MODIFICATIONS OF BIVARIATE BINOMIAL DISTRIBUTION AND CONDITIONAL BIVARIATE ORDER STATISTICS

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MODIFICATIONS OF BIVARIATE BINOMIAL DISTRIBUTION AND CONDITIONAL BIVARIATE ORDER STATISTICS

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We have read the dissertation entitled **"MODIFICATIONS OF BIVARIATE BINOMIAL DISTRIBUTION AND CONDITIONAL BIVARIATE ORDER STATISTICS"** completed by **Gülder KEMALBAY** under supervision of **Prof. Dr. İsmihan Bayramoğlu** and we certify that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

MODIFICATIONS OF BIVARIATE BINOMIAL DISTRIBUTION AND CONDITIONAL BIVARIATE ORDER STATISTICS

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In this thesis, the new trivariate discrete distributions which are modifications of the bivariate binomial distribution are obtained. These new discrete distributions present a theoretical interest as well as can be used in many probability models, especially in distribution theory of conditional bivariate order statistics. The distributional properties of bivariate order statistics are studied and derived under the condition that certain values of the underlying random vectors (X, Y) are truncated and fall in the threshold set $\{(t, s) \in \mathbb{R}^2 : t \leq u, s \leq v\}$, $(u, v) \in \mathbb{R}^2$.

Keywords: Bivariate binomial distribution, bivariate order statistics, conditional distributions.

ÖZ

İKİ DEĞİŞKENLİ BİNOM DAĞILIMININ MODİFİKASYONLARI VE KOŞULLU İKİ DEĞİŞKENLİ SIRA İSTATİSTİKLERİ

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Bu tezde, iki değişkenli binom dağılımının modifikasyonu yapılarak bazı yeni üç değişkenli kesikli dağılımlar bulunmuştur. Bu dağılımlar teorik olarak önemli oldukları kadar bir çok olasılık modellerinde ve özellikle koşullu sıra istatistikleri teorisinin gelişiminde kullanılabilir. Sonlu sayıda rastgele (X, Y) vektörlerin kırpılması ve $\{(t, s) \in \mathbb{R}^2 : t \leq u, s \leq v\}$, $(u, v) \in \mathbb{R}^2$ eşik kümesine düşmesi koşulu altında iki değişkenli sıra istatistiklerinin dağılımı elde edilmiş ve dağılım özellikleri çalışılmıştır.

Anahtar Kelimeler: İki değişkenli binom dağılımı, iki değişkenli sıra istatistikleri, koşullu dağılımlar .

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Chapter 1

Introduction

Let $(X_1, Y_1), ..., (X_n, Y_n)$ be independent copies of the bivariate random vector (X, Y) with joint distribution function $F_{X,Y}(x, y)$. Denote by $X_{r:n}$ and $Y_{s:n}$ the r^{th} and s^{th} order statistics of $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$, respectively. The joint distribution of bivariate order statistics $(X_{r:n}, Y_{s:n})$ can be easily derived from the bivariate binomial distribution, which was first introduced by Aitken and Gonin [3] in connection with the fourfold sampling scheme.

Recently, Bairamov and Gultekin [8] considered the new trivariate and quadrivariate distributions constructed on the basis of the bivariate binomial distribution.

Durante and Jaworski [16] considered the conditional distribution function of random variables (X, Y) given $(X, Y) \in \Re$, where $\Re$ is a Borel set in $\mathbb{R}^2$ with joint distribution function

$$H_{\Re}(x, y) = P\{X \leq x, Y \leq y \mid (X, Y) \in \Re\},$$

and using this conditional distribution, introduced a threshold copula. The threshold copula has interesting and important applications for studying the dependence among financial markets, especially regarding spatial contagion. For more recent result concerning threshold copulas and contagion, see [5], [10], and [30]. For some interesting applications of order statistics and their concomitants, bivariate distributions and copulas, in insurance, see [17], [1] and [32].

It can be seen that the bivariate order statistics are also important for the construction of new bivariate distributions with high correlation. For example, Baker's-type distributions are constructed on the basis of distributions of bivariate order statistics and attract significant interest in the statistical literature: See, e.g., [7] and [23].

The conditional bivariate order statistics can also be used in reliability analysis for studying the mean residual life functions of complex systems. The statistical theory of reliability considers systems that consist of n components, and the lifetimes of these components are assumed to be nonnegative random variables. Recently, Bairamov [6] considered complex systems that consist of n elements, which each contain two or more components, and studied the reliability properties of such systems. Let a system consists of n elements, and assume that each element has two components, (A_i, B_i), $i = 1, 2, ..., n$. Let X_i be the lifetime of the component A_i and Y_i be the lifetime of the component B_i, $i = 1,\ 2, ..., n$. Then, (X_i, Y_i) represents the lifetime of the i^{th} element. Assume that the components of the i^{th} element are dependent, i.e., X_i and Y_i are dependent random variables with joint distribution function $F(x, y)$. As an example, [6] considered $(r, s) - out - of - n$ systems, which function if and only if at least r of the n components $A_1,\ A_2, ..., A_n$ and s of the n components $B_1,\ B_2, ..., B_n$ function. Then, the reliability of such a system is

$$P\{T > t\} = P\{X_{n-r+1:n} > t, Y_{n-s+1:n} > t\},$$

where T is the lifetime of the system and $(X_{r:n}, Y_{s:n})$ is the vector of bivariate r^{th} and s^{th} order statistics constructed from the sample $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$. The mean residual life function of an $(r, s) - out - of - n$ system with intact components at time t is

$$\Phi_{r,s:n}(t) = E\{T - t \mid X_{1:n} > t, Y_{1:n} > t\}$$
$$= E\{T^{(t)}_{r,s:n}\},$$

where $T^{(t)}_{r,s:n}$ is a conditional random variable defined as $T^{(t)}_{r,s:n} \equiv (X_{n-r+1:n} - t, Y_{n-s+1:n} - t \mid \{\text{none of the components has failed at time } t\})$. It is clear that to evaluate $\Phi_{r,s:n}(t)$, we must know the survival function of the conditional random variable $T^{(t)}_{r,s:n}$, i.e., the survival function of conditional order statistics.

In this thesis, we consider the joint distribution of bivariate order statistics $(X_{r:n}, Y_{s:n})$ under the condition that h of the random observations $(X_1, Y_1), ..., (X_n, Y_n)$ are truncated, i.e., they fall in the set $\mathbf{B}_{uv} = \{(t, s) \in \mathbb{R}^2 : t \leq u,\ s \leq v\}$, $(u, v) \in \mathbb{R}^2$, assuming $P\{(X, Y) \in \mathbf{B}_{uv}\} > 0$. This conditional distribution is derived using novel modifications of the bivariate binomial distribution introduced in the chapter of main results of this thesis.

This thesis is organized as follows: In the second chapter, we give a brief introduction to the bivariate binomial distribution and its some extensions. In the third chapter, the joint distribution of bivariate order statistics is presented by the help of bivariate binomial distribution. In the chapter of main results, we consider novel trivariate distributions obtained from bivariate binomial distribution by introducing new events in a fourfold model. Then, using the modified trivariate distributions, the conditional distribution of bivariate order statistics $(X_{r:n}, Y_{s:n})$, $1 \leq r, s \leq n$ constructed from bivariate observations (X_i, Y_i), $i = 1, 2, ..., n$ is derived, where we assume that a certain number of these observations are truncated. As a special case, the distribution function of conditional bivariate extreme order statistics is presented and some dependence properties are obtained. Furthermore, some numerical and graphical results are illustrated.

Chapter 2

Bivariate Binomial Distribution and Its Extensions

The bivariate and multivariate binomial distributions have appeared in several studies as a natural extension of the univariate binomial distribution to the two and higher dimensional case. These distributions arise in many fields of statistics and probability while for example studying order statistics, exceedances, strategic games and reliability theory.

Aitken and Gonin [3] was the first who introduced the bivariate binomial distribution in connection with the fourfold sampling scheme with replacement. They considered a population where each individual can be classified either A or A^c and simultaneously B or B^c. They explained the bivariate binomial distribution with 2×2 contingency table by writing the probabilities of AB, AB^c, A^cB and A^cB^c on each cell and fixing the total sample size n which is allocated into each four cells.

As the limiting form of bivariate binomial distribution, [33], [20] derived the bivariate Poisson distribution.

The canonical expansion of Aitken and Gonin's bivariate binomial distribution was given by [18] related to a series bilinear in Krawtchouk's polynomials. However, [19] showed that the trinomial and bivariate binomial distributions have a similar structure and actually the trinomial distribution is a special case of bivariate binomial distribution. In Aitken and Gonin's model if one assumes that the A and B can not occur at the same time, i.e. $P(AB) = 0$, then bivariate binomial distribution coincides with the trinomial distribution.

Moreover, Aitken and Gonin's model has been studied with different aspects by several authors. The maximum likelihood estimators were given by [22]. [21] studied the conditional distribution, regression functions and conditions for bivariate Poisson and Gaussian limits. [25] discussed the conditional distribution of more general version of Aitken and Gonin's model which was introduced by [18] and [21]. [29] proposed the conditional distributions related to a trivariate binomial distribution.

Also, some characterization of bivariate and multivariate binomial distributions have been studied. [31] established a characterization for multinomial distribution. [15] obtained a characterization for multivariate binomial distribution with univariate marginals. Recently, a characterization of the trivariate binomial distribution based on the distribution of the sum of two trivariate random vectors was given by [12].

[9] proposed a new formulation of the bivariate binomial distribution with the meaning of each of random variables has marginally a binomial distribution and have non-zero correlation. Recently, [8] considered the novel trivariate and quadrivariate distributions constructed on the basis of the bivariate binomial distribution.

In this chapter, we consider the bivariate binomial distribution and its some modifications. As an extension of this probability model, we present the study of [8] where the new class of multivariate discrete distribution with binomial and multinomial marginals is proposed.

2.1 Bivariate Bernoulli Distribution

Let us consider a discrete bivariate random variable (X, Y) taking four possible values $(0,0), (0,1), (1,0), (1,1)$ with probabilities $p_{00}, p_{01}, p_{10}, p_{11}$, respectively. Let us define the joint probabilities as follows

$$P\{X=0, Y=0\} = p_{00}, \quad P\{X=0, Y=1\} = p_{01},$$
$$P\{X=1, Y=0\} = p_{10}, \quad P\{X=1, Y=1\} = p_{11},$$

where $p_{00} + p_{01} + p_{10} + p_{11} = 1$ and marginal probabilities are

$$p_{00} + p_{01} = p_{0+}, \quad p_{10} + p_{11} = 1 - p_{0+} = p_{1+},$$
$$p_{00} + p_{10} = p_{+0}, \quad p_{01} + p_{11} = 1 - p_{+0} = p_{+1}.$$

This probability model can be described in the following 2×2 contingency table:

$X \setminus Y$	0	1	$\sum$
0	p_{00}	p_{01}	p_{0+}
1	p_{10}	p_{11}	p_{1+}
$\sum$	p_{+0}	p_{+1}	1

It is clear that marginal distributions of X and Y follow univariate Bernoulli law with parameters $(p_{10} + p_{11})$ and $(p_{10} + p_{11})$, respectively. Then the expected value and variance of X and Y is given as

$$E(X) = p_{10} + p_{11}, \qquad E(Y) = p_{01} + p_{11},$$
$$Var(X) = (p_{10} + p_{11})(p_{00} + p_{01}), \quad Var(Y) = (p_{01} + p_{11})(p_{00} + p_{10}).$$

The covariance between X and Y is given

$$\begin{aligned} Cov(X,Y) &= p_{11} - (p_{10}+p_{11})(p_{01}+p_{11}) \\ &= p_{00}p_{11} - p_{10}p_{01}, \end{aligned}$$

and the correlation coefficient of (X,Y)

$$\rho(X,Y) = \frac{p_{00}p_{11} - p_{10}p_{01}}{\sqrt{(p_{10}+p_{11})(p_{00}+p_{01})}\sqrt{(p_{01}+p_{11})(p_{00}+p_{10})}},$$

where

$$\rho = \begin{cases} -1 & if \ p_{00} = p_{11} = 0, \\ +1 & if \ p_{01} = p_{10} = 0. \end{cases}$$

The probability generating function of random variables (X,Y) with bivariate Bernoulli distribution is given as

$$\begin{aligned} \Phi(t,s) &= \sum_{i,j=0}^{1} t^i s^j p_{ij} \\ &= p_{11}ts + p_{10}t + p_{01}s + p_{00}. \end{aligned}$$

As in the univariate case, several bivariate distributions with binomial, Poisson, geometric, exponential or gamma marginals are naturally arise from bivariate Bernoulli distribution. For more details about the bivariate distributions generated by the bivariate Bernoulli distribution, one can see [28] we consider the bivariate binomial distribution as sum of the n mutually independent random variables $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ which have identically bivariate Bernoulli distribution.

2.2 Bivariate Binomial Distribution

In this section, we will derive the distribution of the sum of n independent bivariate Bernoulli random variables $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$. Let us denote the sum of random variables by

$$\xi = \sum_{i=1}^{n} X_i \text{ and } \eta = \sum_{i=1}^{n} Y_i.$$

Then we will obtain the probability of $P\{\xi = i, \eta = j\}$ for all i, j satisfying $0 \leq i, j \leq n$ which is the main subject of this section. First, we define the fourfold sampling scheme as follows:

Fourfold Sampling Scheme

Suppose that our population consists of two independent samples and each sample has two individuals, A, A^c and B, B^c, with probabilities $P(AB) = \pi_{11}$, $P(AB^c) = \pi_{12}$, $P(A^cB) = \pi_{21}$ and $P(A^cB^c) = \pi_{22}$, where $\sum_{ij} \pi_{ij} = 1$. More precisely, in this scheme, the event A occurs together with B or B^c and the event B occurs together with A or A^c. Therefore, the possible outcomes of the experiment are AB, AB^c, A^cB and A^cB^c. We will refer to this sampling scheme as the fourfold sampling scheme (or fourfold model).

Now, a bivariate binomial distribution in fourfold sampling scheme can be described as follows:

Under random sampling with replacement n times, let ξ denotes the number of trials in which A appears and η denotes the number of trials in which B appears, respectively. The joint probability mass function of (ξ, η) is given as follows:

$$P(i,j) \equiv P\{\xi = i, \eta = j\} = \sum_{k=max(0,i+j-n)}^{min(i,j)} \frac{n!}{k!(i-k)!(j-k)!(n-i-j+k)!} \times \pi_{11}^{k} \pi_{12}^{i-k} \pi_{21}^{j-k} \pi_{22}^{n-i-j+k}, \tag{2.1}$$

where $i = 0, ..., n$; $j = 0, ..., n$.

Formula (2.1) can be easily explained: If in n trials A appears together with B k times, A and B^c appear together $i-k$ times. Then B appears together with A^c $j-k$ times and B^c appears together with A^c $n-i-j+k$ times. The bivariate binomial distribution can be described symbolically as follows:

$A \setminus B$	B	B^c
A	π_{11} k times AB	π_{12} $i-k$ times AB^c
A^c	π_{21} $j-k$ times A^cB	π_{22} $n-i-j+k$ times A^cB^c

Figure 2.1 Description for Bivariate Binomial Distribution

If the experiment is repeated n times, then k outcomes of the event A can be realized together with B in $\binom{n}{k}$ ways. Then, $i-k$ outcomes of the event A can be observed with B^c in $\binom{n-k}{i-k}$ ways and A^c can be realized together with B in $\binom{n-k-(i-k)}{j-k} = \binom{n-i}{j-k}$ ways. Therefore, in n independent trials, the number of possible cases when A occurs i times and B occurs j times is

$$\binom{n}{k}\binom{n-k}{i-k}\binom{n-i}{j-k} = \frac{n!}{k!(i-k)!(j-k)!(n-i-j+k)!}$$

with probability

$$P(AB)^k P(AB^c)^{i-k} P(AB^c)^{j-k} P(A^cB^c)^{n-i-j+k},$$

where $P(AB) = \pi_{11}$, $P(AB^c) = \pi_{12}$, $P(A^cB) = \pi_{21}$ and $P(A^cB^c) = \pi_{22}$.

For some discussion of the bivariate and multivariate binomial distributions, see [20], [18], [19], [31], [15], [21], [26], and [24].

Note that the bivariate binomial distribution can be obtained from the multinomial distribution if one sets $AB = C_1$, $AB^c = C_2$, $A^cB = C_3$, $A^cB^c = C_4$, $P(C_1) = p_{11}$, $P(C_2) = p_{12}$, $P(C_3) = p_{21}$, and $P(C_4) = p_{22}$. If we denote by ζ_i the number of cases in which C_i occurs out of n repetitions, where $i = 1, 2, 3, 4$, then $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is multinomial, $\xi_1 = \zeta_1 + \zeta_2$ and $\xi_2 = \zeta_1 + \zeta_3$.

It is obvious that the marginal distributions of ξ and η are univariate binomial with parameters $(\pi_{11} + \pi_{12})$ and $(\pi_{11} + \pi_{21})$, respectively.

$$P\{\xi = i\} = \binom{n}{i}(\pi_{11} + \pi_{12})^i(1 - (\pi_{11} + \pi_{12}))^{n-i}, \ i = 0, 1, ..., n;$$
$$P\{\eta = j\} = \binom{n}{j}(\pi_{11} + \pi_{21})^j(1 - (\pi_{11} + \pi_{21}))^{n-j}, \ j = 0, 1, ..., n.$$

Thus, the expected value and variance of ξ and η can be obtained easily:

$$\begin{aligned} E(\xi) &= n(\pi_{11} + \pi_{12}), & E(\eta) &= n(\pi_{11} + \pi_{21}), \\ Var(\xi) &= n(\pi_{11} + \pi_{12})(\pi_{21} + \pi_{22}), & Var(\eta) &= n(\pi_{11} + \pi_{21})(\pi_{12} + \pi_{22}). \end{aligned}$$

The covariance between ξ and η is given by

$$\begin{aligned} Cov(\xi, \eta) &= Cov\left(\sum_{i=1}^{n} X_i \, , \sum_{i=1}^{n} Y_i\right) \\ &= \sum_{i=1}^{n} Cov(X_i, Y_i) \\ &= \sum_{i=1}^{n} [E(X_iY_i) - E(X_i)E(Y_i)] \\ &= n(\pi_{22}\pi_{11} - \pi_{21}\pi_{12}). \end{aligned}$$

Note that if $i \neq j$ then X_i and Y_j are mutually independent random variables.

Thus for $i \neq j$ the expected value $E(X_iY_j) = E(X_i)E(Y_j)$ which implies that $Cov(X_i, Y_j) = 0$. Since the correlation coefficient is invariant under linear transformation, $\rho(\xi, \eta) = \rho(X, Y)$ where (ξ, η) and (X, Y) are a bivariate random vectors of bivariate binomial and bivariate Bernoulli distributions, respectively.

2.2.1 Probability Generating Function of Bivariate Binomial Distribution

To obtain the probability generating function of the random vector (ξ, η) with probability mass function $P(i, j)$ in (2.1), let us define

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\xi = \sum_{r=1}^{n} \gamma_1^r$ and $\eta = \sum_{r=1}^{n} \gamma_2^r$. Since the trials are independent, the probability generating function of the random vector (ξ, η) is given by

$$\begin{aligned} \Phi(t, s) &= \left(\sum_{x_1, x_2 = 0}^{1} t^{x_1} s^{x_2} q_{x_1, x_2} \right)^n \\ &= \left(t^1 s^1 q_{1,1} + t^1 s^0 q_{1,0} + t^0 s^1 q_{0,1} + t^0 s^0 q_{0,0} \right)^n \end{aligned}$$

where

$$q_{x_1, x_2} = P\{\gamma_1^r = x_1,\ \gamma_2^r = x_2\};\ x_1, x_2 = 0, 1.$$

We have

$$q_{1,1} = P(AB) = \pi_{11},$$
$$q_{1,0} = P(AB^c) = \pi_{12},$$
$$q_{0,1} = P(A^cB) = \pi_{21},$$
$$q_{0,0} = P(A^cB^c) = \pi_{22}.$$

Then the probability generating function of bivariate binomial distribution is

$$\Phi(t,s) = \left(\pi_{11}ts + \pi_{12}t + \pi_{21}s + \pi_{22}\right)^n. \tag{2.2}$$

2.2.2 Poisson Approximation for the Bivariate Binomial Distribution

It is well known that univariate Poisson distribution function can be obtained by taking the limit of univariate binomial distribution function as $n \to \infty$ such that $p \to 0$ and $np \to \lambda$. By using this fact, [20] presented the similar derivation of the bivariate Poisson distribution as the limiting form of the bivariate binomial distribution. However, [11] obtained the same function with more difficult way by taking limits of the factorial moment generating functions of the bivariate Bernoulli distribution.

Let us consider the bivariate binomial distribution $P(i,j)$ in (2.1) as $n \to \infty$ such that $\pi_{11}, \pi_{12}, \pi_{21} \to 0$ and $n\pi_{11} \to \lambda_{11}$, $n\pi_{12} \to \lambda_{12}$, $n\pi_{21} \to \lambda_{21}$ where $\pi_{22} = 1 - \pi_{11} - \pi_{12} - \pi_{21}$. Then the limiting form of $P(i,j)$ is given by

$$\begin{aligned}
\lim_{n\to\infty} P(i,j) &\equiv \lim_{n\to\infty} P\{\xi = i, \eta = j\} \equiv p(i,j) \\
&= \lim_{n\to\infty} \sum_{k=\max(0,i+j-n)}^{\min(i,j)} \frac{n(n-1)...(n-i)...(n-i-j+k)!}{k!(i-k)!(j-k)!(n-i-j+k)!} \\
&\times \left(\frac{\lambda_{11}}{n}\right)^{k} \left(\frac{\lambda_{12}}{n}\right)^{i-k} \left(\frac{\lambda_{21}}{n}\right)^{j-k} \left(\frac{\lambda_{22}}{n}\right)^{n-i-j+k}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \sum_{k=\max(0,i+j-n)}^{\min(i,j)} \frac{nn[1-\frac{1}{n}]...n[1-\frac{i}{n}]...n[1-\frac{i+j-k-1}{n}]}{k!(i-k)!(j-k)!} \\
&\times \frac{1}{n^{i+j-k}} (\lambda_{11})^{k} (\lambda_{12})^{i-k} (\lambda_{21})^{j-k} \left(1 - \frac{\lambda_{11}+\lambda_{12}+\lambda_{21}}{n}\right)^{n-i-j+k}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \sum_{k=\max(0,i+j-n)}^{\min(i,j)} 1[1-\frac{1}{n}]...[1-\frac{i}{n}]...[1-\frac{i+j-k-1}{n}] \\
&\times \frac{(\lambda_{11})^{k}}{k!} \frac{(\lambda_{12})^{i-k}}{(i-k)!} \frac{(\lambda_{21})^{j-k}}{(j-k)!} \left(1 - \frac{\lambda_{11}+\lambda_{12}+\lambda_{21}}{n}\right)^{n-i-j+k}
\end{aligned}$$

$$= e^{-(\lambda_{11}+\lambda_{12}+\lambda_{21})} \sum_{k=0}^{\min(i,j)} \frac{(\lambda_{11})^{k}}{k!} \frac{(\lambda_{12})^{i-k}}{(i-k)!} \frac{(\lambda_{21})^{j-k}}{(j-k)!}; \qquad (2.3)$$

$$i,j = 0,1,2,...$$

The probability generating function of bivariate binomial distribution is $\Phi(t,s) = (\pi_{11}ts + \pi_{12}t + \pi_{21}s + \pi_{22})^{n}$. Then taking limit as $n \to \infty$ such that $n\pi_{11} \to \lambda_{11}$, $n\pi_{12} \to \lambda_{12}, n\pi_{21} \to \lambda_{21}$ where $\pi_{22} = 1 - \pi_{11} - \pi_{12} - \pi_{21}$, we obtain the probability generating function of bivariate Poisson distribution which is denoted by $\Phi^{*}(t,s)$:

$$
\begin{aligned}
\Phi^*(t,s) \equiv & \lim_{n\to\infty} \Phi(t,s) \\
&= \lim_{n\to\infty} (\pi_{11}ts + \pi_{12}t + \pi_{21}s + \pi_{22})^n \\
&= \lim_{n\to\infty} \left(\frac{\lambda_{11}}{n}ts + \frac{\lambda_{12}}{n}t + \frac{\lambda_{21}}{n}s + 1 - \frac{\lambda_{11} - \lambda_{12} - \lambda_{21}}{n} \right)^n \\
&= \lim_{n\to\infty} \left(1 - \frac{\lambda_{11} - \lambda_{12} - \lambda_{21 -}\lambda_{11}ts - \lambda_{12}t - \lambda_{21}s}{n} \right)^n \\
&= e^{-(\lambda_{11}+\lambda_{12}+\lambda_{21})+\lambda_{11}ts+\lambda_{12}t+\lambda_{21}s}. \qquad (2.4)
\end{aligned}
$$

2.3 Extensions of the Bivariate Binomial Distribution

Based on the construction of bivariate binomial distribution, Bairamov and Gultekin [8] proposed a new trivariate binomial distribution in fourfold sampling scheme. However, they extended the fourfold sampling scheme to a general model and introduced a new quadrivariate binomial distributions in the general sampling scheme. Moreover, they discussed some possible applications of this new class of discrete distributions in field of lifetesting, exceedances and some strategic games.

2.3.1 Trivariate Binomial Distribution in the Fourfold Model

Consider a fourfold sampling scheme, i.e., suppose that the outcome of the random experiment is one of the events A or A^c and simultaneously one of B or B^c with the probabilities $P(AB) = \pi_{11}$, $P(AB^c) = \pi_{12}$, $P(A^cB) = \pi_{21}$ and $P(A^cB^c) = \pi_{22}$, where $\sum_{ij} \pi_{ij} = 1$.

Under random sampling with replacement n times, let ξ, η and ζ denote the number of occurrences of A, B and AB, respectively. Then the joint probability mass function of trivariate binomial distribution in fourfold sampling scheme can be represented in the following theorem:

Theorem 2.1. *[8] The joint probability mass function of the random vector* (ξ, η, ζ) *is given as*

$$P(i,j,h) \equiv P\{\xi = i, \eta = j, \zeta = h\}$$
$$= \frac{n!}{h!(i-h)!(j-h)!(n-i-j+h)!}\pi_{11}^{h}\pi_{12}^{i-h}\pi_{21}^{j-h}\pi_{22}^{n-i-j+h}, \qquad (2.5)$$

where $h = \max(0, i+j-n), ..., \min(i,j)$; $i = 0, ..., n$; $j = 0, ..., n$.

The trivariate binomial distribution in the fourfold sampling scheme can be described schematically as follows:

$A \setminus B$	B	B^c
A	π_{11} h times AB	π_{12} $i-h$ times AB^c
A^c	π_{21} $j-h$ times A^cB	π_{22} $n-i-j+h$ times A^cB^c

Figure 2.2 Description for Trivariate Binomial Distribution

Unlike the bivariate binomial distribution, both total sample size n and the number of occurrences of AB, i.e. $\zeta = h$ are fixed in the trivariate binomial distribution.

The probability generating function of the random vector (ξ, η, ζ) with probability mass function $P(i, j, h)$ in (2.5) is obtained as follows: Let us define

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } AB \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\xi = \sum_{r=1}^{n} \gamma_1^r$, $\eta = \sum_{r=1}^{n} \gamma_2^r$ and $\zeta = \sum_{r=1}^{n} \gamma_3^r$. Since the trials are independent, the probability generating function of the random vector (ξ, η, ζ) is given by

$$\Phi(t, s, z) = \left(\sum_{x_1, x_2, x_3 = 0}^{1} t^{x_1} s^{x_2} z^{x_3} q_{x_1, x_2, x_3} \right)^n$$

where

$$q_{x_1,x_2,x_3} = P\{\gamma_1^r = x_1,\ \gamma_2^r = x_2,\ \gamma_3^r = x_3\};\ x_1, x_2, x_3 = 0, 1.$$

We have

$$\begin{aligned} q_{1,1,1} &= P(AB(AB)) = \pi_{11}, & q_{1,0,0} &= P(AB^c(AB)^c) = \pi_{12}, \\ q_{1,1,0} &= P(AB(AB)^c) = 0, & q_{0,1,0} &= P(A^cB(AB)^c) = \pi_{21}, \\ q_{1,0,1} &= P(AB^c(AB)) = 0, & q_{0,0,1} &= P(AB(AB)^c) = 0, \\ q_{0,1,1} &= P(A^cB(AB)) = 0, & q_{0,0,0} &= P(A^cB^c(AB)^c) = \pi_{22}. \end{aligned}$$

Then, the probability generating function of trivariate binomial distribution is

$$\Phi(t, s, z) = \left(\pi_{11}tsz + \pi_{12}t + \pi_{21}s + \pi_{22}\right)^n . \tag{2.6}$$

Similar to the bivariate binomial distribution, also a Poisson approximation for trivariate binomial distribution can be given as follows:

Theorem 2.2. *[8]Consider the trivariate binomial distribution in fourfold sampling scheme with joint probability mass function $P(i, j, h)$ when the number of trials is large $n \to \infty$ such that $\pi_{11}, \pi_{12}, \pi_{21} \to 0$ and $n\pi_{11} \to \lambda_{11}$, $n\pi_{12} \to \lambda_{12}$, $n\pi_{21} \to \lambda_{21}$ where $\pi_{22} = 1 - \pi_{11} - \pi_{12} - \pi_{21}$. Then the limiting form of $P(i, j, h)$ is given as follows*

$$\lim_{n\to\infty} P(i,j,h) \equiv \lim_{n\to\infty} P\{\xi = i, \eta = j, \zeta = h\} \equiv p(i,j,h)$$
$$= e^{-(\lambda_{11}+\lambda_{12}+\lambda_{21})} \frac{(\lambda_{11})^h}{h!} \frac{(\lambda_{12})^{i-h}}{(i-h)!} \frac{(\lambda_{21})^{j-h}}{(j-h)!}; \qquad (2.7)$$

$$i, j = 0, 1, 2, ...; \; h = 0, ..., \min(i, j).$$

The distribution given in (2.7) is called a trivariate Poisson distribution.

We conclude this section with an extension of fourfold sampling scheme to a general model such that the results of an experiment are A_iB_j, $i, j = 1, 2, ..., m$. Then, the quadrivariate distribution in a general model is given. For sake of simplicity, we consider the case when $m = 3$.

2.3.2 Quadrivariate Binomial Distribution in the General Model

Bairamov and Gultekin [8] also consider the extension of a fourfold sampling scheme to a general model as follows: Suppose that our population consists of two independent sample and each individual of a sample is being classified as one of the events $A_1, A_2, ..., A_m$ and simultaneously as one of $B_1, B_2, ..., B_m$ with probabilities $P(A_iB_j) = \pi_{ij}$ where $\sum_{ij} \pi_{ij} = 1$; $i, j = 1, 2, ..., m$ and $m \geq 3$. In this general scheme, the outcomes of the experiment are A_iB_j, $i, j = 1, 2, ..., m$.

Without loss of generality, we can consider the general model for $m = 3$. Therefore in this sampling scheme, all possible results of the experiment are the pairs A_1B_1, A_1B_2, A_1B_3, A_2B_1, A_2B_2, A_2B_3, A_3B_1, A_3B_2 and A_1B_3.

Assume that the experiment is repeated n times and trials are independent. Let ξ, η, ζ and ϑ denote the number of occurrences of A_1, B_1, A_1B_2 and A_2B_1, respectively. Then the quadrivariate binomial distribution in the general sampling scheme (for $m = 3$) can be represented in the following theorem:

Theorem 2.3. *[8] The joint probability mass function of the random vector* $(\xi, \eta, \zeta, \vartheta)$ *is given as*

$$\begin{aligned} P(i,j,h,r) &\equiv P\{\xi = i, \eta = j, \zeta = h, \vartheta = r\} \\ &= \sum_{k=\max(0,i+j-n)}^{\min(i-h,j-r)} \frac{n!}{k!h!r!(i-k-h)!(j-k-r)!(n-i-j+k)!} \\ &\times \pi_{11}^{k}\pi_{12}^{h}\pi_{13}^{i-k-h}\pi_{21}^{r}\pi_{31}^{j-k-r}(\pi_{22}+\pi_{23}+\pi_{32}+\pi_{33})^{n-i-j+k}, \end{aligned} \tag{2.8}$$

where $i = h, ..., n-r;\ j = r, ..., n-h;\ h = 0, ..., n-r;\ r = 0, ..., n.$

This model can be described schematically as follows:

$A \setminus B$	B_1	B_2	B_3
A_1	π_{11} k times A_1B_1	π_{12} h times A_1B_2	π_{13} $i-k-h$ times A_1B_3
A_2	π_{21} r times A_2B_1	π_{22} A_2B_2	π_{23} A_2B_3
A_3	π_{31} $j-k-r$ times A_3B_1	π_{32} A_3B_2	π_{33} A_3B_3

Figure 2.3 Description for Quadrivariate Binomial Distribution

The probability generating function of the random vector $(\xi, \eta, \zeta, \vartheta)$ with probability mass function $P(i, j, h, r)$ in (2.8) is obtained as follows: Let us define

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A_1 \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B_1 \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A_1B_2 \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_4^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A_2B_1 \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\xi = \sum_{r=1}^{n} \gamma_1^r$, $\eta = \sum_{r=1}^{n} \gamma_2^r$, $\zeta = \sum_{r=1}^{n} \gamma_3^r$ and $\vartheta = \sum_{r=1}^{n} \gamma_4^r$. Since the trials are independent, the probability generating function of the random vector $(\xi, \eta, \zeta, \vartheta)$ is given by

$$\Phi(t, s, z, w) = \left(\sum_{x_1,x_2,x_3,x_4=0}^{1} t^{x_1} s^{x_2} z^{x_3} w^{x_4} q_{x_1,x_2,x_3,x_4} \right)^n$$

where

$$q_{x_1,x_2,x_3,x_4} = P\{\gamma_1^r = x_1,\ \gamma_2^r = x_2,\ \gamma_3^r = x_3,\ \gamma_4^r = x_4\};\ x_1, x_2, x_3, x_4 = 0, 1.$$

We have

$$
\begin{aligned}
q_{1,1,1,1} &= P(A_1B_1(A_1B_2)(A_2B_1)) = 0\\
q_{0,1,1,1} &= P(A_1^cB_1(A_1B_2)(A_2B_1)) = 0\\
q_{1,1,1,0} &= P(A_1B_1(A_1B_2)(A_2B_1)^c) = 0\\
q_{0,1,1,0} &= P(A_1^cB_1(A_1B_2)(A_2B_1)^c) = 0\\
q_{1,1,0,1} &= P(A_1B_1(A_1B_2)^c(A_2B_1)) = 0\\
q_{0,1,0,1} &= P(A_1^cB_1(A_1B_2)^c(A_2B_1)) = \pi_{21}\\
q_{1,1,0,0} &= P(A_1B_1(A_1B_2)^c(A_2B_1)^c) = \pi_{11}\\
q_{0,1,0,0} &= P(A_1^cB_1(A_1B_2)^c(A_2B_1)^c) = \pi_{31}\\
q_{1,0,1,1} &= P(A_1B_1^c(A_1B_2)(A_2B_1)) = 0\\
q_{0,0,1,1} &= P(A_1^cB_1^c(A_1B_2)(A_2B_1)) = 0\\
q_{1,0,1,0} &= P(A_1B_1^c(A_1B_2)(A_2B_1)^c) = \pi_{12}\\
q_{0,0,1,0} &= P(A_1^cB_1^c(A_1B_2)(A_2B_1)^c) = 0\\
q_{1,0,0,1} &= P(A_1B_1^c(A_1B_2)^c(A_2B_1)) = 0\\
q_{0,0,0,1} &= P(A_1^cB_1^c(A_1B_2)^c(A_2B_1)) = 0\\
q_{1,0,0,0} &= P(A_1B_1^c(A_1B_2)^c(A_2B_1)^c) = \pi_{13}\\
q_{0,0,0,0} &= P(A_1^cB_1^c(A_1B_2)^c(A_2B_1)^c) = \pi_{22}+\pi_{23}+\pi_{32}+\pi_{33}.
\end{aligned}
$$

Then, the probability generating function of quadrivariate binomial distribution is

$$
\begin{aligned}
&\Phi(t,s,z,w)\\
&= \left(\pi_{11}ts+\pi_{12}tz+\pi_{21}sw+\pi_{13}t+\pi_{31}s+\pi_{22}+\pi_{23}+\pi_{32}+\pi_{33}\right)^n. \qquad (2.9)
\end{aligned}
$$

However, The limiting form of quadrivariate binomial distribution is represented as follows:

Theorem 2.4. *[8] Consider the quadrivariate binomial distribution in general* *sampling scheme* (*for* $m = 3$) *with joint probability mass function* $P(i, j, h, r)$ *when the number of trials is large* $n \to \infty$ *such that* $\pi_{11}, \pi_{12}, \pi_{21}, \pi_{13}, \pi_{31} \to 0$ *and* $n\pi_{11} \to \lambda_{11}$, $n\pi_{12} \to \lambda_{12}$, $n\pi_{21} \to \lambda_{21}$, $n\pi_1 \to \lambda_1$, $n\pi_2 \to \lambda_2$ *where* $\pi_1 = \pi_{11} + \pi_{12} + \pi_{13}$ *and* $\pi_2 = \pi_{11} + \pi_{21} + \pi_{31}$. *Then the limiting form of* $P(i, j, h, r)$ *is given as follows*

$$\begin{aligned}
\lim_{n\to\infty} P(i,j,h,r) &\equiv \lim_{n\to\infty} P\{\xi = i, \eta = j, \zeta = h, \vartheta = r\} \equiv p(i,j,h,r) \\
&= e^{-(\lambda_1+\lambda_2-\lambda_{11})} \sum_{k=0}^{\min(i-h,j-r)} \frac{(\lambda_{11})^k}{k!} \frac{(\lambda_{12})^h}{h!} \frac{(\lambda_{21})^r}{r!} \\
&\times \frac{(\lambda_1 - \lambda_{11} - \lambda_{12})^{i-k-h}}{(i-k-h)!} \frac{(\lambda_2 - \lambda_{11} - \lambda_{21})^{j-k-r}}{(j-k-r)!}, \qquad (2.10) \\
& i = h, h+1, ...;\ j = r, r+1, ...;\ h = 0, 1, 2, ...;\ r = 0, 1, 2, ...
\end{aligned}$$

The distribution in (2.10) *is a version of the quadrivariate Poisson distribution.*

Chapter 3

Bivariate Order Statistics

Suppose that $X_1, ..., X_n$ are independent and identically distributed random variables from a population with distribution function $F_X(x)$. The corresponding order statistics are obtained by arranging X_i's in nondecreasing order and is denoted by $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. In literature a lot of attention has been devoted to the study of order statistics. For a comprehensive study for theory and applications of order statistics, we refer to [14], [2] and [4]. Although the order statistics attracted the attention of several authors, many of results dealing with order statistics have been derived for the univariate sample.

In this section, we consider a bivariate random sample $(X_1, Y_1), ..., (X_n, Y_n)$. Let $X_{r:n}$ and $Y_{s:n}$ be the corresponding r^{th} and s^{th} order statistics constructed on the basis of the bivariate observations (X_i, Y_i), $i = 1, 2, ..., n$. Then, we will show that the joint distribution of bivariate order statistics $(X_{r:n}, Y_{s:n})$ can be easily obtained from the bivariate binomial distribution.

3.1 Distribution of a Single Order Statistic

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with distribution functions $F_X(x)$ and $F_Y(y)$, respectively. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$, $Y_{1:n} \leq Y_{2:n} \leq ... \leq Y_{n:n}$ be the corresponding order statistics of the sample X and Y, respectively. The distribution function of $X_{r:n}$ is obtained by binomial distribution as follows:

$$\begin{aligned} F_{X_{r:n}}(x) &= P\{X_{r:n} \leq x\} \\ &= P\{\text{at least } r \text{ of } X_1, X_2, ..., X_n \text{ are less than or equal to } x\} \\ &= \sum_{i=r}^{n} P\{\text{exactly } i \text{ of } X_1, X_2, ..., X_n \text{ are less than or equal to } x\} \\ &= \sum_{i=r}^{n} \binom{n}{i} F_X(x)^i [1 - F_X(x)]^{n-i}. \end{aligned} \tag{3.1}$$

Furthermore, using the identity between a binomial distribution having success probability p and Pearson's (1934) incomplete beta function (or is called regularized incomplete Beta function) such that

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r} dt \tag{3.2}$$

one can write the distribution function of $X_{r:n}$ as follows:

$$
\begin{aligned}
F_{X_{r:n}}(x) &\equiv P\{X_{r:n} \leq x\} \\
&= \sum_{i=r}^{n} \binom{n}{i} F_X(x)^i [1 - F_X(x)]^{n-i} \\
&= \int_0^{F_X(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} dt \\
&= \frac{B_{F_X(x)}(r, n-r+1)}{B(r, n-r+1)} = I_{F_X(x)}(r, n-r+1), \qquad (3.3) \\
& -\infty < x < \infty.
\end{aligned}
$$

where $I_x(a, b)$ $(0 < x < 1)$ is the Pearson's incomplete Beta function which is defined as incomplete Beta function $B_x(a, b)$ over Beta function $B(a, b)$:

$$
B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \ 0 < x < 1,
$$

$$
B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{(a-1)!(b-1)!}{(a+b-1)!}; \ a,b \text{ are positive integers.}
$$

Similarly, the distribution function of $Y_{s:n}$ is given as

$$
\begin{aligned}
F_{Y_{s:n}}(y) &\equiv P\{Y_{s:n} \leq y\} \\
&= \sum_{j=s}^{n} \binom{n}{j} F_Y(y)^j [1 - F_Y(y)]^{n-j} \\
&= \int_0^{F_Y(y)} \frac{n!}{(s-1)!(n-s)!} t^{s-1}(1-t)^{n-s} dt \\
&= \frac{B_{F_Y(y)}(s, n-s+1)}{B(s, n-s+1)} = I_{F_Y(y)}(s, n-s+1), \qquad (3.4) \\
& -\infty < y < \infty.
\end{aligned}
$$

For further discussion, one can see [14] and [4], among others.

3.2 Joint Distribution of Bivariate Order Statistics

Let us assume that $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with distribution functions $F_X(x)$ and $F_Y(y)$, respectively. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a bivariate sample with joint distribution function $F(x, y)$.

Additionally, $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$, $Y_{1:n} \leq Y_{2:n} \leq ... \leq Y_{n:n}$ be the corresponding marginal order statistics derived by arranging the random samples in nondecreasing order of magnitude. Let $X_{r:n}$ and $Y_{s:n}$ be the corresponding r^{th} and s^{th} order statistics constructed on the basis of the bivariate observations (X_i, Y_i), $i = 1, 2, ..., n$. The marginal distribution function of $X_{r:n}$ and $Y_{s:n}$ is given by

$$F_{X_{r:n}}(x) = P\{X_{r:n} \leq x\} = \sum_{i=r}^{n} \binom{n}{i} F_X(x)^i [1 - F_X(x)]^{n-i},$$

$$F_{Y_{s:n}}(y) = P\{Y_{s:n} \leq y\} = \sum_{j=s}^{n} \binom{n}{j} F_Y(y)^j [1 - F_Y(y)]^{n-j}.$$

The joint distribution function of $X_{r:n}$ and $Y_{s:n}$ can be obtained easily from the bivariate binomial distribution if one considers the fourfold model with $A =$

$\{X_i \leq x\}$ and $B = \{Y_i \leq y\}$. Then, we obtain the following cell probabilities

$$P(AB) = P\{X_i \leq x, Y_i \leq y\} = F(x, y)$$
$$P(AB^c) = P\{X_i \leq x, Y_i > y\} = F_X(x) - F(x, y)$$
$$P(A^cB) = P\{X_i > x, Y_i \leq y\} = F_Y(y) - F(x, y)$$
$$P(A^cB^c) = P\{X_i > x, Y_i > y\} = 1 - F_X(x) - F_Y(y) + F(x, y).$$

If ξ and η are the number of occurrences of events A and B in n independent trials of the fourfold experiment, respectively, it is clear that

$$\begin{aligned} P\{X_{r:n} \leq x, Y_{s:n} \leq y\} &= \sum_{i=r}^{n} \sum_{j=s}^{n} P\{\xi = i, \eta = j\} \\ &= \sum_{i=r}^{n} \sum_{j=s}^{n} \sum_{k=a}^{b} \frac{n!}{k!(i-k)!(j-k)!(n-i-j+k)!} \\ &\times \pi_{11}^{k} \pi_{12}^{i-k} \pi_{21}^{j-k} \pi_{22}^{n-i-j+k}, \end{aligned} \tag{3.5}$$

where

$$\pi_{11} = F(x, y)$$
$$\pi_{12} = F_X(x) - F(x, y)$$
$$\pi_{21} = F_Y(y) - F(x, y)$$
$$\pi_{22} = 1 - F_X(x) - F_Y(y) + F(x, y)$$

and

$$a = max(0, i + j - n), b = min(i, j).$$

(see, [13]).

Chapter 4

Main Results

4.1 Modifications of Bivariate Binomial Distribution Under Fourfold Sampling Scheme

In this section, we consider novel modifications of bivariate binomial distributions and obtain new trivariate discrete distributions in fourfold sampling scheme. These distributions are an important class of distributions that are used to derive conditional distribution of bivariate order statistics constructed from a bivariate random sample under the condition that a certain number of observations fall in the given threshold set which will be discussed in the next section. The novel trivariate discrete distributions are of interest for distribution theory. The probability generating functions of these distributions are also derived.

4.1.1 The Modified Bivariate Binomial Distribution

Remember that the fourfold sampling scheme (or fourfold experiment), i.e. suppose that the outcome of the random experiment is one of the events A or A^c and simultaneously one of B or B^c with probabilities $P(AB) = \pi_{11}, P(AB^c) = \pi_{12}$, $P(A^cB) = \pi_{21}$ and $P(A^cB^c) = \pi_{22}$, $\sum_{ij} \pi_{ij} = 1$. In this scheme, the event A occurs together with B or B^c, and the event B occurs together with A or A^c, therefore the outcomes of the experiment are AB, AB^c, A^cB and A^cB^c.

If we repeat the fourfold experiment n times independently, then we will use the expression "in n independent fourfold trials" or "in n independent trials of fourfold experiment".

In this section, we introduce novel trivariate distributions obtained from bivariate binomial distributions by introducing new events in fourfold model. In this set up of fourfold experiment, for further modifications of bivariate binomial distribution we consider the following four cases:

1. We assume that together with A, B, A^c, B^c the event C also can occur in the experiment and $C \subset AB$.
2. We assume that C and D also can occur in the experiment and $C \subset AB$, $D \subset AB^c$.
3. We assume that C and E also can occur in the experiment and $C \subset AB$, $E \subset A^cB$.
4. We assume D, E and F can also occur in the experiment and $D \subset AB^c$, $E \subset A^cB$ and $F \subset A^cB^c$.

According to these four cases, we consider n independent trials of fourfold experiment and define the random variables, ξ, η and ζ as follows:

Definition 4.1.

(a) If $C \subset AB$ then ξ, η and ζ are the number of occurrences of events A, B, C, respectively.

(b) If $C \subset AB$ and $D \subset AB^c$, then ξ, η and ζ are the number of occurrences of events A, B, $C \cup D$, respectively.

(c) If $C \subset AB$ and $E \subset A^cB$, then ξ, η and ζ are the number of occurrences of events A, B, $C \cup E$, respectively.

(d) If $D \subset AB^c$,$E \subset A^cB$ and $F \subset A^cB^c$, then ξ, η and ζ are the number of occurrences of events A, B, $D \cup E \cup F$, respectively.

(d1) If $D \subset AB^c$,$E \subset A^cB$ and $F \subset A^cB^c$, then ξ, η and ζ are the number of occurrences of events A, B, $AB \cup D \cup E \cup F$, respectively.

Note that, the events C, D, E and F are distinct for each cases $(a), (b), (c), (d)$, $(d1)$ and ξ, η and ζ denote distinct random variables for each case, i.e. ξ, η and ζ in (a) are distinctive than ξ, η and ζ in (b) etc. We use such a notation to avoid introducing tremendous number of letters in notations. Therefore, each of cases $(a), (b), (c)$, (d) and $(d1)$ must be considered separately. The joint distributions of random variables ξ, η and ζ for each of the cases $(a), (b), (c), (d), (d1)$ are given in the following Theorems 4.2-4.7, respectively.

Theorem 4.2. *In the fourfold sampling scheme, let $C \subset AB$ and ξ, η, ζ be the number of occurrences of the events A, B, C in n independent trials, respectively (case (a) in Definition 4.1). Then, the joint probability mass function of ξ, η and* *ζ is*

$$
\begin{aligned}
P_1(i,j,h) &\equiv P\{\xi=i,\eta=j,\zeta=h\} \\
&= \sum_{k=a}^{b} C_1(n;h,k,i,j)P(C)^h[P(AB)-P(C)]^{k-h}P(AB^c)^{i-k} \\
&\times P(A^cB)^{j-k}P(A^cB^c)^{n-i-j+k},
\end{aligned} \tag{4.1}
$$

where

$$
\begin{aligned}
C_1(n;h,k,i,j) &= \frac{n!}{h!(k-h)!(i-k)!(j-k)!(n-i-j+k)!}; \\
& a=max(0,i+j-n);\ b=min(i,j);\quad i,j=0,1,...,n; \\
& h=0,...,\min(i,j).
\end{aligned}
$$

Proof. If $\xi=i$, we consider all possible cases of the occurrence of the event A and we indicate these cases as $k=0,1,...,$ then A occurs together with B k times and together with B^c $i-k$ times. $\zeta=h$ indicates that C occurs h times. Because $C\subset AB$, h may be at most $\min(i,j)$ because $\xi=i$, $\eta=j$. Then, $AB\backslash C = AB\cap C^c$ occurs $k-h$ times. $\eta=j$ implies that if B appears together with A^c $j-k$ times, B^c appears together with A^c $n-i-j+k$ times. Schematically, this situation can be described as follows:

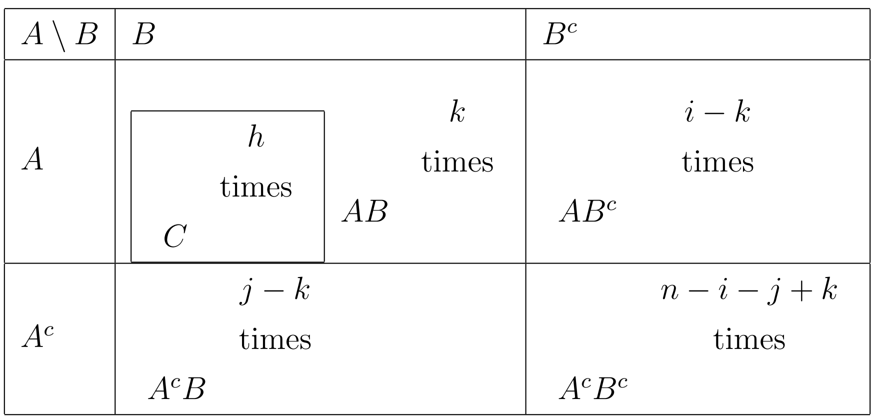


Figure 4.1 Description for $P_1(i,j,h)$

Therefore, it is clear that if we repeat the experiment n times, then h outcomes of the event C can be observed in $\binom{n}{h}$ ways and $k-h$ outcomes of the event $AB\backslash C$ can be realized in $\binom{n-h}{k-h}$ ways. Then, $i-k$ outcomes of the event A can be observed with B^c in $\binom{n-h-(k-h)}{i-k} = \binom{n-k}{i-k}$ ways and A^c can be realized together with B in $\binom{n-k-(i-k)}{j-k} = \binom{n-i}{j-k}$ ways.
Thus in n independent trials, the number of possible cases in which A appears i times, B appears j times and C appears h times is

$$\binom{n}{h}\binom{n-h}{k-h}\binom{n-k}{i-k}\binom{n-i}{j-k} = \frac{n!}{h!(k-h)!(i-k)!(j-k)!(n-i-j+k)!}$$

with probability,

$$P(C)^h[P(AB)-P(C)]^{k-h}P(AB^c)^{i-k}P(A^cB)^{j-k}P(A^cB^c)^{n-i-j+k}.$$

It is clear that $max(0, i+j-n) \leq k \leq min(i,j)$ and $i, j = 0, 1, ..., n;\ h = 0, ..., \min(i,j)$. □

Remark 4.3. *If $C = AB$, then ξ, η, ζ are the number of occurrences of the events A, B, AB in n independent trials, respectively. In this case, from (4.1) we have*

$$\begin{aligned} &P\{\xi=i, \eta=j, \zeta=h\} \\ &\quad = \frac{n!}{h!(i-h)!(j-h)!(n-i-j+h)!}P(AB)^h P(AB^c)^{i-h} \\ &\quad \times P(A^cB)^{j-h}P(A^cB^c)^{n-i-j+h}, \qquad (4.2) \\ &i, j = 0, 1, ..., n; \quad h = \max(0, i+j-n), ..., \min(i,j), \end{aligned}$$

and it is clear that the bivariate binomial distribution is the marginal probability mass function (p.m.f.) of (4.2).

Theorem 4.4. *In fourfold model let $C \subset AB$ and $D \subset AB^c$ and ξ, η and ζ be the number of occurrences of the events A, $B, C \cup D$ in n independent trials, respectively (case (b) in Definition 4.1). Then the joint probability mass function of ξ, η and ζ is*

$$\begin{aligned} P_2(i,j,h) &\equiv P\{\xi = i, \eta = j, \zeta = h\} \\ &= \sum_{k=a}^{b} \sum_{l=0}^{h} C_2(n;k,l,h,i,j) P(C)^l [P(AB) - P(C)]^{k-l} P(D)^{h-l} \\ &\times [P(AB^c) - P(D)]^{i-k-h+l} P(A^cB)^{j-k} P(A^cB^c)^{n-i-j+k} \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} C_2(n;k,l,h,i,j) &= \frac{n!}{l!(k-l)!(h-l)!(i-k-h+l)!(j-k)!(n-i-j+k)!}; \\ a &= max(0, i+j-n);\ b = min(i,j); \quad i,j = 0,1,2,...,n; \\ h &= 0,...,i. \end{aligned}$$

Proof. We know the implications of $\xi = i$ and $\eta = j$ from the proof of Theorem 4.2. Unlike in the previous theorem, $\zeta = h$ i.e., $C \cup D$ occurs h times. Because $C \subset AB$ and $D \subset AB^c$, $C \cup D \subset AB \cup AB^c = A$. Therefore, h can be at most i because $\xi = i$. Then, indicating all possible cases of the occurrence of event C by $l = 0, 1, 2...$, one observes that D occurs $h - l$ times. Hence, $AB \backslash C$ occurs $k - l$ times and $AB^c \backslash D$ occurs $i - k - (h - l)$ times. Then, similar to the proof of Theorem 4.2, all possible cases of the occurrence of the event $\{\xi = i, \eta = j, \zeta = h\}$

can be schematically described as follows:

$A \setminus B$	B	B^c
A	[l times, C] k times, AB	[$h-l$ times, D] $i-k$ times, AB^c
A^c	$j-k$ times, A^cB	$n-i-j+k$ times, A^cB^c

Figure 4.2 Description for $P_2(i, j, h)$

Then, in n independent repeated trials, l outcomes of the event C can be observed in $\binom{n}{l}$ ways and $k-l$ outcomes of the event $AB \backslash C$ can be realized in $\binom{n-l}{k-l}$ ways. Therefore, $h-l$ outcomes of the event D can be realized in $\binom{n-l-(k-l)}{h-l} = \binom{n-k}{h-l}$ ways and $i-k-h+l$ outcomes of the event $AB^c \backslash D$ can be realized in $\binom{n-k-(h-l)}{i-k-h+l}$ ways. Then, A^c can be realized together with B in $\binom{n-k-h+l-(i-k-h+l)}{j-k} = \binom{n-i}{j-k}$ ways.

Thus in n independent trials, the number of possible cases in which A appears i times, B appears j times and $C \cup D$ appears h times is

$$\binom{n}{l}\binom{n-l}{k-l}\binom{n-k}{h-l}\binom{n-k-(h-l)}{i-k-h+l}\binom{n-i}{j-k}$$
$$= \frac{n!}{l!(k-l)!(h-l)!(i-k-h+l)!(j-k)!(n-i-j+k)!}$$

and each case has the same probability,

$$P(C)^l[P(AB) - P(C)]^{k-l}P(D)^{h-l}[P(AB^c) - P(D)]^{i-k-h+l}$$
$$\times P(A^cB)^{j-k}P(A^cB^c)^{n-i-j+k}.$$

It is clear that $max(0, i + j - n) \leq k \leq min(i, j)$ and $i, j = 0, 1, ..., n$; $h = 0, 1, ..., i$. □

Theorem 4.5. *Let $C \subset AB$ and $E \subset A^cB$ in the fourfold sampling scheme. Assume that ξ, η and ζ denote the number of occurrences of the events A, B, $C \cup E$ in n independent trials, respectively (case (c) in Definition 4.1). Then, the joint probability mass function of ξ, η and ζ is*

$$\begin{aligned} P_3(i, j, h) &\equiv P\{\xi = i, \eta = j, \zeta = h\} \\ &= \sum_{k=a}^{b}\sum_{l=0}^{h} C_3(n; k, l, h, i, j)P(C)^l[P(AB) - P(C)]^{k-l}P(AB^c)^{i-k} \\ &\times P(E)^{h-l}[P(A^cB) - P(E)]^{j-k-h+l}P(A^cB^c)^{n-i-j+k}, \end{aligned} \quad (4.4)$$

where

$$C_3(n; k, l, h, i, j) = \frac{n!}{l!(k-l)!(i-k)!(h-l)!(j-k-h+l)!(n-i-j+k)!};$$
$$a = max(0, i + j - n); \quad b = min(i, j); \quad i, j = 0, 1, ..., n;$$
$$h = 0, 1, ..., j.$$

Proof. This theorem can be proved in a manner similar to the proof of Theorem

4.4 using the below schematic representation:

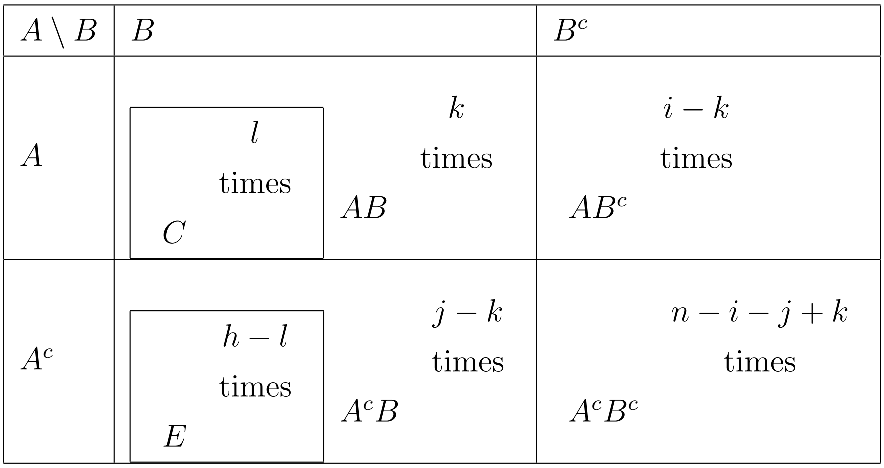


Figure 4.3 Description for $P_3(i, j, h)$

We know the implications of $\xi = i$ and $\eta = j$ from the proof of Theorem 4.2. Unlike in the previous theorem, $\zeta = h$ i.e., $C \cup E$ occurs h times. Because $C \subset AB$ and $E \subset A^cB$, $C \cup E \subset AB \cup A^cB = B$. Therefore, h can be at most j because $\eta = j$. Then, indicating all possible cases of the occurrence of event C by $l = 0, 1, 2...$, one observes that E occurs $h - l$ times. Hence, $AB \backslash C$ occurs $k - l$ times and $A^cB \backslash E$ occurs $j - k - (h - l)$ times. Then, similar to the proof of Theorem 4.4, all possible cases of the occurrence of the event $\{\xi = i, \eta = j, \zeta = h\}$ can be found easily. □

Theorem 4.6. *In the fourfold sampling scheme, let $D \subset AB^c$, $E \subset A^cB$, $F \subset A^cB^c$ and ξ, η, ζ be the number of occurrences of the events A, B, $D \cup E \cup F$ in n independent trials, respectively (case (d) in Definition 4.1). Then, the joint* *probability mass function of* ξ, η *and* ζ *is*

$$\begin{aligned}
P_4(i,j,h) &\equiv P\{\xi = i, \eta = j, \zeta = h\} \\
&= \sum_{k=a}^{b} \sum_{p=0}^{i-k} \sum_{q=0}^{j-k} C_4(n;k,p,q,h,i,j) P(AB)^k \\
&\times P(D)^p [P(AB^c) - P(D)]^{i-k-p} P(E)^q [P(A^cB) - P(E)]^{j-k-q} \\
&\times P(F)^{h-p-q} [P(A^cB^c) - P(F)]^{n-i-j+k-h+p+q}, \qquad (4.5)
\end{aligned}$$

where

$$\begin{aligned}
&C_4(n;k,p,q,h,i,j) \\
&= \frac{n!}{k!p!(i-k-p)!q!(j-k-q)!(h-p-q)!} \\
&\times \frac{1}{(n-i-j+k-h+p+q)!}; \\
&a = max(0, i+j-n); \;\; b = min(i,j); \;\; i,j,h = 0,1,...,n.
\end{aligned}$$

Proof. The schematic representation for this theorem is as follows:

$A \setminus B$	B	B^c
A	k times AB	[p times D] $i-k$ times AB^c
A^c	[q times E] $j-k$ times A^cB	[$h-p-q$ times F] $n-i-j+k$ times A^cB^c

Figure 4.4 Description for $P_4(i,j,h)$

For clarity of explanation, we denote by $\mu(M)$ the number of occurrence of any event M in n independent trials of the fourfold experiment. Because $D \cup E \cup F$ occurs h times, i.e., $\zeta = h$ and $D \cap E \cap F = \varnothing$, $h = \mu(D) + \mu(E) + \mu(F)$, where $\mu(D) = p$, $\mu(E) = q$, $\mu(F) = h - p - q$ are the number of occurrences of the events D, E and F, respectively. Then, the number of occurrences of AB is k, of $AB^c \backslash D$ is $i - k - p$, of $A^cB \backslash E$ is $j - k - q$ and of $A^cB^c \backslash F$ is $n - i - j + k - (h - p - q)$.

The implications of $\xi = i$ and $\eta = j$ are also known from the proof of the first Theorem.

Therefore, it is clear that if we repeat the experiment n times, then k outcomes of the event AB can be observed in $\binom{n}{k}$ ways, p outcomes of the event D can be observed in $\binom{n-k}{p}$ ways and $i - k - p$ outcomes of the event $AB^c \backslash D$ can be realized in $\binom{n-k-p}{i-k-p}$ ways. Then, q outcomes of the event E can be observed in $\binom{n-k-p-(i-k-p)}{q} = \binom{n-i}{q}$ ways and $j - k - q$ outcomes of the event $A^cB \backslash E$ can be realized in $\binom{n-i-q}{j-q-k}$ ways. Finally, $h - p - q$ outcomes of the event F can be realized in $\binom{n-i-q-(j-k-q)}{h-p-q} = \binom{n-i-j+k}{h-p-q}$ ways.

Thus in n independent trials, the number of possible cases in which A appears i times, B appears j times and $D \cup E \cup F$ appears h times is

$$\begin{aligned}
&\binom{n}{k}\binom{n-k}{p}\binom{n-k-p}{i-k-p}\binom{n-i}{q}\binom{n-i-q}{j-k-q}\binom{n-i-j+k}{h-p-q} \\
&= \frac{n!}{k!p!(i-k-p)!q!(j-k-q)!(h-p-q)!} \\
&\times \frac{1}{(n-i-j+k-h+p+q)!}
\end{aligned}$$

and each case has equal probability,

$$\begin{aligned}
&P(AB)^k P(D)^p [P(AB^c) - P(D)]^{i-k-p} P(E)^q \\
&\times [P(A^cB) - P(E)]^{j-k-q} P(F)^{h-p-q} [P(A^cB^c) - P(F)]^{n-i-j+k-h+p+q}.
\end{aligned}$$

It is clear that $a = max(0, i + j - n)$; $b = min(i, j)$; $i, j, h = 0, 1, ..., n$. □

Theorem 4.7. *In the fourfold sampling scheme, let* $D \subset AB^c$, $E \subset A^cB$, $F \subset A^cB^c$ *and* ξ, η, ζ *be the number of occurrences of the events* A, B, $AB \cup D \cup E \cup F$ *in* n *independent trials, respectively (case* $(d1)$ *in Definition 4.1). Then, the joint probability mass function of* ξ, η *and* ζ *is*

$$\begin{aligned} P_{4.1}(i, j, h) &\equiv P\{\xi = i, \eta = j, \zeta = h\} \\ &= \sum_{k=a}^{b} \sum_{p=0}^{i-k} \sum_{q=0}^{j-k} C_{4.1}(n; k, p, q, h, i, j) P(AB)^k \\ &\times P(D)^p [P(AB^c) - P(D)]^{i-k-p} P(E)^q [P(A^cB) - P(E)]^{j-k-q} \\ &\times P(F)^{h-p-q-k} [P(A^cB^c) - P(F)]^{n-i-j+2k-h+p+q}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &C_{4.1}(n; k, p, q, h, i, j) \\ &= \frac{n!}{k!p!(i-k-p)!q!(j-k-q)!(h-p-q-k)!} \\ &\times \frac{1}{(n-i-j+2k-h+p+q)!}; \\ &a = max(0, i + j - n);\ b = min(i, j);\ \ i, j, h = 0, 1, ..., n. \end{aligned}$$

Proof. The proof of this theorem is similar to the proof of Theorem 4.6. The schematic representation for this theorem is as follows:

$A \setminus B$	B	B^c
A	k times AB	[p times D] $i-k$ times AB^c
A^c	[q times E] $j-k$ times A^cB	[$h-p-q-k$ times F] $n-i-j+k$ times A^cB^c

Figure 4.5 Description for $P_{4.1}(i, j, h)$

□

4.1.2 Probability Generating Function of The Modified Bivariate Binomial Distribution

The probability generating function (p.g.f.) of the bivariate binomial distribution is $\Phi(t, s) = (\pi_{11}ts+\pi_{12}t+\pi_{21}s+\pi_{22})^n$. Below, we provide the probability generating functions of the trivariate distributions given in Theorem 4.2-4.7.

Lemma A1. *Consider the fourfold sampling scheme given in case (a) in Definition 4.1. Then, the joint probability generating function of the random vector* (ξ, η, ζ) *with probability mass function (p.m.f.)* $P_1(i, j, h)$ *in (4.1) in Theorem 4.2 is*

$$\Phi_1(t, s, z) = \ (\alpha_1 tsz + \alpha_2 ts + \alpha_3 t + \alpha_4 s + \alpha_5)^n, \tag{4.7}$$

where

$\alpha_1 = P(C), \alpha_2 = P(AB) - P(C), \alpha_3 = P(AB^c), \alpha_4 = P(A^cB)$ and $\alpha_5 = P(A^cB^c)$.

Proof. To derive the joint probability generating functions, let us write

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } C \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$r = 1, 2, ..., n.$$

It is clear that $\xi = \sum_{r=1}^{n} \gamma_1^r$, $\eta = \sum_{r=1}^{n} \gamma_2^r$ and $\zeta = \sum_{r=1}^{n} \gamma_3^r$. Because the trials are independent, the p.g.f. of the random vector (ξ, η, ζ) is

$$\Phi(t, s, z) = \left(\sum_{x_1, x_2, x_3 = 0}^{1} t^{x_1} s^{x_2} z^{x_3} q_{x_1, x_2, x_3} \right)^n \tag{4.8}$$

where

$$q_{x_1,x_2,x_3} = P\{\gamma_1^r = x_1,\ \gamma_2^r = x_2,\ \gamma_3^r = x_3\};\ x_1, x_2, x_3 = 0, 1.$$

We have

$$\begin{aligned}
q_{1,1,1} &= P(ABC) = P(C)\\
q_{1,1,0} &= P(ABC^c) = P(AB) - P(C)\\
q_{1,0,1} &= P(AB^cC) = 0\\
q_{0,1,1} &= P(A^cBC) = 0\\
q_{0,0,1} &= P(A^cB^cC) = 0\\
q_{0,1,0} &= P(A^cBC^c) = P(A^cB)\\
q_{1,0,0} &= P(AB^cC^c) = P(AB^c)\\
q_{0,0,0} &= P(A^cB^cC^c) = P(A^cB^c).
\end{aligned}$$

Then, substituting these values in (4.8) and simplifying, we obtain (4.7). □

The proof of the following lemmas are similar.

Lemma A2. *Consider the fourfold sampling scheme given in case (b) in Definition 4.1. Then, the joint probability generating function of the random vector* (ξ, η, ζ) *with p.m.f.* $P_2(i, j, h)$ *given in (4.3) in Theorem 4.4 is*

$$\Phi_2(t, s, z) = \ (\alpha_1 tsz + \alpha_2 ts + \alpha_3 tz + \alpha_4 t + \alpha_5 s + \alpha_6)^n, \tag{4.9}$$

where

$$\alpha_1 = P(C), \alpha_2 = P(AB) - P(C), \alpha_3 = P(D), \alpha_4 = P(AB^c) - P(D),$$
$$\alpha_5 = P(A^cB) \text{ and } \alpha_6 = P(A^cB^c).$$

Proof. To derive the joint probability generating functions, let us write

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } C \cup D \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$r = 1, 2, ..., n.$$

We have

$$\begin{aligned}
q_{1,1,1} &= P(AB(C \cup D)) = P(C) \\
q_{1,1,0} &= P(AB(C \cup D)^c) = P(AB) - P(C) \\
q_{1,0,1} &= P(AB^c(C \cup D)) = P(D) \\
q_{0,1,1} &= P(A^cB(C \cup D)) = 0 \\
q_{0,0,1} &= P(A^cB^c(C \cup D)) = 0 \\
q_{0,1,0} &= P(A^cB(C \cup D)^c) = P(A^cB) \\
q_{1,0,0} &= P(AB^c(C \cup D)^c) = P(AB^c) - P(D) \\
q_{0,0,0} &= P(A^cB^c(C \cup D)^c) = P(A^cB^c).
\end{aligned}$$

Then substituting these values in (4.8) and simplifying, we obtain (4.9). □

Lemma A3. *Consider the fourfold sampling scheme given in case (c) in Definition 4.1. Then, the joint probability generating function of the random vector* (ξ, η, ζ) *with p.m.f.* $P_3(i, j, h)$ *given in (4.4) in Theorem 4.5 is*

$$\Phi_3(t, s, z) = (\alpha_1 tsz + \alpha_2 ts + \alpha_3 sz + \alpha_4 t + \alpha_5 s + \alpha_6)^n, \quad (4.10)$$

where

$$\alpha_1 = P(C), \alpha_2 = P(AB) - P(C), \alpha_3 = P(E), \alpha_4 = P(AB^c),$$
$$\alpha_5 = P(A^cB) - P(E) \text{ and } \alpha_6 = P(A^cB^c).$$

Proof. To derive the joint probability generating functions, let us write

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } C \cup E \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$r = 1, 2, ..., n.$$

We have

$$q_{1,1,1} = P(AB(C \cup E)) = P(C)$$
$$q_{1,1,0} = P(AB(C \cup E)^c) = P(AB) - P(C)$$
$$q_{1,0,1} = P(AB^c(C \cup E)) = 0$$
$$q_{0,1,1} = P(A^cB(C \cup E)) = P(E)$$
$$q_{0,0,1} = P(A^cB^c(C \cup E)) = 0$$
$$q_{0,1,0} = P(A^cB(C \cup E)^c) = P(A^cB) - P(E)$$
$$q_{1,0,0} = P(AB^c(C \cup E)^c) = P(AB^c)$$
$$q_{0,0,0} = P(A^cB^c(C \cup E)^c) = P(A^cB^c).$$

Then substituting these values in (4.8) and simplifying, we obtain (4.10). □

Lemma A4. *Consider the fourfold sampling scheme given in case (d) in*

Definition 4.1. Then, the joint probability generating function of the random vector (ξ, η, ζ) *with p.m.f.* $P_4(i,j,h)$ *given in (4.5) in Theorem 4.6 is*

$$\Phi_4(t,s,z) = (\alpha_1 ts + \alpha_2 tz + \alpha_3 sz + \alpha_4 t + \alpha_5 s + \alpha_6 z + \alpha_7)^n, \qquad (4.11)$$

where

$$\alpha_1 = P(AB), \alpha_2 = P(D), \alpha_3 = P(E), \alpha_4 = P(AB^c) - P(D),$$
$$\alpha_5 = P(A^cB) - P(E), \alpha_6 = P(F), \alpha_7 = P(A^cB^c) - P(F).$$

Proof. To derive the joint probability generating functions, let us write

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } D \cup E \cup F \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$
$$r = 1, 2, ..., n.$$

We have

$$q_{1,1,1} = P(AB(D \cup E \cup F)) = 0$$
$$q_{1,1,0} = P(AB(D \cup E \cup F)^c) = P(AB)$$
$$q_{1,0,1} = P(AB^c(D \cup E \cup F)) = P(D)$$
$$q_{0,1,1} = P(A^cB(D \cup E \cup F)) = P(E)$$
$$q_{0,0,1} = P(A^cB^c(D \cup E \cup F)) = P(F)$$
$$q_{0,1,0} = P(A^cB(D \cup E \cup F)^c) = P(A^cB) - P(E)$$
$$q_{1,0,0} = P(AB^c(D \cup E \cup F)^c) = P(AB^c) - P(D)$$
$$q_{0,0,0} = P(A^cB^c(D \cup E \cup F)^c) = P(A^cB^c) - P(F).$$

Then substituting these values in (4.8) and simplifying, we obtain (4.11). □

Lemma A4.1. *Consider the fourfold sampling scheme given in case (d1) in Definition 4.1. Then, the joint probability generating function of the random vector* (ξ, η, ζ) *with p.m.f.* $P_{4.1}(i,j,h)$ *given in (4.6) in Theorem 4.7 is*

$$\Phi_{4.1}(t,s,z) = \ (\alpha_1 tsz + \alpha_2 tz + \alpha_3 sz + \alpha_4 t + \alpha_5 s + \alpha_6 z + \alpha_7)^n, \tag{4.12}$$

where

$$\begin{aligned} &\alpha_1 = P(AB), \alpha_2 = P(D), \alpha_3 = P(E), \alpha_4 = P(AB^c) - P(D), \\ &\alpha_5 = P(A^cB) - P(E), \alpha_6 = P(F), \alpha_7 = P(A^cB^c) - P(F). \end{aligned}$$

Proof. To derive the joint probability generating functions, let us write

$$\gamma_1^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } A \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_2^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } B \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_3^r = \begin{cases} 1 & \text{if in the } r^{th} \text{ trial } AB \cup D \cup E \cup F \text{ appears,} \\ 0 & \text{otherwise.} \end{cases}$$

$$r = 1, 2, ..., n.$$

We have

$$
\begin{aligned}
q_{1,1,1} &= P(AB(AB \cup D \cup E \cup F)) = P(AB) \\
q_{1,1,0} &= P(AB(AB \cup D \cup E \cup F)^c) = 0 \\
q_{1,0,1} &= P(AB^c(AB \cup D \cup E \cup F)) = P(D) \\
q_{0,1,1} &= P(A^cB(AB \cup D \cup E \cup F)) = P(E) \\
q_{0,0,1} &= P(A^cB^c(AB \cup D \cup E \cup F)) = P(F) \\
q_{0,1,0} &= P(A^cB(AB \cup D \cup E \cup F)^c) = P(A^cB) - P(E) \\
q_{1,0,0} &= P(AB^c(AB \cup D \cup E \cup F)^c) = P(AB^c) - P(D) \\
q_{0,0,0} &= P(A^cB^c(AB \cup D \cup E \cup F)^c) = P(A^cB^c) - P(F).
\end{aligned}
$$

Then substituting these values in (4.8) and simplifying, we obtain (4.12). □

4.2 Conditional Distribution of Bivariate Order Statistics

In this section, we consider the joint distribution of bivariate order statistics $(X_{r:n}, Y_{s:n})$ under the condition that h of the random observations $(X_1, Y_1), ...,$ (X_n, Y_n) are truncated, i.e., they fall in the set $\mathbf{B}_{uv} = \{(t, s) \in \mathbb{R}^2 : t \leq u,\ s \leq v\}$, $(u, v) \in \mathbb{R}^2$, assuming $P\{(X, Y) \in \mathbf{B}_{uv}\} > 0$. This conditional distribution is derived using novel modifications of the bivariate binomial distribution introduced in the previous chapter.

4.2.1 Conditional Bivariate Order Statistics

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be i.i.d. random variables with distribution function $F_X(x)$ and $F_Y(y)$, respectively. Consider $(X_1, Y_1), ..., (X_n, Y_n)$ be a bivariate sample with joint distribution function $F(x, y)$. Now, we are interested in the conditional joint distribution of bivariate order statistics under the condition that h of the bivariate observations (X_i, Y_i), $i = 1, 2, ..., n$ are truncated and belong to the set

$$\mathbf{B}_{uv} = \{(t, s) \in \mathbb{R}^2 : t \leq u, s \leq v\}, \ (u, v) \in \mathbb{R}^2. \tag{4.13}$$

Lemma 4.8. *Let (X, Y) be a bivariate random vector with joint distribution function $F(x, y)$ and $(X_1, Y_1), ..., (X_n, Y_n)$ be independent copies of (X, Y). If $(X_{r:n}, Y_{s:n})$, $r, s = 1, 2, ..., n$ is the vector of bivariate order statistics and B is any Borel set on $\mathbb{R}^2$, then*

$$\begin{aligned} F_{r,s:n}(x, y \mid u, v) &\equiv P\{X_{r:n} \leq x, Y_{s:n} \leq y \mid \\ &\quad h \textit{ of } (X_1, Y_1), ..., (X_n, Y_n) \textit{ belong to } \mathbf{B}\} \\ &= \frac{1}{\binom{n}{h} P\{(X, Y) \in \mathbf{B}\}^h P\{(X, Y) \in \mathbf{B}^c\}^{n-h}} \\ \times \sum_{i=r}^{n} \sum_{j=s}^{n} P\{&\textit{exactly } i \textit{ of } X\text{'s } \leq x, \ \textit{exactly } j \textit{ of } Y\text{'s } \leq y, \\ &\textit{exactly } h \textit{ of } (X_i, Y_i)\text{'s} \in \mathbf{B}\}, \end{aligned} \tag{4.14}$$

where $B^c = \mathbb{R}^2 \backslash B$ is the complement of B.

Proof. From the conditional probability formula, one has

$$\begin{aligned} &P\{X_{r:n} \leq x, Y_{s:n} \leq y \mid h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}\} \\ &\quad = \frac{P\{X_{r:n} \leq x, Y_{s:n} \leq y, \ h \textit{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}\}}{P\{h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}\}}. \end{aligned} \tag{4.15}$$

Because the random vectors (X_i, Y_i), $i = 1, 2, ..., n$ are assumed to be independent and identically distributed, then from the binomial distribution, one has

$$\begin{aligned} &P\{h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}\} \\ &\quad = \binom{n}{h} P\{(X, Y) \in \mathbf{B}\}^h P\{(X, Y) \in \mathbf{B}^c\}^{n-h}. \end{aligned} \tag{4.16}$$

Now, (4.16) and (4.15) imply (4.14). Thus, the lemma is proved. □

For deriving the conditional distribution function of bivariate order statistics $F_{r,s:n}(x, y \mid u, v)$, we consider the following four possible cases:

$$Case\ a: \ u \leq x, \, v \leq y$$

$$Case\ b: \ u \leq x, \, v > y$$

$$Case\ c: \ u > x, \, v \leq y$$

$$Case\ d: \ u > x, \, v > y$$

Definition of Case a *If $u \leq x$, $v \leq y$, then we denote $A = \{X_i \leq x\}$, $B = \{Y_i \leq y\}$ and $C = \{X_i \leq u, Y_i \leq v\}$. Let ξ be the number of observations (X_i, Y_i), $i = 1, 2, ..., n$, for which $X_i \leq x$, η be the number of observations for which $Y_i \leq y$ and ζ be the number of observations for which $X_i \leq u$ and $Y_i \leq v$. It is clear that $C \subset AB$ and ξ, η, ζ are the number of observations in n independent trials of the fourfold experiment of the events A, B and C, respectively, as in case (a) of Definition 4.1. We have*

$$P(C) = P\{X \leq u, Y \leq v\} = F(u, v) \tag{4.17}$$

$$\begin{aligned} P(AB) - P(C) &= P\{X \leq x, Y \leq y\} - P\{X \leq u, Y \leq v\} \\ &= F(x, y) - F(u, v) \end{aligned} \tag{4.18}$$

$$P(AB^c) = P\{X \leq x, Y > y\} = F_X(x) - F(x, y) \tag{4.19}$$

$$P(A^cB) = P\{X > x, Y \leq y\} = F_Y(y) - F(x, y) \tag{4.20}$$

$$P(A^cB^c) = P\{X > x, Y > y\} = \bar{F}(x, y). \tag{4.21}$$

Theorem 1a *If* $u \le x,\ v \le y,$ *then*

$$\begin{aligned} &F^{(1)}_{r,s:n}(x, y \mid u, v) \\ &\qquad \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\} \\ &\qquad = \frac{1}{\binom{n}{h} F(u,v)^h [1 - F(u,v)]^{n-h}} \\ &\qquad \times \sum_{i=r}^{n} \sum_{j=s}^{n} \sum_{k=a}^{b} C_1(n; h, k, i, j) F(u,v)^h [F(x,y) - F(u,v)]^{k-h} \\ &\qquad \times [F_X(x) - F(x,y)]^{i-k} [F_Y(y) - F(x,y)]^{j-k} \bar{F}(x,y)^{n-i-j+k}, \qquad (4.22) \\ &\quad h = 0, 1, ..., \min(r, s) \text{ and} \\ &F^{(1)}_{r,s:n}(x, y \mid u, v) = 0 \text{ if } \min(r, s) < h \le n, \end{aligned}$$

where

$$C_1(n; h, k, i, j) = \frac{n!}{h!(k-h)!(i-k)!(j-k)!(n-i-j+k)!};$$

$$a = max(0, i + j - n);\ b = min(i, j).$$

Proof. Because $P\{(X, Y) \in \mathbf{B}_{uv}\} = F(u, v)$ and $P\{(X, Y) \in \mathbf{B}^c_{uv}\} = 1 - F(u, v)$, from Lemma 4.8 , we have

$$\begin{aligned} &F^{(1)}_{r,s:n}(x, y \mid u, v) \\ &\qquad \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\} \\ &\qquad = \frac{1}{\binom{n}{h} F(u,v)^h [1 - F(u,v)]^{n-h}} \sum_{i=r}^{n} \sum_{j=s}^{n} P\{\xi = i, \eta = j, \zeta = h\}. \end{aligned}$$

Now, (4.22) easily follows from Theorem 4.2, from the Definition of Case a and the equalities (4.17)-(4.21). For $i = r$, $j = s$, and $h = \min(r, s)$, the probability

$$P\{\xi = i, \eta = j, \zeta = h\}$$

does not vanish. For $i = r+1$, $j = s$, $r < s$, and $h = s+1$, this probability vanishes because $C \subset AB$ and the number of occurrences of C cannot exceed the number of occurrences of AB ($\{\xi = i, \eta = j\}$ implies that the number of occurrences of AB is $min(i, j)$). Therefore, for the values of $(i, j) = (r, s)$, $(r+1, s)$, $(r, s+1), ..., (n, n)$, the value of h will vary from 0 to $\min(r, s)$. □

Definition of Case b *If* $u \leq x$, $v > y$, *then we denote* $A = \{X_i \leq x\}$, $B = \{Y_i \leq y\}$, $C = \{X_i \leq u, Y_i \leq y\}$ *and* $D = \{X_i \leq u, y < Y_i \leq v\}$. *Let* ξ *be the number of observations* (X_i, Y_i), $i = 1, 2, ..., n$, *for which* $X_i \leq x$, η *be the number of observations for which* $Y_i \leq y$ *and* ζ *be the number of observations for which* $X_i \leq u$ *and* $Y_i \leq v$. *It is clear that* $C \subset AB$, $D \subset AB^c$ *and* ξ, η, ζ *are the number of observations in* n *independent trials of the fourfold experiment of the events* A, B *and* $C \cup D$, *respectively, as in case* (b) *of Definition 4.1. We have*

$$P(C) = P\{X \leq u, Y \leq y\} = F(u, y) \tag{4.23}$$

$$\begin{aligned} P(AB) - P(C) &= P\{X \leq x, Y \leq y\} - P\{X \leq u, Y \leq y\} \\ &= F(x, y) - F(u, y) \end{aligned} \tag{4.24}$$

$$P(D) = P\{X \leq u, y < Y \leq v\} = F(u, v) - F(u, y) \tag{4.25}$$

$$\begin{aligned} P(AB^c) - P(D) &= P\{X \leq x, Y > y\} - P\{X \leq u, y < Y \leq v\} \\ &= F_X(x) - F(x, y) - F(u, v) + F(u, y) \end{aligned} \tag{4.26}$$

$$P(A^cB) = P\{X > x, Y \leq y\} = F_Y(y) - F(x, y) \tag{4.27}$$

$$P(A^cB^c) = P\{X > x, Y > y\} = \bar{F}(x, y). \tag{4.28}$$

Theorem 2a *If* $u \le x$, $v > y$, *then*

$$F^{(2)}_{r,s:n}(x,y \mid u,v) \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h\text{of } (X_1,Y_1),...,(X_n,Y_n) \text{ belong to } \mathbf{B}_{uv}\}$$

$$= \frac{1}{\binom{n}{h} F(u,v)^h [1-F(u,v)]^{n-h}} \sum_{i=r}^{n}\sum_{j=s}^{n}\sum_{k=a}^{b}\sum_{l=0}^{h} C_2(n;k,l,h,i,j) F(u,y)^l$$
$$\times [F(x,y) - F(u,y)]^{k-l}$$
$$\times [F(u,v) - F(u,y)]^{h-l} [F_X(x) - F(x,y) - F(u,v) + F(u,y)]^{i-k-h+l}$$
$$\times [F_Y(y) - F(x,y)]^{j-k} \bar{F}(x,y)^{n-i-j+k}, \qquad (4.29)$$

$h = 0,1,...,r$ and

$F^{(2)}_{r,s:n}(x,y \mid u,v) = 0$ if $r < h \le n$,

where

$$C_2(n;k,l,h,i,j) = \frac{n!}{l!(k-l)!(h-l)!(i-k-h+l)!(j-k)!(n-i-j+k)!};$$
$$a = max(0, i+j-n);\ b = min(i,j).$$

Proof. Similar to the proof of Theorem 1a, the proof of this theorem easily follows from Lemma 4.8, Theorem 4.4, Definition 4.1 (b), Definition of Case b, and equalities (4.23)-(4.28).

Definition of Case c. *If* $u > x$, $v \le y$, *then we denote* $A = \{X_i \le x\}$, $B = \{Y_i \le y\}$, $C = \{X_i \le x, Y_i \le v\}$ *and* $E = \{x < X_i \le u, Y_i \le v\}$. *Let* ξ *be the number of observations* (X_i, Y_i), $i = 1,2,...,n$, *for which* $X_i \le x$, η *be the number of observations for which* $Y_i \le y$ *and* ζ *be the number of observations for which* $X_i \le u$ *and* $Y_i \le v$. *It is clear that* $C \subset AB$, $E \subset A^cB$ *and* ξ, η, ζ *are the number of observations in* n *independent trials of the fourfold experiment of the events* A, *B and $C \cup E$, respectively, as in case (c) of Definition 4.1. We have*

$$P(C) = P\{X \le x, Y \le v\} = F(x,v) \tag{4.30}$$

$$\begin{aligned} P(AB) - P(C) &= P\{X \le x, Y \le y\} - P\{X \le x, Y \le v\} \\ &= F(x,y) - F(x,v) \end{aligned} \tag{4.31}$$

$$P(AB^c) = P\{X \le x, Y > y\} = F_X(x) - F(x,y) \tag{4.32}$$

$$P(E) = P\{x < X \le u, Y \le v\} = F(u,v) - F(x,v) \tag{4.33}$$

$$\begin{aligned} P(A^cB) - P(E) &= P\{X > x, Y \le y\} - P\{x < X \le u, Y \le v\} \\ &= F_Y(y) - F(x,y) - F(u,v) + F(x,v) \end{aligned} \tag{4.34}$$

$$P(A^cB^c) = P\{X > x, Y > y\} = \bar{F}(x,y). \tag{4.35}$$

Theorem 3a *If $u > x$, $v \le y$, then*

$F^{(3)}_{r,s:n}(x,y \mid u,v) \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h\text{of } (X_1,Y_1),...,(X_n,Y_n) \text{ belong to } \mathbf{B}_{uv}\}$

$$\begin{aligned} &= \frac{1}{\binom{n}{h} F(u,v)^h [1-F(u,v)]^{n-h}} \sum_{i=r}^{n} \sum_{j=s}^{n} \sum_{k=a}^{b} \sum_{l=0}^{h} C_3(n;k,l,h,i,j) F(x,v)^l \\ &\times [F(x,y) - F(x,v)]^{k-l} [F_X(x) - F(x,y)]^{i-k} \\ &\times [F(u,v) - F(x,v)]^{h-l} [F_Y(y) - F(x,y) \\ &- F(u,v) + F(x,v)]^{j-k-h+l} \bar{F}(x,y)^{n-i-j+k}, \end{aligned} \tag{4.36}$$

$h = 0, 1, ..., s$ and

$F^{(3)}_{r,s:n}(x,y \mid u,v) = 0$ if $s < h \le n$,

where

$$C_3(n; k, l, h, i, j) = \frac{n!}{l!(k-l)!(i-k)!(h-l)!(j-k-h+l)!(n-i-j+k)!};$$
$$a = max(0, i+j-n);\ b = min(i, j).$$

Proof. Similar to the proof of Theorem 2a, the proof of this theorem easily follows from Lemma 4.8, Theorem 4.5, Definition 4.1 (c), Definition of Case c, and the equalities (4.30)-(4.35). □

Definition of Case d *If $u > x$, $v > y$, then we denote $A = \{X_i \le x\}$, $B = \{Y_i \le y\}$, $D = \{X_i \le x, y < Y_i \le v\}$, $E = \{x < X_i \le u, Y_i \le y\}$ and $F = \{x < X_i \le u, y < Y_i \le v\}$. Let ξ be the number of observations (X_i, Y_i), $i = 1, 2, ..., n$, for which $X_i \le x$, η be the number of observations for which $Y_i \le y$ and ζ be the number of observations for which $X_i \le u$ and $Y_i \le v$. It is clear that $D \subset AB^c$, $E \subset A^cB$, $F \subset A^cB^c$ and ξ, η, ζ are the number of observations in n independent trials of the fourfold experiment of the events A, B and $AB \cup D \cup E \cup F$, respectively as in case (d1) of Definition 4.1. We have*

$$P(AB) = P\{X \le x, Y \le y\} = F(x, y) \tag{4.37}$$

$$P(D) = P\{X \le x, y < Y \le v\}$$
$$= F(x, v) - F(x, y) \tag{4.38}$$

$$P(AB^c) - P(D) = P\{X \le x, Y > y\} - P\{X \le x, y < Y \le v\}$$
$$= F_X(x) - F(x, v) \tag{4.39}$$

$$P(E) = P\{x < X \le u, Y \le y\} = F(u, y) - F(x, y) \tag{4.40}$$

$$P(A^cB) - P(E) = P\{X > x, Y \le y\} - P\{x < X \le u, Y \le y\}$$
$$= F_Y(y) - F(u, y) \tag{4.41}$$

$$\begin{aligned} P(F) =& P\{x < X \le u, y < Y \le v\} \\ &= F(u,v) - F(x,v) - F(u,y) + F(x,y) \end{aligned} \tag{4.42}$$

$$\begin{aligned} & P(A^cB^c) - P(F) \\ =& P\{X > x, Y > y\}) - P\{x < X \le u, y < Y \le v\} \\ =& 1 - F_X(x) - F_Y(y) - F(u,v) + F(x,v) + F(u,y). \end{aligned} \tag{4.43}$$

Theorem 4.1a *If* $u > x$, $v > y$, *then*

$F^{(4.1)}_{r,s:n}(x, y \mid u, v) \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h\text{of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\}$

$$\begin{aligned} &= \frac{1}{\binom{n}{h} F(u,v)^h [1 - F(u,v)]^{n-h}} \sum_{i=r}^{n} \sum_{j=s}^{n} \sum_{k=a}^{b} \sum_{p=0}^{i-k} \sum_{q=0}^{j-k} C_{4.1}(n; k, p, q, h, i, j) F(x,y)^k \\ &\times [F(x,v) - F(x,y)]^p [F_X(x) - F(x,v)]^{i-k-p} \\ &\times [F(u,y) - F(x,y)]^q [F_Y(y) - F(u,y)]^{j-k-q} \\ &\times [F(u,v) - F(x,v) - F(u,y) + F(x,y)]^{h-p-q-k} \\ &\times [1 - F_X(x) - F_Y(y) - F(u,v) + F(x,v) + F(u,y)]^{n-i-j+2k-h+p+q}, \end{aligned} \tag{4.44}$$

$h = 0, ..., n,$

where

$$\begin{aligned} & C_{4.1}(n; k, p, q, h, i, j) \\ &= \frac{n!}{k!p!(i-k-p)!q!(j-k-q)!(h-p-q-k)!} \\ &\times \frac{1}{(n-i-j+2k-h+p+q)!}; \\ & a = max(0, i+j-n);\ b = min(i,j). \end{aligned}$$

Proof. Using Lemma 4.8, Definition 4.1 (d1), and Definition of Case d, one has

$$F^{(4.1)}_{r,s:n}(x, y \mid u, v)$$
$$= \frac{1}{\binom{n}{h} F(u,v)^h [1 - F(u,v)]^{n-h}} \sum_{i=r}^{n} \sum_{j=s}^{n} P\{\xi = i, \eta = j, \zeta = h\}.$$

Using Theorem 4.7 and equalities (4.37)-(4.43), we complete the proof. □

Finally, using the results of Theorems 1a-4.1a, the conditional distribution of bivariate order statistics is presented in the following theorem:

Theorem 5 *Let (X, Y) be a bivariate random vector with joint distribution function $F(x, y)$ and $(X_1, Y_1), ..., (X_n, Y_n)$ be independent copies of (X, Y). If $(X_{r:n}, Y_{s:n})$, $r, s = 1, 2, ..., n$, is the vector of bivariate order statistics and $B_{uv} = \{(t, s) \in \mathbb{R}^2 : t \le u, s \le v\}$, $(u, v) \in \mathbb{R}^2$, then*

$$F_{r,s:n}(x, y \mid u, v) \equiv P\{X_{r:n} \le x, Y_{s:n} \le y \mid h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\}$$

$$= \begin{cases} F^{(1)}_{r,s:n}(x, y \mid u, v) & \text{if} \quad u \le x,\ v \le y, \\ F^{(2)}_{r,s:n}(x, y \mid u, v) & \text{if} \quad u \le x,\ v > y, \\ F^{(3)}_{r,s:n}(x, y \mid u, v) & \text{if} \quad u > x,\ v \le y, \\ F^{(4.1)}_{r,s:n}(x, y \mid u, v) & \text{if} \quad u > x,\ v > y, \end{cases}$$

$h = 0, 1, ..., \min(r, s)$.

Remark 4.9. *One can verify the accuracy of the results presented in Theorems 1a-4.1a. Here, we present a different method for deriving the conditional distribution of bivariate order statistics using the properties of extreme order statistics*

$(X_{n:n}, Y_{n:n})$ *as follows: Consider*

$$\begin{aligned}
&F_{n,n:n}(x, y \mid u, v) \\
&\qquad = P\{X_{n:n} \le x, Y_{n:n} \le y \mid h \text{ of } (X_1, Y_1), ..., \\
&\qquad (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\} \\
&\qquad = \frac{1}{\binom{n}{h} F(u,v)^h [1 - F(u,v)]^{n-h}} \\
&\times P\{X_{n:n} \le x, Y_{n:n} \le y, h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\}. \qquad (4.45)
\end{aligned}$$

Because $X_{n:n} \le x$, *implies that all* $X's$ *are less than or equal to* x, *we can write*

$$\begin{aligned}
&P\{X_{n:n} \le x, Y_{n:n} \le y, h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\} \\
&\qquad = \sum_{j_1, j_2, ..., j_n}^{n} P\{X_{n:n} \le x, Y_{n:n} \le y, (X_{j_1}, Y_{j_1}) \in \mathbf{B}_{uv}, ..., (X_{j_h}, Y_{j_h}) \in \mathbf{B}_{uv}, \\
&(X_{j_{h+1}}, Y_{j_{h+1}}) \in \mathbf{B}^c_{uv}, ..., (X_{j_n}, Y_{j_n}) \in \mathbf{B}^c_{uv}\} \\
&\qquad = \sum_{j_1, j_2, ..., j_n}^{n} P\{X_{n:n} \le x, Y_{n:n} \le y, (X_1, Y_1) \in \mathbf{B}_{uv}, ..., (X_h, Y_h) \in \mathbf{B}_{uv} \\
&(X_{h+1}, Y_{h+1}) \in \mathbf{B}^c_{uv}, ..., (X_n, Y_n) \in \mathbf{B}^c_{uv}\} \\
&\qquad = \binom{n}{h} P\{X_{n:n} \le x, Y_{n:n} \le y, (X_1, Y_1) \in \mathbf{B}_{uv}, ..., (X_h, Y_h) \in \mathbf{B}_{uv},
\end{aligned}$$

$$
\begin{aligned}
&(X_{h+1}, Y_{h+1}) \in \mathbf{B}_{uv}^c, ..., (X_n, Y_n) \in \mathbf{B}_{uv}^c\} \\
&\qquad = \binom{n}{h} P\{X_1 \le x, ..., X_n \le x,\ Y_1 \le y, ..., Y_n \le y, \\
&\quad (X_1, Y_1) \in \mathbf{B}_{uv}, ..., (X_h, Y_h) \in \mathbf{B}_{uv}, (X_{h+1}, Y_{h+1}) \in \mathbf{B}_{uv}^c, ..., (X_n, Y_n) \in \mathbf{B}_{uv}^c\} \\
&\qquad = \binom{n}{h} P\{X \le x, Y \le y, (X, Y) \in \mathbf{B}_{uv}\}^h P\{X \le x, Y \le y, (X, Y) \in \mathbf{B}_{uv}^c\}^{n-h} \\
&\qquad = \binom{n}{h} P\{X \le x, Y \le y, X \le u, Y \le v\}^h \\
&\quad \times P\{X \le x, Y \le y, (X \le u, Y > v \cup X > u, Y \le v \cup X > u, Y > v)\}^{n-h} \\
&\qquad = \binom{n}{h} \left[P\{X \le \min(x, u), Y \le \min(y, v)\}\right]^h \\
&\quad \times [P\{X \le \min(x, u), v < Y \le y\} + P\{u < X \le x, Y \le \min(y, v)\} \\
&\quad + P\{u < X \le x, v < Y \le y\}]^{n-h}. \qquad (4.46)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&F_{n,n:n}(x, y\,|u, v) \\
&= \frac{1}{F(u, v)^h [1 - F(u, v)]^{n-h}} \left[F(\min(x, u), \min(y, v)\right]^h \\
&\times [F(\min(x, u), y) - F(\min(x, u), v) + F(x, \min(y, v)) - F(u, \min(y, v)) \\
&+ F(u, v) - F(u, y) - F(x, v) + F(x, y)]^{n-h}. \qquad (4.47)
\end{aligned}
$$

If $u \le x$, $v \le y$, then we obtain

$$\begin{aligned}
&P\{X_{n:n} \le x, Y_{n:n} \le y, h \textit{ of } (X_1,Y_1),...,(X_n,Y_n) \textit{ belong to } \mathbf{B}_{uv}\}\\
&\qquad = \binom{n}{h}[P\{X \le u, Y \le v\}]^h\\
&\times[P\{X \le u, v < Y \le y\} + P\{u < X \le x, Y \le v\}\\
&\ +P\{u < X \le x, v < Y \le y\}]^{n-h}\\
&\qquad = \binom{n}{h}F(u,v)^h\\
&\qquad \times [F(u,y) - F(u,v) + F(x,v) - F(u,v)\\
&\qquad + F(u,v) - F(u,y) - F(x,v) + F(x,y)]^{n-h}\\
&\qquad = \binom{n}{h}F(u,v)^h[F(x,y) - F(u,v)]^{n-h}. \qquad (4.48)
\end{aligned}$$

Thus taking into account (4.48) in (4.45), we obtain

$$\begin{aligned}
&F^{(1)}_{n,n:n}(x,y\,|u,v)\\
&= \frac{[F(x,y) - F(u,v)]^{n-h}}{[1 - F(u,v)]^{n-h}}. \qquad (4.49)
\end{aligned}$$

Now, let $r = s = n$ in Theorem 1a. Then, it can be easily verified that $F_{n,n:n}(x,y \mid u,v)$ in Theorem 1a equals to (4.49).

Similarly, we obtain the joint distribution of conditional extreme order statistics $(X_{n:n}, Y_{n:n})$ for the other cases:

If $u \le x$, $v > y$

$$\begin{aligned}
&F^{(2)}_{n,n:n}(x,y\,|u,v)\\
&= \frac{F(u,y)^h[2F(x,y) - F(u,y) - F(x,v)]^{n-h}}{F(u,v)^h[1 - F(u,v)]^{n-h}}. \qquad (4.50)
\end{aligned}$$

If $u > x,\, v \le y$

$$F^{(3)}_{n,n:n}(x, y\,|u, v) \\ = \frac{F(x, v)^h[2F(x, y) - F(u, y) - F(x, v)]^{n-h}}{F(u, v)^h[1 - F(u, v)]^{n-h}}. \quad (4.51)$$

If $u > x,\, v > y$

$$F^{(4.1)}_{n,n:n}(x, y\,|u, v) \\ = \frac{F(x, y)^h[3F(x, y) - 2F(x, v) - 2F(u, y) + F(u, v)]^{n-h}}{F(u, v)^h[1 - F(u, v)]^{n-h}}. \quad (4.52)$$

Therefore, we can represent the joint distribution of conditional extreme order statistics $(X_{n:n}, Y_{n:n})$ using the results obtained in Remark 4.9 as follows:

Theorem 4.10. *Let (X, Y) be a bivariate random vector with joint distribution function $F(x, y)$ and $(X_1, Y_1), ..., (X_n, Y_n)$ be independent copies of (X, Y). If $(X_{n:n}, Y_{n:n})$ is the vector of bivariate maximum order statistics and $B_{uv} = \{(t, s) \in \mathbb{R}^2 : t \le u, s \le v\}$, $(u, v) \in \mathbb{R}^2$, then*

$$F_{n,n:n}(x, y\,|u, v) \equiv P\{X_{n:n} \le x, Y_{n:n} \le y \mid h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\}$$

$$= \begin{cases} F^{(1)}_{n,n:n}(x, y \mid u, v) & \text{if} \quad u \le x,\, v \le y, \\ F^{(2)}_{n,n:n}(x, y \mid u, v) & \text{if} \quad u \le x,\, v > y, \\ F^{(3)}_{n,n:n}(x, y \mid u, v) & \text{if} \quad u > x,\, v \le y, \\ F^{(4.1)}_{n,n:n}(x, y \mid u, v) & \text{if} \quad u > x,\, v > y, \end{cases}$$

$h = 0, 1, ..., n$.

4.2.1.1 Marginal Distributions of Conditional Extreme Order Statistics $(X_{n:n}, Y_{n:n})$

In this section we are interested in the marginal distribution functions of bivariate extreme order statistics $(X_{n:n}, Y_{n:n})$ under the condition that h of the bivariate observations (X_i, Y_i), $i = 1, 2, ..., n$ are truncated, i.e. fall in the set $\mathbf{B}_{uv}$.

Let us denote the marginal distribution function of $(X_{n:n})$ under condition that h of $(X_1, Y_1), ..., (X_n, Y_n)$ belong to $\mathbf{B}_{uv}$ by $F_{X_{n:n}}(x\,|u, v\,)$ and denote the marginal distribution function of $(Y_{n:n})$ under condition that h of $(X_1, Y_1), ..., (X_n, Y_n)$ belong to $\mathbf{B}_{uv}$ by $F_{Y_{n:n}}(y\,|u, v\,)$.

Then, by taking the limit of conditional joint distribution $F_{n,n:n}(x, y\,|u, v\,)$ as $y \to \infty$, one can easily find the marginal distribution function $F_{X_{n:n}}(x\,|u, v\,)$ Similarly, by taking the limit of $F_{n,n:n}(x, y\,|u, v\,)$ as $x \to \infty$, one can obtain the $F_{Y_{n:n}}(y\,|u, v\,)$:

$$\begin{aligned}
&\lim_{y\to\infty} F_{n,n:n}(x, y\,|u, v\,) \\
&\equiv \lim_{y\to\infty} P\{X_{n:n} \leq x, Y_{n:n} \leq y\,|h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\,\} \\
&\equiv F_{X_{n:n}}(x\,|u, v\,) \qquad (4.53)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to\infty} F_{n,n:n}(x, y\,|u, v\,) \\
&\equiv \lim_{x\to\infty} P\{X_{n:n} \leq x, Y_{n:n} \leq y\,|h \text{ of } (X_1, Y_1), ..., (X_n, Y_n) \text{ belong to } \mathbf{B}_{uv}\,\} \\
&\equiv F_{Y_{n:n}}(y\,|u, v\,). \qquad (4.54)
\end{aligned}$$

Remember that we consider the four possible cases to obtain the conditional distribution function of bivariate extreme order statistics $F_{n,n:n}(x, y \mid u, v)$:

$Case\ a: \ u \leq x,\, v \leq y$

$Case\ b: \ u \leq x,\, v > y$

$Case\ c: \ u > x,\, v \leq y$

$Case\ d: \ u > x,\, v > y$

Let us consider the $Case\ a)$ and $Case\ c)$ as $y \to \infty$. Because $v \leq y$, we obtain $v < \infty$ if $y \to \infty$. Similarly, let us consider the $Case\ b)$ and $Case\ d)$ as $y \to \infty$. Because $v > y$, we obtain $v = \infty$ if $y \to \infty$.

Therefore, for deriving the $F_{X_{n:n}}(x\,|u,v) \equiv \lim_{y\to\infty} F_{n,n:n}(x,y\,|u,v)$ we have four possible cases:

$i)\quad u \leq x,\, v < \infty$

$ii)\ \ u \leq x,\, v = \infty$

$iii)\ u > x,\, v < \infty$

$iv)\ u > x,\, v = \infty$

Here, we obtain the marginal distribution of $(X_{n:n})$ under condition that h of $(X_1, Y_1), ..., (X_n, Y_n)$ belong to $\mathbf{B}_{uv}$ as follows:

If $u \leq x,\, v \leq y$

$$
\begin{aligned}
F^{(1)}_{X_{n:n}}(x\,|u,v) &\equiv \lim_{y\to\infty} F^{(1)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{y\to\infty}\left(\frac{[F(x,y)-F(u,v)]^{n-h}}{[1-F(u,v)]^{n-h}}\right) \\
&= \frac{[F(x)-F(u,v)]^{n-h}}{[1-F(u,v)]^{n-h}}, \qquad (4.55)\\
u \le x,\ v < \infty.
\end{aligned}
$$

If $u \le x,\ v > y$

$$
\begin{aligned}
F^{(2)}_{X_{n:n}}(x\,|u,v) &\equiv \lim_{y\to\infty} F^{(2)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{y\to\infty}\left(\frac{F(u,y)^h[2F(x,y)-F(u,y)-F(x,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right) \\
&= \frac{[F(x)-F(u)]^{n-h}}{[1-F(u)]^{n-h}}, \qquad (4.56)\\
u \le x,\ v = \infty.
\end{aligned}
$$

If $u > x,\ v \le y$

$$
\begin{aligned}
F^{(3)}_{X_{n:n}}(x\,|u,v) &\equiv \lim_{y\to\infty} F^{(3)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{y\to\infty}\left(\frac{F(x,v)^h[2F(x,y)-F(u,y)-F(x,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right) \\
&= \frac{F(x,v)^h[2F(x)-F(u)-F(x,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}, \qquad (4.57)\\
u > x,\ v < \infty.
\end{aligned}
$$

If $u > x$, $v > y$

$$\begin{aligned}
F^{(4.1)}_{X_{n:n}}(x\,|u,v) &\equiv \lim_{y\to\infty} F^{(4.1)}_{n,n:n}(x,y\,|u,v)\\
&= \lim_{y\to\infty}\left(\frac{F(x,y)^h[3F(x,y)-2F(x,v)-2F(u,y)+F(u,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right)\\
&= \frac{F(x)^h[F(x)-F(u)]^{n-h}}{F(u)^h[1-F(u)]^{n-h}}, \qquad (4.58)\\
&u > x,\ v=\infty.
\end{aligned}$$

Now, we represent the marginal distribution of $(X_{n:n})$ under condition that h of $(X_1,Y_1),...,(X_n,Y_n)$ belong to $\mathbf{B}_{uv}$ in the following theorem:

Theorem 4.11. *Let (X,Y) be a bivariate random vector with joint distribution function $F(x,y)$ and $(X_1,Y_1),...,(X_n,Y_n)$ be independent copies of (X,Y). If $(X_{n:n},Y_{n:n})$ is the vector of bivariate maximum order statistics with joint distribution function $F_{n,n:n}(x,y\,|u,v)$ given in Theorem 4.10 and $B_{uv}=\{(t,s)\in\mathbb{R}^2 : t\le u, s\le v\}$, $(u,v)\in\mathbb{R}^2$, then*

$$F_{X_{n:n}}(x\,|u,v) \equiv \lim_{y\to\infty} F_{n,n:n}(x,y\,|u,v)$$

$$= \begin{cases} F^{(1)}_{X_{n:n}}(x\,|u,v) & \text{if} \quad u\le x,\ v<\infty,\\ F^{(2)}_{X_{n:n}}(x\,|u,v) & \text{if} \quad u\le x,\ v=\infty,\\ F^{(3)}_{X_{n:n}}(x\,|u,v) & \text{if} \quad u> x,\ v<\infty,\\ F^{(4.1)}_{X_{n:n}}(x\,|u,v) & \text{if} \quad u> x,\ v=\infty,\end{cases}$$

$h=0,1,...,n.$

Analogously, let us consider the *Case* a) and *Case* b) as $x \to \infty$. Because $u \leq x$, we obtain $u < \infty$ if $x \to \infty$. Also, let us consider the *Case* c) and *Case* d) as $x \to \infty$. Because $u > x$, we obtain $u = \infty$ if $x \to \infty$.

Therefore, for deriving the $F_{Y_{n:n}}(y\,|u,v) \equiv \lim_{x\to\infty} F_{n,n:n}(x,y\,|u,v)$ we have four possible cases:

i) $u < \infty,\ v \leq y$

ii) $u < \infty,\ v > y$

iii) $u = \infty,\ v \leq y$

iv) $u = \infty,\ v \leq y$

Now, we obtain the marginal distribution of $(Y_{n:n})$ under condition that h of $(X_1, Y_1), ..., (X_n, Y_n)$ belong to $\mathbf{B}_{uv}$ as follows:

If $u \leq x,\ v \leq y$

$$\begin{aligned} F^{(1)}_{Y_{n:n}}(y\,|u,v) &\equiv \lim_{x\to\infty} F^{(1)}_{n,n:n}(x,y\,|u,v) \\ &= \lim_{x\to\infty}\left(\frac{[F(x,y)-F(u,v)]^{n-h}}{[1-F(u,v)]^{n-h}}\right) \\ &= \frac{[F(y)-F(u,v)]^{n-h}}{[1-F(u,v)]^{n-h}}, \\ & u < \infty,\ v \leq y. \end{aligned} \tag{4.59}$$

If $u \leq x,\ v > y$

$$
\begin{aligned}
F^{(2)}_{Y_{n:n}}(y\,|u,v) &\equiv \lim_{x\to\infty} F^{(2)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{x\to\infty}\left(\frac{F(u,y)^h[2F(x,y)-F(u,y)-F(x,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right) \\
&= \frac{F(u,y)^h[2F(y)-F(u,y)-F(v)]^{n-h}}{F(u,v)^h[1-F(u)]^{n-h}}, \qquad (4.60)\\
& u<\infty,\ v>y.
\end{aligned}
$$

If $u>x,\ v\le y$

$$
\begin{aligned}
F^{(3)}_{Y_{n:n}}(y\,|u,v) &\equiv \lim_{x\to\infty} F^{(3)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{x\to\infty}\left(\frac{F(x,v)^h[2F(x,y)-F(u,y)-F(x,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right) \\
&= \frac{[F(y)-F(v)]^{n-h}}{[1-F(v)]^{n-h}}, \qquad (4.61)\\
& u=\infty,\ v\le y.
\end{aligned}
$$

If $u>x,\ v>y$

$$
\begin{aligned}
F^{(4.1)}_{Y_{n:n}}(y\,|u,v) &\equiv \lim_{x\to\infty} F^{(4.1)}_{n,n:n}(x,y\,|u,v) \\
&= \lim_{x\to\infty}\left(\frac{F(x,y)^h[3F(x,y)-2F(x,v)-2F(u,y)+F(u,v)]^{n-h}}{F(u,v)^h[1-F(u,v)]^{n-h}}\right) \\
&= \frac{F(y)^h[F(y)-F(v)]^{n-h}}{F(v)^h[1-F(v)]^{n-h}}, \qquad (4.62)\\
& u=\infty,\ v\le y.
\end{aligned}
$$

Now, we represent the marginal distribution of $(X_{n:n})$ under condition that h of $(X_1, Y_1), ..., (X_n, Y_n)$ belong to $\mathbf{B}_{uv}$ in the following theorem:

Theorem 4.12. *Let (X, Y) be a bivariate random vector with joint distribution function $F(x, y)$ and $(X_1, Y_1), ..., (X_n, Y_n)$ be independent copies of (X, Y). If $(X_{n:n}, Y_{n:n})$ is the vector of bivariate maximum order statistics with joint distribution function $F_{n,n:n}(x, y\,|u, v)$ given in Theorem 4.10 and $B_{uv} = \{(t, s) \in \mathbb{R}^2 : t \leq u, s \leq v\}$, $(u, v) \in \mathbb{R}^2$, then*

$$F_{Y_{n:n}}(y\,|u, v) \equiv \lim_{x\to\infty} F_{n,n:n}(x, y\,|u, v)$$

$$= \begin{cases} F^{(1)}_{Y_{n:n}}(y\,|u, v) & \text{if} \quad u < \infty,\ v \leq y, \\ F^{(2)}_{Y_{n:n}}(y\,|u, v) & \text{if} \quad u < \infty,\ v > y, \\ F^{(3)}_{Y_{n:n}}(y\,|u, v) & \text{if} \quad u = \infty,\ v \leq y, \\ F^{(4.1)}_{Y_{n:n}}(y\,|u, v) & \text{if} \quad u = \infty,\ v \leq y, \end{cases}$$

$h = 0, 1, ..., n.$

4.2.2 Some Dependence Results of Conditional Extreme Order Statistics $(X_{n:n}, Y_{n:n})$

In this section, we present some results concerning dependence between conditional bivariate extreme order statistics. In particular, Pearson's correlation coefficient of conditional bivariate order statistics $(X_{n:n}, Y_{n:n})$ has been calculated. In the case of the underlying distribution is Farlie-Gumbel-Morgenstern (FGM), the corresponding dependence analysis involving association parameter α is given.

Let $F(x, y) = F_X(x)F_Y(y)\{1+\alpha(1-F_X(x))(1-F_Y(y))\}$ be the Farlie-Gumbel-Morgenstern distribution with uniform marginal distributions $F_X(x) = x$, $F_Y(y) = y$, $0 \leq x, y \leq 1$.

Therefore using the equalities 4.55-4.58, one can easily obtain the marginal distribution function $F_{X_{n:n}}(x\,|u,v)$ as follows:

If $u \leq x,\ v < \infty$

$$F^{(1)}_{X_{n:n}}(x\,|u,v) = \frac{[x - uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}{[1 - uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}.$$

If $u \leq x,\ v = \infty$

$$F^{(2)}_{X_{n:n}}(x\,|u,v) = \frac{[x-u]^{n-h}}{[1-u]^{n-h}}.$$

If $u > x,\ v < \infty$

$$F^{(3)}_{X_{n:n}}(x\,|u,v) = \frac{[xv\{1+\alpha(1-x)(1-v)\}]^{h}[2x - u - xv\{1+\alpha(1-x)(1-v)\}]^{n-h}}{[uv\{1+\alpha(1-u)(1-v)\}]^{h}[1 - uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}.$$

If $u > x,\ v = \infty$

$$F^{(4.1)}_{X_{n:n}}(x\,|u,v) = \frac{x^{h}[x-u]^{n-h}}{u^{h}[1-u]^{n-h}}.$$

By similar consideration, the marginal distribution function $F_{Y_{n:n}}(y\,|u,v)$ can be easily derived using the equalities 4.59-4.62 as follows:

If $u < \infty,\ v \leq y$

$$F^{(1)}_{Y_{n:n}}(y\,|u,v) = \frac{[y - uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}{[1 - uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}.$$

If $u < \infty$, $v > y$

$$F^{(2)}_{Y_{n:n}}(y\,|u,v) = \frac{[uy\{1+\alpha(1-u)(1-y)\}]^h[2y-v-uy\{1+\alpha(1-u)(1-y)\}]^{n-h}}{[uv\{1+\alpha(1-u)(1-v)\}]^h[1-uv\{1+\alpha(1-u)(1-v)\}]^{n-h}}.$$

If $u = \infty$, $v \leq y$

$$F^{(3)}_{Y_{n:n}}(y\,|u,v) = \frac{[y-v]^{n-h}}{[1-v]^{n-h}}.$$

If $u = \infty$, $v \leq y$

$$F^{(4.1)}_{Y_{n:n}}(y\,|u,v) = \frac{y^h[y-v]^{n-h}}{v^h[1-v]^{n-h}}.$$

However, the probability density functions are defined as $\partial F_{X_{n:n}}(x\,|u,v)/\partial x$, $\partial F_{Y_{n:n}}(y\,|u,v)/\partial y$ and denoted by $f_{X_{n:n}}(x\,|u,v)$, $f_{Y_{n:n}}(y\,|u,v)$, respectively. For simplicity, let us denote the conditional random variables as follows:

$$X^*_{n:n} \equiv (X_{n:n}\,|h \text{ of } (X_1,Y_1),...,(X_n,Y_n) \text{ belong to } \mathbf{B}_{uv})$$
$$Y^*_{n:n} \equiv (Y_{n:n}\,|h \text{ of } (X_1,Y_1),...,(X_n,Y_n) \text{ belong to } \mathbf{B}_{uv})$$

Therefore, the expected value and variance of $X^*_{n:n}$ and $Y^*_{n:n}$ can be easily calculated through probability density functions. For calculating covariance between X^* and Y^*, we can use Hoeffding's formula which represents covariance in terms of distribution functions:

$$Cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(x,y) - F_X(x)F_Y(y)]dxdy. \tag{4.63}$$

Therefore, the covariance between X^* and Y^* is given by

$$
\begin{aligned}
&Cov(X^*_{n:n}, Y^*_{n:n}) \\
&= \iint\limits_{D} \left[F_{n,n:n}(x, y\,|u, v) - F_{X_{n:n}}(x\,|u, v)F_{Y_{n:n}}(y\,|u, v)\right] dxdy \\
&= \int_u^1 \int_v^1 \left[F^{(1)}_{n,n:n}(x, y\,|u, v) - F^{(1)}_{X_{n:n}}(x\,|u, v)F^{(1)}_{Y_{n:n}}(y\,|u, v)\right] dxdy + \\
&\int_u^1 \int_0^v \left[F^{(2)}_{n,n:n}(x, y\,|u, v) - F^{(2)}_{X_{n:n}}(x\,|u, v)F^{(2)}_{Y_{n:n}}(y\,|u, v)\right] dxdy + \\
&\int_0^u \int_v^1 \left[F^{(3)}_{n,n:n}(x, y\,|u, v) - F^{(3)}_{X_{n:n}}(x\,|u, v)F^{(3)}_{Y_{n:n}}(y\,|u, v)\right] dxdy + \\
&\int_0^u \int_0^v \left[F^{(4.1)}_{n,n:n}(x, y\,|u, v) - F^{(4.1)}_{X_{n:n}}(x\,|u, v)F^{(4.1)}_{Y_{n:n}}(y\,|u, v)\right] dxdy
\end{aligned}
$$

where D is the domain of the integral,

$$
\begin{aligned}
D = \{&x, y\,|u \le x, v \le y \cup u \le x, v > y \cup u > x, v \le y \cup u > x, v > y; \\
&x, y, u, v \in [0, 1]\}.
\end{aligned}
$$

In the following table, we represent some numerical results of $(X_{n:n}, Y_{n:n})$ related to expected value, variance, covariance and Pearson's correlation coefficient.

Table 4.1 Some Dependence Results of $(X_{n:n}, Y_{n:n})$

	$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
$E(X^*_{n:n})$	1.706063	1.700992	1.685666
$E(Y^*_{n:n})$	1.741799	1.734802	1.718757
$Var(X^*_{n:n})$	4.394958	4.370221	4.296420
$Var(Y^*_{n:n})$	4.603365	4.570730	4.493932
$Cov(X^*_{n:n}, Y^*_{n:n})$	0.001145	0.000136	-0.003059
$\rho(X^*_{n:n}, Y^*_{n:n})$	0.000255	0.000030	-0.000696

It can be observed from the Table 4.1 that for this class of bivariate distributions Pearson's correlation coefficient increases if dependence parameter α increases.

4.3 Numerical Results and Graphics Related to Conditional Bivariate Order Statistics

Let $F(x, y) = F_X(x)F_Y(y)\{1 + \alpha(1 - F_X(x))(1 - F_Y(y))\}$ be the Farlie-Gumbel-Morgenstern (FGM) distribution and $F_X(x) = x$, $F_Y(y) = y$, $0 \leq x, y \leq 1$. This class of distributions has a simple analytical form and is suitable for calculations. Below, we provide some numerical results and graphs of the conditional distribution of bivariate order statistics given in Theorems 1a-4.1a and Theorem 5 in the case of the underlying distribution is FGM. All numerical results presented here are obtained by using a powerful mathematical software Wolfram Mathematica 7.

4.3.1 Numerical Results for Conditional r^{th} and s^{th} Order Statistics

It can be observed from the following Table 4.2 that for fixed values of x, y, n, r, s, h, u, v (case $u \leq x,\ v \leq y$) the function $F^{(1)}_{r,s:n}(x, y\,|u, v)$ decreases with respect to dependence parameter α, namely $F^{(1)}_{r,s:n}(x, y\,|u, v)$ is a decreasing function. Other cases can be analyzed similarly. This is supported by the graph of function $F^{(1)}_{r,s:n}(x, y\,|u, v)$ with respect to association parameter α, given in Figure 4.6.

i) Theorem 1a. $u \leq x, v \leq y$:

Table 4.2 Some numerical results of $F^{(1)}_{r,s:n}(x, y\,|u, v)$

for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 3,\ s = 2,\ h = 2.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(1)}_{r,s:n}$	$F^{(1)}_{r,s:n}$	$F^{(1)}_{r,s:n}$
	0.3	0.7	0.531549	0.637989	0.825002
	0.3	0.8	0.531549	0.637989	0.825002
	0.3	0.9	0.531549	0.637989	0.825002
$n = 10$	0.5	0.7	0.968258	0.975471	0.988142
$r = 3$	0.5	0.8	0.968258	0.975471	0.988142
$s = 2$	0.5	0.9	0.968258	0.975471	0.988142
$h = 2$	0.7	0.7	0.999467	0.999588	0.999801
	0.7	0.8	0.999467	0.999588	0.999801
	0.7	0.9	0.999467	0.999588	0.999801
	0.9	0.7	0.999999	0.999999	0.999999
	0.9	0.8	0.999999	0.999999	0.999999
	0.9	0.9	0.999999	0.999999	0.999999

ii) Theorem 2a. $u \leq x, v > y$:

Table 4.3 Some numerical results of $F^{(2)}_{r,s:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 3,\ s = 2,\ h = 3.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(2)}_{r,s:n}$	$F^{(2)}_{r,s:n}$	$F^{(2)}_{r,s:n}$
	0.4	0.2	0.691544	0.692651	0.626379
	0.4	0.3	0.905233	0.904104	0.867047
	0.4	0.4	0.981521	0.980683	0.969238
$n = 10$	0.4	0.5	0.998422	0.998264	0.99682
$r = 3$	0.6	0.2	0.691544	0.692651	0.626379
$s = 2$	0.6	0.3	0.905233	0.904104	0.867047
$h = 3$	0.6	0.4	0.981521	0.980683	0.969238
	0.6	0.5	0.998422	0.998264	0.99682
	0.8	0.2	0.691544	0.692651	0.626379
	0.8	0.3	0.905233	0.904104	0.867047
	0.8	0.4	0.981521	0.980683	0.969238
	0.8	0.5	0.998422	0.998264	0.99682

iii) Theorem 3a. $u > x, v \leq y$:

Table 4.4 Some numerical results of $F^{(3)}_{r,s:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 3,\ s = 2,\ h = 2.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(2)}_{r,s:n}$	$F^{(2)}_{r,s:n}$	$F^{(2)}_{r,s:n}$
	0.4	0.2	0.691544	0.692651	0.626379
	0.4	0.3	0.905233	0.904104	0.867047
	0.4	0.4	0.981521	0.980683	0.969238
$n = 10$	0.4	0.5	0.998422	0.998264	0.99682
$r = 3$	0.6	0.2	0.691544	0.692651	0.626379
$s = 2$	0.6	0.3	0.905233	0.904104	0.867047
$h = 3$	0.6	0.4	0.981521	0.980683	0.969238
	0.6	0.5	0.998422	0.998264	0.99682
	0.8	0.2	0.691544	0.692651	0.626379
	0.8	0.3	0.905233	0.904104	0.867047
	0.8	0.4	0.981521	0.980683	0.969238
	0.8	0.5	0.998422	0.998264	0.99682

iv) Theorem 4.1a. $u > x, v > y$:

Table 4.5 Some numerical results of $F^{(4.1)}_{r,s:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 3,\ s = 2,\ h = 5.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(4.1)}_{r,s:n}$	$F^{(4.1)}_{r,s:n}$	$F^{(4.1)}_{r,s:n}$
	0.1	0.2	0.231141	0.229566	0.154363
	0.1	0.3	0.272180	0.276014	0.229362
	0.1	0.4	0.280915	0.287194	0.261350
$n = 10$	0.1	0.5	0.281849	0.288585	0.268146
$r = 3$	0.15	0.2	0.489626	0.481675	0.348019
$s = 2$	0.15	0.3	0.577337	0.578870	0.504351
$h = 5$	0.15	0.4	0.596121	0.602165	0.567672
	0.15	0.5	0.598141	0.605041	0.580471
	0.2	0.2	0.699618	0.684485	0.521360
	0.2	0.3	0.825730	0.82221	0.741853
	0.2	0.4	0.852847	0.855081	0.827730
	0.2	0.5	0.855771	0.859112	0.844456

4.3.2 Graphs of $F_{r,s:n}(x, y\,|u, v)$ as a Function of Dependence Parameter α

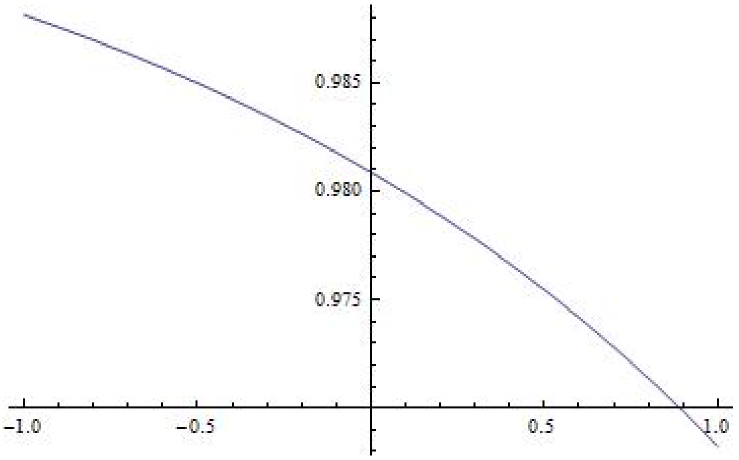


Figure 4.6 Graph of $F^{(1)}_{r,s:n}(x, y\,|u, v)$ as a function of α in Theorem 1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 2,\ x = 0.3,\ y = 0.9.$

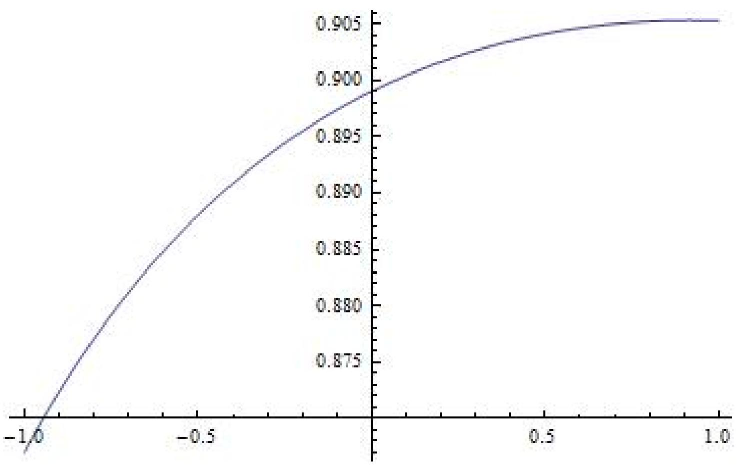


Figure 4.7 Graph of $F^{(2)}_{r,s:n}(x, y\,|u, v)$ as a function of α in Theorem 2a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 3,\ x = 0.4,\ y = 0.3.$

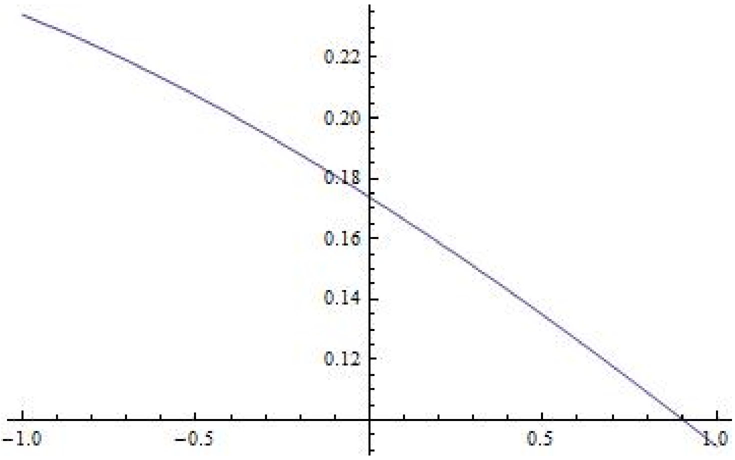


Figure 4.8 Graph of $F^{(3)}_{r,s:n}(x, y\,|u, v)$ as a function of α in Theorem 3a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 2,\ x = 0.3,\ y = 0.9.$

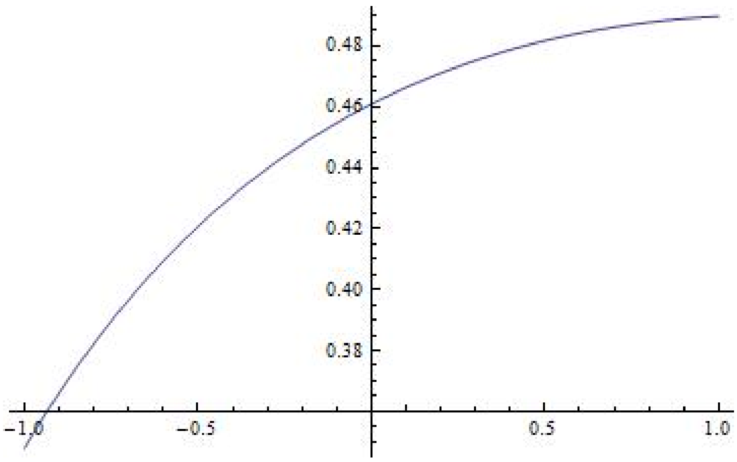


Figure 4.9 Graph of $F^{(4.1)}_{r,s:n}(x, y\,|u, v)$as a function of α in Theorem 4.1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 3,\ x = 0.4,\ y = 0.3.$

4.3.3 Graphical Illustrations for $F_{r,s:n}(x, y\,|u, v)$

i) Theorem 1a. $u \leq x, v \leq y$:

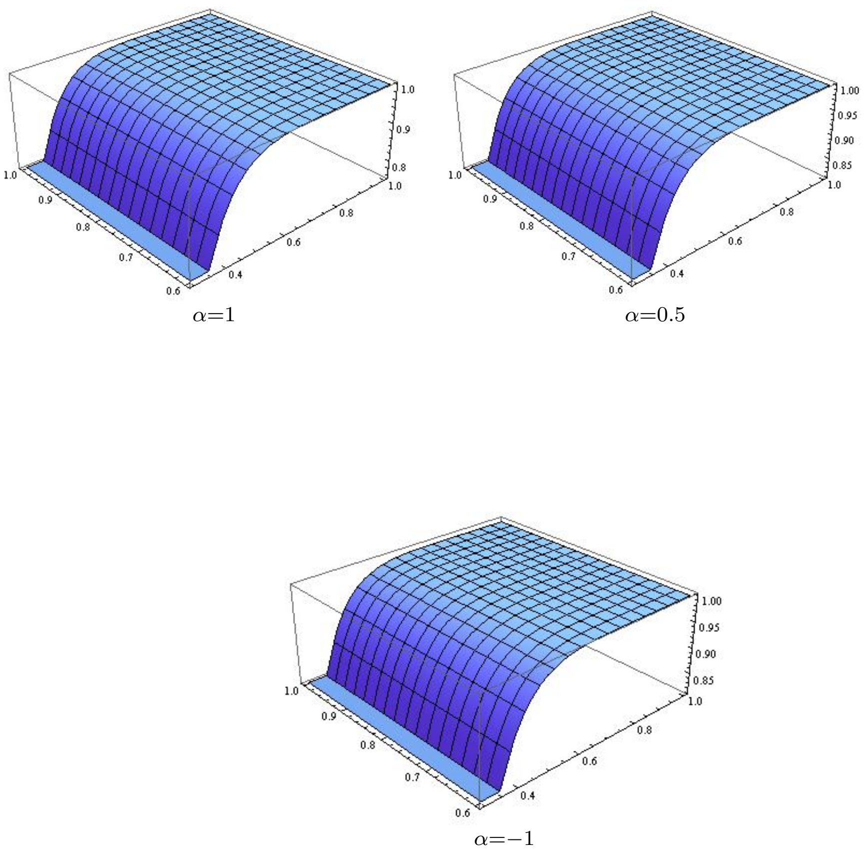
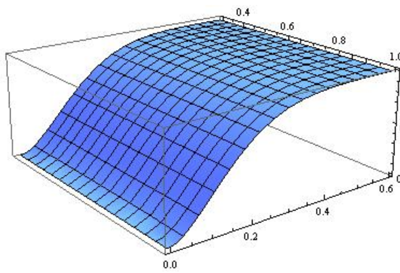


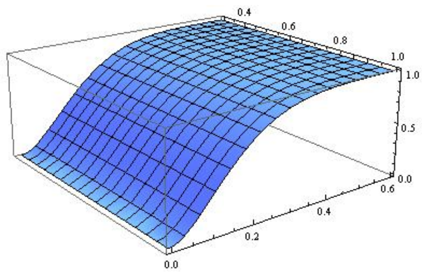
Figure 4.10 Graph of $F^{(1)}_{r,s:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 2.$

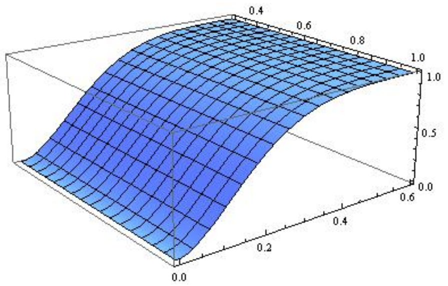
ii) Theorem 2a. $u \leq x, v > y$:



$\alpha=1$



$\alpha=0.5$

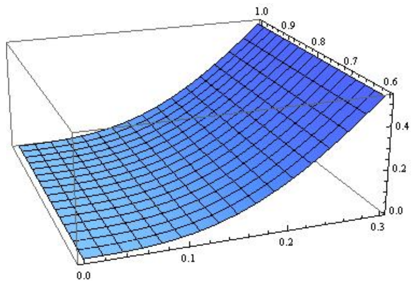


$\alpha=-1$

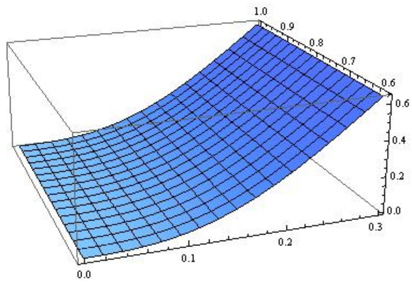
Figure 4.11 Graph of $F^{(2)}_{r,s:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 3.$

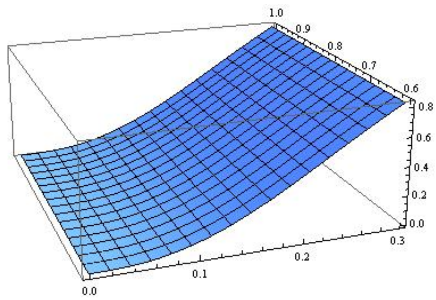
iii) Theorem 3a. $u > x, v \leq y$:



α=1



α=0.5



α=−1

Figure 4.12 Graph of$F^{(3)}_{r,s:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 2.$

iv) Theorem 4.1a. $u > x, v > y$:

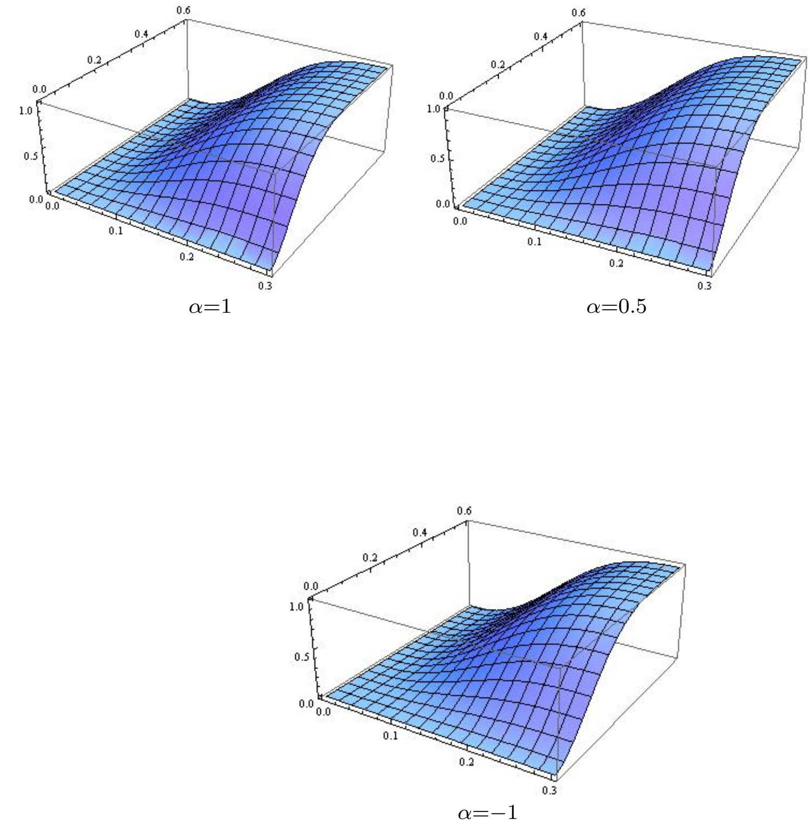
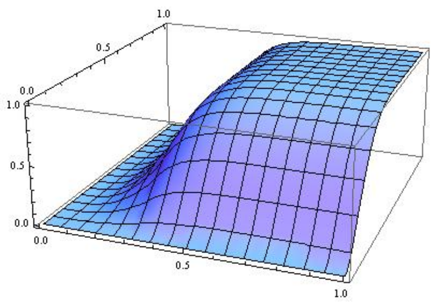


Figure 4.13 Graph of $F_{r,s:n}^{(4.1)}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 5.$

4.3.4 Piecewise Graphs of $F_{r,s:n}(x, y\,|u, v)$

v) Theorem 5.



α=1

Figure 4.14 Graph of $F_{r,s:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 3,\ s = 2,\ h = 2.$

4.3.5 Numerical Results for Conditional 1^{th} and 1^{th} Order Statistics

i) Theorem 1a. $u \leq x, v \leq y$:

Table 4.6 Some numerical results of $F^{(1)}_{1,1:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 1,\ h = 1.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(1)}_{1,1:n}$	$F^{(1)}_{1,1:n}$	$F^{(1)}_{1,1:n}$
	0.3	0.7	0.999999	0.999999	0.999999
	0.3	0.8	0.999999	0.999999	0.999999
	0.3	0.9	0.999999	0.999999	0.999999
$n = 10$	0.5	0.7	0.999999	0.999999	0.999999
$r = 1$	0.5	0.8	0.999999	0.999999	0.999999
$s = 1$	0.5	0.9	0.999999	0.999999	0.999999
$h = 1$	0.7	0.7	0.999999	0.999999	0.999999
	0.7	0.8	0.999999	0.999999	0.999999
	0.7	0.9	0.999999	0.999999	0.999999
	0.9	0.7	0.999999	0.999999	0.999999
	0.9	0.8	0.999999	0.999999	0.999999
	0.9	0.9	0.999999	0.999999	0.999999

ii) Theorem 2a. $u \leq x, v > y$:

Table 4.7 Some numerical results of $F^{(2)}_{1,1:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 1,\ h = 1.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(2)}_{1,1:n}$	$F^{(2)}_{1,1:n}$	$F^{(2)}_{1,1:n}$
	0.4	0.2	0.844397	0.862576	0.892458
	0.4	0.3	0.952994	0.959967	0.971108
	0.4	0.4	0.989264	0.99098	0.993696
$n = 10$	0.4	0.5	0.99851	0.998727	0.999071
$r = 1$	0.6	0.2	0.844397	0.862576	0.892458
$s = 1$	0.6	0.3	0.952994	0.959967	0.971108
$h = 1$	0.6	0.4	0.989264	0.99098	0.993696
	0.6	0.5	0.99851	0.998727	0.999071
	0.8	0.2	0.844397	0.862576	0.892458
	0.8	0.3	0.952994	0.959967	0.971108
	0.8	0.4	0.989264	0.99098	0.993696
	0.8	0.5	0.99851	0.998727	0.999071

iii) Theorem 3a. $u > x, v \leq y$:

Table 4.8 Some numerical results of $F^{(3)}_{1,1:n}(x, y\,|u, v)$

for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 1,\ h = 1.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(3)}_{1,1:n}$	$F^{(3)}_{1,1:n}$	$F^{(3)}_{1,1:n}$
	0.1	0.7	0.480560	0.532353	0.636512
	0.1	0.75	0.480560	0.532353	0.636512
	0.1	0.8	0.480560	0.532353	0.636512
$n = 10$	0.1	0.85	0.480560	0.532353	0.636512
$r = 1$	0.15	0.7	0.664358	0.710839	0.798576
$s = 1$	0.15	0.75	0.664358	0.710839	0.798576
$h = 1$	0.15	0.8	0.664358	0.710839	0.798576
	0.15	0.85	0.664358	0.710839	0.798576
	0.2	0.7	0.810484	0.842851	0.900659
	0.2	0.75	0.810484	0.842851	0.900659
	0.2	0.8	0.810484	0.842851	0.900659
	0.2	0.85	0.810484	0.842851	0.900659

iv) Theorem 4.1a. $u > x, v > y$:

Table 4.9 Some numerical results of $F_{1,1:n}^{(4.1)}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 1,\ h = 5.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F_{1,1:n}^{(4.1)}$	$F_{1,1:n}^{(4.1)}$	$F_{1,1:n}^{(4.1)}$
	0.1	0.2	0.869051	0.862873	0.784964
	0.1	0.3	0.897066	0.89554	0.862091
	0.1	0.4	0.900266	0.899761	0.878633
$n = 10$	0.1	0.5	0.900445	0.900035	0.880408
$r = 1$	0.15	0.2	0.945465	0.939053	0.869758
$s = 1$	0.15	0.3	0.976066	0.974625	0.953394
$h = 5$	0.15	0.4	0.97957	0.979221	0.971155
	0.15	0.5	0.979767	0.979520	0.973044
	0.2	0.2	0.962762	0.956460	0.891586
	0.2	0.3	0.993964	0.992697	0.976679
	0.2	0.4	0.99754	0.997379	0.994693
	0.2	0.5	0.99774	0.997684	0.996603

4.3.6 Graphs of $F_{1,1:n}(x, y\,|u, v)$ as a Function of Dependence Parameter α

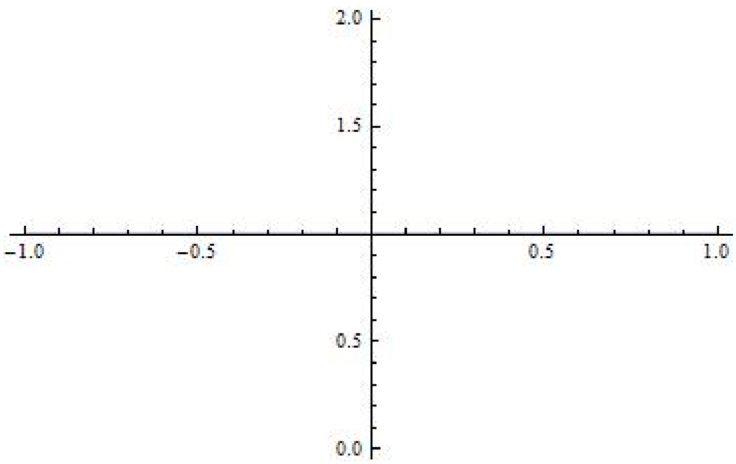


Figure 4.15 Graph of $F^{(1)}_{1,1:n}(x, y\,|u, v)$ as a function of α in Theorem 1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1,\ x = 0.3,\ y = 0.9.$

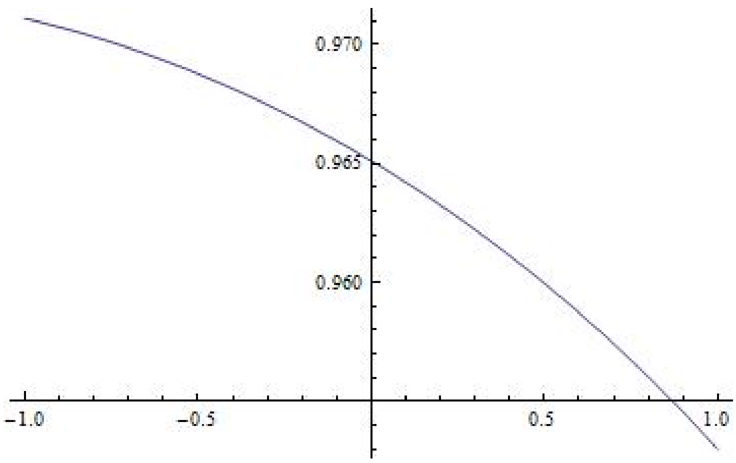


Figure 4.16 Graph of $F^{(2)}_{1,1:n}(x, y\,|u, v)$ as a function of α in Theorem 2a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1,\ x = 0.4,\ y = 0.3.$

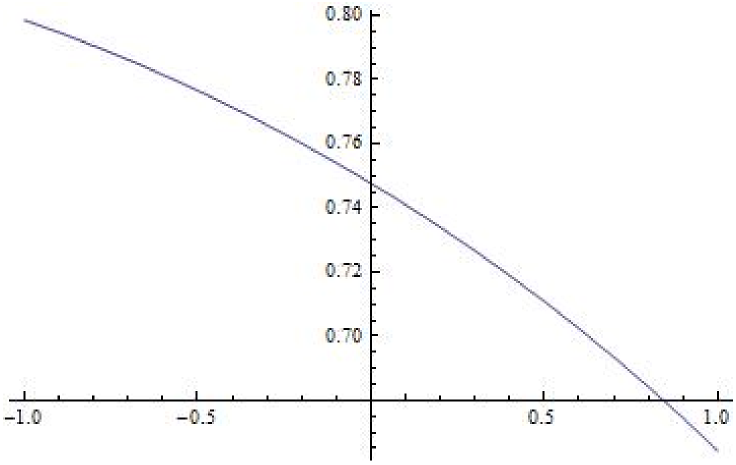


Figure 4.17 Graph of $F^{(3)}_{1,1:n}(x, y\,|u, v)$ as a function of α in Theorem 3a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1,\ x = 0.15,\ y = 0.75.$

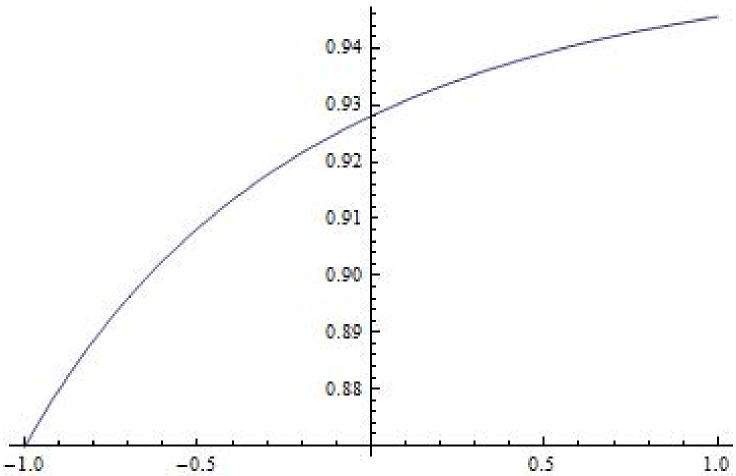


Figure 4.18 Graph of $F^{(4.1)}_{1,1:n}(x, y\,|u, v)$ as a function of α in Theorem 4.1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1,\ x = 0.15,\ y = 0.2.$

4.3.7 Graphical Illustrations for $F_{1,1:n}(x, y\,|u, v)$

i) Theorem 1a. $u \le x, v \le y$:

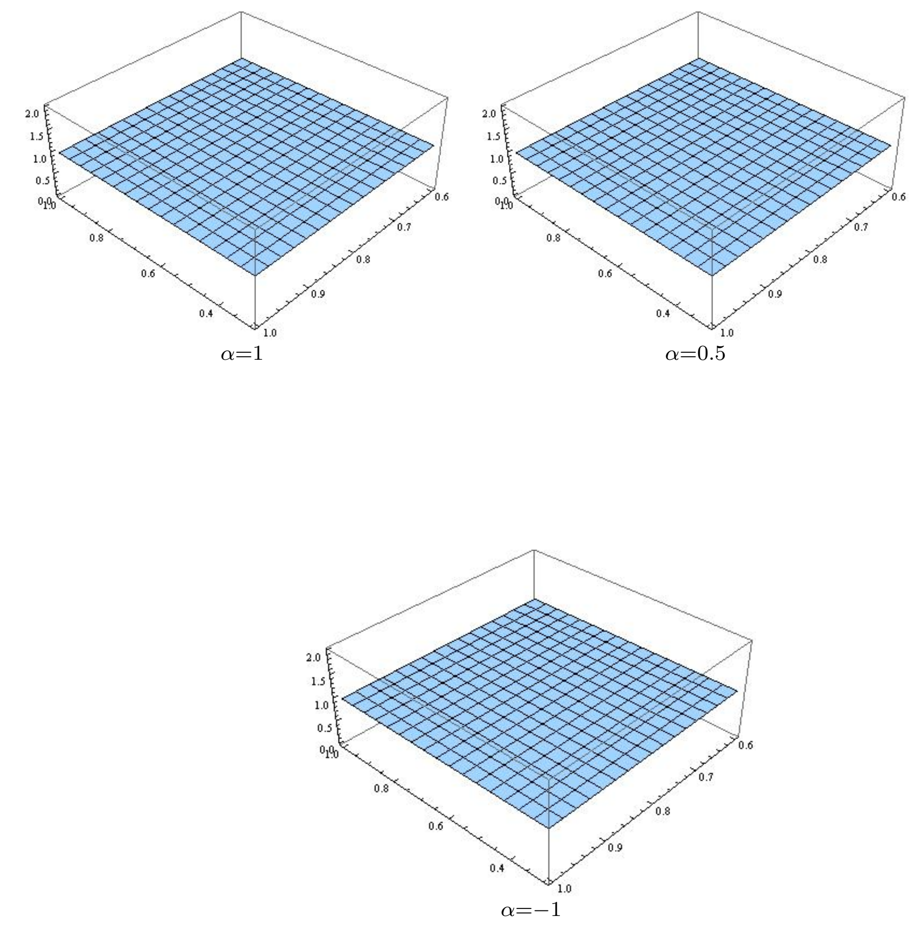


Figure 4.19 Graph of $F^{(1)}_{1,1:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1.$

ii) Theorem 2a. $u \leq x, v > y$:

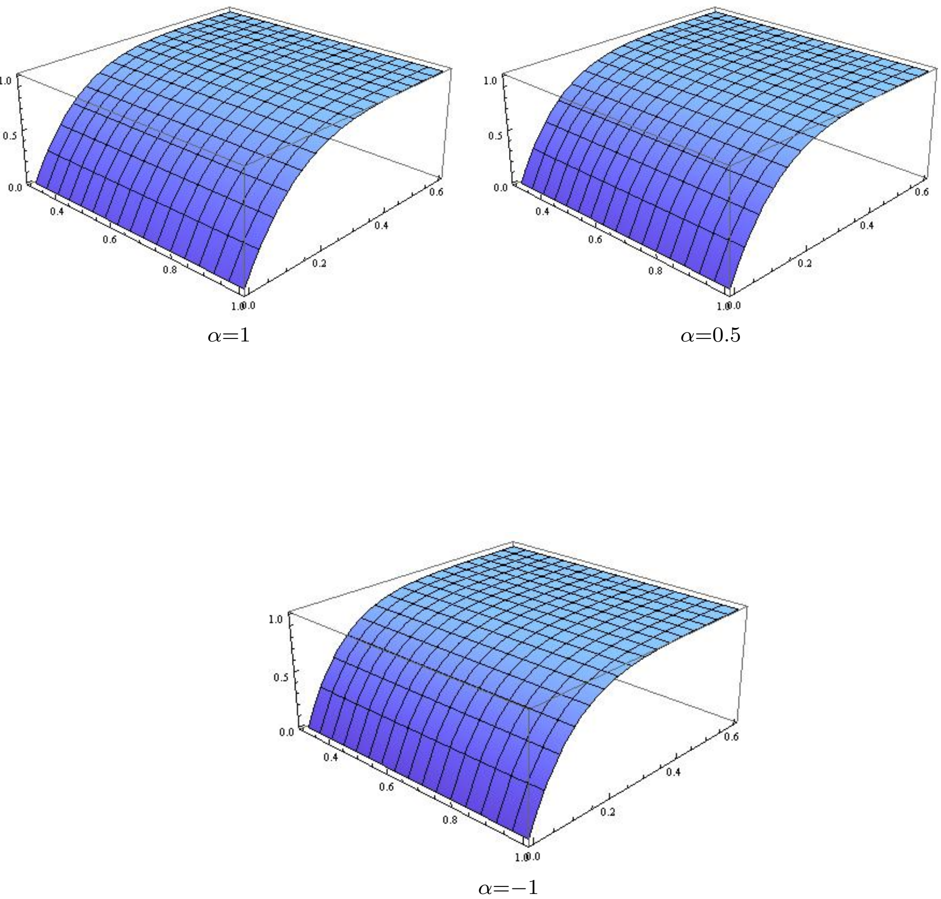
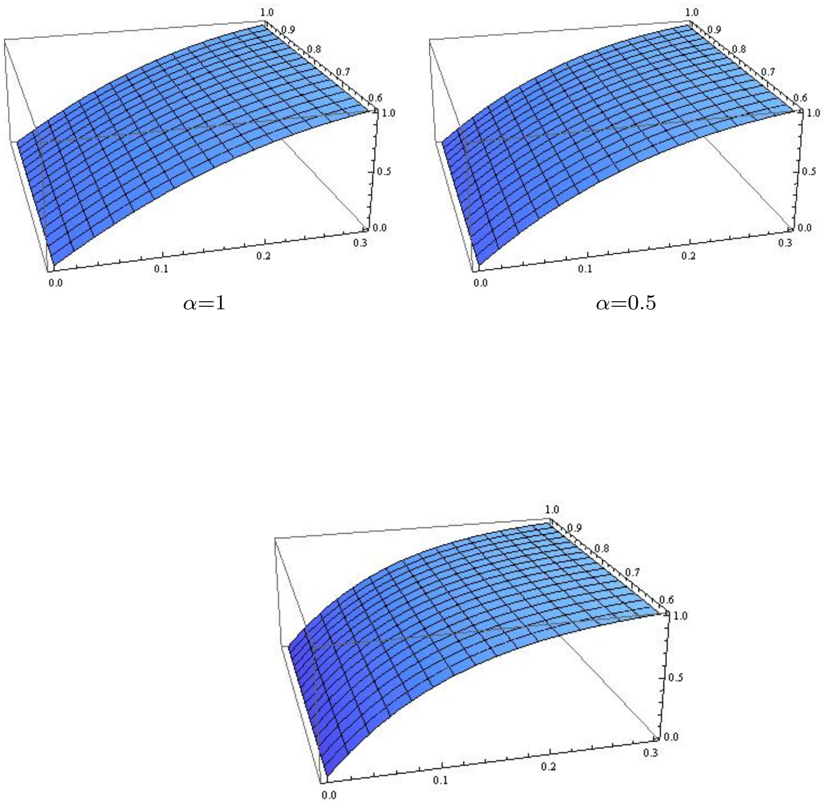


Figure 4.20 Graph of $F^{(2)}_{1,1:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1.$

iii) Theorem 3a. $u > x, v \leq y$:



α=−1

Figure 4.21 Graph of $F^{(3)}_{1,1:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1.$

iv) Theorem 4.1a. $u > x, v > y$:

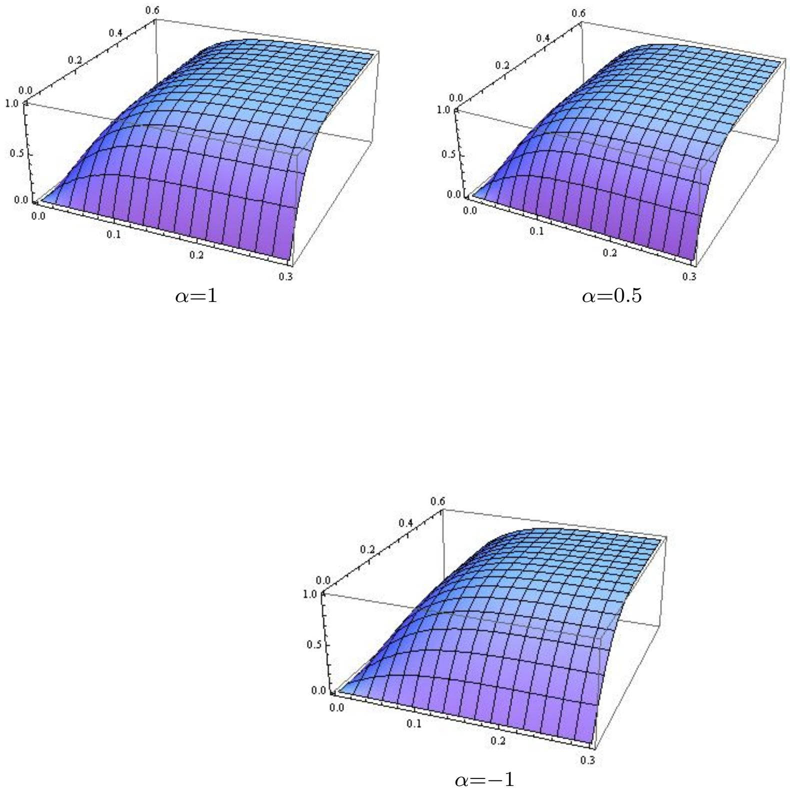
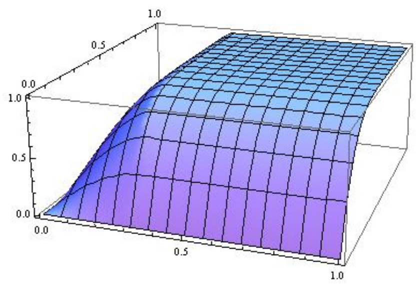


Figure 4.22 Graph of $F_{1,1:n}^{(4.1)}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1.$

4.3.8 Piecewise Graphs of $F_{1,1:n}(x, y\,|u, v)$

v) Theorem 5.



α=1

Figure 4.23 Graph of$F_{1,1:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1.$

4.3.9 Numerical Results for Conditional 1^{th} and n^{th} Order Statistics

i) Theorem 1a. $u \leq x, v \leq y$:

Table 4.10 Some numerical results of $F^{(1)}_{1,n:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 10,\ h = 1.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(1)}_{1,n:n}$	$F^{(1)}_{1,n:n}$	$F^{(1)}_{1,n:n}$
	0.3	0.7	0.011726	0.014046	0.022292
	0.3	0.8	0.066641	0.073624	0.095397
	0.3	0.9	0.285728	0.298138	0.333408
$n = 10$	0.5	0.7	0.011726	0.014046	0.022292
$r = 1$	0.5	0.8	0.066641	0.073624	0.095397
$s = 10$	0.5	0.9	0.285728	0.298138	0.333408
$h = 1$	0.7	0.7	0.011726	0.014046	0.022292
	0.7	0.8	0.066641	0.073624	0.095397
	0.7	0.9	0.285728	0.298138	0.333408
	0.9	0.7	0.011726	0.014046	0.022292
	0.9	0.8	0.066641	0.073624	0.095397
	0.9	0.9	0.285728	0.298138	0.333408

ii) Theorem 2a. $u \leq x, v > y$:

Table 4.11 Some numerical results of $F^{(2)}_{1,n:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 10,\ h = 1.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(2)}_{1,n:n}$	$F^{(2)}_{1,n:n}$	$F^{(2)}_{1,n:n}$
	0.4	0.2	7.5×10^{-9}	1.9×10^{-8}	1.1×10^{-7}
	0.4	0.3	5.8×10^{-7}	1.3×10^{-6}	5.5×10^{-6}
	0.4	0.4	0.000012	0.000025	0.000089
$n = 10$	0.4	0.5	0.000169	0.000265	0.000727
$r = 1$	0.6	0.2	7.5×10^{-9}	1.9×10^{-8}	1.1×10^{-7}
$s = 10$	0.6	0.3	5.8×10^{-7}	1.3×10^{-6}	5.5×10^{-6}
$h = 1$	0.6	0.4	0.000012	0.000025	0.000089
	0.6	0.5	0.000169	0.000265	0.000727
	0.8	0.2	7.5×10^{-9}	1.9×10^{-8}	1.1×10^{-7}
	0.8	0.3	5.8×10^{-7}	1.3×10^{-6}	5.5×10^{-6}
	0.8	0.4	0.000012	0.000025	0.000089
	0.8	0.5	0.000169	0.000265	0.000727

iii) Theorem 3a. $u > x, v \leq y$:

Table 4.12 Some numerical results of $F^{(3)}_{1,n:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 10,\ h = 6.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(3)}_{1,n:n}$	$F^{(3)}_{1,n:n}$	$F^{(3)}_{1,n:n}$
	0.1	0.7	0.129180	0.139156	0.163967
	0.1	0.75	0.193866	0.204892	0.230570
	0.1	0.8	0.280186	0.291513	0.315722
$n = 10$	0.1	0.85	0.392469	0.40302	0.422609
$r = 1$	0.15	0.7	0.137154	0.148409	0.180361
$s = 10$	0.15	0.75	0.205635	0.218211	0.252738
$h = 6$	0.15	0.8	0.297017	0.310136	0.344992
	0.15	0.85	0.415906	0.428424	0.460475
	0.2	0.7	0.138515	0.150062	0.184036
	0.2	0.75	0.207623	0.220557	0.257598
	0.2	0.8	0.299838	0.313380	0.351279
	0.2	0.85	0.419815	0.432812	0.468452

iv) Theorem 4.1a. $u > x, v > y$:

Table 4.13 Some numerical results of $F^{(4.1)}_{1,n:n}(x, y\,|u, v)$

for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 1,\ s = 10,\ h = 8.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(4.1)}_{1,n:n}$	$F^{(4.1)}_{1,n:n}$	$F^{(4.1)}_{1,n:n}$
	0.1	0.2	0.000014	8.9×10^{-6}	1.1×10^{-7}
	0.1	0.3	0.000596	0.000428	0.000019
	0.1	0.4	0.007747	0.006253	0.000911
$n = 10$	0.1	0.5	0.051674	0.047535	0.020394
$r = 1$	0.15	0.2	0.000014	9.2×10^{-6}	1.1×10^{-7}
$s = 10$	0.15	0.3	0.000611	0.000441	0.000021
$h = 8$	0.15	0.4	0.007949	0.006441	0.000985
	0.15	0.5	0.053077	0.049001	0.02174
	0.2	0.2	0.000014	9.2×10^{-6}	1.2×10^{-7}
	0.2	0.3	0.000612	0.000442	0.000021
	0.2	0.4	0.007967	0.006459	0.000996
	0.2	0.5	0.053204	0.049144	0.021932

4.3.10 Graphs of $F_{1,n:n}(x, y\,|u, v)$ as a Function of Dependence Parameter α

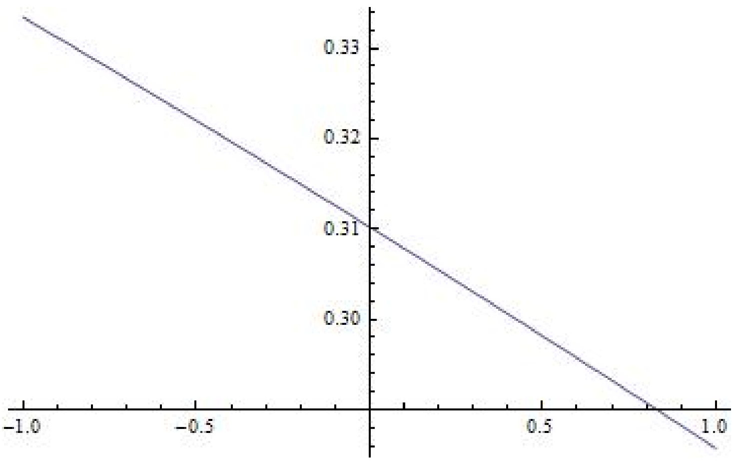


Figure 4.24 Graph of $F^{(1)}_{1,n:n}(x, y\,|u, v)$ as a function of α in Theorem 1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 1,\ x = 0.3,\ y = 0.9.$

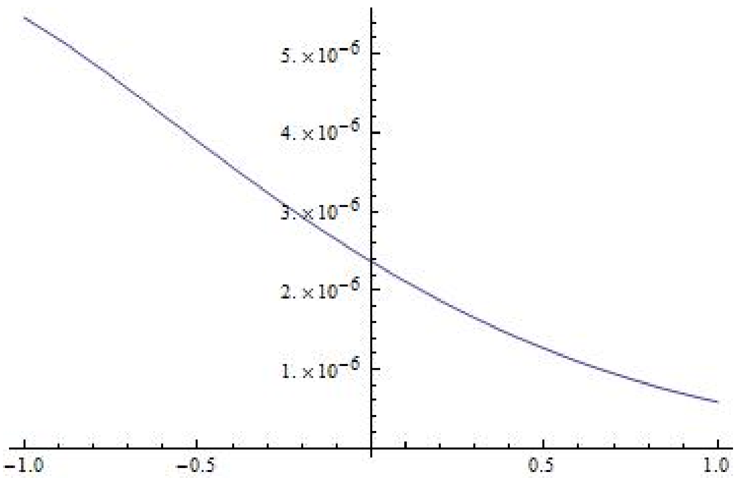


Figure 4.25 Graph of $F^{(2)}_{1,n:n}(x, y\,|u, v)$ as a function of α in Theorem 2a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 1,\ x = 0.4,\ y = 0.3.$

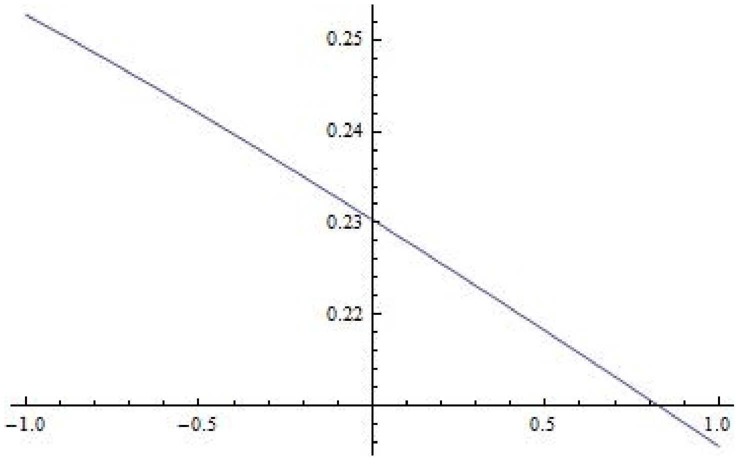


Figure 4.26 Graph of $F^{(3)}_{1,n:n}(x, y\,|u, v)$ as a function of α in Theorem 3a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 6,\ x = 0.15,\ y = 0.75.$

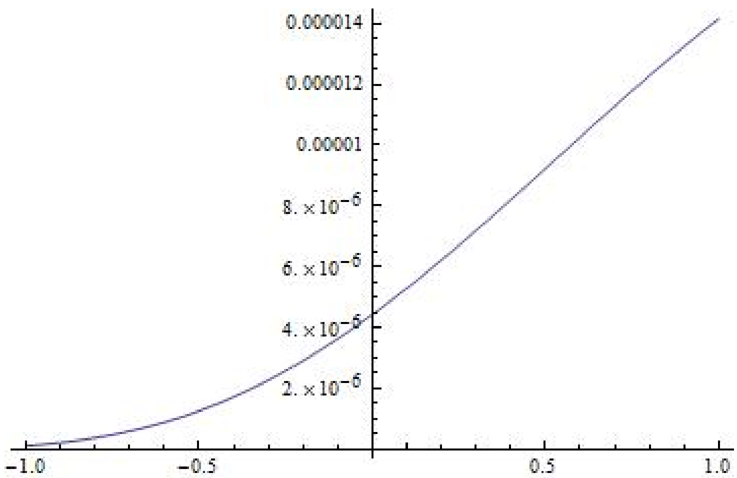


Figure 4.27 Graph of $F^{(4.1)}_{1,n:n}(x, y\,|u, v)$ as a function of α in Theorem 4.1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 8,\ x = 0.15,\ y = 0.2.$

4.3.11 Graphical Illustrations for $F_{1,n:n}(x, y\,|u, v)$

i) Theorem 1a. $u \leq x, v \leq y$:

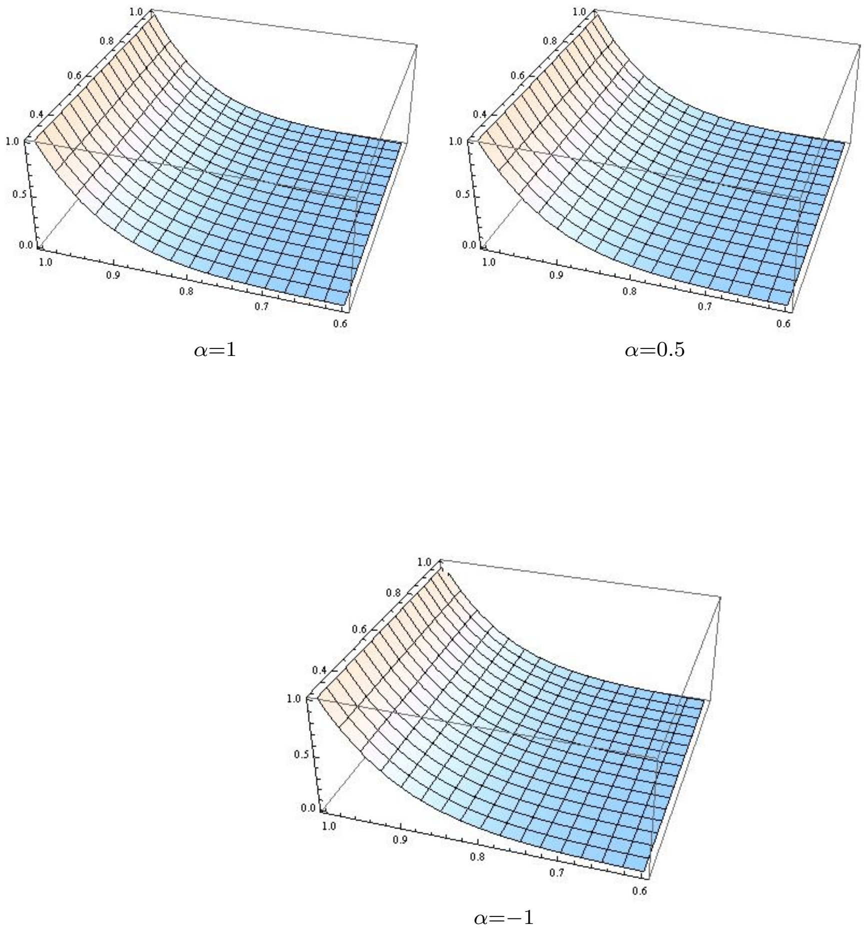


Figure 4.28 Graph of $F^{(1)}_{1,n:n}(x, y\,|u, v)$;
$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 1.$

ii) Theorem 2a. $u \leq x, v > y$:

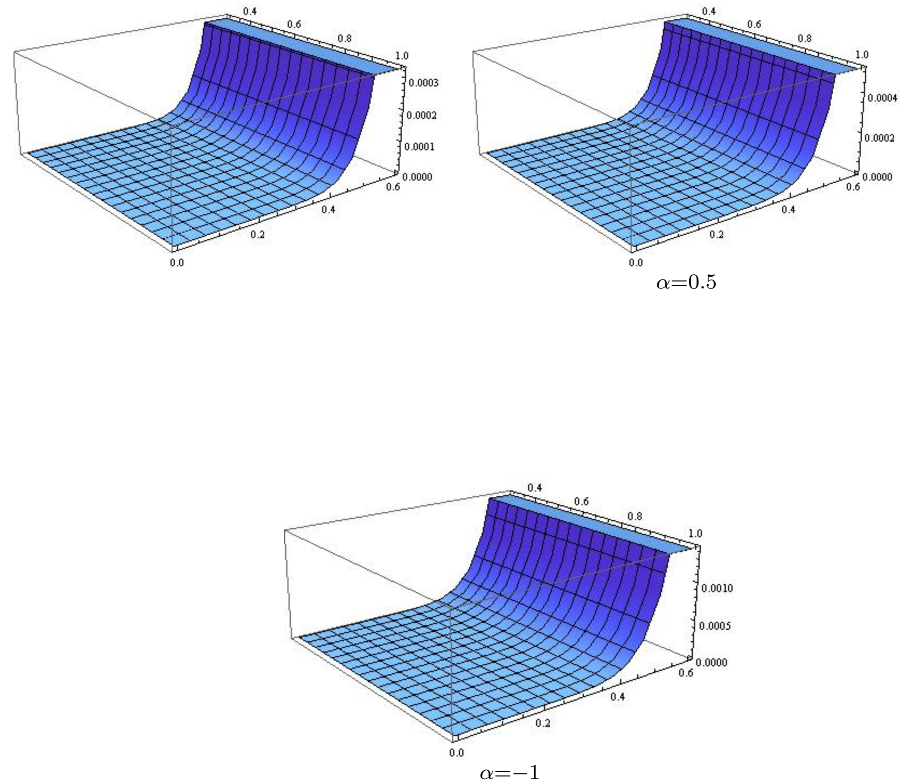


Figure 4.29 Graph of $F^{(2)}_{1,n:n}(x, y\,|u, v)$;
$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 1.$

iii) Theorem 3a. $u > x, v \leq y$:

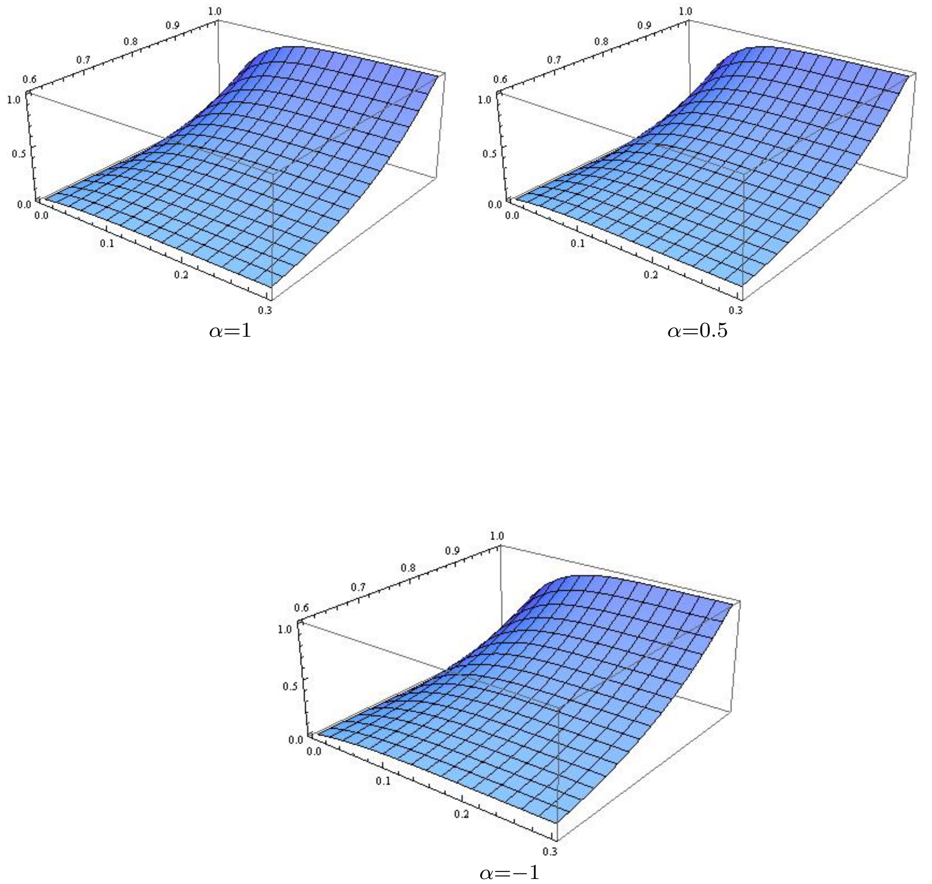
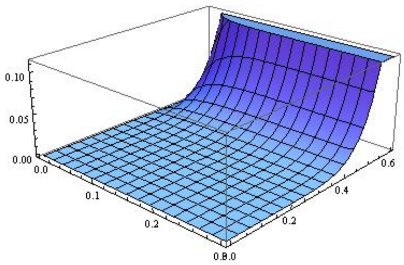


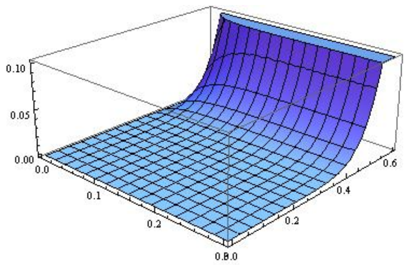
Figure 4.30 Graph of $F^{(3)}_{1,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 6.$

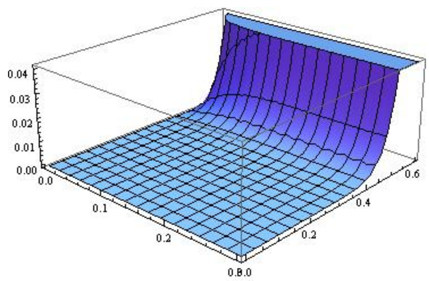
iv) Theorem 4a. $u > x, v > y$:



α=1



α=0.5



α=−1

Figure 4.31 Graph of $F^{(4.1)}_{1,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 8.$

4.3.12 Piecewise Graphs of $F_{1,n:n}(x, y\,|u, v)$

v) Theorem 5.

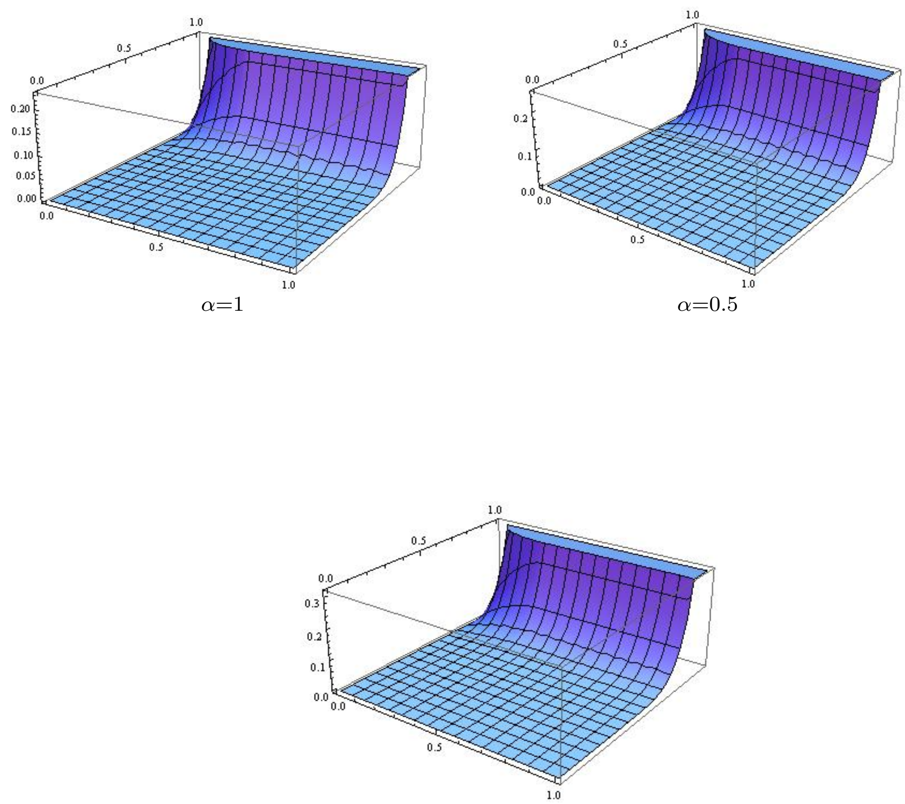


Figure 4.32 Graph of $F_{1,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 10,\ h = 1.$

4.3.13 Numerical Results for Conditional n^{th} and n^{th} Order Statistics

i) Theorem 1a. $u \leq x, v \leq y$:

Table 4.14 Some numerical results of $F^{(1)}_{n,n:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 10,\ s = 10,\ h = 6.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(1)}_{n,n:n}$	$F^{(1)}_{n,n:n}$	$F^{(1)}_{n,n:n}$
	0.3	0.7	8.9×10^{-7}	1.3×10^{-6}	3.1×10^{-6}
	0.3	0.8	9.9×10^{-6}	0.000018	0.000061
	0.3	0.9	0.000033	0.000076	0.000379
$n = 10$	0.5	0.7	0.002501	0.002145	0.001385
$r = 10$	0.5	0.8	0.005502	0.005335	0.004909
$s = 10$	0.5	0.9	0.009793	0.010771	0.013722
$h = 6$	0.7	0.7	0.024251	0.022216	0.017439
	0.7	0.8	0.049605	0.047783	0.043193
	0.7	0.9	0.087442	0.089110	0.093651
	0.9	0.7	0.087442	0.089110	0.093651
	0.9	0.8	0.183934	0.186059	0.191785
	0.9	0.9	0.340065	0.344358	0.355942

ii) Theorem 2a. $u \leq x, v > y$:

Table 4.15 Some numerical results of $F^{(2)}_{n,n:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 10,\ s = 10,\ h = 6.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(2)}_{n,n:n}$	$F^{(2)}_{n,n:n}$	$F^{(2)}_{n,n:n}$
	0.4	0.2	4.8×10^{-9}	1.7×10^{-9}	6.6×10^{-12}
	0.4	0.3	1.9×10^{-7}	8.1×10^{-8}	1.1×10^{-9}
	0.4	0.4	2.3×10^{-7}	1.1×10^{-6}	4.8×10^{-8}
$n = 10$	0.4	0.5	0.000014	8.4×10^{-6}	1.1×10^{-6}
$r = 10$	0.6	0.2	2.3×10^{-7}	1.1×10^{-7}	1.1×10^{-9}
$s = 10$	0.6	0.3	9.5×10^{-6}	5.1×10^{-6}	1.7×10^{-7}
$h = 6$	0.6	0.4	0.000122	0.000073	6.8×10^{-6}
	0.6	0.5	0.000809	0.000562	0.000131
	0.8	0.2	9.2×10^{-7}	5.8×10^{-7}	1.7×10^{-8}
	0.8	0.3	0.000042	0.000029	2.3×10^{-6}
	0.8	0.4	0.000583	0.000442	0.000084
	0.8	0.5	0.004181	0.003552	0.001499

iii) Theorem 3a. $u > x, v \leq y$:

Table 4.16 Some numerical results of $F^{(3)}_{n,n:n}(x, y\,|u, v)$
for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 10,\ s = 10,\ h = 6.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(3)}_{n,n:n}$	$F^{(3)}_{n,n:n}$	$F^{(3)}_{n,n:n}$
	0.1	0.7	1.6×10^{-11}	2.4×10^{-11}	3.1×10^{-11}
	0.1	0.75	6.3×10^{-11}	1.1×10^{-10}	1.8×10^{-10}
	0.1	0.8	1.5×10^{-10}	3.0×10^{-10}	6.4×10^{-10}
$n = 10$	0.1	0.85	2.7×10^{-10}	6.7×10^{-10}	1.8×10^{-9}
$r = 10$	0.15	0.7	9.1×10^{-10}	1.3×10^{-9}	2.0×10^{-9}
$s = 10$	0.15	0.75	3.6×10^{-9}	6.1×10^{-9}	1.2×10^{-8}
$h = 6$	0.15	0.8	9.0×10^{-9}	1.7×10^{-8}	4.2×10^{-8}
	0.15	0.85	1.7×10^{-8}	3.9×10^{-8}	1.2×10^{-7}
	0.2	0.7	1.6×10^{-8}	2.3×10^{-8}	4.1×10^{-8}
	0.2	0.75	6.5×10^{-8}	1.1×10^{-7}	2.4×10^{-7}
	0.2	0.8	1.6×10^{-7}	3.1×10^{-7}	8.4×10^{-7}
	0.2	0.85	3.1×10^{-7}	6.9×10^{-7}	2.3×10^{-6}

iv) Theorem 4.1a. $u > x, v > y$:

Table 4.17 Some numerical results of $F^{(4.1)}_{n,n:n}(x, y\,|u, v)$ for $u = 0.3,\ v = 0.6,\ n = 10,\ r = 10,\ s = 10,\ h = 6.$

			$\alpha = 1$	$\alpha = 0.5$	$\alpha = -1$
	x	y	$F^{(4.1)}_{n,n:n}$	$F^{(4.1)}_{n,n:n}$	$F^{(4.1)}_{n,n:n}$
	0.1	0.2	0.000	0.000	0.000
	0.1	0.3	0.000	0.000	0.000
	0.1	0.4	0.000	0.000	0.000
$n = 10$	0.1	0.5	0.000	0.000	0.000
$r = 10$	0.15	0.2	0.000	0.000	0.000
$s = 10$	0.15	0.3	0.000	0.000	0.000
$h = 6$	0.15	0.4	0.000	0.000	0.000
	0.15	0.5	0.000	0.000	0.000
	0.2	0.2	0.000	0.000	0.000
	0.2	0.3	0.000	0.000	0.000
	0.2	0.4	0.000	0.000	0.000
	0.2	0.5	0.000	0.000	0.000

4.3.14 Graphs of $F_{n,n:n}(x, y\,|u, v)$ as a Function of Dependence Parameter α

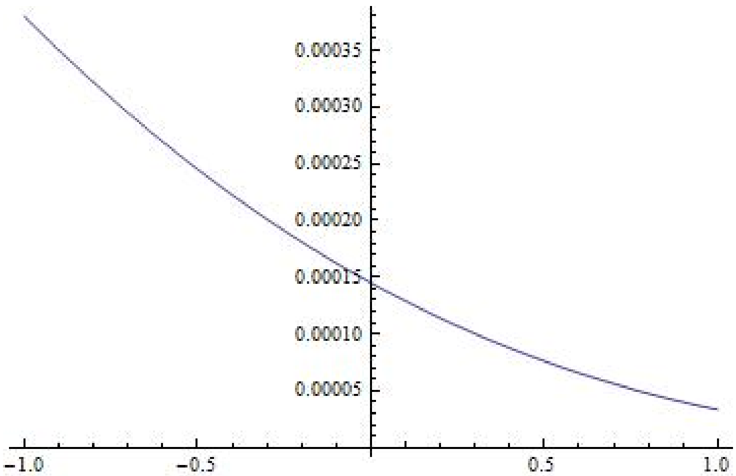


Figure 4.33 Graph of $F^{(1)}_{n,n:n}(x, y\,|u, v)$ as a function of α in Theorem 1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6,\ x = 0.3,\ y = 0.9.$

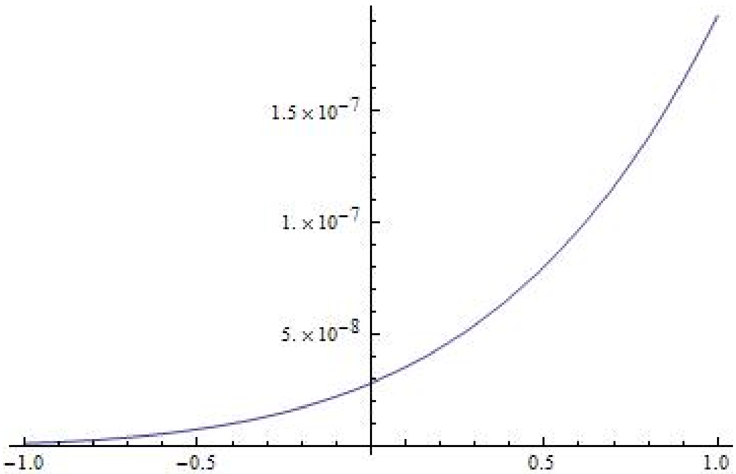


Figure 4.34 Graph of $F^{(2)}_{n,n:n}(x, y\,|u, v)$ as a function of α in Theorem 2a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 1,\ s = 1,\ h = 1,\ x = 0.4,\ y = 0.3.$

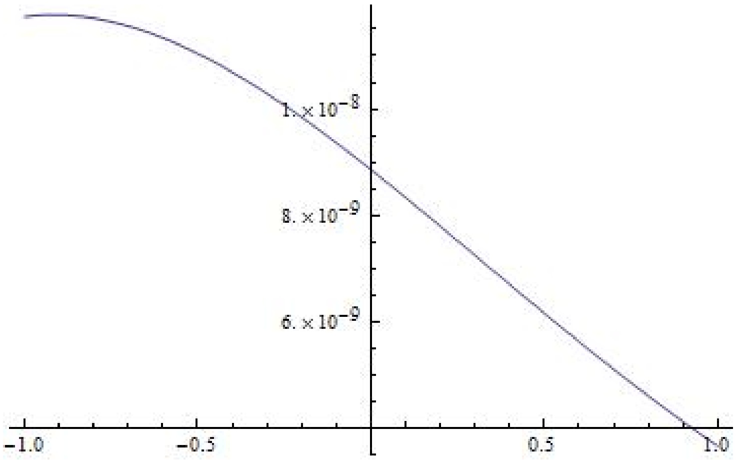


Figure 4.35 Graph of $F^{(3)}_{n,n:n}(x, y\,|u, v)$ as a function of α in Theorem 3a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6,\ x = 0.15,\ y = 0.75.$

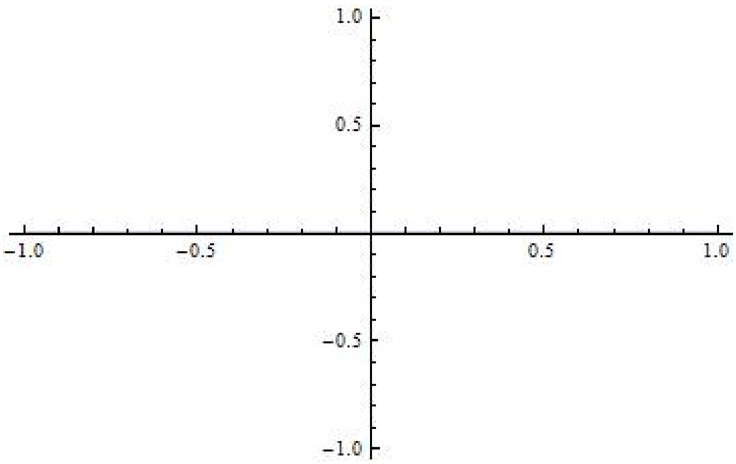
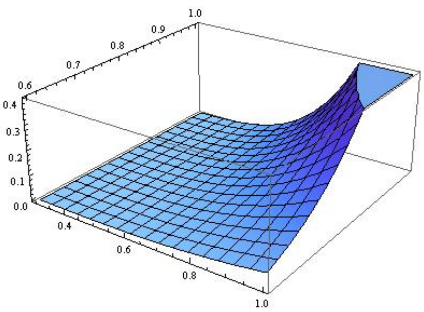


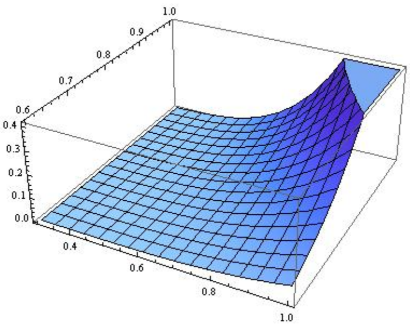
Figure 4.36 Graph of $F^{(4.1)}_{n,n:n}(x, y\,|u, v)$ as a function of α in Theorem 4.1a; $n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6,\ x = 0.15,\ y = 0.2.$

4.3.15 Graphical Illustrations for $F_{n,n:n}(x, y\,|u, v)$

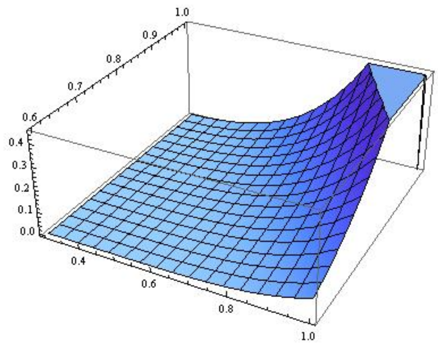
i) Theorem 1a. $u \le x, v \le y$:



α=1



α=0.5



α=−1

Figure 4.37 Graph of $F^{(1)}_{n,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6.$

ii) Theorem 2a. $u \leq x, v > y$:

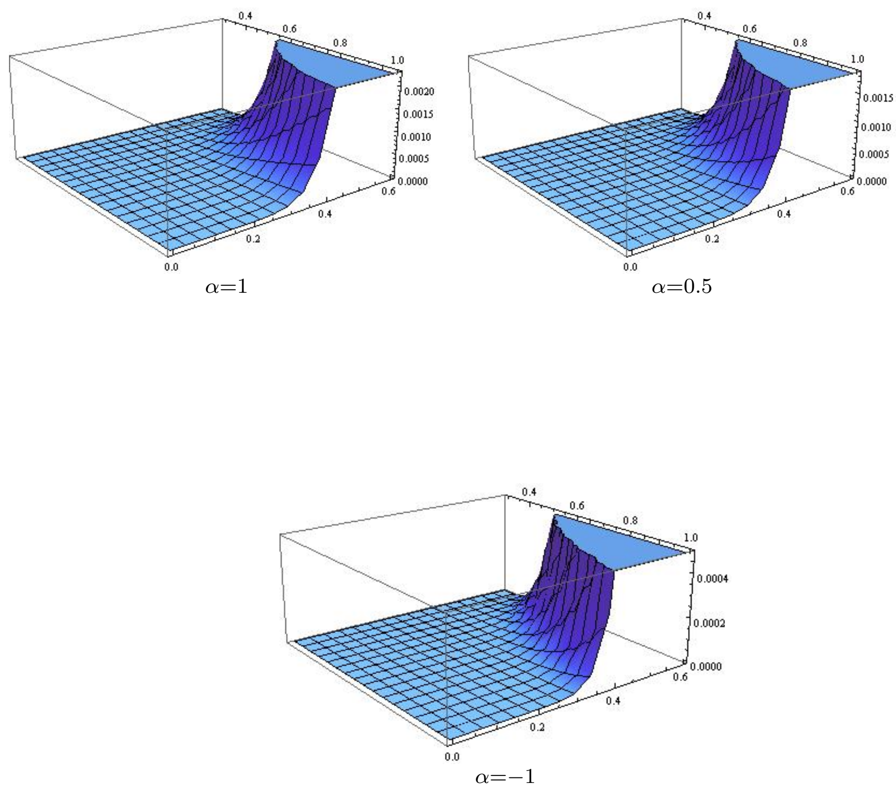


Figure 4.38 Graph of $F^{(2)}_{n,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6.$

iii) Theorem 3a. $u > x, v \leq y$:

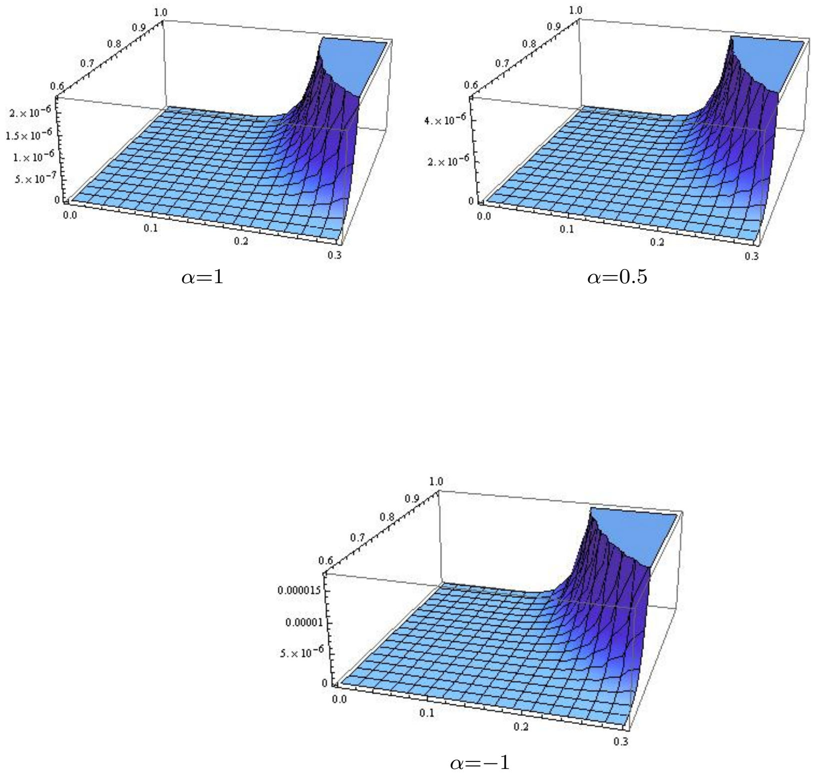


Figure 4.39 Graph of $F^{(3)}_{n,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6.$

iv) Theorem 4a. $u > x, v > y$:

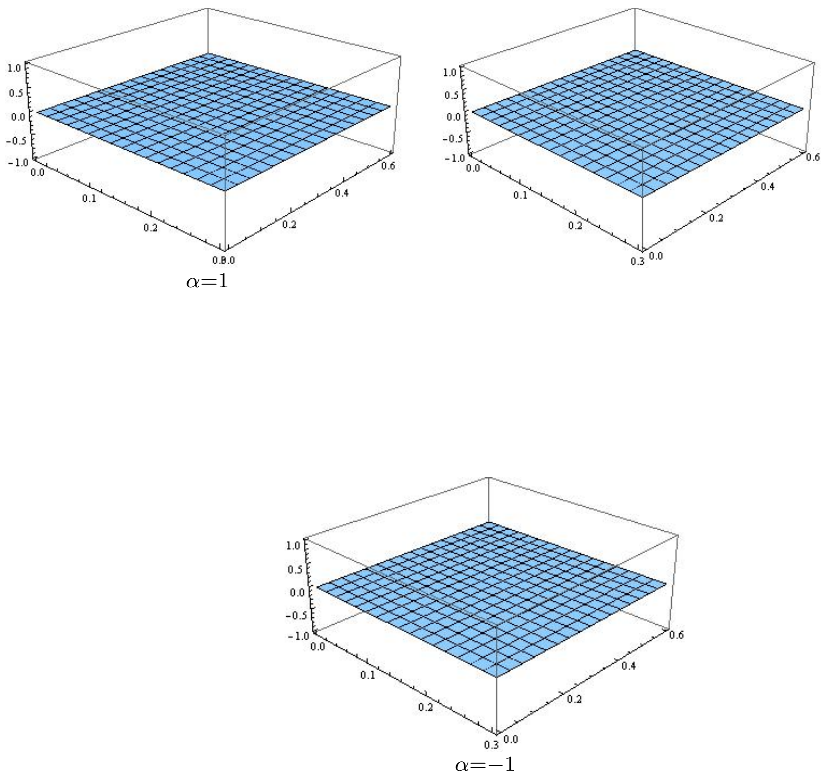
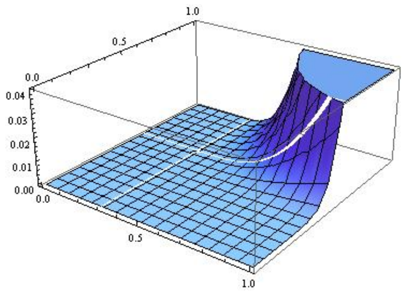


Figure 4.40 Graph of $F^{(4.1)}_{n,n:n}(x, y\,|u, v)$;

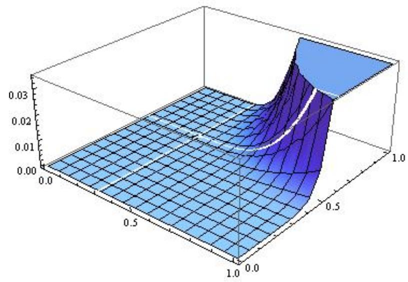
$n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6.$

4.3.16 Piecewise Graphs of $F_{n,n:n}(x, y\,|u, v)$

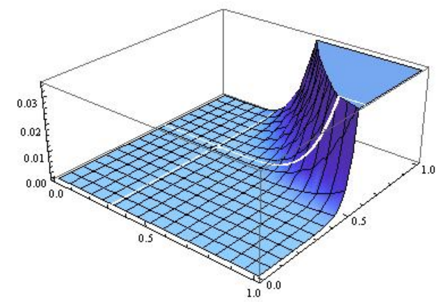
v) Theorem 5.



α=1



α=0.5



α=−1

Figure 4.41 Graph of $F_{n,n:n}(x, y\,|u, v)$;

$n = 10,\ u = 0.3,\ v = 0.6,\ r = 10,\ s = 10,\ h = 6.$

Chapter 5

Conclusions

In this thesis, we introduce new modifications of bivariate binomial distribution. These modifications are trivariate discrete distributions that are important probability models and can be used in distribution theory of conditional bivariate order statistics. Conditional bivariate order statistics are constructed from the bivariate random sample under condition that a certain number of observations fall in the given threshold set. The new distributions obtained in this work present theoretical interest in probability theory, statistics and can be used in many fields of applications of probability and statistics. The probability generating functions of these distributions are also derived. Furthermore, the marginal distributions of bivariate conditional order statistics for some special cases are obtained. The dependence structure of conditional bivariate order statistics is studied by using Pearson's correlation coefficient as a measure of linear dependence. Some numerical results concerning the distribution function of conditional bivariate order statistics calculated by using Wolfram Mathematica is also presented. Graphical representations for distribution functions of conditional bivariate order statistics is also provided. The results presented in this study can also be applied widely for reliability analysis of complex systems and studying the dependence among financial markets in crises and other extreme situations.

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