SIGNATURE-BASED RELIABILITY ANALYSIS OF SYSTEMS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

SIGNATURE-BASED RELIABILITY ANALYSIS OF SYSTEMS

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The concept of system signature is a useful tool in a variety of applications including the evaluation of the reliability characteristics of systems, and the comparison of the performance of competing systems. In this thesis, we aim to study the reliability properties of the generalization of consecutive type systems in the context of system signature. In particular, we will obtain signature-based expressions for reliability characteristics of systems and investigate their properties. Stochastic comparison of different systems will also be studied. A combinatorial formula, which calculates the exact reliability of a generalized system, is given. Signature based analysis are illustrated and numerics are provided.

Keywords: Consecutive k-out-of-n:F system; Consecutive k-within-m-out-of-n:F system; m-consecutive-k-out-of-n:F; Exchangeable lifetimes; Failure rate; Mean residual life; Mixtures; Signature; Stochastic ordering.

ÖZ

SİSTEM İMZASINA DAYALI GÜVENİLİRLİK ANALİZİ

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Sistem imzası kavramı, sistemlerin karakteristiklerinin güvenilirlik değerlendirmesini ve hesaplama sistemlerinin performans karşılaştırmasını içeren çeşitli uygulamalarda kullanılabilecek yararlı bir araçtır. Bu tezde, sistem imzası kavramı dâhilinde genelleştirilmiş ardıl sistemlerin güvenilirlik karakteristiklerini çalışılmayı amaçlamaktayız. Bu çalışma kapsamında, sistemlerin güvenilirlik karakteristiklerine yönelik imzaya dayalı ifadeler elde edilecek ve özellikleri araştırılacaktır. Farklı sistemlerin stokastik karşılaştırması ayrıca bu çalışma kapsamına dâhil edilecektir. Genelleştirilmiş sistemin tam güvenilirlik oranını hesaplayan kombinasyonel formül bu çalışmada sunulmuştur. Sistem imzası kullanılarak yapılan analizler tablolarla görselleştirilmiş ve sayısal değerler verilmiştir.

Anahtar Kelimeler: Ardıl n'den k'lı F sistemler; Ardıl n'den m içinde k'lı F sistemler; Simetrik bağımlı yaşam zamanları; Bozulma oranı; Ortalama Yaşam süreleri; Karışık sistemler; İmza; Stokastik Sıralama.

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.

To memory of my beloved grandma,

AYŞE MELAHAT ÖZDİLEK

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Chapter 1

Introduction

Developing technology leads to increase in the number of complex systems as well as complexity inside of the systems. Remarkable achievements made communications, electrics and electronics systems became more and more sophisticated. Thus, the scientists such as engineers and applied probabilists have an interest in these systems. Overall, new methods and techniques have been developed to secure the maximum effectiveness of such systems. In the mean of these developments, reliability theory was introduced and the first journal, which is aware of this, was IEEE-Transaction on Reliability (1963).

One can define reliability in terms of probability such that a mechanism which implements its functions sufficiently under a specific time and conditions. In terms of device, it can be defined as a system consisting of many components or performance of a component in a system. Probability theory can be used for analyzing the reliability of components besides the reliability of systems consisting of these components. Since the performance of system heavily depends on the performance of each components, the reliability of a system can be called as a function of reliability of its components.

Evaluating reliability has a crucial importance at every steps of processing and controlling engineering systems. For evaluating the system's reliability, one should specify the structure of the system that defines the rule(s) of the operation and relations between the system components. Early works on system reliability have focused on binary system modeling.

In the literature, various reliability models have been defined and studied under different assumptions on components. Undoubtedly, the simplest reliability structures are series and parallel models. A series system with n components operates if all components operate. A parallel system of n components operates if at least one component is in a working state. A k-out-of-n:F system, which consists of n components, fails if at least k of n components fail. On the other hand, k-out-of-n:G system, which consists of n components, functions if at least k of n components operate. A linear consecutive k-out-of-n:F system as a generalization of series and parallel systems, consists of n linearly ordered components such that the system fails if at least k consecutive components fail. A linear consecutive k-out-of-n:F system usually has much higher reliability then the series systems and is less expensive then the parallel systems. As a dual of consecutive k-out-of-n:F system, a consecutive k-out-of-n:G system with n components operates if at least k consecutive components operate. Consecutive type systems have been used to model telecommunication and oil pipeline systems, and vacuum systems in accelerators. Recent discussions on consecutive k-out-of-n systems appear in the works of Yun et al. [88], Xiao et al. [87], Eryilmaz [27], Navarro and Eryilmaz [65], Eryilmaz [28], Eryilmaz [29]. An excellent review of such systems and their generalizations is presented in Kuo and Zuo [58].

Throughout this thesis, the binary state systems are assumed which means that the system and its components may either work or fail. Thus the state of each component or system is a discrete random variable with two possible outcomes. In nonseries systems, it is not necessary that all components must operate for functioning of systems. So the relationship between components and system are investigated by coherent systems. One can easily define a system containing relevant components with a nondecreasing structure function as coherent system. If X_i denotes the state of the ith component in the system, then

$$x_i = \begin{cases} 1 & \text{if the } i\text{th component functions} \\ 0 & \text{if the } i\text{th component fails} \end{cases}$$

for $i = 1, 2, \ldots, n$, where n is the number of components in the system. Similarly, ϕ, which denotes the state of the system, can be defined as

$$\phi(x_1, x_2, \ldots, x_n) = \phi(\vec{x}) = \begin{cases} 1 & \text{if system functions} \\ 0 & \text{if system fails} \end{cases}$$

The function $\phi(\vec{x})$, which is called structure function of system, shows the state of system as a function of states of components. The component i is said the be irrelevant if and only if

$$\phi(1_i, \vec{x}) = \phi(0_i, \vec{x}) \text{ for all } (\cdot_i, \vec{x}) = (x_1, x_2, \ldots, x_{i-1}, \ldots, x_{i+1}, \ldots, x_n)$$

If there exists at least one $\vec{x}$ satisfying $(1_i, \vec{x}) = 1$ and $(0, \vec{x}) = 0$, it can be said that component is relevant. In words if the state of system is not affected by the state of ith component then ith component is irrelevant to the system. Below we provide the definition of coherent system. For a detailed description and properties of coherent systems we refer to Barlow and Proschan [10] as well as Kuo and Zuo [58].

Definition. A system of components is coherent if

$$\begin{array}{ll} i. & \text{its structure function is increasing,} \\ ii. & \text{each component is relevant.} \end{array} \tag{1.1}$$

According to this definition, the following conditions must be satisfied.

1. $\phi(\mathbf{0}) = 0$ which means system is failed when all components are failed.
2. $\phi(\mathbf{1}) = 1$ which means system is operating when all components operate.
3. $\mathbf{x} < \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y})$ which means improvement of any component does not decrease the performance of the system.
4. For every component i, there exists a component state vector such that the state of component i dictates the state of the system.

System	*Structure Function*	*Reliability*
Series	$\phi(\vec{x}) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, ..., x_n)$	$P(\sum_{i=1}^{n} X_i = n)$
Parallel	$\phi(\vec{x}) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, ..., x_n)$	$P(\sum_{i=1}^{n} X_i \geq 1)$
k-out-of-n:F	$\phi(\vec{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i > n-k \\ 0, & \sum_{i=1}^{n} x_i \leq n-k \end{cases}$	$P(\sum_{i=1}^{n} X_i > n-k)$
k-out-of-n:G	$\phi(\vec{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i \geq k \\ 0, & \sum_{i=1}^{n} x_i < k \end{cases}$	$P(\sum_{i=1}^{n} X_i \geq k)$
Consecutive k-out-of-n:F	$\phi(\vec{x}) = \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} (1 - x_j))$	$P(L_n^0 < k)$
Consecutive k-out-of-n:G	$\phi(\vec{x}) = 1 - \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} x_j)$	$P(L_n^1 \geq k)$

Table 1.1: Structure functions and reliabilites of various coherent structures

Reliability of a coherent system consisting of n components can be defined as

$$R = P(\phi(\vec{X}) = 1).$$

Similarly the reliability of the ith component of this system is defined as

$$P(X_i = 1) = p_i \quad \text{for } i = 1, 2, ..., n.$$

In Table 1.1 we present the structure functions and reliabilities of various coherent structures consisting of n components. L_n^1 and L_n^0 denote the lengths of longest success and failure runs in $\vec{x}$, respectively.

For example; let the states of $n = 12$ components be $\vec{x} = (011110100011)$. Then we have $L_{10}^1 = 4$ and $L_{10}^0 = 3$. For a detailed description of the longest run random variables we refer to Balakrishnan and Koutras [9] as well as Fu and Lou [42].

As a generalization of k-out-of-n:F and consecutive k-out-of-n:F systems, the consecutive k-within-m-out-of-n:F system consisting of n linearly ordered components such that the system fails if and only if there are m consecutive components which include among them at least k failed components. For an illustration, let the states of $n = 10$ components be $\vec{x} = (0011010011)$. Then the system is in a failure state if $m = 4$ and $k = 3$ while it is in a functioning state when $m = 5$ and $k = 4$. This system was first introduced by Griffith [46]. This model includes consecutive k-out-of-n:F and k-out-of-n:F systems when $m = k$ and $m = n$, respectively.

In this thesis, we investigate the signature-based analysis of consecutive k-within-m-out-of-n:F systems with exchangeable components ordered in a line or a circle. We will evaluate the reliability characteristics of systems, and compare the performance of systems in the context of system signature. Stochastic comparison of different systems will also be studied. Maximum number of failed components is calculated by a new approximation. The approximate results are compared with the simulated and exact results for the various values of n; m; k. For the ease of understanding, some examples are presented in this study.

Chapter 2

System Signature

One of the most important lifetime characteristic of a coherent system is the survival function defined by

$$R(t) = P(T > t),$$

where T denotes the lifetime of a coherent system.

The evaluation of the function $R(t)$ is of special importance not only for computing the survival probabilities but also for evaluating the other reliability characteristics such as hazard rate, and mean residual lifetime.

Let T_i denote the lifetime of the ith component in a coherent system with the structure function ϕ and lifetime T. Then

$$T = \phi(T_1, T_2, ..., T_n).$$

Define

$$X_i(t) = \begin{cases} 1 & \text{if } T_i > t \\ 0 & \text{if } T_i \leq t \end{cases}, \quad i = 1, 2, ..., n$$

It is clear that the binary stochastic process $X_i(t)$ represents the state of the ith component at time t. The survival function $R(t)$ can be investigated by the help of $X_i(t)$s. For example, the survival function of a k-out-of-n:F system can be written as

$$R(t) = P(\sum_{i=1}^{n} X_i(t) > n - k).$$

Similarly, the survival function of a consecutive k-out-of-n:F system can be expressed as

$$R(t) = P(L_n^0(t) < k),$$

where $L_n^0(t)$ denotes the longest run of 0s(failures) in $X_1(t), X_2(t), ..., X_n(t)$(see, e.g. Eryılmaz [29]).

A general representation for the survival function of coherent systems can be given in terms of signature. Let T be the lifetime of a coherent system consisting of independent and identical components with the lifetimes $T_1, T_2, ..., T_n$ and common c.d.f. $F(t) = P(T_i \leq t)$, $i = 1, 2, ..., n$. The signature of this system is defined as the probability vector $(p_1, p_2, ..., p_n)$, with

$$p_i = P(T = T_{i:n}) \quad \text{for } i = 1, ..., n, \tag{2.1}$$

where $T_{1:n} \leq T_{2:n} \leq ... \leq T_{n:n}$ are the order statistics associated with $T_1, T_2, ..., T_n$. Equivalently, we have

$$p_i = \frac{\#\text{ of orderings for which the } i\text{th failure causes system failure}}{n!} \tag{2.2}$$

for $i = 1, ..., n$.

A component state vector $\vec{X}$ is a path vector if $\phi(\vec{X}) = 1$. Let $r_i(n)$ be the number of path sets of the system containing exactly i working components. Define

$$a_i(n) = \binom{n}{i}^{-1} r_i(n), \quad i = 1, 2, ..., n, \tag{2.3}$$

through the system of equations

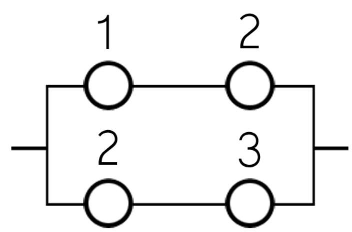
$$a_i(n) = \sum_{j=n-i+1}^{n} p_j, \quad i = 1, 2, ..., n, \tag{2.4}$$

or equivalently,

$$p_i = a_{n-i+1}(n) - a_{n-i}(n), \quad i = 1, 2, ..., n. \tag{2.5}$$

(see, e.g. Boland,[11])
That is, the signature of a system can be obtained by computing $r_i(n)$. The problem of finding $r_i(n)$ is combinatorial one. Specifically, determination of the total number of binary sequences satisfying certain conditions which depend on the structure of a system.

Example 2.1. Let us find the signature of the following consecutive 2-out-of-3:G system



We can define the system lifetime T as follows

$$T = \max(\min(T_1, T_2), \min(T_2, T_3)).$$

There are 3! orderings of the component lifetimes which are given as follows.

Ordering	T
$T_1 < T_2 < T_3$	$T_{2:3}$
$T_1 < T_3 < T_2$	$T_{2:3}$
$T_2 < T_1 < T_3$	$T_{1:3}$
$T_2 < T_3 < T_1$	$T_{1:3}$
$T_3 < T_1 < T_2$	$T_{2:3}$
$T_3 < T_2 < T_1$	$T_{2:3}$

Then we have

$$\begin{aligned} p_1 &= P(\underbrace{T_2 < T_1 < T_3}_{T=T_{1:3}}) + P(\underbrace{T_2 < T_3 < T_1}_{T=T_{1:3}}) = \frac{2}{6} \\ p_2 &= 1 - p_1 = \frac{4}{6} \\ p_3 &= 0 \end{aligned}$$

so the signature is $p = (\frac{1}{3}, \frac{2}{3}, 0)$.

According to the following Theorem the survival function of any coherent system can be written as a linear combination of the survival functions of order statistics, or equivalently survival functions of i-out-of-n:F systems.

Theorem 2.1 (Samaniego [79]) *Let T be the system lifetime and let $T_1, T_2, ..., T_n$ be independent and identically distributed component lifetimes of a coherent system of order n, and. Then*

$$P(T > t) = \sum_{i=1}^{n} p_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j} = \sum_{i=1}^{n} p_i P(T_{i:n} > t). \tag{2.6}$$

The equation (2.6) is proven by Navarro and Rychlik in 2007 [72] when the lifetimes $T_1, T_2, ..., T_n$ have an absolutely continuous exchangeable joint distribution, i.e. the joint distribution (survival function) of $T_1, T_2, ..., T_n$ is invariant under permutation of the variables. In addition to this the equation (2.6) is represented by minimal and maximal signatures which are also very useful tools for considering reliability properties of coherent systems (Kochar et al. [55], Navarro et al. [70], Navarro et al. [74], Eryilmaz ([31], [32]), Navarro and Rychlik [73], Navarro et al. [75]). This representation can also be used for system design in economic models (Dugas and Samaniego [25]). You can see the book of Samaniego [80] for a more comprehensive review of the signature concept and its applications.

Navarro et al. [71] discusses about representation of any coherent system in terms of generalized mixture of series or parallel systems. More explicitly, if $T_1, ..., T_n$ are exchangeable lifetimes, then the reliability function of a coherent system T can be expressed as either

$$P\left\{T>t\right\}=\sum_{i=1}^{n}\alpha_{i}P\left\{T_{1:i}>t\right\}, \tag{2.7}$$

or

$$P\left\{T>t\right\}=\sum_{i=1}^{n}\beta_{i}P\left\{T_{i:i}>t\right\}, \tag{2.8}$$

where the vectors of coefficients $(\alpha_1,\alpha_2,...,\alpha_n)$, and $(\beta_1,\beta_2,...,\beta_n)$ satisfying $\sum_{i=1}^{n}\alpha_i = 1, \sum_{i=1}^{n}\beta_i = 1$ are named as minimal and maximal signatures, respectively . The random variables $T_{1:i}$ and $T_{i:i}$ denote the lifetimes of series and parallel systems with i components, respectively, $1 \leq i \leq n$.

The minimal and maximal signature formulation of any coherent system ϕ is proven by Eryilmaz [31] and it can be expressed as

$$\alpha_i=\binom{n}{i}\sum_{j\in A_i(\phi)}(-1)^{i+j-n}\binom{i}{n-j}\frac{r_{n-j}(n)}{\binom{n}{j}}, \tag{2.9}$$

and

$$\beta_i=\binom{n}{i}\sum_{j\in B_i(\phi)}(-1)^{i-j+1}\binom{i}{j}\frac{r_{n-j}(n)}{\binom{n}{j}}, \tag{2.10}$$

where $A_i(\phi)$ and $B_i(\phi)$ are the sets which depend on the structure ϕ.

Chapter 3

Consecutive-k Systems

The consecutive system is also named as the consecutive-k-out-of-n:F system in the literature. It consists of n linearly ordered components and it fails if and only if some k consecutive components fail. The system firstly introduced by Kontoleon [56] under the name of the r-successive-out-of-n:F system, but only an enumerating algorithm was given. Chiang and Niu [19] motivated the system with some real applications and showed computed reliability of system with some recursive equations. Following this Derman, Lieberman and Ross [24] made two fundamental contributions on consecutive systems. The first is extension of linear system into circular system that n components arranged into a cycle and the second and the most important contribution, is that they introduced the optimal design aspect into the problem. Shanthikumar [81] and Hwang [51] first applied time complexity analysis on proposed algorithms for computing the system reliabilities; which concerns the theoretical computer science aspect of the consecutive system problem. On the other hand, Chao, Lin and Fu ([17], [41]) studied the asymptotic behaviour of the system when n tends to infinity, ensuring the probability theory plays huge role.

There are many variations and generalizations of the consecutive systems, such as circular consecutive-k-out-of-n:F system [24], weighted-consecutive-k-out-of-n:F system ([14], [86]), f-or-consecutive-k-out-of-n:F system [15] and consecutive-k-r-out-of-n:DFM systems [57], etc. In recent years, there were many studies on these systems. A detailed chronological survey study about the reliability of consecutive k-out-of-n:F and related systems is presented in Chao et al [18]. This survey focuses on developments in the area occurring between 1980 and 1995.

The consecutive k-within-m-out-of-n:F system is one of the widely studied related systems, which consists of n linearly ordered components such that the system fails if, and only if, there are m consecutive components containing k failed components. In the literature, k-within-consecutive-m-out-of-n:F and consecutive k-out-of-m-from-n:F can also been used as alternative names for this system. This model includes consecutive k-out-of-n:F and k-out-of-n:F systems for $m = k$ and $m = n$ respectively, and these models can be applied in quality control and radar detection. Various distinctive properties of the consecutive k-within-m-out-of-n:F systems with independent components have already been discussed. Habib and Szantai [49] found out the bounds for this system. The reliability of a consecutive k-within-m-out-of-n:F system which involves IID(independent and identically distributed) components has been studied by Lomonosov [60]. An algorithm to compute the reliability of a multi-state consecutive k-within-m-out-of-n:G system has been proposed by Habib et al [50]. This multi-state system is the generalization of a consecutive k-within-m-out-of-n:G which is dual of the consecutive $m-k+1$-within-m-out-of-n:F. In recent times, Eryilmaz et al. [37] discussed the lifetime distribution of consecutive k-within-m-out-of-n:F system involving exchangeable components by use of Samaniego's signature. Consecutive k-within-m-out-of-n:F systems can also be described as a series system of $(n-m+1)$-dependent k-out-of-m:F systems.

Griffith [46] introduced another remarkable generalization named as the m-consecutive-k-out-of-n:F system containing n linearly ordered components such that the system fails if, and only if, there are at least m nonoverlapping runs of k consecutive failed components ($n \geq mk$). The m-consecutive-k-out-of-n:F system is the generalization of consecutive k-out-of-n:F and m-out-of-n:F systems for $m = 1$ and $k = 1$, respectively. This system has been recently studied in ([3], [45],[64]). Agarwal et al. [3] explained this system by using of graphical illustrations. Some authors have attempted to propose specific rules to combine the consecutive-type systems. The functioning principle of these systems is dependent on the consecutively failed/working components and the total number of failed/working components. The (n, f, k):F system, which is an example of this type of systems, involves n ordered components and fails if, and only if, there are at least k consecutive failed components, or at least f failed components in the (n, f, k):F system ([82], [15]). This system has been employed by Sun and Liao [82] to display automatic payment systems in banks. This reliability model has been named as the combined f -out-of-n:F and consecutive k-out-of-n:F system and in Zuo et al. [91] applied this model to compute the reliability of an industrial system. Cui et al. [21] introduced a n, f, k :F system consisting of n components linearly(circularly) ordered, and the system fails if, and only if, there are at least k consecutive failed components and there are at least f failed components. The recursive formulas for evaluating the reliability of (n, f, k):F and n, f, k :F systems and their duals for INID(independent and non identically distributed) components have been discussed in their article. The dual of the (n, f, k):F system has been discussed by Gera [43]. Demir [23] pointed out explicit reliability formulas for the systems which consist of Markov-dependent components. The same system involving exchangeable dependent components has been studied via a signature-based analysis by Eryilmaz [28] . Guo et al. [48] presented a $(n, f, k(i, j))$:F ($n, f, k(i, j)$:F) system which consists of n components linearly (circularly) ordered. This system fails if, and only if, there exist at least f failed components or (and) at least k consecutive failed components among components $i, i + 1, ..., j - 1, j$.

Zhao et al. [89] is firstly introduced consecutive-k systems with sparse. They define sparse d as if there are no failed components between two failed components and d working components between those two failed components. This system involves n components ordered in a line and fails if there are at least k consecutive failures with sparse d. There are many application of system such as cold-standby repairable, communications systems. It has been generalized as m-consecutive-k-out-of-n:F system with sparse d and a (n, f, k):F system with sparse d. Cui and Xie [20] studied a system containing N modules with the ith module combined of n_i components in parallel, which is the general case of consecutive k-out-of-n:F and (n, f, k):F systems

Recently, Eryilmaz and Zuo [40] introduced a new model named as a constrained consecutive (k, d)-out-of-n:G system which turns into the consecutive k-out-of-n:G system for $d = 0$. This model can be applied to evaluate the constrained binary sequences in communication systems involving magnetic and optical recording media.

Another commonly used system is a circular m-consecutive-k-out-of-n:F system with non-overlapping runs consists of n components which are circularly ordered such that system fails if, and only if there are at least m non-overlapping runs of k consecutive failed components. This system has been also introduced by Griffith [46] in 1986. Later, Boland & Papastavridis [12], Papastavridis [77], Makri & Philippou [61], Agarwal et al. [2], and Eryilmaz et al. [38] studied on this system. Similarly, a circular m-consecutive-k-out-of-n:F system with overlapping runs consists of n components, which are circularly ordered, such that the system fails if, and only if there are at least m overlapping runs of k consecutive failed components. This system has been studied by Agarwal & Mohan [1] and Eryilmaz [35]. Overlapping means that runs have common components. These linear and circular system models generalize the consecutive-k-out-of-n:F system which is firstly introduced by Chiang & Niu [19] in 1981 and then studied by Bollinger & Salvia [13], Shanthikumar [81], Derman et al [24]. Recent discussions on such systems are in Gera [44], Levitin & Dai [59] and Eryilmaz [34].

The paper titled "Review of recent advances in reliability of consecutive k-out-of-n and related systems", which was written by Eryilmaz [30] , touches upon the recent developments between 1995 and 2010 on consecutive k-out-of-n systems by collecting the results of the probabilistic characteristics of the systems. Eryilmaz has discussed the extensions and generalizations of consecutive k-out-of-n systems which have many different applications in engineering fields.

Chapter 4

Signature-based Anaylsis For Linear Case

A consecutive k-within-m-out-of-n:F system, which is one of the generalized consecutive-k systems, involves n ordered components in a line. This system does not function if and only if m consecutive components consists of at least k failed components ($1 < k \leq m \leq n$). It is possible to observe various applications for such systems in practice (see, e.g. Chang et al. [16]). A consecutive k-within-m-out-of-n:F system turns into consecutive k-out-of-n:F (this system fails if and only if at least k consecutive components fail) in the condition of $m = k$. But it turns into k-out-of-n:F (this system fails if and only if at least k components fail) systems if m equals to n. There are numerous articles for finding the reliability of this system. However, reliability characteristics of this system is not a concept which has been comprehensively studied. Several findings regarding the evaluation of the survival function of this system have been obtained in Papastavridis [76], Iyer [52] and Eryilmaz et al [37].

In this section, signature-based analysis of consecutive k-within-m-out-of-n:F systems consisting of exchangeable components is studied. A sequence of lifetimes $T_1, ..., T_n$ is exchangeable if for each n,

$$P\left\{T_1 \leq t_1, ..., T_n \leq t_n\right\} = P\left\{T_{\pi(1)} \leq t_1, ..., T_{\pi(n)} \leq t_n\right\},$$

for any permutation $\pi = (\pi(1), ..., \pi(n))$ of $\{1, ..., n\}$, i.e. the joint distribution (or survival function) of $T_1, ..., T_n$ is symmetric in $t_1, ..., t_n$. The exchangeability refers that the components in the system are identically distributed but they influence each other.

The representation (2.6) also holds for the case when $T_1, ..., T_n$ have an absolutely continuous exchangeable joint distribution (Navarro and Rychlik [72]). Since the distribution of components does not effect the signature, we can say that the system consisting of exchangeable components has exactly the same signature vector of a system consisting independent and identically distributed components. On the other hand, Navarro et al. [74] proved that absolute continuity assumption is not really necessary for evaluating the equation (2.6). See Navarro et al. [74] and Navarro et al. [75] for the details.

This chapter is grouped as follows. In Section 4.1, we study failure rate and mean residual life functions. The signature-based analysis of consecutive k-within-m-out-of-n:F system is discussed and some reliability characteristic such as failure rate and mean residual life functions are found with the help of minimal and maximal signatures. In Section 4.2,we reach some stochastic ordering outcomes by which we formulate the survival function and mean time to failure of the system. Numerical results are obtained.

4.1 Aging characteristics

Consecutive k-within-m-out-of-n:F system can be reformed as a combination of $n-m+1$ number of k-out-of-m:F subsystems as dependent series systems. Meanwhile, lifetime of this system is expressed as

$$T_{k,m:n} = \min(T_{k:m}^{(1)}, T_{k:m}^{(2)}, ..., T_{k:m}^{(n-m+1)}), \tag{4.1}$$

where $T_{k:m}^{(j)}$ denotes the lifetime of k-out-of-m:F system of components with the lifetimes $T_j, T_{j+1}, ..., T_{j+m-1}, 1 \leq j \leq n-m+1$.

It is obvious that,a consecutive k-within-m-out-of-n:F system will never fail if and only if at least $n-k+1$ components work. Hence, the signature vector has the form $(0,...,0,p_k,p_{k+1},...,p_n)$, and

$$P\left\{T_{k,m:n}>t\right\}=\sum_{i=k}^{n}p_iP\left\{T_{i:n}>t\right\}.$$

$N(j,k,m,n)$ defines the total number of linearly ordered n components consisting of j failed and $n-j$ working components, which also consists less than k failed components in any consecutive m components. If then, $r_i(n)=N(n-i,k,m,n)$ denotes the number of path sets of consecutive k-within-m-out-of-n:F system with i working components (Eryilmaz [31]). $N(j,k,m,n)$ number can only be calculated just in the case of $k=2$. Therefore, we can just obtain an explicit formula only for the signature of linear consecutive 2-within-m-out-of-n:F, system, which is shown by

$$p_i=\binom{n}{i}^{-1}\left[\frac{n-i+1}{i}\binom{n-(i-2)(m-1)}{i-1}-\binom{n-(i-1)(m-1)}{i}\right], \quad (4.2)$$

(Eryilmaz [31]).

Proposition 4.1 *For* $2\leq m\leq n$,

(a) The signature of consecutive 2*-within-m-out-of-n:F system has the form* $(0,p_2,...,p_a,0,...0)$ *with* $p_2,...,p_a>0$, *where* $a=[(n+2m-1)/m]$ *and* $[x]$ *shows the integer part of* x.

(b) For $2m\geq n$, *the signature of consecutive* 2*-within-m-out-of-n:F system has the form* $(0,p,1-p,0,...0)$, *where* $1-p=\binom{n-m+1}{2}/\binom{n}{2}$, *and hence*

$$P\left\{T_{2,m:n}>t\right\}=pP\left\{T_{2:n}>t\right\}+(1-p)P\left\{T_{3:n}>t\right\}.$$

Proof. The first part readily follows from the property of the combinatorial terms involved in (4.2). The second part is an immediate consequence of part (a). □

A MATLAB code is written to evaluate the signature of consecutive k-within-m-out-of-n:F system for different values of k, m, and n. By using equations (2.3) and (2.5) an algorithm is built which computes the signature in a reasonable CPU time for $n < 20$. In Tables (4.1)-(4.2) we illustrated the signature of the consecutive k-within-m-out-of-n:F systems, which is obtained by using different values of m and k in the case of $n = 5$ and $n = 10$. The MATLAB code is available upon request. Note that, an algorithm is written by Navarro and Rubio [69] to calculate the signatures of all type of coherent systems with five components.

The mixture signature expression of consecutive k-within-m-out-of-n:F system is evaluated by Eryilmaz [31] in below.

$$\alpha_i = \binom{n}{i} \sum_{j=n-i}^{z(n,m,k)} \frac{N(j,k,m,n)}{\binom{n}{j}} (-1)^{i+j-n} \binom{i}{n-j}, \tag{4.3}$$

for $n - z(n,m,k) \leq i \leq n$, ($\alpha_i = 0$ for $1 \leq i < n - z(n,m,k)$) and

$$\beta_i = \binom{n}{i} \sum_{j=0}^{\min(i,z(n,m,k))} \frac{N(j,k,m,n)}{\binom{n}{j}} (-1)^{i-j+1} \binom{i}{j}.$$

Thus using the minimal signature we also have

$$P\left\{T_{k,m:n} > t\right\} = \sum_{i=n-z(n,m,k)}^{n} \alpha_i P\left\{T_{1:i} > t\right\}. \tag{4.4}$$

It should be figured that the positive integer $z(n,m,k)$ literally is the maximum number of failed components when the system can still function. By the following Lemma, we can easily calculate it.

Lemma 4.2 *For* $1 < k \leq m \leq n$,

$$z(n,m,k) = \begin{cases} n - \left[\frac{n}{m}\right](m-k+1) & \text{if } n - m\left[\frac{n}{m}\right] < k \\ (k-1)(1+\left[\frac{n}{m}\right]) & \text{if } n - m\left[\frac{n}{m}\right] \geq k. \end{cases}$$

Proof. Let us divide n cells into $\left[\frac{n}{m}\right]$ nonoverlapping windows of size m. To get the maximum number of failed components we put failed components to the first $k-1$ cells of each window. Thus the nonoverlapping windows include totally $(k-1)\left[\frac{n}{m}\right]$ failed components. If the remaining $n-m\left[\frac{n}{m}\right]$ cells is less than k, then all remaining cells can be filled with failed components and hence

$$z(n,m,k) = (k-1)\left[\frac{n}{m}\right] + n - m\left[\frac{n}{m}\right],$$

if $n-m\left[\frac{n}{m}\right] < k$, and if $n-m\left[\frac{n}{m}\right] \geq k$, then

$$z(n,m,k) = (k-1)\left[\frac{n}{m}\right] + k - 1.$$

The proof is completed. □

The failure rate of an absolutely continuous lifetime random variable T is defined by $r(t) = f(t)/S(t)$ for t such that $S(t) > 0$, where $S(t) = P\{T > t\}$ and $f(t) = -S'(t)$. Using the minimal and maximal signature expressions given above, the failure rate function of consecutive k-within-m-out-of-n:F system can be obtained from

$$r_{k,m:n}(t) = \frac{\sum_{i=n-z(n,m,k)}^{n} \alpha_i f_{1:i}(t)}{\sum_{i=n-z(n,m,k)}^{n} \alpha_i P\{T_{1:i} > t\}} = \frac{\sum_{i=k}^{n} p_i f_{i:n}(t)}{\sum_{i=k}^{n} p_i P\{T_{i:n} > t\}},$$

where $f_{i:n}(t)$ is the probability density function of $T_{i:n}, 1 \leq i \leq n$ and

$$\begin{aligned} P\{T_{i:n} > t\} &= 1 - \sum_{j=i}^{n} (-1)^{j-i} \binom{j-1}{i-1} \binom{n}{j} P\{T_{j:j} \leq t\} \\ &= \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} P\{T_{1:j} > t\} \end{aligned}$$

(see, e.g. David and Nagaraja [22]).

The minimal signature expression stated in (4.4) is helpful to obtain to study obtaining the asymptotic analysis of the failure rate function

Theorem 4.3 (Navarro and Hernandez [67]) *Let S be a survival function such that*

$$S(t) = \sum_{i=1}^{n} \omega_i S_i(t), \tag{4.5}$$

for all $t \geq 0$, where $S_1(t), ..., S_n(t)$ are survival functions and $\omega_1, ..., \omega_n$ are real numbers such that $\sum_{i=1}^{n} \omega_i = 1$. Let $r_i(t)$ be the failure rate function corresponding to $S_i(t), i = 1, ..., n$. If

$$\lim_{t\to\infty} \inf \frac{r_i(t)}{r_1(t)} > 1, \qquad \lim_{t\to\infty} \sup \frac{r_i(t)}{r_1(t)} < \infty,$$

for $i = 2, 3, ..., n$, then $\lim_{t\to\infty} \frac{r(t)}{r_1(t)} = 1$, where $r(t)$ is the failure rate function corresponding to $S(t)$.

The proof of the below Proposition can be easily obtained from (4.4) and Theorem 4.3.

Proposition 4.4 *If $T_1, ..., T_n$ are exchangeable, and for $i = n-z(n,m,k)+1, ..., n$*

$$\lim_{t\to\infty} \inf \frac{r_{1:i}(t)}{r_{1:n-z(n,m,k)}(t)} > 1, \;\; \lim_{t\to\infty} \sup \frac{r_{1:i}(t)}{r_{1:n-z(n,m,k)}(t)} < \infty,$$

then

$$\lim_{t\to\infty} \frac{r_{k,m:n}(t)}{r_{1:n-z(n,m,k)}(t)} = 1.$$

Corollary 4.5 *If $T_1, ..., T_n$ are i.i.d. with common failure rate function $r(t)$, then for $1 < k \leq m \leq n$,*

$$\lim_{t\to\infty} \frac{r_{k,m:n}(t)}{r(t)} = n - z(n,m,k).$$

Proof. Because $T_1, ..., T_n$ are i.i.d. the failure rate function of $T_{1:i}$ is $r_{1:i}(t) = ir(t)$.

For $i = n - z(n,m,k) + 1, ..., n$,

$$\liminf_{t\to\infty} \frac{r_{1:i}(t)}{r_{1:n-z(n,m,k)}(t)} = \frac{i}{n - z(n,m,k)} > 1,$$

and

$$\limsup_{t\to\infty} \frac{r_{1:i}(t)}{r_{1:n-z(n,m,k)}(t)} = \frac{i}{n - z(n,m,k)} < \infty.$$

Thus the proof follows from Proposition 4.4. □

Remark (1)**.** If z_ϕ denotes the maximum number of failed components in a working system with lifetime $\phi(T_1, ..., T_n)$ and $T_1, ..., T_n$ are IID with common failure rate function $r(t)$, then

$$\lim_{t\to\infty} \frac{r_\phi(t)}{r(t)} = n - z_\phi,$$

where $r_\phi(t)$ is the failure rate function associated with the coherent structure ϕ.

Remark (2)**.** In fact, the result presented in Corollary 4.5 can also be obtained from Theorem 5.3 of Samaniego ([80], page 66) which states that if $r(t)$ has limit r as $t \to \infty$, then a coherent system with failure rate $r_T(t)$ and signature $\mathbf{p} = (p_1, ..., p_n)$ satisfies $r_T(t) \to (n - K + 1)r$ as $t \to \infty$, where $K = \max\{i \mid p_i > 0\}$.

Remark (3)**.** Comparing the results given in Corollary 4.5 and Remark 2, for consecutive k-within-m-out-of-n:F system we observe that

$$K = \max\{i \mid p_i > 0\} = 1 + z(n,m,k),$$

and hence

$$P\{T_{k,m:n} > t\} = \sum_{i=k}^{1+z(n,m,k)} p_i P\{T_{i:n} > t\}.$$

Protection of the increasing failure rate (IFR) property of a system is one the important concept in system safety and reliability theory. A sufficient condition is given by Triantafyllou and Koutras [84] for the nonpreservation of the IFR property of coherent systems. The minimum number of working components in a functioning coherent system and its signature are denoted as n_0 $(1 \leq n_0 < n)$

and $\mathbf{p} = (p_1, ..., p_n)$, respectively. The coherent system can not protect the IFR property, when $p_{i_0} > (n - i_0)p_{i_0+1}$ for $i_0 = n - n_0$,. Indeed, i_0 is the maximum number of failed components in a functioning coherent system, above equation can be rewritten as follows

$$p_{z_\phi} > (n - z_\phi)p_{z_\phi+1}. \tag{4.6}$$

Let $n = 5, m = 4$ and $k = 2$. Then from Table 4.1 the signature of consecutive 2-within-4-out-of-5:F system is $(0, 0.9, 0.1, 0, 0)$ and $z(n, m, k) = 2$. Since $p_2 > (5-2)p_3$, it is found that consecutive 2-within-4-out-of-5:F system does not preserve the IFR property. The following result is immediate from part (b) of Proposition 4.1 and the condition (4.6).

Proposition 4.6 *For $2m \geq n$, consecutive 2-within-m-out-of-n:F system does not preserve the IFR property if $n > (n - m)(n - m + 1)$.*

The mean residual lifetime (MRL) function, defined by $m_T(t) = E(T - t \mid T > t)$, plays an important role in reliability. It can be computed from

$$M_T(t) = \frac{1}{S(t)} \int_t^\infty S(x)dx.$$

In last years, there are interesting papers on this concept. For instance, for parallel system containing n independent and identical components with lifetimes $T_1, T_2, \ldots, T_n$, Bairamov et al. [7] introduced a MRL function as

$$\Psi_n(t) = E(T_{n:n} - t | T_{1:n} > t)$$

which shows the conditional expectation of the remaining lifetime of the system, given that at time t all components are functioning. This result is also developed for a k-out-of-n system by Asadi and Bayramoglu ([5], [6]). MRL functions of a k-out-of-n system with independent but not identically distributed components' lifetimes is studied by Sadegh ([78]) and Bairamov and Gurler ([8]). Note that, the above definition is different from ordinary definition MRL function of the system. For more detailed discussion we refer to Tavangar and Bairamov [83]. The MRL of consecutive k-within-m-out-of-n:F system can be computed from

$$M_{k,m:n}(t) = E(T_{k,m:n} - t \mid T_{k,m:n} > t) = \frac{\sum_{i=n-z(n,m,k)}^{n} \alpha_i P\{T_{1:i} > t\} M_{1:i}(t)}{\sum_{i=n-z(n,m,k)}^{n} \alpha_i P\{T_{1:i} > t\}},$$

where $M_{1:i}(t)$ is the MRL of series system of i components, i.e. $M_{1:i}(t) = E(T_{1:i} - t \mid T_{1:i} > t)$.

Theorem 4.7 (Navarro and Hernandez [68]) *If (4.5) holds and the mean residual life functions $M_1, M_2, ..., M_n$ of $S_1, S_2, ..., S_n$ respectively, satisfy*

$$\lim_{t\to\infty} \inf \frac{M_1(t)}{M_i(t)} > 1, \qquad \lim_{t\to\infty} \sup \frac{M_1(t)}{M_i(t)} < \infty,$$

for $i = 2, 3, ..., n$, then the mean residual life function M_T of S satisfies $\lim_{t\to\infty} \frac{M_T(t)}{M_1(t)} = 1$.

Proposition 4.8 *If $T_1, ..., T_n$ are exchangeable, and for $i = n-z(n,m,k)+1, ..., n$*

$$\lim_{t\to\infty} \inf \frac{M_{1:n-z(n,m,k)}(t)}{M_{1:i}(t)} > 1, \quad \lim_{t\to\infty} \sup \frac{M_{1:n-z(n,m,k)}(t)}{M_{1:i}(t)} < \infty,$$

then

$$\lim_{t\to\infty} \frac{M_{k,m:n}(t)}{M_{1:n-z(n,m,k)}(t)} = 1.$$

When less than r $(< n)$ components fail at time t in a system containing n component, this system can still function. So, The residual lifetime of such system can be express as the conditional random variable $\{T - t \mid T_{r:n} > t\}$. As Khaledi and Shaked [54] figured that when the rth component will fail but the system is still functioning, then the user has two choice such as either a maintenance or rebuilding the system. For the systems having signature in the form of $(0, ..., 0, p_s, p_{s+1}, ..., p_n)$, is showed by Khaledi and Shaked [54],

$$P\{T - t > x \mid T_{r:n} > t\} = \sum_{i=s}^{n} p_i P\{T_{i:n} > t + x \mid T_{r:n} > t\}.$$

Therefore for the linear consecutive k-within-m-out-of-n:F system we have

$$P\left\{T_{k,m:n}-t>x \mid T_{r:n}>t\right\}=\sum_{i=k}^{1+z(n,m,k)} p_i P\left\{T_{i:n}>t+x \mid T_{r:n}>t\right\},$$

and

$$E\left(T_{k,m:n}-t \mid T_{r:n}>t\right)=\sum_{i=k}^{1+z(n,m,k)} p_i E\left(T_{i:n}-t \mid T_{r:n}>t\right),$$

for $r \leq k$. For $i \geq k$, the conditional probability $P\left\{T_{i:n}>t+x \mid T_{r:n}>t\right\}$ can be computed from

$$P\left\{T_{i:n}>s \mid T_{r:n}>t\right\}=\frac{1}{P\left\{T_{r:n}>t\right\}} \sum_{a=n-i+1}^{n} \sum_{b=\max(a,n-r+1)}^{n} \binom{n}{b}\binom{b}{a} p_{b,b-a,n}(t,s),$$

where

$$\begin{aligned} p_{j,m,n}(t,s)=P\{&T_1>s,...,T_{j-m}>s,T_{j-m+1}\in(t,s],...,T_j\in(t,s],\\ &T_{j+1}\leq t,...,T_n\leq t\}. \end{aligned}$$

If $T_1,...,T_n$ are exchangeable then

$$\begin{aligned} p_{j,m,n}(t,s)=\ &\bar{F}(\underbrace{s,...,s}_{j-m\text{ times}},\underbrace{t,...,t}_{m\text{ times}})-\sum_{i=1}^{m}(-1)^{i+1}\binom{m}{i}\bar{F}(\underbrace{s,...,s}_{j-m+i\text{ times}},\underbrace{t,...,t}_{m-i\text{ times}})\\ &-\sum_{i=1}^{n-j}(-1)^{i+1}\binom{n-j}{i}\bar{F}(\underbrace{s,...,s}_{j-m\text{ times}},\underbrace{t,...,t}_{i+m\text{ times}})\\ &+\sum_{i_1=1}^{m}\sum_{i_2=1}^{n-j}(-1)^{i_1+i_2+2}\binom{m}{i_1}\binom{n-j}{i_2}\bar{F}(\underbrace{s,...,s}_{j-m+i_1\text{ times}},\underbrace{t,...,t}_{m-i_1+i_2\text{ times}}) \end{aligned}$$

(Eryilmaz [33]). Thus the mean residual life function $E\left(T_{k,m:n}-t \mid T_{r:n}>t\right)$ can be computed using the joint survival function of components' lifetimes.

4.2 Stochastic ordering results

The signature of a system has been found to be useful for comparing systems in terms of various stochastic orderings. Assume that T and Z are two lifetime random variables with survival functions $\bar{F}(t)$ and $\bar{G}(t)$, respectively. If $\bar{F}(t) \leq \bar{G}(t)$ for all t. T is stochastically smaller than Z (denoted by $T \leq_{st} Z$). Assume that $r_T(t)$ and $r_Z(t)$ are the hazard rate functions of lifetime random variables T and Z, respectively. If $r_T(t) \geq r_Z(t)$ for all t,, then T is smaller than Z in hazard rate (hr) ordering and shown as $T \leq_{hr} Z$. Let $f(t)$ and $g(t)$ denote the density functions of T and Z, respectively. If $f(t)/g(t)$ is decreasing for all t, then T is smaller than Z in likelihood ratio ordering and we write $T \leq_{lr} Z$.

Consider two discrete distributions $\mathbf{p} = (p_1, ..., p_n)$ and $\mathbf{q} = (q_1, ..., q_n)$,

(a) $\mathbf{p} \leq_{st} \mathbf{q}$ if $\sum_{j=i}^{n} p_j \leq \sum_{j=i}^{n} q_j$ for all $i = 1, 2, ..., n$,

(b) $\mathbf{p} \leq_{hr} \mathbf{q}$ if $\sum_{j=i}^{n} p_j / \sum_{j=i}^{n} q_j$ is decreasing in i,

(c) $\mathbf{p} \leq_{lr} \mathbf{q}$ if p_i/q_i is decreasing in i, when $p_i, q_i > 0$.

Assume that $\mathbf{p} = (p_1, ..., p_n)$ and $\mathbf{q} = (q_1, ..., q_n)$ are the signatures of coherent systems $T = \phi(T_1, ..., T_n)$ and $Z = \psi(T_1, ..., T_n)$,respectively containing n i.i.d. components. It is proven that if $\mathbf{p} \leq_{st} \mathbf{q}$, then $T \leq_{st} Z$ (Kochar et al. [55]). This findings are also true for the systems containing common exchangeable components (Navarro et al. [70]). It is also proved that the following stochastic comparison for the systems consist of common exchangeable components.

(a) If $T_{i:n} \leq_{hr} T_{i+1:n}$ for $i = 1, ..., n-1$ and $\mathbf{p} \leq_{hr} \mathbf{q}$, then $T \leq_{hr} Z$.

(b) If $T_{i:n} \leq_{lr} T_{i+1:n}$ for $i = 1, ..., n-1$ and $\mathbf{p} \leq_{lr} \mathbf{q}$, then $T \leq_{lr} Z$.

Theorem 4.9 *Let $T_1, ..., T_n$ be exchangeable. Then for $1 < k \leq m \leq n$,*

(a) $T_{k,m:n} \leq_{st} T_{k:m}^{(i)}, i = 1, 2, ..., n-m+1,$

(b) $T_{k:n} \leq_{st} T_{k,m:n}$,

(c) $T_{k,m:n} \leq_{st} T_{m,m:n}$.

(d) $T_{2,m+1:n} \leq_{st} T_{2,m:n}$, *for* $2m \geq n$ *and* $2 \leq m < n$.

Proof. The proof of part (a) can be easily seen from (4.1). The form of signature vectors of k-out-of-n:F system and consecutive k-within-m-out-of-n:F system are as $\mathbf{p} = (0, ..., 0, 1_k, 0, ..., 0)$ and $\mathbf{q} = (0, ..., 0, q_k, q_{k+1}, ..., q_n)$, respectively. Since $\mathbf{p} \leq_{st} \mathbf{q}$ we have $T_{k:n} \leq_{st} T_{k,m:n}$. For part (c), one can define $X_i(t) = 1$ if $T_i \leq t$, and $X_i(t) = 0$ if $T_i > t$. Hence, $m \geq k$

$$\begin{aligned} P\{T_{k,m:n} > t\} &= P\left\{\sum_{i=1}^{m} X_i(t) < k, \sum_{i=2}^{m+1} X_i(t) < k, ..., \sum_{i=n-m+1}^{n} X_i(t) < k\right\} \\ &\leq P\left\{\sum_{i=1}^{m} X_i(t) < m, \sum_{i=2}^{m+1} X_i(t) < m, ..., \sum_{i=n-m+1}^{n} X_i(t) < m\right\} \\ &= P\{T_{m,m:n} > t\} \end{aligned}$$

which means that the lifetime of consecutive k-within-m-out-of-n:F system is stochastically smaller than the lifetime of consecutive m-out-of-n:F system and so part (c) is proved. Part (d) follows from Proposition 1 and the result of Navarro et al. [70] □

From part (c) of Theorem 4.9 we readily observe that $T_{k,m:n}$ is stochastically increasing in k, i.e. $T_{k,m:n} \leq_{st} T_{k+1,m:n}$. In a similar vein it can be easily seen that $T_{k,m:n}$ is stochastically decreasing in m and n, i.e. $T_{k,m+1:n} \leq_{st} T_{k,m:n}$ and $T_{k,m:n+1} \leq_{st} T_{k,m:n}$.

Similar results can also be obtained in terms of hazard rate and likelihood ratio orderings. Using the results of Navarro et al. [70] we have $T_{k:n} \leq_{hr} T_{k,m:n}$, and $T_{2,m+1:n} \leq_{hr} T_{2,m:n}$ for $2m \geq n$ and $2 \leq m < n$ when $T_{i:n} \leq_{hr} T_{i+1:n}$ for $i = 1, ..., n-1$. Similarly, if $T_{i:n} \leq_{lr} T_{i+1:n}$ for $i = 1, ..., n-1$ then $T_{k:n} \leq_{lr} T_{k,m:n}$, and $T_{2,m+1:n} \leq_{lr} T_{2,m:n}$ for $2m \geq n$ and $2 \leq m < n$.

Using Theorem 4.9 we can find the below simple approximations for the survival function and mean time to failure (MTTF) of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. That type of approximations are quite useful especially for large values of n where the computation is difficult.

Corollary 4.10 *If $T_1, ..., T_n$ are exchangeable, then the survival function and the MTTF of consecutive k-within-m-out-of-n:F system can be approximated respectively by*

$$P\{T_{k,m:n} > t\} \simeq \frac{1}{2}\left(P\{T_{k:n} > t\} + P\{T_{k:m} > t\}\right),$$

and

$$E(T_{k,m:n}) \simeq \frac{1}{2}\left(E(T_{k:n}) + E(T_{k:m})\right),$$

for $1 < k \leq m \leq n$.

Proof. If $T_1, ..., T_n$ are exchangeable, then

$$T_{k:m}^{(1)} \overset{d}{=} T_{k:m}^{(2)} \overset{d}{=} ... \overset{d}{=} T_{k:m}^{(n-m+1)},$$

and hence from part (a) of Theorem 4.9 we have $P\{T_{k,m:n} > t\} \leq P\{T_{k:m} > t\}$. On the other hand, from part (b) of Theorem 4.9 one obtains $P\{T_{k,m:n} > t\} \geq P\{T_{k:n} > t\}$. Thus the proof is completed by averaging lower and upper bounds. □

Let $\mathbf{p}$ and $\mathbf{q}$ be the signature vectors of two coherent systems $T = \phi(T_1, ..., T_n)$ and $Z = \psi(T_1, ..., T_n)$, containing common exchangeable components. Consider $\mathbf{p} = (0, ..., 0, p_s, p_{s+1}, ..., p_n)$ and $\mathbf{q} = (0, ..., 0, q_s, q_{s+1}, ..., q_n)$. Zhang [90] showed that if $\mathbf{p} \leq_{st} \mathbf{q}$, then for $r \leq s$, $\{T - t \mid T_{r:n} > t\} \leq_{st} \{Z - t \mid T_{r:n} > t\}$.

In addition, the below results can be shown as Theorem 4.9 and Corollary 4.10.

Theorem 4.11 *Let $T_1, ..., T_n$ be exchangeable. Then for $1 < k \leq m \leq n$ and $r \leq k$,*

(a) $\{T_{k,m:n} - t \mid T_{r:n} > t\} \leq_{st} \left\{T_{k:m}^{(i)} - t \mid T_{r:n} > t\right\}, i = 1, 2, ..., n-m+1,$

(b) $\{T_{k:n} - t \mid T_{r:n} > t\} \leq_{st} \{T_{k,m:n} - t \mid T_{r:n} > t\}$,

(c) $\{T_{k,m:n} - t \mid T_{r:n} > t\} \leq_{st} \{T_{m,m:n} - t \mid T_{r:n} > t\}$.

(d) $\{T_{2,m+1:n} - t \mid T_{r:n} > t\} \leq_{st} \{T_{2,m:n} - t \mid T_{r:n} > t\}$, *for* $2m \geq n$, $2 \leq m < n$ *and* $r \leq 2$.

Corollary 4.12 *If* $T_1, ..., T_n$ *are exchangeable, then for* $1 < k \leq m \leq n$ *and* $r \leq k$

$$\begin{aligned} &P\{T_{k,m:n} - t > x \mid T_{r:n} > t\} \\ &\simeq \frac{1}{2}\left(P\{T_{k:n} - t > x \mid T_{r:n} > t\} + P\{T_{k:m} - t > x \mid T_{r:n} > t\}\right), \end{aligned}$$

and

$$E\left(T_{k,m:n} - t \mid T_{r:n} > t\right) \simeq \frac{1}{2}\left(E\left(T_{k:n} - t \mid T_{r:n} > t\right) + E\left(T_{k:m} - t \mid T_{r:n} > t\right)\right).$$

4.3 An example

Consider a consecutive k-within-m-out-of-n:F system that components have an exchangeable Lomax distribution with the joint survival function

$$\bar{F}(t_1, ..., t_n) = \left(1 + \sum_{i=1}^{n} t_i\right)^{-\alpha},$$

for $\alpha > 0, t_i > 0, i = 1, ..., n$. Then $P\{T_{1:j} > t\} = (1 + jt)^{-\alpha}$, and

$$P\{T_{i:n} > t\} = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} (1 + jt)^{-\alpha}.$$

For $\alpha > 1$,

$$E(T_{i:n}) = \frac{1}{\alpha - 1} \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \frac{1}{j},$$

$1 \leq i \leq n$.

m	k	$\mathbf{p}$
2	2	$(0, 0.4, 0.5, 0.1, 0)$
3	2	$(0, 0.7, 0.3, 0, 0)$
	3	$(0, 0, 0.3, 0.5, 0.2)$
4	2	$(0, 0.9, 0.1, 0, 0)$
	3	$(0, 0, 0.7, 0.3, 0)$
	4	$(0, 0, 0, 0.4, 0.6)$

Table 4.1: Signature of consecutive k-within-m-out-of-5:F systems

In Table 4.3, approximate (A) values of $P\{T_{k,m:n} > t\}$ along with the exact (E) or simulated (S) values- exact values when $n = 10, 15$ and simulated values when $n = 30, 50$- are presented. After 50000 trials, simulation results are obtained. The performance of the approximation surely depends on the values of n, m, and k. For instance; when m come closer to n, we get better approximation. Thus, to find a more accurate approximation we have to consider $P\{T_{k:n} > t\}$ and $P\{T_{k:m} > t\}$ as a function of n, m, and k, which will be studied later.

m	k	$\mathbf{p}$
2	2	$(0, 0.2, 0.3333, 0.3, 0.1429, 0.0238, 0, 0, 0, 0)$
3	2	$(0, 0.3778, 0.4556, 0.1619, 0.0047, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.0667, 0.1667, 0.2667, 0.2857, 0.1810, 0.0333, 0, 0)$
4	2	$(0, 0.5333, 0.4333, 0.0333, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.1883, 0.3405, 0.3571, 0.1143, 0.0048, 0, 0, 0)$
	4	$(0, 0, 0, 0.0333, 0.1095, 0.2143, 0.3095, 0.2667, 0.0667, 0)$
5	2	$(0, 0.6667, 0.3333, 0, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.3333, 0.4286, 0.2381, 0, 0, 0, 0, 0)$
	4	$(0, 0, 0, 0.1190, 0.2867, 0.3571, 0.2381, 0, 0, 0)$
	5	$(0, 0, 0, 0, 0.0238, 0.0952, 0.2143, 0.3333, 0.3333, 0)$
6	2	$(0, 0.7778, 0.2222, 0, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.5, 0.4048, 0.0952, 0, 0, 0, 0, 0)$
	4	$(0, 0, 0, 0.2619, 0.4206, 0.2698, 0.0476, 0, 0, 0)$
	5	$(0, 0, 0, 0, 0.1032, 0.2873, 0.3762, 0.2111, 0.0222, 0)$
	6	$(0, 0, 0, 0, 0, 0.0238, 0.1095, 0.2667, 0.4, 0.2)$
7	2	$(0, 0.8667, 0.1333, 0, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.6667, 0.3048, 0.0286, 0, 0, 0, 0, 0)$
	4	$(0, 0, 0, 0.4525, 0.4286, 0.1143, 0.0048, 0, 0, 0)$
	5	$(0, 0, 0, 0, 0.2619, 0.4476, 0.2571, 0.0333, 0, 0)$
	6	$(0, 0, 0, 0, 0, 0.1190, 0.3476, 0.4, 0.1333, 0)$
	7	$(0, 0, 0, 0, 0, 0, 0.0333, 0.1667, 0.4, 0.4)$

m	k	$\mathbf{p}$
8	2	$(0, 0.9333, 0.0667, 0, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.8167, 0.1786, 0.0048, 0, 0, 0, 0, 0)$
	4	$(0, 0, 0, 0.6667, 0.3095, 0.0238, 0, 0, 0, 0)$
	5	$(0, 0, 0, 0, 0.5, 0.4286, 0.0714, 0, 0, 0)$
	6	$(0, 0, 0, 0, 0, 0.3333, 0.5, 0.1667, 0, 0)$
	7	$(0, 0, 0, 0, 0, 0, 0.1833, 0.4833, 0.3333, 0)$
	8	$(0, 0, 0, 0, 0, 0, 0, 0.0667, 0.3333, 0.6)$
9	2	$(0, 0.9778, 0.0222, 0, 0, 0, 0, 0, 0, 0)$
	3	$(0, 0, 0.9333, 0.0667, 0, 0, 0, 0, 0, 0)$
	4	$(0, 0, 0, 0.8667, 0.1333, 0, 0, 0, 0, 0)$
	5	$(0, 0, 0, 0, 0.7778, 0.2222, 0, 0, 0, 0)$
	6	$(0, 0, 0, 0, 0, 0.6667, 0.3333, 0, 0, 0)$
	7	$(0, 0, 0, 0, 0, 0, 0.5333, 0.4667, 0, 0)$
	8	$(0, 0, 0, 0, 0, 0, 0, 0.3778, 0.6222, 0)$
	9	$(0, 0, 0, 0, 0, 0, 0, 0, 0.2, 0.8)$

Table 4.2: (Continued) Signature of consecutive k-within-m-out-of-10:F systems.

n	m	k	E	A	n	m	k	S	A
10	3	2	0.8555	0.8623	30	6	3	0.8214	0.7767
	4	2	0.8266	0.8486		8	4	0.8937	0.8386
	4	3	0.9595	0.9419		10	4	0.8546	0.8284
	5	3	0.9425	0.9364		15	4	0.7932	0.7961
15	5	3	0.9133	0.8901	50	10	3	0.6387	0.6610
	6	3	0.8920	0.8832		15	3	0.5694	0.6147
	6	4	0.9677	0.9419		20	5	0.7597	0.7516
	10	4	0.9237	0.9244		25	5	0.7201	0.7242

Table 4.3: Survival function of consecutive k-within-m-out-of-n:F system with exchangeable Lomax components when $\alpha = 1$ and $t = 0.1$.

Chapter 5

Signature-based Anaylsis For Circular Case

In last years, usage of reliability concept in consecutive systems have been increased. These type of systems are mainly used for modelling telecommunication and transportation systems(Chang et al. [16]). By considering some assumptions and criteria, reliability analyses of such systems have been discussed. We can categorize a system according to the formation of its components (as either linear or circular) and operating principle (as either failure (F) or good (G) system). Thus, a circular consecutive k-within-m-out-of-n:F system involves n components in a circle fails if and only if at least k components fail among m consecutive components ($1 < k \leq m \leq n$). For $m = k$ and $m = n$, the circular consecutive k-within-m-out-of-n:F system transforms into circular consecutive k-out-of-n:F (this system fails if and only if at least k consecutive components fail) and k-out-of-n:F system (this system fails if and only if at least k components fail), respectively. One can find numerous publications about these systems e.g. Papastavridis [76], Iyer [52], Eryilmaz et al. [37], Eryilmaz, S. [30], Triantafyllou and Koutras [85] and Kan et al [53].

In this chapter, we will discuss the signature-based anaylsis of circular consecutive k-within-m-out-of-n:F system containing exchangeable components.

5.1 Aging Characteristics

Circular consecutive k-within-m-out-of-n:F system can be reformed as a combination of n number of k-out-of-m:F subsystems as dependent series systems. Meanwhile, lifetime of this system is expressed as

$$T_{k,m:n} = \min\left(Z_{k:m}^{(1)}, Z_{k:m}^{(2)}, \ldots, Z_{k:m}^{(n)}\right), \tag{5.1}$$

where $Z_{k:m}^{(j)}$ expresses the lifetime of k-out-of-m:F system with components

$$\begin{cases} T_j, T_{j+1}, \ldots, T_{j+m-1}, & \text{if } 1 \leq j \leq n-m+1, \\ T_j, T_{j+1}, \ldots, T_n, T_1, \ldots, T_{j+m-1-n}, & \text{if } n-m+1 < j \leq n. \end{cases}$$

For instance, let $n = 4$, $m = 3$ and $k = 2$ then

$$T_{2,3:4} = \min\left(Z_{2:3}^{(1)}, Z_{2:3}^{(2)}, Z_{2:3}^{(3)}, Z_{2:3}^{(4)}\right),$$

where

$Z_{2:3}^{(1)}$ is the second smallest in T_1, T_2, T_3 (or the lifetime of 2-out-of-3:F system with components T_1, T_2, T_3),

$Z_{2:3}^{(2)}$ is the second smallest in T_2, T_3, T_4 (or the lifetime of 2-out-of-3:F system with components T_2, T_3, T_4),

$Z_{2:3}^{(3)}$ is the second smallest in T_3, T_4, T_1 (or the lifetime of 2-out-of-3:F system with components T_3, T_4, T_1) and

$Z_{2:3}^{(4)}$ is the second smallest in T_4, T_1, T_2 (or the lifetime of 2-out-of-3:F system with components T_4, T_1, T_2).

It is obvious that,a circular consecutive k-within-m-out-of-n:F system will never fail if, and only if, at least $n-k+1$ components work. Hence, the signature vector has the form $(0, \ldots, 0, p_k, p_{k+1}, \ldots, p_n)$ and

$$P\{T_{k,m:n} > t\} = \sum_{i=k}^{n} p_i P\{T_{i:n} > t\}.$$

$N_C(j, k, m, n)$ defines the total number of circularly ordered n components consisting of j failed and $n - j$ working components, which also consists less than k failed components in any consecutive m components. If then, $r_i(n) = N_C(n - i, k, m, n)$ (Kuo and Zuo [58]) denotes the number of path sets of circular consecutive k-within-m-out-of-n:F system with i working components. $N_C(j, k, m, n)$ number can only be calculated just in the case of $k = 2$. Therefore, we can just obtain an explicit formula only for the signature of circular consecutive 2-within-m-out-of-n:F, system, which is shown by

$$p_i = \binom{n}{i}^{-1} \left[\frac{n-i+1}{i} \frac{n}{n-(i-1)(m-1)} \binom{n-(i-1)(m-1)}{i-1} - \frac{n}{n-i(m-1)} \binom{n-i(m-1)}{i} \right], \tag{5.2}$$

(Eryilmaz[31]).

The different values of k, m, and n have been used in MATLAB code to evaluate the signature of circular consecutive k-within-m-out-of-n:F system. The algorithm in MATLAB code has been improved by using of Eqs. (2.3) and (2.5). The signature is calculated in an acceptable CPU time for $n < 20$,. In Tables 5.1,5.2 and 5.3, it is illustrated below that the signature of the consecutive k-within-m-out-of-n:F systems, which is obtained by using different values of m and k in the case of $n = 5$, $n = 6$ and $n = 10$. The MATLAB code is accessible if requested.

k	m	n	$\mathbf{p}$
2	2	5	$(0, 1/2, 1/2, 0, 0)$
	3		$(0, 1, 0, 0, 0)$
	4		$(0, 1, 0, 0, 0)$
	5		$(0, 1, 0, 0, 0)$
3	3	5	$(0, 0, 1/2, 1/2, 0)$
	4		$(0, 0, 1, 0, 0)$
	5		$(0, 0, 1, 0, 0)$
4	4	5	$(0, 0, 0, 1, 0)$
	5		$(0, 0, 0, 1, 0)$

Table 5.1: Signature of circular consecutive k-within-m-out-of-5:F systems

k	m	n	$\mathbf{p}$
2	2	6	$(0, 4/10, 5/10, 1/10, 0, 0)$
	3		$(0, 4/5, 1/5, 0, 0, 0)$
	4		$(0, 1, 0, 0, 0, 0)$
	5		$(0, 1, 0, 0, 0, 0)$
	6		$(0, 1, 0, 0, 0, 0)$
3	3	6	$(0, 0, 3/10, 5/10, 2/10, 0)$
	4		$(0, 0, 9/10, 1/10, 0, 0)$
	5		$(0, 0, 1, 0, 0, 0)$
	6		$(0, 0, 1, 0, 0, 0)$
4	4	6	$(0, 0, 0, 2/5, 3/5, 0)$
	5		$(0, 0, 0, 1, 0, 0)$
	6		$(0, 0, 0, 1, 0, 0)$
5	5	6	$(0, 0, 0, 0, 1, 0)$
	6		$(0, 0, 0, 0, 1, 0)$

Table 5.2: Signature of circular consecutive k-within-m-out-of-6:F systems

k	m	n	$\mathbf{p}$
2	2	10	$(0, 56/252, 91/252, 75/252, 28/252, 2/252, 0, 0, 0, 0)$
	3		$(0, 16/36, 17/36, 3/36, 0, 0, 0, 0, 0, 0)$
	4		$(0, 2/3, 1/3, 0, 0, 0, 0, 0, 0, 0)$
	5		$(0, 8/9, 1/9, 0, 0, 0, 0, 0, 0, 0)$
	6		$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
	7		$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
	8		$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
	9		$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
	10		$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
3	3	10	$(0, 0, 7/84, 17/84, 26/84, 24/84, 10/84, 0, 0, 0)$
	4		$(0, 0, 63/252, 105/252, 82/252, 2/252, 0, 0, 0, 0)$
	5		$(0, 0, 21/42, 19/42, 2/42, 0, 0, 0, 0, 0)$
	6		$(0, 0, 5/6, 1/6, 0, 0, 0, 0, 0, 0)$
	7		$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
	8		$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
	9		$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
	10		$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
4	4	10	$(0, 0, 0, 6/126, 19/126, 35/126, 45/126, 21/126, 0, 0)$
	5		$(0, 0, 0, 8/42, 17/42, 15/42, 2/42, 0, 0, 0)$
	6		$(0, 0, 0, 60/126, 65/126, 1/126, 0, 0, 0, 0)$
	7		$(0, 0, 0, 37/42, 5/42, 0, 0, 0, 0, 0)$
	8		$(0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$
	9		$(0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$
	10		$(0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$

Table 5.3: Signature of circular consecutive k-within-m-out-of-10:F systems

5	5	10	$(0,0,0,0,5/126,19/126,39/126,49/126,14/126,0)$
	6		$(0,0,0,0,25/126,59/126,42/126,0,0,0)$
	7		$(0,0,0,0,25/42,17/42,0,0,0,0)$
	8		$(0,0,0,0,125/126,1/126,0,0,0,0)$
	9		$(0,0,0,0,1,0,0,0,0,0)$
	10		$(0,0,0,0,1,0,0,0,0,0)$
6	6	10	$(0,0,0,0,0,4/84,17/84,35/84,28/84,0)$
	7		$(0,0,0,0,0,24/84,53/84,7/84,0,0)$
	8		$(0,0,0,0,0,37/42,5/42,0,0,0)$
	9		$(0,0,0,0,0,1,0,0,0,0)$
	10		$(0,0,0,0,0,1,0,0,0,0)$
7	7	10	$(0,0,0,0,0,0,3/36,13/36,20/36,0)$
	8		$(0,0,0,0,0,0,7/12,5/12,0,0)$
	9		$(0,0,0,0,0,0,1,0,0,0)$
	10		$(0,0,0,0,0,0,1,0,0,0)$
8	8	10	$(0,0,0,0,0,0,0,2/9,7/9,0)$
	9		$(0,0,0,0,0,0,0,1,0,0)$
	10		$(0,0,0,0,0,0,0,1,0,0)$
9	9	10	$(0,0,0,0,0,0,0,0,1,0)$
	10		$(0,0,0,0,0,0,0,0,1,0)$

Table 5.3: Signature of circular consecutive k-within-m-out-of-10:F systems

Eryilmaz [31] obtained the following representations for the minimal and maximal signatures of circular consecutive k-within-m-out-of-n:F system.

$$\alpha_i = \binom{n}{i} \sum_{j=n-i}^{z(n,m,k)} (-1)^{i+j-n} \binom{i}{n-j} \frac{N_C(j,k,m,n)}{\binom{n}{j}} \tag{5.3}$$

for $n - z(n,m,k) \leq i \leq n$, ($\alpha_i = 0$ for $1 \leq i < n - z(n,m,k)$) and

$$\beta_i = \binom{n}{i} \sum_{j=0}^{\min(i,z(n,m,k)} (-1)^{i-j+1} \binom{i}{j} \frac{N_C(j,k,m,n)}{\binom{n}{j}}.$$

Thus, using the minimal signature we also have

$$P\left\{T_{k,m:n} > t\right\} = \sum_{i=n-z(n,m,k)}^{n} \alpha_i P\left\{T_{1:i} > t\right\}. \tag{5.4}$$

By using following Lemma, we can obtain $z(n,m,k)$ which is actually the maximum number of failed components such that the system still functions.

Lemma 5.1 *For* $1 < k \leq m \leq n$,

$$z(n,m,k) = \left\lfloor \frac{n(k-1)}{m} \right\rfloor$$

where $\lfloor x \rfloor$ *denotes the greatest integer less than* x.

Proof. There are totally n overlapping windows of size m. Each window can contain at most $k-1$ failed components. So the proportion of failed components in a window is at most $\frac{k-1}{m.}$. Since there are n windows, then

$$z(n,m,k) = \left\lfloor \frac{n(k-1)}{m} \right\rfloor.$$

Thus, the proof is complete. □

It can be seen from the tables 5.1-5.3 that for some values of n, m and k the signature of circular consecutive k-within-m-out-of-n:F system coincides with the signature of k-out-of-n:F system other than the case $n = m$.

Lemma 5.2 *For $m \geq \left\lfloor \frac{n(k-1)}{k} \right\rfloor + 1$, a circular consecutive k-within-m-out-of-n:F system turns into k-out-of-n:F system.*

Proof. By using Lemma 5.1, for a circular consecutive k-within-m-out-of-n:F system the maximum number of failed components such that system can still work is $\left\lfloor \frac{n(k-1)}{m} \right\rfloor$. For k-out-of-n:F system, the maximum number of failed component such that system can still work is $k-1$

$$\min_{m} \left\lfloor \frac{n(k-1)}{m} \right\rfloor = k-1.$$

The minimum m which satisfies the above equation is $\left\lfloor \frac{n(k-1)}{k} \right\rfloor + 1$. Thus, the proof is complete. □

Remark (Eryilmaz and Kan [36]). If z_ϕ is the maximum number of failed components such that a coherent system with lifetime $\phi(T_1, \ldots, T_n)$ can still work and $T_1, \ldots, T_n$ are IID with common failure rate function $r(t)$, then

$$\lim_{t \to \infty} \frac{r_\phi(t)}{r(t)} = n - z_\phi,$$

where $r_\phi(t)$ is the failure rate function associated with the coherent structure ϕ.

Now we will examine the IFR property of circular consecutive k-within-m-out-of-n:F system for some values of k,m and n. Let $n = 6$ $m = 4$ and $k = 3$. Then from Table 5.2 the signature of circular consecutive 3-within-4-out-of-6:F system is $(0, 0, 9/10, 1/10, 0, 0)$ and $z(n, m, k) = 3$. By using equation (4.6), $p_3 > (6-3)p_4$, we conclude that circular consecutive 3-within-4-out-of-6:F system does not preserve the IFR property (Triantafyllou and Koutras [85]).

5.2 Stochastic Ordering Results

As it is mentioned in previous chapter's section 4.2, comparison of systems have great importance. In this section, we will use similar expression as in section 4.2 but this time the coherent system is circular consecutive k-within-m-out-of-n:F system

Theorem 5.3 *Let $T_1, \ldots, T_n$ be exchangeable. Then for $1 < k \leq m \leq n$:*

(a) $T_{k,m:n} \leq_{st} Z^{(i)}_{k:m}$, $i = 1, 2, \ldots, n$;,

(b) $T_{k:n} \leq_{st} T_{k,m:n}$; where $T_{k:n}$ denotes the lifetime of the k-out-of-n:F system.

(c) $T_{k,m:n} \leq_{st} T_{m,m:n}$;

Proof. The proof of part (a) can be easily seen from (5.1). The form of signature vectors of k-out-of-n:F system and circular consecutive k-within-m-out-of-n:F system are as $\mathbf{p} = (0, 0, \ldots, 0, 1_k, 0, \ldots, 0)$ and $\mathbf{q} = (0, 0, \ldots, 0, q_k, q_{k+1}, \ldots, q_n)$, respectively. Since $\mathbf{p} \leq_{st} \mathbf{q}$ we have $T_{k:n} \leq_{st} T_{k,m:n}$. In part (c), one can define $X_i(t) = 1$ if $T_i \leq t$, and $X_i(t) = 0$ if $T_i > t$. Hence, $m \geq k$

$$
\begin{aligned}
P\{T_{k,m:n} > t\} = P\Bigg\{ & \sum_{i=1}^{m} X_i(t) < k, \sum_{i=2}^{m+1} X_i(t) < k, \ldots, \sum_{i=n-m+1}^{n} X_i(t) < k, \\
& \sum_{i=n-m+2}^{n} X_i(t) + \sum_{i=1}^{1} X_i(t) < k, \ldots, \sum_{i=n}^{n} X_i(t) + \sum_{i=1}^{m-1} X_i(t) < k \Bigg\} \\
\leq P\Bigg\{ & \sum_{i=1}^{m} X_i(t) < m, \sum_{i=2}^{m+1} X_i(t) < m, \ldots, \sum_{i=n-m+1}^{n} X_i(t) < m, \\
& \sum_{i=n-m+2}^{n} X_i(t) + X_1(t) < m, \ldots, X_n(t) + \sum_{i=1}^{m-1} X_i(t) < m \Bigg\} \\
= P\{ & T_{m,m:n} > t\}
\end{aligned}
$$

which means that the lifetime of circular consecutive k-within-m-out-of-n:F system is stochastically smaller than the lifetime of circular consecutive m-out-of-n:F system and so part (c) is proved. □

By using part (c) of Theorem 5.3, on the right-hand side of equation when we substitute $k+1$ into k, we can observe that $T_{k,m:n}$ is stochastically increasing in k, i.e., $T_{k,m:n} \leq_{st} T_{k+1,m:n}$. Similarly it immediately follows that $T_{k,m:n}$ is stochastically decreasing in m and n, i.e., $T_{k,m+1:n} \leq_{st} T_{k,m:n}$ and $T_{k,m:n+1} \leq T_{k,m:n}$. One can prove the hazard rate and likelihood ratio orderings similarly.

By using Theorem 5.3 we get the following relations for the survival function and mean time to failure (MTTF) of circular consecutive k-within-m-out-of-n:F system consisting of exchangeable components. For large values of n calculations are getting difficult. So these type of relations are valuable for evaluating the reliability of such complex systems.

Corollary 5.4 *If $T_1, T_2, \ldots, T_n$ are exchangeable, then the survival function and the MTTF of circular consecutive k-within-m-out-of-n:F system can be rewritten as the convex combination of k-out-of-n:F and circular consecutive k-out-of-m:F systems.*

$$P\{T_{k,m:n} > t\} = w_{k,m,n} P\{T_{k:n} > t\} + (1 - w_{k,m,n}) P\{T_{m,m:n} > t\}$$

and

$$E(T_{k,m:n}) = w_{k,m,n} E(T_{k:n}) + (1 - w_{k,m,n}) E(T_{m,m:n})$$

for $w_{k,m,n} \in [0,1]$ and $1 < k \leq m \leq n$.

Proof. If $T_1, T_2, \ldots, T_n$ are exchangeable, then

$$Z_{k:m}^{(1)} \stackrel{d}{=} Z_{k:m}^{(2)} \stackrel{d}{=} \ldots \stackrel{d}{=} Z_{k:m}^{(n)},$$

and since from part (b) of Theorem 5.3 we have $P\{T_{k,m:n} > t\} \leq P\{T_{k:n} > t\}$. In addition, from part (c) of Theorem 5.3 we find $P\{T_{k,m:n} > t\} \geq P\{T_{m,m:n} > t\}$.Hence, the proof is done. □

5.3 An Example

Consider a circular consecutive k-within-m-out-of-n:F system that components have an multivariate exchangeable Pareto distribution with the joint survival function

$$\bar{F}_\alpha(t_1, t_2, \ldots, t_n) = (\sum_{i=1}^{n} t_i - n + 1)^{-\alpha}$$

for $\alpha > 0, t_i > 1, i = 1, \ldots, n$. Then $P\{T_{1:j} > t\} = (1 + j(t-1))^{-\alpha}$, and

$$P\{T_{i:n} > t\} = 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} (1 - (1 + j(t-1))^{-\alpha}).$$

For $\alpha > 1$,

$$E(T_{i:n}) = \frac{1}{\alpha - 1} \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \frac{1}{j},$$

$1 \leq i \leq n$.

In tables 5.4- 5.8 different approximate values $(R_{sim}, R_{0.5}, R_w)$ of $P\{T_{k,m:n} > t\}$ along with the exact values (R_{exact}) when $n = 8, 12, 15$ and in table 5.9 and 5.10 different approximate values $(R_{sim}, R_{0.5}, R_w)$ of $P\{T_{k,m:n} > t\}$ when $n = 30$ are presented. Moreover, in tables 5.11, 5.12 and figure 5.1 exact,simulated and approximate values of mean time to failure $(MTTF_{exact}, MTTF_{sim}, MTTF_{0.5}, MTTF_w)$ for specific values of n, m and k along with the plots of hazard rate of the system are given. After $100,000$ trials, simulation results are obtained. Surely, the performance of the first approximation $(R_{0.5})$ mainly depends on the values of n, m,and k. For instance, when m comes closer to n, we are getting the better approximation. Thus, to find more accurate approximation, we have to consider $P\{T_{k:n} > t\}$ and $P\{T_{m,m:n} > t\}$ as a function of n, m and k. By using the formulas of the maximum number of failed component a better approximation, denoted by(R_w), for $k + 1 \leq m \leq \left\lfloor \frac{n(k-1)}{k} \right\rfloor$

$$R_w \tilde{=} w_{k,m,n} P\{T_{k:n} > t\} + (1 - w_{k,m,n}) P\{T_{m,m:n} > t\}$$

where

$$w_{k,m,n} = \begin{cases} 0.5 & \text{if } k+1 = \left\lfloor \frac{n(k-1)}{k} \right\rfloor \\ \frac{5(m-k-1)-e^{(0.9336-0.0052n)}(m-\left\lfloor \frac{n(k-1)}{k} \right\rfloor)}{5(\left\lfloor \frac{n(k-1)}{k} \right\rfloor -k-1)} & \text{if } k+1 \neq \left\lfloor \frac{n(k-1)}{k} \right\rfloor \end{cases}$$

is found. For a better approximation of R_w, selecting the proper value of w is important. We obtain w as above. A circular consecutive k-within-m-out-of-n:F system converts into circular consecutive k-out-of-n:F for $m = k$ and k-out-of-n:F system for $m \geq \left\lfloor \frac{n(k-1)}{k} \right\rfloor + 1$ which is proved in Lemma 5.2. Thus, weight formula has been found for $k+1 \leq m \leq \left\lfloor \frac{n(k-1)}{k} \right\rfloor$. When m equals to lower and upper bounds ($k+1 = m = \left\lfloor \frac{n(k-1)}{k} \right\rfloor$), weight has to be 0.5 which shows that two systems, circular consecutive k-out-of-n:F and k-out-of-n:F, have equal weights. When m equals to only the lower bound by using regression analysis on exact reliability values of the systems weight can be found as $\frac{e^{(0.9336-0.0052n)}}{5}$. Further, when m is equal to only the upper bound, weight is 1. At the intervals of these bounds, weight formula has a linear equation with a slope of $\frac{1-e^{0.9336-0.0052n}}{\left\lfloor \frac{n(k-1)}{k} \right\rfloor -(k+1)}$. Shortly, w can be used as a leverage between these two systems. It can be immediately seen that $0.5 \leq w_{k,m,n} \leq 1$. So by Corollary 5.4, $w_{k,m,n}$ can be used.

α	t	R_{exact}	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.8228	0.8241	0.8389	0.8389
	1.3	0.4924	0.4934	0.5215	0.5215
	1.5	0.3273	0.3260	0.3547	0.3547
	1.7	0.2363	0.2353	0.2600	0.2600
	1.9	0.1805	0.1810	0.2002	0.2002
2	1.1	0.7493	0.7507	0.7697	0.7697
	1.3	0.3637	0.3653	0.3954	0.3954
	1.5	0.2083	0.2083	0.2333	0.2333
	1.7	0.1337	0.1334	0.1527	0.1527
	1.9	0.0929	0.0929	0.1073	0.1073

Table 5.4: Simulation and Approximations for the survival function when $n = 6,\ m = 3,\ k = 2$.

α	t	R_{exact}	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.7812	0.7790	0.7799	0.7815
	1.3	0.4283	0.4268	0.4387	0.4408
	1.5	0.2711	0.2705	0.2840	0.2857
	1.7	0.1904	0.1908	0.2014	0.2027
	1.9	0.1428	0.1436	0.1526	0.1537
2	1.1	0.6943	0.6969	0.6974	0.6996
	1.3	0.2982	0.2971	0.3130	0.3151
	1.5	0.1598	0.1598	0.1723	0.1736
	1.7	0.0989	0.0991	0.1090	0.1100
	1.9	0.0670	0.0669	0.0749	0.0756

Table 5.5: Simulation and Approximations for the survival function when $n = 8,\ m = 3,\ k = 2$.

α	t	R_{exact}	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.8845	0.8844	0.9262	0.8863
	1.3	0.5204	0.5194	0.6547	0.5389
	1.5	0.3345	0.3342	0.4757	0.3575
	1.7	0.2358	0.2354	0.3616	0.2577
	1.9	0.1772	0.1773	0.2869	0.1970
2	1.1	0.8176	0.8157	0.8840	0.8219
	1.3	0.3753	0.3760	0.5340	0.3998
	1.5	0.2021	0.2022	0.3411	0.2268
	1.7	0.1249	0.1251	0.2353	0.1457
	1.9	0.0845	0.0844	0.1705	0.1012

Table 5.6: Simulation and Approximations for the survival function when $n = 12,\ m = 8,\ k = 4$.

α	t	R_{exact}	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.9954	0.9971	0.9933	0.9917
	1.3	0.8667	0.8683	0.8659	0.8467
	1.5	0.6900	0.6882	0.7177	0.6866
	1.7	0.5495	0.5490	0.5927	0.5589
	1.9	0.4462	0.4443	0.4980	0.4647
2	1.1	0.9904	0.9994	0.9832	0.9808
	1.3	0.7817	0.7823	0.7915	0.7630
	1.5	0.5505	0.5479	0.5956	0.5578
	1.7	0.3942	0.3948	0.4559	0.4189
	1.9	0.2925	0.2922	0.3564	0.3235

Table 5.7: Simulation and Approximations for the survival function when $n = 15,\ m = 10,\ k = 8$.

α	t	R_{exact}	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.9879	0.9856	0.9916	0.9872
	1.3	0.7916	0.7929	0.8663	0.7974
	1.5	0.5899	0.5903	0.7151	0.6059
	1.7	0.4505	0.4513	0.5943	0.4725
	1.9	0.3557	0.3568	0.4966	0.3787
2	1.1	0.9761	0.9741	0.9859	0.9754
	1.3	0.6783	0.6790	0.7891	0.6891
	1.5	0.4371	0.4349	0.5959	0.4607
	1.7	0.2967	0.2961	0.4551	0.3236
	1.9	0.2126	0.2126	0.3563	0.2387

Table 5.8: Simulation and Approximations for the survival function when $n = 15,\ m = 12,\ k = 8$.

α	t	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.9314	0.8585	0.8507
	1.3	0.5706	0.5856	0.5689
	1.5	0.3685	0.4307	0.4153
	1.7	0.2592	0.3300	0.3172
	1.9	0.1942	0.2629	0.2521
2	1.1	0.8837	0.7955	0.7838
	1.3	0.4184	0.4742	0.4564
	1.5	0.2251	0.3129	0.2990
	1.7	0.1366	0.2182	0.2079
	1.9	0.0918	0.1597	0.1520

Table 5.9: Simulation and Approximations for the survival function when $n = 30,\ m = 10,\ k = 6$.

α	t	R_{sim}	$R_{0.5}$	R_w
1.5	1.1	0.9865	0.9254	0.9302
	1.3	0.8094	0.6759	0.6920
	1.5	0.6087	0.5239	0.5412
	1.7	0.4633	0.4200	0.4359
	1.9	0.3666	0.3432	0.3571
2	1.1	0.9818	0.8793	0.8870
	1.3	0.6986	0.5748	0.5941
	1.5	0.4560	0.4081	0.4257
	1.7	0.3081	0.3011	0.3153
	1.9	0.2206	0.2287	0.2400

Table 5.10: Simulation and Approximations for the survival function when $n = 30,\ m = 10,\ k = 8$.

k	m	n	MTTF_{exact}	MTTF_{sim}	$\text{MTTF}_{0.5}$	MTTF_w
2	3	6	0.8333	0.8333	0.9168	0.9168
2	3	8	0.6785	0.6802	0.7195	0.7239
4	5	10	1.4134	1.4230	1.2622	1.4794
4	6	10	1.1357	1.1351	1.4665	1.2209
4	6	12	1.0582	1.0584	1.2894	1.1768
4	8	12	0.8176	0.8173	1.2839	0.9046
6	8	12	1.6094	1.6113	2.0424	1.8187
6	10	15	1.1348	1.1279	1.7484	1.3046
6	12	15	0.9840	0.9831	1.7455	0.9758
8	12	15	1.5353	1.5356	2.3794	1.6965
8	10	15	1.9513	1.9587	2.3753	2.1850
8	10	30	–	1.5372	1.5835	1.6450
6	10	30	–	0.8881	1.1892	1.1451
6	12	30	–	0.7319	1.1895	1.0510
8	12	30	–	1.2090	1.5978	1.5287

Table 5.11: MTTF of circular consecutive k-within-m-out-of-n:F systems for $\alpha = 1.5$

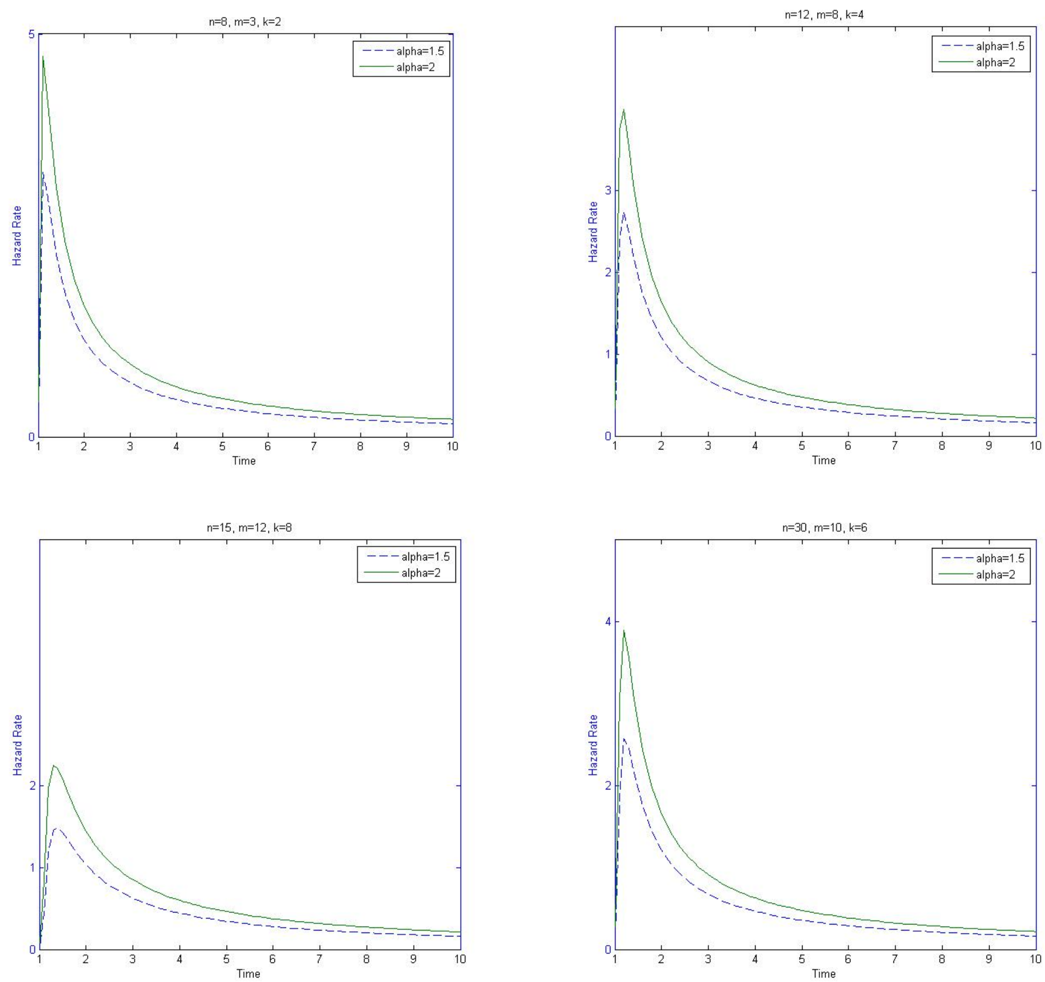


Figure 5.1: Hazard rate of the system for the multivariate exchangeable Pareto distribution

k	m	n	MTTF_{exact}	MTTF_{sim}	$\text{MTTF}_{0.5}$	MTTF_{w}
2	3	6	0.4166	0.4184	0.4581	0.4581
2	3	8	0.3392	0.3388	0.3598	0.3620
4	5	10	0.7067	0.7055	0.7288	0.7373
4	6	10	0.5678	0.5677	0.7319	0.6097
4	6	12	0.5291	0.5273	0.6439	0.5878
4	8	12	0.4088	0.4098	0.6442	0.4529
6	8	12	0.8047	0.8027	1.0217	0.9097
6	10	15	0.5674	0.5675	0.8718	0.6513
6	12	15	0.4920	0.4924	0.8697	0.4892
8	12	15	0.7676	0.7662	1.1905	0.8485
8	10	15	0.9756	0.9741	1.1872	1.0921
8	10	30	–	0.7886	0.7975	0.8286
6	10	30	–	0.4415	0.5952	0.5731
6	12	30	–	0.3664	0.5971	0.5275
8	12	30	–	0.6011	0.7967	0.7623

Table 5.12: MTTF of circular consecutive k-within-m-out-of-n:F systems for $\alpha = 2$

Chapter 6

Generalized m-cons.-k-out-of-n:F System

A circular m-consecutive-k-out-of-n:F system with non-overlapping runs consists of n components which are circularly ordered such that system fails if, and only if there are at least m non-overlapping runs of k consecutive failed components. The application areas of such system are oil pipeline systems, vacuum system in an electron accelerator, computer ring networks and microwave stations of a telecom network. For instance, consider a microwave signal transmitting system combined of many stations which are ordered linearly or circularly. The system fails if and only if at least k adjacent stations fail in the system. This chapter mentions about more generalized version of m-consecutive-k-out-of-n:F system in circular case, that is named as circular m-consecutive-k,l-out-of-n:F system where the linear case is introduced by Eryilmaz & Mahmoud [39]. For $l = 0$ and $l = k-1$, this system turns into circular m-consecutive-k-out-of-n:F system with non-overlapping runs and circular m-consecutive-k-out-of-n:F system with overlapping runs,respectively. For $m = 1$, ordinary consecutive-k-out-of-n:F system is obtained. The circular m-consecutive-k,l-out-of-n:F system combined of n circularly ordered components such that the system fails if and only if there are at least m l-overlapping runs of k consecutive components ($n \geq m(k-l)+l, l < k$). This system has wider applications in specific areas such as infrared detecting

and bank automatic payment systems. For instance, consider a system which is constructed by the circularly arranged identical and independent transmitters. The main principle of the given system is collecting the sufficient amount of information and transferring it. Basic functioning principle for the system can be defined as follows:
The information is gathered by the k consecutive transmitters. New information can be provided with the reuse of the given number of these transmitters which is denoted by "l". When at least m blocks of k consecutive working transmitters collect information then it is transferred. This is an example for G-system.On the other hand, for F systems, the system fails if and only if there are at least m blocks of length k consecutive disfunctioning transmitters, consisting l reused ones. Consider a system contains 16 transmitters that are circularly ordered (where 1st component and 16th components are adjacent) and 0's and 1's represent functioning and disfunctioning transmitters respectively. For m=3 and k=4 the following line up 1111101111110110 will function for $l = 0, 1$ and will fail for $l = 2, 3$.
In this chapter, the number of path sets including a certain number of working components in a circular arrangement is obtained by using binomial distribution of order k for l-overlapping runs of length k, studied by Aki & Hirano [4] and Makri & Philippou [62]. After derivation of this formula, reliability and system signature of this system has been computed. All proofs of this chapter are given in the Appendix.

6.1 Reliability Evaluation

Assume that a system contains independent and identically distributed components having common reliability p, then the reliability of whole system can be computed as

$$R = \sum_{i=n-z_\phi}^{n} r_i(n) p^i (1-p)^{n-i} \tag{6.1}$$

where $r_i(n)$ denotes the number of path sets of a system including i working components and z_ϕ is the minimum number of working components such that the system can still work successfully.

Lemma 6.1 *For a circular m-consecutive-k,l-out-of-n:F system,*

$$k_\phi = k + (m-1)(k-l)$$

and

$$z_\phi = n - 2 - \left\lfloor \frac{n-k-(m-1)(k-l)-1}{k} \right\rfloor$$

where $n \geq m(k-l)+l, l < k$ *and* $\lfloor x \rfloor$ *denotes the integer part of* x.

Lemma 6.2 (Makri et al.[63]) *Let* $C(\beta;\alpha;r-\alpha;m_1-1,m_2-1)$ *be the number of allocations of* β *indistinguishable balls into* r *distinguishable cells,* α *specified of which have capacity* m_1-1 *and each of the rest* $r-\alpha$ *has capacity* m_2-1*.Then*

$$C(\beta;\alpha;r-\alpha;m_1-1,m_2-1) = \sum_{j_1=0}^{\left\lfloor \frac{\beta}{m_1} \right\rfloor} \sum_{j_2=0}^{\left\lfloor \frac{\beta-m_1j_1}{m_2} \right\rfloor} (-1)^{j_1+j_2} \binom{\alpha}{j_1} \binom{r-\alpha}{j_2} \times \binom{\beta-m_1j_1-m_2j_2+r-1}{r-1}$$

Note that $C(n-i;i;k-1)$ represents the number of ways of $n-i$ success can be placed into i linear cells with no cell receiving more than $k-1$. Considering cyclic arrangement, each such arrangements gives n arrangements by rotation. But the set of the $nC(n-i;i;k-1)$ arrangements is partitioned into sets of i like arrangements. So, in circular case

$$C^c(n-i;i;k-1) = \frac{n}{i} C(n-i;i;k-1)$$

where $C^c(n-i,i,k-1)$ denotes the number of cyclic arrangement of $n-i$ successes and i cells such that each cell contains at most $k-1$ consecutive success. Now, by using Lemma 6.2, $r_i(n)$ can be calculated for the circular m-consecutive-k,l-out-of-n:F system.

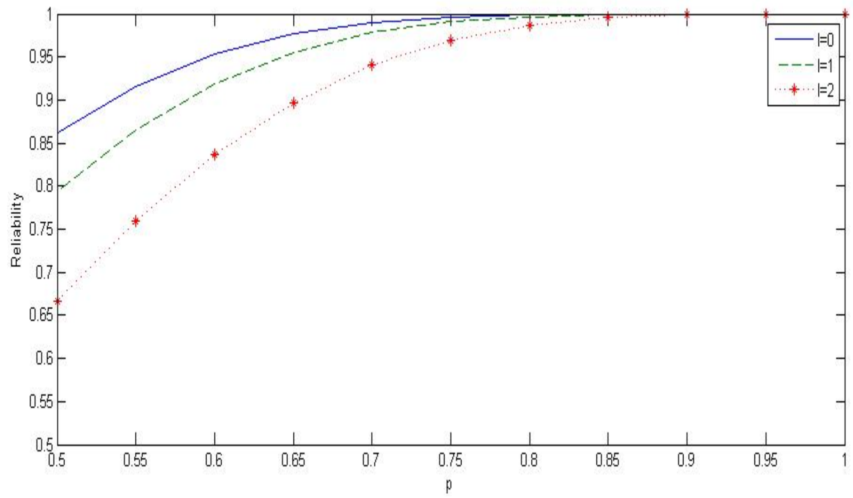


Figure 6.1: Reliability of the circular 2-consecutive-3,l-out-of-10:F system for different values of l

Theorem 6.3 *The number of path sets of circular m-consecutive-k,l-out-of-n:F system including i working components is*

$$r_i(n) = \frac{n}{i}\Bigg[C(n-i, i, 0, k-1, k-1) + \sum_{s=1}^{m-1}\sum_{a=1}^{\min(i,s)}\binom{i}{a}\binom{s-1}{a-1}C(n-i-al-s(k-l), i-a, k-l-1, k-1)\Bigg]$$

for $n - z_\phi \leq i \leq n$.

By using Theorem, one can compute the reliability of the circular m-consecutive-k,l-out-of-n:F system. In Fig. 6.1, it can be easily seen that the reliability of the circular 2-consecutive-3,l-out-of-10:F system decreases by l and it is bounded by circular 2-consecutive-3,l-out-of-10:F system with overlapping and non-overlapping runs for $l = 0, 1, 2$. For example, one can illustrate the computation of reliability not only by using $r_i(n)$ but also by hand for the values $n = 5$, $m = 2$, and $k = 2$. In table 6.1, the possible binary states are listed for

	Number of working components				
$l = 0$	1	2	3	4	5
	–	00111	00011	00001	00000
		01011	00101	00010	
		10011	01001	00100	
		01101	10001	01000	
		10101	00110	10000	
		11001	01010		
		01110	10010		
		10110	01100		
		11010	10100		
		11100	11000		
$r_i(5)$	0	10	10	5	1
$l = 1$	0	1	2	3	4
	–	01011	00011	00001	00000
		01101	00101	00010	
		10101	01001	00100	
		10110	10001	01000	
		11010	00110	10000	
			01010		
			10010		
			01100		
			10100		
			11000		
$r_i(5)$	0	5	10	5	1

Table 6.1: All possible binary sequences

both $l = 0$ and $l = 1$.

For $l = 0$, the reliability of circular 2-consecutive-2,0-out-of-5:F system is

$$R = 10p^2(1-p)^3 + 10p^3(1-p)^2 + 5p^4(1-p) + p^5$$

which can also be computed by using equation (6.1) for $r_i(5) = (0, 10, 10, 5, 1)$. By using same way, we can calculate the reliability of circular 2-consecutive-2,1-out-of-5:F system. The functioning states are shown in table, therefore the reliability of circular 2-consecutive-2,1-out-of-5:F system is

$$R = 5p^2(1-p)^3 + 10p^3(1-p)^2 + 5p^4(1-p) + p^5$$

n	m	k	l	$\mathbf{p}$
10	2	3	0	$(0, 0, 0, 0, 0, 0.119, 0.381, 0.5, 0, 0)$
			1	$(0, 0, 0, 0, 0.0397, 0.2222, 0.4881, 0.25, 0, 0)$
			2	$(0, 0, 0, 0.476, 0.1508, 0.3492, 0.3690, 0.0833, 0, 0)$
10	3	3	0	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$
			1	$(0, 0, 0, 0, 0, 0, 0.0833, 0.5833, 0.3333, 0)$
			2	$(0, 0, 0, 0, 0.0397, 0.1508, 0.4762, 0.3333, 0, 0)$
10	2	4	0	$(0, 0, 0, 0, 0, 0, 0, 0.3333, 0.6667, 0)$
			1	$(0, 0, 0, 0, 0, 0, 0.0833, 0.4722, 0.4444, 0)$
			2	$(0, 0, 0, 0, 0, 0.0476, 0.2024, 0.5278, 0.2222, 0)$
			3	$(0, 0, 0, 0, 0.0397, 0.1518, 0.3095, 0.5, 0, 0)$
12	2	4	0	$(0, 0, 0, 0, 0, 0, 0, 0.0606, 0.2667, 0.4909, 0.1818, 0)$
			1	$(0, 0, 0, 0, 0, 0, 0.0152, 0.1182, 0.3576, 0.5091, 0, 0)$
			2	$(0, 0, 0, 0, 0, 0.013, 0.0628, 0.203, 0.4303, 0.2909, 0, 0)$
			3	$(0, 0, 0, 0, 0.0152, 0.0628, 0.1494, 0.2939, 0.3515, 0.1273, 0, 0)$
12	3	3	0	$(0, 0, 0, 0, 0, 0, 0, 0, 0.1818, 0.5455, 0.2727, 0)$
			1	$(0, 0, 0, 0, 0, 0, 0.0152, 0.1545, 0.5030, 0.3273, 0, 0)$
			2	$(0, 0, 0, 0, 0.0152, 0.0628, 0.21, 0.3788, 0.3333, 0, 0, 0)$

Table 6.2: The signatures of circular m-consecutive-k,l-out-of-n:F system for some values

which can verify that the computations are true with the corresponding $r_i(5) = (0, 5, 10, 5, 1)$ for $l = 1$.

6.2 Numerical Examples

Applying the formula (2.6) for the structure of circular m-consecutive-k,l-out-of-n:F system, we can restate it as follows

$$P(T > t) = \sum_{i=k_\phi}^{z_\phi+1} p_i P(T_{i:n} > t) \tag{6.2}$$

In Table 6.2, the signatures of circular m-consecutive-k,l-out-of-n:F system for some values are presented

By using the formula (6.2) the reliability of circular 2-consecutive-3,l-out-of-10:F system can be written as

$$
\begin{aligned}
P(T^0_{3,2:10} > t) &= 0.0119P(T_{6:10} > t) + 0.381P(T_{7:10} > t) + 0.5P(T_{8:10} > t) \\
P(T^1_{3,2:10} > t) &= 0.0397P(T_{5:10} > t) + 0.2222P(T_{6:10} > t) + 0.4881P(T_{7:10} > t) + \\
&\quad 0.25P(T_{8:10} > t) \\
P(T^2_{3,2:10} > t) &= 0.476P(T_{4:10} > t) + 0.1508P(T_{5:10} > t) + 0.3492P(T_{6:10} > t) + \\
&\quad 0.3690P(T_{7:10} > t) + +0.0833P(T_{8:10} > t)
\end{aligned}
$$

and mean time to failure (MTTF) of the system are as

$$
\begin{aligned}
E(T^0_{3,2:10}) &= 0.0119E(T_{6:10}) + 0.381E(T_{7:10}) + 0.5E(T_{8:10}) \\
E(T^1_{3,2:10}) &= 0.0397E(T_{5:10}) + 0.2222E(T_{6:10}) + 0.4881E(T_{7:10}) + 0.25E(T_{8:10}) \\
E(T^2_{3,2:10}) &= 0.476E(T_{4:10}) + 0.1508E(T_{5:10}) + 0.3492E(T_{6:10}) + 0.3690E(T_{7:10}) + \\
&\quad 0.0833E(T_{8:10})
\end{aligned}
$$

The signature of a system has been found to be useful for comparing systems in terms of various stochastic orderings. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be, respectively, the signatures of coherent systems $T = \phi(T_1, \ldots, T_n)$ and $Z = \psi(T_1, \ldots, T_n)$, both used on n i.i.d. components. In 1999 Kochar et al. [55] proved that if $\mathbf{p} \leq_{st} \mathbf{q}$, then $T \leq_{st} Z$. By using table 6.2, it is easy to see that signature of circular m-consecutive-k,l-out-of-n:F system is stochastically less or equal than the signature of circular m-consecutive-k,$l-1$-out-of-n:F system for $l = 1, \ldots, k-1$. Hence,

$$T^{(l)}_{k,m:n} \leq_{st} T^{(l-1)}_{k,m:n} \text{ for } l = 1, \ldots, k-1$$

Assuming T_i's are s-independent and have common exponential distribution function with mean 1, the expected value of the ith smallest component is equal to

$$E(T_{i:n}) = \sum_{j=1}^{i} \frac{1}{n-j+1}$$

n	m	k	l	$MTTF$
10	2	3	0	1.2325
			1	1.1056
			2	0.9389
10	3	3	0	1.9290
			1	1.5679
			2	1.1512
10	2	4	0	1.7623
			1	1.6234
			2	1.4448
			3	1.2067
12	2	4	0	1.5699
			1	1.4032
			2	1.2798
			3	1.1224
12	3	3	0	1.6790
			1	1.3335
			2	1.0305

Table 6.3: MTTF of circular m-consecutive-k,l-out-of-n:F system for some values

for $i = 1, 2, \ldots, n$. By using this expectation one can easily compute the MTTF of circular m-consecutive-k,l-out-of-n:F system for some values of n, m, k, l as follows: We see that the MTTF is increasing in m and k and decreasing in n and l which is consistent with figure 6.1.

Chapter 7

Summary and Conclusions

In this thesis, the concept of system signature, which is firstly introduced by Samaniego [79], is used for evaluating of the reliability characteristics of generalized version of consecutive-k systems, One of the generalized system is consecutive k-within-m-out-of-n:F system containing n linearly ordered components and fails if and only if there are m consecutive components which include among them at least k failed components. For $m = k$ and $m = n$, this system turns into consecutive k-out-of-n:F (this system fails if and only if at least k consecutive components fail) and k-out-of-n:F (this system fails if and only if at least k components fail) systems, respectively. We obtained signature-based analysis for both linear and circular consecutive k-within-m-out-of-n:F system consisting of exchangeable components. Stochastic comparison of different systems are also studied.

The other generalized system that we considered is named as circular m-consecutive-k,l-out-of-n:F system combined of n circularly ordered components such that the system fails if and only if there are at least m l-overlapping runs of k consecutive components ($n \geq m(k-l)+l, l < k$). For $l = 0$ and $l = k-1$, this system turns into circular m-consecutive-k-out-of-n:F system with non-overlapping runs and circular m-consecutive-k-out-of-n:F system with overlapping runs, respectively. For $m = 1$, ordinary consecutive-k-out-of-n:F system is obtained. The parameter l is a leverage in this system which provides that the reliability of this system is bounded by overlapping ($0 < l < k, k > 1$) and non-overlapping ($l = 0$) cases. We calculate $r_i(n)$ which denotes the number of path sets of circular m-consecutive-k,l-out-of-n:F system including i working components. A combinatorial formula, which calculates the exact reliability of this system, is given. Signature based analysis are illustrated and numerics are provided.

Appendix A

Proofs

Proof of Lemma 6.1. A circular m-consecutive-k,l-out-of-n:F system combined of n circularly ordered components such that the system fails iff there are at least m l-overlapping runs of k consecutive components ($n \geq m(k-l)+l, l<k$). So, the minimum number of failed components such that system fails can be found as

$$k_\phi = k + (m-1)(k-l)$$

For finding the maximum number of failed components such that system still functions, which is denoted by z_ϕ, we can consider a binary sequence of runs which are cyclicly arranged as follows

$$0\underbrace{11\ldots1}_{x}0\underbrace{11\ldots}_{n-x-2}$$

where x denotes the number of failed components such that total number of l-overlapping runs is $m-1$, and the remaining part $n-x-2$ runs can obtain a maximum of

$$n-x-2-\left\lfloor\frac{n-x-2}{k}\right\rfloor$$

failures. The maximum value of x is $k+(m-1)(k-l)-1$ so

$$\begin{aligned} z_\phi = & \quad n-k-(m-1)(k-l)+1-2-\left\lfloor \tfrac{n-k-(m-1)(k-l)+1-2}{k} \right\rfloor \\ = & \quad n-2-\left\lfloor \tfrac{n-1-m(k-l)-l}{k} \right\rfloor \end{aligned}$$

□

Proof of Theorem 6.3. Consider a binary sequence of runs which are cyclicly arranged as follows

$$\underbrace{11\ldots1}_{x_1}0\underbrace{1\ldots1}_{x_2}0\ldots0\underbrace{1\ldots1}_{x_i}0$$

such that for $s=0,1,\ldots,m-1$,

$$x_1+x_2+\ldots+x_i=n-i \tag{A.1}$$

where

$$\left\lfloor \frac{x_1-l}{k-l} \right\rfloor + \ldots + \left\lfloor \frac{x_a-l}{k-l} \right\rfloor = s$$

for a of $x_j \times s \geq k$ and $i-a$ of $x_j \times s < k$. and $\lfloor \frac{x_i-l}{k-l} \rfloor$ denotes the number of l-overlapping runs of length k in the ith failure run.

By using Theorem 4.1 of Makri et al. [63], the number of path sets of a circular m-consecutive-k,l-out-of-n system can be calculated as follows

$$r_i(n) = \frac{n}{i}\sum_{s=0}^{m-1}\sum_{a}\binom{i}{a}N(i,a,k,l,s,n)$$

where $N(i,a,k,l,s,n)$ denotes the number of integer solution to (A.1). Let $y_j = x_j - l$ for $j=1,2,\ldots,a$ and $y_j = x_j$ for $j=a+1,a+2,\ldots,i$.

Then (A.1) is equivalent to

$$y_1+y_2+\ldots+y_i=n-i-al \tag{A.2}$$

st

$$\left\lfloor \frac{y_1 - l}{k - l} \right\rfloor + \ldots + \left\lfloor \frac{y_a - l}{k - l} \right\rfloor = s$$

$y_1, y_2, \ldots, y_a \geq k - l$ and $0 \leq y_{a+1}, y_{a+2}, \ldots, y_i < k$.
Let $\left\lfloor \frac{y_j}{k-l} \right\rfloor = z_j$ for $j = 1, 2, \ldots, a$. Then (A.2) is equivalent to

$$y_1 + y_2 + \ldots + y_i = n - i - al \tag{A.3}$$

st

$$z_j(k - l) \leq y_j < z_j(k - l) + k - l \text{ for } j = 1, 2, \ldots, a$$
$$0 \leq y_j < k \text{ for } j = a + 1, a + 2, \ldots, i$$

and

$$z_1 + z_2 + \ldots + z_a = s \tag{A.4}$$

st

$$z_i > 0 \text{ for } i = 1, 2, \ldots, a.$$

The number of integer solutions to (A.4) is $\binom{s-1}{a-1}$ and by using Lemma 2.1 of Makri et al. [63] and taking $u_j = y_j - (k-l)z_j$ for $j = 1, 2, \ldots, a$ we can obtain the number of integer solutions to (A.3) as $N(i, a, k, l, s, n) = \binom{s-1}{a-1} C(n - i - al - s(k-l), i-a, k-l-1, k-1)$ for $s = 1, 2, \ldots, m-1$. For $s = 0$, the number of integer solutions can be obtained as $N(i, a, k, l, s, n) = C(n - i, i, 0, k - 1, k - 1)$. $\square$

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