RELIABILITY ANALYSIS OF CONSECUTIVE-k SYSTEMS IN A STRESS-STRENGTH SETUP

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RELIABILITY ANALYSIS OF CONSECUTIVE-k SYSTEMS IN A STRESS-STRENGTH SETUP

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M.S. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **“RELIABILITY ANALYSIS OF CONSECUTIVE-*k* SYSTEMS IN A STRESS-STRENGTH SETUP”** completed by **FATİH AKICI** under supervision of **Prof. Dr. Serkan Eryılmaz** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

RELIABILITY ANALYSIS OF CONSECUTIVE-k SYSTEMS IN A STRESS-STRENGTH SETUP

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This thesis is concerned with the study of reliability of consecutive-k systems in a stress-strength setup. The exact reliability expression for the linear consecutive-k-out-of-n:F system is obtained whenever the first c components of the system are subjected to one kind of a random stress and the remaining $n-c$ components are subjected to another kind of a random stress. The maximum likelihood and minimum variance unbiased estimators of the reliability are also obtained.

Keywords: Consecutive k-out-of-n system, Point estimator, Reliability, Stress-strength reliability.

ÖZ

ARDIL-k SİSTEMLERİN ETKİ-DAYANIKLILIK KURULUMU ALTINDA GÜVENİLİRLİK ANALİZİ

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Bu tez ardıl-k sistemlerin güvenilirliklerinin etki-dayanıklılık kurulumu altında incelenmesi üzerinedir. Doğrusal ardıl n'den k'lı:F sistemin tam güvenilirlik ifadesi, sistemin ilk c bileşeninin bir tip rastgele etkiye, kalan $n-c$ bileşeninin bir başka tip rastgele etkiye maruz kaldığı bir durum için ortaya konulmuştur. İlgili güvenilirliğin en çok olabilirlik ve en küçük varyanslı yansız tahmin edicileri de elde edilmiştir.

Anahtar Kelimeler: Ardıl n'den k'lı sistem, Etki-dayanıklılık güvenilirliği, Güvenilirlik, Nokta tahmin edicisi.

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To my family

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Chapter 1

Introduction

Linear or circular consecutive-k-out-of-n:F(G) systems are the systems composed of n linearly or circularly ordered components such that the system fails (works) if and only if at least k consecutive components of it are failed (working). F is the acronym for *failure* and G for *good*. Based on the definition, one can clearly infer that a consecutive-1-out-of-n:F(G) system is a series (parallel) system, while a consecutive n-out-of-n:F(G) system is a parallel (series) system. A consecutive k-out-of-n:F system (henceforth (C,k, n:F)) is *dual* of a consecutive-k-out-of-n:G system (henceforth (C,k, n:G)). Consecutive-k-out-of-n systems are used in modelling oil pipeline systems, microwave stations in telecom networks, vacuum systems of electron accelerators etc. Consider an agricultural irrigation system of length 200 meters carrying a lake's water to an inclined farm such that the system has 10 pums which are located 20 meters far from each other. Assume that each pump is able to send the water 40 meters away. It is clearly seen that if 1 pump does not work, its preceding neighbour can send the water to the next pump but if any consecutive 2 pumps fail, then the water cannot be carried to the high points of the farm so the system fails. This system is an example of (C,2,10:F) system.

Let ξ_i be a binary state random variable corresponding to the ith component of the system, where $\xi_i = 1(0)$ denotes the functioning (failing) state of the ith component, $i = 1, 2, \ldots, n$. A (C,k, n:F) system can be considered as the series arrangement of the parallel subsystems of all k consecutive components, or k-windows. Similarly, a (C,k, n:G) system can be considered as the parallel arrangement of the series subsystems of k-windows. Therefore the structure functions of (C,k, n:F) and (C,k, n:G) systems are given respectively by

$$\phi_F(\xi_1, \xi_2, \ldots, \xi_n) = \prod_{j=1}^{n-k+1} \left(1 - \prod_{i=j}^{j+k-1} (1 - \xi_i) \right),$$

$$\phi_G(\xi_1, \xi_2, \ldots, \xi_n) = 1 - \prod_{j=1}^{n-k+1} \left(1 - \prod_{i=j}^{j+k-1} \xi_i \right).$$

Assume that we have two (C,k, n:F) systems of $n = 10$ components and let k be equal to 3. Let the following two realizations be obtained based on the states of components of the systems.

$$1101000111 \tag{1.1}$$

$$1001001001 \tag{1.2}$$

The first system fails since there are 3 consecutively failed components (the 5th, 6th and 7th ones), while the second system operates properly despite the fact that it has more failed components than the first one.

In the literature, a number of reliability expressions for (C,k, n:F) systems have been obtained under the assumption of i.i.d. components. In this case, the reliability of (C,k, n:F) system is obtained as

$$R_{n,k} = \sum_{i=0}^{n} N(i,k,n) p^{n-i} (1-p)^i,$$

where p is the reliability of each component and $N(i,k,n)$ denotes the number of ways to arrange i failed components in a line such that no k or more failed components are consecutive. So the problem is reduced to the problem of finding $N(i,k,n)$. This number is given by

$$N(i,k,n) = \sum_{j=0}^{\lfloor i/k \rfloor} (-1)^j \binom{n-i+1}{j} \binom{n-jk}{n-i},$$

(see, e.g. Kuo and Zuo (2003)).

Actually when analysing the reliability of a (C,k, n:F) system, one realizes that the reliability should be expressed in a slightly different form as

$$R_{n,k} = \sum_{i=0}^{M} N(i,k,n) p^{n-i} (1-p)^i,$$

where

$$M = \begin{cases} n - \frac{n+1}{k} + 1, & \text{if } n+1 \text{ is a multiple of } k\ , \\ n - \left\lfloor \frac{n+1}{k} \right\rfloor, & \text{if } n+1 \text{ is not a multiple of } k\ . \end{cases}$$

The problem again reduces to evaluating the same quantity $N(i,k,n)$. For many other solutions and approaches to obtaining $N(i,k,n)$ under the independence assumption, see Kuo and Zuo (2003). However, the most successful solution to this variable is given by using minimal and maximal signatures in the cases of i.i.d. and exchangeable components. For this, see Navarro et al. (2007), Eryılmaz (2010a) and Eryılmaz (2010b).

When $2k \geq n$, the following simpler expression of system reliability is given by Kuo and Zuo (2003):

$$R_{n,k} = \begin{cases} 1, & 0 \leq n < k \ , \\ 1 - q^k - (n-k)pq^k, & k \leq n < 2k \ . \end{cases}$$

In a dynamic setup, components of a system are assumed to have two states: surviving up to time t or failing before t. Let the survival probability of a component be denoted by $R(t) = P\{T > t\}$, where T represents the lifetime of that component. The dynamic reliability expression -or survival function- $R_{n,k}(t)$ of a (C,k, n:F) system under the i.i.d. assumption is given by replacing p with $R(t)$ in the static reliability equation as

$$R_{n,k}(t) = \sum_{i=0}^{n} N(i,k,n) R(t)^{n-i} (1 - R(t))^i,$$

The upper limit of the summation and the quantity $N(i,k,n)$ are again a matter of interest and their solutions are the same as in the static case. The mean time to failure of a (C,k, n:F) system composed of independent and identical components is given by Kuo and Zuo (2003) in the following two forms.

$$MTTF = \sum_{i=0}^{M} N(i,k,n) \sum_{j=0}^{i} (-1)^j \binom{i}{j} E(T_{1,n-i+j}),$$

and

$$MTTF = \sum_{i=0}^{M} \frac{N(i,k,n)}{\binom{n}{i}} E(T_{i+1,n} - T_{i,n}),$$

where $T_{i,n}$ denotes the ith order statistic of a sample of size n from CDF $F(t)$ and M is defined above.

Dependence among components of any kind of a system is a more realistic assumption because of the common production and operating environments in which the components take place together. Therefore it is seen in the literature that combinatorial methods are not helpful for the reliability analysis of consecutive-k systems having dependent components. In the dependence case, runs-based approaches have been more successful to obtain reliability expressions of this kind of systems.

Based on the definition, the probability of survival, that is, the reliability of a (C,k, n:F) system can be defined by the longest run random variable:

$$R_{n,k} = P\{L_n^0 < k\},$$

where L_n^0 denotes the longest run of failures in $\xi_1, \xi_2, \ldots, \xi_n$. The reliability of a (C,k, n:G) system can be defined similarly:

$$R_{n,k} = P\{L_n^1 \geq k\},$$

where L_n^1 denotes the longest run of successes in $\xi_1, \xi_2, \ldots, \xi_n$.

In the dynamic setup, the lifetime of a (C,k, n:F) system composed of dependent components is given by Eryılmaz (2009) as

$$T_{n,k} = min(T^{[1:k]}, T^{[2:k+1]}, \ldots, T^{[n-k+1:n]}),$$

where $T^{[j:m]} = max(T_j, \ldots, T_m)$, for $1 \leq j < m \leq n$. For $2k \geq n$ the survival function of a (C,k, n:F) system is obtained as

$$R_{n,k}(t) = P(T_{n,k} > t) = 1 - \sum_{i=k}^{n} \left[P(T^{[i-k+1:i]} \leq t) - P(T^{[i-k+1:i+1]} \leq t) \right],$$

where for convenience $P(T^{[n-k+1:n+1]} \leq t) = 0$.

It is recorded in Eryılmaz (2009) that the reliability $R_{n,k}(t)$ is always a lower bound for the survival function of a (C,k, n:F) system for any k.

(C,k, n:F) system was first introduced by Kontoleon(1980) and have been very popular since that time. Kuo and Zuo (2003) is the most extensive study of the topic. For understanding the longest run statistic, the reader should see Balakrishnan and Koutras (2002) as well as Fu and Lou (2003). See also Fu et al. (2003), Boland and Samaniego (2004), Eryılmaz (2007) and Navarro and Eryılmaz (2007). Griffith (1986) considered consecutive-k-within-m-out-of-n:F systems, which is a generalization of (C,k, n:F) systems, which becomes a (C,k, n:F) and a k-out-of-n:F system when m is selected as k and n, respectively. After that time, this generalization of the topic has also been well-studied. (C,k, n:F) systems have also been used in a combined form with k-out-of-n systems, see Zuo et al. (2000). Since it is not easy to derive exact formulas for the reliability of (C,k, n:F) systems, obtaining bounds for this reliability has been very attractive. For various bounds for the system reliability, see Eryılmaz et al. (2009) and the references therein. For the design problem, see Yun et al. (2007).

We proceed our study by explaining the "Stress-Strength Reliability" concept.

Chapter 2

Stress-Strength Reliability

The stress-strength reliability has been widely studied in the literature. The elementary setup of the stress-strength reliability is as follows. Let the components of the system are exposed to an external force called stress, while they have an internal ability to endure the stress, which is called strength. Strength and stress are called load and resistance in hydrosystems engineering. Let X be a stress applied to the system having a random strength Y. The state of the system is defined by the following binary random variable.

$$\xi = \begin{cases} 1, & if\ Y > X, \\ 0, & if\ Y \leq X. \end{cases}$$

So, the reliability of the system is

$$R = E[\xi] = P\{Y > X\}.$$

For a simple illustration of the stress-strength reliability of a one-component system, assume that the cumulative distribution functions of the strength random variable Y and the stress random variable X be

$$\begin{aligned} F(x) &= P(Y \leq x) = 1 - exp(-\theta x) \\ G(x) &= P(X \leq x) = 1 - exp(-\alpha x), x > 0, \theta > 0, \alpha > 0, \end{aligned}$$

respectively. Then the reliability is found as

$$\begin{aligned} R = P\{Y > X\} &= \int_0^\infty P\{Y > X|X = x\} dF_X(x) \\ &= \int_0^\infty exp(-\theta x) d(1 - exp(-\alpha x)) \\ &= \frac{1}{1 + \frac{\theta}{\alpha}}. \end{aligned}$$

For a brief and extensive review of the topic, see Johnson (1988) and Kotz et al. (2003). The importance of these type of models arises from their possibility of being applied to many different areas from chemicals to process control, see Nadarajah and Kotz (2006). In the literature these models are studied for one-component systems, that is, one stress and one strength random variables are taken into consideration, evaluation and estimation of the probability $P(Y > X)$ is studied. For example see Pal et al. (2005), Ali and Woo (2005), Kundu and Gupta (2006), Baklizi (2008a) and (2008b). An et al. (2008) studied the reliability of a component which has discrete strength and stress variables by using universal generating functions. Banerjee and Biswas (2003) formulated the stress-strength problem in a regression seup.

In a more realistic model, it is logical to assume a system of n components, of which ith component is subjected to a stress denoted by X_i, and has the strength Y_i. The states of components are defined in terms of stress and strength random variables as follows.

$$\xi_i = \begin{cases} 1, & if\ Y_i > X_i, \\ 0, & if\ Y_i \le X_i, \end{cases}$$

for $i = 1, 2, \ldots, n$.

The reliability of the ith component is defined as

$$E[\xi_i] = P\{Y_i > X_i\}.$$

The reliability of systems in stress-strength scheme has been studied under various assumptions. Bhattacharyya and Johnson (1974) gave the MVU estimation of a k-out-of-n system and obtained its asymptotic distribution. They also obtained the uniformly most accurate unbiased confidence interval for the system reliability. Paul and Uddin (1974) evaluated system reliability under the assumption of non-identical strengths and a common stress. Eryılmaz (2008a) assumed a common stress random variable X applied to all the components, while n_1 of the components are of one kind and their strengths are independent and identically distributed as $F^{(1)}$, the remaining $n - n_1$ of the components are of another kind and their strengths are independent and identically distributed as $F^{(2)}$. Actually in this setup n_1 is considered to be a change point of the strengths of the components. He obtained the exact stress-strength reliability $R^G_{n,k}$ of a (C,k, n:G) system for the case $2k \ge n = n_1 + n_2$ as

$$R^G_{n,k} = P\{L_n \ge k\} = \sum_{j=k}^{n} p(j, k),$$

where

$$p(j,k) = \begin{cases} g(0,0,1,0) - \sum_{m=0}^{k-1} g(m,0,2,0), & \text{if } k \leq j \leq n_1 - 1, \\ g(0,0,0,1) - \sum_{m=j-n_1}^{k-1} g(m+n_1-j, j-n_1, 1, 1) & \\ - \sum_{m=0}^{j-n_1-1} g(0,m,0,2), & \text{if } n_1 \leq j \leq n-1, \\ 1 - \sum_{m=n-n_1}^{k-1} g(m+n_1-n, n-n_1, 1, 0) & \\ - \sum_{m=0}^{n-n_1-1} g(0,m,0,1), & \text{if } j = n, \end{cases}$$

and $g(a,b,c,d)$ is defined as

$$\begin{aligned} g(a,b,c,d) &= E_{F_X}[(1-F^{(1)}(X))^a(1-F^{(2)}(X))^b(F^{(1)}(X))^c(F^{(2)}(X))^d] \\ &= \int (1-F^{(1)}(x))^a(1-F^{(2)}(x))^b(F^{(1)}(x))^c(F^{(2)}(x))^d dF_X(x). \end{aligned}$$

He stated as a corollary that the reliability $R^G_{n,n}$ of a series system is given under this setup as

$$R^G_{n,n} = g(n_1, n_2, 0, 0),$$

and the reliability $R^G_{n,1}$ of parallel systems are given as

$$R^G_{n,1} = 1 - g(0, 0, n1, n2).$$

Given that the stress and strength random variables are exponentially distributed, he also obtained the MVU estimate of the system reliability of a (C,k,n:G) system. He also suggested a nonparametric MVU estimator of the system reliability for this case. Eryılmaz (2008b) constituted a multivariate stress-strength model for a coherent system of n independent subsystems, each of which composed of m dependent elements by using conditional orderings. The term conditional orderings was first proposed by Bairamov (2006) for multivariate observations as follows. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n \in \mathbf{S} \subseteq \mathbf{R}^m$ be i.i.d. random vectors with m-variate c.d.f. F($\mathbf{x}$) and p.d.f. f($\mathbf{x}$), where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$

and $\mathbf{S}$ is the support of $\mathbf{X}$. Consider the real-valued function $\mathbf{N}(\mathbf{x}) : R^m \to R$, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, which is continuous in its arguments satisfying $\mathbf{N}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in R^m$ with $\mathbf{N}(\mathbf{x})$=0 if and only if $\mathbf{x}$=$\mathbf{0}$, where $\mathbf{0}$ is the m-vector of 0's. $\mathbf{N}(\mathbf{X}_1), \mathbf{N}(\mathbf{X}_2), \ldots, \mathbf{N}(\mathbf{X}_n)$ are i.i.d. random variables with c.d.f. $P\{\mathbf{N}(\mathbf{X}_i) \leq x\}, x \in R$. The function $\mathbf{N}(\mathbf{x})$ introduces partial ordering among $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ and $\mathbf{X}_1$ is named to be conditionally less than $\mathbf{X}_2$ if $\mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{X}_2)$, which is denoted by $\mathbf{X}_1 \prec \mathbf{X}_2$. Letting $\mathbf{X}^{(r)}$ denote the rth smallest among $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ with respect to $\mathbf{N}(\mathbf{x})$ then $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(n)}$ are called conditionally $\mathbf{N}$-ordered statistics. Eryılmaz (2008b) considered a coherent system of n independent components, which are in fact subsystems consisting of m dependent elements, that is, $\mathbf{Y}_i = (Y_i^1, Y_i^2, \ldots, Y_i^m)$, $i = 1, 2, \ldots, n$ denote the random strength vector of the ith component. He also considered that the elements are subjected to a common random stress $\mathbf{X} = (X_1, X_2, \ldots, X_m)$. The reliabilities of a system when the components are connected in series, parallel and k-out-of-n followingly.

The reliability R_S of a system when its components are connected is series is given as

$$R_S = P\{\mathbf{Y} \prec \mathbf{X}^{(1)}\} = P\{\mathbf{N}(\mathbf{Y}) \leq \mathbf{N}(\mathbf{X}^{(1)})\} = \int \ldots \int \prod_{j=1}^{n} (1 - h_j(\mathbf{y}))g(\mathbf{y})d\mathbf{y},$$

while the reliability function R_P is given as follows when the components are connected in parallel

$$R_P = P\{\mathbf{Y} \prec \mathbf{X}^{(n)}\} = P\{\mathbf{N}(\mathbf{Y}) \leq \mathbf{N}(\mathbf{X}^{(n)})\} = 1 - \int \ldots \int \prod_{j=1}^{n} h_j(\mathbf{y})g(\mathbf{y})d\mathbf{y},$$

and as a generalization of these two, when the components form a k-out-of-n system,

$$
\begin{aligned}
R_{n,k} &= P\{\mathbf{Y} \prec \mathbf{X}^{(n-k+1)}\} = P\{\mathbf{N(Y)} \leq \mathbf{N}(\mathbf{X}^{(n-k+1)})\} \\
&= \int \dots \int u(\mathbf{x}) \sum_{i=1}^{n} \sum_{P_i} \prod_{l=1}^{n-k} h_{j_l}(\mathbf{x}) \prod_{l=n-k+2}^{n} (1 - h_{j_l}(\mathbf{x})) f_i(\mathbf{x}) d\mathbf{x},
\end{aligned}
$$

where $u(\mathbf{x}) = P\{\mathbf{N(Y)} \leq \mathbf{N(X)}\}$ is the structural function associated with $\mathbf{Y}$.

Eryılmaz (2008c) generalized the binary setup to a multi-state setup by using a function $\xi = K(Y - X)$, $K : \mathbb{R} \to S$ which is called "kernel function". Selecting the range S to be a continuous or discrete set, one can obtain a continuous or discrete definition of the state random variable ξ. Selecting S as $\{0, 1\}$ yields us to the usual binary setup. A multi-state setup of system reliability arises from the logical need to assign not only failure and success to the component states, but also many states between proper failure and proper success to the component states. Therefore in real life situations the state of a system can be located at a place between [0,1] interval. Consider two *new better than used* light bulbs. Let currently both of them be working but the first one be newer than the second one. Assigning 1 to both of their states would yield a shortness in the reliability analysis. The two bulbs are currently working but in fact the second one is *more near to failure* than the first one is. Therefore assigning a larger number to the state of the first light bulb will be more rational. Let K be a continuous function having an inverse K^{-1} and let $S = [0, 1]$. The event $\{\xi > K(0)\} \equiv \{Y > X\}$ denotes the survival of the system and the survival level of the system is determined by the values on the interval ($K(0)$,1]. The system reliability under this setup is expressed as $P\{\xi > s\}$. For reliability analysis of a multi-state system, the probability $P\{\xi > s\}$ is evaluated for $s > K(0)$ and $s \leq K(0)$. Letting X and Y have continuous c.d.f.'s F and G respectively, the reliability of the system, that is, the probability of the system state to be above state s is evaluated as

$$P\{\xi > s\} = \int\int_{K(y-x)>s} dF(x)dG(y).$$

For an illustration, let $F(x) = 1 - exp(-\theta x)$ and $G(x) = 1 - exp(-\alpha x), x \geq 0$. Then for $0 < s < 1$, the reliability $R(s)$ of the system is found as

$$R(s) = P(\xi > s) = \begin{cases} 1 - \frac{\alpha}{\alpha+\theta} exp(\theta K^{-1}(s)), & \text{if } s \leq K(0), \\ \frac{\theta}{\alpha+\theta} exp(-\alpha K^{-1}(s)), & \text{if } s > K(0), \end{cases}$$

and $P(\xi > 0) = 1$ and $P(\xi > 1) = 0$.

Eryılmaz(2008c) also computed the probability of a system to be in a specific state in the case of a discrete selection of the support S as $S = 0, 1, \ldots, M$ under the exponentially distributed stress and strength random variables assumption as follows.

Let $S_1 = \{0, 1, \ldots, M_1\} \subset S$ represent the states determining the levels of failure and $S/S1$ denote the set of states representing the levels of operation. Define the intervals $I_0 = (a_0, a_1], I_1 = (a_1, a_2], \ldots, I_{M_1} = (a_{M_1}, 0], I_{M_1+1} = (0, a_{M_1+1}], I_{M_1+2} = (a_{M_1+1}, a_{M_1+2}], \ldots, I_M = (a_{M-1}, a_M)$, where $-\infty \equiv a_0 < a_1 < \ldots < a_{M_1} < 0 < a_{M_1+1} < \ldots < a_M \equiv \infty$. The kernel function K can be constructed as

$$\begin{aligned} K(x) &= i \Leftrightarrow x \in I_i, i = 0, 1, \ldots, M_1 - 1, \\ K(x) &= M_1 \Leftrightarrow x \in I_{M_1}, \\ K(x) &= M_1 + 1 \Leftrightarrow x \in I_{M_1+1}, \\ K(x) &= i \Leftrightarrow x \in I_i, i = M_1 + 2, \ldots, M. \end{aligned}$$

Now consider the exponential setup below.

Let $F(x) = 1 - exp(-\theta x)$ and $G(x) = 1 - exp(-\alpha x), x \geq 0$. Then

$$P(\xi = i) = \begin{cases} \frac{\alpha}{\alpha+\theta}(exp(\theta a_{i+1}) - exp(-\theta a_i), & \text{if } i = 0, 1, \ldots, M_1 - 1, \\ \frac{\theta}{\alpha+\theta}(exp(-\alpha a_{i-1}) - exp(-\alpha a_i), & \text{if } i = M_1 + 2, \ldots, M - 1, M, \end{cases}$$

and

$$P\{\xi = M_1\} = \frac{\alpha}{\alpha + \theta}(1 - exp(\theta a_{M_1})),$$

$$P\{\xi = M_1 + 1\} = \frac{\theta}{\alpha + \theta}(1 - exp(-\alpha a_{M_1+1})).$$

Eryılmaz (2010) studied stress-strength reliability of coherent structures and obtained exact reliability expression in terms of the number of path sets of the structure under exchangeability assumption. Another assumption in that paper was the common stress assumption, which made the analysis more difficult because of making the states of components dependent on each other. He also obtained a lower bound and approximation for the system reliability and gave MLE and MVU estimates of it.

The stress and strength random variables have also been modelled by stochastic processes in the literature.This scheme yields a dynamic (time-dependent) analysis of the stress-strength model. In this setup, $X(t)$ and $Y(t)$ is considered as the stress experienced at time t and the strength of the system at time t. Letting $Z(t) = Y(t) - X(t)$, the lifetime T of the system is expressed as $T = inf\{t : t \geq 0, Z(t) \leq 0\}$ and the reliability $R(t_0)$ of the system at time $t_0 > 0$ is $R(t_0) = P(T > t_0) = P(inf_{0<t\leq t_0} Z(t) > 0)$. The ones who considered this setup first, Basu and Ebrahimi (1983) assumed $X(t)$ and $Y(t)$ be two independent Brownian motions with mean value functions $\mu_1 t$ and $\mu_2 t$ and with covariance kernels $\sigma_1^2 min(s,t)$ and $\sigma_2^2 min(s,t)$, resulting in $Z(t)$ also be a Brownian motion. Whitmore (1990) made some comments about and extensions of the stochastic reliability concept. Ebrahimi and Ramallingam (1993) generalized this model and by using the stopped processes $X^*(t)$ and $Y^*(t)$ they showed that $Z^*(t) = Y^*(t) - X^*(t)$ is a homogenous Markov process. Basu and Lingham (2003) contributed to this model by estimating the parameters of the model in a Bayesian perspective.

In this thesis, we will deal with a non-dynamic system having a change point. This means that the behaviour of the components change at a certain point. One can consider a change point either in the stress variables or in the strength variables or in both of them. Existence of the change point c for either case differentiates the reliabilities of the components which are located before the change point than the ones located after it. We say that a point c is the change point of the system when the reliabilities of the components are in the form

$$E[\xi_i] = P(\xi_i = 1) = \begin{cases} p_1, & \text{if } 1 \leq i \leq c, \\ p_2, & \text{if } c+1 \leq i \leq n, \end{cases}$$

for $p_1 \neq p_2$.

We consider a system composed of n components of which strengths $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed, and the first c components of the system are subject to the stress X_1 and the remaining $n - c$ components of the system are subject to the stress X_2, where X_1 and X_2 are independent and nonidentical random variables. Because of this scheme, the point c is called the change point of the system.

The following example will utilize the change point model considered in this study. Starting from 2003, Turkey has been buying gas from Russia via the *Blue Stream* gas pipeline. This pipeline starts from Novorossiysk, Russia, passes across the Black Sea, enters into Turkey from Samsun and ends in Ankara. Once gas is processed in the gas station and leaves the Russian shore, the pipes go approximately 2140 meters under the sea, and in Samsun it leaves the sea and enters soil, which is a dryer environment. Clearly, the pipeline experiences traveling in two different environments: water and soil. The pipes in different environments are subject to different stress factors, such as hydrosulfur, pressure, temperature, corrosion and oxidation. Even if the pipes are identical, their reliabilities will be different because of these different stress factors. In this example, the pipe corresponding to Samsun can be considered as a change point of the system.

We now define formally the reliabilities of components of the change point model which we will study in this thesis.

Definition. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random strengths of the components of a system. Let X_1 and X_2 be two random stresses such that the components 1,2,...,c are subject to the random stress X_1, and the components $c+1, c+2, \ldots, n$ are subject to the random stress X_2. Define the state random variables of the components as follows.

$$\xi_i = \begin{cases} 1, & \text{if } Y_i > X_1, \\ 0, & \text{if } Y_i \le X_1, \end{cases}$$

for $i = 1, 2, \ldots, c$, and

$$\xi_i = \begin{cases} 1, & \text{if } Y_i > X_2, \\ 0, & \text{if } Y_i \le X_2, \end{cases}$$

for $i = c+1, c+2, \ldots, n$.

Then the reliability of each component in the first group is

$$E[\xi_i] = P(Y_i > X_1),$$

for $i = 1, 2, \ldots, c$,

and the reliability of each component in the second group is

$$E[\xi_i] = P(Y_i > X_2),$$

for $i = c+1, c+2, \ldots, n$.

It should be noted that the random variables $\xi_1, \xi_2, \ldots, \xi_n$ are dependent which makes the reliability analysis difficult.

Chapter 3

Consecutive-k Systems in Stress-Strength Setup

In this chapter we present main contributions of this study. We obtain exact reliability expression and its maximum likelihood and uniformly minimum variance unbiased estimators.

We have mentioned in the introduction that the longest run statistic have been efficiently used by the researchers to study the reliability of consecutive-k-out-of-n systems. With the use of longest run statistic, Eryılmaz (2009) obtained exact reliability formulas for consecutive-k-out-of-n systems when $2k \geq n$ without making any assumption on the dependence among the components, i.e. the components were considered to be arbitrarily dependent. He also calculated some important reliability characteristics such as mean time to failure, failure rate and mean residual lifetime functions and also obtained approximations to the reliability function which is also an important problem in reliability analysis. For the case $2k \geq n$, Eryılmaz (2009) showed that, the reliability $R_{n,k}$ of a (C,k, n:F) system is represented as

$$\begin{aligned} R_{n,k} &= 1 - \sum_{i=k}^{n} P(\xi_{i-k+1} = 0, \ldots, \xi_i = 0) \\ &\quad + \sum_{i=k}^{n-1} P(\xi_{i-k+1} = 0, \ldots, \xi_{i+1} = 0), \end{aligned} \tag{3.1}$$

which is an important contribution on which we construct a change point model for a stress-strength setup. We present our main findings below.

3.1 Exact Reliability Expression

Now we deal with our main problem. Assume that the strengths $Y_1, Y_2, \ldots, Y_n$ are independently and identically distributed as F, the first stress X_1 is distributed as G_1, and the second stress X_2 is distributed as G_2. To find the reliability of the system, we should evaluate (3.1) under these assumptions. That is, we should find the quantity $\sum_{i=k}^{n} P(\xi_{i-k+1} = 0, \ldots, \xi_i = 0)$ using the following arguments.

1. If $c < i - k + 1$:

$$\begin{aligned} P(\xi_{i-k+1} = 0, \ldots, \xi_i = 0) &= P(Y_{i-k+1} \leq X_2, \ldots, Y_i \leq X_2) \\ &= \int_0^\infty (F(x))^k dG_2(x) \end{aligned}$$

2. If $i - k + 1 \leq c < i$:

$$\begin{aligned} P(\xi_{i-k+1} = 0, \ldots, \xi_i = 0) &= P(Y_{i-k+1} \leq X_1, \ldots, Y_c \leq X_1, \\ &\quad Y_{c+1} \leq X_2, \ldots, Y_i \leq X_2) \\ &= \int_0^\infty \int_0^\infty (F(x_1))^{c-i+k} (F(x_2))^{i-c} \\ &\quad dG_1(x_1) dG_2(x_2) \end{aligned}$$

3. If $c \geq i$:

$$\begin{aligned} P(\xi_{i-k+1} = 0, \ldots, \xi_i = 0) &= P(Y_{i-k+1} \leq X_1, \ldots, Y_i \leq X_1) \\ &= \int_0^\infty (F(x))^k dG_1(x) \end{aligned}$$

Therefore

$$\sum_{i=k}^{n} P(\xi_{i-k+1}=0,\ldots,\xi_i=0) = \sum_{i=k}^{c}\int_0^\infty (F(x))^k dG_1(x)$$
$$+ \sum_{i=max(c+1,k)}^{min(c+k-1,n)} \int_0^\infty \int_0^\infty (F(x_1))^{c-i+k}(F(x_2))^{i-c} dG_1(x_1)dG_2(x_2)$$
$$+ \sum_{i=c+k}^{n}\int_0^\infty (F(x))^k dG_2(x) \tag{3.2}$$

Likewise, we should find $\sum_{i=k}^{n-1} P(\xi_{i-k+1}=0,\ldots,\xi_{i+1}=0)$ by using

1. If $c < i-k+1$:

$$\begin{aligned} P(\xi_{i-k+1}=0,\ldots,\xi_{i+1}=0) &= P(Y_{i-k+1}\le X_2,\ldots,Y_{i+1}\le X_2) \\ &= \int_0^\infty (F(x))^{k+1} dG_1(x) \end{aligned}$$

2. If $i-k+1 \le c < i+1$:

$$\begin{aligned} P(\xi_{i-k+1}=0,\ldots,\xi_{i+1}=0) &= P(Y_{i-k+1}\le X_1,\ldots,Y_c\le X_1, \\ &\quad Y_{c+1}\le X_2,\ldots,Y_{i+1}\le X_2) \\ &= \int_0^\infty\int_0^\infty (F(x_1))^{c-i+k}(F(x_2))^{i-c+1} \\ &\quad dG_1(x_1)dG_2(x_2) \end{aligned}$$

3. If $c \ge i+1$:

$$\begin{aligned} P(\xi_{i-k+1}=0,\ldots,\xi_i=0) &= P(Y_{i-k+1}\le X_1,\ldots,Y_{i+1}\le X_1) \\ &= \int_0^\infty (F(x))^{k+1} dG_1(x) \end{aligned}$$

Hence

$$\sum_{i=k}^{n-1} P(\xi_{i-k+1}=0,\ldots,\xi_{i+1}=0) = \sum_{i=k}^{c-1} \int_0^\infty (F(x))^{k+1} dG_1(x)$$
$$+ \sum_{i=max(c,k)}^{min(c+k-1,n-1)} \int_0^\infty \int_0^\infty (F(x_1))^{c-i+k}(F(x_2))^{i-c+1} dG_1(x_1)dG_2(x_2)$$
$$+ \sum_{i=c+k}^{n-1} \int_0^\infty (F(x))^{k+1} dG_2(x) \tag{3.3}$$

Using (3.2) and (3.3) in (3.1), the reliability expression of the (C,k, n:F) system for $2k \geq n$ under the abovementioned stress-strength setup is found to be

$$\begin{aligned} R_{n,k} &= 1 - \sum_{i=k}^{c} \int_0^\infty (F(x))^k dG_1(x) \\ &- \sum_{i=max(c+1,k)}^{min(c+k-1,n)} \int_0^\infty \int_0^\infty (F(x_1))^{c-i+k}(F(x_2))^{i-c} dG_1(x_1)dG_2(x_2) \\ &- \sum_{i=c+k}^{n} \int_0^\infty (F(x))^k dG_2(x) + \sum_{i=k}^{c-1} \int_0^\infty (F(x))^{k+1} dG_1(x) \\ &+ \sum_{i=max(c,k)}^{min(c+k-1,n-1)} \int_0^\infty \int_0^\infty (F(x_1))^{c-i+k}(F(x_2))^{i-c+1} dG_1(x_1)dG_2(x_2) \\ &+ \sum_{i=c+k}^{n-1} \int_0^\infty (F(x))^{k+1} dG_2(x), \end{aligned} \tag{3.4}$$

where $\sum_{i=a}^{i=b} x_i \equiv 0$ if $a > b$, for $2k \geq n$. Note that selecting $c = 0$ or $c = n$, one obtains the reliability of a (C,k,n:F) system of which components have i.i.d. random strengths and are subjected to a common random stress X_2 or X_1, respectively, for $2k \geq n$.

3.2 Maximum Likelihood and Minimum Variance Unbiased Estimation

At the very beginning of this subchapter, we obtain the exact reliability expression for the (C,k,n:F) system under the assumption that strength and stress random variables have exponential distributions. After that, maximum likelihood estimator (MLE) and minimum variance unbiased estimator (UMVUE) for the reliability function in the exponential distribution case is obtained.

Let the distributions of strengths Y_i, $i = 1, 2, \ldots, n$, and stresses X_1 and X_2 be

$$\begin{aligned} F(x) = P(Y_i \leq x) &= 1 - exp(-\theta x), \\ G_1(x) = P(X_1 \leq x) &= 1 - exp(-\alpha x), \\ G_2(x) = P(X_2 \leq x) &= 1 - exp(-\beta x), x > 0, \theta > 0, \alpha > 0, \beta > 0. \end{aligned}$$

The exact reliability formula given in (3.4) for this case is obtained as follows.

$$\begin{aligned} R_{n,k} &= 1 - \sum_{i=k}^{c} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{1}{1 + j\frac{\theta}{\alpha}} \\ &- \sum_{i=max(c+1,k)}^{min(c+k-1,n)} \sum_{j=0}^{c-i+k} \sum_{l=0}^{i-c} \binom{c-i+k}{j} \binom{i-c}{l} (-1)^{j+l} \frac{1}{1 + j\frac{\theta}{\alpha}} \frac{1}{1 + l\frac{\theta}{\beta}} \\ &- \sum_{i=c+k}^{n} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{1}{1 + j\frac{\theta}{\beta}} + \sum_{i=k}^{c-1} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j \frac{1}{1 + j\frac{\theta}{\alpha}} \\ &+ \sum_{i=max(c,k)}^{min(c+k-1,n-1)} \sum_{j=0}^{c-i+k} \sum_{l=0}^{i-c+1} \binom{c-i+k}{j} \binom{i-c+1}{l} (-1)^{j+l} \frac{1}{1 + j\frac{\theta}{\alpha}} \frac{1}{1 + l\frac{\theta}{\beta}} \\ &+ \sum_{i=c+k}^{n-1} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j \frac{1}{1 + j\frac{\theta}{\beta}} \qquad (3.5) \end{aligned}$$

Thanks to the invariance property of maximum likelihood estimators, one can easily obtain the maximum likelihood estimator of the system reliability by substituting θ, α and β with their maximum likelihood estimators. For this, let $Y_1, Y_2, \ldots, Y_m$, $X_1^{(1)}, X_2^{(1)}, \ldots, X_{m_1}^{(1)}$ and $X_1^{(2)}, X_2^{(2)}, \ldots, X_{m_2}^{(2)}$ be samples of size m, m_1 and m_2 from strength and stress distributions, respectively. Here, one can obtain the MLE of the system reliability by substituting θ with $1/\bar{Y} = m/\sum_{i=1}^{m} Y_i$, α with $1/\bar{X}_1 = m_1/\sum_{i=1}^{m_1} X_i^{(1)}$, and β with $1/\bar{X}_2 = m_2/\sum_{i=1}^{m_2} X_i^{(2)}$, since these sample means are the MLE estimators of the parameters of exponential random variables. See for example Casella and Berger (2002).

Considering the reliability expression given in (3.5), one can obtain the Uniformly Minimum Variance Unbiased Estimator of the system reliability just by obtaining the unbiased estimator of

$$\tau_{j,l}(\theta, \alpha, \beta) = \frac{1}{1 + j\frac{\theta}{\alpha}} \frac{1}{1 + l\frac{\theta}{\beta}}, \tag{3.6}$$

since the reliability of the system is a linear combination of it. Therefore for unbiased estimation of the system reliability, it is sufficient to estimate it. For this, we will use Rao-Blackwell Theorem.

Theorem 3.1 *Let $Y_1, Y_2 \ldots, Y_m$, $X_1^{(1)}, X_2^{(1)}, \ldots, X_{m_1}^{(1)}$ and $X_1^{(2)}, X_2^{(2)}, \ldots, X_{m_2}^{(2)}$ be random samples from*

$$\begin{aligned} F(x) &= 1 - exp(-\theta x), \\ G_1(x) &= 1 - exp(-\alpha x), \\ G_2(x) &= 1 - exp(-\beta x), x > 0, \theta > 0, \alpha > 0, \beta > 0, \end{aligned}$$

respectively. Then the MVU estimate of $\tau_{j,l}(\theta, \alpha, \beta)$ is given by

$$\tau_{j,l}\widehat{(\theta,\alpha,}\beta) = Q_{p_1,p_2}(j,l;V_1,V_2;m,m_1,m_2),$$

where $p_1 = min(\frac{V_1}{j},1), p_2 = min(\frac{V_2}{l},1), V_1 = \frac{T}{T^{(1)}}, V_2 = \frac{T}{T^{(2)}}, T = \sum_{i=1}^{m} Y_i, T^{(1)} = \sum_{i=1}^{m_1} X_i^{(1)}, T^{(2)} = \sum_{i=1}^{m_2} X_i^{(2)},$ *and* $Q_{p_1,p_2}(j,l;V_1,V_2;m,m_1,m_2) = \int_0^{p_2}\int_0^{p_1}[1 - \frac{js^{(1)}}{V_1}]^{m-1}[1 - \frac{ls^{(2)}}{V_2}]^{m-1}(m_1-1)(1-s^{(1)})^{m_1-2}(m_2-1)(1-s^{(2)})^{m_2-2}ds^{(1)}ds^{(2)}.$

Proof. An unbiased estimate of $\tau_{j,l}(\theta,\alpha,\beta)$ is

$$h(Y_1,X_1^{(1)},X_1^{(2)}) = \begin{cases} 1, & \text{if } Y_1 > jX_1^{(1)} \quad \text{and } Y_2 > lX_1^{(2)}, \\ 0, & \text{otherwise,} \end{cases}$$

which can be seen from

$$\begin{aligned} \tau_{j,l}(\theta,\alpha,\beta) &= \frac{1}{1+j\frac{\theta}{\alpha}}\frac{1}{1+l\frac{\theta}{\beta}} \\ &= \int\int (1-F(jx_1))(1-F(lx_2))dG_1(x_1)dG_2(x_2) \\ &= P\{Y_1 > jX_1^{(1)}\}P\{Y_2 > lX_1^{(2)}\} \end{aligned}$$

The unique MVU estimate of $\tau_{j,l}(\theta,\alpha,\beta)$ is

$$\tau_{j,l}\widehat{(\theta,\alpha,}\beta) = P\{Y_1 > jX_1^{(1)}, Y_2 > lX_1^{(2)}|\mathbf{T}\},$$

where $\mathbf{T} = (T, T^{(1)}, T^{(2)})$, since $\mathbf{T}$ is a complete sufficient statistic.

Let $S_1 = \frac{Y_1}{T}$, $S_2 = \frac{Y_2}{T}$, $S^{(1)} = \frac{X_1^{(1)}}{T^{(1)}}$, $S^{(2)} = \frac{X_1^{(2)}}{T^{(2)}}$. Note that

$$\begin{aligned} f_{S_1,S_2,S^{(1)},S^{(2)}}(s_1,s_2,s^{(1)},s^{(2)}|\mathbf{T}{=}\mathbf{t}) &= (m-1)^2(m_1-1)(m_2-1)(1-s_1)^{m-2} \\ &\quad (1-s_2)^{m-2}(1-s^{(1)})^{m_1-2}(1-s^{(2)})^{m_2-2}, \end{aligned}$$

$$0 < s_i < 1, 0 < s^{(i)} < 1, i = 1,2.$$

$$\begin{aligned} \tau_{j,l}\widehat{(\theta,\alpha,\beta)} &= P\{Y_1 > jX_1^{(1)}, Y_2 > lX_1^{(2)}|\mathbf{T}\} \\ &= \int\int P\{S_1 > \frac{js^{(1)}}{V_1}, S_2 > \frac{ls^{(2)}}{V_2}|\mathbf{T}, S^{(1)} = s^{(1)}, S^{(2)} = s^{(2)}\} \\ &\quad dF_{S^{(1)}}(s^{(1)})dF_{S^{(2)}}(s^{(2)}) \\ &= \int_0^{p_2}\int_0^{p_1}[1-\frac{js^{(1)}}{V_1}]^{m-1}[1-\frac{ls^{(2)}}{V_2}]^{m-1}(m_1-1)(1-s^{(1)})^{m_1-2} \\ &\quad (m_2-1)(1-s^{(2)})^{m_2-2}ds^{(1)}ds^{(2)}. \end{aligned}$$

Now, the problem is to find the limits of the integrals. It is obvious that

$$0 \leq \frac{js^{(1)}}{V_1} < S_1 \leq 1,$$

so, for $j \neq 0$

$$s^{(1)} < \frac{V_1}{j}, \text{if } \frac{V_1}{j} < 1,$$
$$s^{(1)} < 1, \text{if } \frac{V_1}{j} \geq 1.$$

Therefore

$$p_1 = \begin{cases} \frac{V_1}{j}, & \text{if } \frac{V_1}{j} < 1, \\ 1, & \text{if } \frac{V_i}{j} \geq 1, \end{cases}$$

for $j \neq 0$ and $p_1 = 1$ for $j = 0$.

The same logic works for the limits of s_2 and we evaluate that

$$p_2 = \begin{cases} \frac{V_2}{l}, & \text{if } \frac{V_2}{l} < 1, \\ 1, & \text{if } \frac{V_2}{l} \geq 1, \end{cases}$$

for $l \neq 0$ and $p_2 = 1$ for $l = 0$. □

3.3 Numerical Results

We now provide numerical results for the exact reliability expression ($R_{n,k}$) given in (3.5), its maximum likelihood estimator ($\hat{R}_{n,k}$) which is obtained by substituting the parameters θ, α and β in (3.5) with $1/\bar{Y}$,$1/\bar{X}_1$ and $1/\bar{X}_2$, and its minimum variance unbiased estimator ($\tilde{R}_{n,k}$) given in Theorem (3.1). The mean squared errors of the two estimators are also presented for performance comparison.

c	θ	α	β	$R_{k,n}$	$\hat{R}_{k,n}$	$\tilde{R}_{k,n}$	$MSE(\hat{R}_{k,n})$	$MSE(\tilde{R}_{k,n})$
2	1	5	9	0.9405	0.9304	0.9246	0.0023	0.0019
	5	1	9	0.2154	0.2120	0.2015	0.0050	0.0051
	9	1	5	0.1169	0.1271	0.1183	0.0033	0.0030
1	1	5	9	0.9682	0.9617	0.9681	0.0010	0.0007
	5	1	9	0.6711	0.6868	0.6935	0.0124	0.0136
	9	1	5	0.3711	0.3738	0.3665	0.0109	0.0111

Table 1. Exact reliability, its MLE and MVU estimators, and mean squared errors of the estimators when $n = 3$ and $k = 2$.

c	θ	α	β	$R_{k,n}$	$\hat{R}_{k,n}$	$\tilde{R}_{k,n}$	$MSE(\hat{R}_{k,n})$	$MSE(\tilde{R}_{k,n})$
1	1	5	9	0.9900	0.9873	0.9917	0.0002	0.0001
	5	1	9	0.7873	0.7671	0.7799	0.0131	0.0142
	9	1	5	0.4530	0.4555	0.4495	0.0193	0.0212
4	1	5	9	0.9692	0.9556	0.9667	0.0015	0.0010
	5	1	9	0.2390	0.2485	0.2369	0.0090	0.0089
	9	1	5	0.1360	0.1398	0.1299	0.0031	0.0029

Table 2. Exact reliability, its MLE and MVU estimators, and mean squared errors of the estimators when $n = 5$ and $k = 3$.

For obtaining these numerical values, a Monte Carlo simulation study is carried out in MATLAB. Code is available on request. 500 realizations of the given functions have been simulated, and not more because of the high computational complexity of the functions. The tables contain the values of the exact and estimated reliabilities of $(C, 2, 3 : F)$ and $(C, 3, 5 : F)$ systems, respectively. We calculated the results for $m = m_1 = m_2 = 10$, under various selections of c, θ, α, β. The cases $c = 0$ and $c = n$ would correspond to the reliability of a (C,k, n:F) system having i.i.d. component strengths and is subjected to a unique stress, these selections would not represent a system having a change point in its stress random variables. Therefore we avoided these selections so as not to step out of the change point concept, however simulation under these selections is also possible for comparisons with the results existing in the literature. For every selection of n, k and c we tried to see the performance of the estimators whenever the parameter θ of the strength random variables is lower than, in between, and greater than the parameters α and β of the stress random variables. It is seen from the tables that both of the two estimators of the system reliability are plausible because of their quite low mean squared errors. We observe that there is no significant difference between the two estimators.

Chapter 4

Coherent Systems in Stress-Strength Setup

In this chapter, we express the reliability of any coherent system as a linear combination of reliabilities of series and parallel subsystems, by using minimal and maximal signatures.

Consider a coherent system of n components. Let ξ_i denote the state of ith component of the system, which takes value 1 in case of functioning and 0 in case of a fail. The structure function Φ of the system takes value 1 in case of functioning and 0 in case of a fail. So the reliability of a coherent system is

$$R_\Phi = P(\Phi(\xi_1, \xi_2, \ldots, \xi_n) = 1).$$

Let the random variables ξ_i, $i = 1, 2, \ldots, n$, be exchangeable. Then

$$\begin{aligned} &P(\xi_1 = 0, \xi_2 = 0, \ldots, \xi_r = 0, \xi_{r+1} = 1, \xi_{r+2} = 1, \ldots, \xi_n = 1) \\ &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} P(\xi_1 = 1, \xi_2 = 1, \ldots, \xi_{n-r+i} = 1) \\ &= \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} P(\xi_1 = 0, \xi_2 = 0, \ldots, \xi_{r+i} = 0) \end{aligned}$$

Navarro et al. (2007) showed that the reliability of any coherent system can be expressed as a linear combination of the reliabilities of series subsystems or parallel subsystems. Based on their definition, we can represent the stress-strength reliability of a coherent system in terms of the reliability of series or parallel subsystems. Let the components of a coherent system have i.i.d. strength random variables $Y_1, Y_2, \ldots, Y_n$ and be subject to a common stress X. The state random variables ξ_i of components take value 1 if $Y_i > X$ and 0 if $Y_i \leq X$. It is obvious that $\xi_1, \xi_2, \ldots, \xi_n$ are exchangeable dependent. Therefore the reliability of any coherent system in this setup can be expressed by one of the following forms.

$$R_\Phi = \sum_{i=1}^{n} \alpha_i P(Y_{1:i} > X) \quad (4.1)$$

$$= \sum_{i=1}^{n} \beta_i P(Y_{i:i} > X), \quad (4.2)$$

where the coefficient vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$ satisfying $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{i=1}^{n} \beta_i = 1$ are called minimal and maximal signatures, respectively. These representations of system reliability is useful when i-dimensional exchangeable survival function or distribution function is given.

By using the number of minimal path sets, the minimal and maximal signatures are obtained by Eryılmaz (2010b) as

$$\alpha_i = \sum_{j \in A_i} (-1)^{i+j-n} \binom{j}{n-i} r_{n-j}(n),$$

and

$$\beta_i = \sum_{j \in B_i} (-1)^{i-j+1} \binom{n-j}{n-i} r_{n-j}(n),$$

where $r_{n-j}(n)$ denotes the number of path sets of the structure Φ having exactly $n-j$ working components and A_i and B_i are the appropriate sets depending on the structure of the system. Thus from (4.1) and (4.2) we observe that it is enough to evaluate the reliability of series or parallel systems for computing the stress-strength reliability for any coherent structure. A detailed discussion is given in Eryılmaz (2010a).

We finally give the minimal and maximal signatures for the (C,k, n:F) system. Based on Eryılmaz (2010a), it is true for a (C,k, n:F) system that

$$\alpha_i = \sum_{j=n-i}^{M} (-1)^{i+j-n} \binom{j}{n-i} r_{n-j}(n)$$

and

$$\beta_i = \sum_{j=n-i}^{M} (-1)^{i-j+1} \binom{n-j}{n-i} r_{n-j}(n),$$

where

$$r_{n-j}(n) = \sum_{i=0}^{min([j/k],n-j+1)} (-1)^i \binom{n-j+1}{i} \binom{n-ik}{n-j},$$

and $M = n+1-\frac{n+1}{k}$ if $n+1$ is a multiple of k and $M = n - [\frac{n+1}{k}]$ if not, where $[x]$ is the integer part of x.

Chapter 5

Summary and Conclusions

In this thesis, we studied the reliability of consecutive-k-out-of-n:F systems in a stress-strength setup in a way that the components have independent and identically distributed strengths and the first c of them are exposed to one type of a random stress, and the remaining $n-c$ are exposed to another type of a random stress. Since the states of the components are neither independent nor exchangeable, we studied the reliability for the case $2k \geq n$ which is mathematically tractable. We obtained the exact reliability formula for this case. We presented the maximum likelihood and the minimum variance unbiased estimators for the system reliability. When we compare these two estimators, we realized that both of them have very low mean squared errors. We pointed out that the performances of the two estimators are not singificantly different, therefore it is recommended to use the maximum likelihood estimator because of its lower computational complexity. A further research problem can be the evaluation of system reliability under more change points, i.e. more different stress random variables. Another point that can be considered is assuming the component strengths to be non-i.i.d. random variables.

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