileşenden en az k tanesinin bozulması durumunda bozulmaktadır. Bu çalışmada; simetrik bağımlı bileşenlerden oluşan ardıl n'den m içinde k'lı:F sistemlerin güvenilirlik özellikleri çalışılmaktadır. İlgili sistemin güvenilirlik fonksiyonu için sınırlar ve yaklaşımlar bulunmuştur. Sistem imzasının Monte Carlo tahmin edicisi elde edilmiş ve bu tahmin edici güvenilirlik fonksiyonunun yaklaşık değerini elde etmek için kullanılmıştır. Elde edilen sonuçlar çok değişkenli Pareto dağılımı ile örneklenmiştir.

Anahtar Kelimeler: Simetrik bağımlı yaşam zamanları; Monte Carlo simülasyonu; Hareketli sıra istatistikleri; Çok değişkenli Pareto dağılımı; Sistem imzası.

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To my family

Contents

Chapter 1

Introduction

The reliability is defined as the probability that a system will perform satisfactorily for at least a given period of time when used under stated conditions. Also reliability known as the ability to perform a required function under given conditions for a given time or time interval, often expressed as a probability. We can define reliability in terms of device, such as performance of a component or a system which consists of components.

Reliability evaluation has a vital importance at all stages of processing and controlling engineering systems. For evaluating the system's reliability, one should specify the structure of the system that defines the rule(s) of the operation and relations between the system components. Early works on system reliability have focused on binary system modeling.

In a binary system modeling, the system and its components may either work or fail. Thus the state of each component or system is a discrete random variable with two possible outcomes. In nonseries systems, it is not necessary that all components must operate for functioning of systems. So the relationship between components and systems are investigated by coherent systems.

If X_i denotes the state of the ith component in the system. Then

$$X_i = \begin{cases} 1 & \text{if } i\text{th component functions,} \\ 0 & \text{if } i\text{th component fails,} \end{cases} \tag{1.1}$$

for $i = 1, 2, ..., n$, where n is the number of components in the system. Similarly, ϕ, which shows the state of system, can be defined as

$$\phi(X_1, X_2, ..., X_n) = \begin{cases} 1 & \text{if system functions,} \\ 0 & \text{if system fails.} \end{cases} \tag{1.2}$$

The function $\phi(\vec{X})$, which is called the structure function of system, shows the state of system as a function of states of components. The component i is said to be irrelevant if and only if

$$\phi(1_i, \vec{X}) = \phi(0_i, \vec{X}) \text{ for all } (._i, \vec{X}) = (X_1, X_2, ..., X_{i-1}, ., X_{i+1}, ..., X_n).$$

If there exists at least one $\vec{X}$ satisfying $\phi(1_i, \vec{X}) = 1$ and $\phi(0_i, \vec{X}) = 0$ it can be said that component is relevant. In words if the state of system can not be affected by the state of ith component then ith component is irrelevant to the system. Below we provide the definitions of coherent system and its dual. For a detailed description and properties of coherent systems we refer to Barlow and Proschan (1975) as well as Kuo and Zuo (2003).

Definition. A system of components is coherent if

$$\begin{array}{ll} i. & \text{its structure function is increasing,} \\ ii. & \text{each component is relevant.} \end{array} \tag{1.3}$$

According to this definition, the following conditions must be satisfied.

1. $\phi(\mathbf{0}) = 0$ which means system is failed when all components are failed.
2. $\phi(\mathbf{1}) = 1$ which means system is operating when all components operate.
3. $\mathbf{x} < \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y})$ which means improvement of any component does not decrease the performance of the system.

4. For every component i, there exists a component state vector such that the state of component i dictates the state of the system.

Reliability of a coherent system consisting of n components can be defined as

$$R = P(\phi(\vec{X}) = 1).$$

Similarly the reliability of the ith component of this system is defined as

$$P(X_i = 1) = p_i \quad \text{for } i = 1, 2, ..., n.$$

Definition. Given structure function ϕ, its dual ϕ^D is given by

$$\phi^D(\vec{X}) = 1 - \phi(1 - \vec{X}),$$

where $(1 - \vec{X}) = (1 - X_1, 1 - X_2, ..., 1 - X_n)$.

In the literature, various reliability models have been defined and studied under different assumptions on components. Undoubtedly, the simplest reliability structures are series and parallel models. A series system with n components operates if all components operate. A parallel system of n components operates if at least one component is in a working state. A k-out-of-n:F system, which consists of n components, fails iff at least k of n components fail. On the other hand, k-out-of-n:G system, which consists of n components, functions iff at least k of n components operate. A linear consecutive k-out-of-n:F system as a generalization of series and parallel systems, consists of n linearly ordered components such that the system fails iff at least k consecutive components fail. A linear consecutive k-out-of-n:F system usually has much higher reliability then the series systems and is less expensive then the parallel systems. As a dual of consecutive k-out-of-n:F system, a consecutive k-out-of-n:G system with n components operates iff at least k consecutive components operate. Consecutive type systems have been used to model telecommunication and oil pipeline systems, and vacuum systems in accelerators. Recent discussions on consecutive k-out-of-n systems appear in the works of Yun et al. (2007), Xiao et al.(2007), Eryılmaz (2007), Navarro and

Eryılmaz (2007), Eryılmaz (2008), Eryılmaz(2009). An excellent review of such systems and their generalizations is presented in Kuo and Zuo (2003).

In Table 1 we present the structure functions and reliabilities of varios coherent structures consisting of n components.

System	*Structure Function*	*Reliability*
Series	$\phi(\vec{X}) = \prod_{i=1}^{n} X_i = \min(X_1, X_2, ..., X_n)$	$P(\sum_{i=1}^{n} X_i = n)$
Parallel	$\phi(\vec{X}) = \coprod_{i=1}^{n} X_i = \max(X_1, X_2, ..., X_n)$	$P(\sum_{i=1}^{n} X_i \geq 1)$
k-out-of-n:F	$\phi(\vec{X}) = \begin{cases} 1, & \sum_{i=1}^{n} X_i > n-k \\ 0, & \sum_{i=1}^{n} X_i \leq n-k \end{cases}$	$P(\sum_{i=1}^{n} X_i > n-k)$
k-out-of-n:G	$\phi(\vec{X}) = \begin{cases} 1, & \sum_{i=1}^{n} X_i \geq k \\ 0, & \sum_{i=1}^{n} X_i < k \end{cases}$	$P(\sum_{i=1}^{n} X_i \geq k)$
Consecutive k-out-of-n:F	$\phi(\vec{X}) = \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} (1 - X_j))$	$P(L_n^0 < k)$
Consecutive k-out-of-n:G	$\phi(\vec{X}) = 1 - \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} X_j)$	$P(L_n^1 \geq k)$

Table 1. Structure functions and reliabilites of various coherent structures

In Table 1, L_n^1 and L_n^0 denote the lengths of longest success and failure runs in $\vec{X}$, respectively.

For example; let the states of $n = 10$ components be $\vec{X} = (0111010011)$. Then we have $L_{10}^1 = 3$ and $L_{10}^0 = 2$. For a detailed description of the longest run random variables we refer to Balakrishnan and Koutras (2002) as well as Fu and Lou (2003).

As a generalization of k-out-of-n:F and consecutive k-out-of-n:F systems, the consecutive k-within-m-out-of-n:F system consisting of n linearly ordered components such that the system fails if and only if there are m consecutive components which include among them at least k failed components. For an illustration, let the states of $n = 10$ components be $\vec{X} = (1100101100)$. Then the system is in a failure state if $m = 4$ and $k = 3$ while it is in a functioning state when $m = 5$ and $k = 4$. This system was first introduced by Griffith (1986). This model includes consecutive k-out-of-n:F and k-out-of-n:F systems when $m = k$ and $m = n$, respectively.

In this study, we investigate the reliability properties of consecutive k-within-m-out-of-n:F systems with exchangeable components. We obtain bounds and approximations for the survival function of this system. The performance of the bounds and approximations is evaluated using Monte Carlo estimator of system signature.

Chapter 2

The signature of coherent systems

One of the most important lifetime characteristic of a coherent system is the survival function defined by

$$R(t) = P(T > t),$$

where T denotes the lifetime of a coherent system.

The evaluation of the function $R(t)$ is of special importance not only for computing the survival probabilities but also for evaluating the other reliability characteristics such as hazard rate, and mean residual lifetime.

Let T_i denote the lifetime of the ith component in a coherent system with the structure function ϕ and lifetime T. Then

$$T = \phi(T_1, T_2, ..., T_n).$$

Define

$$X_i(t) = \begin{cases} 1 & \text{if } T_i > t \\ 0 & \text{if } T_i \leq t \end{cases}, \quad i = 1, 2, ..., n$$

It is clear that the binary stochastic process $X_i(t)$ represents the state of the ith component at time t. The survival function $R(t)$ can be investigated by the help of $X_i(t)$s. For example, the survival function of a k-out-of-n:F system can be written as

$$R(t) = P(\sum_{i=1}^{n} X_i(t) > n - k).$$

Similarly, the survival function of a consecutive k-out-of-n:F system can be expressed as

$$R(t) = P(L_n^0(t) < k),$$

where $L_n^0(t)$ denotes the longest run of 0s(failures) in $X_1(t), X_2(t), ..., X_n(t)$(see, e.g. Eryılmaz (2009)).

A general representation for the survival function of coherent systems can be given in terms of signature. Let T be the lifetime of a coherent system consisting of independent and identical components with the lifetimes $T_1, T_2, ..., T_n$ and common c.d.f. $F(t) = P(T_i \leq t)$, $i = 1, 2, ..., n$. The signature of this system is defined as the probability vector $(p_1, p_2, ..., p_n)$, with

$$p_i = P(T = T_{i:n}) \quad \text{for } i = 1, ..., n, \tag{2.1}$$

where $T_{1:n} \leq T_{2:n} \leq ... \leq T_{n:n}$ are the order statistics associated with $T_1, T_2, ..., T_n$. Equivalently, we have

$$p_i = \frac{\#\text{ of orderings for which the } i\text{th failure causes system failure}}{n!} \tag{2.2}$$

for $i = 1, ..., n$.

A component state vector $\vec{X}$ is a path vector if $\phi(\vec{X}) = 1$. Let $r_i(n)$ be the number of path sets of the system containing exactly i working components. Define

$$a_i(n) = \binom{n}{i}^{-1} r_i(n), \quad i = 1, 2, ..., n, \tag{2.3}$$

through the system of equations

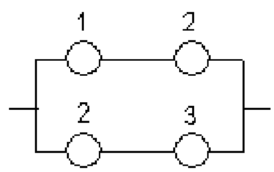
$$a_i(n) = \sum_{j=n-i+1}^{n} p_j(n), \quad i = 1, 2, ..., n, \tag{2.4}$$

or equivalently,

$$p_i(n) = a_{n-i+1}(n) - a_{n-i}(n), \quad i = 1, 2, ..., n. \tag{2.5}$$

That is, the signature of a system can be obtained by computing $r_i(n)$. The problem of finding $r_i(n)$ is combinatorial one. Specifically, determination of the total number of binary sequences satisfying certain conditions which depend on the structure of a system.

Example 2.1. Let us find the signature of the following consecutive 2-out-of-3:G system



We can define the system lifetime T as follows

$$T = \max(\min(T_1, T_2), \min(T_2, T_3)).$$

There are 3! orderings of the component lifetimes which are given as follows.

Ordering	T
$T_1 < T_2 < T_3$	$T_{2:3}$
$T_1 < T_3 < T_2$	$T_{2:3}$
$T_2 < T_1 < T_3$	$T_{1:3}$
$T_2 < T_3 < T_1$	$T_{1:3}$
$T_3 < T_1 < T_2$	$T_{2:3}$
$T_3 < T_2 < T_1$	$T_{2:3}$

Then we have

$$\begin{aligned} p_1 &= P(\underbrace{T_2 < T_1 < T_3}_{T=T_{1:3}}) + P(\underbrace{T_2 < T_3 < T_1}_{T=T_{1:3}}) = \frac{2}{6} \\ p_2 &= 1 - p_1 = \frac{4}{6} \\ p_3 &= 0 \end{aligned}$$

so the signature is $p = (\frac{1}{3}, \frac{2}{3}, 0)$.

According to the following theorem the survival function of any coherent system can be written as a linear combination of the survival functions of order statistics, or equivalently survival functions of i-out-of-n:F systems.

Theorem 2.1 ([13]) *Let $T_1, T_2, ..., T_n$ be the i.i.d. component lifetimes of a coherent system of order n, and let T be the system lifetime. Then*

$$P(T > t) = \sum_{i=1}^{n} p_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j} = \sum_{i=1}^{n} p_i P(T_{i:n} > t). \tag{2.6}$$

Navarro and Rychlik (2007) proved that the representation (2.6) also holds in the case whenever the lifetimes $T_1, T_2, ..., T_n$ have an absolutely continuous exchangeable joint distribution, i.e. the joint distribution (survival function) of $T_1, T_2, ..., T_n$ is invariant under permutation of the variables.

2.1 Signature of consecutive k-out-of-n systems

A linear consecutive k-out-of-n:F system as a generalization of series and parallel systems, consists of n linearly ordered components such that the system fails iff at least k consecutive components fail. A typical path of consecutive k-out-of-$n : F$ systems including i working components is given by

$$1\underbrace{\ldots}_{y_1}1 0\underbrace{\ldots}_{x_1}0 1\underbrace{\ldots}_{y_2}1 0\underbrace{\ldots}_{x_2}0 \ldots\ldots\ldots 0\underbrace{\ldots}_{x_r}0 1\underbrace{\ldots}_{y_{r+1}}1,$$

where $0 < x_j < k$, $j = 1, \ldots, r$ and $y_1 \geq 0, y_2 > 0, \ldots, y_r > 0, y_{r+1} \geq 0$. Thus the number of path sets of the system containing exactly i working components is given by

$$r_i(n) = \sum_{r=0}^{n} N(i, r, k, n),$$

where $N(i, r, k, n)$ denotes the number of simultaneous integer solutions for the systems

$$x_1 + x_2 + \cdots + x_r = n - i$$

such that $0 < x_j < k$, $j = 1, \ldots, r$ and

$$y_1 + y_2 + \cdots + y_{r+1} = i$$

such that $y_1 \geq 0, y_2 > 0, \ldots, y_r > 0, y_{r+1} \geq 0$.

After some combinatorial calculations $N(i, r, k, n)$ is obtained as

$$N(i, r, k, n) = \binom{i+1}{r} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{n-i-(k-1)j-1}{r-1}$$

In Table 2 we provide signatures of consecutive k-out-of-n:F systems for some choices of k and n.

n	k	$\mathbf{p}$
4	2	$(0, \frac{1}{2}, \frac{1}{2}, 0)$
5	2	$(0, \frac{4}{10}, \frac{5}{10}, \frac{1}{10}, 0)$
5	3	$(0, 0, \frac{3}{10}, \frac{5}{10}, \frac{2}{10})$
6	2	$(0, \frac{5}{15}, \frac{7}{15}, \frac{3}{15}, 0, 0)$
6	3	$(0, 0, \frac{2}{10}, \frac{4}{10}, \frac{4}{10}, 0)$

Table 2. Signatures of consecutive k-out-of-n:F systems for some choices of k and n.

Chapter 3

Reliability of consecutive k–within-m-out-of-n:F systems

As we stated before, a consecutive k-within-m-out-of-n:F system consisting n linearly ordered components such that the system fails if and only if there are m consecutive components which include among them at least k failed components. The names k-within-consecutive-m-out-of-n:F and consecutive k-out-of-m-from-n:F have also been used for this system in the literature.

As we mentioned before there are many application areas of these systems such as quality control systems, and radar detection. Various bounds and approximations for the reliability of consecutive k-within-m-out-of-n:F system consisting of independent components have been proposed in the literature. For example, Sfakianakis et al. (1992) provided lower and upper bounds for the reliability of such systems which consist of independent identical components. Iyer (1992) studied the distribution of the lifetime of this system with independent exponentially distributed component lifetimes. Papastavridis and Koutras (1993) presented upper and lower bounds for the reliability of a linear and circular systems consisting of independent nonidentical components. Habib and Szantai (2000) improved the bounds obtained by Sfakianakis et al. (1992) by applying higher order Boole-Bonferroni bounds. Recently, Habib et al. (2007) presented an algorithm to compute the reliability of multi-state consecutive k-within-m-out-of-n:G system which is the generalization of consecutive k-within-m-out-of-n:G system to the multi-state case.

Dependence among component lifetimes emerges from the common random production and operating environments. Analysis of systems that consist of dependent components might be difficult especially whenever the system has a complex structure. We study the reliability properties of consecutive k-within-m-out-of-n:F system which consists of exchangeable components. Systems with exchangeable components have been widely studied in the literature. See e.g. Shanthikumar (1985), Papastavridis (1989), Navarro et al. (2005), Bassan and Spizzichino (2005), Navarro and Rychlik (2007), Navarro (2008).

We first provide some notations and properties that will be used throught this chapter. In section 3.2 we provide bounds and approximations for the survival function of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. Then we develop a method based on Samaniego's signature for simulating the reliability characteristics of the corresponding system. Finally we provide numerical illustrations whenever the lifetimes of components have exchangeable Pareto distribution.

3.1 Notations and preliminaries

- $T_{k:m}^{(j)}$ kth smallest among $T_j, T_{j+1}, ..., T_{j+m-1}$, $k \leq m, 1 \leq j \leq n-m+1$;
- $Y_i(t) = \begin{cases} 1 & \text{if } T_i \leq t \\ 0 & \text{if } T_i > t \end{cases}$
- A_j the event of $\left\{T_{k:m}^{(j)} > t\right\}$;
- $T_{k,m:n}$ the lifetime of consecutive k-within-m-out-of-n:F system, $1 \leq k \leq m \leq n$;

- $R_{k,m:n}(t) = P\{T_{k,m:n} > t\}$ the survival function of consecutive k-within-m-out-of-n:F system

The main goal of this thesis is to study the reliability properties of consecutive k-within-m-out-of-n:F system with exchangeable lifetimes. A sequence of lifetimes $T_1, T_2, ..., T_n$ is exchangeable if for each n,

$$P\{T_1 \leq t_1, ..., T_n \leq t_n\} = P\left\{T_{\pi(1)} \leq t_1, ..., T_{\pi(n)} \leq t_n\right\},$$

for any permutation $\pi = (\pi(1), ..., \pi(n))$ of $\{1, 2, ..., n\}$, i.e. the joint distribution (survival function) of $T_1, T_2, ..., T_n$ is symmetric in $t_1, t_2, ..., t_n$. The results obtained in this study readily hold for a system with i.i.d. lifetimes since a sequence of independent, identically distributed (i.i.d.) lifetimes is exchangeable.

Consecutive k-within-m-out-of-n:F system can be represented as a series system of $n - m + 1$ dependent k-out-of-m:F systems. That is,

$$T_{k,m:n} = \min(T_{k:m}^{(1)}, T_{k:m}^{(2)}, ..., T_{k:m}^{(n-m+1)}), \tag{3.1}$$

where $T_{k:m}^{(j)}$ shows the lifetime of k-out-of-m:F system of components with the lifetimes $T_j, T_{j+1}, ..., T_{j+m-1}$, $1 \leq j \leq n - m + 1$. It is clear that the random variables $T_{k:m}^{(1)}, T_{k:m}^{(2)}, ..., T_{k:m}^{(n-m+1)}$ have the common terms and this makes the problem of finding the exact reliability difficult especially whenever $T_1, T_2, ..., T_n$ are dependent which is the case in this study. The random variables $T_{k:m}^{(j)}$, $1 \leq j \leq n-m+1$ are known as moving order statistics in the literature. Although the theory of usual order statistics has been well developed in the literature not much work has been done for moving order statistics. We may refer to David and Nagaraja (2003, p.140) for limited results on moving order statistics.

Using (3.1), the survival function of consecutive k-within-m-out-of-n:F system can be written as

$$R_{k,m:n}(t) = P\{T_{k,m:n} > t\} = P\left\{T_{k:m}^{(1)} > t, T_{k:m}^{(2)} > t, ..., T_{k:m}^{(n-m+1)} > t\right\}.$$

Consider the random variable $S_m^{(j)}(t) = \sum_{i=j}^{j+m-1} Y_i(t)$ which denotes the total number of failed components among $T_j, T_{j+1}, ..., T_{j+m-1}$ at time t. By the exchangeability we have

$$\begin{aligned} P\left\{S_m^{(j)}(t) = s\right\} &= P\left\{S_m^{(1)}(t) = s\right\} \\ &= \binom{m}{s} \sum_{i=0}^{m-s} (-1)^i \binom{m-s}{i} P\left\{T_1 \le t, ..., T_{s+i} \le t\right\} \\ &= \binom{m}{s} \sum_{i=0}^{s} (-1)^i \binom{s}{i} P\left\{T_1 > t, ..., T_{m-s+i} > t\right\} \end{aligned} \tag{3.2}$$

The latter equations can be obtained using Theorem 2.1 of George and Bowman (1995). For simplicity hereafter we will use the following notation.

$$f(a,b) = \sum_{i=0}^{a} (-1)^i \binom{a}{i} P\left\{T_1 \le t, ..., T_{b+i} \le t\right\},$$

and

$$g(a,b) = \sum_{i=0}^{a} (-1)^i \binom{a}{i} P\left\{T_1 > t, ..., T_{b+i} > t\right\}.$$

With the notation given above, equation (3.2) can be rewritten as

$$P\left\{S_m^{(j)}(t) = s\right\} = \binom{m}{s} f(m-s, s) = \binom{m}{s} g(s, m-s).$$

3.2 Bounds and approximations for the survival function

In this section, we evaluate the probability

$$R_{k,m:n}(t) = P\left\{\bigcap_{i=1}^{n-m+1} A_i\right\}, \tag{3.3}$$

using various inequalities. We first obtain a lower bound using the second order Bonferroni inequality which is also known as Hunter-Worsley inequality (Hunter (1976), Worsley (1982)). This variant of Bonferroni inequality has been found to be very quick and useful for the reliability evaluation of consecutive k-within-m-out-of-n:F system consisting of i.i.d. components (Habib and Szantai, 2000).The proofs of the following Theorems are presented in Appendix.

Theorem 3.1 *Let $(T_1, T_2, ..., T_n)$ be an exchangeable random vector representing the lifetimes. Then for $1 \leq k \leq m \leq n$,*

$$R_{k,m:n}(t) \geq 1 - (n-m+1)P\left\{T_{k:m}^{(1)} \leq t\right\} + (n-m)P\left\{T_{k:m}^{(1)} \leq t,\ T_{k:m}^{(2)} \leq t\right\},$$

where

$$P\left\{T_{k:m}^{(1)} \leq t\right\} = \sum_{s=k}^{m} \binom{m}{s} f(m-s,\ s) = \sum_{s=k}^{m} \binom{m}{s} g(s,\ m-s),$$

and

$$P\left\{T_{k:m}^{(1)} \leq t,\ T_{k:m}^{(2)} \leq t\right\} = \binom{m-1}{k-1} f(m-k,\ k+1) + \sum_{l=k}^{m-1} \binom{m-1}{l} f(m-l-1,\ l), \tag{3.4}$$

or in terms of the joint survival function

$$\begin{aligned} P\left\{T_{k:m}^{(1)} \leq t,\ T_{k:m}^{(2)} \leq t\right\} &= \tbinom{m-1}{m-k}\left[g(k-1,\ m-k) - 2g(k-1,\ m-k+1)\right. \\ &\quad \left. + g(k-1,\ m-k+2)\right] \\ &\quad + \sum_{l=k}^{m-1} \tbinom{m-1}{l} g(l, m-l-1). \end{aligned} \tag{3.5}$$

The probability given in (3.4) ((3.5)) can be easily calculated if the joint distribution (survival) function of lifetimes of the components is given.

An approximation formula for the survival function can also be obtained using the following product-type approximation formula (see, e.g. Costigan (1996)).

$$R_{k,m:n}(t) = P\left\{\bigcap_{i=1}^{n-m+1} A_i\right\} \simeq \frac{\prod_{i=2}^{n-m+1} P\left\{A_{i-1}A_i\right\}}{\prod_{i=2}^{n-m} P\left\{A_i\right\}} = \frac{\left[P\left\{A_1A_2\right\}\right]^{n-m}}{\left[P\left\{A_1\right\}\right]^{n-m-1}}, \tag{3.6}$$

where the last equation follows from exchangeability. The probabilities in (3.6) can be easily evaluated using the equations given in Theorem 3.1. For example,

$$\begin{aligned} P\left\{A_1A_2\right\} &= P\left\{T_{k:m}^{(1)} > t, T_{k:m}^{(2)} > t\right\} \\ &= 1 - P\left\{T_{k:m}^{(1)} \leq t\right\} - P\left\{T_{k:m}^{(2)} \leq t\right\} + P\left\{T_{k:m}^{(1)} \leq t, T_{k:m}^{(2)} \leq t\right\} \end{aligned} \tag{3.7}$$

It should be noted that the probability given by (3.7) with $m = n-1$ is actually the exact survival function of consecutive k-within-$(n-1)$-out-of-n:F system. The following Theorem provides an upper bound for the reliability of consecutive k-within-m-out-of-n system consisting of exchangeable components.

Theorem 3.2 *Let $(T_1, T_2, ..., T_n)$ be an exchangeable random vector. Then for $1 \le k \le m \le n$,*

$$\begin{aligned} R_{k,m:n}(t) &\le \sum_{j_1,j_2,...,j_r=0}^{k-1} \binom{m}{j_1} ... \binom{m}{j_r} f\left(r.m - \sum_{i=1}^{r} j_i, \sum_{i=1}^{r} j_i\right) \\ &= \sum_{j_1,j_2,...,j_r=0}^{k-1} \binom{m}{j_1} ... \binom{m}{j_r} g\left(\sum_{i=1}^{r} j_i, r.m - \sum_{i=1}^{r} j_i\right), \end{aligned}$$

where $r = \left[\frac{n}{m}\right]$.

3.3 Simulation based on Samaniego's signature

The system with exchangeable components has the same signature vector with the system with i.i.d. components. This is crucial for the development of our simulation. Simulation of the lifetime of consecutive k-within-m-out-of-n:F system without using this fact needs to generate random vectors from the distribution $F(t_1, t_2, ..., t_n) = P\{T_1 \le t_1, ..., T_n \le t_n\}$. This is not an easy task. Therefore we first obtain the Monte Carlo estimates of the signature of consecutive k-within-m-out-of-n:F system consisting of i.i.d. components and then use these estimates to estimate the survival function of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. That is, the estimator of survival function is given by

$$\hat{R}_{k,m:n}(t) = \sum_{i=1}^{n} \hat{p}_i P\{T_{i:n} > t\}, \tag{3.8}$$

where $\hat{p}_i$ is the Monte Carlo estimate of the ith element of the signature vector and

$$\begin{aligned} P\left\{T_{i:n}>t\right\} &= 1-\sum_{j=i}^{n}(-1)^{j-i}\binom{j-1}{i-1}\binom{n}{j}P\left\{T_{j:j}\leq t\right\} \\ &= 1-\sum_{j=n-i+1}^{n}(-1)^{j-n+i-1}\binom{j-1}{n-i}\binom{n}{j}P\left\{T_{1:j}\leq t\right\}, \end{aligned}$$

where $T_{1:j} = \min(T_1,...,T_j)$ and $T_{j:j} = \max(T_1,...,T_j)$. We readily have $P\left\{T_{1:j}\leq t\right\} = 1-P\left\{T_1>t,...,T_j>t\right\}$, $P\left\{T_{j:j}\leq t\right\} = P\left\{T_1\leq t,...,T_j\leq t\right\}$.

In Table 3 we present the order statistic representation of the lifetime of consecutive 2-within-3-out-of-4:F system by writing out all possible permutations of T_1, T_2, T_3, T_4. From Table 3 we compute

$$\begin{aligned} p_1 &= P\left\{T_{2,3:4}=T_{1:4}\right\}=0, \\ p_2 &= P\left\{T_{2,3:4}=T_{2:4}\right\}=20/24, \\ p_3 &= P\left\{T_{2,3:4}=T_{3:4}\right\}=4/24, \\ p_4 &= P\left\{T_{2,3:4}=T_{4:4}\right\}=0. \end{aligned}$$

In the Table 4, we present the Monte Carlo estimate $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, ..., \hat{p}_n)$ for various values of n, m, and k.

Ordering	$T_{2,3:4}$	Ordering	$T_{2,3:4}$
$T_1 < T_2 < T_3 < T_4$	$T_{2:4}$	$T_3 < T_1 < T_4 < T_2$	$T_{2:4}$
$T_1 < T_2 < T_4 < T_3$	$T_{2:4}$	$T_3 < T_1 < T_2 < T_4$	$T_{2:4}$
$T_1 < T_3 < T_4 < T_2$	$T_{2:4}$	$T_3 < T_2 < T_1 < T_4$	$T_{2:4}$
$T_1 < T_3 < T_2 < T_4$	$T_{2:4}$	$T_3 < T_2 < T_4 < T_1$	$T_{2:4}$
$T_1 < T_4 < T_3 < T_2$	$T_{3:4}$	$T_3 < T_4 < T_2 < T_1$	$T_{2:4}$
$T_1 < T_4 < T_2 < T_3$	$T_{3:4}$	$T_3 < T_4 < T_1 < T_2$	$T_{2:4}$
$T_2 < T_1 < T_3 < T_4$	$T_{2:4}$	$T_4 < T_1 < T_3 < T_2$	$T_{3:4}$
$T_2 < T_1 < T_4 < T_3$	$T_{2:4}$	$T_4 < T_1 < T_2 < T_3$	$T_{3:4}$
$T_2 < T_3 < T_4 < T_1$	$T_{2:4}$	$T_4 < T_2 < T_1 < T_3$	$T_{2:4}$
$T_2 < T_3 < T_1 < T_4$	$T_{2:4}$	$T_4 < T_2 < T_3 < T_1$	$T_{2:4}$
$T_2 < T_4 < T_1 < T_3$	$T_{2:4}$	$T_4 < T_3 < T_1 < T_2$	$T_{2:4}$
$T_2 < T_4 < T_3 < T_1$	$T_{2:4}$	$T_4 < T_3 < T_2 < T_1$	$T_{2:4}$

Table 3. Order statistic representation of consecutive 2-within-3-out-of-4:F system.

n	m	k	$\hat{\mathbf{p}}$
4	3	2	$(0, 0.8320, 0.1700, 0)$
10	3	2	$(0, 0.3855, 0.4611, 0.1646, 0.0049, 0, 0, 0, 0, 0)$
10	7	2	$(0, 0.8683, 0.1323, 0, 0, 0, 0, 0, 0, 0)$
10	7	5	$(0, 0, 0, 0, 0.2594, 0.4464, 0.2523, 0.0350, 0, 0)$
15	7	5	$(0, 0, 0, 0, 0.0481, 0.1447, 0.2498, 0.2901,$ $0.2264, 0.0345, 0, 0, 0, 0, 0)$
15	10	4	$(0, 0, 0, 0.4610, 0.4055, 0.1206, 0.0102, 0, 0, 0,$ $0, 0, 0, 0, 0)$
20	10	7	$(0, 0, 0, 0, 0, 0, 0.0133, 0.0547, 0.1336, 0.2139,$ $0.2612, 0.2189, 0.1068, 0, 0, 0, 0, 0, 0, 0)$
20	10	9	$(0, 0, 0, 0, 0, 0, 0, 0, 0.0011, 0.0043, 0.0150, 0.0456,$ $0.0965, 0.1670, 0.2359, 0.2488, 0.1720, 0, 0, 0)$

Table 4. Monte Carlo estimates of system signature for some choices of n,m, and k.

Via the same simulation method we can also approximate the other reliability characteristics of consecutive k-within-m-out-of-n:F system. For example mean time to failure (MTTF) of the system can be estimated from

$$\hat{E}\left(T_{k,m:n}\right) = \sum_{i=1}^{n} \hat{p}_i E(T_{i:n}).$$

3.4 Numerical results

In this section we present some numerical results when $(T_1, T_2, ..., T_n)$ has a multivariate Pareto distribution whose survival function is

$$\bar{F}_a(t_1, ..., t_n) = \left(\sum_{i=1}^{n} t_i - n + 1 \right)^{-a}, \quad a > 0, t_i > 1, i = 1, ..., n.$$

It is easy to see that $(T_1, ..., T_n)$ is exchangeable, and

$$\begin{aligned} P\left\{T_{1:j} \leq t\right\} &= 1 - P\left\{T_1 > t, ..., T_j > t\right\}, \\ &= 1 - \bar{F}_a(t, ..., t), \\ &= 1 - \left(j(t-1)+1\right)^{-a}. \end{aligned}$$

Thus

$$P\left\{T_{i:n} > t\right\} = 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \left(1 - \left(j(t-1)+1\right)^{-a}\right).$$

On the other hand, if $a > 1$, then $E(T_{1:j}) = \frac{1}{j(a-1)}$, and hence

$$E\left(T_{i:n}\right) = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \frac{1}{j(a-1)}.$$

We were able to compute the precise value of $\mathbf{p}$ for small values of n generating all the permutations of numbers from 1 up to n. The precise value of $\mathbf{p}$ for $n = 5$, $m = 3$, $k = 2$ is found to be $\mathbf{p} = (0, \frac{84}{120}, \frac{36}{120}, 0, 0)$. This allows computation of the exact value of the survival function for $n = 5$ as provided in Table 5. This table also includes the bounds and approximations for the survival function. From Table 5 it can be observed that the approximation based on simulation is rather effective, which suggests (3.8) could be used as a reference value for larger n where the computation of the exact value is not possible.

The simulation results along with the bounds and approximations for the survival function are presented in Tables 6,7,8 and 9 for $n = 15$, $m = 12$, $k = 8$; $n = 15$, $m = 10$, $k = 8$; $n = 30$, $m = 10$, $k = 8$, and $n = 30$, $m = 10$, $k = 6$ respectively. In these tables, $\tilde{R}_{k,m:n}(t)$ denotes the approximation computed from (3.6) and $\hat{R}_{k,m:n}(t)$ shows the simulated reliability given in (3.8). LB and UB denote the lower and upper bounds given in Theorem 3.1 and Theorem 3.2, respectively. We also compute $(LB + UB)/2$ as an alternative approximation. The performance of the approximation computed from (3.6) is relatively effective if m is close enough to n or/and k is close enough to m. That is, the closer m to n or/and the closer k to m the better approximation. The approximation computed from $(LB + UB)/2$ seems stronger for larger n when m and k are fixed. We also observe that for fixed a, the bounds and approximations perform better for smaller values of t (or equivalently for highly reliable structures).

a	t	$R_{k,m:n}(t)$	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{LB+UB}{2}$
1.5	1.1	0.8760	0.8735	0.8771	0.8725	0.9329	0.9027
	1.3	0.5861	0.5786	0.5871	0.5711	0.7187	0.6449
	1.5	0.4138	0.4061	0.4132	0.3945	0.5547	0.4746
	1.7	0.3099	0.3031	0.3102	0.2903	0.4404	0.3653
	1.9	0.2426	0.2369	0.2424	0.2242	0.3593	0.2917
2.0	1.1	0.8205	0.8170	0.8192	0.8150	0.8999	0.8574
	1.3	0.4644	0.4571	0.4638	0.4450	0.6179	0.5314
	1.5	0.2873	0.2813	0.2874	0.2656	0.4300	0.3478
	1.7	0.1935	0.1892	0.1930	0.1737	0.3127	0.2432
	1.9	0.1388	0.1356	0.1389	0.1217	0.2366	0.1791

Table 5. Bounds, approximations, and exact value for the survival function when $n = 5$, $m = 3$, $k = 2$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{LB+UB}{2}$
1.5	1.1	0.9927	0.9858	0.9927	0.9960	0.9944
	1.3	0.8340	0.8473	0.8332	0.8772	0.8552
	1.5	0.6441	0.6524	0.6422	0.7066	0.6744
	1.7	0.5032	0.5205	0.5007	0.5675	0.5341
	1.9	0.4034	0.4122	0.4006	0.4635	0.4321
2.0	1.1	0.9852	0.9900	0.9852	0.9917	0.9885
	1.3	0.7358	0.7493	0.7342	0.7973	0.7657
	1.5	0.4979	0.5052	0.4946	0.5709	0.5327
	1.7	0.3481	0.3562	0.3444	0.4134	0.3789
	1.9	0.2544	0.2592	0.2508	0.3091	0.2800

Table 6. Bounds and approximations for the survival function when $n = 15$, $m = 12$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{LB+UB}{2}$
1.5	1.1	0.9973	0.9869	0.9973	0.9990	0.9982
	1.3	0.8964	0.9055	0.8955	0.9430	0.9192
	1.5	0.7366	0.7386	0.7326	0.8247	0.7787
	1.7	0.6000	0.6200	0.5932	0.7043	0.6487
	1.9	0.4951	0.5120	0.4867	0.6015	0.5441
2.0	1.1	0.9942	0.9841	0.9942	0.9979	0.9960
	1.3	0.8260	0.8371	0.8235	0.8989	0.8612
	1.5	0.6090	0.6374	0.6009	0.7237	0.6623
	1.7	0.4501	0.4698	0.4387	0.5690	0.5038
	1.9	0.3416	0.3571	0.3291	0.4508	0.3900

Table 7. Bounds and approximations for the survival function when $n = 15$, $m = 10$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{LB+UB}{2}$
1.5	1.1	0.9920	0.9925	0.9920	0.9973	0.9947
	1.3	0.7700	0.8169	0.7528	0.8841	0.8184
	1.5	0.5249	0.6394	0.4563	0.7095	0.5829
	1.7	0.3710	0.4873	0.2599	0.5664	0.4131
	1.9	0.2760	0.3967	0.1420	0.4601	0.3010
2.0	1.1	0.9832	0.9898	0.9831	0.9941	0.9886
	1.3	0.6409	0.7281	0.5974	0.8045	0.7009
	1.5	0.3629	0.4782	0.2328	0.5702	0.4015
	1.7	0.2228	0.3334	0.0478	0.4083	0.2280
	1.9	0.1485	0.2434	0.0000	0.3024	0.1512

Table 8. Bounds and approximations for the survival function when $n = 30$, $m = 10$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{LB+UB}{2}$
1.5	1.1	0.9214	0.9619	0.9192	0.9677	0.9434
	1.3	0.4587	0.5997	0.3555	0.6793	0.5174
	1.5	0.2463	0.3965	0.0596	0.4655	0.2666
	1.7	0.1540	0.2789	0.0000	0.3384	0.1692
	1.9	0.1066	0.2107	0.0000	0.2589	0.1295
2.0	1.1	0.8609	0.9084	0.8539	0.9402	0.8970
	1.3	0.2981	0.4506	0.1088	0.5374	0.3231
	1.5	0.1257	0.2422	0.0000	0.3100	0.1550
	1.7	0.0666	0.1513	0.0000	0.1976	0.0988
	1.9	0.0407	0.1035	0.0000	0.1360	0.0680

Table 9. Bounds and approximations for the survival function when $n = 30$, $m = 10$, $k = 6$.

Chapter 4

Summary and Conclusions

In the main part of this thesis, we studied the reliability of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. Bounds and approximations based on the probabilities associated with moving order statistics were provided for the survival function of this system. The formulas have been represented both in terms of the joint c.d.f. and the joint survival function of $T_1, T_2, ..., T_n$ so that the computations can be easily performed if either the joint c.d.f. or joint survival is known.

A simulation study based on Samaniego's signature was also performed to estimate the system reliability. The proposed method does not need to generate random vectors from the joint distribution of $T_1, T_2, ..., T_n$, a difficult task in Monte Carlo simulation. By this method we can also estimate the other reliability characteristics of systems consisting of exchangeable components.

The performance of the approximations is satisfactory under particular selections of k, m, and n. The results obtained are readily applicable for consecutive k-out-of-n:F $(m = k)$ and k-out-of-n:F $(m = n)$ systems which consist of exchangeable components. With a slight modification similar results can also be obtained for consecutive k-within-m-out-of-n:G systems.

Chapter 5

Appendix

Proof of Theorem 3.1: According to the Hunter-Worsley variant of Bonferroni inequality we have

$$P\left\{\bigcup_{i=1}^{n} C_i\right\} \leq \sum_{i=1}^{n} P\left\{C_i\right\} - \sum_{i=1}^{n-1} P\left\{C_i C_{i+1}\right\}.$$

Using this inequality for (3.3) one obtains

$$R_{k,m:n}(t) \geq 1 - \sum_{i=1}^{n-m+1} P\left\{T_{k:m}^{(i)} \leq t\right\} + \sum_{i=1}^{n-m} P\left\{T_{k:m}^{(i)} \leq t, T_{k:m}^{(i+1)} \leq t\right\}.$$

By the exchangeability we have

$$R_{k,m:n}(t) \geq 1-(n-m+1)P\left\{T_{k:m}^{(1)} \leq t\right\}+(n-m)P\left\{T_{k:m}^{(1)} \leq t, T_{k:m}^{(2)} \leq t\right\}. \quad (5.1)$$

The probabilities in (5.1) can be computed using the following equations.

$$P\left\{T_{k:m}^{(1)} \leq t\right\} = P\left\{\sum_{i=1}^{m} Y_i(t) \geq k\right\}$$

$$= \sum_{s=k}^{m} \binom{m}{s} f(m-s, s) = \sum_{s=k}^{m} \binom{m}{s} g(s, m-s), \quad (5.2)$$

and

$$P\left\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\right\} = P\left\{\sum_{i=1}^{m} Y_i(t) \ge k, \sum_{i=2}^{m+1} Y_i(t) \ge k\right\}$$
$$= P\left\{Y_1(t) + \sum_{i=2}^{m} Y_i(t) \ge k, \sum_{i=2}^{m} Y_i(t) + Y_{m+1}(t) \ge k\right\}$$
$$= \sum_{l} P\left\{Y_1(t) \ge k-l, Y_{m+1}(t) \ge k-l, \sum_{i=2}^{m} Y_i(t) = l\right\}. \tag{5.3}$$

Consider the probability in (5.3). It is clear that

$$P\left\{Y_1(t) \ge k-l, Y_{m+1}(t) \ge k-l, \sum_{i=2}^{m} Y_i(t) = l\right\}$$
$$= \begin{cases} P\left\{\sum\limits_{i=2}^{m} Y_i(t) = l\right\} & \text{if } k \le l \\ P\left\{Y_1(t) = 1, Y_{m+1}(t) = 1, \sum\limits_{i=2}^{m} Y_i(t) = l\right\} & \text{if } k = l+1 \\ 0 & \text{if } k > l+1. \end{cases}$$

Thus

$$P\left\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\right\} = P\left\{Y_1(t) = 1, Y_{m+1}(t) = 1, S_{m-1}^{(2)}(t) = k-1\right\} + \sum_{l=k}^{m-1} P\left\{S_{m-1}^{(2)}(t) = l\right\} \tag{5.4}$$

$$= \binom{m-1}{k-1} \sum_{i=0}^{m-k} (-1)^i \binom{m-k}{i} P\left\{T_1 \le t, ..., T_{k+i+1} \le t\right\} \tag{5.5}$$
$$+ \sum_{l=k}^{m-1} \binom{m-1}{l} \sum_{i=0}^{m-l-1} (-1)^i \binom{m-l-1}{i} P\left\{T_1 \le t, ..., T_{l+i} \le t\right\}.$$

Therefore the proof of (3.4) is completed.

For the proof of (3.5) we need to write (5.4) in terms of joint survival function (or $g(a,b)$). It is clear that

$$\begin{aligned}
& P\left\{Y_1(t)=1, Y_{m+1}(t)=1, S^{(2)}_{m-1}(t)=k-1\right\} \\
= \; & P\left\{E_{k,m}\right\} - P\left\{E_{k,m}\cap\{T_1>t\}\right\} \\
& -P\left\{E_{k,m}\cap\{T_{m+1}>t\}\right\} + P\left\{E_{k,m}\cap\{T_1>t\}\cap\{T_{m+1}>t\}\right\},
\end{aligned} \tag{5.6}$$

where $E_{k,m}$ denotes the event of $\{m-k$ of $T_2, T_3, ..., T_m$ are greater than $t\}$. Thus we have

$$\begin{aligned}
P\left\{E_{k,m}\right\} &= \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\left\{T_1>t,...,T_{m-k+i}>t\right\} \\
&= \binom{m-1}{m-k}g(k-1,m-k),
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
P\left\{E_{k,m}\cap\{T_1>t\}\right\} &= P\left\{E_{k,m}\cap\{T_{m+1}>t\}\right\} \\
&= \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\left\{T_1>t,...,T_{m-k+i+1}>t\right\} \\
&= \binom{m-1}{m-k}g(k-1,m-k+1),
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
& P\left\{E_{k,m}\cap\{T_1>t\}\cap\{T_{m+1}>t\}\right\} \\
= \; & \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\left\{T_1>t,...,T_{m-k+i+2}>t\right\} \\
= \; & \binom{m-1}{m-k}g(k-1,m-k+2).
\end{aligned} \tag{5.9}$$

Using (5.7)-(5.9) in (5.6) and considering (5.6) in (5.4), the proof of (3.5) is completed.

Proof of Theorem 3.2: It is clear that

$$\begin{aligned}
R_{k,m:n}(t) &= P\left\{T^{(1)}_{k:m}>t, T^{(2)}_{k:m}>t, ..., T^{(n-m+1)}_{k:m}>t\right\} \\
&\leq P\left\{T^{(1)}_{k:m}>t, T^{(m+1)}_{k:m}>t, ..., T^{(s)}_{k:m}>t\right\},
\end{aligned}$$

where $s = \left(\left[\frac{n}{m}\right] - 1\right) \cdot m + 1$. Since the order statistics $T_{k:m}^{(1)}, T_{k:m}^{(m+1)}, ..., T_{k:m}^{(s)}$ are nonoverlapping (they do not have the common terms) we have

$$\begin{aligned}
& P\left\{T_{k:m}^{(1)} > t, T_{k:m}^{(m+1)} > t, ..., T_{k:m}^{(s)} > t\right\} \\
= \; & P\left\{\sum_{i=1}^{m} Y_i(t) < k, \sum_{i=m+1}^{2m} Y_i(t) < k, ..., \sum_{i=s}^{s+m-1} Y_i(t) < k\right\} \\
= \; & \sum_{j_1, j_2, ..., j_r = 0}^{k-1} P\left\{\sum_{i=1}^{m} Y_i(t) = j_1, \sum_{i=m+1}^{2m} Y_i(t) = j_2, ..., \sum_{i=s}^{s+m-1} Y_i(t) = j_r\right\} \\
= \; & \sum_{j_1, j_2, ..., j_r = 0}^{k-1} \binom{m}{j_1} ... \binom{m}{j_r} P\left\{T_1 \le t, ..., T_{j_1+...+j_r} \le t,\right. \\
& \left. T_{j_1+...+j_r+1} > t, ..., T_{s+m-1} > t\right\}.
\end{aligned}$$

The proof is completed noting that

$$\begin{aligned}
& P\left\{T_1 \le t, ..., T_{j_1+...+j_r} \le t, T_{j_1+...+j_r+1} > t, ..., T_{s+m-1} > t\right\} \\
= \; & \sum_{i=0}^{s+m-1-(j_1+...+j_r)} (-1)^i \binom{s+m-1-(j_1+...+j_r)}{i} P\left\{T_1 \le t, ..., T_{j_1+...+j_r+i} \le t\right\} \\
= \; & \sum_{i=0}^{j_1+...+j_r} (-1)^i \binom{j_1+...+j_r}{i} P\left\{T_1 > t, ..., T_{s+m-1-(j_1+...+j_r)+i} > t\right\}.
\end{aligned}$$

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