CAPACITATED DYNAMIC ECONOMIC LOT-SIZING PROBLEM WITH PERISHABLE ITEMS

GÜL IŞIK

FEBRUARY 2014

CAPACITATED DYNAMIC ECONOMIC LOT-SIZING PROBLEM WITH PERISHABLE ITEMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
IZMIR UNIVERSITY OF ECONOMICS

BY

GÜL IŞIK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

FEBRUARY 2014

Approval of the Graduate School of Natural and Applied Sciences

(Prof. Dr. Cüneyt GÜZELİŞ)
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

(Assoc. Prof. Dr. Arslan ÖRNEK)
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

(Asst. Prof. Dr. Zeynep SARGUT)
Supervisor

Examining Committee Members
(Title and Name in alphabetical
order of last name)

Asst. Prof. Dr. Muhittin Hakan DEMİR
Assoc. Prof. Dr. Deniz Türsel ELİİYİ
Asst. Prof. Dr. Zeynep SARGUT

ABSTRACT

CAPACITATED DYNAMIC ECONOMIC LOT-SIZING PROBLEM WITH PERISHABLE ITEMS

Işık, Gül

M.Sc. in Intelligent Engineering Systems
Graduate School of Natural and Applied Sciences

Supervisor: Asst. Prof. Dr. Zeynep Sargut

February 2014, 98 pages

Competition in manufacturing industry forces companies to produce products with a better way while trying to minimize costs or maximize expected benefits in production. Therefore, there are many available researches in this area to decide the answers of some questions such as how much or when they should produce to achieve their aims.

In this study, dynamic economic lot sizing models for perishable products are considered when demand is deterministic and variable. Capacitated and uncapacitated dynamic lot sizing models are investigated additionally. A mathematical model is developed to minimize total costs which include production, holding and backlogging costs. Stock deterioration rate in the model depends on the age of products and also production costs, inventory holding costs, and backlogging costs are assumed as general concave functions. Moreover, holding costs and backlogging costs are age-dependent. The structural properties of the optimal solutions are analyzed and these are used for developing an algorithm which gives an approximate solution for this kind of problems. The solutions of the algorithm are compared with the solutions of GAMS solver. Then, the performance of the algorithm is discussed.

Keywords: Perishable Inventory, Capacitated Dynamic Lot Sizing Problem, Dynamic Programming

ÖZ

DAYANIKSIZ ÜRÜNLER İÇİN KAPASİTELİ DİNAMİK EKONOMİK SİPARİŞ VERME MODELİ

Işık, Gül

Akıllı Mühendislik Sistemleri Yüksek Lisans Programı
Fen Bilimleri Enstitüsü

Tez Danışmanı: Yard. Doç. Dr. Zeynep Sargut

Şubat 2014, 98 sayfa

Üretim endüstrisindeki rekabet, şirketleri maliyetleri azaltmaya yada beklenen kazanımları attırmaya çalışırken daha iyi bir yolla üretim yapmaya zorlar. Bu nedenle, düşünülen hedefe ulaşmak, nezaman ve nekadar üretim yapılmalı sorularının cevabına karar vermek için bu alanda bir çok mevcut çalışmalar ve araştırmalar bulunmaktadır.

Bu çalışmada, çabuk bozulan yani dayanıksız ürünler için talep bilindiğinde fakat talep değişken olduğunda dinamik ekonomik sipariş verme modeli üzerinde durulmuştur. Buna ek olarak, kapasiteli ve kapasitesiz üretimlerde dinamik ekonomik sipariş verme modeli incelenmiştir. Üretim yapmaktan, envanter tutmaktan ve biriken siparişlerden dolayı oluşan maliyetleri azaltmak amacıyla matematiksel bir model oluşturulmuştur. Bu modelde, stok bozulma oranı ürünün yaşına bağlı olup, üretim maliyetleri, envanter tutma maliyetleri ve biriken siparişlerin maliyetleri konkav olarak kabul edilmiştir. Ayrıca, envanter tutma ve biriken sipariş maliyetleri de ürünün yaşına bağlıdır. İdeal (optimum) çözümlerin yapısal özellikleri incelenmiş, yeni geliştirilen ve yaklaşık sonuç veren algoritmada kullanılmıştır. Algotimanın sonuçları, GAMS çözücüsünün verdiği sonuçlar ile karşılaştırılmıştır. Bu şekilde algoritmanın performansı incelenmiştir.

Anahtar kelimeler: Çabuk Bozulan Envanter, Kapsiteli Ekonomik Sipariş Verme Problemi, Dinamik Programlama.

ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to my advisor Asst. Prof. Dr. Zeynep Sargut for her guidance, contribution, academic and personal support, and patience during my studies in the Izmir University of Economics for master thesis.

I would like to express my gratitude to Asst. Prof. Dr. Mahmut Ali Gökçe for his support and contribution in my MSc years in the Izmir University of Economics. Additionally, I would like to appreciate to him for his advice and comments while enrolling in Intelligent Production Systems Master Program.

I also owe my success to Assoc. Prof. Dr. Deniz Türsel Eliiiyi, Asst. Prof. Dr. Erdinç Öner and Asst. Prof. Dr. Selin Özpeynirci in my MSc years. I am very grateful for their contributions and suggestions.

My family, Muzaffer Işık and Aygül Işık, have provided all their support during these years. I would like to thank and express my gratitude to them for their endless love, being with me and trusting me in all my life. I cannot achieve success without them.

Lastly, I am very grateful to Baran Özel, Ece Kanman, Can Kemaloğlu and Günsu Askan for their morale support and encouragement. Moreover, I would like to thank to Burcu Çelik, Burak Gökgür and Oktay Karabağ for their friendships and also their supports whenever I needed.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

After the industrial revolution, manufacturing has become a complex process in the competitive world. Graves (2002) states that volume of manufacturing, variety and quality of products, and also manufacturing operations have changed correspondingly.

According to Akartunalı (2007) in the growing economy, cost reduction on complex manufacturing operations has become very important and production planning provides important effects in a positive way for manufacturers and companies in this market. The growth has attracted attention, thus many scientific researches, different models and solution methods have been developed and applied for production planning.

While production planning plays an important role for reducing cost or maximizing profit, inventory management also gains importance. Yano and Lee (1995) claims that inventory management should be considered with production planning, since it sets the inventory activities, operating policy and procedures which are designed to maximize expected gain and customer satisfaction with using the least inventory investment. Therefore, many inventory models are generated and solution approaches are developed to control and to improve inventory activities efficiently. Inventory models can be categorized in many ways. In an inventory model, demands can be certain or uncertain. If demand is certain per period, it means that demand is deterministic but it can be variable or constant. When demand is variable per period, production or procurement quantities vary per period. Uncertain demand models are called stochastic demand models. Other important property is about capacity. According to Yano and Lee (1995) for some models there exists production capacity while some models do not include the capacity so in capacitated production the production amount is limited because of some conditions such as working hours or the number of workers. Moreover, in inventory models the aim which can be considered as performance measure is generally minimization of total cost or maximization of total profit. Some models consider a single product whereas others consider multiple products and also planning horizon can be finite or infinite. Furthermore, unsatisfied demand can be allowed by backlogging for some of them however, for some companies backlogging is not allowed according to their policies. Last important property is product types which are specified as perishable or non-perishable products.

Manufacturing facilities can face some problems while planning their production. Unexpected situations like broken machines or capacity problems such as number of workers or number of machines and also problems with raw material suppliers can cause some delays in production. Furthermore, this may lead to delay in satisfying demands on time so some companies rare to ask customers for backlogging not to lose their customers completely although it causes dissatisfaction of customers. Therefore, backlogging is an important activity in production planning. Backordering and backlogging are the terms that have same meaning in the literature according to Wu et al. (2011).

Prastacos (1981) defines non-perishable products as products that do not decay for long time. Furniture, electronic devices, clothes can be considered as non-perishable products. However, perishable products are kinds of products which have a limited shelf life or limited time for use or consumption. The time period when the product keeps its quality well is called as life time or shelf life for that product. If life time of a product expires, that product cannot be used after its life time. Dairy products, some foods, and medical drugs can be examples for perishable products. Additionally, while foods such as meat, milk or egg have a short life time, medical drugs or other chemicals have longer life time for use but they can also be considered as perishable products.

Age of a product is defined according to the length of period between its production period and current period. Hsu and Lowe (2001) states that inventory costs may not be related with the age of products which is called as age-dependent while considering non-perishable products. Since the quality and the quantity of the products do not change over time. However, this cannot be considered for the perishable products. When inventory is perishable, quantity and the quality of products change per period according to perishability rate of products and they deteriorate based on periods or ages. Thus, period-pair-dependent or age-dependent inventory cost should be determined for perishable products.

Age-dependent inventory is related with the age of products according to Hsu (2003). Deterioration rate is the percentage of the items perished at the end of the given period. Deterioration rate also depends on the difference between production period and period that the product is used. When the length of time that the product kept in the inventory increases, the deterioration rate increases. When it passes its shelf life, deterioration rate becomes 1. Period-pair dependent inventory depends both production period and age of the products. Deterioration rate may not be only related with the age of goods. For example, some products like meat or milk can deteriorate faster in summer then in winter. Therefore, the deterioration rate may depend on both production period and the age of stock.

Backordering cost is related to the degree of the customer dissatisfaction and the loss of goodwill. Backlogging a product for a long time causes customer dissatisfaction and the length of time increases, the customer dissatisfaction increases more. Therefore, backlogging cost which is penalty cost should not be a linear fashion and it should also be period-pair dependent and it should increase more than increment of delay time.

In this thesis, a production facility that produces a single perishable product with known demand which is dynamic over a finite planning horizon is considered. However, demand is variable (dynamic) and cost functions also change over time. They are not fixed for every interval so dynamic lot sizing is scope of this project. For every interval the production is limited and it cannot pass the production limit. This can be explained as the production capacity and it restricts the production. In other words, it puts a capacity constraint to the problem.

Our problem is to determine the amount of production at each period to meet all demand at the end of the planning horizon so that the total cost of production, inventory and backlogging is minimized. As the result of production amounts the backlogging and inventory values are determined. Backlogging can be needed because of the limited production capacity and deterioration of products kept at inventory and also cost minimization requires some backlogging. Moreover, in this study we assume that the production cost, inventory cost and backlogging cost functions are general non-decreasing concave functions. A dynamic programming based approach is devised to solve this problem.

Chapter 2 focuses on dynamic economic lot sizing problem and the studies of this problem in the literature. Moreover, versions of dynamic economic lot sizing problem are mentioned.

In Chapter 3, a new mathematical model for the uncapacitated problem is introduced when the products are perishable and also the backlogging and inventory costs are general concave. A former formulation exists in Hsu (2003). We updated it so that the model has a network flow representation. Also, the related literature is explained and compared with our study. The network of our problem is drawn and explained. Experimental results are discussed. Moreover, some theorems and properties related with our study are introduced. Finally, the dynamic programming algorithm which can be applicable for our study is mentioned.

The studies for capacitated production problem are in Chapter 4. The mathematical model is updated by adding the capacities. The empirical analysis is performed to understand the structure of the optimal solution. We prove many properties of the optimal solution, one of which is that dynamic programming algorithm to solve our problem and explain our dynamic programming algorithm. We devise an algorithm that finds a good solution for a sub-problem when the production decisions are known. Our algorithm finds the optimal solution or near optimal solutions, therefore it is an approximation algorithm. We run many tests for our algorithm and include many examples to explain how it works. We also tested the performance of the algorithm. We tried six types of demand patterns and explore its effect on performance of the algorithm.

Finally, the study is concluded in Chapter 5. We discuss the experimental results in this section.

CHAPTER 2

DYNAMIC ECONOMIC LOT-SIZING PROBLEM

According to Sargut (2006), the objective of Dynamic Economic Lot Sizing Problem (DLSP) is to minimize the total cost which includes ordering (production) and inventory costs during a finite planning horizon. Firstly, Wagner and Whitin (1958) has studied in DLSP and developed the one of the simplest and earliest dynamic lot-sizing model in the literature. This model includes the production costs which have fixed-cost structure and inventory holding costs which are linear, and also production has been considered as uncapacitated. Then, many following variations of DLSP have been performed. First variations are about the convex and concave cost structures. Other variations of DLSP are about the form of inventory held at the facility that may be considered as combinations of holding and backlogging. Moreover, the examples of models where the total cost is minimized can be seen in the literature. Additionally, some models where the profit is maximized have been developed. These models are considered as the models with one location, but they can be modified in order to optimize the decisions of a multi-level system. In multi-level system, costumer does not directly get the products from manufacturer since there are some levels (wholesaler, retailer, etc.) between them. In each level inventory is held. Therefore, decisions of multi-level system are more complicated and the transportation decisions should be included between the levels.

Wagner and Whitin (1958) has developed their models without the production capacity constraint while other many researchers have developed the capacitated versions of DLSP. In this case, capacities can be stationary or non-stationary (time-varying). In addition, Bitran and Yanasse (1982) states that DLSP with non-stationary capacities is very complex and NP-hard. Veinott (1963) has studied in general concave production and holding costs in DLSP. Furthermore, Zangwill (1969) and Zangwill (1968) have generated a network approach and showed the problem as a concave cost network flow problem and also Zangwill (1969) has extended the work of Veinott (1963) by adding backlogging in the model. This study has illustrated that optimal solution is an extreme point solution with an arborescent structure (as tree) which means that each node has only one positive incoming arc in the network.

Generally, finding minimum cost is the main interest of companies and the researchers. In the literature, there can be seen many instances and studies which consider the flow of products from the manufacturer to consumers as network flows. Zangwill (1968) performed an analysis for determining minimum cost for certain types of concave cost networks. Since, considering costs as a linear function is often unrealistic due to set-up costs or quantity discounts so concave cost functions are more realistic. Some theorems that characterize the extreme points for certain single product or multi-product networks have been developed according to single source to single destination, acyclic single source to multiple destinations and multiple sources to single destination.

Concavity can be explained that every line which is drawn by joining any two points selected on a function should be on or below the function. The formal definition can be derived by a function *f* of a single variable.

f: $X \rightarrow R$ is concave if for any $x, y \in X$, for all $\lambda \in (0, 1)$,

$$f(\lambda y + (1 - \lambda) x) \geq \lambda f(y) + (1 - \lambda) f(x).$$

f: X → R is strictly concave if for any x, y ∈ X with x ≠ y, for all λ ∈ (0, 1),

$$f(\lambda y + (1 - \lambda)x) > \lambda f(y) + (1 - \lambda) f(x).$$

A function f is concave if the line between any two points on the function always lies on or below the function itself which is illustrated in Figure 1.

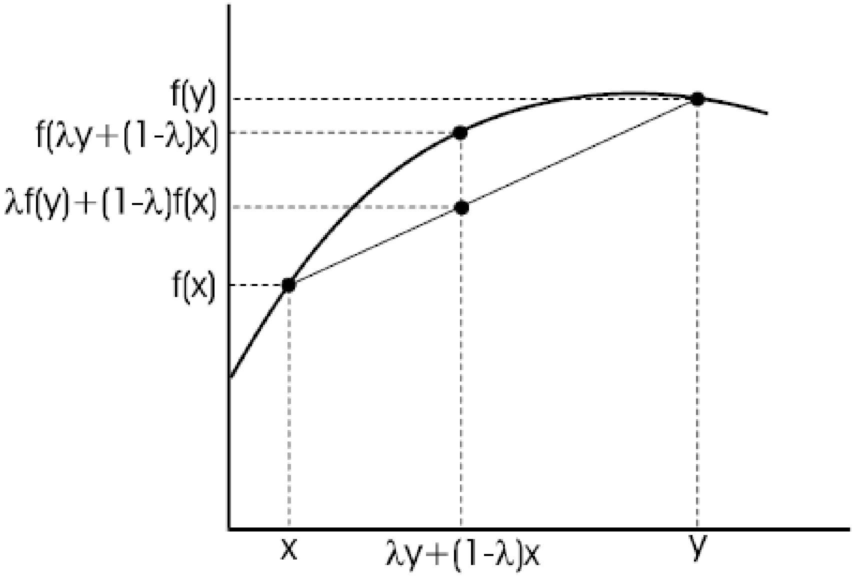


Figure 1 A Nondecreasing Concave Function

In this thesis, the cost structure is nondecreasing concave since when the product quantity increase, production costs, holding costs, and also backlogging costs increase accordingly.

CHAPTER 3

DYNAMIC ECONOMIC LOT-SIZING PROBLEM WITH PERISHABLE ITEMS

3.1. Related Literature

In literature, there are available many researches for perishable lot-sizing model. Hsu (2000a) has studied about dynamic economic lot sizing model for perishable products when deterioration depends on age of stock and the inventory and production costs general concave. Dynamic programming algorithm is developed for solving the problems in polynomial time. The network of the problem has been illustrated in Hsu (2000a) while backlogging is not allowed.

In the model of Hsu (2000a), planning horizon consists of n periods, demand is known per period for a single perishable product. Products deteriorate according to deterioration rate. There is no backlogging in the model. Production and holding costs are assumed as nondecreasing and concave. The aim of the model in Hsu (2000a) is minimizing the total costs.

Figure 2 shows the constructed network in Hsu (2000a). Node F is single supply node. The arcs between node N and S represent the flow of demand. Node S is the demand nodes per period. Also, the arcs between nodes N represent the inventory which is carried per period.

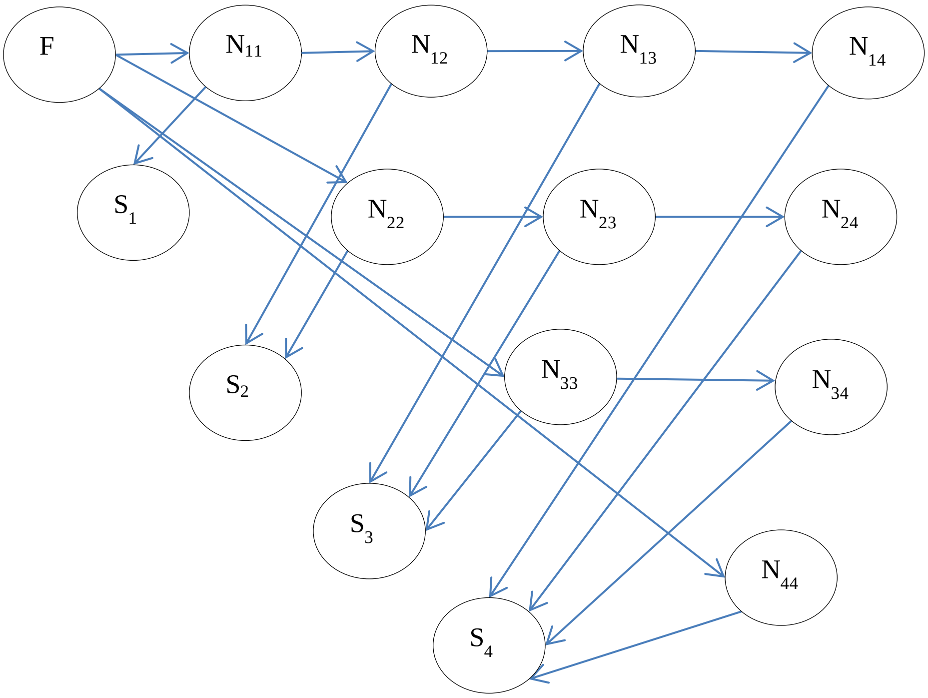


Figure 2 Network Flow of Hsu (2000a)

In the literature, the most related work our study is Hsu (2003). In Hsu (2003), the economic lot sizing model for perishable products with age-dependent inventory and backlogging costs have been considered. For this age-dependent inventory and backordering costs problem a mathematical model and a dynamic programming algorithm have been performed. Also the algorithm has been used for some special cases of their problem.

Decision variables in Hsu (2003) are given below.

x_t is the amount of production in period t.

$c_t(x_t)$ is the production cost for producing x_t units in period t.

y_{it} is the amount of inventory which is produced in period i and held at the beginning of period t.

$H_{it}(y_{it})$ is the carrying cost of y_{it} units in the inventory in period t.

α_{it} is the proportion of y_{it} which is lost in period t.

Z_{kt} is the amount of unfilled period k demand at the end of period t.

$P_{kt}(Z_{kt})$ is that the penalty cost of leaving Z_{kt} unfilled in period t.

M_{it} is the amount of period i production used to satisfy the demand in period t.

The mathematical model is given below.

Minimize $$\sum_{\mathrm{t}=1}^{\mathrm{n}}\left[c_t(x_t) + \sum_{i=1}^{t} H_{it}(y_{it}) + \sum_{k=1}^{t} P_{kt}(Z_{kt})\right] \qquad (2.1)$$

Subject to:

$$\mathrm{x_t} - \sum_{\mathrm{i=1}}^{\mathrm{t}} \mathrm{M_{it}} = \mathrm{y_{tt}} \qquad 1 \leq \mathrm{t} \leq \mathrm{n} \qquad (2.2)$$

$$\left(1-\alpha_{i,t-1}\right) \mathrm{y_{i,t-1}} - \mathrm{M_{it}} = \mathrm{y_{it}} \qquad 1 \leq \mathrm{i} < t \leq n \qquad (2.3)$$

$$Z_{k,t-1} - M_{tk} = Z_{kt} \qquad 1 \leq \mathrm{k} < t \leq n \qquad (2.4)$$

$$\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{M_{it}} = \mathrm{d_t} \qquad 1 \leq \mathrm{t} \leq \mathrm{n} \qquad (2.5)$$

$$\mathrm{x_t}, \mathrm{y_{it}}, \mathrm{Z_{it}} \geq 0, \quad 1 \leq \mathrm{i} < t \leq n; \quad M_{kt} \geq 0, \quad 1 \leq \mathrm{k} \leq \mathrm{n},\ 1 \leq \mathrm{t} \leq \mathrm{n} \qquad (2.6)$$

The objective minimizes the total cost of production, holding and backlogging. Constraints (2.2) calculate the remaining amount of product at the end of period t after satisfying the demand of period t. Constraints (2.3) take into account the perishability of the products. The inventory from previous period perishes according to loss rate and some of the remaining part is used to satisfy the demand of next period. The remainder is calculated accordingly. Constraints (2.4) calculate the remaining unfilled demand of period k in period t after satisfying some part of demand in period t. Constraints (2.5) state that total amount of items used to satisfy the demand of period t should be equal to the total demand of period t. Final constraints (2.6) enforce non-negativity on all decision variables.

Tables 1 and 2 summarize the properties of the related literature and our study respectively. The information about the type of product, problem types, time complexity of the problem and the cost functions such as production, holding and backlogging can be seen explicitly. In these studies, there are some special cases and the properties of these have been given in Table 1.

All papers in the Table 1 consider the production cost functions as general concave and nondecreasing. While Hsu (2000a) and Hsu (2003) include perishable products, Hsu and Lowe (2001) consider non-perishable products. In Hsu (2000a), there is no backlogging and the inventory holding cost function is general concave and age-dependent. Hsu and Lowe (2001) and Hsu (2003) allow backlogging. Hsu and Lowe (2001) consider backlogging cost functions as period-pair-dependent and inventory holding costs are covered as both period-pair-dependent and independent while Hsu (2003) considers general concave age-dependent backlogging cost functions and inventory holding costs are considered as general concave and age-dependent.

In Table 2, all costs functions are general concave; moreover, the holding and backlogging costs are considered as age-dependent.

Table 1 The Summary of the Related Literature

	Hsu (2000a)	**Hsu and Lowe (2001)**	**Hsu (2003)**
Production cost function	General concave and nondecreasing	General concave and nondecreasing	General concave and nondecreasing
Perishability	$\alpha_{it} \geq \alpha_{jt}$ where $i \leq j$	No perishable items	$\alpha_{it} \geq \alpha_{jt}$ where $i \leq j$
Backlogging cost function	No backlogging	Period-pair dependent	General concave age- dependent
Inventory holding cost function	General concave age- dependent	Period-pair dependent /independent	General concave age-dependent
Running time of the algorithm	$O(n^4)$	$O(n^3)$	$O(n^4)$

Table 2 The Property of our Problem

	Our Problem
Production cost function	General concave and nondecreasing
Perishability	$\alpha_{it} \geq \alpha_{\mathrm{j}t}$ where $\mathrm{i} \leq \mathrm{j}$
Backlogging cost function	General concave age-dependent
Inventory holding cost function	General concave age-dependent

Chan et al. (2002) considers economic lot-sizing models (ELS) with modified all-unit discount freight ordering cost function and linear holding cost function. They showed that this problem is NP-hard and suggest an approximation algorithm that assumes ZIP which implies that a positive inventory cannot be carried to a production period and which may not hold in any optimal solution.

Chu et al. (2005) considers a more general problem by considering general economies of scale cost functions and perishability. They assume age-dependent deterioration rate and holding cost. Since the problem is NP-hard, they propose an approximation scheme that solves the problem approximately and it is guaranteed to provide a solution which is not more than 1.52 times of the objective function value.

3.2. Mathematical Model

We assume production cost, backlogging and inventory holding costs are nondecreasing concave and age-dependent. Products are perishable and they deteriorate according to their perishability rate. A new mathematical model generated is related to the model and the problem in Hsu (2003). We eliminated one constraint and we defined new decision variables by using different notation for demand flow. By the help of this, the number of constraints is decreased and also the model can be defined as a network flow model.

We define our model as below.

Decision Variables

x_t is the production volume at the beginning of period t.

Z_{it} is the amount of demand in period t which is going to be satisfied from production in period i. When $i > t$, it means backlog.

y_{it} is the inventory left at the beginning of period t from the production in period i.

Parameters and Functions

d_t is the demand in period *t.*

α_{it} is the proportion of loss for y_{it} during period *t* or deterioration rate during period *t.*

c_t is the production cost function of production at period t.

B_{it} is the penalty cost function or backlogging cost function, the cost of satisfying the demand of period t from production in period i, where $i > t.$

H_{it} is the cost function of holding inventory during period *t* of items produced in period *i*.

Assumptions

1. The production, inventory and penalty cost functions are assumed to be non-decreasing and concave with
   $c_t\,(0) = 0,\ H_{it}(0) = 0, \text{and}\ \ B_{it}(0) = 0.$
2. Inventory and penalty cost functions are considered as age-dependent. In age dependent costs, the costs are related with the age of products so time of periods that products are carried at the inventory is important. This can be shown as $H_{12}(\mathrm{x}) = \ H_{23}(\mathrm{x})$.
3. We assume zero inventory at the beginning and the end of the horizon.
4. We assume that there is no backlogged demand at the beginning period and also no remaining backlogged item at the end of the

horizon. In other words, all demand is satisfied at the end of the horizon.

To summarize the differences between the model in Hsu (2003) and our model are given below.

Model-wise difference:

- $\mathrm{Z_{it}}$ in our model is used for both decision variables $M_{it}\ and\ z_{kt}$. It is enough and sufficient to express the meaning of both two decision variables related with satisfied and unsatisfied demand.
- We eliminated the constraint related with unsatisfied demand which is $z_{k,t-1} - M_{tk} = z_{kt}$ where $1 \leq \mathrm{i} < \ t \leq n$.

Representation-wise difference:

- Our model has a network flow representation whereas the model in Hsu (2003) does not have any.

Additionally, to identify the deterioration rate which is in parameters, an example is shown below.

$\frac{1}{(1-\alpha_{11})(1-\alpha_{12})(1-\alpha_{13})}$ units of product should be produced at period 1 for one unit to be used in period 3. Therefore, to meet demand, the production amount should be increased as much as the fraction of loss rate. This can be shown with an example. Let's assume the deterioration rates are $\alpha_{11} = 0.05$, $\alpha_{12} = 0.2$, and $\alpha_{13} = 0.5$. α_{11} is the proportion of loss of products kept at the inventory at the end of period 1. α_{12} is the proportion of loss of products which have been produced at the beginning of the period 1 and kept at the inventory at the end of the period 2. α_{13} is same as α_{11} and α_{12}. It is the proportion of loss products waiting at the inventory 3 periods. Since, in this case products have been produced in period 1 and they have been kept at the inventory until at the end of period 3. If we want to meet one unit of demand of period 3 from the production of period 1, we have to produce

2.632 units at the beginning of the period 1. In other words, $\frac{1}{(1-0.05)(1-0.2)(1-0.5)} = 2.632$.

Our mathematical model of the problem is given below.

$$\textbf{Minimize} \quad \sum_{t=1}^{n}\left[c_t(x_t) + \sum_{i=1}^{t} H_{it}(y_{it}) + \sum_{i=t+1}^{n} B_{it}(Z_{it})\right] \qquad \text{(P)}$$

Subject to:

$$x_t - \sum_{i=1}^{t} Z_{ti} = y_{tt} \qquad 1 \le t \le n \qquad (3.1)$$

$$\left(1-\alpha_{i,t-1}\right) y_{i,t-1} - Z_{it} = y_{it} \qquad 1 \le i < t \le n \qquad (3.2)$$

$$\sum_{i=1}^{n} Z_{it} = d_t \qquad 1 \le t \le n \qquad (3.3)$$

$$y_{it} \ge 0 \qquad 1 \le i \le t \le n \qquad (3.4)$$

$$Z_{it} \ge 0 \qquad 1 \le i, t \le n \qquad (3.5)$$

$$x_t \ge 0 \qquad 1 \le t \le n \qquad (3.6)$$

This problem can be formulated as a minimum cost flow problem with flow loss. The incoming flow to a node is greater than or equal to the outgoing flow. A 4-period problem is represented in Figure 3. The notation in Hsu (2000a) is followed while drawing the network. However, we have extra backward arcs that represent backlogging.

In Figure 3, node F represents the source for production. N_{it} nodes are created for every paired periods (i,t) while $1 \le i \le t \le n$. Nodes represented by S are demand nodes where the period number appears as a subscript. The arcs show the flow of the products. The arc between (F, N_{tt}) for every t gives the production at period t. This can be considered as x_t. For every pair of i and t, there is an arc between $(N_{it}, N_{i,t+1})$ which gives the inventory holding at period t from production at period i and this is indicated as y_{it}. For every t, the arc between (N_{it}, S_t) are demand satisfaction arcs and does not have an extra cost and also it is shown as Z_{it}. However, the arcs between $(N_{tt}\ , S_k)$ for every t and k $1 \leq k \leq t$ are the backlogging arcs. This can be illustrated as Z_{tk} where $i > t$.

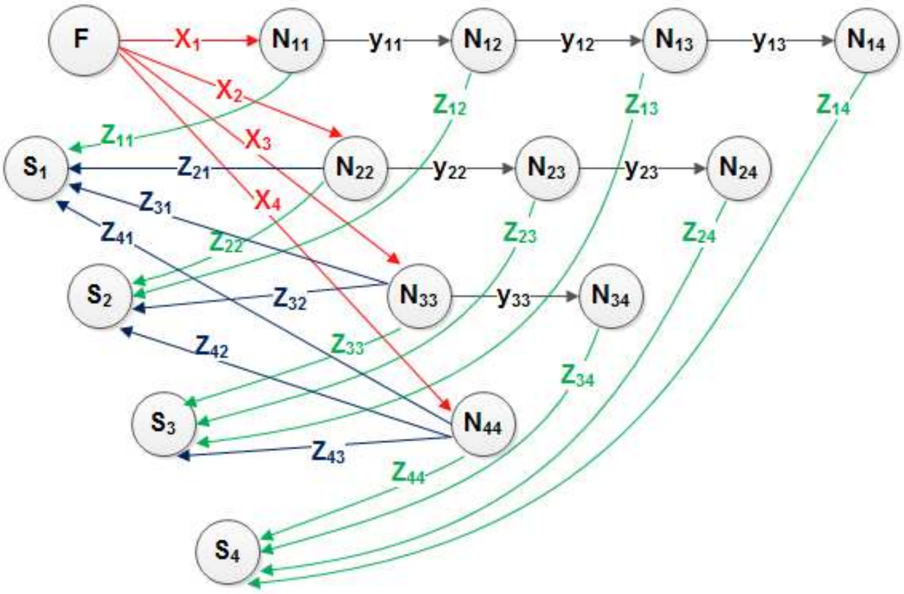


Figure 3 Network Flow Representation for DLSP Model

3.3. Solution Method

In Winston (1993), Dynamic Programming (DP) is defined as a method used to solve complex decision problems. The method is found by Richard Bellman in 1953. It is a recursive method. From sub-problems of a whole problem the suboptimal solutions are found. Then, the sub-problems are expanded and new suboptimal solutions are found for new expanded part while using previous optimal solution. The expansion of the problem lasts until problem turns into the original problem. Therefore, the stopping condition is met when the problem is solved.

For solving our problem, the problem is divided into smaller parts or sub-problems and combination of these parts is considered to obtain a solution method for this kind of problems. To decide on the small parts (components), Zero Inventory Property and Interval Division Property have been explored.

Zero-Inventory Property (ZIP): A positive inventory cannot be carried into a production period. This means that in a given period we can have at most one of the following: inventory from previous periods or production. In other words,

$$x_t y_{it} = 0 \qquad\qquad \forall t,\ i \le t$$

In Wagner and Whitin (1958) this property is used to obtain a dynamic programming based solution and define the parts of the optimal solution.

Interval Division Property (IDP): Production in a period satisfies the demand of some consecutive periods. Let j_1, j_2, ..., j_k be the production periods in the ascending order and i_1, i_2, ..., i_k be the end of the consecutive demand periods in the ascending order, then production in period j_m satisfies the demand of the periods from i_{m-1} to i_m.

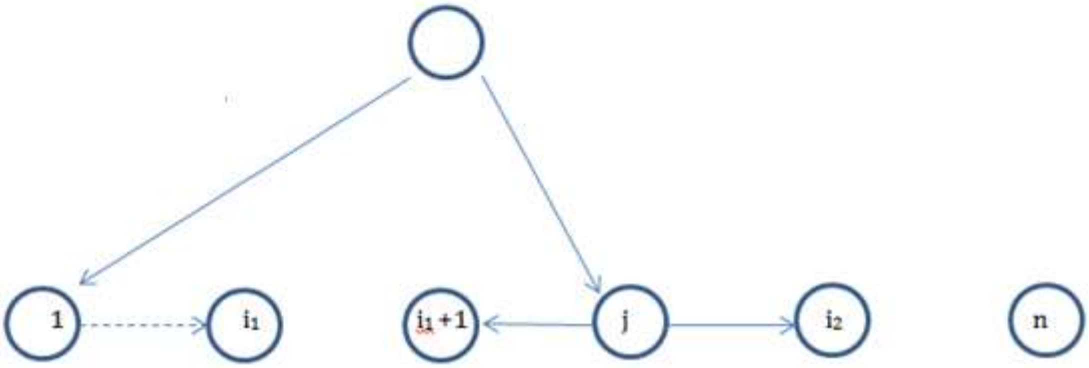


Figure 4 Interval Division Property (IDP)

An example is shown in Figure 4. There are 2 production periods and n periods for demands. Production at period 1 satisfies the demand of periods from 1 to i_1 in consecutive order and production at j satisfies the remaining demand from period $i_1 + 1$ to period i_2. This divided periods and productions into intervals and make them groups between each other.

Florian and Klein (1971) has developed an algorithm with general concave inventory cost functions with or without backlogging and IDP can be seen in these problems. Florian and Klein (1971) claims that optimal plans consist of independent components. In these components, firstly the inventory level is not zero for every period but the inventory level of last period is zero. Secondly, if the production level is positive, it should be at capacity with the exception that at most one period production level can be less than the capacity that is shown as ε. It can be calculated by the following equation. Therefore, let us to refer that d_i is demand of period i and K is the capacity. Then, ε can be found as:

$$\varepsilon = \sum_{i=1}^{n} d_i - \left\lfloor \frac{\sum_{i=1}^{n} d_i}{K} \right\rfloor K$$

Sargut (2006) states that IDP can be useful when inventory and backlogging costs are concave. Moreover, total minimum cost is calculated by adding minimum cost for each component.

Table 3 and 4 are detailed versions of Table 1 and 2 respectively. Tables include details in terms of Interval Division Property and Zero Inventory Property. All papers in the Table 3 have some special cases where IDP holds. Table 4 includes the general properties of our study. IDP and ZIP do not hold for any optimal solution in our study. It can be seen in following examples since the optimal solution shows that these properties may not hold in some problems.

Table 3 The Properties of the Models in Related Literature in more Detail

	Hsu (2000a)	Hsu and Lowe (2001)	Hsu (2003)
Production cost function	General concave and nondecreasing	General concave and nondecreasing	General concave and nondecreasing
Perishability	$\alpha_{it} \geq \alpha_{\mathrm{j}t}$ where $\mathrm{i} \leq \mathrm{j}$	No perishable items	$\alpha_{it} \geq \alpha_{\mathrm{j}t}$ where $\mathrm{i} \leq \mathrm{j}$
Backlogging cost function	No backlogging	Period-pair dependent	General concave age-dependent
Inventory holding cost function	General concave age- dependent	Period-pair dependent /independent	General concave age-dependent
IDP for general case	Yes	No	No
ZIP for general case	No	No	No
Running time of Algorithm	O(n^4)	O(n^3)	O(n^4)
Special cases (IDP holds)	General concave and age dependent inventory costs	General concave, period pair dependent backlogging and inventory cost	General concave, non-decreasing demand, age dependent inventory and backlogging cost
Running time of special instances	O(n^2)	O(n^3)	O(n^2)

Table 4 The Property of our Problem with Detail

	Our Problem
Production cost function	General concave and nondecreasing
Perishability	$\alpha_{it} \geq \alpha_{\mathrm{j}t}$ where $\mathrm{i} \leq \mathrm{j}$
Structure of the backlogging cost function	General concave age-dependent
Structure of the inventory holding cost function	General concave age-dependent
IDP and ZIP for general case	No

3.4. EMPIRICAL ANALYSIS

This section includes some example problems and their solutions.

Example 3.1: Let us assume there are five periods such as January, February, March, April, and May. All demand values are equal to 10. The production costs are $c_1(x) = 10000x,\ c_2(x) = 2000x,\ c_3(x) = 100x,\ c_4(x) = 3000x,\ c_5(x) = 10x$. The holding cost is linear $H_{it}(y) = 100(t - i + 1)y.$ The following equations show the penalty cost functions which are $B_{i+1,i}(z) = 200z,\ B_{i+2,i}(z) = 400z,\ B_{i+3,i}(z) = 600z$. The deterioration rate matrix (α_{it}) is given below. N/A means not defined.

	January	February	March	April	May
January	N/A	0.3	0.8	1	1
February	N/A	N/A	0.4	0.2	1
March	N/A	N/A	N/A	0.5	0.25
April	N/A	N/A	N/A	N/A	0.45
May	N/A	N/A	N/A	N/A	N/A

The unique optimal solution for the problem is that $d_1,\ d_2,\ d_3,\ d_4$ are satisfied from production at period 3, it can be seen that the demand of periods 1 and 2 are met by backlogging from period 3; furthermore, the products which is produced in period 3 and available in the inventory in period 4 are used to meet the demand of period 4. Finally d_5 is satisfied in period 5. Thus, it is seen that $x_3 = 40,\ x_5 = 10$. Using GAMS, it can be solved within seconds. The solution shows that IDP and ZIP are provided. Since, period 3 which is one of the production periods satisfies the demand of period 1, 2, 3, and 4 in a consecutive way. Also, inventory does not carried into production periods.

Example 3.2: Consider one instance of a 100-period problem with age-dependent inventory cost function and the demand values are generated uniformly in the interval (50, 500). The holding costs increase linearly by age that is $H_{it}(y) = 100(t - i + 1)y.$ The backlogging cost also increases linearly by age. The following equation shows the penalty cost function that is $B_{it}(z) = 200(t - i)z$. The production cost constraints are set uniformly in the interval (110, 1100). The deterioration rate matrix (α_{it}) is given below.

$\alpha_{i,i+1}$ = 0.05, $\alpha_{i,i+2}$ = 0.2, α_{it} = 1 when $(t - i) \geq 3$.

The optimal solution for the example 3.2 shows that the demand of periods is met consecutively by consecutive production periods. Therefore, it can be said that IDP are provided but ZIP does not hold for this example. Since, for some periods inventory is carried to production periods.

Hsu (2000a) showed that zero-inventory property may not hold for any optimal solution for perishable items while interval division property can hold. Additionally, the same examples in Hsu (2003) are valid to show that for some problem instances both ZIP and IDP are not satisfied. We used same examples to show that ZIP or IDP may not hold.

Example 1 in Hsu (2003) shows that IDP may not hold. Suppose there are six periods where $d_1 = d_2 = d_5 = d_6 = 10,\ d_3 = 20,$ and $d_4 = 4.$ The production costs are $C_1(x) = x,\ C_2(x) = 4x,\ C_3(x) = 6x,\ C_4(x) = 8x,\ C_5(x) = x,\ C_6(x) = 4x$.

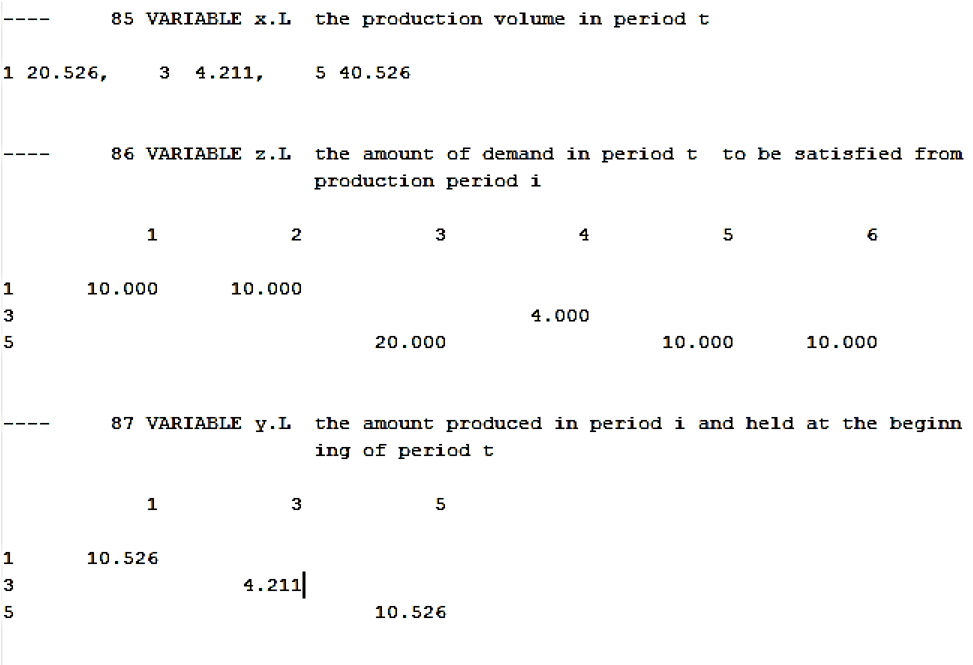
$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 5;$

$H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 4;$

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2.$

The backorder costs are $B_{kt}(z) = 30$ for all $k \leq t$ and $z > 0.$

In the optimal solution, d_1 and d_2 are met by production in period 1. Demand in period 4 is satisfied by period 3. Moreover, demand in periods 3, 5, and 6 are met by production in period 5. 3, 5, and 6 are not consecutively indexed demand periods. Additionally, ZIP holds in the optimal solution which can be seen in Figure 5. Since, the inventory is not carried to the production periods which are period 1, 3 and 5.



```
----     85 VARIABLE x.L  the production volume in period t

1 20.526,    3  4.211,    5 40.526


----     86 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      10.000      10.000
3                                            4.000
5                                20.000                  10.000      10.000


----     87 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           3           5

1      10.526
3                   4.211
5                                10.526
```

Figure 5 GAMS Solution for Example 1 of Hsu (2003)

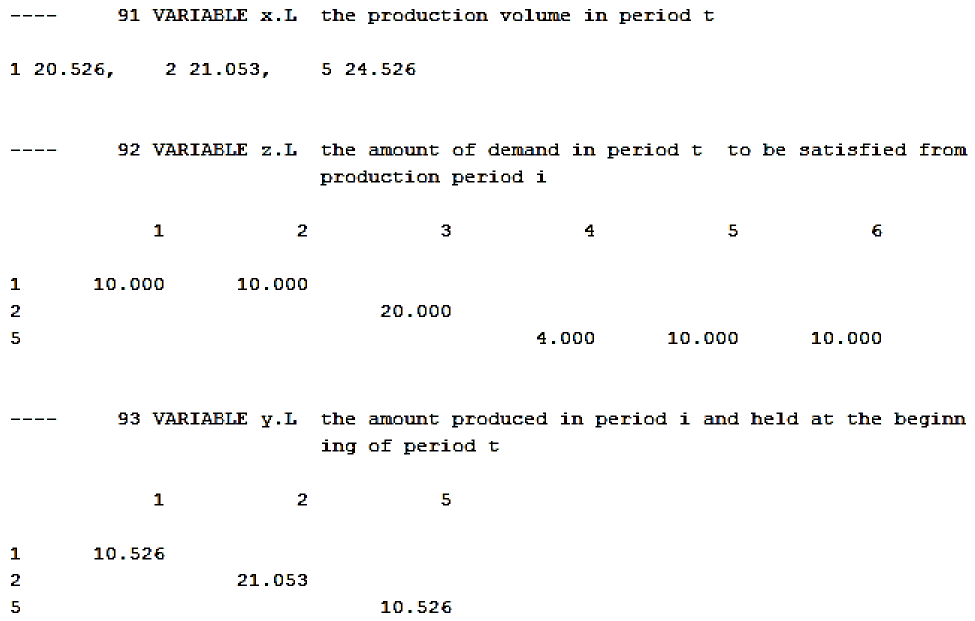
In Example 2 in HSU (2003) ZIP does not hold but IDP in the optimal solution holds. Consider the problem in Example 1 and suppose:

$B_{kk}(z) = 2z,$ for $1 \leq k \leq 5;$

$B_{k,k+1}(z) = 10z,$ for $1 \leq k \leq 4;$

$B_{k,k+t}(z) = +\infty,$ for $1 \leq k \leq 3$ and $t \geq 2.$

In optimal solution, it is seen that production in period 1 satisfies demand in first and second periods. Period 2 satisfies the third period. Furthermore, demand of period 4, 5, and 6 are met by production in period 5. It can be seen in Figure 6 that production periods are met demand in consecutive order so IDP holds. However, there exists production in period 2 although inventory is carried to period 2. Therefore, ZIP does not hold.



```
----     91 VARIABLE x.L  the production volume in period t

1 20.526,    2 21.053,    5 24.526


----     92 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      10.000      10.000
2                              20.000
5                                           4.000      10.000      10.000


----     93 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2           5

1      10.526
2                  21.053
5                              10.526
```

Figure 6 GAMS Solution for Example 2 of Hsu (2003)

3.5. The Structure of the Optimal Solution for Uncapacitated Production Problem

In this section, the structural properties for uncapacitated problems are mentioned. Some theorems help to find or analyse the solutions by checking whether the conditions of the theorems hold. Therefore, Theorems 1 and 2 in Hsu (2003) are explained below. We find the same solution with the solution of the model in Hsu (2003) by solving with our model. It means that model in Hsu (2003) and our model are alternative models and the theorems about the structure are valid for both.

Theorem 1: There exists an optimal solution to (P) where for each t, there is a unique *i* such that $Z_{it} = d_t$, where $1 \leq i \leq t$.

Proof: Each period's demand is satisfied by the production at only one period. The objective is concave and the feasible region is a compact polyhedral. According to Bazaraa and Shetty (1979), the optimal solution will be an extreme point in the feasible region. For a network flow problem extreme point solution corresponds correspond to an arborescent flow in the network (Zangwill, 1968). In other words, every node will have only one incoming arc with positive flow except the source node. ◻

However, in capacitated production problem there may be instances where Theorem 1 may not be valid. Since, for this kind of problems the production of period i may not be enough to meet the period t demand because of the limited production capacity. Thus, there may be more than one production period to satisfy a specific period demand.

In the optimal solution for uncapacitated problems, it is observed that if the holding cost and backlogging cost increase when the difference between production period and period that product is used increases and if it is also known that the demand of a period k is satisfied by a production in period i, the production periods before period i such as period i-1 cannot satisfy demand of period k or demand of further periods of period k. Since, if the holding costs and backlogging costs are considered, satisfying demand of any period after k using production in any period before i without using this theorem increases total costs which is out of objective.

The conditions and details are explained below in Theorem 2 and Figure 7.

Theorem 2: There exists an optimal solution to (P) where if $\mathrm{i} < j$ are two production periods and $Z_{jk} = d_k$, for some $\mathrm{k} \geq \mathrm{j}$, then $Z_{it} = 0$ for all *t*, where $\mathrm{k} \leq \mathrm{t} \leq \mathrm{n}$.

Proof: Assume that in the optimal solution demand in period k is satisfied via production in period j, where $\mathrm{i} < j \leq k$. Let x_i^+ be the production quantity of period i in the optimal solution and y_{il}^+ be the inventory left at the beginning of period l from the production in period *i*. For any $i < j$ we can say that everything is kept the same except period k is satisfied from period i instead of period j.

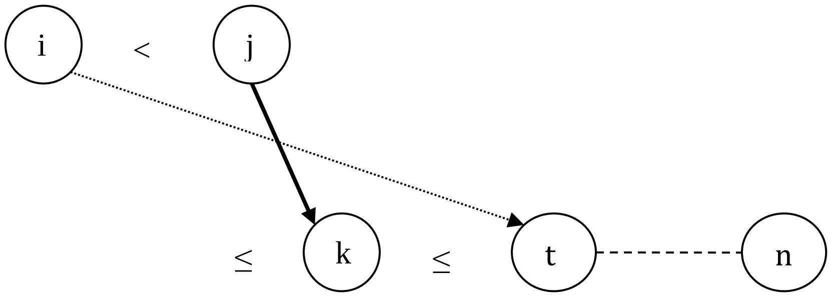


Figure 7 The Intersecting Demand Satisfaction with Holding

Let us assume in the optimal solution period k is satisfied from production at period j and p periods after period k ($k < r_1 < r_2 ... < r_p = t$) are satisfied by production at period i. The proof is done by showing that if they are satisfied from period j, we will have a better objective value.

That is:

$$c_i(x_i^+) + \sum_{l=i}^{t-1} H_{il}(y_{il}^+) + c_j(x_j^+) + \sum_{l=j}^{t-1} H_{jl}(y_{jl}^+) >$$

$$c_i(x_i^+ - \sum_{m=r_1}^{r_p} A_{im}^i d_m) + \sum_{l=i}^{k-1} H_{il}(y_{il}^+ - \sum_{m=r_1}^{r_p} A_{lm}^i d_m) + c_j(x_j^+ + \sum_{m=r_1}^{r_p} A_{jm}^j d_m) + \sum_{l=j}^{k-1} H_{jl}(y_{jl}^+ + \sum_{m=r_1}^{r_p} A_{lm}^j d_m)$$

The details of the proof without backlogging are given in Hsu (2000a). The proof for the case with backlogging is given in Hsu (2000b). □

3.6. Dynamic Programming Algorithm

Hsu (2003) presents that a dynamic programming algorithm which can be applied when the conditions in Theorem 1 and Theorem 2 are valid. Therefore, we can also adopt the same algorithm. Since the problem under consideration is the same.

$V(i)$ refers the optimal value of the problem when we consider periods from 1 to i. $V(0) \equiv 0,$ and $V(n)$ is the optimal value of the whole planning horizon.

To find the optimal value of the whole planning horizon, the sub-parts of the problem and their optimal values should be found. Therefore, $P(r,i)$ which is the optimal value of the problem for periods from r to i with at most one production period should be found.

The backward recursion of DP can be written as below.

$$V(i) = \min_{1 \le r \le i}\{V(r-1) + P(r,i)\}.$$

The following calculations are needed to find out $P(r,i)$.

A_{kt}^{i} is the amount of production that must be made at period i to satisfy one unit demand of period t from the beginning of period k.

$$A_{kt}^{i} = \frac{1}{(1-\alpha_{ik})(1-\alpha_{i,k+1})\ldots\ldots(1-\alpha_{i,t-1})} = \frac{1}{\prod_{l=k}^{t-1}(1-\alpha_{il})} \qquad i \le k < t.$$

$F_i(p,q,r)$ shows the amount produced at period i and kept at period p for demand of periods from q to r.

d_k is the demand at period k.

$$F_i(p,q,r) = \begin{cases} \sum_{k=q}^{r} A_{pk}^{i}\ d_k & if\ i < q \\ \sum_{k=max\{i,q\}}^{r} A_{pk}^{i} d_k + \sum_{k=q}^{max\{i,q\}-1} d_k & if\ q \le i \end{cases}.$$

$TH_i(q,r)$ is the holding cost for demands from q to r when produced at period i where $i \le q \le r$.

$$TH_i(q,r) = \sum_{l=i}^{q-1} H_{il}\big(F_i(l,q,r)\big) + \sum_{l=q}^{r} H_{il}\big(F_i(l,l+1,r)\big) \qquad l = i \ldots r$$

$TB_i(q,r)$ is the backordering cost for demands from q to r when produced at period i where $i > q \geq r$.

$TB_i(q,r) = \sum_{l=q}^{r} B_{il}(d_l)$.

$P(r,i) = \min_k\{P^k(r,i)\}$ is the total cost of producing at period k for demands of periods from r to i. Therefore,

$$P^k(r,i) = \begin{cases} c_k\big(F_k(k,r,i)\big) + TH_k(r,i) & k < r < i \\ c_k\big(F_k(k,r,i)\big) + TH_k(k,i) + TB_k(r,k) & r < k < i \\ c_k\big(F_k(k,r,i)\big) + TB_k(r,i) & r < i < k \end{cases}$$

According to Hsu (2003) the computational complexity of recursion is that $O(n^2)$. Since, there are n nodes and n^2 arcs and also finding shortest path on this network takes $O(n^2)$ time (Ahuja et al., 1993). All $TB_i(q,r)$ values can be computed in $O(n^4)$ time as well as $TH_i(q,r)$. To obtain $P(r,i)$ additional $O(n^3)$ effort is required. The overall computational complexity of the DP algorithm is found $O(n^4)$ by the summation which is $O(n^2) + O(n^4) + O(n^3) + O(n^4) = O(n^4)$

CHAPTER 4

CAPACITATED DYNAMIC ECONOMIC LOT-SIZING PROBLEM WITH PERISHABLE ITEMS

Uncapacitated problems are computationally easier than the capacitated problems. Systems have capacity because number of shifts, number of workers and machines used in production, and also other many factors restrict time and amount of production. Florian and Klein (1971) has developed their study including capacity limit which is one of the main difference from Wagner and Whitin (1958). The problem studied by Florian and Klein (1971) is harder than the problem in Wagner and Whitin (1958) since optimal solution in Wagner and Whitin (1958) satisfies Zero-Inventory Ordering Policy but ZIP does not necessarily lead to an optimal solution for the capacitated version. Moreover, product types such as perishable products make problem harder. Hsu (2000a) has studied problems with perishable products. Hsu (2000a) claims that these problems are much harder since number of nodes in network representation increases and inventory cost calculation is much more complex. This situation is valid both capacitated and uncapacitated production.

4.1. Mathematical Model

In the capacitated version of the model, a capacity constraint (4.6) has been added which implies the production per period cannot exceed the production limit per period. Backlogging is allowed. Inventory holding and backlogging costs are both concave and age-dependent. Furthermore, perishability depends on periods and production costs are concave and nondecreasing. This study has stationary production capacities. In addition, Bitran and Yanasse (1982) shows that non-stationary one is NP-hard even with easy cost structures.

Decision Variables

x_t is the production volume at the beginning of period t.

Z_{it} is the amount of demand in period t which is going to be satisfied from production in period i. If $i > t$, it means there exists backlogging.

y_{it} is the inventory left at the beginning of period t from the production in period i.

Parameters and Functions

d_t is demand in period *t.*

α_{it} is the proportion of y_{it} lost during period *t.*

c_t is the cost function of production at period t.

B_{it} is the penalty cost function for backlogging, the cost of satisfying the demand of period t from production in period i, where $i > t.$

H_{it} is the cost of holding inventory during period *t* of items produced in period *i*.

K is the fixed production capacity for each period.

Assumptions

1. The production, inventory and penalty cost functions are assumed to be non-decreasing and concave where
   $c_t\,(0) = 0,\ H_{it}(0) = 0,$ and $B_{it}(0) = 0.$
2. Inventory and penalty cost functions are considered as age-dependent.
3. We assume zero inventories at the beginning and the end of the horizon.
4. Moreover, we assume that there is no backlogged demand at the beginning period and also no remaining backlogged item at the end of the horizon. In other words, all demand is satisfied at the end of the horizon.
5. Let us define $\triangle f(x, \beta) = f(x + \beta) - f(x)$.

   For any x, y, β ≥ 0

   $\triangle B_{it}(x, \beta) \geq \triangle B_{jt}(y, \beta)$, for $1 \leq t < j \leq i \leq n$

   $\triangle H_{it}(x, \beta) \geq \triangle H_{jt}(y, \beta)$, for $1 \leq i \leq j \leq t \leq n$

   This assumption states that there is a nondecreasing marginal holding/backordering cost with respect to the age of inventory/backorder, namely (t-i) or (i-t). This assumption shows that when the carried or backlogged products are increased by β, the backlogging and holding costs will increase through number of periods. If periods are increased for a specific amount of products while backlogging or holding, the costs for backlogging and holding is increased for that specific amount. Since, holding inventory longer increases the costs and this is same as for backlogging.

6. Assumption on the deterioration rate $\alpha_{i,t} > \alpha_{j,t}$ where $i \leq j$.

$$\textbf{Minimize} \quad \sum_{t=1}^{n}\left[c_t(x_t) + \sum_{i=1}^{t} H_{it}(y_{it}) + \sum_{i=t+1}^{n} B_{it}(Z_{it})\right] \qquad (P1)$$

Subject to:

$$x_t - \sum_{i=1}^{t} Z_{ti} = y_{tt} \qquad 1 \le t \le n \qquad (4.1)$$

$$(1-\alpha_{i,t-1})\ y_{i,t-1} - Z_{it} = y_{it} \qquad 1 \le i < t \le n \qquad (4.2)$$

$$\sum_{i=1}^{n} Z_{it} = d_t \qquad 1 \le t \le n \qquad (4.3)$$

$$y_{it} \ge 0 \qquad 1 \le i \le t \le n \qquad (4.4)$$

$$Z_{it} \ge 0 \qquad 1 \le i, t \le n \qquad (4.5)$$

$$0 \le x_t \le K \qquad 1 \le t \le n \qquad (4.6)$$

4.2. Structural Properties of the Optimal Solution

According to Theorem 1, the demand of a period is satisfied from production at one period. However, this theorem does not hold in capacitated production case. Since the capacity restricts the amount of production and insufficient amount may affect the meeting demand so that production at many periods can be used to satisfy the demand for a period.

Theorem 3: In the optimal solution of problem (P) for each period t, there is at most one i with $Z_{it} > 0$ and $0 < x_i < K$.

Proof: When demand of a period is satisfied by more than one period we obtain cycles with positive flow. Since we have concave costs, we can find a better solution, where there is no cycle. We will explain this in a reduced network where the inventory arcs are eliminated but they are still there between an x value and z values. The first level arcs are the positive production arcs and the second level arcs are the positive z values. In the reduced network below demand of period t is satisfied by the periods i, j, and k.

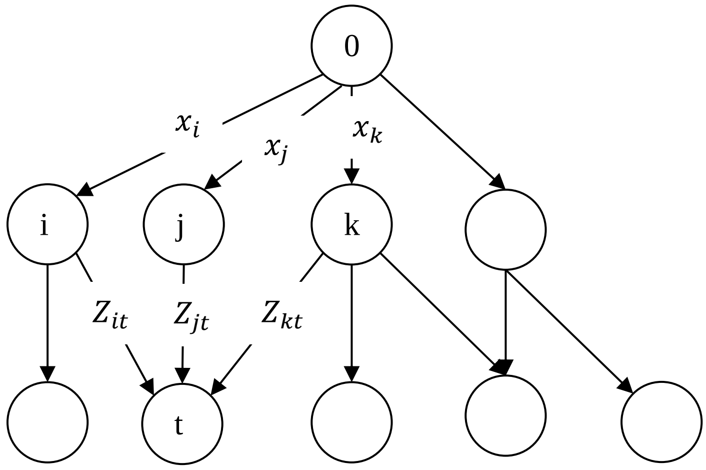


Figure 8 A Reduced Network

If two of the production periods have positive x values less than the capacity, we can improve the solution by equating one of them to capacity/zero and decreasing/increasing the other one by some amount. At one ∂ value, where at least one of the values of x_i, z_{it}, x_j, z_{jt} hits zero is the best solution because of the concave cost structure. □

Any optimal solution is composed of at least one component. Based on Theorem 3, we name a component by its range of production periods and the demand periods. Its property is that some consecutive demand periods are satisfied via some production periods and it may not be connected. Production periods start with the next period of the previous component's last production period and ends with its last positive production period. In between we can have zero production periods. Demand periods are the consecutive periods where their demand is satisfied. Two examples of the component (1, 4, 1, 5) are given in Figures 9 and 10. The first one is not connected, whereas the second one is connected. A component may be degenerate as can be seen in the first reduced network in Figure 9.

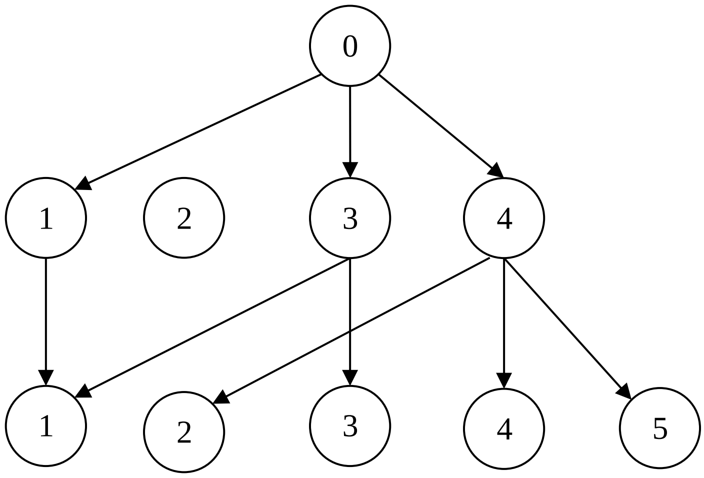


Figure 9 A Disconnected Component

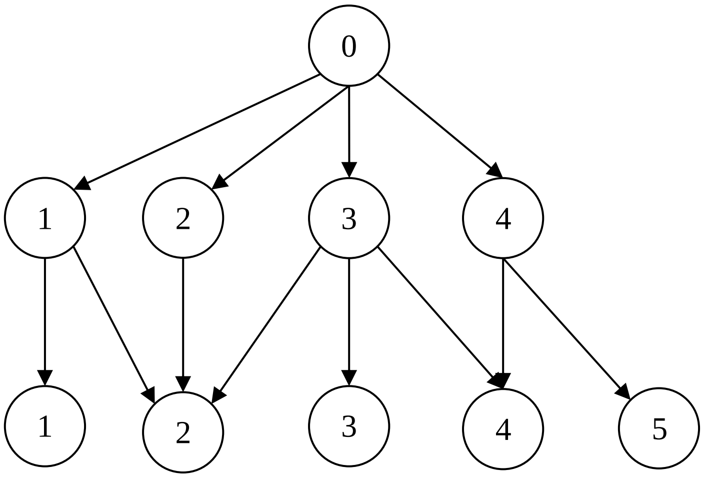


Figure 10 A Connected Component

Suppose that a component is composed of the production periods from s to m and the demand periods from p to r.

We define component as a set of consecutive production periods and a set of consecutive demand periods. Let component (s, m, p, r) represents the production periods from s to m where s ≤ m and the demand periods from p to r where p ≤ r.

The demand of periods which are p to r is fully satisfied by the production periods in the component. It means that there is no incoming inventory/backlog into the component as well as outgoing inventory/backlog from the component. Moreover, the total production up to a production period t_1, where s≤ t_1 <m is less than or more than the amount needed to satisfy the demand up to a period t_2, where p ≤ t_2 < r. The next component starts with production period m+1 and ends with the last positive production period.

Theorem 4: In a given component (s, m, p, r) there will be at least θ periods with production at full capacity and one period with production quantity $0 < \epsilon < K$ and at other periods production amount is equal to zero, where

$$\theta = \left\lfloor \frac{\sum_{k=p}^{r} d_k}{\mathrm{K}} \right\rfloor$$

We consider number of full production periods equal to $\theta = \left\lfloor \frac{\sum_{k=p}^{r} d_k}{\mathrm{K}} \right\rfloor$ and $\theta + 1$. Because of perishability, integer part of division is not enough to satisfy the demand.

Proof: In a connected part of that component we obtain many cycles with more than one positive flow. If two of the production periods in the connected component have positive x values less than the capacity, we can improve the solution by making one of them equal to capacity/zero and decreasing/increasing the other one by some amount because of the concave cost structure.

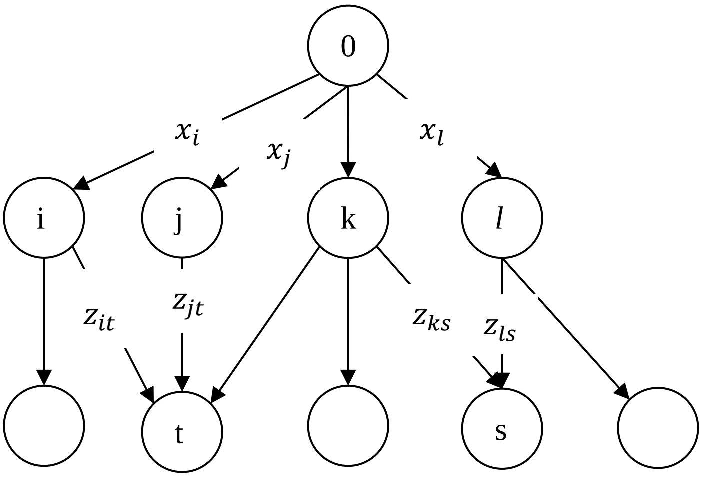


Figure 11 The Reduced Network for the Proof of Theorem 4

Suppose that the values of x_i and x_l are between 0 and K. At least one of the values of x_i, z_{it}, x_j, z_{jt} hits zero is the best solution because of the concave cost structure. □

Theorem 5: In an optimal solution we do not have intersecting z arcs such as

a. (t, r_1) and (k, p_2) where $t \leq p_2 < r_1 \leq k$.

b. (t, r_1) and (k, p_2) where $t < k \leq p_2 < r_1$.

c. (t, r_1) and (k, p_2) where $p_2 < r_1 < t < k$ with the extra assumption

$$\Delta B_{i+k,t}(x, \beta) - \Delta B_{it}(y, \beta) > \Delta B_{j+k,t}(z, \beta) - \Delta B_{jt}(\text{u}, \beta)$$

where $i > j > t$ and $k,\ x,\ y,\ z,\ u,\ \beta > 0$.

d. (t, r_1) and (k, p_2) where $p_2 < t \leq r_1 < k$ with the extra assumption

$$\Delta B_{it}(x, \beta) > \Delta B_{ij}(y, \beta) + \Delta B_{jt}(z, \beta)$$

where $i > j > t$ and $x,\ y,\ z,\ \beta\ \geq 0$.

e. (t, r_1) and (k, p_2) where $t < p_2 \leq k < r_1$ with the extra assumption

$$\Delta H_{it}(x, \beta) > \Delta H_{ij}(y, \beta) + \Delta H_{jt}(z, \beta)$$

where $i < j < t$ and $x,\ y, z, \beta \geq 0$

Proof: We will prove it by contradiction. For part a, consider the following situation in the optimal solution. Suppose that $z_{tr_1} \geq z_{kp_2}$. Now consider this updated solution.

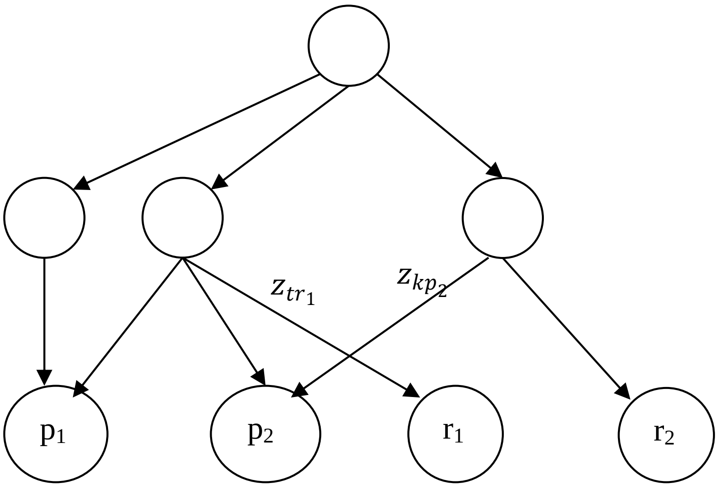


Figure 12 First Reduced Network for the Proof of Theorem 5

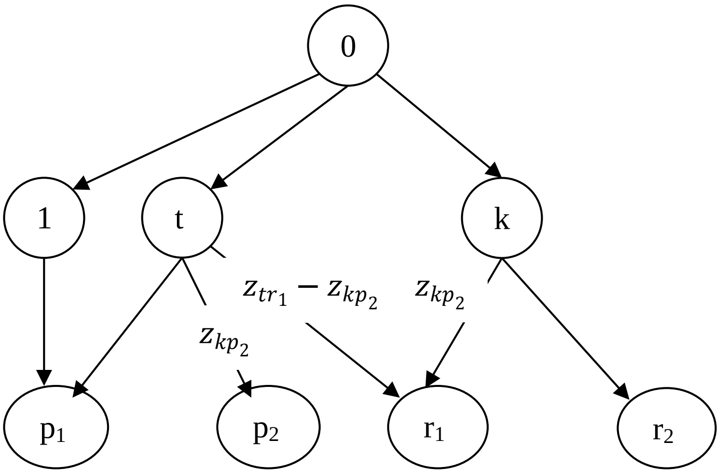


Figure 13 Second Reduced Network for the Proof of Theorem 5

While total production at periods t and k are kept below the capacity and constant as possible

- The backlogging cost change of sending z_{kp_2} units from k to r_1 instead of k to p_2 is negative.
- The holding cost change of decreasing the amount from t to r_1 by z_{kp_2} and increasing the amount from t to p_2 by the same amount is negative.
- Moreover the production amount will decrease at period t.

Therefore, we can eliminate this situation one by one until we do not have any.

For part b, Figures 14 and 15 are below.

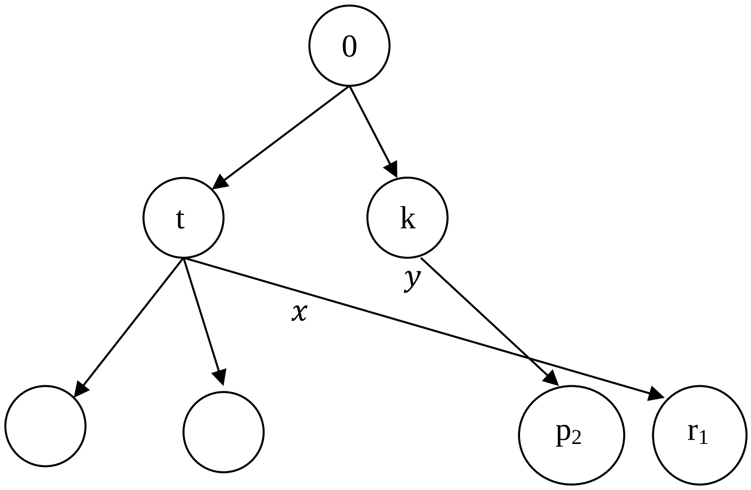


Figure 14 First Reduced Network for the Part b of Theorem 5

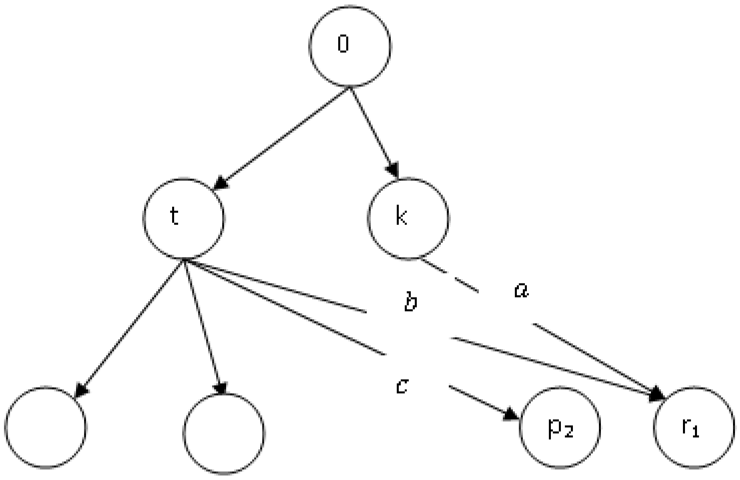


Figure 15 Second Reduced Network for the Part b of Theorem 5

$$a = y\frac{A_{kp_2}^{k}}{A_{kr_1}^{k}}$$

b = $x - y\, A_{kp_2}^{k}\ / A_{kr_1}^{k}$

c = y

We send just y units from t to p_2.

The change in the cost can be summarized as below.

1. Carry extra $y\frac{A^k_{kp_2}}{A^k_{kr_1}}$ units for ages 1, 2, ..., r_1-k+1
2. Carry $y\,A^k_{kp_2}\,/A^k_{kr_1}$ units less for ages 1, 2, .., r_1-t+1
3. Carry y units less for ages 1, 2, ..., p_2-k+1
4. Carry extra y units from 1, 2, ..., p_2-t+1

The joint effect of 1 and 2 is negative since it corresponds to carrying less than y units for age r_1-k+2 to r_1-t+1.

The joint effect of 3 and 4 is also negative since we carry y units less for age p_2-t+2 to p_2-k+1. Moreover production at period t will decrease. Total effect will be negative. There we will not have this kind of intersection in an optimal solution.

For part c we have the following figure which is Figure 16.

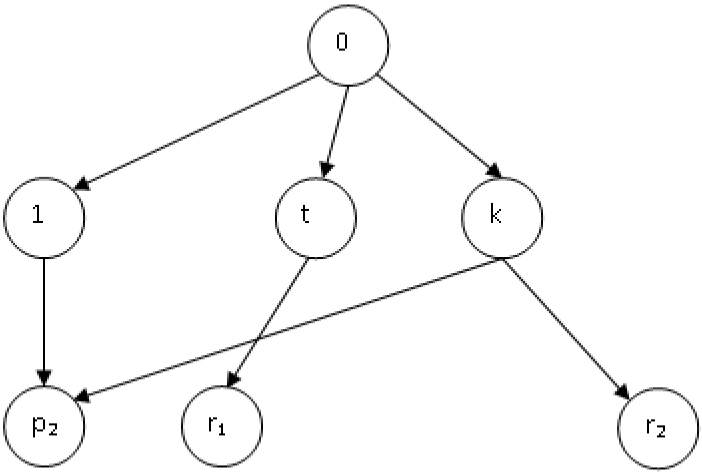


Figure 16 Reduced Network for the Part c of Theorem 5

For part d we have Figure 17, we can prove similarly and the production at period t will decrease.

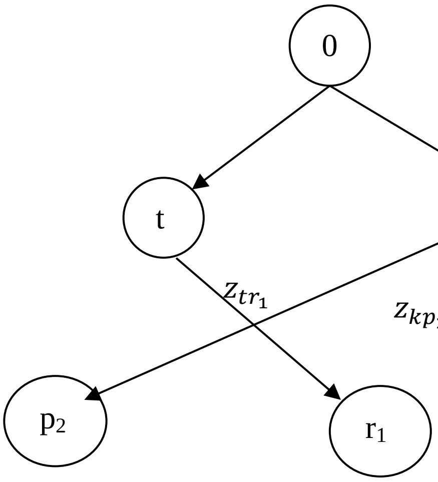


Figure 17 Reduced Network for the Part d of Theorem 5

For part e we have Figure 18, we can prove similarly and the production at period t will decrease.

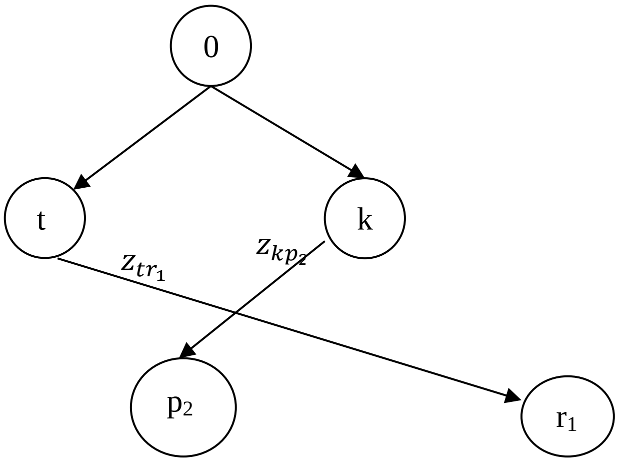


Figure 18 Reduced Network for the Part e of Theorem 5

There is no assumption that prevents the following case. ($t=r_1$)

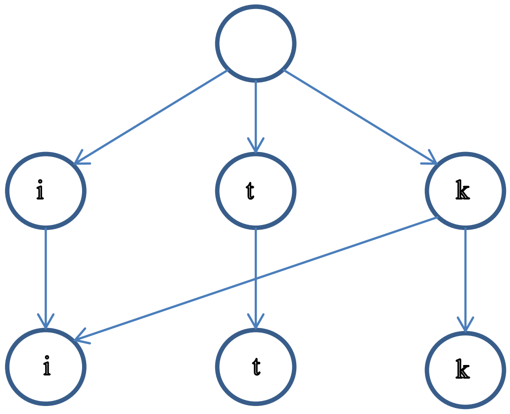


Figure 19 A Possible Network in Optimal Solution

□

Theorems 3 and 4 and also part a, b and c of Theorem 5 are valid with the beginning assumption which is

For any x, y, $\beta \geq 0$

$\triangle B_{it}(x, \beta) \geq \triangle B_{jt}(y, \beta)$, for $1 \leq t < j \leq i \leq n$

$\triangle H_{it}(x, \beta) \geq \triangle H_{jt}(y, \beta)$, for $1 \leq i \leq j \leq t \leq n$

However, Theorems 3 and 4 and also all parts of Theorem 5 are valid when the extra assumptions in the Theorem 5 are applied.

From now on we consider a component by consecutive set of production periods and consecutive set of demand periods, where the next component has later demand and production periods then the previous component. A component can be disconnected in that case we say that it is degenerate.

4.3. EMPIRICAL ANAYSIS

Four data sets have been generated with different capacities to analyse the problem. Number of periods, the amount of demand per periods and production costs are the same but the production capacity, holding costs and the penalty costs are different in the examples. Thus, there are 6 periods where $d_1 = 15,\ d_2 = 10,\ d_3 = 20,\ d_4 = 5,\ d_5 = 30$, and $d_6 = 1$. The costs of production are $C_1(x) = x,\ C_2(x) = 4x,\ C_3(x) = 6x,\ C_4(x) = 8x,\ C_5(x) = x$, $C_6(x) = 4x$.

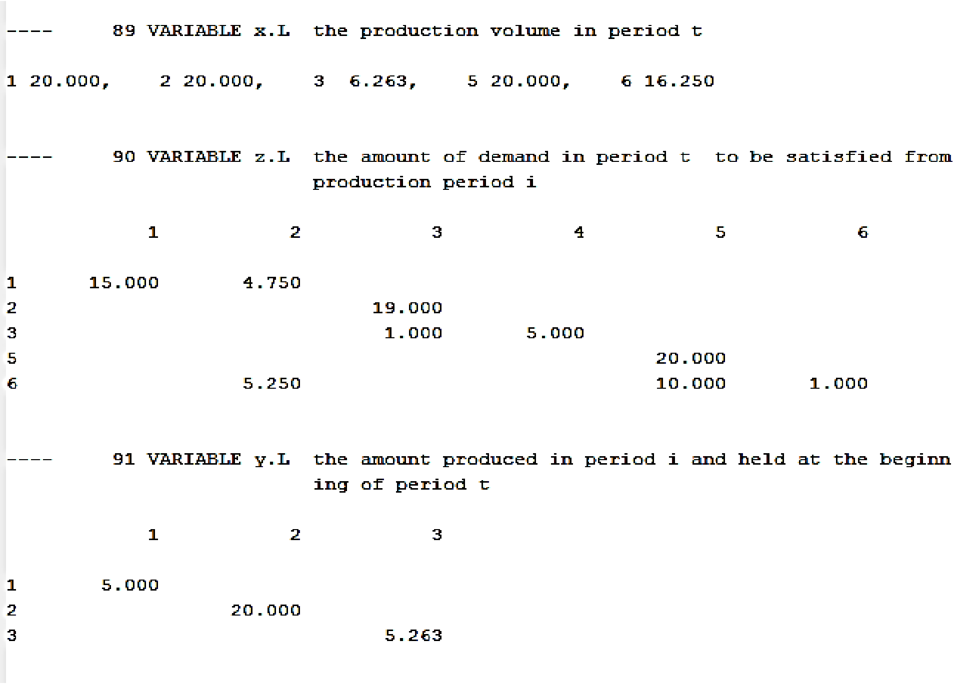
Data Set 1:

K=20

$B_{i+1,i}(y) = 5y;\ \ B_{i+2,i}(y) = 7y;\ B_{i+3,i}(y) = 10y,$

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6;\ \ H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5,$

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.



```
----     89 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    3  6.263,    5 20.000,    6 16.250


----     90 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      15.000       4.750
2                              19.000
3                               1.000       5.000
5                                                      20.000
6                   5.250                              10.000       1.000


----     91 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2           3

1       5.000
2                  20.000
3                               5.263
```

Figure 20 Optimal Solution of Data Set 1 in GAMS

The problem is solved using GAMS. The optimal solution is reported in Figure 20. According to this solution, IDP and ZIP do not hold for the optimal solution.

Demand periods are not satisfied by production periods in a consecutive order. Since $B_{i+k,i}(y)$ where $k \geq 4$ is assumed to be 0. However, it should be as $B_{i+4,i}(y) > B_{i+k,i}(y)$ where $1 \leq k \leq 3$. The solution consists of 2 components. Demand periods 1, 2, 5, and 6 belong to a component and other demand periods 3 and 4 are belong to other component.

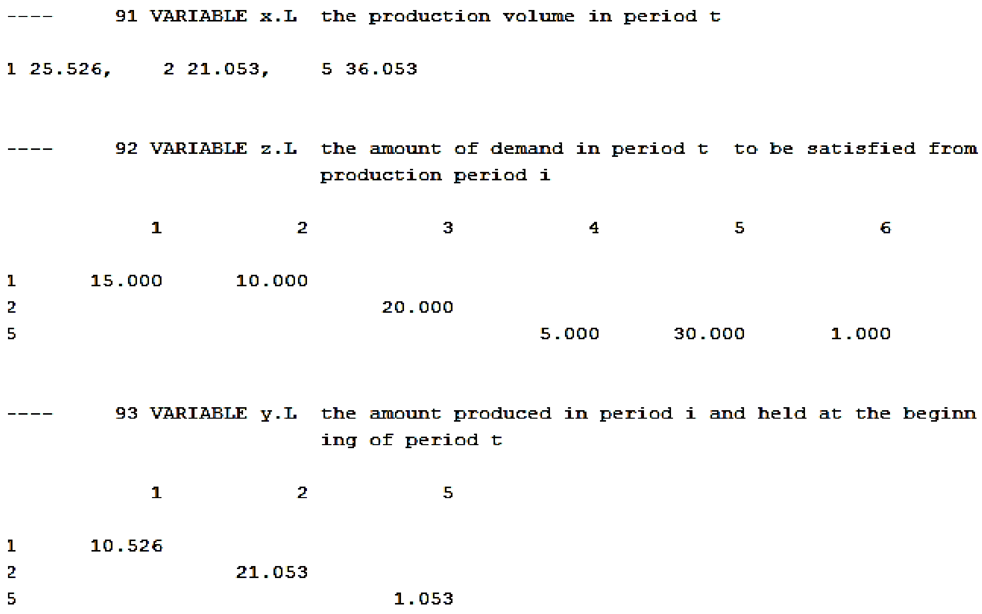
Data Set 2:

K=2000

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y,$

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6;\ H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5,$

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.



```
----      91 VARIABLE x.L  the production volume in period t

1 25.526,     2 21.053,     5 36.053


----      92 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      15.000      10.000
2                              20.000
5                                           5.000      30.000       1.000


----      93 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            1           2           5

1      10.526
2                  21.053
5                               1.053
```

Figure 21 Optimal Solution of Data Set 2 in GAMS

In the optimal solution, IDP holds where the penalty costs are increasing by age. That means demand periods are consecutive order whereas, ZIP does not hold in the problem.

Data Set 3:

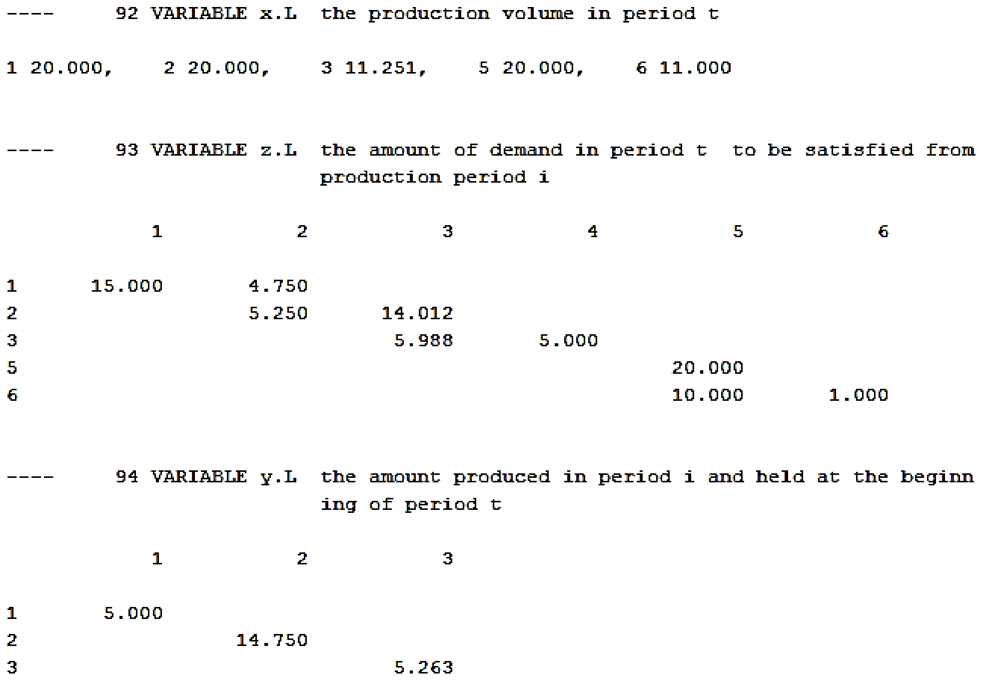
K=20

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y,$

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6;\ \ H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5,$

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.

In this example, it can be said that ZIP does not hold. Because, if Figure 22 is observed, the inventory form periods 1 and 2 is carried to production periods 2 and 3. According to ZIP the inventory cannot be carried to production periods.



```
----     92 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    3 11.251,    5 20.000,    6 11.000


----     93 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250      14.012
3                               5.988       5.000
5                                                      20.000
6                                                      10.000       1.000


----     94 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2           3

1       5.000
2                  14.750
3                               5.263
```

Figure 22 Optimal Solution of Data Set 3 in GAMS

Data Set 4:

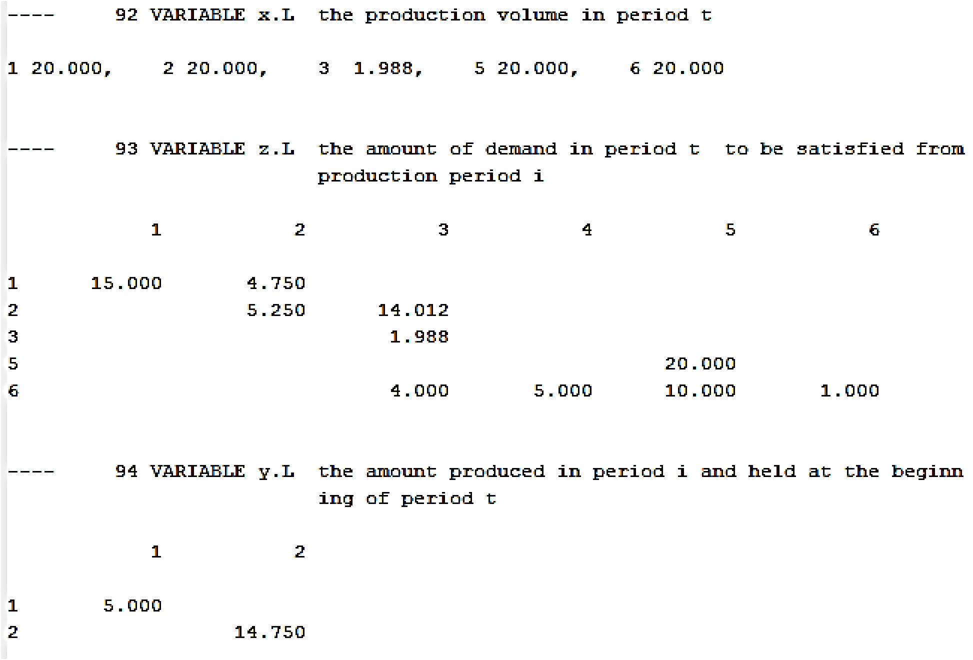
K=20,

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y,$

$H_{ii}(y) = 5y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6$; $H_{i,i+1}(y) = y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5$,

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.

For this example, ZIP does not hold. Since, inventory from periods 1 and 2 is carried to production periods 2 and 3.



```
----     92 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    3  1.988,    5 20.000,    6 20.000


----     93 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250      14.012
3                               1.988
5                                                      20.000
6                               4.000       5.000      10.000       1.000


----     94 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2

1       5.000
2                  14.750
```

Figure 23 Optimal Solution of Data Set 4 in GAMS

The optimal solution may change upon changing backlogging costs. Since, when the backlogging costs are changed, some different optimal solutions are obtained. Especially, IDP and Theorem 1 do not hold in Data Set 1. Since the production periods does not divided the periods in a consecutive order to meet the demand and also in some periods such as period 2, 3 and 5 the demand is not satisfied by only one periods production.

Additionally, Theorem 1 does not hold in Data Set 3 but Interval Division Property (IDP) is provided in both Data Set 2 and Data Set 3. In the example named Data Set 4, IDP does not hold since the holding costs are not formed as $H_{ii}(y) < H_{i,i+1}(y)$ and also ZIP is not provided.

4.4. The Feasibility Test

When we compare total demand and total production capacity of the problem for whole planning horizon, we can basically understand problem feasibility if the product is non-perishable. The problem with non-perishable product and allowing backlogging is feasible when $\sum_{i=1}^{n} d_i \leq nK$. However, if the product is perishable for the problem, feasibility checking is not easy because the amount of inventory decreases when time passes. Thus a basic table calculation can be helpful for checking feasibility. Thus, let us to show an example to check the feasibility.

Example 4.4.1: Assume that demands for six periods are 15, 10, 20, 5, 30, and 1 respectively. The loss rate of product for waiting a period in inventory is 0.05, the loss rate for two periods is 0.20 and the loss rate for three and more periods is 1. Backlogging is allowed and production capacity is 20 units.

$H_{ii}(y) = 5y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6$;

$H_{i,i+1}(y) = y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5$,

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.

Table 5 Test of Feasibility Example 4.4.1

to from	P 1 15	P 2 10	P 3 20	P 4 5	P 5 30	P 6 1	Max. amount of production
P 1	15	4.75					20
P 2		5.25	14				20
P 3			6	5	6,64		20
P 4					20		20
P 5					3,36	1	4,42
P 6							-

The table shows the amount satisfied either from production or inventory for each period. That refers Z values in the model for the problem. We assume production at each period at full capacity until unneeded. In the table, full capacity production until period 5 provides a feasible solution for the problem.

Here we use a first-in-first-out (FIFO) policy. We observe that amount produced in a period is used starting from the first needed period until it runs out. Since in this problem there is no objective function to find an optimal solution so optimality checking and the constraints (holding cost for holding inventory) which restricts the problem to achieve the optimality is not scope of this example. Therefore, if there are products in the inventory, firstly they are used to meet demand.

At first period 20 units are produced and 15 units of them are used for first period demand. The remaining 5 units are carried to second period but 5% of carried product are spoiled so 4.75 units are used for demand of period 2. Period 2 production meets the demand of period 2 and 3. The production of period 3 satisfies remaining part of demand of period 3, the total demand of period 4 and the part of demand of period 5. Therefore, some of remaining part of the production of period 3 is carried two periods and according to loss rate some products are perished. In period 5, there is no need to full capacity production. Since the production in period 4 satisfies the some part of demand of period 5 so 4.42 units are enough for demand of period 5 and 6. Moreover, it is already assumed that at the beginning period there is no inventory from previous periods and there is no inventory left at the end of periods.

4.5. Dynamic Programming Formulation

$V(m,r)$ is the minimum cost of satisfying the demand of periods 1,…,r by the production at periods 1,…,m. We write the following backward recursion.

$$V(m,r) = \min_{\substack{1\le s\le m \\ 1\le p\le r}} \{V(s-1,p-1) + P(s,m,p,r)\}.$$

$P(s,m,p,r)$ is the minimum cost of satisfying the demand of periods p,…,r by the production at periods s,…,m. In other words, it is the cost of the component (s, m, p, r).

We are looking for $V(n,n)$ as the optimal solution value of P.

This algorithm runs in $O(n^4)$ time if we know the cost of each component. Because this dynamic programming formulation corresponds to a shortest path problem on a network with $O(n^2)$ nodes and $O(n^4)$ arcs. The shortest path can be solved in the order of arcs time.

4.6. Calculating the Cost of a Component

4.6.1. Number of Z Arcs in a Component

N = number of positive production periods in the component + the number of demand periods in the component –1

To be connected we need to have N positive z values or z arcs in the reduced network. For a disconnected component we may have less N arcs.

$N = \theta_{pr} + 1 + r - p + 1 - 1 = \theta_{pr} + r - p + 1$

The minus 1 part is needed not to create a cycle in the part without the production arcs. θ_{pr} is the number of full production periods needed to satisfy demand periods from p to r.

For the component (s, m, p, r) we have $\binom{\mathrm{m-s+1}}{\theta_{\mathrm{pr}}}(\mathrm{m} - \mathrm{s} + 1 - \theta_{\mathrm{pr}})$ possible combinations of full production and zero production assignments. This value takes its maximum value s = 1 and $\theta_{\mathrm{pr}} = \left\lceil \frac{\mathrm{m-s}}{2} \right\rceil$.

Let us to consider a component (1, 6, 1, 6) and assume that the component has 3 full production period and 2 zero production periods. The remaining period produces between zero and the capacity. Therefore, we can say that the number of possible combinations of assignments is $\binom{6}{3}(3)$. $\binom{6}{3}$ gives the number of combinations for full capacity periods and this is multiplied by 3 because of the remaining period which is ε can be one of the other three periods. In that case, the number of combinations gets its maximum value. Since, if θ and s increase, the number of combination decreases. However, we ignore a component which has more than 2 empty periods where there is not production. If the number of combinations increases, the running time of algorithm also increases and if there are more than 2 empty periods in a component, in this case without these periods the production satisfies the demand so problem can be considered as uncapacitated. Therefore, number of total possible combinations in this study can be at most $\binom{\mathrm{n}}{\mathrm{n-3}}3$. This equals at most $O(n^3)$.

For a given combination, we have to find the best objective value of a combination. Thereafter, we need to the best one among all combinations to find the optimal solution of a component. Let $P_i(s, m, p, r)$ be the minimum cost of the ith combination then we can write the following equation.

$$P(s, m, p, r) = min_i P_i(s, m, p, r)$$

Now we will show how we calculate a good solution of a given combination in n(θ   + p – r + 1) time, which is $O(n^2)$. After our empirical analysis and Theorem 5, we propose an algorithm that finds a near optimal solution for a given component and combination.

4.6.2. Finding a Good Solution for component (s, m, p, r) for a Particular Combination

In this thesis, an algorithm is generated to solve the combinations of a component. Also, this thesis mainly focuses to solve these combinations or sub-problems. After solving possible combinations with our algorithm, the best results within solutions are selected for final solution.

We have converted the full capacity periods and zero production periods information into best z values for the component. The remaining period will be the ε period. In other words, identifying the full capacity and zero production periods helps identifying the best z values for the component and the only period that is not identified as a full or zero production period is the ε period.

We use a method similar to finding an initial solution to transportation problem such as the Northwest-Corner method. (For the Northwest-Corner Method see Winston, 1993).

Our algorithm finds a good solution for each combination of a component. By proving Theorem 5, it is wise to stay close to diagonal as much as possible and avoid intersections.

Initial Part:

First of all, we start from first demand and production period and then the production period is checked if it is ε or not. If production amount is not ε, we compare following unsatisfied demand and following unused production. After comparison, the smaller value is inserted to the table. Then, the inserted value is subtracted from both unsatisfied demand and unused production and we check if there is a positive remaining value in row or columns from comparison. If there is a positive remaining value in unused production, we calculate the loss and we use this value for further comparisons. Until period of ε, the algorithm tends to satisfy its current period demand first and then it satisfies demands for previous periods if exists any unsatisfied demand of these periods. When there are two or more periods for backlogging, the algorithm satisfies firstly the earlier period with unsatisfied demand which is at the beginning. In problem with concave cost function, this movement generally behaves from left to right of the table. Then, if any unused production remains, it is carried to next period.

When production amount is ε , it starts from last period to previous periods. If ε is in last period or there are no remaining periods, then calculate ε value by checking backorders and holding periods according to how many periods are satisfied by ε. If ε is not in last period, we insert smaller value of unsatisfied demand and unused production in the intersection of row and column. When the production period satisfies its own unsatisfied demand, the algorithm moves to previous production period for comparison. If the production period cannot satisfy its own demand, the algorithm checks whether there exists any unused production below. If such production exists, the algorithm uses that unused production first to satisfy demand via backlogging. If backlogging is not possible, it moves to previous production period and that production period firstly satisfy this further unsatisfied demand and then it satisfies its own demand. Therefore, algorithm satisfies demand of periods from right to left of rows up to its own period respectively in this case. It performs in this way until the skipped period where production amount is ε reached.

When the skipped period is reached, the algorithm starts to check whether any remaining unused production exists or not. If exits, algorithm puts the remaining products from nearest unsatisfied demand period to furthest periods. At the end, ε value is calculated by checking backorders and holding periods according to how many periods are satisfied by ε.

Second Part:

According to these steps, algorithm creates an initial solution and it looks for improvements. The improvement is provided by checking the values in the diagonal cells of the table. If there exists backlogged products and also there are not any carried products for period of ε, the algorithm starts to process as described again. However, in this case the value of ε is known. Therefore, when the period of ε is reached, the algorithm satisfies the demand of period of ε from its known production and then it continues to insert remaining unused productions to the nearest periods. This provides that unused productions are used for satisfying their own periods primarily and this sometimes helps to reduce costs because of cost structure. Finally, the results of the initial solution and second part solution are checked. The better one is used for the final solution.

Assume that P is the current production period and D is the current demand period. We know which period produces ε and which periods produce at full capacity.

Steps of the Algorithm:

Initial Part:

Step 1: Start from the first period of the component (s, m, p, r) that is P=s, D=p.

Step 2: If the production amount of P is equal to ε, go to Step 10.

Step 3: If the production amount of P is not equal to ε. Compare unsatisfied demand for D and unused production amount of P.

Step 4: Insert the smaller value to the cell (P, D) in the table.

Step 5: Update unsatisfied demand for D and unused production amount of P by subtracting the inserted value.

Step 6: If unsatisfied demand for D is 0 and unused production for P is 0 then there is no solution so go to the next period.

Step 7: If unused production for P is 0, move to next production and satisfy firstly its own period demand. Then, satisfy unused demand of previous periods from left to right if exist.

Step 8: If unused production for P is greater than 0, unused production for P is multiplied by loss rate to update P.

Step 9: Go to Step 2.

Step 10: If the production amount is ε, start from last period to previous periods. P=m, D=r.

Step 11: Compare unsatisfied demand for D and unused production amount of P.

Step 12: Insert the smaller values to the cell (P, D) in the table if there is no further unsatisfied demand.

Step 13: Update demand and production periods by subtracting the inserted value from unsatisfied demand for D and unused production amount of P.

Step 14: If unsatisfied demand for D is 0, then move to previous production period to satisfy its own demand.

Step 15: If unsatisfied demand for D is greater than 0, check if there is remaining unused production below. If exists, use them to satisfy demand via backlogging. If does not exists, move to previous production period and satisfy first demand of this period and then continue until its own period demand is satisfied.

Step 16: If period of ε is reached, insert all remaining unused production to the nearest period for unsatisfied demand.

Step 17: Calculate value of ε.

Step 18: $\varepsilon = a + b + e$ where

a is the total production amount to meet unsatisfied demands which equal to a when D=P.

b is the total production amount to meet unsatisfied backlogged demands which equal to b when D < P.

e is the total production amount to meet unsatisfied demands which equal to $e(1-\alpha_{ii})(1-\alpha_{i,i+1})\ldots(1-\alpha_{i,i+n})$ when D >P and n > 0.

Step 18.1: Calculate the total cost. This is the summation of production, inventory and backlogging costs.

Step 18.2: Calculate production cost. Production amounts are multiplied by production costs for each period. Let $c_i(x_i)$ refers the production cost for producing x_i unit at period i.

$$\sum_{i=S}^{m} c_i(x_i)$$

Step 18.3: Calculate inventory costs. Let $H_{it}(y_{it})$ refers the holding cost for holding y_{it} in the inventory. y_{it} is also kept in the right columns of the table. Moreover, it can be found in above of the diagonal line of the table by adding the loss. The equations for obtaining y values from z and x are also available in section 4.7 below.

$$\sum_{i=k}^{m} H_{it}(y_{it}) \quad \text{where i < t}$$

Step 18.4: Calculate backlogging costs. Let $B_{it}(Z_{it})$ refers the backlogging costs for backlogged product. Z_{it} is found in below of diagonal line of the table.

$$\sum_{i=k}^{m} B_{it}(Z_{it}) \quad \text{where i > t}$$

Second Part:

If period of ε satisfies demand of previous periods without satisfying its own period demand totally, algorithm performs the second stage.

Step 1: Start the algorithm with the value of ε computed in the initial part of the algorithm

Step 2: Use the production for demand of period of ε.

Step 3: Use the remaining unused production for unsatisfied demand periods according to nearest to furthest.

Example A:

We consider the problem when K =20,

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y$,

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6;\ \ H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5,$

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2.$

Demand values are 15, 10, 20, 5, 1, and 30. Production costs per period are 10, 40, 60, 80, 10, and 40 per units respectively. Total demand is 81 units; therefore we need at least 4 full capacity production periods. We consider some combinations and apply our method. Then, we solve with GAMS by giving the conditions. The bold periods are the full production periods and the circled one is the ε period.

The cost per unit per period according to parameters above is given in Table 6. In the table each cell gives the corresponding z value. Besides this, if we draw a diagonal line to the table, the values in lower triangle give the numbers of backordered products. Moreover, the cells of diagonal line of the table have the minimum unit cost of Z per row.

Table 6 The Unit Cost of Z for index pairs

	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
Period 1	10	11,6	20,7	-	-	-
Period 2	45	40	43,15	60,21	-	-
Period 3	67	65	60	64,21	86,51	-
Period 4	90	87	85	80	85,26	112,82
Period 5	24	20	17	15	10	11,6
Period 6	60	54	50	47	45	40

- **Combination 1 of Example A**

Table 7 Solution Table of Combination 1 of Example A:

To / From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i+1}$	$y_{i,i+2}$
P 1	15	4,75					~~20~~ ~~5~~	0	0
P 2		5,25	14				~~20~~ ~~14,75~~	0	0
P 3							0	-	-
P 4			6	5	1	6,04	~~20~~ ~~15~~ ~~9~~	7,55	0
P 5						3,96	ε=4,16	4,16	0
P 6						20	~~20~~	0	0
Unsatisfied Demand	~~15~~	~~10~~ ~~5,25~~	~~20~~ ~~6~~	~~5~~	~~1~~	~~30~~ ~~10~~ ~~3,96~~	-	-	-

Based on table, the flow of z values can be observed by checking which periods satisfy which periods demand.

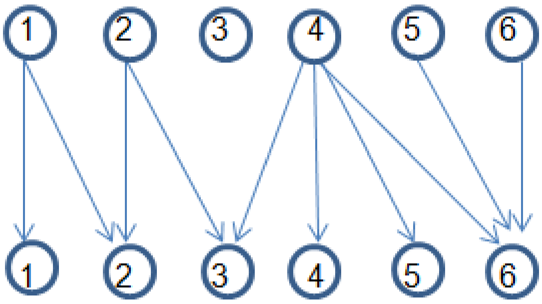
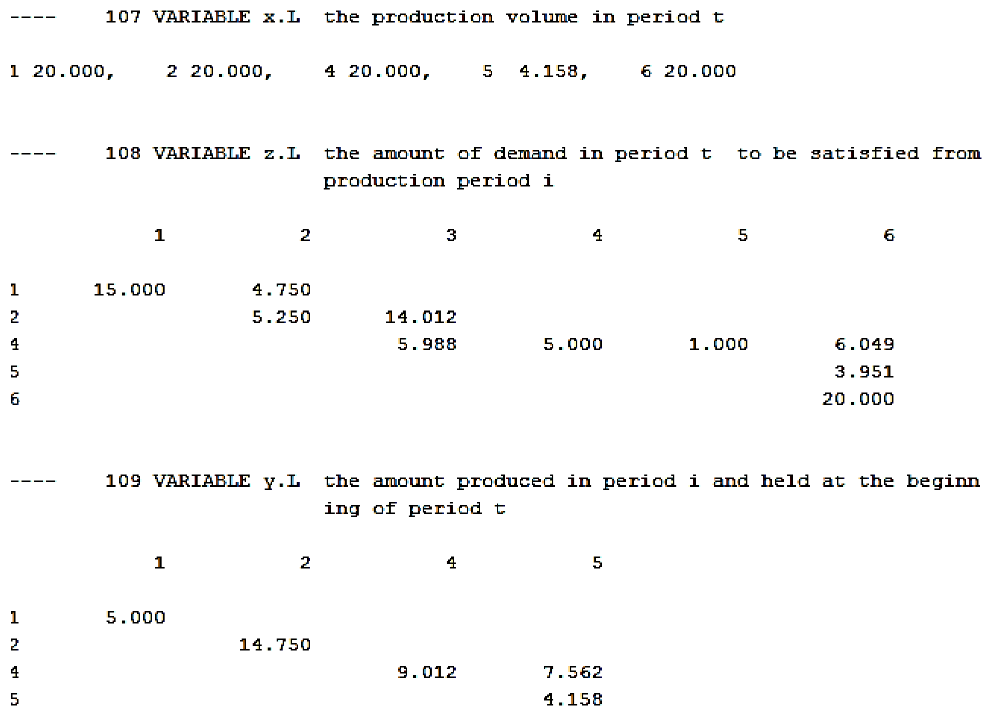


Figure 24 The Network Flow of Combination 1-A Based on Algorithm

In Figure 24, period 1 satisfies both demand of period 1 and period 2. Period 2 satisfies both demand of period 2 and 3. In period 4, demand of periods 3, 4, 5, and 6 are satisfied. Demand of period 6 is satisfied by production periods 5 and 6.



```
----    107 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    4 20.000,    5  4.158,    6 20.000

----    108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250      14.012
4                               5.988       5.000       1.000       6.049
5                                                                   3.951
6                                                                  20.000

----    109 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2           4           5

1       5.000
2                  14.750
4                               9.012       7.562
5                                           4.158
```

Figure 25 GAMS Result of Combination 1-A

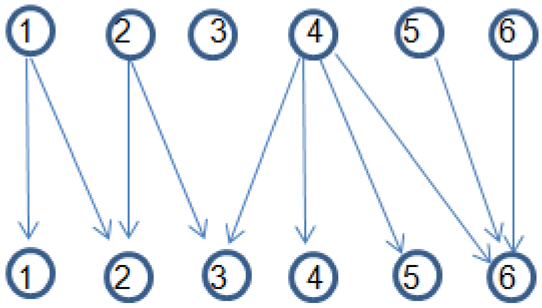


Figure 26 The Network Flow of Combination 1-A according to GAMS

Optimal solution identified by GAMS software gives the optimal objective function value which is the minimizing the costs gives 3542. Moreover, based on optimal solution, the flow of z values are shown in Figure 26. The flow of z values are same for both optimal solution identified by GAMS software and the algorithm. Decimal values are round up in the table of algorithm. Therefore, there may be slight differences between values.

- **Combination 2 of Example A:**

Table 8 Solution Table of Combination 2 of Example A

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	15	4,75					~~20~~ ~~5~~	5	0
P 2		5,25	14				~~20~~ ~~14,75~~	14,75	0
P 3			5,21				ε=5,21	-	-
P 4			0,79	5	1	10	~~20~~ ~~6,84~~ ~~5,79~~ ~~0,79~~	14,21	12,5
P 5							0	0	0
P 6						20	~~20~~	0	0
Unsatisfied Demand	~~15~~	~~10~~ ~~5,25~~	~~20~~ ~~6~~ ~~5, 21~~	~~5~~	~~1~~	~~30~~ ~~10~~	-	-	-

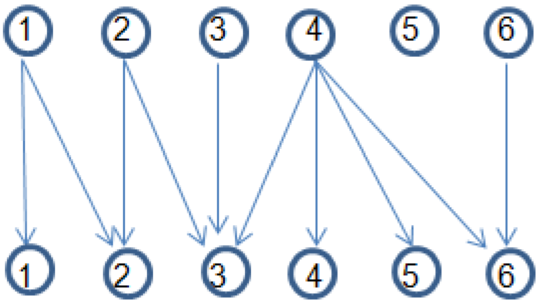
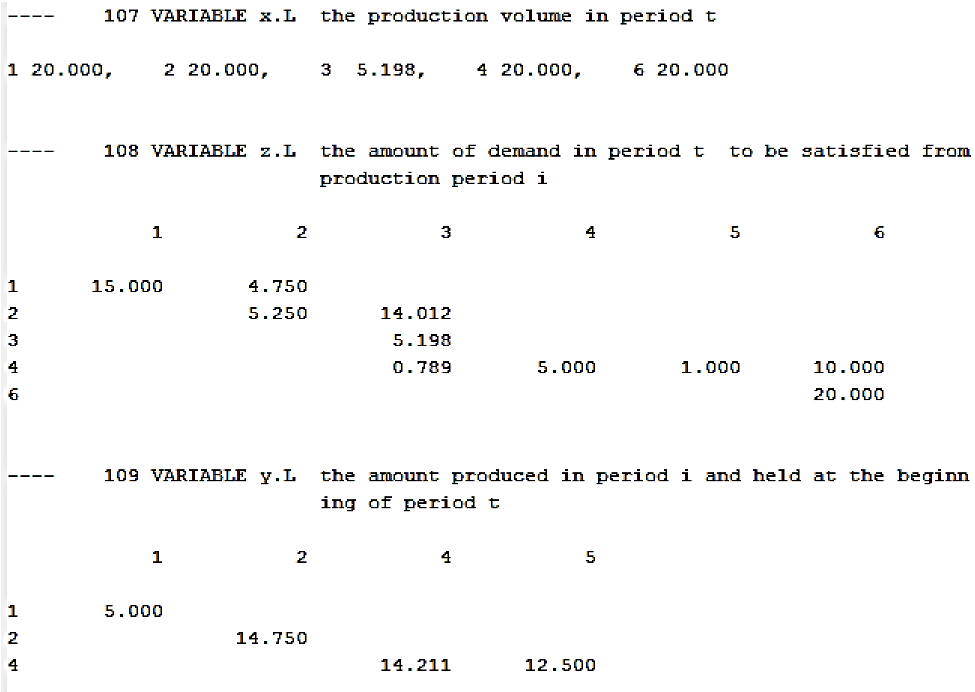


Figure 27 The Network Flow of Combination 2-A based on Algorithm

According to network flow in combination 2 of example A, demand of period 1 is satisfied by production period 1. Demand of period 2 is satisfied by the production periods 1 and 2. Periods 2, 3, and 4 satisfy the demand of period

3. Moreover, production period 4 satisfies also demand of period 4, 5 and 6. Some demand of period 6 is satisfied by period 6.



```
----     107 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    3  5.198,    4 20.000,    6 20.000


----     108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250      14.012
3                               5.198
4                               0.789       5.000       1.000      10.000
6                                                                  20.000


----     109 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            1           2           4           5

1       5.000
2                  14.750
4                              14.211      12.500
```

Figure 28 GAMS Result of Combination 2 - A

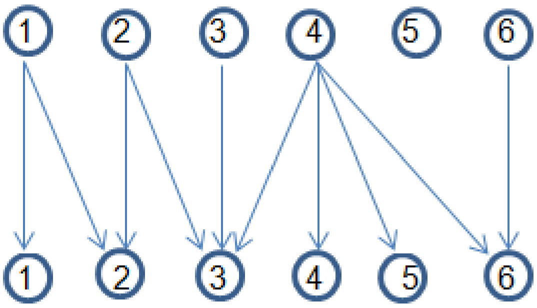


Figure 29 The Network Flow of Combination 2-A according to GAMS

When Table 8 and Figure 28 are compared and also Figures 27 and 29 are observed, the algorithm solution performs as optimal solution identified by GAMS software. Both optimal solution and algorithm find the same objective value. The difference in decimal values is slight since decimal values are round up to provide simple calculation in the algorithm.

- **Combination 3 of Example A:**

Table 9 Solution Table of Combination 3 of Example A

To From	P1	P2	P3	P4	P 5	P 6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P1	15	4,75					~~20~~ ~~5~~	5	0
P2		5,25	14				~~20~~ ~~14,75~~	14,75	0
P3							0	-	-
P4			2,5				ε=2,5	0	0
P5			3,5	5	1	10	~~20~~ ~~9,5~~ ~~8,5~~ ~~3,5~~	10,5	0
P6						20	~~20~~	0	0
Unsatisfied demand	~~15~~	~~10~~ ~~5,25~~	~~20~~ ~~6~~ ~~2,5~~	~~5~~	~~1~~	~~30~~ ~~10~~	-	-	-

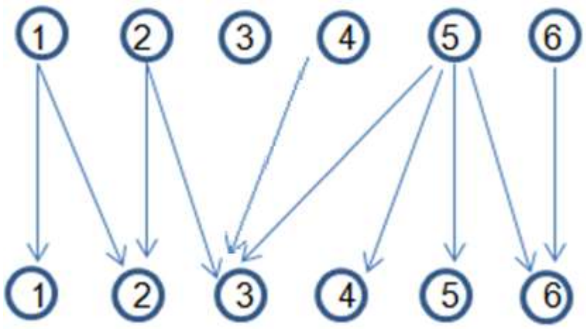


Figure 30 Initial Network Flow of Combination 3-A based on Algorithm

In this example, the period of ε satisfies the demand by backlogging without using the production for its own period. Therefore, by knowing the value of ε the second part of the algorithm performs.

Table 10 The Improved Solution Table of Combination 3 of Example A

To From	P1	P2	P3	P4	P 5	P 6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	15	4,75					~~20~~ ~~5~~	5	0
P 2		5,25	14				~~20~~ ~~14,75~~	14,75	0
P 3							0	-	-
P 4				2,5			ε=2,5	0	0
P 5			6	2.5	1	10	~~20~~ ~~9,5~~ ~~8,5~~ ~~6~~	10,5	0
P 6						20	~~20~~	0	0
Unsatisfied demand	~~15~~	~~10~~ ~~5,25~~	~~20~~ ~~6~~	~~5~~	~~1~~	~~30~~ ~~10~~	-	-	-

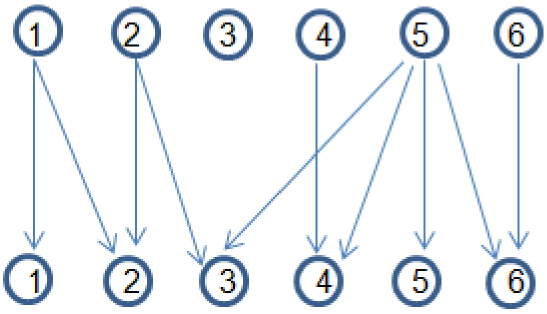
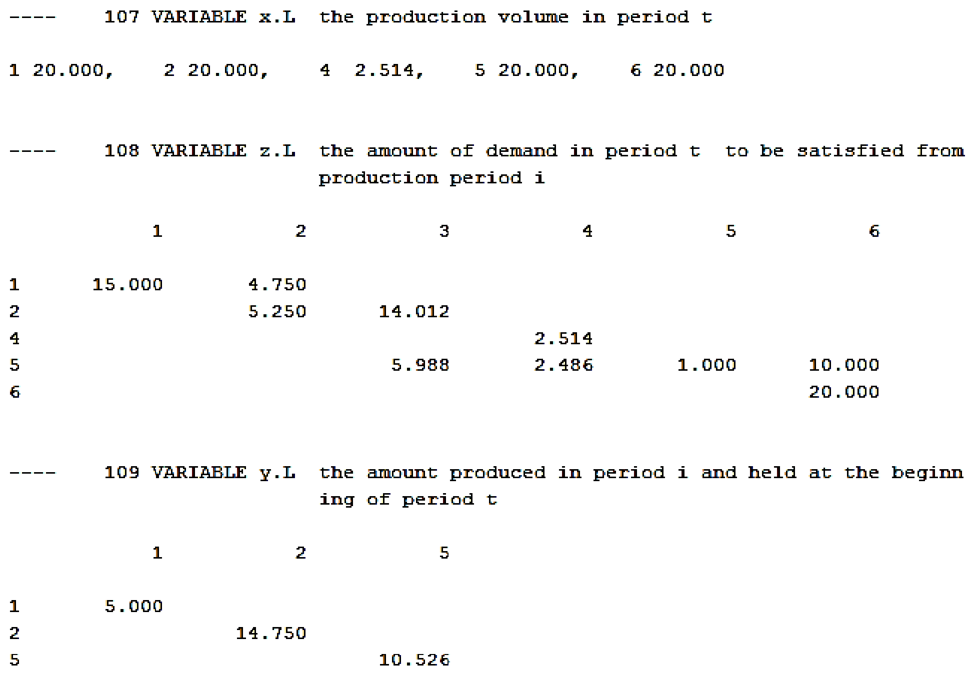


Figure 31 The Network Flow of Combination 3-A according to algorithm



```
----    107 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    4  2.514,    5 20.000,    6 20.000


----    108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250      14.012
4                                           2.514
5                               5.988       2.486       1.000      10.000
6                                                                  20.000


----    109 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            1           2           5

1       5.000
2                  14.750
5                              10.526
```

Figure 32 GAMS Result of Combination 3-A

The objective function value identified by GAMS software is 2285 approximately and our algorithm finds it different initially. If the optimal solution found by GAMS software and our improved table are checked, it is seen that the second part of the algorithm finds the optimal solution. Only decimal values change it slightly because of rounding up. Moreover, it can be said that Theorem 2 does not hold in both optimal solution and our algorithm.

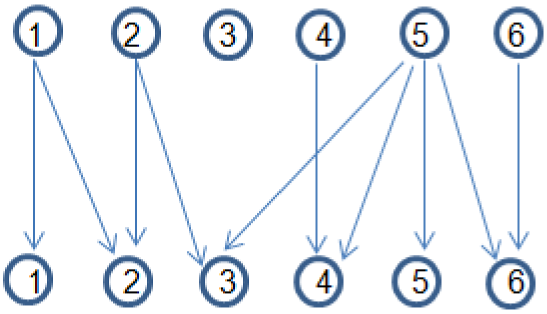


Figure 33 The Network Flow of Combination 3-A according to GAMS

- **Combination 4 of Example A:**

Table 11 Solution Table of Combination 4 of Example A

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	1,5						ε=1,5	0	0
P 2	10	10					~~20~~ ~~10~~	0	0
P 3							0	-	-
P 4	3,5		11,5	5			~~20~~ ~~15~~ ~~3,5~~	0	0
P 5			8,5		1	10	~~20~~ ~~9,5~~ ~~8,5~~	10,5	0
P 6						20	20	0	0
Unsatisfied demand	15	10	20	5	1	30	-	-	-

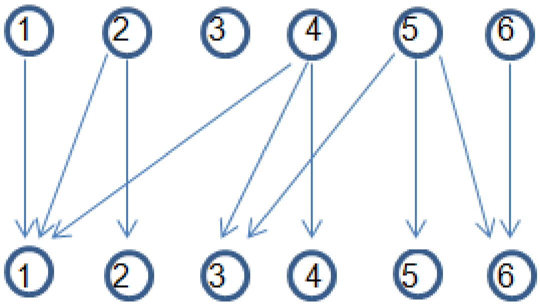
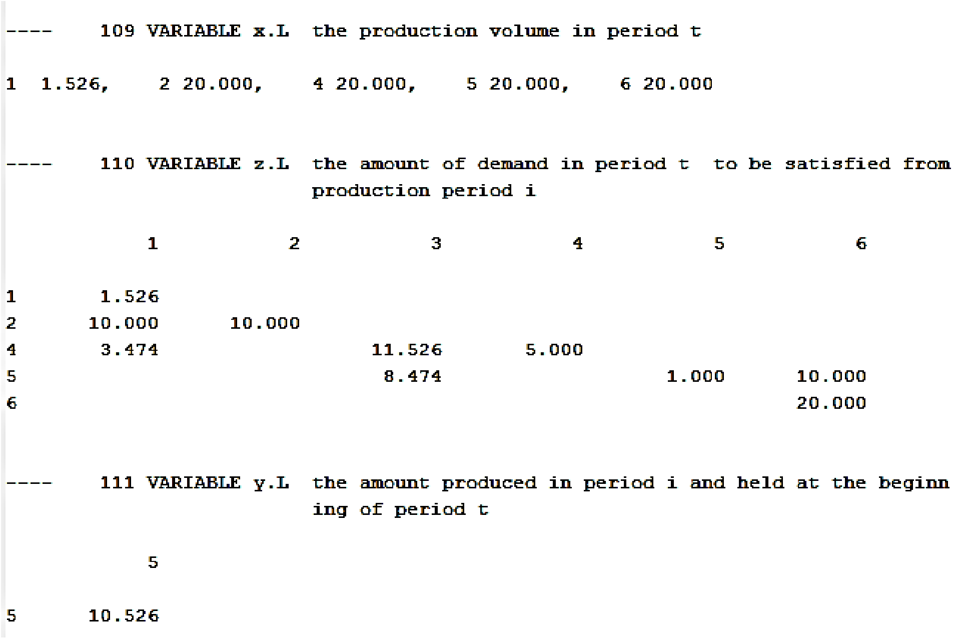


Figure 34 The Network Flow of Combination 4-A based on algorithm

The result of the algorithm shows that production periods 1, 2 and 4 are used to satisfy the demand of period 1 and also productions in period 2, 4, 5 and 6 satisfy their demand of period. Moreover, some of production in period 4 and 5 are used for period 3. Additionally, period 5 satisfies the demand of period 6.



```
----    109 VARIABLE x.L  the production volume in period t

1  1.526,    2 20.000,    4 20.000,    5 20.000,    6 20.000


----    110 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1       1.526
2      10.000      10.000
4       3.474                  11.526       5.000
5                               8.474                   1.000      10.000
6                                                                  20.000


----    111 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            5

5      10.526
```

Figure 35 GAMS Result of Combination 4-A

When optimal solution and algorithm are compared, it is seen that both have same solution.

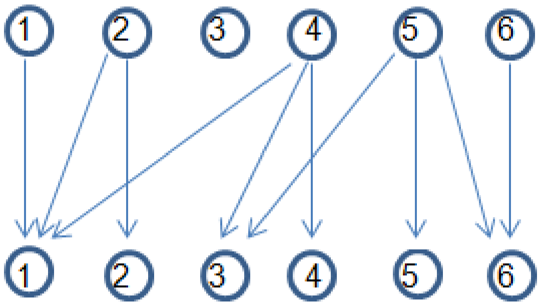


Figure 36 The Network Flow of Combination 4-A according to GAMS

Example B

We consider the parameters for this problem as:

K=20;

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y$,

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6$; $H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5$,

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$

$c_1 = 10,\ c_2 = 40,\ c_3 = 60,\ c_4 = 80\ c_5 = 10,\ c_6 = 40.$

Demands are different and 30, 5, 18, 10, 1, and 10 respectively. Thus, there are 3 full capacity production periods because of division of demand and capacity.

- **Combination 1 of Example B:**

Table 12 Solution Table of Combination 1 of Example B

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	10	5	4,75				~~20~~ ~~15~~ ~~5~~	5	0
P 3							0	0	0
P 4			13,25	1,5			ε=14,75	0	0
P 5				8,5	1	10	~~20~~ ~~9,5~~ ~~8,5~~	10,5	0
P 6							0	0	0
Unsatisfied demand	30	5	18	10	1	10	-	-	-

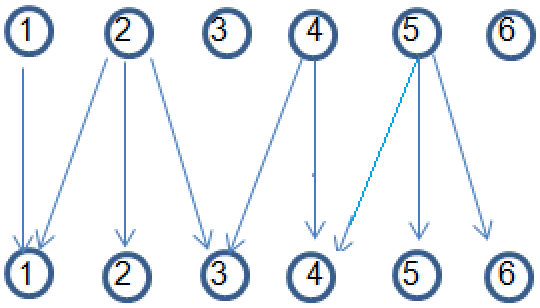


Figure 37 The Initial Network Flow of Combination 1-B based on Algorithm

In the table, period of ε which is period 4 satisfies the demand of period 3 by backlogging so the second part of algorithm performs to search a better solution.

Table 13 The Improved Solution Table of Combination 1 of Example B

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	10	5	4,75				~~20~~ ~~15~~ ~~5~~	5	0
P 3							0	0	0
P 4			4,75	10			ε=14,75	0	0
P 5			8,5		1	10	~~20~~ ~~9,5~~ ~~8,5~~	10,5	0
P 6							0	0	0
Unsatisfied demand	30	5	18	10	1	10	-	-	-

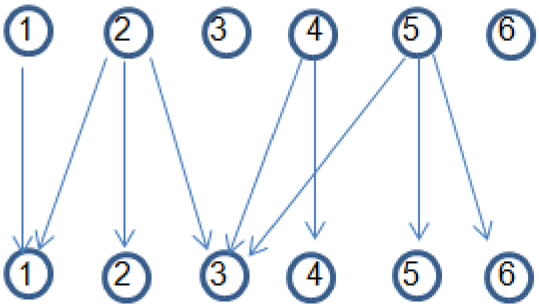
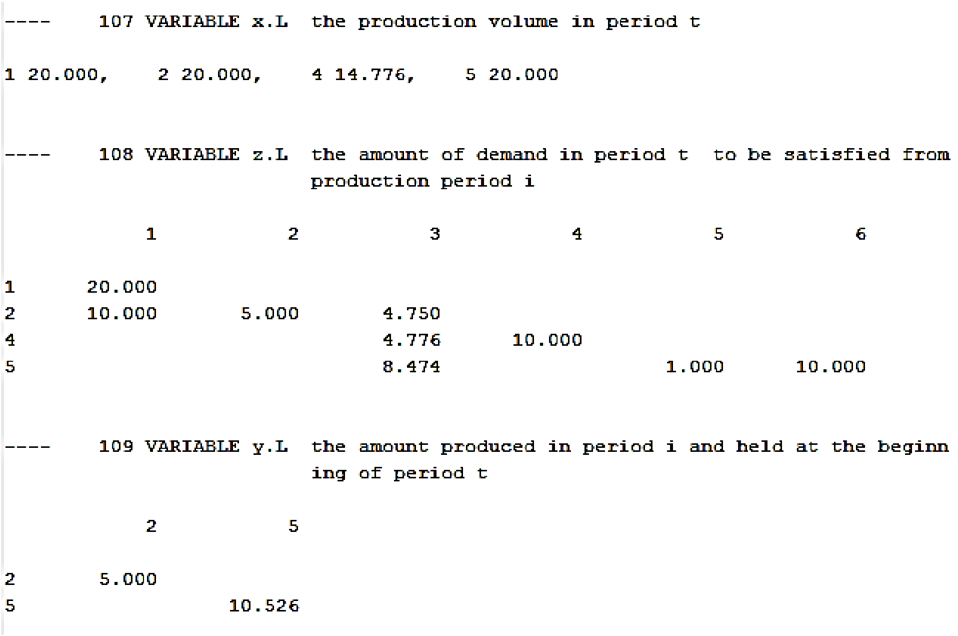


Figure 38 The Network Flow of Combination 1- B based on Algorithm

Second part of the algorithm helps to reach the optimal solution when network flows and solutions are compared. In this case, period 4 is firstly satisfies its own period demand and then it sends unused production to period 3 for backlogging.



```
----    107 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    4 14.776,    5 20.000


----    108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      20.000
2      10.000       5.000       4.750
4                               4.776      10.000
5                               8.474                   1.000      10.000


----    109 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            2           5

2       5.000
5                  10.526
```

Figure 39 GAMS Result of Combination 1-B

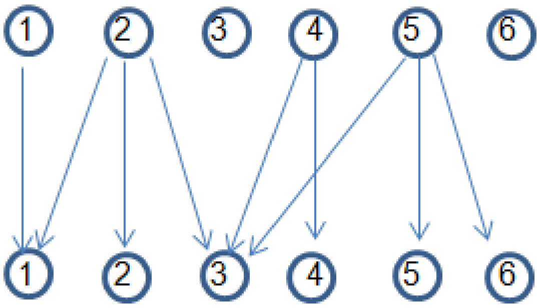


Figure 40 The Network Flow of Combination 1- B according to GAMS

Network flows of optimal solution and algorithm are same. The demand of period 1 is satisfied by production of periods 1 and 2. Period 2 also satisfies the demand of periods 2 and 3. Some part of demand in period 3 is satisfied by production of period 4 and 5. Moreover, period 4 is also used for period 4. Remaining part of production in period 5 is used for demand of periods 5 and 6.

- **Combination 2 of Example B:**

Table 14 Solution Table of Combination 2 of Example B

To From	*P1*	*P2*	*P3*	*P4*	*P5*	*P6*	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	10	4,5					ε=14,5	0	0
P 3							0	-	-
P 4		0,5	9,5	10			~~20~~ ~~10~~ ~~0,5~~	0	0
P 5			8,5		1	10	~~20~~ ~~9,5~~ ~~8,5~~	10,5	0
P 6							0	-	-
Unsatisfied demand	30	5	18	10	1	10	-	-	-

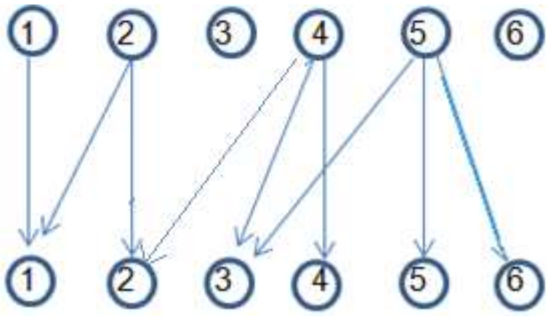


Figure 41 The Initial Network Flow of Combination 2-B based on Algorithm

As it can be seen in table the production of period of ε satisfies the previous period demand although demand of its own period is not satisfied totally. Therefore, the second part of the algorithm checks for the better solution if exists.

Table 15 The Improved Solution Table of Combination 2 of Example B

To / From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	9,5	5					ε=14,5	0	0
P 3							0	-	-
P 4	0,5		9,5	10			~~20~~ ~~10~~ ~~0,5~~	0	0
P 5			8,5		1	10	~~20~~ ~~9,5~~ ~~8,5~~	10,5	0
P 6							0	-	-
Unsatisfied demand	30	5	18	10	1	10	-	-	-

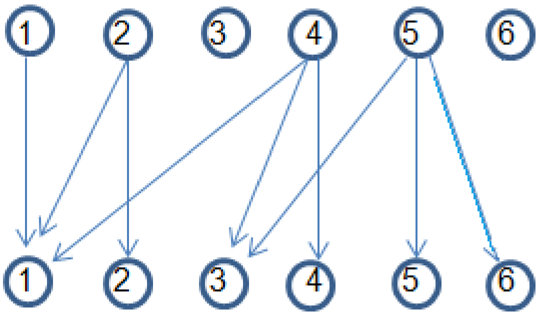
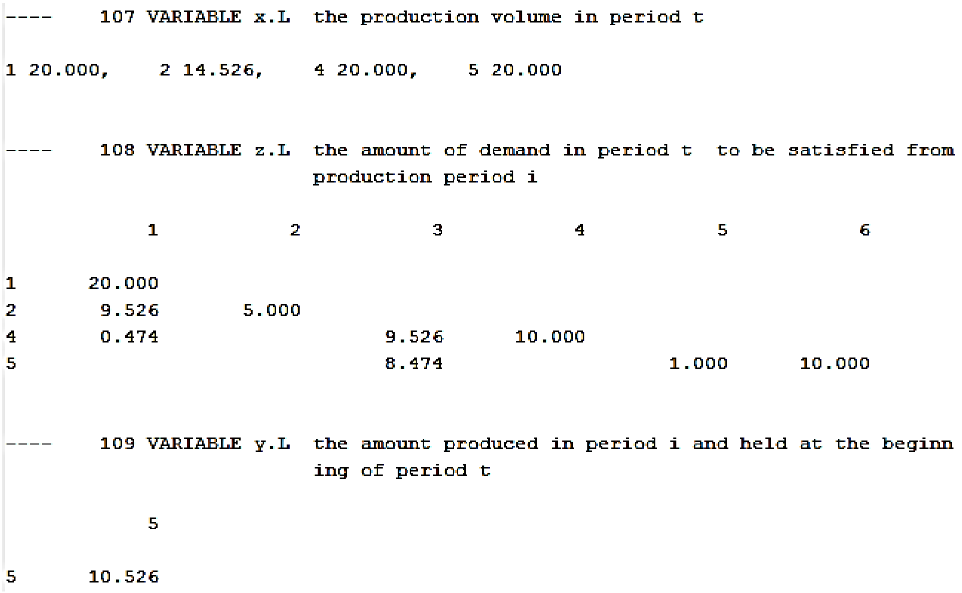


Figure 42 The Network Flow of Combination 2-B based on Algorithm

In combination 2 of example B, the demand of period 1 is satisfied by the production in periods 1, 2, and 4. Production of periods 2, 4 and 5 also satisfy their own demands. Moreover, period 3 is satisfied by some part of production in periods 4 and 5. Finally period 6 is satisfied by the production period 5.



```
----     107 VARIABLE x.L  the production volume in period t

1 20.000,    2 14.526,    4 20.000,    5 20.000


----     108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      20.000
2       9.526       5.000
4       0.474                   9.526      10.000
5                               8.474                   1.000      10.000


----     109 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            5

5      10.526
```

Figure 43 GAMS Result of Combination 2-B

If the Figure 43 and Table 15 are compared, it is seen that the production amount for period ε and the solutions are the same.

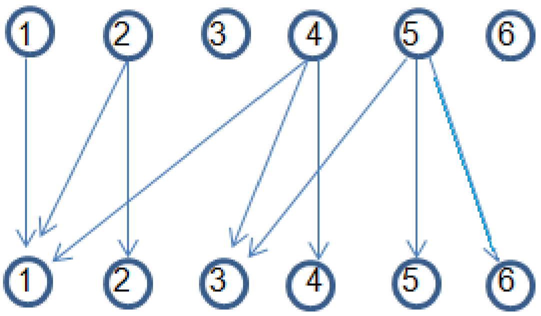


Figure 44 The Network Flow of Combination 2-B according to GAMS

- **Combination 3 of Example B:**

Table 16 Solution Table of Combination 3 of Example B

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	10	4					ε=14	0	0
P 3							0	-	-
P 4							0	0	0
P 5		1	18		1		~~20~~ ~~19~~ ~~1~~	0	0
P 6				10		10	~~20~~ ~~10~~	0	0
Unsatisfied demand	30	5	18	10	1	10	-	-	-

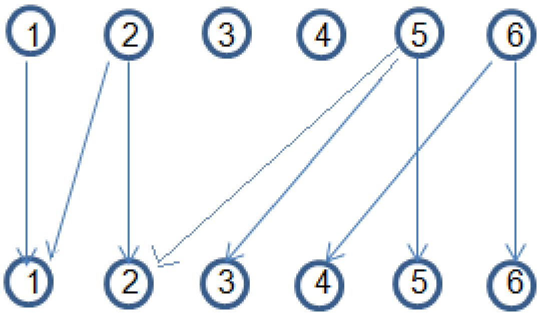


Figure 45 The Initial Network Flow of Combination 3-B based on Algorithm

Initially in the algorithm production in period 2 is used for demand of periods 1 and 2. However, period 2 performs backlogging without satisfying the demand of own period totally. Therefore, second part of the algorithm searches for better solution if exists.

Table 17 The Improved Solution Table of Combination 3 of Example B

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	20						~~20~~	0	0
P 2	9	5					ε=14	0	0
P 3							0	-	-
P 4							0	-	-
P 5	1		18		1		~~20~~ ~~19~~ ~~1~~	0	0
P 6				10		10	~~20~~ ~~10~~	0	0
Unsatisfied demand	30	5	18	10	1	10	-	-	-

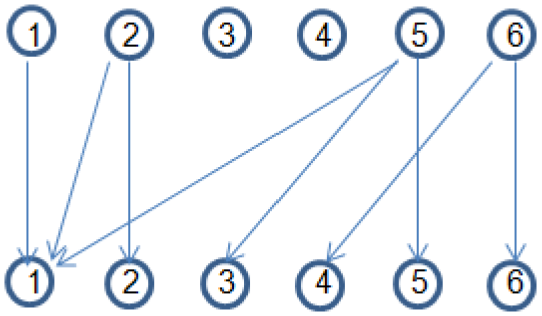
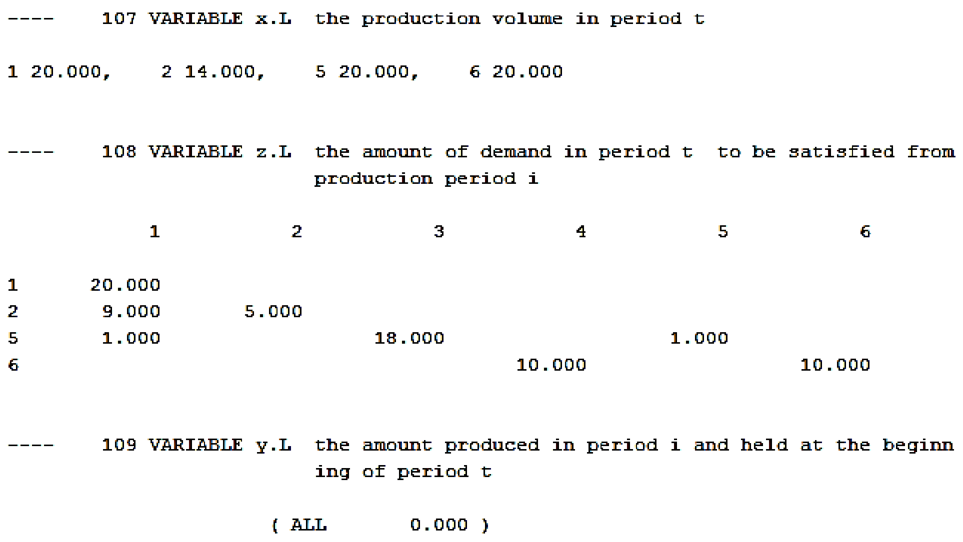


Figure 46 The Network Flow of Combination 3-B based on Algorithm

Keeping values in the diagonal line of the table sometimes helps to minimize total costs for this type of cost structure. The algorithm already tends to provide this aim in its progress. In this case, second part of the algorithm forces the period especially partial production period to satisfy its own period firstly. If the diagonal line is checked, it is seen that the algorithm put values in diagonal line of the table as much as possible. In addition, when the solutions are observed for this sample problem, the algorithm finds the optimal solution.



```
----     107 VARIABLE x.L  the production volume in period t

1 20.000,    2 14.000,    5 20.000,    6 20.000


----     108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      20.000
2       9.000       5.000
5       1.000                  18.000                   1.000
6                                          10.000                  10.000


----     109 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

                       ( ALL       0.000 )
```

Figure 47 GAMS Result of Combination 3-B

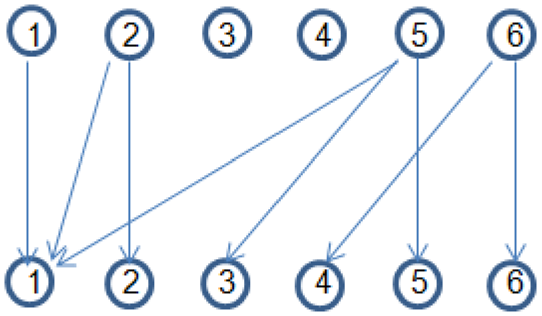


Figure 48 The Network Flow of Combination 3-B according to GAMS

When the final result of algorithm and optimal solution identified by GAMS software are investigated, it is observed that algorithm can find the optimal solution with the second part of it. Based on network flows, periods 1, 2, 5 and 6 are used for satisfying demands of their periods. Moreover, period 2 and 5 perform backlogging to satisfy the demand of period 1. Period 3 is satisfied by period 5 and period 4 is satisfied by period 6 with backlogging.

- **Combination 4 of Example B:**

Table 18 Solution Table of Combination 4 of Example B

To From	P1	P2	P3	P4	P5	P6	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	14,5						ε=14,5	0	0
P 2							0	0	0
P 3	2		18				~~20~~ ~~2~~	-	-
P 4	10			10			~~20~~ ~~10~~	0	0
P 5	3,5	5			1	10	~~20~~ ~~9,5~~ ~~8,5~~ ~~3,5~~	10,5	0
P 6							0	0	0
Unsatisfied demand	30	5	18	10	1	10	-	-	-

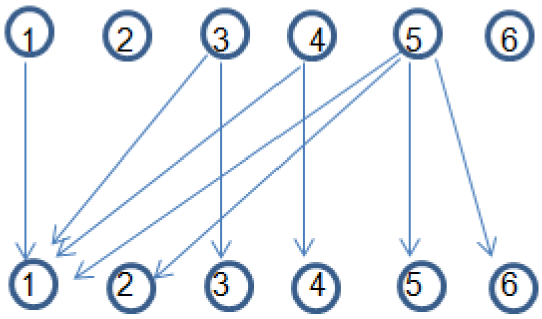
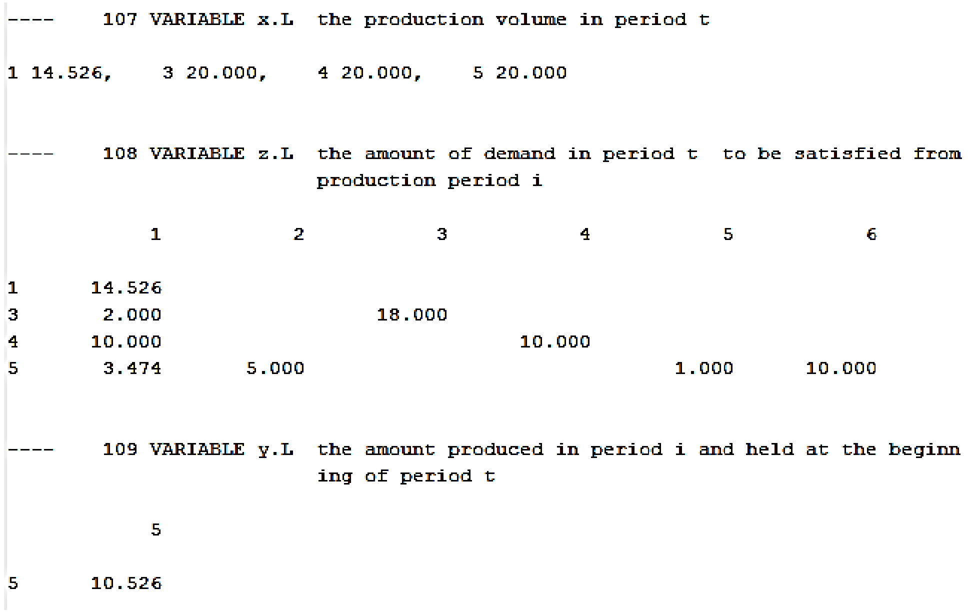


Figure 49 The Network Flow of Combination 4-B based on Algorithm

In this case, algorithm finds the optimal solution at first. Some part of demand in period 1 is satisfied by period 1 and remaining part is satisfied by periods 3, 4 and 5 by backlogging. Production periods 3, 4, and 5 are also used for satisfying their own demands. Moreover, period 5 satisfies the demand of period 2 by backlogging and it satisfies the demand of period 6 by holding inventory.



```
----     107 VARIABLE x.L  the production volume in period t

1 14.526,    3 20.000,    4 20.000,    5 20.000


----     108 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      14.526
3       2.000                  18.000
4      10.000                              10.000
5       3.474       5.000                                   1.000      10.000


----     109 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            5

5      10.526
```

Figure 50 GAMS Result of Combination 4-B

Optimal objective function value is 3368 and it is same with algorithm objective. It can be seen and calculated from Table 18 and Figure 50.

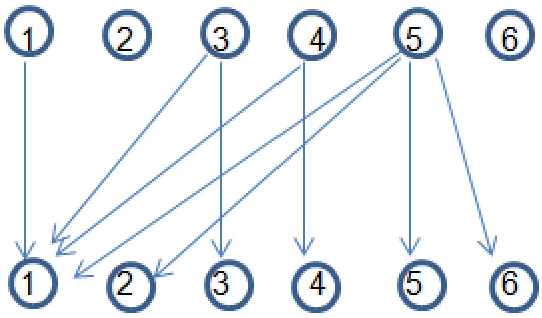


Figure 51 The Network Flow of Combination 4-B according to GAMS

4.7. Obtaining Y Values for Given X and Z

Additionally, constraints (1) and (2) in model (P) can be combined and y can be written in terms of x and z. If we know the values of the decision variables x and z, y values can be calculated.

$$y_{it} = \left\{ \begin{array}{ll} \left(x_i - \sum_{k=1}^{i} Z_{ik}\right)\prod_{k=i}^{t}(1-\alpha_{ik}) - \left(\sum_{k=i}^{t-2} Z_{i,k+1} \prod_{m=k+2}^{t}(1-\alpha_{im})\right) - Z_{it} & if\ i<t \\ x_i - \sum_{k=1}^{i} Z_{ik} & if\ i=t \end{array} \right\}$$

Proof:

The inventory left at the beginning of period t is found the equation below.

$$\mathrm{x_t} - \sum_{\mathrm{i}=1}^{\mathrm{t}} \mathrm{Z_{ti}} = \mathrm{y_{tt}} \qquad \text{where} \qquad 1 \leq \mathrm{t} \leq \mathrm{n}$$

Thus, y_{tt} can be calculated by $y_{tt} = (x_t - Z_{tt} - Z_{t,t-1} - Z_{t,t-2} \dots - Z_{t1})$.

Then, the inventory left at the beginning of next period from the production at period i is found by the equation below.

$$\mathrm{y_{it}} = \left(1 - \alpha_{i,t-1}\right)\ \mathrm{y_{i,t-1}} - \mathrm{Z_{it}} \qquad \text{where} \quad 1 \leq \mathrm{i} < t \leq n$$

Thus, when the values are written into the equation, $y_{i,t+1}$ can be calculated by $y_{t,t+1} = \left(x_t - Z_{tt} - Z_{t,t-1} - Z_{t,t-2} \ldots - Z_{t1}\right)(1 - \alpha_{tt}) - Z_{t,t+1}$.

Moreover, the inventory left at the beginning of forward period from the production at period i is found by the equation which is below.

$$y_{t,t+2} = \left(\left(x_t - Z_{tt} - Z_{t,t-1} - Z_{t,t-2} \ldots - Z_{t1}\right)\left(1 - \alpha_{t,t}\right) - Z_{t,t+1}\right)\left(1 - \alpha_{t,t+1}\right) - Z_{t,t+2}$$

If we continue in the same fashion, we will obtain the desired result.

As a result, the condensed form is obtain from the combination of constraints 4.1 and 4.2 in our model.□

Example derivation for obtaining y values: The inventory left at the beginning of period 5 from the production in period 3 is processed by stages.

First of all, the production is done for period 3. Then the demand for period 1, period 2, and period 3 are satisfied if there is backlogging. After this, the inventory left from period 3 is transformed to period 4 with some loss because keeping the product for one period makes some of them deteriorates. When period 4 is reached, the demand is met from the third period production which is inventory at hand. Then, the remaining products after meeting the demand are kept for period 5 with some loss. However, in this case the loss rate is different and high. Since keeping the products in inventory two periods causes a larger amount perished products than keeping them one period would. At last, the inventory left at the beginning of period 5 from the production in period 3 is found in this way.

Firstly, we calculate the inventory left at the beginning of period 3.

$$y_{33} = (x_3 - Z_{31} - Z_{32} - Z_{33})$$

Then, the inventory left at the beginning of period 3 is carried to period 4. However, some part of it is perished during period 3. Thus, the left at the beginning of period 4 is found after some of them are used for demand of period 4.

$$y_{34} = (x_3 - Z_{31} - Z_{32} - Z_{33})(1 - \alpha_{33}) - Z_{34}$$

Finally, the inventory left at the beginning of period 4 is carried to period 5. Again, some of inventory is perished according to deterioration rate. After satisfying demand of period 5, the inventory left at the beginning of period 5 is found.

$$y_{35} = \big((x_3 - Z_{31} - Z_{32} - Z_{33})(1 - \alpha_{33}) - Z_{34}\big)(1 - \alpha_{34}) - Z_{35}$$

A more condensed form of this equation is shown as:

$$y_{35} = \left(x_3 - \sum_{k=1}^{3} Z_{3k}\right)\prod_{k=3}^{5}(1 - \alpha_{3k}) - \left(\sum_{k=3}^{5-2} Z_{3,k+1} \prod_{m=k+2}^{5} (1 - \alpha_{3m})\right) - Z_{35}$$

The example can be extended for the inventory left at the beginning of period 6 from the production in period 3 as below.

$$y_{36} = \left(x_3 - \sum_{k=1}^{3} Z_{3k}\right)\prod_{k=3}^{6}(1 - \alpha_{3k}) - \left(\sum_{k=3}^{6-2} Z_{3,k+1} \prod_{m=k+2}^{6} (1 - \alpha_{3m})\right) - Z_{36}$$

Therefore, it can be seen that this provides the main equation which combines constraints 4.1 and 4.2 above.

Example C:

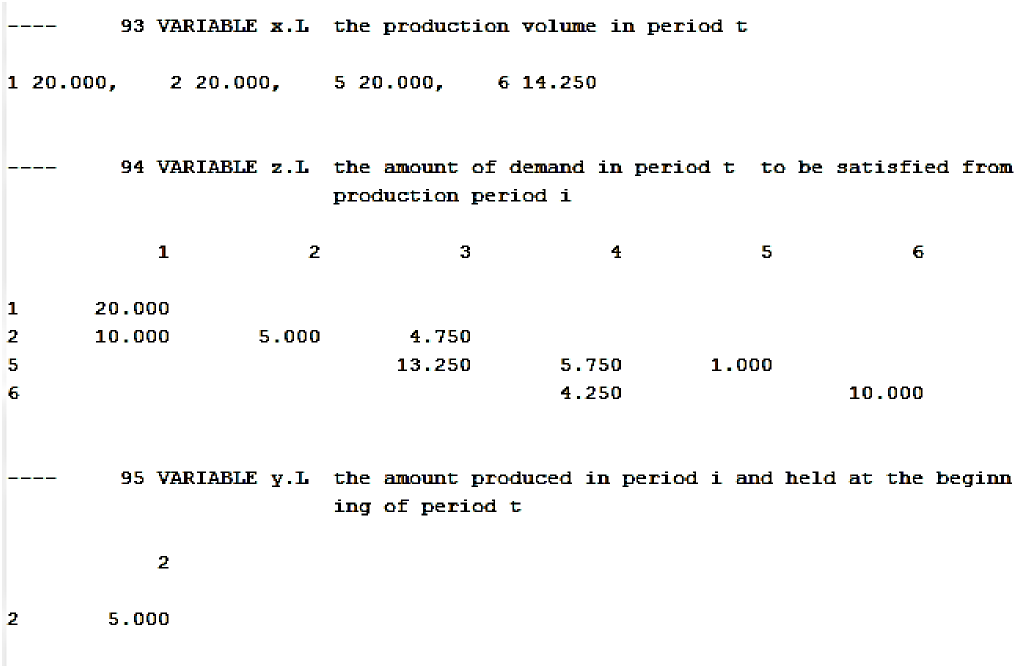
For example, demands according to periods are considered as follows $d_1 = 30, d_2 = 5,\ d_3 = 18,\ d_4 = 10,\ d_5 = 1, d_6 = 10$. The production costs are $c_1 = 10,\ c_2 = 40,\ c_3 = 60,\ c_4 = 80,\ c_5 = 10,\ c_6 = 40$. While the capacity limit per period which is K = 20, and the backlogging and holding costs with deterioration rate are as below:

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y$;

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6$; $H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5$;

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$.

Optimal objective function value of the problem which is the minimizing total costs equals to 1976 approximately.



```
----     93 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    5 20.000,    6 14.250


----     94 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                          production period i

            1           2           3           4           5           6

1      20.000
2      10.000       5.000       4.750
5                              13.250       5.750       1.000
6                                           4.250                  10.000


----     95 VARIABLE y.L  the amount produced in period i and held at the beginn
                          ing of period t

            2

2       5.000
```

Figure 52 GAMS Result of Example C

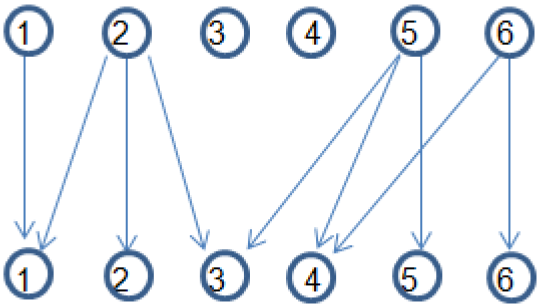


Figure 53 The Network Flow of Example C according to GAMS

Here conditions of Theorem 2 does not hold in the optimal solution identified by GAMS software since period 5 is satisfied from period 5 at the same time period 4 is satisfied by period 6. There is an intersection of z arcs, which was not observed in uncapacitated case.

Based on Theorem 4 we can say that the optimal solution is composed of a number of connected components. We can identify these parts by (n, m, p, r), where the production periods are n, …,m and the demand periods are p,…r. This means demand of the periods from p,…r are satisfied from the production in periods n, …,m. This part is connected meaning that some production periods satisfy multiple periods and some periods are satisfied by multiple production periods and when the flow of demand is drawn, it is seen that there is not any period which can be separated from other periods.

Therefore, Theorem 2 may not hold in the capacitated case. Meaning that if $\mathrm{i} < \mathrm{j}$ are two production periods and $Z_{jk} > 0$, for some $\mathrm{k} \geq \mathrm{j}$, then there can be $Z_{it} > 0$ for all *t*, where k ≤ t ≤ n.

Example D:

Through this example, it is shown that the number of production periods and demand periods do not have to be equal in an optimal solution. The number of production periods may be more than the number of demand periods in a component for some cases.

Assume that demands are 17, 10 and 5 units for period 1, 2 and 3 respectively. Production capacity is 20 units per period. By Theorem 4, there should be at least 2 full capacity production periods and remaining units should be produced at another suitable period which produces partial production between zero and the capacity.

The production costs per unit are determined as below.

$c_1 = 10,\ c_2 = 40,\ c_3 = 60, c_4 = 80.$

Moreover, the backlogging costs per unit are considered as follows.

$B_{i+1,i} = 5, B_{i+2,i} = 7,\ B_{i+3,i} = 10\ .$

The holding costs per period are $H_{i+1,i} = 1,\ H_{i+2,i} = 5.$

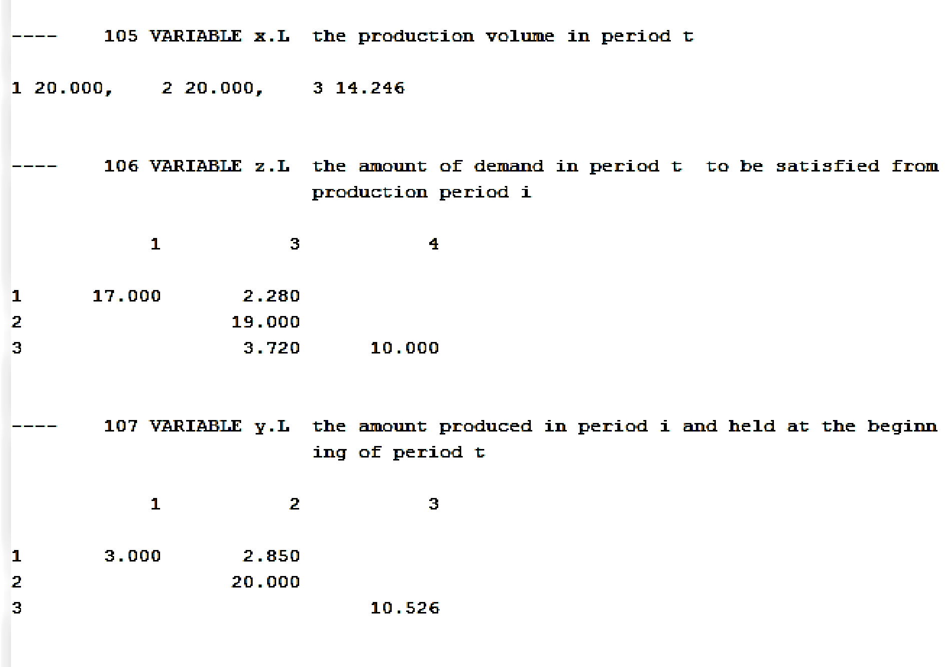
$\alpha_{ii} = 0.05$ for $1 \leq i \leq 6;$ $\alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5;$

$\alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$

- **Combination 1 of Example D:**

Table 19 Solution Table of Combination 1 of Example D

To From	**P1**	**P2**	***P3***	P4	Unused Production Amount	$y_{i,i}$	$y_{i,i+1}$
P 1	17		2,28		~~20~~ ~~3~~ ~~2,85~~	3	2,85
P 2			19		~~20~~ ~~19~~	20	0
P 3			3,72	10	ε=14,22	0	0
P 4					0	0	0
Unsatisfied Demand	~~17~~	0	~~25~~ ~~22,72~~ ~~3,72~~	~~10~~	-	-	-



```
----     105 VARIABLE x.L  the production volume in period t

1 20.000,    2 20.000,    3 14.246


----     106 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           3           4

1      17.000       2.280
2                  19.000
3                   3.720      10.000


----     107 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            1           2           3

1       3.000       2.850
2                  20.000
3                              10.526
```

Figure 54 GAMS Result of Combination 1-D

Optimal solution identified by GAMS software and solution of algorithm are consistent. The objective value is 1902.

Example E:

In this example, demand values are 15, 10, 20, 5, 1, and 30. Moreover, the production costs are assumed that $C_1(x) = x,\ C_2(x) = 4x,\ C_3(x) = 6x,\ C_4(x) = 8x,\ C_5(x) = x,\ C_6(x) = 4x$. Also, K= 20 units per period.

Backordering costs and holding costs are period-pair dependent and also the cost matrices and deterioration rates are given below.

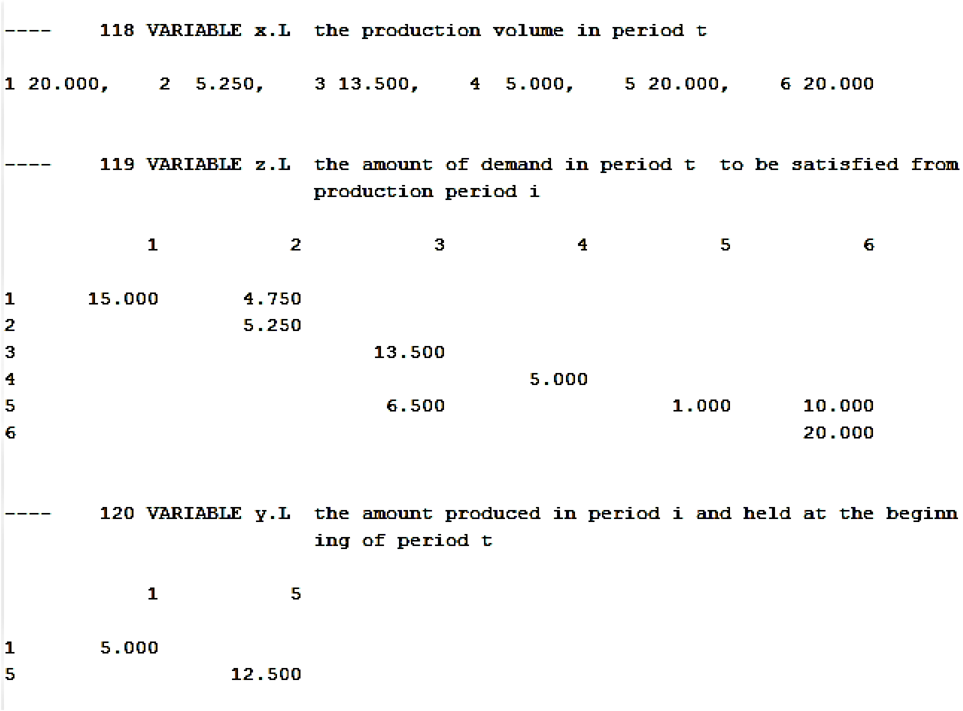
The solution is included and it shows that Theorem 2 may not be hold in the solution for problems with period-pair dependent cost structure.

$$\alpha = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0.05 & 0.20 & 1 & 1 & 1 & 1 \\ 2 & \mathrm{N/A} & 0.10 & 0.25 & 1 & 1 & 1 \\ 3 & \mathrm{N/A} & \mathrm{N/A} & 0.15 & 0.50 & 1 & 1 \\ 4 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 0.30 & 0.40 & 1 \\ 5 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 0.20 & 0.20 \\ 6 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 0.10 \end{array}$$

$$H_{it}(y) = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 25 & 125 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 2 & \mathrm{N/A} & 50 & 125 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 3 & \mathrm{N/A} & \mathrm{N/A} & 30 & 40 & \mathrm{N/A} & \mathrm{N/A} \\ 4 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 15 & 200 & \mathrm{N/A} \\ 5 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 26 & 100 \\ 6 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & 40 \end{array}$$

$$B_{it}(y) = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 2 & \sqrt{100y} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 3 & \sqrt{200y} & \sqrt{196y} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 4 & \sqrt{400y} & \sqrt{250y} & \sqrt{120y} & \mathrm{N/A} & \mathrm{N/A} & \mathrm{N/A} \\ 5 & \sqrt{625y} & \sqrt{300y} & \sqrt{200y} & \sqrt{169y} & \mathrm{N/A} & \mathrm{N/A} \\ 6 & \sqrt{700y} & \sqrt{400y} & \sqrt{350y} & \sqrt{256y} & \sqrt{196y} & \mathrm{N/A} \end{array}$$

The feasible solution identified by GAMS solver shows that a demand period is satisfied by multiple production periods but Theorem 4 does not hold in the feasible solution. Since, there are two components and for each component there is not a unique partial production period which is between zero and the capacity. However, it should be considered that the result identified by GAMS solver below is not an optimal solution.



```
----     118 VARIABLE x.L  the production volume in period t

1 20.000,    2  5.250,    3 13.500,    4  5.000,    5 20.000,    6 20.000


----     119 VARIABLE z.L  the amount of demand in period t  to be satisfied from
                           production period i

            1           2           3           4           5           6

1      15.000       4.750
2                   5.250
3                              13.500
4                                           5.000
5                               6.500                   1.000      10.000
6                                                                  20.000


----     120 VARIABLE y.L  the amount produced in period i and held at the beginn
                           ing of period t

            1           5

1       5.000
5                  12.500
```

Figure 55 Feasible Solution for Example E in GAMS

Example F

Through this basic example, how dynamic programming algorithm will behave or solve a problem is mentioned. Consider a 3-period problem.

$C_1(x) = x,\ C_2(x) = 4x,\ C_3(x) = 6x,\ C_4(x) = 8x,\ C_5(x) = x,\ C_6(x) = 4x$, K=15,

$B_{i+1,i}(y) = 5y,\ B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y,$

$H_{ii}(y) = y, \quad \alpha_{ii} = 0.05 \quad \text{for } 1 \le i \le 6;$

$H_{i,i+1}(y) = 5y, \quad \alpha_{i,i+1} = 0.2 \quad \text{for } 1 \le i \le 5,$

$H_{i,i+k}(y) = +\infty, \quad \alpha_{i,i+k} = 1 \quad \text{for } 1 \le i \le 3 \text{ and } k \ge 2.$

Demands are 10, 12 and 15 units per period respectively. Therefore, we are looking for minimum total cost which is V(3,3).

Dynamic Programming divides a problem into components with the equation below.

$$V(m,r) = \min_{\substack{1\leq s\leq m \\ 1\leq p\leq r}}\{V(s-1,p-1)+P(s,m,p,r)\}.$$

Moreover, the possible combinations of each component are solved by the algorithm which does not guaranteed to find the optimal solution. Then, best combinations which give the minimum cost are selected for final solution.

To find V(3,3) we have three alternative ways since some of them are infeasible which are shown as bold type. Before to figure out the value of V(3,3), V(2,2) should be explored firstly. In equations below, to find the value of V(2,2), V(1,1) should be explored at first since we know the value of V(0,0) which is zero and the components value can be find by the algorithm step by step.

- V(3,3)= $\min\begin{pmatrix} \mathrm{V}(2,2)+\mathrm{P}(3,3,3,3) \\ \mathbf{V}(\mathbf{1},\mathbf{2})+\mathbf{P}(\mathbf{2},\mathbf{3},\mathbf{3},\mathbf{3}) \\ \mathbf{V}(\mathbf{2},\mathbf{1})+\mathbf{P}(\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{3}) \\ \mathrm{V}(1,1)+\mathrm{P}(2,3,2,3) \\ \mathrm{V}(0,0)+\mathrm{P}(1,3,1,3) \end{pmatrix}$

- V(2,2)= $\min\begin{pmatrix} \mathrm{V}(1,1)+\mathrm{P}(2,2,2,2) \\ \mathrm{V}(0,0)+\mathrm{P}(1,2,1,2) \end{pmatrix}$

- V(1,1)= (V(0,0)+P(1,1,1,1))

The value of components identified by our algorithm:
P(1,1,1,1) is 100.
P(2,2,2,2) is 600.
P(3,3,3,3) is 900.
Component (2,3,2,3) has two combinations since one period produces at capacity, other period produces partial. So, P(2,3,2,3) is the minimum value of combinations which are 1332 and 1380.
Component (1,3,1,3) has three combinations since any two periods produce at capacity and one of them produces partial. So, P(1,3,1,3) is the minimum value of combinations 1221, 1585, and 1345.

Finally, the component (1,2,1,2) has two combinations and P(1,2,1,2) is the minimum value of 445 and 685.
To select best components, we should solve the equations below.

V(1,1)= (V(0,0)+P(1,1,1,1)) = 0+100= 100

$$V(2,2)= min\begin{pmatrix}V(1,1)+P(2,2,2,2)\\ V(0,0)+P(1,2,1,2)\end{pmatrix}=\begin{cases}100+600=700\\ 0+445=445\end{cases}$$

$$V(3,3)= min\begin{pmatrix}V(2,2)+P(3,3,3,3)\\ V(1,1)+P(2,3,2,3)\\ V(0,0)+P(1,3,1,3)\end{pmatrix}=\begin{cases}445+900=1345\\ 100+1332=1432\\ 0+1221=1221\end{cases}$$

In the end, 1221 is selected by dynamic programming since the component (1,3,1,3) gives the minimum results. The algorithm which gives minumum result is below and it gives the optimal solution whwn it is compared with the result of GAMS solver. The only difference is occured because of rounding decimal values up.

Table 20 The Basic Table of the Algorithm for Example F

	P1	P2	P3	Production amount
P1	10	4,75		15
P2		7,25	7,36	15
P3			7,64	$\varepsilon = 7{,}64$
Demand	10	12	15	

4.8. Experimental Results and Algorithm Performance

Theoretically, computational complexity of the algorithm is O(n^9) time, since there are O(n^4) components, each component has O(n^3) possible combinations and the cost calculation for component and combination pair takes O(n^2) time. This running time is a very high upper bound. O(n^3) is calculated considering components with at most two empty periods. We ignore other combinations which have more than 2 empty periods.

However, the algorithm generally terminates in fewer steps than the worst case we provided. Since, some components are infeasible so we do not consider all of them and also for cost calculation the algorithm generally does not have to check all cells one by one in the table while assigning values.
Based on experimentation, the algorithm can be considered as well but it cannot always find the optimal solution. Therefore, we can say the algorithm serves an approximation solution for these complex problems. 6 data sets are generated to analyse the algorithm performances. The results are available in Tables 21, 22, 23, 24, 25 and 26. For all data sets, data are common except demands and at most 6-period problem is considered. Data sets differ by the demand values. The deterioration rate depends on age. Holding and backlogging costs are also nondecreasing concave and age-dependent. Capacity is constant per period. Moreover, production costs are assumed general concave. Thus, the parameters for all data sets are considered as below.

- Demand for data set 1 is in increasing order where demands are 10, 15, 17, 20, 25, and 27 units per period respectively.

- Demand of data set 2 is in decreasing order and demands are 21, 17, 15, 12, 10, and 8 units per period respectively.

- Demands are 16 units and 17 units per period for data set 3 and 4 respectively.

- Demand of data set 5 is 13, 14, 15, 16, 19, and 25 units per period respectively. Demands are in increasing order for data set 1 but these values are assigned to analyse the performance of the algorithm when there exists a period where there is not any production. Since according to these values, for some combinations there exists a period where there is no production.

- Demand of data set 6 is 15, 10, 20, 5, 1, and 30 units per period. Demand values fluctuate in this data set so they are not in increasing or decreasing order.

The average gap 1 (GAP1) in the following tables is the average of percentage of gap based on total costs. The average gap 2 (GAP2) in the following tables is the average of percentage of gap based on holding, backlogging and production in period of ε costs.

$$\mathrm{GAP1}_i = \left(\frac{Result_i\ of\ algorithm - Optimal\ result_i}{Optimal\ result_i}\right)100$$

$$\mathrm{GAP2}_i = \left(\frac{(Result_i\ of\ algorithm - production\ costs\ _i) - (\ Optimal\ result_i - production\ costs\ _i)}{(\ Optimal\ result_i - production\ costs\ _i)}\right)100$$

The production costs in equation GAP2 include the costs of production periods where there exists full production. Production in period ε is not included in the equation GAP2. Since the production cost for this period may change and affect the total cost because of possibility of varying from solution to solution.

The following parameters are common for all data sets.

K=20

$B_{i+1,i}(y) = 5y, B_{i+2,i}(y) = 7y, B_{i+3,i}(y) = 10y, B_{i+4,i}(y) = 14y, B_{i+5,i}(y) = 20y$,

$H_{ii}(y) = y,\ \alpha_{ii} = 0.05$ for $1 \leq i \leq 6$; $H_{i,i+1}(y) = 5y,\ \alpha_{i,i+1} = 0.2$ for $1 \leq i \leq 5$,

$H_{i,i+k}(y) = +\infty, \alpha_{i,i+k} = 1$ for $1 \leq i \leq 3$ and $k \geq 2$

$c_1 = 10,\ c_2 = 40,\ c_3 = 60,\ c_4 = 80\ c_5 = 10,\ c_6 = 40$

Table 21 Result of Data Set 1 – increasing demand

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	4	4	-	-
5x5	5	5	-	-
6x6	6	6	-	-

Table 22 Result of Data Set 2 – decreasing demand

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	4	4	-	-
5x5	20	20	-	-
6x6	30	26	0,175	2,36

Table 23 Result of Data Set 3 – constant demand 1

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	4	4	-	-
5x5	5	5	-	-
6x6	30	24	0,135	1,107

Table 24 Result of Data Set 4 – constant demand 2

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	4	4	-	-
5x5	5	5	-	-
6x6	6	5	0,19	5,03

Table 25 Result of Data Set 5 –increasing demand 2

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	12	12	-	-
5x5	20	20	-	-
6x6	6	6	-	-

Table 26 Result of Data Set 6 – fluctuate demand

Component size	Number of Combinations	How many times optimal solution is identified	Average GAP1 (%) If not optimal	Average GAP2 (%) If not optimal
2x2	2	2	-	-
3x3	3	3	-	-
4x4	4	4	-	-
5x5	5	5	-	-
6x6	6	5	0.19	7.2

In tables 21, 22, 23, 24, 25, and 26 component numbers and sizes, number of cases where algorithm finds optimal solution, and the average gap based on total costs between solution of algorithm and optimal solution if exists and also average gap between algorithm and optimal solution while considering holding, backordering and production for period of ε costs are shown.

According to results, the algorithm performance can be considered as promising. The average gap is shown based on percentage. According to total demand, if there exists a period where there is no production, it can be said that the algorithm sometimes gives the approximation solution. However, when the demand is increasing order and there does not exist a period where production is not available, it can be said that the algorithm is promising. Since the average gap has been occurred generally if there exists a period where there is no production. The average gaps based on total cost in the tables are less than 1%. However, if it is considered that full capacity production periods are constant, they don’t change for both algorithm and optimal solution. Therefore, if the holding, backlogging and the production for period of ε costs are compared for both algorithm and optimal solution results, the gap increases. Moreover, it can be said that according to results we could not find a demand pattern to get optimal solution all the time with our algorithm.

Additionally, it is observed that the algorithm performs well when the holding costs are expensive as much as backordering costs. In some cases, the costs may be considered in this way. Since the type of warehouses for perishable products especially for foods should be cold one and the cost of this type of warehouses may be expensive so holding inventory costs may be expensive as much as paying penalty for backlogging.

CHAPTER 5

CONCLUSION AND FUTURE WORK

In this study, dynamic lot sizing models are analysed. Products are considered as single and perishable. Furthermore, all costs are determined as nondecreasing concave. Backordering costs are included and assumed as age-dependent. Moreover, inventory holding costs are also age-dependent and deterioration rate differs based on age.

Besides general case, the problems that are uncapacitated and capacitated versions of general case are investigated. A mathematical model is developed to minimize total costs for both capacitated and uncapacitated versions.

Additionally, it is observed that some structural properties that can be exploited for developing a solution approach for uncapacitated problems are not valid for capacitated problems. We define a different sub-problem and identify the structural properties of the optimal solution and the sub-problems. We show that DP is a valid approach to solve the sub-problems. We devise an algorithm that finds good solutions to sub-problems, some of which are optimal. Objective values for the sub-problems found by the algorithm and GAMS are analysed. The algorithm finds optimal solution in most cases. Therefore, the algorithm can be promising for complex problems. However, there are some cases the algorithm cannot reach the optimal solution.

Therefore, it is appeared that this dynamic programming algorithm gives the approximate solution for the problems with these sub-problems.

In experimental results, we considered data sets with different demand patterns to investigate the behaviour of our algorithm for finding optimal solution all the time but according to results of data sets we could not find such a relationship between the demand pattern and the performance of the algorithm.

Finally, this study can be continued for future work by improving the algorithm to obtain an optimal solution for every time. As an improvement, after algorithm solves a problem and finds an initial solution, the position of some z values in the table can be changed by checking the cost table. This means that some additional steps may be included in the algorithm and these steps may help to find a cycle in the table of initial solution. Shifting some values as a cycle in the table by considering cost table may reduce the initial total costs and may help to find the optimal solution in the end.

REFERENCES

Ahuja, R.K., Magnanti, T.L. and Orlin, J.B. (1993). Network Flows: Theory, Algorithms and Applications. New Jersey: Prentice Hall.

Akartunalı, K. (2007). “Computational Methods for Big Bucket Production Planning Problems: Feasible Solutions and Strong Formulations,” PhD Dissertation, Industrial Engineering, University of Wisconsin, MADISON.

Bazaraa, M.S. and Shetty, C.M. (1979). Nonlinear Programming: Theory and Algorithms. New York: John Wiley & Sons, Inc.

Bitran, G.R. and Yanasse, H.H. (1982). “Computational Complexity of the Capacitated Lot Size Problem,” *Management Science*, 28(10), 1174-1186.

Chan, L.M.A., Muriel, A., Shen, Z.J. and Simchi-Levi, D. (2002). “On the Effectiveness of Zero-Inventory-Ordering Policies for the Economic Lot-Sizing Model with a Class of Piecewise Linear Cost Structure,” *Operations Research,* 50(6), 1058-1067.

Chu, L.Y., Hsu, V.N. and Shen, Z.J.M. (2005). “An Economic Lot-Sizing Problem with Perishable Inventory and Economies of Scale Costs: Approximation Solutions and Worst Case Analysis,” *Naval Research Logistics*, 52(6), 536–548.

Florian, M. and Klein, M. (1971). “Deterministic Production Planning with Concave Costs and Capacity Constraints,” *Management Science*, 18(1), 12-22.

Graves, S.C. (2002). Manufacturing Planning and Control. In: Pardalos, P., Resende, M. ed., *Handbook of Applied Optimization*, Oxford University Press.

Hsu, V.N. (2003). “An Economic lot Size Model for Perishable Products with Age-Dependent Inventory and Backorder Costs,” *IIE Transactions,* 35, 775-780.

Hsu, V.N. (2000a). “Dynamic Economic Lot Size Model with Perishable Inventory,” *Management Science*, 46(8), 1159-1169.

Hsu, V.H. (2000b). “An Economic Lot Size Model for Perishable Products with Age-Dependent Inventory and Backorder Costs,” Working paper, School of Management, George Mason University.

Hsu, V.N. and Lowe T.J. (2001). “Dynamic Economic Lot Size Models with Period-Pair-Dependent Backorder and Inventory Costs,” *Operations Research*, 49(2), 316-321.

Prastacos, G. P. (1981). “Allocation of a Perishable Product Inventory,” *Operations Research*, 29(1), 95-107.

Sargut, F.Z. (2006). “Efficient Approaches to Integtated Requirements Planning Problem in Supply Chain Optimization,” PhD Dissertation, University of Florida.

Veinott, A.F. (1963). Unpublished notes, Report, Stanford University, Stanford, California.

Wagner, H.M. and Whitin T.M. (1958). “Dynamic Version of the Economic Lot Size Model,” *Management Science*, 5, 89–96.

Winston, W.L. (1993). Operations Research Application and Algorithms. Belmont, California: Duxbury Press.

Wu, T., Shi, L., Akartunalı, K., and Geunes, J. (2011). “An optimization framework for solving capacitated multi-level lot-sizing problems with backlogging,” *European Journal of Operational Research*, *214*(2), 428–441.

Yano, C.A. and Lee, H.L. (1995). “Lot Sizing with Random Yields- A Review,” *Operations Research,* 43(2), 311-334.

Zangwill, W.I. (1968). “Minimum Concave Cost Flows in Certain Networks,” *Management Science*, 14, 429–450.

Zangwill, W.I. (1969). "A Backlogging Model and a Dynamic Economic Lot Size Production System - A Network Approach," *Management Science*, 15, 506–527.