MARSHALL-OLKIN TYPE SHOCK MODELS AND THEIR APPLICATIONS

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JUNE 2015

MARSHALL-OLKIN TYPE SHOCK MODELS AND THEIR APPLICATIONS

A DISSERTATION SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL

AND APPLIED SCIENCES OF

IZMIR UNIVERSITY OF ECONOMICS

BY

CEMAL MURAT ÖZKUT

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

JUNE 2015

Ph.D. DISSERTATION EXAMINATION RESULT FORM

Approval of the Graduate School of Natural and Applied Sciences

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ABSTRACT

MARSHALL-OLKIN TYPE SHOCK MODELS AND THEIR APPLICATIONS

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June 2015

In traditional Marshall-Olkin type shock models and their modifications, there are three type of shocks that arrive at random times. These shocks destroy the components of a system which has two or more components. In this thesis, we assume that if the magnitude of the shock exceeds some predefined threshold, then the corresponding component is destroyed; otherwise it continues to survive. It is obvious that, this approach is different from classical Marshall-Olkin type shock models. More precisely, we assume that the shock time and the magnitude of the shock are dependent random variables with given bivariate distribution. The magnitude of shock is an important factor that should be taken into account. Hence, this approach is more flexible for modeling many real life applications of shock models. In this work, new class of bivariate distributions involving the joint distributions of shock times and their magnitudes are obtained. Dependence properties of new bivariate distributions are studied. For different examples of underlying bivariate distributions of lifetimes and shock magnitudes, the joint distributions of lifetimes of the components were investigated. The multivariate extension of the proposed model is also discussed. The proposed model is a mixture of a singular distribution function, and an absolutely continuous function, which makes it difficult to obtain maximum likelihood estimators(MLE) of the unknown parameters. Using Expectation-Maximization(EM) algorithm, we analyzed data sets, for both bivariate and multivariate Marshall-Olkin type distribution with magnitude shock effect models. Also, asymptotic confidence intervals of the unknown parameters of both bivariate and multivariate proposed models are constructed.

Keywords: Marshall-Olkin distribution, shock models, dependence, maximum likelihood estimation, EM algorithm.

ÖZ

MARSHALL-OLKIN TİPİ ŞOK MODELLERİ VE UYGULAMALARI

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Klasik Marshall-Olkin tipi şok modelleri ve bu modellerin modifikasyonlarında, iki ya da daha fazla bileşen içeren sistem farklı kaynaklar tarafından rastgele zamanlarda üretilen şoklara maruz kalır ve sistemin ilgili bileşenleri yok olur. Marshall-Olkin tipi şok modellerinden farklı olarak, üretilen şokun şiddetinin önceden belirlenen eşik değerden yüksek olduğu taktirde ilgili bileşenin imha edileceğini aksi takdirde bileşenin çalışmaya devam edeceğini varsaydık. Daha iyi anlatmak gerekirse, şok zamanı ve şiddetinin bağımlı iki değişkenli dağılıma sahip olduğunu varsaydık. Şokların şiddetlerinin dikkate alınması gerekliliği yaklaşımı bize şok modellerin gerçek yaşam uygulamalarında ortaya çıkan gereksinimleri karşlamamıza izin veriyor. Bu tez çalışmasında, şok zamanı ve şok şiddetinin ortak dağılımını içeren yeni iki değişkenli dağılım sınıfı elde edildi. Yeni iki değişkenli dağılımın bağımlılık özellikleri çalışıldı. Bileşenlerin yaşam süreleri ve şok zamanlarının ikili dağımlarının verildiği farklı örnekler için, bileşenlerin yaşam sürelerinin ortak dağılımları incelendi. Ayrıca önerilen modelin genişletilmiş çok değişkenli modeli ayrıca tartışıldı. Önerilen modelin tekil dağılım fonksiyonu ve tamamıyla sürekli fonksiyonun kombinasyonu şeklinde olması ortak dağılımın bilinmeyen parametrelerinin en çok olabilirlik tahmin edicilerini bulmayı zorlaştırmaktadır. Bu yüzden, beklenti maximizasyonu algoritması kullanarak önerilen ikili ve çoklu modellerin veri setlerini inceledik. Ayrıca, önerilen ikili ve çoklu modellerin bilinmeyen parametrelerinin asimptotik güven aralıkları oluşturuldu.

Anahtar Kelimeler: Marshall-Olkin dağılımları, şok modelleri, bağımlılık, en büyük olabilirlik kestirimi, Beklenti ençoklama algoritması.

ACKNOWLEDGEMENT

I would like to thank everyone who has contributed to this challenging and rewarding experience. First of all, I would like to express my great appreciation to my supervisor, Prof. Dr. İsmihan Bayramoğlu for his patience, academic guidance and many interesting discussions on scientific and real life problems. Also, I would like to thank him for his help and support whenever I need. It has been an honor and pleasure to know and to work with him.

I would like to express my gratitude to Assoc. Prof. Dr. Güvenç Arslan and Assoc. Prof. Dr. Aylin Alin for being in my committee and for their helpful advices and interesting comments which resulted in the improvement of this thesis.

I want to thank Assoc. Prof. Dr. Gözde Yazgı Tütüncü for her endless help, her dearest friendship, encouragement and support. Whenever I am in trouble, she always tries to do whatever she can.

I also want to thank my colleagues Cihangir Kan and Ceki Franko for their great friendship. They always near me and share my troubles. It was great pleasure to share the office and work with them. I hope this friendship lasts forever.

I would like to special thank my parents Ferhan and Ali ihsan Özkut and to my sisters Pınar Ülgen and Çağlar Özkut and my brother Mert Ülgen for their support and encouragement. They have helped me in difficult times and listen me without any complaint.

Last but not at least, I would like to thank my dear girl friend Eylem Bal. Her love and support without any complaint encourage me to complete this thesis.

To my family and in memory of my grandmother Yaşar TAMUK...

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Chapter 1

Introduction

In classical Marshall-Olkin model [26], a system consisting of two components is subject to three different shocks. These shocks are produced by different sources at a random time. The properties of the shocks can be described as follows: a shock from the first source affects the first component, a shock from the second source affects the second component, and a shock from the third source effects both components. According to this model, the corresponding component (or components) dies when a shock occurs. In 1967, Marshall and Olkin [26] considered the independent exponential shock times and derived the joint survival function of the components of the system.

Recently, there have been numerous papers which deal with the extensions of Marshall-Olkin distributions. Ryu [37] extended Marshall and Olkin's bivariate exponential distribution to the new bivariate absolutely continuous distribution. This new distribution does not have lack memory property and has advantage of identifying the shock arrival rates individually. Also, their impacts are provided.

Marshall and Olkin [27] introduced a new family of distributions, established by adding a new parameter to an existing distribution. Through this model, the new family of distributions can be constructed from the existing family with survival function $\overline{F}(x)$, by setting the new survival function as

$$\overline{H}(x) = \frac{\alpha \overline{F}(x)}{1 - \overline{\alpha}\overline{F}(x)}, \qquad -\infty < x < \infty, \, 0 < \alpha < \infty$$

where $\overline{\alpha} = 1 - \alpha$. If this method is applied twice, it produce no new distribution. This is because the model has a stability property. Another advantage of the method is the flexibility in the new distribution. Various extended Marshall-Olkin families of distributions can be obtained by using this method. For instance, Thomas and Jose [41] considered Marshall-Olkin semi-Pareto distribution and Marshall-Olkin Pareto distribution. In this study, the new family of distributions have Pareto marginals. Also, some characteristic properties of proposed distribution were investigated. By using a compound distribution with mixing exponential distribution, Ghitany et al. [11] and Ghitany et al. [10] constructed respectively Marshall-Olkin extended Weibull distribution and Marshall-Olkin extended Lomax distribution. These distributions were used in the analysis of randomly censored data. Also, efficient estimates of the parameters were obtained. Jayakumar and Thomas [15] studied Marshall-Olkin distributions with three parameters, which is the generalization of the family of two parameters Marshall-Olkin distributions. Recently in 2011, Jose et al. [16] constructed Marshall-Olkin distributions with Weibull marginals and discussed some properties of this distribution. Li and Pellerey [23] studied the aging properties of the generalized Marshall-Olkin distributions. On the other hand, some authors have tried to generalize the bivariate exponential distribution for constructing new distributions because such distributions can be used affectively in the analysis of lifetime data. In 1999, Gupta and Kundu [13] introduced three parameters (location, scale and shape) generalized exponential distributions. They also analyzed the theoretical properties of this family. By using this generalized exponential distribution, Sarhan and Balakrishnan [39] defined a new bivariate distribution. However, explicit form of the marginal distributions cannot be derived in their model. They also discussed the mixture of the proposed bivariate distributions. The results of Sarhan and Balakrishnan [39] were modified by Kundu and Gupta [19] by using different marginal distributions. In their modification, the marginal distributions are generalized exponential distributions. Afterward, Kundu and

Gupta [20] extended the Sarhan-Balakrihnan distribution by adding a new shape parameter. This method has an advantage over existing distributions in the sense that it allows the marginal distributions to be more flexible.

In classical Marshall-Olkin exponential distribution and all its modifications, there are two types of shocks. One is called fatal shock and it destroys the corresponding component. Another is called non-fatal shock, and when the shock occurs, the corresponding component may not be destroyed, but has a chance of surviving, and each shock to a component represents an independent opportunity for failure with respective probability.

These two different shocks can be also thought as Poisson processes. Both approaches lead to classical Marshall-Olkin bivariate exponential distribution. In addition, there are many application fields, such as life testing, reliability, economics finance and insurance, in which the bivariate exponential distributions play a key role. These important probability models can be used in the lack of the independence assumption, which is questionable or unrealistic in applications.

In the field of financial mathematics, if one deals with the portfolio credit risk modeling, the unknown future lifetimes of the d credit risky assets are random variables. Mai [25] considers an appropriate stochastic model for these unknown lifetimes, and investigates collateralized debt obligations. These are financial contracts-typically traded between globally active financial institutions, who recommend insurance to investors, instead of the default of credit risky assets. In many cases, defaults have been exposed to exogenous shocks, which can be interpreted as economic crises affecting one, two, three or more assets that have credit risks. The appropriate model for collateralized debt obligations pricing is the Marshall-Olkin distribution.

Although there has been many researches about modifications of the Marshall-Olkin distribution obtained from different shock models, there has been little attention to shock magnitudes. However, there is a need to consider shock models. These are not always fatal in practical applications, mainly depending on shock magnitudes. For example, in economic crises, affecting one or more credit-risky assets at a time, the magnitude of exogenous economic shocks causing the default of assets may be different, i.e. small or big shocks may occur. If the shock is sufficiently big, then there is a failure but if it is not, the effect of the shock can be ignored. Therefore, it will be reasonable to consider magnitudes of shocks arriving at a random time. The times of arrival of shocks and their magnitudes should be considered as stochastically dependent random variables with given joint distribution function.

In this thesis, the new Marshall-Olkin type shock model is introduced. As mentioned above, in many practical applications there is a need to consider the magnitude of a shock in order to model real problems. Accordingly, in this work, we assume that the shocks coming at random times have different magnitudes. The shock times and their magnitudes are assumed to be dependent random variables with given joint distribution (or survival) function. The structure of the classical Marshall-Olkin distribution is changed by this new assumption, i.e. the distribution function obtained in this work depends on bivariate joint distribution of shock times and their magnitudes. Therefore, according to the real life conditions appeared in applications, this model involves some known bivariate distribution functions and we obtain a broad class of a new type of Marshall-Olkin bivariate distribution. These distributions are different from existing modifications of Marshall-Olkin exponential distribution. Also, the proposed model allows us to use various marginal distributions regarding the nature of the considered problem. The new Marshall-Olkin type distributions also present an independent interest for generating new bivariate distributions, thus we discuss the dependence properties of the new bivariate survival function obtained from this model. It should be noted that this distribution is not continuous, i.e it has both absolutely continuous and singular part similar to the classical Marshall-Olkin distribution. For more details, refer to [33].

In the literature, there are several papers about the estimation of the unknown parameter of the Marshall-Olkin distribution. Since the Marshall-Olkin distribution has absolutely continuous and singular parts, the usual techniques are not considered suitable for deriving the maximum likelihood estimation (MLE) of the unknown parameter. Several suggestions and approximations methods have been suggested in the literature. For example, Arnold [2] obtained the maximum likelihood and method of moments estimates of unknown parameters. Bhattacharyya and Johnson [8] discussed the alternative maximum likelihood estimation to Arnold [2]. Method of moment and maximum likelihood estimation for the parameters of the Marshall-Olkin distribution is also studied by [7] and they compared results with the results of [2]. Proschan and Sullo [36] studied MLE for the multivariate Marshall-Olkin model. Pena and Gupta [35] obtained Bayesian estimation result. Recently, Karlis [17] and Kundu and Dey [18] investigated the estimation of unknown parameters of Marshall-Olkin model. They consider estimating unknown parameters problem as a missing value problem, and they used Expectation-Maximization algorithm to compute MLEs of these kinds of distributions.

A further highlight of this thesis is finding the maximum likelihood estimator of the unknown parameter of the proposed Marshall-Olkin type distribution with magnitude shock effect. Throughout the thesis, we use MOMSE instead of Marshall-Olkin type distribution with magnitude shock effect. Since MOMSE has both singular and absolutely continuous part, statistical inference is not an easy task because of the complex structure of its density function, similar to the classical Marshall-Olkin distribution. The another contribution of the present dissertation is to overcome this drawback and find the maximum likelihood estimator of the proposed model by using the Expectation-Maximization(EM) algorithm. In order to implement EM algorithm, the problem of finding maximum likelihood estimator is treated as a missing value problem. Briefly, in the EM algorithm, in the 'E'-step, the missing values are replaced with the their expected values. Then, by using expected values, a 'pseudo-log-likelihood' function is constructed and in the 'M'-step, maximum likelihood estimates for a sample is calculated by maximizing 'pseudo-log-likelihood' function. Next, EM algorithm is analyzed on some data sets which involve both bivariate and multivariate MOMSE. Also, asymptotic confidence intervals of the unknown parameters of both bivariate and multivariate distributions are constructed.

The rest of the thesis is organized as follows. The classical Marshall-Olkin distribution and its properties are presented in Chapter 2. The construction of the new Marshall Olkin shock model is given in Chapter 3. For some different underlying bivariate distributions of lifetimes and shock magnitudes, the joint distributions of lifetimes of the components are investigated in Chapter 4. Dependence properties of the joint distribution of the lifetimes of the components, and stochastic comparisons of different shock models are also discussed in Chapter 5. Then, the result of the maximum likelihood estimation of the bivariate Gumbel distribution of MOMSE is given in Chapter 6. Finally, in Chapter 7 the multivariate extension of the proposed model and its propoerties is studied.

Chapter 2

Classical Marshall-Olkin Distributions

In 1967, Marshall-Olkin [26] introduced a two-component system and the components are subject to shocks at a random time. According to this model, there are three types of shocks that are produced by three sources. A shock from the source 1 affects the first component, a shock from the source 2 affects the second component and a shock from the source 3 affects both components simultaneously. Suppose that $X_j,\ j = 1, 2$ denotes the life length of the component j and $T_i,\ i = 0, 1, 2$ denotes the waiting time for the shock i. Let $T_i,\ i = 0, 1, 2$ be independent exponential random variables having parameters $\theta_i > 0$. For that reason, the lifetime of the component j can be represent as

$$
\begin{aligned}
X_1 &= \min(T_1, T_0) \\
X_2 &= \min(T_2, T_0)
\end{aligned}
$$

Under this setup, the joint survival function of X_1 and X_2 can be found as follows:

$$
\begin{aligned}
P(X_1 > t_1, X_2 > t_2) &= P(\min(T_1, T_0) > t_1, \min(T_2, T_0) > t_2) \\
&= P(T_1 > t_1, T_0 > t_1, T_2 > t_2, T_0 > t_2) \\
&= P(T_1 > t_1, T_2 > t_2, T_0 > \max(t_1, t_2))
\end{aligned}
$$

Taking into account that T_i have exponential distribution with parameter θ_i, we have

$$P(X_1 > t_1, X_2 > t_2) = exp(-\theta_1 t_1 - \theta_2 t_2 - \theta_0 \max(t_1, t_2)) \tag{2.1}$$

for every $t_1 \geq 0$ and $t_2 \geq 0$.
In a similar way, the marginal survival function of X_j can be derived as:

$$\begin{aligned} P(X_j > t_j) &= P(\min(T_j, T_0) > t_j) \\ &= P(T_j > t_j, T_0 > t_j) \\ &= P(T_j > t_j)P(T_0 > t_j) \\ P(X_j > t_j) &= exp(-t_j(\theta_j + \theta_0)) \end{aligned}$$

that has exponential distribution with parameter $(\theta_j + \theta_0)$.
The crucial point of the model is that, as the failure of the components can be affected simultaneously, the event $X_1 = X_2$ can occur with positive probability. As a result, Marshall-Olkin distribution is not absolutely continuous with respect to the Lebesgue measure in $\Re^2$ and has a singular part on the line $x_1 = x_2$.

Theorem 2.1 ([26]) *In Marshall-Olkin model, the joint survival function (2.1) has both absolutely continuous and singular part. That is*

$$\bar{F}(t_1, t_2) = \frac{\theta_1 + \theta_2}{\theta} \bar{F}_a(t_1, t_2) + \frac{\theta_0}{\theta} \bar{F}_s(t_1, t_2),$$

where singular part is

$$\bar{F}_s(t_1, t_2) = exp(-\theta \max(t_1, t_2))$$

and absolutely continuous part is

$$\bar{F}_a(t_1, t_2) = \frac{\theta}{\theta_1 + \theta_2} exp(-\theta_1 t_1 - \theta_2 t_2 - \theta_0 max(t_1, t_2)) - \frac{\theta_0}{\theta_1 + \theta_2} exp(-\theta \max(t_1, t_2)),$$

where $\theta = \theta_1 + \theta_2 + \theta_0$.

The joint density function of the Marshall-Olkin distribution is given by

$$f(t_1, t_2) = \begin{cases} \theta_1(\theta_2 + \theta_0)\bar{F}(t_1, t_2), & t_2 > t_1 \\ \theta_2(\theta_1 + \theta_0)\bar{F}(t_1, t_2), & t_1 > t_2 \\ \theta_0\bar{F}(t_1, t_2), & t_1 = t_2. \end{cases}$$

In classical and extended versions of the Marshall-Olkin distribution obtained from different shock models, magnitude of the shocks are not taken into consideration. On the other hand, we need to consider shock models, which are not always fatal depending on shock magnitudes in real life problems. For example, in economic crises, affecting one or more credit-risky assets at a time the magnitude of exogenous economic shocks causing the default of assets may be different, that is small shocks as well as big shocks may occur. When the shocks have enough magnitude, their influence may be forceful. On the contrary, if the shocks do not have enough magnitude, then they are not able to affect. Hence, it will be reasonable to consider magnitudes of shocks arriving at a random time. The times of arrival of shocks and the magnitudes should be considered as stochastically dependent random variables with given joint distribution function.

In the next chapter, a new Marshall-Olkin type shock model is introduced. As mentioned above, it is more suitable for modeling real life problems when we consider the shocks with their magnitudes. Therefore, we assume that the shocks coming in random times have magnitudes and the shock times and their magnitudes are dependent random variables with given joint distribution (or survival) function. Because of this assumption, a new structure is presented which is different from the classical Marshall-Olkin distribution, that is the distribution function obtained in this work depends on bivariate joint distribution of shock times and their magnitudes. Hence, in accordance with following to the real life conditions appearing in applications, we model some well known bivariate distribution functions and obtain a wide class of a new type of Marshall-Olkin bivariate distributions. These distributions are different from existing Marshall-Olkin bivariate distribution and their modifications, and allow to use various marginal distributions regarding the nature of the considered problem. The new Marshall-Olkin type distributions present also an independent interest for generating new bivariate distributions, that is why we discuss the dependence properties of the new bivariate survival function obtained from this model.

Chapter 3

MOMSE

Consider a system with two components and let X_1 and X_2 denote the life length of the component 1 and component 2, respectively. A shock from the first source which comes at random time T_1 affects the component 1 and this shock has magnitude D_1. Similarly, a shock from the second source which comes at random time T_2 affects the component 2 and this shock has magnitude D_2 and a shock from the third source comes at random time T_0 with the magnitude D_0 affects both components simultaneously. We assume that the random variables T_i and D_i, $i = 0, 1, 2$ are stochastically dependent and the affect of the shocks for both components depends on the upper-threshold d, that is if the magnitude of the shock is greater than d, then the corresponding component will be destroyed, otherwise the component has chance to survive. Let us denote the joint distribution function of the random vector $(T_i, D_i), i = 0, 1, 2$ by $F_{T_i,D_i}(t, d)$, respectively. Also, we assume that the bivariate random vectors (T_1, D_1), (T_2, D_2), and (T_0, D_0) are independent. According to this set up, we can define independent conditional random variables as follows

$$
\begin{aligned}
V_1 &\equiv (T_1 | D_1 > d) \\
V_2 &\equiv (T_2 | D_2 > d) \\
V_0 &\equiv (T_0 | D_0 > d) \qquad (3.1)
\end{aligned}
$$

that is,

V_i:is the time of arrival of the shock which has a magnitude greater than $d, i = 0, 1, 2$. In other words, V_i is the time of arrival of the fatal shock for the corresponding component, $i = 0, 1, 2$.

Thus, the life lengths X_1 and X_2 of the components 1 and 2 can be written respectively as follows

$$X_1 = \min(V_1, V_0)$$
$$X_2 = \min(V_2, V_0)$$

Theorem 3.1 *The joint survival function of X_1 and X_2 is*

$$\overline{F}_{X_1,X_2}(t_1, t_2) = \frac{\overline{F}_{T_1,D_1}(t_1, d)\overline{F}_{T_2,D_2}(t_2, d)\overline{F}_{T_0,D_0}(\max(t_1, t_2), d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \tag{3.2}$$

Proof. By using (3.1) we can write joint survival function of X_1 and X_2 as follows

$$\begin{aligned} P(X_1 > t_1, X_2 > t_2) &= P(\min(V_1, V_0) > t_1, \min(V_2, V_0) > t_2) \\ &= P(V_1 > t_1, V_0 > t_1, V_2 > t_2, V_0 > t_2) \\ &= P(V_1 > t_1)P(V_2 > t_2)P(V_0 > \max(t_1, t_2)) \\ &= P(T_1 > t_1 | D_1 > d)P(T_2 > t_2 | D_2 > d)P(T_0 > \max(t_1, t_2) | D_0 > d) \\ &= \frac{P(T_1 > t_1, D_1 > d)P(T_2 > t_2, D_2 > d)P(T_0 > \max(t_1, t_2), D_0 > d)}{P(D_1 > d)P(D_2 > d)P(D_0 > d)} \end{aligned}$$

and in terms of joint and marginal survival functions it can be expressed as

$$\overline{F}_{X_1,X_2}(t_1, t_2) = \frac{\overline{F}_{T_1,D_1}(t_1, d)\overline{F}_{T_2,D_2}(t_2, d)\overline{F}_{T_0,D_0}(\max(t_1, t_2), d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)}$$

□

Corollary 3.2 *Marginal survival functions of X_1 and X_2 are*

$$\overline{F}_{X_1}(t_1) = \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_0,D_0}(t_1,d)}{\overline{F}_{D_1}(d)\overline{F}_{D_0}(d)}$$

and

$$\overline{F}_{X_2}(t_2) = \frac{\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(t_2,d)}{\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)}$$

Proof.

$$\begin{aligned}
P(X_1 > t_1) &= P(\min(V_1, V_0) > t_1) \\
&= P(V_1 > t_1, V_0 > t_1) \\
&= P(V_1 > t_1)P(V_0 > t_1) \\
&= P(T_1 > t_1|D_1 > d)P(T_0 > t_1|D_0 > d) \\
&= \frac{P(T_1 > t_1, D_1 > d)P(T_0 > t_1, D_0 > d)}{P(D_1 > d)P(D_0 > d)} \\
\overline{F}_{X_1}(t_1) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_0,D_0}(t_1,d)}{\overline{F}_{D_1}(d)\overline{F}_{D_0}(d)}
\end{aligned}$$

Marginal survival function of X_2 can be found in a similar way. □

It is seen that the joint survival function of MOMSE involves the survival function of the random variables T_1 and T_2 and also the marginal survival functions of the random variables D_1 and D_2.

Chapter 4

Some Special Bivariate Distributions Examples

In this chapter, we investigate the MOMSE distribution for some different underlying joint distributions of random variables (T_i, D_i), $i = 0, 1, 2$. To be more precise, in particular cases, as a joint bivariate distribution of $(T_i, D_i), i = 0, 1, 2$ we consider bivariate Gumbel exponential distribution, Farlie-Gumbel-Morgenstern distribution with exponential marginals, the bivariate Pareto distribution, the bivariate Logistic distribution, the bivariate exponential distribution and also bivariate Lomax distribution. The reason of adopting these particular distributions is that all of them have been successfully used in modeling, reliability and life time analysis. Detailed description of each model is given in the corresponding subsections. Note that the MOMSE distribution like the Marshall-Olkin distribution has absolutely continuous and singular parts.

4.1 Bivariate Gumbel Exponential Distribution

Consider the bivariate Gumbel distribution

$$F_{Z_1,Z_2}(z_1, z_2) = 1 - e^{-z_1} - e^{-z_2} + e^{-(z_1+z_2+\theta z_1 z_2)},\ z_1, z_2 > 0 \text{ and } 0 \leq \theta \leq 1. \quad (4.1)$$

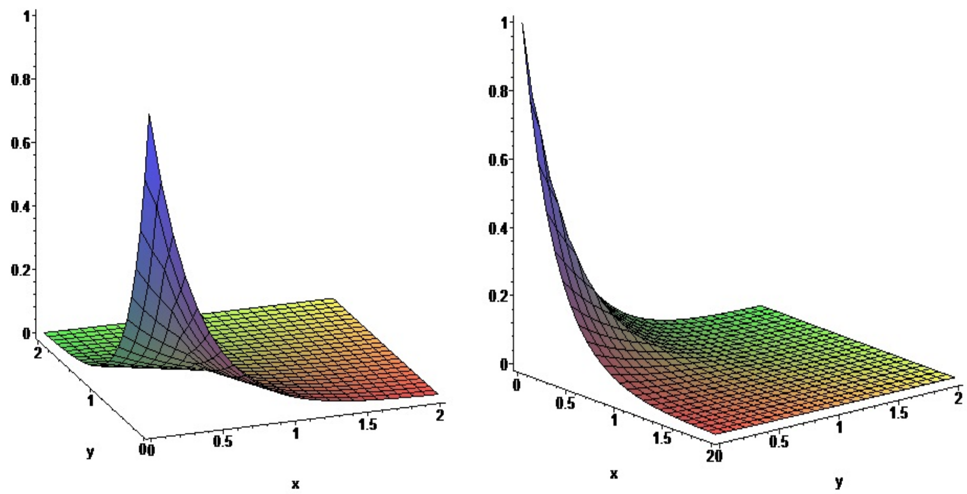

Figure 4.1: Graph of $\overline{F}_{X_1,X_2}(t_1, t_2)$ given in (4.3), for $\theta = 0.5, d = 1$.

The marginal distributions are exponential. This distribution was introduced by Gumbel in 1960. It has applications in many areas including competing risks, extreme values, failure times and reliability. For further properties, refer to [12] and [5]. Let (T_1, D_1),(T_2, D_2) and (T_0, D_0) be the random vectors with bivariate Gumbel distribution function. Using (4.1), the joint survival function of (T_i, D_i) can be written as

$$\overline{F}_{T_i,D_i}(t_i, d) = e^{-(t_i+d+\theta t_i d)}, i = 0, 1, 2. \tag{4.2}$$

Then using (4.2) in Theorem 3.1, joint survival function of MOMSE can be found as follows

$$\begin{aligned}\overline{F}_{X_1,X_2}(t_1, t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1, d)\overline{F}_{T_2,D_2}(t_2, d)\overline{F}_{T_0,D_0}(\max(t_1, t_2), d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \\ &= \frac{e^{-(t_1+d+\theta t_1 d)}e^{-(t_2+d+\theta t_2 d)}e^{-(\max(t_1,t_2)+d+\theta \max(t_1,t_2)d)}}{e^{-3d}} \\ \overline{F}_{X_1,X_2}(t_1, t_2) &= e^{-t_1(1+\theta d)}e^{-t_2(1+\theta d)}e^{-\max(t_1,t_2)(1+\theta d)} \qquad (4.3)\end{aligned}$$

Note that if $d = 0$, then the MOMSE changes into the usual Marshall-Olkin bivariate distribution. In Figure 4.1, the graph of $\overline{F}_{X_1,X_2}(t_1, t_2)$ given in (4.3) for particular values of $\theta = 0.5$ and $d = 1$ is presented.

Theorem 4.1 *The joint survival function of MOMSE distribution given in (4.3)* *has both absolutely continuous and singular part. That is,*

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \alpha\overline{F}_a(t_1,t_2) + (1-\alpha)\,\overline{F}_s(t_1,t_2)$$

where

$$\alpha = \frac{2}{3}$$

$$\overline{F}_a(t_1,t_2) = \frac{1}{2}e^{-3(\max(t_1,t_2)+\max(t_1,t_2)d\theta)}\left(3e^{|t_1-t_2|(1+d\theta)}-1\right)$$

$$\overline{F}_s(t_1,t_2) = e^{-3(\max(t_1,t_2)+\max(t_1,t_2)d\theta)}$$

Proof. For $t_1 < t_2$

$$\begin{aligned}\frac{\partial^2\overline{F}(t_1,t_2)}{\partial t_1\partial t_2} &= f_1(t_1,t_2)\\ &= 2(1+\theta d)^2 e^{-(t_1+2t_2)(1+\theta d)}\end{aligned}$$

and for $t_2 < t_1$

$$\begin{aligned}\frac{\partial^2\overline{F}(t_1,t_2)}{\partial t_1\partial t_2} &= f_2(t_1,t_2)\\ &= 2(1+\theta d)^2 e^{-(2t_1+t_2)(1+\theta d)}\end{aligned}$$

$$f_a(t_1,t_2) = \frac{1}{\alpha}\begin{cases} f_1(t_1,t_2) & ,t_1 < t_2\\ f_2(t_1,t_2) & ,t_2 < t_1\end{cases}$$

$$\alpha\overline{F}_a(t_1,t_2) = \int_{t_1}^{\infty}\int_{t_2}^{\infty} f(u,v)\,dvdu$$

For $t_1 < t_2$

$$\overline{F}_a(t_1,t_2) = \frac{1}{\alpha}\left(\underbrace{\int_{t_2}^{\infty}\int_{t_1}^{v} f_1(u,v)\,dudv}_{1} + \underbrace{\int_{t_2}^{\infty}\int_{t_2}^{u} f_2(u,v)\,dvdu}_{2}\right)$$

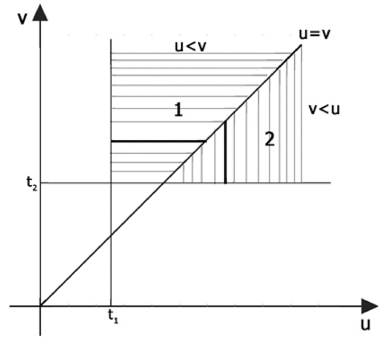

Figure 4.2: Graphical representation of the region of $u < v, u > v$ and $u = v$.

$$\overline{F}_a(t_1, t_2) = \frac{1}{\alpha}\left(\frac{1}{3}e^{-3t_2(1+d\theta)}(3e^{-(t_1-t_2)(1+d\theta)} - 1)\right)$$

with a symmetric expression when $t_1 > t_2$. Combining both cases

$$\overline{F}_a(t_1, t_2) = \frac{1}{\alpha}\left(\frac{1}{3}e^{-3\max(t_1,t_2)(1+d\theta)}(3e^{|t_1-t_2|(1+d\theta)} - 1)\right)$$

Since

$$\overline{F}_a(0, 0) = 1$$

$$\alpha = \frac{2}{3}$$

With α and $\overline{F}_a$, the singular part $\overline{F}_s$ can be obtained by subtraction. That is,

$$\overline{F}_s(t_1, t_2) = \frac{\overline{F}_{X_1,X_2}(t_1, t_2) - \alpha\overline{F}_a(t_1, t_2)}{1 - \alpha}$$

$$\overline{F}_s(t_1, t_2) = e^{-3\max(t_1,t_2)(1+d\theta)}$$

So,

$$\overline{F}_{X_1,X_2}(t_1, t_2) = \frac{2}{3}\left(\frac{1}{2}e^{-3\max(t_1,t_2)(1+d\theta)}(3e^{|t_1-t_2|(1+d\theta)} - 1)\right) + \frac{1}{3}e^{-3\max(t_1,t_2)(1+d\theta)}$$

□

4.2 Farlie-Gumbel-Morgenstern Distribution

Consider the Farlie-Gumbel-Morgenstern (FGM) bivariate exponential distribution

$$F_{Z_1,Z_2}(z_1,z_2) = F_{Z_1}(z_1)F_{Z_2}(z_2)\left[1+\alpha\left(1-F_{Z_1}(z_1)\right)\left(1-F_{Z_2}(z_2)\right)\right] \tag{4.4}$$

where $-1 \leq \alpha \leq 0$ and $Z_i \sim Exp\,(1)$. This distribution was used to model the joint distribution of two adjacent intervals in a Markov-dependent point process. We refer to [3], [4], [21] and [5] for more properties. Let (T_1, D_1),(T_2, D_2) and (T_0, D_0) be the random vectors with the Farlie-Gumbel-Morgenstern (FGM) bivariate exponential distribution. Let $\omega = \min(t_1, t_2)$ and $\beta = \max(t_1, t_2)$ then from Theorem 3.1 joint survival function of MOMSE can be written as

$$\begin{aligned}
\overline{F}_{X_1,X_2}(t_1,t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \\
\overline{F}_{X_1,X_2}(t_1,t_2) &= e^{-3d-2t_1-2t_2-2\beta}\left(e^{d+t_1}+\alpha\left(e^d-1\right)\left(e^{t_1}-1\right)\right) \\
&\quad \times\left(e^{d+t_2}+\alpha\left(e^d-1\right)\left(e^{t_2}-1\right)\right) \\
&\quad\quad \times\left(e^{d+\beta}+\alpha\left(e^d-1\right)\left(e^{\beta}-1.\right)\right)
\end{aligned} \tag{4.5}$$

Absolutely continuous and singular part can be found similarly by using Theorem 4.1. That is,

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{2}{3}\overline{F}_a(t_1,t_2) + \frac{1}{3}\overline{F}_s(t_1,t_2),$$

where

$$\begin{aligned}
\overline{F}_a(t_1,t_2) &= \frac{1}{2}e^{-3d-2(t_1+t_2)-4\beta}(\alpha(1-e^d)(1-e^{\beta})+e^{d+\beta})^2 \\
&\times(3e^{2\beta}\alpha(1-e^d-e^{\omega}+e^{d+\omega}) \\
&\quad -e^{2\omega}\alpha(1-e^d-e^{\beta}+e^{d+\beta})+3e^{d+t_1+t_2+\beta}-e^{d+t_1+t_2+\omega}),
\end{aligned}$$

$$\overline{F}_s(t_1,t_2) = e^{-3(d+2\beta)}\left(e^{d+\beta}+\alpha\left(e^d-1\right)\left(e^{\beta}-1\right)\right)^3.$$

Below we provide a graph of $\overline{F}_{X_1,X_2}(t_1,t_2)$ with respect to d for some special values of t_1 and t_2. It is observed that the probability of $X_1 > 1.25$ and $X_2 > 1.5$

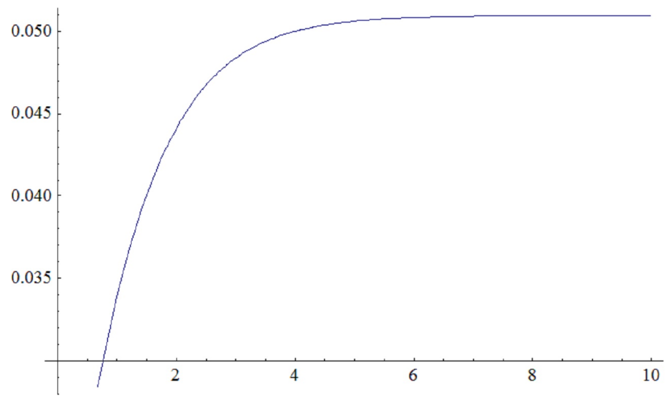

Figure 4.3: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.5) with respect to d for $t_1 = 1.25, t_2 = 1.5$ and $\theta = 0.7$.

increases as d increases.

4.3 Bivariate Pareto Distribution

Consider the second kind bivariate Pareto distribution. The joint survival function is

$$\overline{F}_{Z_1,Z_2}(z_1,z_2) = (1+z_1+z_2)^{-c}, \;\; z_1, z_2 \geq 0. \tag{4.6}$$

Bivariate Pareto distributions are very popular in many areas such as modeling of performance measures for general systems, reliability, and modeling of daily exchange rate data. For more details one may refer to [29], [14], [30], [34] and [5]. Let (T_1, D_1),(T_2, D_2) and (T_0, D_0) be the random vectors with the bivariate Pareto distribution. Let $\omega = \min(t_1,t_2)$, $\beta = \max(t_1,t_2)$ and the parameter $c = 2$, then using (4.6) in Theorem 3.1,joint survival function of MOMSE can be found as follows

$$\begin{aligned}\overline{F}_{X_1,X_2}(t_1,t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_3}(d)} \\ &= \frac{(1+t_1+d)^{-2}(1+t_2+d)^{-2}(1+\max(t_1,t_2)+d)^{-2}}{(1+d)^{-6}} \\ &= \frac{(1+d)^6}{(1+d+t_1)^2(1+d+t_2)^2(1+d+\beta)^2} \end{aligned} \tag{4.7}$$

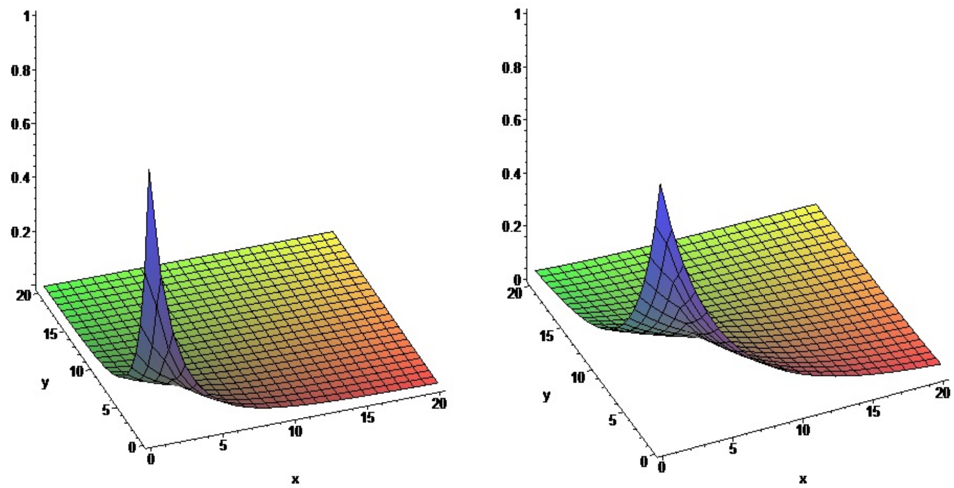

Figure 4.4: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.7), for $d=5$ and $d=15$.

which has both absolutely continuous and singular part. That is,

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{2}{3}\overline{F}_a(t_1,t_2) + \frac{1}{3}\overline{F}_s(t_1,t_2),$$

where

$$\overline{F}_a(t_1,t_2) = \frac{(1+d)^6\left(\frac{3(1+d+\beta)^2}{(1+d+\omega)^2}-1\right)}{2\,(1+d+\beta)^6},$$

$$\overline{F}_s(t_1,t_2) = \frac{(1+d)^6}{(1+d+\beta)^6}.$$

Further studies show that, if c>2, then calculations become complicated. Because the order of polynomial expressions, which are to be integrated, has increased . Other examples also show that the role of parameters in the model are important and they should be considered regarding to the nature of the model.

4.4 Bivariate Logistic Distribution

Consider the bivariate Logistic distribution. We have the following joint distribution and survival function, respectively

$$F(x,y) = (1 + e^{-x} + e^{-y})^{-1}$$

$$\bar{F}(x,y) = 1 - (1 + e^{-x})^{-1} - (1 + e^{-y})^{-1} + (1 + e^{-x} + e^{-y})^{-1}. \tag{4.8}$$

For further information, refer to [1] and [5]. Let (T_1, D_1),(T_2, D_2) and (T_0, D_0) be the random vectors with bivariate Logistic distribution function. Let $\omega = \min(t_1, t_2)$ and $\beta = \max(t_1, t_2)$. Using (4.8) in Theorem 3.1, joint survival function of MOMSE can be found as follows

$$\begin{aligned}\overline{F}_{X_1,X_2}(t_1,t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \\ &= \frac{8(1+2e^d)^3(e^d+e^\omega+2e^{d+\omega})(e^d+e^\beta+2e^{d+\beta})^2}{(1+3e^d)^3(1+e^\omega)(e^d+e^\omega+e^{d+\omega})(1+e^\beta)^2(e^d+e^\beta+e^{d+\beta})^2}.\end{aligned} \tag{4.9}$$

Using Theorem 4.1, we can easily obtain both absolutely continuous and singular part. That is,

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{2}{3}\overline{F}_a(t_1,t_2) + \frac{1}{3}\overline{F}_s(t_1,t_2)$$

where

$$\begin{aligned}\bar{F}_a(t_1,t_2) = &\frac{4(1+2e^d)^3(e^d+e^\beta+2e^{d+\beta})^2(-e^{2\omega}(1+e^d)(e^d+e^\beta+2e^{d+\beta}))}{(1+3e^d)^3(1+e^\omega)(e^d+e^\omega+e^{d+\omega})(1+e^\beta)^3(e^d+e^\beta+e^{d+\beta})^3} \\ &+ \frac{4(1+2e^d)^3(e^d+e^\beta+2e^{d+\beta})^2e^d(2e^d+3e^{2\beta}(1+e^d)+2e^\beta(1+2e^d))}{(1+3e^d)^3(1+e^\omega)(e^d+e^\omega+e^{d+\omega})(1+e^\beta)^3(e^d+e^\beta+e^{d+\beta})^3} \\ &+ \frac{4(1+2e^d)^3(e^d+e^\beta+2e^{d+\beta})^2(e^\omega(1+2e^d)(2e^d+3e^{2\beta}(1+e^d)+2e^\beta(1+2e^d)))}{(1+3e^d)^3(1+e^\omega)(e^d+e^\omega+e^{d+\omega})(1+e^\beta)^3(e^d+e^\beta+e^{d+\beta})^3}\end{aligned}$$

$$\bar{F}_s(t_1,t_2) = \frac{8(1+2e^d)^3(e^d+e^\beta+2e^{d+\beta})^3}{(1+3e^d)^3(1+e^\beta)^3(e^d+e^\beta+e^{d+\beta})^3}$$

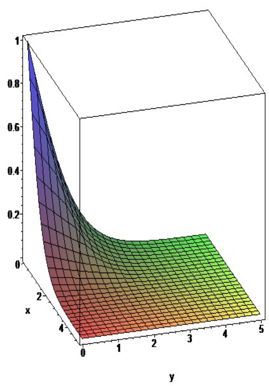

Figure 4.5: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.9) for $d=1$

4.5 Bivariate Exponential Distribution

Consider the bivariate Exponential distribution. We have the following joint survival function

$$\overline{F}_{Z_1,Z_2}(z_1,z_2) = \left(e^{z_1}+e^{z_2}-1\right)^{-1} \tag{4.10}$$

This distribution is the special case of the Ali-Mikhail-Haq distribution. Bivariate Exponential distribution is studied in the reliability analysis. For further information we refer to [1] and [5]. Let (T_1,D_1),(T_2,D_2) and (T_0,D_0) be the random vectors with bivariate Exponential distribution function. Let $\omega=\min(t_1,t_2)$ and $\beta=\max(t_1,t_2)$. Using (4.10) in Theorem 3.1, joint survival function of MOMSE can be found as follows

$$\begin{aligned}
\overline{F}_{X_1,X_2}(t_1,t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \\
&= \frac{\left(e^{t_1}+e^d-1\right)^{-1}\left(e^{t_2}+e^d-1\right)^{-1}\left(e^{\max(t_1,t_2)}+e^d-1\right)^{-1}}{\left(e^d\right)^{-3}} \\
&\times\left(1+2\max(t_1,t_2)+3d+4\max(t_1,t_2)d\right)^{-3} \\
&= \frac{\left(e^{3d}\right)}{\left(e^{t_1}+e^d-1\right)\left(e^{t_2}+e^d-1\right)\left(e^{\max(t_1,t_2)}+e^d-1\right)}. \qquad (4.11)
\end{aligned}$$

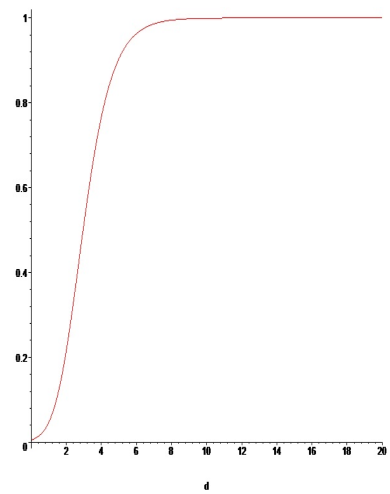

Figure 4.6: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.11) with respect to d for $t_1 = 1.25, t_2 = 2$.

Using Theorem 4.1, we can easily obtain both absolutely continuous and singular part. That is,

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{2}{3}\overline{F}_a(t_1,t_2) + \frac{1}{3}\overline{F}_s(t_1,t_2)$$

where

$$\overline{F}_a(t_1,t_2) = \frac{e^{3d}\left(-2+2e^d-e^\omega+3e^\beta\right)}{2\left(-1+e^d+e^\omega\right)\left(-1+e^d+e^\beta\right)^3}$$

$$\overline{F}_s(t_1,t_2) = \frac{\left(e^{3d}\right)}{\left(-1+e^d+e^\beta\right)^3}$$

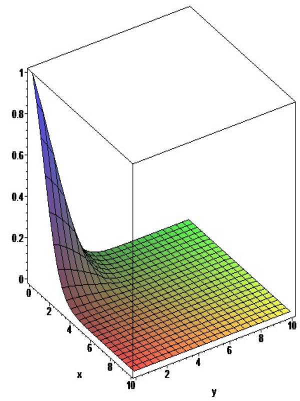

Figure 4.7: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.11), for $d = 2$.

4.6 Bivariate Lomax Distribution

Consider the bivariate Lomax distribution. We have the following joint survival function

$$\overline{F}_{Z_1,Z_2}(z_1,z_2) = (1 + az_1 + bz_2 + \theta z_1 z_2)^{-c}, \; a,b,c > 0 \text{ and } 0 \leq \theta \leq (c+1)ab \tag{4.12}$$

The name of this distribution is also known as Durling distribution. Bivariate Lomax distribution is studied in the reliability analysis. For further information we refer to [31], [38], [22] and [5]. Let (T_1, D_1),(T_2, D_2) and (T_0, D_0) be the random vectors with bivariate Lomax distribution function with the parameters $a = 2$, $b = 3$, $\theta = 4$ and $c = 3$. Let $\omega = \min(t_1, t_2)$ and $\beta = \max(t_1, t_2)$ using (4.12) in Theorem 3.1, joint survival function of MOMSE can be found as follows

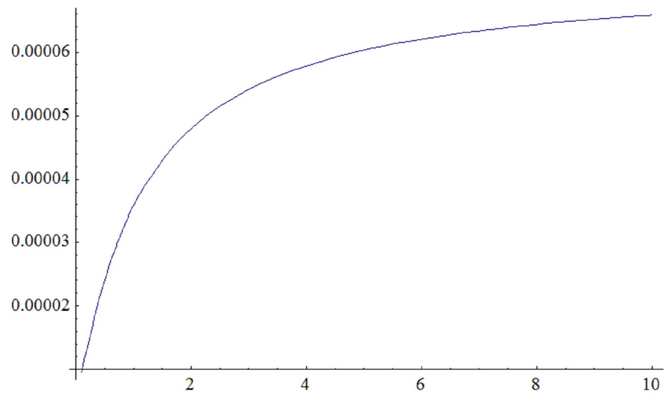

Figure 4.8: Graph of survival function $\overline{F}_{X_1,X_2}(t_1,t_2)$ given in (4.13) with respect to d for $t_1 = 1.25, t_2 = 1.5$.

$$\begin{aligned}
\overline{F}_{X_1,X_2}(t_1,t_2) &= \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)} \\
&= \frac{(1+2t_1+3d+4t_1d)^{-3}(1+2t_2+3d+4t_2d)^{-3}}{(1+3d)^{-9}} \\
&\times (1+2\max(t_1,t_2)+3d+4\max(t_1,t_2)d)^{-3} \\
&= \frac{(1+3d)^9}{(1+3d+2t_1+4dt_1)^3(1+3d+2t_2+4dt_2)^3(1+3d+2\beta+4\beta d)^3}.
\end{aligned} \tag{4.13}$$

Using Theorem (4.1, we can easily obtain both absolutely continuous and singular part. That is,

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{2}{3}\overline{F}_a(t_1,t_2) + \frac{1}{3}\overline{F}_s(t_1,t_2)$$

where

$$\overline{F}_a(t_1,t_2) = \frac{(1+3d)^9\left(-1+\frac{3(1+3d+2\beta+4d\beta)^3}{(1+2\omega+d(3+4\omega))^3}\right)}{2(1+2\beta+d(3+4\beta))^9}$$

$$\overline{F}_s(t_1,t_2) = \frac{(1+3d)^9}{(1+3d+2\beta+4d\beta)^9}$$

In the next Corollary, marginal probability density function (pdf) of X_i obtained easily by using Corollary (3.2).

Corollary 4.2 *Marginal survival functions of X_1 and X_2 can be written as*

$$\overline{F}_{X_i}(t_i) = \frac{(1+3d)^6}{(1+3d+2t_i+4dt_i)^6},\ i=1,2.$$

Marginal pdf of X_i can be derived as follows

$$f_{X_i}(t_i) = -\frac{\partial}{\partial t_i}\overline{F}_{X_i}(t_i)$$

$$f_{X_i}(t_i) = \frac{6\,(1+3d)^6\,(2+4d)}{(1+3d+2t_i+4dt_i)^7}.$$

Theorem 4.3 *The conditional pdf of X_i, given $X_j = t_j$, denoted by $f_{i|j}(t_i|t_j)(i \neq j = 1,2)$, is given by*

$$f_{i|j}(t_i|t_j) = \begin{cases} f_{i|j}^{(1)}(t_i|t_j) & \text{if } t_i > t_j \\ f_{i|j}^{(2)}(t_i|t_j) & \text{if } t_i < t_j \\ f_{i|j}^{(0)}(t_i|t_j) & \text{if } t_i = t_j \end{cases},$$

where

$$f_{i|j}^{(1)}(t_i|t_j) = \frac{6\,(1+2d)\,(1+3d)^3\,(1+2t_j+d(3+4t_j))^3}{(1+3d+2t_i+d(3+4t_i))^7}$$

$$f_{i|j}^{(2)}(t_i|t_j) = \frac{3\,(1+3d)^3\,(2+4d)}{(1+3d+2t_i+4dt_i)^4}$$

$$f_{i|j}^{(0)}(t_i|t_j) = \frac{(1+2d)\,(1+3d)^3\,(1+3d+2t_j+4dt_j)^7}{(2+4d)\,(1+2t_j+d(3+4t_j))^{10}}$$

Proof. Proof follows from the following well known definition of conditional density.

$$f_{i|j}(t_i|t_j) = \frac{f_{X_i,X_j}(t_i,t_j)}{f_{X_j}(t_j)}$$

□

By using the Corollary (4.2) and Theorem (4.3), we can easily obtain covariance and correlation coefficient of the components X_1 and X_2 in the next Corollary.

Corollary 4.4

$$\begin{aligned} E\left[X_i\right] &= \int_0^{\infty} t_i \frac{6\left(1+3d\right)^6\left(2+4d\right)}{\left(1+3d+2t_i+4dt_i\right)^7} \\ &= \frac{1+3d}{10(1+2d)} \end{aligned}$$

$$\begin{aligned} E\left[X_i^2\right] &= \int_0^{\infty} t_i^2 \frac{6\left(1+3d\right)^6\left(2+4d\right)}{\left(1+3d+2t_i+4dt_i\right)^7} \\ &= \frac{\left(1+3d\right)^2}{40(1+2d)^2} \end{aligned}$$

$$\begin{aligned} Var\left[X_i\right] &= E\left[X_i^2\right] - \left(E\left[X_i\right]\right)^2 \\ &= \frac{3\left(1+3d\right)^2}{200(1+2d)^2} \end{aligned}$$

$$\begin{aligned} E\left[X_1X_2\right] &= \int\int t_1t_2 f_{12}(t_1,t_2)dt_1dt_2 \\ &= \underbrace{\int_0^{\infty}\int_0^{t_2} t_1t_2 f_1(t_1,t_2)dt_1dt_2}_{t_1<t_2} + \underbrace{\int_0^{\infty}\int_0^{t_1} t_1t_2 f_2(t_1,t_2)dt_1dt_2}_{t_1>t_2} + \underbrace{\int_0^{\infty} t_1^2 f_0(t_1,t_1)dt_1}_{t_1=t_2} \\ &= \frac{\left(1+3d\right)^2}{70\left(1+2d\right)^2} \end{aligned}$$

Hence the covariance is given by

$$\begin{aligned} Cov\left[X_1,X_2\right] &= E\left[X_1X_2\right] - E\left[X_1\right]E\left[X_2\right] \\ &= \frac{3\left(1+3d\right)^2}{700\left(1+2d\right)^2} \end{aligned}$$

and the correlation is

$$\begin{aligned}\rho[X_1, X_2] &= \frac{Cov[X_1, X_2]}{\sqrt{Var[X_1]\,Var[X_2]}} \\ &= \frac{2}{7}\end{aligned}$$

From Corollary (4.4), we get desired result such that $0 \leq \rho[X_1, X_2] \leq 1$.

Chapter 5

Dependency and Comparison Results

In this chapter, we discuss the dependency properties of the new proposed bivariate survival function MOMSE. First, we need to give some definition and dependency properties in order to obtain our results.

Definition. [32] Let X and Y be continuous random variables. X and Y are right corner set increasing ($RCSI\,[X,Y]$), if

$$P\,[X > t_1, Y > t_2 | X > \bar{t}_1, Y > \bar{t}_2]$$

is nondecreasing in $\bar{t}_1$ and in $\bar{t}_2$ for all t_1 and t_2. The following theorem can be found in [32], page 198.

Theorem 5.1 *Let X and Y be continuous random variables with joint distribution function H. Then*

$$RCSI\,[X,Y] \iff \overline{H}\,(t_1,t_2)\,\overline{H}\,(\bar{t}_1,\bar{t}_2) \geq \overline{H}\,(t_1,\bar{t}_2)\,\overline{H}\,(\bar{t}_1,t_2)$$

for all $t_1, t_2, \bar{t}_1, \bar{t}_2$ in $[-\infty, \infty]$ such that $t_1 \leq \bar{t}_1$ and $t_2 \leq \bar{t}_2$.

Theorem 5.2 *MOMSE satisfies the $RCSI\,[X_1, X_2]$ property.*

Proof. From (3.2) we have

$$\overline{F}_{X_1,X_2}(t_1,t_2) = \frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)}{\overline{F}_{D_1}(d)\overline{F}_{D_2}(d)\overline{F}_{D_0}(d)}.$$

By using Theorem (5.1), we need to show that

$$\begin{aligned}
&\frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)\overline{F}_{T_1,D_1}(\overline{t_1},d)\overline{F}_{T_2,D_2}(\overline{t_2},d)}{\overline{F}^2_{D_1}(d)\overline{F}^2_{D_2}(d)\overline{F}^2_{D_0}(d)}\times \\
&\overline{F}_{T_0,D_0}(\max(\overline{t_1},\overline{t_2}),d) \\
&-\frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_2,D_2}(\overline{t_2},d)\overline{F}_{T_0,D_0}(\max(t_1,\overline{t_2}),d)\overline{F}_{T_1,D_1}(\overline{t_1},d)\overline{F}_{T_2,D_2}(t_2,d)}{\overline{F}^2_{D_1}(d)\overline{F}^2_{D_2}(d)\overline{F}^2_{D_0}(d)}\times \\
&\overline{F}_{T_0,D_0}(\max(\overline{t_1},t_2),d) \\
&\geq 0.
\end{aligned}$$

Namely, we need to check, whether

$$\begin{aligned}
&\left[\frac{\overline{F}_{T_1,D_1}(t_1,d)\overline{F}_{T_1,D_1}(\overline{t_1},d)\overline{F}_{T_2,D_2}(t_2,d)\overline{F}_{T_2,D_2}(\overline{t_2},d)}{\overline{F}^2_{D_1}(d)\overline{F}^2_{D_2}(d)\overline{F}^2_{D_0}(d)}\right]\times \\
&[\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)\overline{F}_{T_0,D_0}(\max(\overline{t_1},\overline{t_2}),d)- \\
&\overline{F}_{T_0,D_0}(\max(t_1,\overline{t_2}),d)\overline{F}_{T_0,D_0}(\max(\overline{t_1},t_2),d)] \\
&\geq 0
\end{aligned}$$

or not. Since the first factor is always is nonnegative, we need to analyze the second term. That is, we need to verify

$$\begin{aligned}
&\overline{F}_{T_0,D_0}(\max(t_1,t_2),d)\overline{F}_{T_0,D_0}(\max(\overline{t_1},\overline{t_2}),d)- \\
&\overline{F}_{T_0,D_0}(\max(t_1,\overline{t_2}),d)\overline{F}_{T_0,D_0}(\max(\overline{t_1},t_2),d) \overset{?}{\geq} 0.
\end{aligned}$$

Hence, for $t_1 \leq \overline{t_1}$ and $t_2 \leq \overline{t_2}$ the following six cases need to be analyzed :

Case 1. $t_1 \leq \overline{t_1} \leq t_2 \leq \overline{t_2}$

Case 2. $t_1 \leq t_2 \leq \overline{t_1} \leq \overline{t_2}$

Case 3. $t_1 \leq t_2 \leq \overline{t_2} \leq \overline{t_1}$

Case 4. $t_2 \leq t_1 \leq \overline{t_2} \leq \overline{t_1}$

Case 5. $t_2 \leq \overline{t_2} \leq t_1 \leq \overline{t_1}$
Case 6. $t_2 \leq t_1 \leq \overline{t_1} \leq \overline{t_2}$

Case 1 $\Rightarrow t_1 \leq \overline{t_1} \leq t_2 \leq \overline{t_2}$
It is clear that

$$\overline{F}_{T_0,D_0}(t_2,d)\overline{F}_{T_0,D_0}(\overline{t_2},d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\overline{F}_{T_0,D_0}(t_2,d) = 0.$$

Case 2 $\Rightarrow t_1 \leq t_2 \leq \overline{t_1} \leq \overline{t_2}$
We have to check

$$\overline{F}_{T_0,D_0}(t_2,d)\overline{F}_{T_0,D_0}(\overline{t_2},d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\overline{F}_{T_0,D_0}(\overline{t_1},d) \overset{?}{\geq} 0 \text{ or}$$
$$\overline{F}_{T_0,D_0}(\overline{t_2},d)\left[\overline{F}_{T_0,D_0}(t_2,d) - \overline{F}_{T_0,D_0}(\overline{t_1},d)\right] \overset{?}{\geq} 0.$$

Since $\overline{t_1} \geq t_2$, then clearly $\overline{F}_{T_0,D_0}(\overline{t_1},d) \leq \overline{F}_{T_0,D_0}(t_2,d)$ and

$$\overline{F}_{T_0,D_0}(\overline{t_2},d)\left[\overline{F}_{T_0,D_0}(t_2,d) - \overline{F}_{T_0,D_0}(\overline{t_1},d)\right] \geq 0.$$

Case 3 $\Rightarrow t_1 \leq t_2 \leq \overline{t_2} \leq \overline{t_1}$
We need to show that whether below inequality is hold or not.

$$\overline{F}_{T_0,D_0}(t_2,d)\overline{F}_{T_0,D_0}(\overline{t_1},d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\overline{F}_{T_0,D_0}(\overline{t_1},d) \overset{?}{\geq} 0 \text{ or}$$
$$\overline{F}_{T_0,D_0}(\overline{t_1},d)\left[\overline{F}_{T_0,D_0}(t_2,d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\right] \overset{?}{\geq} 0.$$

Since $\overline{t_2} \geq t_2$, then clearly $\overline{F}_{T_0,D_0}(\overline{t_2},d) \leq \overline{F}_{T_0,D_0}(t_2,d)$ and

$$\overline{F}_{T_0,D_0}(\overline{t_1},d)\left[\overline{F}_{T_0,D_0}(t_2,d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\right] \geq 0.$$

Case 4 $\Rightarrow t_2 \leq t_1 \leq \overline{t_2} \leq \overline{t_1}$

We have to analyze this condintion in order to satisfy below inequality

$$\overline{F}_{T_0,D_0}(t_1,d)\overline{F}_{T_0,D_0}(\overline{t_1},d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\overline{F}_{T_0,D_0}(\overline{t_1},d) \overset{?}{\geq} 0 \text{ or}$$
$$\overline{F}_{T_0,D_0}(\overline{t_1},d)\left[\overline{F}_{T_0,D_0}(t_1,d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\right] \overset{?}{\geq} 0.$$

Since $\overline{t_2} \geq t_1$, then clearly $\overline{F}_{T_0,D_0}(\overline{t_2},d) \leq \overline{F}_{T_0,D_0}(t_1,d)$ and

$$\overline{F}_{T_0,D_0}(\overline{t_1},d)\left[\overline{F}_{T_0,D_0}(t_1,d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\right] \geq 0.$$

Case 5 $\Rightarrow t_2 \leq \overline{t_2} \leq t_1 \leq \overline{t_1}$
It is clear that

$$\overline{F}_{T_0,D_0}(t_1,d)\overline{F}_{T_0,D_0}(\overline{t_1},d) - \overline{F}_{T_0,D_0}(t_1,d)\overline{F}_{T_0,D_0}(\overline{t_1},d) = 0.$$

Case 6 $\Rightarrow t_2 \leq t_1 \leq \overline{t_1} \leq \overline{t_2}$
Finally, we need to check

$$\overline{F}_{T_0,D_0}(t_1,d)\overline{F}_{T_0,D_0}(\overline{t_2},d) - \overline{F}_{T_0,D_0}(\overline{t_2},d)\overline{F}_{T_0,D_0}(\overline{t_1},d) \overset{?}{\geq} 0 \text{ or}$$
$$\overline{F}_{T_0,D_0}(\overline{t_2},d)\left[\overline{F}_{T_0,D_0}(t_1,d) - \overline{F}_{T_0,D_0}(\overline{t_1},d)\right] \overset{?}{\geq} 0.$$

Since $\overline{t_1} \geq t_1$, then clearly $\overline{F}_{T_0,D_0}(\overline{t_1},d) \leq \overline{F}_{T_0,D_0}(t_1,d)$ and

$$\overline{F}_{T_0,D_0}(\overline{t_2},d)\left[\overline{F}_{T_0,D_0}(t_1,d) - \overline{F}_{T_0,D_0}(\overline{t_1},d)\right] \geq 0.$$

Consequently, all cases are verified clearly, then we can say that MOMSE satisfies the $RCSI\,[X_1, X_2]$ property. □

In the following sections, some properties of the stochastic comparison and some ageing properties of the proposed conditional random variables are given.

5.1 Stochastic Comparison

Stochastic orders among random variables play an important role in statistics, especially in reliability theory. There are many kinds of stochastic orders in the literature. In this section, we focus on only the usual stochastic order.

Definition. Let U_1 and U_2 be two random variables. If

$$P(U_1 > t) \leq P(U_2 > t) \text{ for all } t \in (-\infty, \infty), \tag{5.1}$$

then U_1 is said to be smaller than U_2 in the usual stochastic order. It is denoted by $U_1 \leq_{st} U_2$. For more information about the stochastic order, we refer to [40].

Theorem 5.3 ([40],part a, page 273) *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors. If $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$ and $\boldsymbol{g}$:$\mathbb{R}^n \to \mathbb{R}^k$ is any k-dimensional increasing (decreasing) function, for any positive integer k, then the k-dimensional vectors $\boldsymbol{g(X)}$ and $\boldsymbol{g(Y)}$ satisfy $\boldsymbol{g(X)} \leq_{st} (\geq_{st})\boldsymbol{g(Y)}$.*

Let $\{V_1, V_2, V_0\}$ be independent conditional random variables as defined in (3.1) and $\{W_1, W_2, W_0\}$ be another set of independent conditional random variables defined as $W_i \stackrel{d}{=} V_i$, where $\stackrel{d}{=}$ stands for equality in distribution. Let the random vectors $\mathbf{X}$ and $\mathbf{Y}$ be defined as follows:

$$\mathbf{X} = (X_1, X_2) \text{ and } \mathbf{Y} = (Y_1, Y_2)$$

where $X_1 = \min\{V_1, V_0\}$, $X_2 = \min\{V_2, V_0\}$, $Y_1 = \min\{W_1, W_0\}$ and $Y_2 = \min\{W_2, W_0\}$.

Theorem 5.4 *If $V_i \leq_{st} W_i$ for $i = 0, 1, 2$, then $\mathbf{X} \leq_{st} \mathbf{Y}$.*

Proof. Since V_i and W_i, $i = 0, 1, 2$, are independent random variables, $V_i \leq_{st} W_i$ imply $(V_1, V_2, V_0) \leq_{st} (W_1, W_2, W_0)$. Then the assertion of the theorem follows from Theorem 5.3. □

5.2 Residual Life and Ageing Properties

Let $\mathbf{X}_t = [(X_1 - t, X_2 - t)|X_1 > t, X_2 > t]$ denote the residual life of $\mathbf{X}$ given that the components have survived at time t. Denote by $\overline{F}_i(x|t)$ the residual life function of ith component given that it survives at time t, i.e.

$$\overline{F}_i(x|t) = P\{X_i - t > x \mid X_i > t\}, i = 1, 2.$$

Then it is easy to see that

$$\overline{F}_i(x|t) = \frac{\overline{F}_i(t+x)}{\overline{F}_i(t)},$$

$\overline{F}_i(t) > 0$. For more information about residual life, we refer to [6]. The survival function of $\mathbf{X}_t$ is given in the following theorem.

Theorem 5.5 *Let* $\mathbf{x} = (x_1, x_2)$ *and* $(V_i)_t = [V_i - t \mid V_i > t]\,, i = 0, 1, 2$. *The survival function of* $\mathbf{X}_t$ *is*

$$P(\mathbf{X}_t > \mathbf{x}) = P(\min\{(V_1)_t, (V_0)_t\} > x_1, \min\{(V_2)_t, (V_0)_t\} > x_2).$$

Proof. Indeed, we have

$$\begin{aligned}
P(\mathbf{X}_t > \mathbf{x}) &= P(X_1 - t > x_1, X_2 - t > x_2|X_1 > t, X_2 > t) \\
&= P(\min\{V_1 - t, V_0 - t\} > x_1, \min\{V_2 - t, V_0 - t\} > x_2|V_i > t, i = 0, 1, 2) \\
&= \frac{P(V_1 > x_1 + t, V_0 > x_1 + t, V_2 > x_2 + t, V_0 > x_2 + t)}{P(V_1 > t, V_2 > t, V_0 > t)} \\
&= \frac{P(V_1 > x_1 + t, V_2 > x_2 + t, V_0 > \max(x_1, x_2) + t)}{P(V_1 > t, V_2 > t, V_0 > t)} \\
&= \frac{\overline{F}_1(t + x_1)}{\overline{F}_1(t)} \frac{\overline{F}_2(t + x_2)}{\overline{F}_2(t)} \frac{\overline{F}_0(t + \max(x_1, x_2))}{\overline{F}_0(t)} \\
&= P((V_1)_t > x_1)P((V_2)_t > x_2)P((V_0)_t > \max(x_1, x_2)) \\
&= P(\min\{(V_1)_t, (V_0)_t\} > x_1, \min\{(V_2)_t, (V_0)_t\} > x_2).
\end{aligned}$$

□

Definition. A nonnegative random vector $\mathbf{X} = (X_1, X_2)$ is said to be bivariate new better than used (BNBU) if $\mathbf{X} \geq_{st} \mathbf{X}_t$ for all $t \geq 0$.

Corollary 5.6 *If V_i is new better than used (NBU) for $i = 0, 1, 2$, then $\mathbf{X}$ is BNBU.*

Proof. By Theorem 1.A.30(b) of [40] page 15, if V_i is NBU then $V_i \geq_{st} (V_i)_t$ for all $t > 0$ and $i = 0, 1, 2$. Since $\min\{x, y\}$ is an increasing function with respect to x and y, then from Theorem 6.B.16(a) of [40] page 273, we get

$$\begin{aligned} \mathbf{X} &= (\min\{V_1, V_0\}, \min\{V_2, V_0\}) \\ &\geq_{st} (\min\{(V_1)_t, (V_0)_t\}, \min\{(V_2)_t, (V_0)_t\}) = \mathbf{X}_t. \end{aligned}$$

□

Chapter 6

Maximum Likelihood Estimation

This chapter deals with finding the unknown parameter θ of the MOMSE. Because if sampling is from a population described by a pdf $f(x|\theta)$, knowledge of the parameter θ, provides knowledge of the entire population. In the literature, parameter estimator of the Marshall-Olkin type distribution is widely studied. There are four methods which can be used to find estimators. These methods are the method of moments, maximum likelihood estimators, bayes estimators and EM algorithm. The EM algorithm is different from other methods because of its nature. This method is specifically designed to find the maximum likelihood estimator. In this thesis, only maximum likelihood estimator and EM algorithm are considered. Due to the fact that, MOMSE has both singular and absolutely continuous parts similar to the classical Marshall-Olkin distribution. Statistical inference is not an easy task because of the complex structure of its density function. To overcome this drawback and find the maximum likelihood estimator of the proposed model, we use the EM algorithm. First, we introduce the EM algorithm, then we apply this algorithm to the proposed MOMSE model.

6.1 The EM Algorithm

An expectation–maximization (EM) algorithm is an iterative method for finding maximum likelihood estimates of unknown parameters. Generally, this algorithm is preferred when the problem is considered as a missing data problem. For more details about this algorithm, we refer to [28] and [9]. According to setup of the MOMSE model, the vector $\mathbf{X}_i = (X_{1i}, X_{2i})$ is considered as observed data. On the other hand, the missing data consists of the random vector $\mathbf{V}_i = (V_{0i}, V_{1i}, V_{2i})$, which is non-observable. In the E-step, we calculate the conditional expectation of $\mathbf{V}_i$ given $\mathbf{X}_i$, and then use these results to calculate the MLEs for a sample.
Consider a random bivariate sample $\{(t_{11}, t_{21}), \ldots, (t_{1n}, t_{2n})\}$. Let n_0, n_1 and n_2 denote the number of observations for which $t_{1i} = t_{2i}, t_{1i} < t_{2i}$ and $t_{1i} > t_{2i}$, respectively. More precisely,

$$\begin{aligned} I_0 &= \{i : t_{1i} = t_{2i}\} \quad n_0 = |I_0| \\ I_1 &= \{i : t_{1i} < t_{2i}\} \quad n_1 = |I_1| \\ I_2 &= \{i : t_{1i} > t_{2i}\} \quad n_2 = |I_2| \end{aligned}$$

and $n = n_0 + n_1 + n_2$. For a given sample of observations, the log-likelihood function of the MOMSE model with bivariate Gumbel distribution function can be written as

$$\begin{aligned} L(\theta_0, \theta_1, \theta_2) &= n_1 \ln\left[(1+\theta_1 d)(2 + d(\theta_0 + \theta_2))\right] + n_2 \ln\left[(1+\theta_2 d)(2 + d(\theta_0 + \theta_1))\right] \\ &+ n_0 \ln(1+\theta_0 d) - (1+\theta_1 d)\sum_{i=1}^{n} t_{1i} - (1+\theta_2 d)\sum_{i=1}^{n} t_{2i} - (1+\theta_0 d)\sum_{i=1}^{n} \max(t_{1i}, t_{2i}) \end{aligned} \tag{6.1}$$

Suppose $t_1 < t_2$, that is, $X_1 = \min(V_1, V_0) < \min(V_2, V_0) = X_2$. We have only two inequalities

$$V_1 < V_0 < V_2 \text{ and } V_1 < V_2 < V_0$$

which satisfy this condition among all the permutations of V_0, V_1 and V_2. Also, it is obvious that $V_0 \geq \max(t_1, t_2)$. We need to find the conditional density function V_0, V_1 and V_2.

First, consider the probability under the condition that components X_1 and X_2 are affected at time t_1 and t_2, respectively. More precisely,

$$\lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} P\left(t_0 \leq V_0 \leq t_0 + \Delta t_0 | X_1 = t_1, X_2 = t_2, \theta\right)$$

From the inequality $V_1 < V_0 < V_2$.

$$\begin{aligned} t_1 &= X_1 = \min(V_1, V_0) = V_1 \text{ and} \\ t_2 &= X_2 = \min(V_2, V_0) = V_0 = t_0 \end{aligned}$$

It is clear that $t_0 = \max(t_1, t_2)$. Under this setup conditional density function of V_0 can be obtained as

$$\begin{aligned} f_{V_0}(t_0|t_1, t_2, \theta) &= \frac{f_{V_1}(s) f_{V_0}(t_0) \bar{F}_{V_2}(t_0)}{f_{X_1,X_2}(t_1, t_2)} \\ &= \frac{1 + \theta_0 d}{2 + (\theta_0 + \theta_2) d}, \end{aligned}$$

where $\bar{F}_Y(z)$ denotes the survival function of the random variable Y and $s = \min(t_1, t_2)$. Now, consider the inequality $V_1 < V_2 < V_0$. Since $t_0 > \max(t_1, t_2)$, the conditional density function of V_0 is

$$\begin{aligned} f_{V_0}(t_0|t_1, t_2, \theta) &= \frac{f_{V_1}(s) f_{V_2}(t_2) f_{V_0}(t_0)}{f_{X_1,X_2}(t_1, t_2)} \\ &= \frac{e^{-(t_0 - t_2)(1 + \theta_0 d)}(1 + \theta_0 d)(1 + \theta_2 d)}{2 + (\theta_2 + \theta_0) d} \end{aligned}$$

Hence,

$$f_{V_0}(t_0|t_1, t_2, \theta) \begin{cases} \frac{1+\theta_0 d}{2+(\theta_0+\theta_2)d} & , t_0 = t_2 \\ \frac{e^{-(t_0-t_2)(1+\theta_0 d)}(1+\theta_0 d)(1+\theta_2 d)}{2+(\theta_2+\theta_0)d} & , t_0 > t_2 \end{cases}$$

The conditional expectation can be computed as

$$
\begin{aligned}
E(V_0|t_1, t_2, \theta) &= \left(\frac{1 + \theta_0 d}{2 + (\theta_0 + \theta_2)d}\right) t_2 + \\
&\int_{t_2}^{\infty} s \left(\frac{e^{-(t_0 - t_2)(1+\theta_0 d)}(1 + \theta_0 d)(1 + \theta_2 d)}{2 + (\theta_2 + \theta_0)d}\right) ds \\
&= \frac{t_2(1 + \theta_0 d)}{2 + (\theta_0 + \theta_2)d} + \frac{(1 + \theta_2 d)\left[1 + t_2(1 + \theta_0 d)\right]}{\left[2 + (\theta_0 + \theta_2)d\right](1 + \theta_0 d)}
\end{aligned}
$$

Similarly, one can easily show that

$$
E(V_2|t_1, t_2, \theta) = \frac{t_2(1 + \theta_2 d)}{2 + (\theta_0 + \theta_2)d} + \frac{(1 + \theta_0 d)\left[1 + t_2(1 + \theta_2 d)\right]}{\left[2 + (\theta_0 + \theta_2)d\right](1 + \theta_2 d)}
$$

In addition, it is easily to see that $E(V_1|t_1, t_2, \theta) = t_1$. On the other hand, these conditional expectation of the random variables V_0, V_1 and V_2 can be easily derived for the case $t_1 > t_2$. Now, consider the last case $t_1 = t_2$, i.e. $X_1 = \min(V_0, V_1) = \min(V_0, V_2) = X_2$. In this case, again there are two inequalities

$$
V_0 < V_1 < V_2 \text{ and } V_0 < V_2 < V_1
$$

Using these inequalities, for $t_0 > t_1 = t_2$ conditional density function of the random variable V_1 is

$$
\begin{aligned}
f_{V_1}(t_0|t_1, t_2, \theta) &= \frac{f_{V_1}(t_0) f_{V_0}(t_1) \bar{F}_{V_2}(t_1)}{f_{X_1, X_2}(t_1, t_2)} \\
&= \frac{(1 + \theta_1 d)(1 + \theta_0 d) e^{-(t_0 - t_1)(1+\theta_1 d)}}{2 + (\theta_1 + \theta_2)d}
\end{aligned}
$$

and the conditional expectation of the random variable V_1 is

$$
E(V_1|t_1, t_2, \theta) = \frac{1 + t_1(1 + \theta_1 d)}{(1 + \theta_1 d)}
$$

Similarly, one can obtain the conditional expectation of the random variable V_2 as follows

$$
E(V_2|t_1, t_2, \theta) = \frac{1 + t_1(1 + \theta_2 d)}{(1 + \theta_2 d)}
$$

In this case, it is clear that $E(V_0|t_1,t_2,\theta)=t_1$. For all cases, conditional expectations of the random variables V_0, V_1 and V_2 are given in Table 6.1.

	$E(V_0\|t_1,t_2,\theta)$	$E(V_1\|t_1,t_2,\theta)$
$t_1<t_2$	$\frac{t_2(1+\theta_0 d)^2+(1+\theta_2 d)[1+t_2(1+\theta_0 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_0 d)}$	t_1
$t_1>t_2$	$\frac{t_1(1+\theta_0 d)^2+(1+\theta_1 d)[1+t_1(1+\theta_0 d)]}{[2+(\theta_0+\theta_1)d](1+\theta_0 d)}$	$\frac{t_1(1+\theta_1 d)^2+(1+\theta_0 d)[1+t_1(1+\theta_1 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_1 d)}$
$t_1=t_2$	t_1	$\frac{1+t_1(1+\theta_1 d)}{(1+\theta_1 d)}$

	$E(V_2\|t_1,t_2,\theta)$
$t_1<t_2$	$\frac{t_2(1+\theta_2 d)^2+(1+\theta_0 d)[1+t_2(1+\theta_2 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_2 d)}$
$t_1>t_2$	t_2
$t_1=t_2$	$\frac{1+t_1(1+\theta_2 d)}{(1+\theta_2 d)}$

Table 6.1: The conditional expectations of the random variables V_0, V_1 and V_2.

The algorithm can be described as follows:
E-step: After kth iteration, for $i=1,\ldots,n$, calculate the pseudo-values $a_i = E(V_{0i}|t_1,t_2,\theta^{(k)}), b_i = E(V_{1i}|t_1,t_2,\theta^{(k)})$ and $c_i = E(V_{2i}|t_1,t_2,\theta^{(k)})$ by using the data and the vector parameters $\theta^{(k)} = (\theta_0^{(k)}, \theta_1^{(k)}, \theta_2^{(k)})$ which are obtained at the kth step.

M-step: Since the shock times T_i, $i=0,1,2$ are exponentially distributed with parameter $(1+d\theta_i)$, the maximum likelihood estimates from the exponential distribution are obtained using the pseudo-values a_i, b_i and c_i. Hence, the estimates are updated by

$$\theta_0^{(k+1)} = \frac{1}{d}\left(\frac{n}{\sum a_i}-1\right) \quad \theta_1^{(k+1)} = \frac{1}{d}\left(\frac{n}{\sum b_i}-1\right) \quad \theta_2^{(k+1)} = \frac{1}{d}\left(\frac{n}{\sum c_i}-1\right)$$

The iterating process should stop when some convergence criterion is satisfied, otherwise process returns to the E-step.

6.2 Data Analysis for Bivariate Case

For illustrating the EM algorithm to obtain the unknown parameters of the MOMSE, we generated data from a MOMSE using θ_0, θ_1, θ_2 and d as 0.4, 0.3, 0.6 and 3, respectively. (see Table 6.2).

t_1	t_2	t_1	t_2
0.2327	0.2327	0.0423	0.0454
0.0622	0.5326	0.1656	0.1656
0.2123	0.3321	0.6828	0.1787
0.0705	0.2253	0.1621	0.1621
1.0792	0.5612	0.0523	0.0771
0.6471	0.0423	0.2964	0.2964
0.0095	0.0982	0.0255	0.3624
0.2936	0.2936	0.1716	0.1716
0.0976	0.0115	0.3564	0.2575
0.1223	0.1223	0.1669	0.1669
0.1228	0.8988	0.4179	0.4344
0.1304	0.1312	0.0055	0.3758
0.3826	0.1618	0.0946	0.0646
0.0562	1.6152	0.1557	0.1440
0.2868	0.0262	0.5715	0.5715
0.2029	0.1026	0.5590	0.0397
0.2714	0.5140	0.0648	0.0648
0.0406	0.1988	0.2898	0.2898
0.1307	0.3028	0.7291	0.4573
0.1692	0.1005	0.0056	0.0056

Table 6.2: Simulated data from MOMSE model.

We use some initial values for θ to start the EM algorithm and use the convergence criteria as $|\frac{(L_k - L_{k-1})}{L_{k-1}}| < 10^{-12}$ (L_k denotes the the log-likelihood value at the kth step). Table 6.3 presents the initial values and number of iterations until convergence. For example, if we start the algorithm using $\theta_0^{(1)} = 0.1, \theta_1^{(1)} = 0.1$ and $\theta_2^{(1)} = 0.1$ as initial values, iteration stops after 46 steps. Similarly, when the algorithm starts from the values $\theta_0^{(1)} = 0.7, \theta_1^{(1)} = 0.4$ and $\theta_2^{(1)} = 0.1$, the algorithm converges to the same point after 47 iterations.

In all cases, we derive MLEs of θ_0, θ_1 and θ_2 as 0.2499, 0.4422 and 0.3084

$\theta_1^{(1)}$	$\theta_2^{(1)}$	$\theta_0^{(1)}$	number of iteration
0.1	0.1	0.1	46
0.3	0.3	0.3	44
0.5	0.1	0.5	47
0.7	0.3	0.1	46
0.6	0.8	0.9	40
0.1	0.5	0.1	43
0.1	0.1	0.5	48
0.5	0.1	0.1	40
0.5	0.2	0.8	47
0.4	0.1	0.7	47

Table 6.3: Number of iterations until convergence.

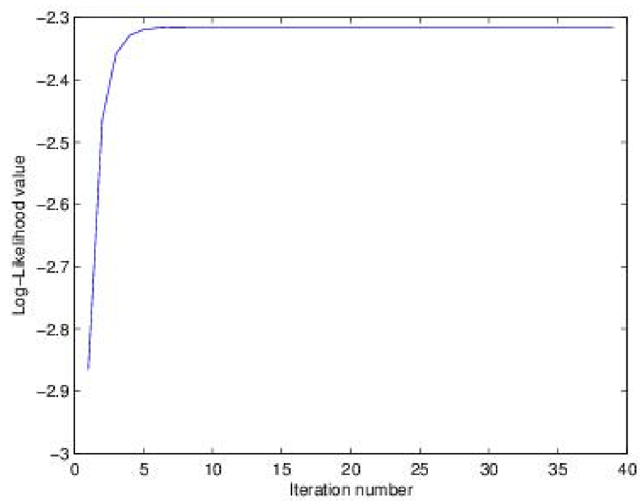

Figure 6.1: Value of Log-Likelihood function at different iteration.

respectively and corresponding log-likelihood value -2.3156. It can be seen from Figure 6.1 that log-likelihood function is nondecreasing. In 1982, Louis [24] suggested a technique for computing the asymptotic confidence interval by using the observed Fisher information matrix obtained from the EM algorithm. By using this technique, we have also computed the 90% confidence intervals of θ_1, θ_2 and θ_0 as follows (0.0134,0.4864), (0.1467,0.7377) and (0.0470, 0.5698), respectively. Detailed procedure can be seen in Appendix section.

Now, consider two components which are connected in series. This model is called a censored bivariate MOMSE. Let us denote $Z = (X_1, X_2)$, $\delta_1 = I(X_1 < X_2)$ and

$\delta_2 = I(X_1 > X_2)$ with $I(A)$ denoting the indicator function of the event. In this model, the random vector observed on system failure is (Z, δ_1, δ_2). See [35] for more details about properties of a censored model with classical Marshall-Olkin distribution. The random variable Z has the following survival distribution

$$P(Z > t) = \exp(-t\,[(1 + d\theta_1) + (1 + d\theta_2) + (1 + d\theta_0)])$$

Bivariate random vector (δ_1, δ_2) has a multinomial distribution $(1; \frac{1+d\theta_1}{3+d(\theta_0+\theta_1+\theta_2)}, \frac{1+d\theta_2}{3+d(\theta_0+\theta_1+\theta_2)})$. Joint density of the vector (Z, δ_1, δ_2) can be written with respect to the product of the Lebesgue measure on $\mathbb{R}^+ = [0, \infty)$ and counting measure on $M = \{(0,0), (0,1), (1,0)\}$ as

$$f(z, m_1, m_2) = [(1 + d\theta_1) + (1 + d\theta_2) + (1 + d\theta_0)] \times$$
$$\exp(-t\,[(1 + d\theta_1) + (1 + d\theta_2) + (1 + d\theta_0)])\times$$
$$\left(\frac{1 + d\theta_1}{3 + d(\theta_0 + \theta_1 + \theta_2)}\right)^{m_1} \left(\frac{1 + d\theta_2}{3 + d(\theta_0 + \theta_1 + \theta_2)}\right)^{m_2} \left(\frac{1 + d\theta_0}{3 + d(\theta_0 + \theta_1 + \theta_2)}\right)^{m_0}$$

for (Z, δ_1, δ_2)=$(z, m_1, m_2)\in \mathbb{R}^+ \times M$ and $m_0 = 1 - m_1 - m_2$. In this model, maximum likelihood estimation can be found using EM algorithm. First, it is necessary to have conditional expectations similar with the usual model. It can be clearly seen that z_i substitutes in all cases t_1 and t_2 in Table 6.1. We can use EM algorithm after deriving conditional expectations for all cases, as can be seen in Table 6.4.

	$\delta_1 = 1, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = 1$
$E(V_0 \| z, \delta_1, \delta_2, \theta)$	$\frac{z(1+\theta_0 d)^2+(1+\theta_2 d)[1+z(1+\theta_0 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_0 d)}$	$\frac{z(1+\theta_0 d)^2+(1+\theta_1 d)[1+z(1+\theta_0 d)]}{[2+(\theta_0+\theta_1)d](1+\theta_0 d)}$
$E(V_1 \| z, \delta_1, \delta_2, \theta)$	z	$\frac{z(1+\theta_1 d)^2+(1+\theta_0 d)[1+z(1+\theta_1 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_1 d)}$
$E(V_2 \| z, \delta_1, \delta_2, \theta)$	$\frac{z(1+\theta_2 d)^2+(1+\theta_0 d)[1+z(1+\theta_2 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_2 d)}$	z

	$\delta_1 = 0, \delta_2 = 0$
$E(V_0 \| z, \delta_1, \delta_2, \theta)$	z
$E(V_1 \| z, \delta_1, \delta_2, \theta)$	$\frac{1+z(1+\theta_1 d)}{(1+\theta_1 d)}$
$E(V_2 \| z, \delta_1, \delta_2, \theta)$	$\frac{1+z(1+\theta_2 d)}{(1+\theta_2 d)}$

Table 6.4: The conditional expectations of random variables V_0, V_1 and V_2 for the series system with two components.

Chapter 7

Multivariate Extension of MOMSE

If the system has more than two components, the extension of MOMSE model can be constructed similar to the classical Marshall-Olkin distribution,. Firstly, let us consider a system having three components. Let X_1, X_2 and X_3 denote the life length of the first, second and third components, respectively. A shock from the first source at random time T_1 with the magnitude D_1 affects component 1, a shock from the second source at random time T_2 with the magnitude D_2 affects component 2, and a shock from the third source at random time T_3 with the magnitude D_3 affects component 3. Another source, at random time T_0 with the magnitude D_0 affects all of the components. We assume that the random variables T_a and D_a, $a \in \{0, 1, 2, 3\}$ are stochastically dependent, and the affect of the shocks for all components depends on the upper-threshold d. More precisely, if the magnitude of the shock is greater than d, then the corresponding component is destroyed, otherwise it survives. Denote the joint distribution function of the random vector (T_a, D_a) by $F_{T_a,D_a}(t, d)$, $a \in \{0, 1, 2, 3\}$. We assume that the bivariate random vectors (T_1, D_1), (T_2, D_2), (T_3, D_3), and (T_0, D_0) are independent. According to this setup, we can define independent conditional random variables as follows:

$$V_1 = T_1 | D_1 > d$$
$$V_2 = T_2 | D_2 > d$$
$$V_3 = T_3 | D_3 > d$$
$$V_0 = T_0 | D_0 > d$$

that is, V_a : is a time of arrival of the shock which has magnitude greater than d. Thus the life lengths $X_1, X_2$and X_3 of components 1, 2 and 3 can be written respectively as follows

$$X_1 = min(V_1, V_0)$$
$$X_2 = min(V_2, V_0)$$
$$X_3 = min(V_3, V_0)$$

Theorem 7.1 *The joint survival function of X_1, X_2 and X_3 is*

$$\bar{F}_{X_1,X_2,X_3}(t_1,t_2,t_3) \equiv P\{X_1 > t_1, X_2 > t_2, X_3 > t_3\} = \frac{\prod_{i=1}^{3} \bar{F}_{T_i,D_i}(t_i,d)\bar{F}_{T_0,D_0}(\max(t_1,t_2,t_3),d)}{\prod_{i=0}^{3} \bar{F}_{D_i}(d)}$$

Proof. Indeed, we have

$$
\begin{aligned}
P(X_1 > t_1, X_2 > t_2, X_3 > t_3) &= P(\min(V_1, V_0) > t_1, \\
\min(V_2, V_0) > t_2, \min(V_3, V_0) > t_3) \\
&= P(V_1 > t_1, V_2 > t_2, V_3 > t_3, V_0 > \max(t_1, t_2, t_3)) \\
&= P(T_1 > t_1 | D_1 > d) P(T_2 > t_2 | D_2 > d) P(T_3 > t_3 | D_3 > d) \times \\
P(T_0 > \max(t_1, t_2, t_3) | D_0 > d) \\
&= \frac{\prod_{i=1}^{3} \bar{F}_{T_i, D_i}(t_i, d) \bar{F}_{T_0, D_0}(\max(t_1, t_2, t_3), d)}{\prod_{i=0}^{3} \bar{F}_{D_i}(d)}
\end{aligned}
$$

□

Now, consider the system having r components. Using similar idea one can easily derive the joint survival function of X_1, X_2,..., X_r as follows

$$
\bar{F}_{X_1, X_2, \ldots, X_r}(t_1, t_2, \ldots, t_r) = \frac{\prod_{i=1}^{r} \bar{F}_{T_i, D_i}(t_i, d) \bar{F}_{T_0, D_0}(\max(t_1, t_2, \ldots, t_r), d)}{\prod_{i=0}^{r} \bar{F}_{D_i}(d)}
$$

Let $(T_1, D_1), (T_2, D_2), \ldots, (T_r, D_r)$ and (T_0, D_0) be the random vectors with bivariate Gumbel distribution function with parameter $\theta_i, i = 0, 1, \ldots, r$. Then the joint survival function of X_1, X_2,...,X_r is

$$
P(X_1 > t_1, X_2 > t_2, \ldots, X_r > t_r) = e^{-\sum_{i=1}^{r} (1+\theta_i d) t_i} e^{-(1+\theta_0 d) \max(t_1, t_2, \ldots, t_r)}
$$

Similar to the bivariate case, the joint density function is not absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^r$, and has singular parts. Hence we have to separate all the possible cases to derive joint density function as follows

$$f(t_1, t_2, \ldots, t_r) = \begin{cases} (1+\theta_0 d)\bar{F}_{X_1,X_2,\ldots,X_r}(t_1, t_2, \ldots, t_r) \\ \qquad \text{if } t_1 = t_2 = \cdots = t_r > 0 \\ (2+(\theta_0+\theta_j)d)\bar{F}_{X_1,X_2,\ldots,X_r}(t_1, t_2, \ldots, t_r)\prod_{\substack{i=1 \\ i\neq j}}^{r}(1+\theta_i d) \\ \qquad \text{if } t_j = \max(t_1, t_2, \ldots, t_r), j = 1, \ldots, r \\ \bar{F}_{X_1,X_2,\ldots,X_r}(t_1, t_2, \ldots, t_r)(1+\theta_0 d)\prod_{m=1}^{w}(1+\theta_{i_m} d) \\ \qquad \text{if } t_{i_1}, t_{i_2}, \ldots, t_{i_w} < t_{j_1} = t_{j_2} = \ldots = t_{j_l} = t_0 \end{cases} \tag{7.1}$$

7.1 EM Algorithm for Multivariate MOMSE

For the construction of the EM algorithm, we need to derive the conditional expectations for all cases given in formula (7.1). Let $\mathbf{T}$ and θ denote the observed data and vector of the unknown parameters respectively. Consider the first case where all t_i's are equal, i.e. $t_1 = t_2 = \cdots = t_r$. Similar to bivariate case, it is clear that $E(V_0|\mathbf{T}, \theta) = t_1$, while for $i = 1, \ldots, r$, $E(V_i|\mathbf{T}, \theta) = \frac{1+t_1(1+\theta_i d)}{(1+\theta_i d)}$. Now, consider the case that there is a value t_j which is greater than all other t_i values,.i.e. $t_j = \max(t_1, t_2, \ldots, t_r)$, $j = 1, \ldots, r$ and $t_i \neq t_j$. Again similar to the bivariate case $E(V_0|\mathbf{T}, \theta) = \frac{t_j(1+\theta_0 d)}{2+(\theta_0+\theta_j)d} + \frac{(1+\theta_j d)[1+t_j(1+\theta_0 d)]}{[2+(\theta_0+\theta_j)d](1+\theta_0 d)}$ and $E(V_j|\mathbf{T}, \theta) = \frac{t_j(1+\theta_j d)}{2+(\theta_0+\theta_j)d} + \frac{(1+\theta_0 d)[1+t_j(1+\theta_j d)]}{[2+(\theta_0+\theta_j)d](1+\theta_j d)}$. Finally, there is a case which is the combination of the first and second cases. Some shocks could come before the common shock which affects all the live components at time t_0. More precisely, $t_{i_1}, t_{i_2}, \ldots, t_{i_w} < t_{j_1} = t_{j_2} = \ldots = t_{j_l} = t_0$ for some $w+l = r$. Similar to previous cases for the values of the left hand side of the inequality $E(V_{i_m}|\mathbf{T}, \theta) = t_{i_m}$ where $m = 1, \ldots, w$. For the values of the right hand side of the inequality $E(V_{j_p}|\mathbf{T}, \theta) = \frac{1+t_0(1+\theta_{j_p} d)}{(1+\theta_{j_p} d)}$ and $E(V_0|\mathbf{T}, \theta) = t_0$ where $p = 1, \ldots, l$. After constructing all conditional expectations values, the EM algorithm can be applied as follows.

E-step: After kth iteration, for $i = 1, \ldots, n$ and $j = 0, \ldots, r$ calculate the pseudo-values $a_{ji} = E(V_{ji}|\mathbf{T}_i, \theta^{(k)})$ by using the data and the vector $\theta^{(k)}$ of parameters $\theta_j^{(k)}$, $j = 0, \ldots, r$ which are obtained at the kth step.

M-step: Since the shock times T_j, $j = 0, 1, ..., r$ are exponentially distributed with parameter $(1+d\theta_j)$, the maximum likelihood estimates from the exponential distribution are obtained using the pseudo-values a_{ji}. Hence, the estimates are updated by

$$\theta_j^{(k+1)} = \frac{1}{d}\left(\frac{n}{\sum_{i=1}^{n} a_{ji}} - 1\right).$$

Iterating process should stop when some convergence criterion is satisfied, otherwise process returns to the E-step.

7.2 Data Analysis for Trivariate Case

To illustrate how the EM algorithm obtains the unknown parameters of the multivariate MOMSE, we use a generated data set from a trivariate MOMSE. First, we need to derive the density function as

$$f(t_1, t_2, t_3) = \begin{cases} (1+\theta_0 d)\bar{F}_{X_1,X_2,X_3}(t_1,t_2,t_3) \text{ if } t_1 = t_2 = t_3 > 0 \\ (2+(\theta_0+\theta_1)d)\bar{F}_{X_1,X_2,X_3}(t_1,t_2,t_3)(1+\theta_2 d)(1+\theta_3 d) \\ \qquad \text{if } t_1 = \max(t_1,t_2,t_3) \\ (2+(\theta_0+\theta_2)d)\bar{F}_{X_1,X_2,X_3}(t_1,t_2,t_3)(1+\theta_1 d)(1+\theta_3 d) \\ \qquad \text{if } t_2 = \max(t_1,t_2,t_3) \\ (2+(\theta_0+\theta_3)d)\bar{F}_{X_1,X_2,X_3}(t_1,t_2,t_3)(1+\theta_1 d)(1+\theta_2 d) \\ \qquad \text{if } t_3 = \max(t_1,t_2,t_3) \\ (1+\theta_0 d)(1+\theta_1 d)\bar{F}_{X_1,X_2,\ldots,X_s}(t_1,t_2,t_3) \text{ if } t_1 < t_2 = t_3 \\ (1+\theta_0 d)(1+\theta_2 d)\bar{F}_{X_1,X_2,\ldots,X_s}(t_1,t_2,t_3) \text{ if } t_2 < t_1 = t_3 \\ (1+\theta_0 d)(1+\theta_3 d)\bar{F}_{X_1,X_2,\ldots,X_s}(t_1,t_2,t_3) \text{ if } t_3 < t_1 = t_2 \end{cases} \tag{7.2}$$

then using formula (7.2) one can construct the conditional expectations as Table 7.1 and Table 7.2.

Case	$E(V_0\|\mathbf{T},\theta)$	$E(V_1\|\mathbf{T},\theta)$
$t_1 = t_2 = t_3$	t_1	$\frac{1+t_1(1+\theta_1 d)}{(1+\theta_1 d)}$
$t_1 = \max(t_1,t_2,t_3)$	$\frac{t_1(1+\theta_0 d)^2+(1+\theta_1 d)[1+t_1(1+\theta_0 d)]}{[2+(\theta_0+\theta_1)d](1+\theta_0 d)}$	$\frac{t_1(1+\theta_1 d)^2+(1+\theta_0 d)[1+t_1(1+\theta_1 d)]}{[2+(\theta_0+\theta_1)d](1+\theta_1 d)}$
$t_2 = \max(t_1,t_2,t_3)$	$\frac{t_2(1+\theta_0 d)^2+(1+\theta_2 d)[1+t_2(1+\theta_0 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_0 d)}$	t_1
$t_3 = \max(t_1,t_2,t_3)$	$\frac{t_3(1+\theta_0 d)^2+(1+\theta_3 d)[1+t_3(1+\theta_0 d)]}{[2+(\theta_0+\theta_3)d](1+\theta_0 d)}$	t_1
$t_1 < t_2 = t_3$	t_2	t_1
$t_2 < t_1 = t_3$	t_1	$\frac{1+t_1(1+\theta_1 d)}{(1+\theta_1 d)}$
$t_3 < t_1 = t_2$	t_1	$\frac{1+t_1(1+\theta_1 d)}{(1+\theta_1 d)}$

Table 7.1: The conditional expectations of the random variables V_0 and V_1 of the trivariate MOMSE model.

Case	$E(V_2\|\mathbf{T},\theta)$	$E(V_3\|\mathbf{T},\theta)$
$t_1 = t_2 = t_3$	$\frac{1+t_1(1+\theta_2 d)}{(1+\theta_2 d)}$	$\frac{1+t_1(1+\theta_3 d)}{(1+\theta_3 d)}$
$t_1 = \max(t_1,t_2,t_3)$	t_2	t_3
$t_2 = \max(t_1,t_2,t_3)$	$\frac{t_2(1+\theta_2 d)^2+(1+\theta_0 d)[1+t_2(1+\theta_2 d)]}{[2+(\theta_0+\theta_2)d](1+\theta_2 d)}$	t_3
$t_3 = \max(t_1,t_2,t_3)$	t_2	$\frac{t_3(1+\theta_3 d)^2+(1+\theta_0 d)[1+t_3(1+\theta_3 d)]}{[2+(\theta_0+\theta_3)d](1+\theta_3 d)}$
$t_1 < t_2 = t_3$	$\frac{1+t_2(1+\theta_2 d)}{(1+\theta_2 d)}$	$\frac{1+t_3(1+\theta_3 d)}{(1+\theta_3 d)}$
$t_2 < t_1 = t_3$	t_2	$\frac{1+t_3(1+\theta_3 d)}{(1+\theta_3 d)}$
$t_3 < t_1 = t_2$	$\frac{1+t_2(1+\theta_2 d)}{(1+\theta_2 d)}$	t_3

Table 7.2: The conditional expectations of the random variables V_2 and V_3 of the trivariate MOMSE model.

We use a generated data set from a trivariate MOMSE model (Table 7.3). In this data set generation, we use threshold (d) and the paramters vector $(\theta_0, \theta_1, \theta_2, \theta_3)$ as 5 and (0.35,0.55,0.45,0.5), respectively.

We looked the relative change of the loglikelihood value and algorithm terminated when this relative change is smaller than 10^{-12}. In this example, if we use (1,1,1,1) as the initial values $((\theta_0^{(1)}, \theta_1^{(1)}, \theta_2^{(1)}, \theta_3^{(1)}))$, EM algorithm converged after 36 iterations, and similarly if we use $\theta_0^{(1)} = 0.3, \theta_1^{(1)} = 0.3, \theta_2^{(1)} = 0.3$ and $\theta_3^{(1)} = 0.3$, algorithm converged after 33 iterations. We obtained following estimates of $(\theta_0, \theta_1, \theta_2, \theta_3)$ as (0.2996, 0.2358, 0.3911, 0.3197) and the corresponding loglikelihood is 6.3075 . We have also computed the 90% confidence intervals of

t1	t2	t3	t1	t2	t3
0.069659	0.044139	0.031041	0.226058	0.226058	0.17106
0.16356	0.16356	0.068778	0.253304	0.237616	0.253304
0.465493	0.040193	0.024975	0.25822	0.219427	0.290669
0.077449	0.077449	0.077449	0.410006	0.195078	0.410006
0.05118	0.05118	0.05118	0.268739	0.09944	0.268739
0.200402	0.40109	0.002241	0.247194	0.104735	0.215409
0.036085	0.036085	0.036085	0.223542	0.01098	0.381222
0.139267	0.062295	0.500786	0.210235	0.210235	0.15786
0.033714	0.033714	0.033714	0.174155	0.174155	0.174155
0.169444	0.169444	0.169444	0.199255	0.199255	0.141193
0.302772	0.324536	0.094643	0.041135	0.041135	0.041135
0.023782	0.311853	0.493157	0.617889	0.457925	0.172914
0.274754	0.022399	0.340186	0.003366	0.110081	0.466479
0.19749	0.065659	0.09662	0.379499	0.011359	0.173795
0.06028	0.496606	0.496606	0.093903	1.02822	0.345047

Table 7.3: Simulated data from trivariate MOMSE with parameters vector (0.35, 0.55, 0.45, 0.5) and threshold 5.

θ_0, θ_1,θ_2 and θ_3 as follows (0.2732, 0.7255), (0.1224, 0.6637), (0.3581, 0.9453) and (0.2561, 0.8097), respectively.

Chapter 8

Conclusion

There is a need to consider shock magnitudes in many practical applications of Marshall-Olkin shock models . For example, in financial applications, if one deals with shocks affecting the financial markets and causing financial crisis, some of the companies can survive if the magnitudes of shocks do not exceed a predefined threshold. This threshold may depend on the financial resistance of the company, flexibility and the capability to survive in extremal situations. Similar necessity can be encountered in investigations of earthquakes. The shocks with small seismic intensity are not catastrophic. There is also a need to consider shocks with magnitudes in biological and medical applications. For instance, in the case of usage of chemical preparations under needed dosage the treatment may not be effective. Hence, there is a need to consider a new Marshall-Olkin shock model where the magnitudes of the shocks are taken into account. In this thesis, under some restrictive assumptions on independence and distributions of considered random variables, we introduce a shock model where the shocks coming from different sources can destroy the components of the system if the magnitudes of the shocks exceed predefined threshold. The corresponding bivariate survival functions are presented and their stochastic properties are investigated. Some special examples with well known bivariate underlying distributions of shock times and magnitudes are presented. In addition, we study statistical inference for proposed model based on the maximum likelihood. Since MOMSE has both singular and absolutely continuous part, statistical inference is not an easy task because of the complex structure of its density function similar to the classical Marshall-Olkin distribution. Another contribution of the present dissertation is to overcome this drawback and find the maximum likelihood estimator of the proposed model by using the EM algorithm. In order to implement the EM algorithm, the problem of the finding maximum likelihood estimator is treated as a missing value problem. The EM algorithm is applied to some data sets which involve both bivariate and multivariate extensions of MOMSE. Also, asymptotic confidence intervals of the unknown parameters of both bivariate and multivariate distributions are constructed.

Appendix A

Observed Fisher Information Matrix

In this chapter, we study how observed fisher information matrix can be constructed. Let the matrix **H** and the vector **G** denote the Hessian matrix and gradient vector of the log-likelihood function. We can obtain the observed fisher information matrix as $\mathbf{H} - \mathbf{G}\mathbf{G}^T$. As an example for bivariate MOMSE model, we can construct matrix **H** and the vector **G** as follows:

$$\mathbf{H} = \begin{pmatrix} -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0^2} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0\partial\theta_1} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0\partial\theta_2} \\ -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0\partial\theta_1} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_1^2} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\ -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0\partial\theta_2} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} & -\frac{\partial^2 L(\theta_0,\theta_1,\theta_2)}{\partial\theta_2^2} \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \frac{\partial L(\theta_0,\theta_1,\theta_2)}{\partial\theta_0} \\ \frac{\partial L(\theta_0,\theta_1,\theta_2)}{\partial\theta_1} \\ \frac{\partial L(\theta_0,\theta_1,\theta_2)}{\partial\theta_2} \end{pmatrix}$$

$$H_{11} = \frac{n_0 d^2}{(1+\hat{\theta}_0 d)^2} + \frac{n_1 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_2))^2} + \frac{n_2 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_1))^2}$$

$$H_{12} = H_{21} = \frac{n_2 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_1))^2}$$

$$H_{13} = H_{31} = \frac{n_1 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_2))^2}$$

$$H_{22} = \frac{n_1 d^2}{(1+\hat{\theta}_1 d)^2} + \frac{n_2 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_1))^2}$$

$$H_{23} = H_{32} = 0$$

$$H_{33} = \frac{n_1 d^2}{(2+d(\hat{\theta}_0+\hat{\theta}_2))^2} + \frac{n_2 d^2}{(1+\hat{\theta}_2 d)^2}$$

$$G_1 = \frac{n_1 d}{(2+d(\hat{\theta}_0+\hat{\theta}_2))} + \frac{n_2 d}{(2+d(\hat{\theta}_0+\hat{\theta}_1))} + \frac{n_0 d}{(1+\hat{\theta}_0 d)} - d\sum_{i=1}^{n} \max(t_{1i}, t_{2i})$$

$$G_2 = \frac{n_1 d}{(1+\hat{\theta}_1 d)} + \frac{n_2 d}{(2+d(\hat{\theta}_0+\hat{\theta}_1))} - d\sum_{i=1}^{n} t_{1i}$$

$$G_3 = \frac{n_1 d}{(2+d(\hat{\theta}_0+\hat{\theta}_2))} + \frac{n_2 d}{(1+\hat{\theta}_2 d)} - d\sum_{i=1}^{n} t_{2i}$$

For more information, we refer to [24].

BIBLIOGRAPHY

[1] M.M. Ali, N.N. Mikhail and M.S. Haq, A class of bivariate distributions including the bivariate logistic, *Journal of Multivariate Analysis* **8** (1978), 405-412.

[2] B. Arnold, Parameter estimation for a multivariate exponential distribution, *Journal of the American Statistical Association* **63** (1968), 848-852.

[3] I. Bairamov and S. Kotz, Dependence structure and symmetry of Huang-Kotz FGM distributions and their extensions, *Metrika* **56** (2002), no. 1, 55-72.

[4] I. Bairamov, S. Kotz and M. Bekci, New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics, *Journal of Applied Statistics* **28** (2001), no. 5, 521-536.

[5] N. Balakrishnan and C.D. Lai, *Continuos Bivariate Distributions*, Springer, 2nd edn. Dordrecht, 2009.

[6] E.R. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing. New York, NY, USA: Holt, Rinehart & Winston*, 1975.

[7] B.M. Bemis, L. Bain and J.J. Higgins, Estimation and hypothesis testing for the parameters of a bivariate exponential distribution, *Journal of the American Statistical Association* **67** (1972), 927-929.

[8] G.K. Bhattacharyya and R.A. Johnson, Maximum likelihood estimation and hypothesis testing in the bivariate exponential model of Marshall and Olkin,

Technical Report No. 276, Department of Statistics, University of Wisconsin 1971.

[9] G.E. Dinse, Non-parametric estimation of partially incomplete time and types of failure data, *Biometrics* **38** (1982), 417-431.

[10] M.E. Ghitany, F.A. Al-Awadhi and L.A. Alkhalfan, Marshall-Olkin extended Lomax distribution and its application to censored data, *Communication in Statistics-Theory and Methods* **36** (2007), 1855-1866.

[11] M.E. Ghitany, E.K. Al-Hussaini and R.A. Al-Jarallah, Marshall-Olkin extended Weibull distribution and its application to censored data, *Journal of Applied Statistics* **32** (2005), 1025-1034.

[12] E.J. Gumbel, Bivariate exponential distributions, *Journal of the American Statistical Association* **55** (1960), 698-707.

[13] R.D. Gupta and D. Kundu, Generalized exponential distribution, *Australian&New Zealand Journal of Statistics* **41** (1999), 173-188.

[14] D.D. Hanagal, Note on estimation of reliability under bivariate Pareto stress-strength model, *Statistical Papers* **38** (1997), 453-459.

[15] K. Jayakumar and M. Thomas, On a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution, *Statistical Papers* **49** (2007), 421-439.

[16] K.K. Jose, M.M. Ristic and A. Joseph, Marshall-Olkin bivariate Weibull distributions and processes, *Statistical Papers* **52** (2011), 789-798.

[17] D. Karlis, ML estimation for multivariate shock models via EM algorithm, *Annals of the Institute of Statistical Mathematics* **55** (2003), 817-830.

[18] D. Kundu and A.K. Dey, Estimating the parameters of the Marshall-Olkin bivariate Weibull distribution by EM algorithm, *Computational Statistics and Data Analysis* **53** (2009), 956-965.

[19] D. Kundu and R.D. Gupta, Bivariate generalized exponential distribution, *Journal of Multivariate Analysis* **100** (2009), 581-593.

[20] D. Kundu and R.D. Gupta, Modified Sarhan-Balakrishnan singular bivariate distribution, *Journal of Statistical Planning and Inference* **140** (2010), 526-538.

[21] C.D. Lai, Morgenstern's bivariate distribution and its application to point process, *Journal of Mathematical Analysis and Applicaitons* **65** (1978), 247-256.

[22] C.D. Lai, M. Xie and I.G. Bairamov, Dependence and ageing properties of bivariate Lomax distribution, *In System and Bayesian Reliability: Essays in Honor of Prof. R.E. Barlow on His 70th Birthday, Y. Hayakawa, T. Irony, and M. Xie (eds.)* (2001) 243-256, World Scientific, Singapore.

[23] X.H. Li and F. Pellerey, Generalized Marshall-Olkin distributions and related bivariate aging properties, *Journal of Multivariate Analysis* **102** (2011), 1399-1409.

[24] T.A. Louis, Finding the observed information matrix when using the EM algorithm, *Journal of the Royal Statistical Society Series B* **44** (1982), 226-233.

[25] J.F. Mai, Extendibility of Marshall-Olkin distributions via Lévy subordinators and an application to portfolio credit risk. *Short summary of dissertation at the Technische Universität München* 2010.

[26] A.W. Marshall and I. Olkin, A multivariate exponential distribution, *Journal of the American Statistical Association* **62** (1967), 30-44.

[27] A.W. Marshall and I. Olkin, A new method of adding a parameter to a family of distributions with application to exponential and Weibull families, *Biometrika* **84** (1997), 641-652.

[28] G. McLachlan and N. Krishnan, *The EM Algorithm and Extensions*, Wiley, Chichester 1997.

[29] S. Nadarajah and S. Kotz, Performance measures for some bivariate Pareto distributions, *International Journal of General Systems* **35** (2006), 387-393.

[30] J. Navarro, J.M. Ruiz and C.J. Sandoval, Properties of systems with two exchangeable Pareto components, *Statistical Papers* **49** (2008), 177-190.

[31] T.K. Nayak, Multivariate Lomax Distribution: Properties and Usefulness in Reliability Theory, *Journal of Applied Probability* **24** (1987), 170-177.

[32] R.B. Nelsen, *An Introduction to Copulas*, Springer, New York 2006.

[33] M. Ozkut and I. Bayramoglu, On Marshall-Olkin type distribution with effect of shock magnitude, *Journal of Computational and Applied Mathematics***271** (2014), 150-162.

[34] E.N. Papadakis and E.G. Tsionas, Multivariate Pareto distributions: inference and financial applications, *Communications in Statistics-Theory and Methods* **39** (2010), 1013-1025.

[35] E.A. Pena and A.K. Gupta, Bayes estimation for the Marshall Olkin exponential distribution, *Journal of Royal Statistical Society Series B* **52** (1990), 379-389.

[36] F. Proschan and P. Sullo, Estimating the parameters of a multivariate exponential distribution, *Journal of the American Statistical Association* **71** (1976), 465-472.

[37] K.W. Ryu, An Extention of Marshall and Olkin Bivariate Exponential Distribution, *Journal of the American Statistical Association* **88** (1993), 1458-1465.

[38] P.G. Sankaran and U.N. Nair, A bivariate Pareto model and its applications to reliability, *Naval Research Logistics* **40** (1993), 1013-1020.

[39] A. Sarhan and N. Balakrishnan, A new class of bivariate distributions and its mixture, *Journal of the Multivariate Analysis* **98** (2007), 1508-1527.

[40] M. Shaked and J.G. Shanthikumar, *Stochastic Orders*, Springer, New York 2007.

[41] A. Thomas and K.K. Jose, Bivariate semi-Pareto minificaiton processes, *Metrika* **59** (2004), 305-313.

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