RELIABILITY OF SYSTEMS WITH A COLD STANDBY COMPONENT

CEKİ FRANKO

MAY 2016

RELIABILITY OF SYSTEMS WITH A COLD STANDBY COMPONENT

A DISSERTATION SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL

AND APPLIED SCIENCES OF

IZMIR UNIVERSITY OF ECONOMICS

BY

CEKİ FRANKO

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MAY 2016

Ph.D. DISSERTATION EXAMINATION RESULT FORM

Approval of the Graduate School of Natural and Applied Sciences

____

Prof. Dr. İsmihan Bayramoğlu
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

____

Assoc. Prof. Dr. G. Yazgı Tütüncü
Head of Department

We have read the dissertation entitled **"RELIABILITY OF SYSTEMS WITH A COLD STANDBY COMPONENT"** completed by **CEKİ FRANKO** under supervision of **Assoc. Prof. Dr. G. Yazgı Tütüncü** and we certify that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

____

Assoc. Prof. Dr. G. Yazgı Tütüncü
Supervisor

Examining Committee Members **Date:**____________________

Assoc. Prof. Dr. G. Yazgı Tütüncü
Dept. of Mathematics, İzmir University of Economics____________________

Prof. Dr. İsmihan Bayramoğlu
Dept. of Mathematics, İzmir University of Economics____________________

Prof. Dr. Serkan Eryılmaz
Dept. of Industrial Engineering, Atılım University____________________

Prof. Dr. Murat Fadıloğlu
Dept. of Industrial Engineering, Yaşar University____________________

Assoc. Prof. Dr. Selma Gürler
Dept. of Statistics Dokuz Eylül University ____________________

ABSTRACT

RELIABILITY OF SYSTEMS WITH A COLD STANDBY COMPONENT

CEKİ FRANKO
Ph.D. in Applied Mathematics and Statistics
Graduate School of Natural and Applied Sciences
Supervisor: Assoc. Prof. Dr. G. Yazgı Tütüncü
May 2016

In this thesis, the influence of a cold standby component to a coherent system and weighted k-out-of-n:G systems consisting of two different types of component are studied. A general method for computing the system reliability of coherent systems having a cold standby component is proposed. Moreover system reliability calculations of weighted k-out-of-n:G systems consisting of two different types of component and a cold standby is presented. Reliability and mean time to failure of different structured systems have been computed. Numerical examples and different optimal system design problems are solved to show the applicability of the proposed method in real life.

Keywords: reliability; cold standby component; coherent system; weighted-k-out-of-n:G system.

ÖZ

SOĞUK YEDEK BİLEŞENE SAHİP SİSTEMLERİN GÜVENİRLİLİĞİ

CEKİ FRANKO
Uygulamalı Matematik ve İstatistik, Doktora
Fen Bilimleri Enstitüsü
Tez Danışmanı: Doç. Dr. G. Yazgı Tütüncü
Mayıs 2016

Bu tezde, soğuk yedek bileşenin, uyumlu sistemler ve iki farklı tip bileşenden oluşan ağırlıklı n'den-k'lı sistemler üzerine etkisi çalışıldı. Uyumlu sistemlerde güvenirliliğin hesaplanması için genel bir yöntem önerildi. Buna ek olarak, iki farklı tip komponentten oluşan ve bir soğuk yedek komponente sahip ağırlıklı n'den-k'lı sistemlerde güvenilirlik hesapları yapıldı. Farklı yapıya sahip sistemler için güvenilirlik ve ortalama yaşam süresi hesaplandı. Önerilen yöntemlerin gerçek hayatta uygulanabilirliğini göstermek için sayısal örnekler ve farklı optimal sistem tasarım problemleri çözüldü.

Anahtar Kelimeler: güvenilirlik; soğuk yedek bileşen; uyumlu sistem; ağırlıklı n'den-k'lı sistem.

ACKNOWLEDGEMENT

I would like to start by giving my gratitude to my advisor Assoc. Dr. Gözde Yazgı Tütüncü. She was the one who encouraged me to start my academic career. She has been always beside me, helping and guiding, from the moment I graduated from university. She has always respected my ideas, therefore she set me free for what I wish to study while helping me throughout all the process. Her insightful remarks and suggestions have helped me to think through and overcome my problems. I could not finish my thesis without her. Moreover, I want to thank her for the days she spent on reading and commenting on the thesis. I would like to thank Prof. Dr. İsmihan Bayramoğlu and Prof. Dr. Serkan Eryılmaz for dedicating their valuable time to guide me throughout my study. Their ideas and studies have always been a great example for me. I would like to give my special thanks to İzmir University of Economics for giving me chances to use all the resources I need to complete my thesis. I want to sincerely thank to my colleagues and my best friends Cihangir Kan, Cemal Murat Özkut and Femin Yalçın for their support and help whenever I needed. My heartfelt gratitude is to my lovely wife Esra Franko, she has been beside me from the beginning of the process listening to my complaints giving me advice and most importantly believing in me whenever I doubt myself. I thank to my mother Ester Franko, my father Leon Franko and my brother Moris Franko for their unconditional love and support. This thesis is also their achievement since I owe them every thing. Last, I would like to thank The Scientific and Technological Research Council of Turkey (TÜBİTAK) for the scholarship they gave me to help me to complete the PhD programme. I hope my thesis will help and guide many researchers.

To my family...

TABLE OF CONTENTS

LIST OF TABLES

Chapter 1

Introduction

Survival or lifetime analysis is a branch of statistics which mainly interested in the expected lifetime of living organism or non living systems. In engineering when the main focus of the analysis is lifetime of a system or components it is often called as reliability analysis. Reliability can be defined as the ability to perform a task satisfactorily in a specific time or time interval. At the same time it is the probability that the task will be performed in a given time. Reliability analysis has been widely performed in different areas of engineering for a variety of system structures. Coherent and weighted k-out-of-n systems are of special importance in engineering due to their applicability in real life. The main goal of analyzing coherent and weighted k-out-of-n systems is to reduce the cost of system setup as well as to increase system reliability. There are different ways to increase system reliability. One of them is to improve the reliability of each component in the system and the other one is to change the system design which may lead to increase in the system reliability. Both of these methods are effective in improving the system reliability however they have may have high application costs or impossible to implement. In order to improve each component's reliability they should be renewed or repaired which is the costly part. Moreover changing the system design may not be possible for many cases because of the area limitations and several system regulations. In this case the only solution that makes sense is to use redundant components. There are two different types of redundancy in system design. One of them is to equip a functioning system with more components than needed to prevent the system from failing because of the unexpected failures of the components. The main downside of this precaution is again the rise in the cost of the system setup. Another disadvantage is that the efficiency of each individual component in the system will decrease. Because of these reasons standby redundancy can be effectively used to increase reliability without increasing the system cost and decreasing component performance. Standby components do not become active until the failure of a component will lead to system failure hence they do not affect the performance of the active components. There are three different type of standby components which are hot,warm and cold standby component and they will be discussed in more details in Chapter 5. In this thesis the effect of a single cold standby component, to reliability of coherent systems with identical components and to weighted k-out-of-n systems containing two different type of components, is investigated. In the literature reliability and reliability properties of systems with a cold standby component is investigated for only k-out-of-n systems having independent and identical components. The contribution of this thesis is to propose the generalization of the reliability analysis for all types of coherent systems with a single cold standby component. Thus a method has been proposed to calculate the reliability for all coherent systems having independent and identical components. Moreover this is the first study in the literature in which optimum system design has been made by finding the reliability of systems having independent but non identical components and a cold standby component. The rest of the thesis is organized as follows. Order statistics and their properties are presented in Chapter 2. The definition of reliability, coherent systems and their connection with system signature is given in Chapter 3. Weighted k-out-of-n systems and their working principle is pointed out in Chapter 4. Finally in Chapter 5 standby systems and main findings of this thesis which are reliability analysis of coherent systems with a cold standby component and weighted k-out-of-n systems containing two different types of components and a cold standby component are presented.

Chapter 2

Order Statistics

The random variables which can be interpreted as results of an experiment measuring values of a certain random variable arranged in order of magnitude, are called order statistics. Order statistics have been extensively used in statistical inference, reliability theory and statistical process control. Order statistics have wide applications in many areas where the use of an ordered sample is important. Order statistics are sufficient statistics, hence they contain all the information about the sample. Moreover since most statistics derived from order statistics have the distribution-free property, it is widely used in non-parametric statistical methods. Another important aspect is that order statistics can be used in several applications of reliability theory. For example, lifetime of a component or the whole system can be represented by order statistics. Therefore the concept of order statistics takes a major place in life-time analysis.

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with cumulative distribution function (cdf) $F(x) = P(X \leqslant x)$. Suppose that the elements of this sample are arranged in order of magnitude and $X_{1:n}$ denotes the smallest; $X_{2:n}$ denotes the second smallest; etc. and $X_{n:n}$ denotes the largest of the set $X_1, X_2, \ldots, X_n$. Then $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denotes the original random sample arranged in increasing order of magnitude, and these are called the order statistics of the sample $X_1, X_2, \ldots, X_n$. We call $X_{i:n}$ for $1 \leq \ldots \leq n$ the ith order statistic. The subject of order statistics deals with the distributional properties of $X_{i:n}$ itself, and some functions of the subset of the n order statistics and their applications is well known from classical statistical theory, that the natural estimate of an unknown distribution function is the empirical distribution function, which is a function of order statistics. Therefore, many important statistics in estimation theory and hypothesis testing appear to be an integral functional of the empirical distribution function, and can be expressed in terms of order statistics. Order statistics do not change their order under probability integral transformation, namely if $U_{i:n} = F(X_{i:n})$ $i = 1, 2, ..., n$ then $U_1 \leq U_2 \leq \ldots \leq U_n$ Due to distribution free properties, they are widely used in nonparametric interval estimation and hypothesis testing. Order statistics and their properties have been extensively studied since early part of the last century, and recent years have seen a particularly rapid growth of studies. For more detailed information one can see the books [1], [2], [3], [13], [14].

2.1 Distribution of Order Statistics From I.I.D Random Variables

Let $X_1, X_2, ..., X_n$ be a sample of size n from the population with c.d.f. F. The order statistics obtained by arranging the random sample $X_1, X_2, ..., X_n$ in increasing order of magnitude are represented either

$$X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$$

or

$$X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}.$$

The distribution function of r^{th} order statistic is

$$F_r(x) = P\left\{X_{r:n} \leq x\right\} = \sum_{i=r}^{n} \binom{n}{i} F^i(x) \left(1 - F(x)\right)^{n-i}. \tag{2.1}$$

If F is absolutely continuous with pdf f, then (2.1) can be written also as follows

$$\begin{aligned} F_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du \\ &= \frac{1}{B(r, n-r+1)} I_{F(x)}(r, n-r+1), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1} dt, \\ I_p(a,b) &= \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt, \end{aligned}$$

and $\frac{1}{B(r,n-r+1)} = \frac{n!}{(r-1)!(n-r)!}$.
Formula (2.1) yields true for discrete, absolutely continuous and continuous except countable number of points (having countable number points of discontinuity). Formula (2.2) is true only for absolutely continuous distribution. Given the realizations of the n order statistics to be $X_{1:n} < X_{2:n} < ... < X_{n:n}$, the original random variables X_i are restrained to take on the values $X_{i:n}$ $(i = 1, 2, .., n)$ which by symmetry assigns equal probability to each of the $n!$ permutations of (1,2,...,n). Therefore, the joint density function of all n order statistics is

$$f_{1,2,...,n}(x_1, x_2, ..., x_n) = \begin{cases} n! \prod_{i=1}^n f(x_i) & \text{if } \ x_1 < x_2 < ... < x_n \\ \\ \quad 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The joint pdf of two or more order statistics can be obtained by integrating from (2.3) as well as by using continuous total probability formula.
The joint pdf of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is

$$f_{r,s}(x,y) = \begin{cases} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F^{r-1}(x) & \\ \times(F(y)-F(x))^{s-r-1}(1-F(y))f(x)f(y) & \text{if } x<y \\ & \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

The joint pdf of order statistics $X_{r_1:n}, X_{r_2:n}, ..., X_{r_k:n}$ is

$$f_{r_1,r_2,...,r_k}(x_1,x_2,...,x_n)$$

$$= \begin{cases} \frac{n!}{r_1!r_2!...r_k!} F^{r_1}(x_1)\left[F(x_2)-F(x_1)\right]^{r_2-r_{1-1}} & \\ \times\left[F(x_3)-F(x_2)\right]^{r_3-r_2-1} ... \left[1-F(x_k)\right]^{n-r_k} & \\ \times f(x_1)...f(x_k) & \text{if } x_1<x_2<...<x_n \\ & \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

If F is a discrete distribution function, then the joint c.d.f. of $X_{r:n}$ and $X_{s:n}$ is

$$F_{r,s}(x,y) = \begin{cases} \sum_{i=r}^{n} \sum_{\max(0,s-i)}^{n-i} \frac{n!}{i!j!(n-i-j)!} F^{i}(x) & \\ \times[F(y)-F(x)]^{j}[1-F(y)]^{n-i-j} & \text{if } x<y \\ & \\ F_s(y) & \text{otherwise} \end{cases} \tag{2.6}$$

and the pmf of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s}(x,y) = F_{r,s}(x,y) - F_{r,s}(x-1,y) - F_{r,s}(x,y-1) + F_{r,s}(x-1,y-1), \ x \leq y. \tag{2.7}$$

Definition. Let $X_{1:n}, ..., X_{n:n}$ be order statistics based on the sample X_1, $X_2, ...,$

X_n. Then

$$Y_1 = X_{1:n}, Y_2 = X_{2:n} - X_{1:n},\ , ..., Y_n = \ X_{n:n} - X_{n-1:n}$$

are called spacings.

Suppose that $X_{1:n}, ..., X_{n:n}$ are order statistics based on the sample X_1, $X_2, ...,$ X_n with d.f.$F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$. Then the spacings

$$Y_1 = X_{1:n}, Y_2 = X_{2:n} - X_{1:n},\ , ..., Y_n = \ X_{n:n} - X_{n-1:n}$$

are independent, furthermore the random variables

$$\begin{aligned} &Z_1 = n\lambda X_{1:n}, Z_2 = (n-1)\lambda(X_{2:n} - X_{1:n})\ , ..., \\ &Z_r = \ (n-r+1)\lambda(X_{r:n} - X_{r-1:n}), ..., Z_n = \ \lambda(X_{n:n} - X_{n-1:n}) \end{aligned}$$

are i.i.d. with the common c.d.f $F(x) = 1 - \exp(-x)$, $x \geq 0$.
If n units are placed under test of solidity and $X_1, X_2, ..., X_n$ represent the life lengths these units, then the lengths of time intervals $X_{r:n} - X_{r-1:n}$,$r = 1, 2, ..., n$ between two failures are independent and identically distributed random variables when the common distribution of X_1, $X_2, ...,$ X_n is exponential.

Theorem 2.1 *Let* $X_{1:n}, X_{2:n}, ..., X_{n:n}$ *be order statistics of the sample* $X_1, X_2, ..., X_n$ *with absolutely continuous c.d.f* F *and p.d.f* f*. Then*

$$\{(X_{r+1:n}, X_{r+1:n}, ..., X_{n:n}) \mid X_{r:n} = x\} \overset{d}{=} (Y_{1:n-r}, Y_{2:n-r}, ..., Y_{n-r:n-r}),$$

where $Y_{1:n-r}, Y_{2:n-r}, ..., Y_{n-r:n-r}$ *are order statistics from the sample* $Y_1, Y_2, ..., Y_{n-r}$ *size* $n-r$*, and*

$$Y \overset{d}{=} X \mid X > x$$

$$\textit{the p.d.f of } Y \textit{ is } f_Y(u) = \begin{cases} 0 & \textit{if } u \leq x \\ \frac{f(u)}{1-F(x)} & \textit{otherwise} \end{cases}$$

For more detailed information on ordered statisitcs one can see the books [1], [13] and [14].

2.2 Distribution Of Order Statistics From Mixed Random Variables

Let $T_1^{(1)}, \ldots, T_{n_1}^{(1)}$ be independent and identically distributed (i.i.d) random variables with cumulative distribution function (c.d.f) $F(t)$. Furthermore let $T_1^{(2)}, \ldots, T_{n_2}^{(2)}$ be i.i.d random variables with cdf $G(t)$. Assume that these two collections of random variables are independent of each other and they represent lifetimes of two different types of components. Let us denote by $\{T_1, \ldots, T_n\}$ the $n = n_1 + n_2$ lifetimes of components in a system combined from $n_1 T^{(1)}$s and $n_2 T^{(2)}$s. Denote by $T_{r:n}$ $r = 1, \ldots, n$ the rth order statistics of the combined sample. Bairamov and Parsi [6] derived the distribution of $T_{r:n}$ as follows

$$\begin{aligned} H_{(r)}(x) =& P(T_{r:n} \leq x) \\ =& \sum_{i=r}^{n} \sum_{j=\max(0,n_1+i-n)}^{\min(i,n_1)} \binom{n_1}{j} \binom{n_2}{i-j} F(x)^j (1-F(x))^{n_1-j} G(x)^{i-j} (1-G(x))^{n_2-i+j}. \end{aligned}$$

The p.d.f of $T_{r:n}$ is given by

$$\begin{aligned} h_{(r)}(x) = & \sum_{i=\max(0,n_1+r-1-n)}^{\min(r-1,n_1-1)} \binom{n_1}{1} \binom{n_1-1}{i} \binom{n_2}{r-1-i} F(x)^i (1-F(x))^{n_1-1-i} \\ & \times G(x)^{r-1-i} (1-G(x))^{n_2-r+i+1} f(x) + \\ & \sum_{i=\max(0,n_1+r-n)}^{\min(r-1,n_1)} \binom{n_2}{1} \binom{n_1}{i} \binom{n_2-1}{r-1-i} F(x)^i (1-F(x))^{n_1-i} \\ & \times G(x)^{r-1-i} (1-G(x))^{n_2-r+i} g(x). \end{aligned}$$

2.3 Residual Lifetimes of Remaining Components From I.I.D Random Variables

Bairamov and Arnold [5] defined the residual lifelengths of the remaining functioning components following the kth failure in the system. In addition they discuss the joint distribution of these exchangeable random variables and identify the sufficient conditions that guarantee independence of the residual lifelengths. Consider an $(n-k+1)$-out-of-n system which will function successfully until k of the components have failed. Consequently, if we denote the lifetimes of the individual components by $T_1, T_2, \ldots T_n$ then the lifetime of the $n-k+1$ out of n system will be represented by the kth order statistic $T_{k:n}$ After an $n-k+1$ out of n system fails (i.e. after the kth failure has been observed), it is often reasonable to stop the system and rescue the functioning components to use in other systems. On the other hand if the system must function without a break the common procedure is to use standby components to prevent the failure of the system hence the system will continue to function with the remaining components together with the standby components. In the modeling of failure times for components of the system with i.i.d components, we assume that the failure of one component does not affect the functioning of the remaining ones. The classical theory of $n-k+1$ out of n systems assumes that the n lifetimes $T_1, T_2, \ldots, T_n$ of the components of the system are independent and identically distributed (i.i.d.) with common absolutely continuous distribution function F and corresponding density f. With this setup, the time of the first failure will be the first order statistic $T_{1:n}$ and the subsequent times between failures can be identified with the spacings $T_{i:n}-T_{i-1:n}, i = 2, 3, \ldots, n$.Note that even under the classical assumption that the original lifetimes were i.i.d., it will turn out that the residual lifetimes of the unfailed components will be exchangeable, but typically not independent. They will be conditionally independent given the time of the k'th failure, but we are not assuming that the time of that failure is known, or equivalently we do not know the time at which the system was switched on, we just know it has stopped functioning because k failures have occurred. Note that if we put the rescued components into a new system, we will need to consider that the lifetime of the

components in this new system are identically distributed but this time they are dependent.
For any $k \in \{1, 2, ..., n\}$ we will use the notation $T_1^{(k)}$, $T_2^{(k)}$, ..., $T_{n-k}^{(k)}$ to denote the residual lifetimes of the $n-k$ components still functioning at the time of the kth failure. For each k, we may define

$$T_{1:n-k}^{(k)} = \min\{T_1^{(k)}, T_2^{(k)}, \ldots, T_{n-k}^{(k)}\}.$$

Upon reflection, it is evident that these $T_{1:n-k}^{(k)}$'s simply represent an alternative description of the spacings of the order statistics of the original sample $T_1, T_2, \ldots, T_n$. Thus

$$T_{k+1:n} - T_{k:n} = T_{1:n-k}^{(k)}$$

and

$$T_{k-1:n} = T_{1:n} + T_{1:n-1}^{(1)} + T_{1:n-2}^{(2)} + \ldots + T_{1:n-k}^{(k)}.$$

If we are given $T_{k:n} = x$, then the conditional distribution of the subsequent order statistics $T_{k+1:n}, \ldots, T_{n:n}$ is the same as the distribution of order statistics of a sample of size $n-k$ from the distribution F truncated below at x. If we denote by $Y_i^{(k)}, i = 1, 2, \ldots, n-k$ the randomly ordered values of $T_{k+1:n}, \ldots, T_{n:n}$, then given $T_{k:n} = x$, these $Y_i^{(k)}$'s will be i.i.d. with common survival function $\bar{F}(x+y)/\bar{F}(x)$. The residual lifetimes after k failures, $T_1^{(k)}, \ldots, T_{n-k}^{(k)}$, may be represented as

$$T_i^{(k)} = Y_i^{(k)} - T_{k:n}, \quad i = 1, 2, \ldots, n-k$$

2.4 Residual Lifetimes of Remaining Components From Mixed Random Variables

In this section we will define the residual lifetimes of the remaining components from two different independent sets combined together. Let $T_1^{(1)}, \ldots, T_{n_1}^{(1)}$ be independent and identically distributed (i.i.d) random variables with cumulative distribution function (c.d.f) $F(t)$. Furthermore let $T_1^{(2)}, \ldots, T_{n_2}^{(2)}$ be i.i.d random variables with cdf $G(t)$. Assume that these two collections of random variables are independent of each other and they represent lifetimes of two different types of components. Let us denote by $\{T_1, \ldots, T_n\}$ the $n = n_1 + n_2$ lifetimes of components in a system combined from n_1 of $T^{(1)}$s and n_2 of $T^{(2)}$s. Denote by $T_{r:n}$ $r = 1, \ldots, n$ the rth order statistics of the combined sample. Let M be a random variable showing the number of failed components of type 1 at the time of rth failure. If we are given $T_{r:n} = x$ and $M = m$ then the conditional distribution of the subsequent order statistics from the first sample $T_{m+1:n_1}^{(1)}, \ldots, T_{n_1:n_1}^{(1)}$ is the same as the distribution of order statistics of a sample of size $n_1 - m$ from the distribution F truncated below at x. Similarly the conditional distribution of the subsequent order statistics from the second sample $T_{r-m+1:n_2}^{(2)}, \ldots, T_{n_2:n_2}^{(2)}$ is the same as the distribution of order statistics of a sample of size $n_2 - r + m$ from the distribution G truncated below at x. If we denote respectively the remaining lifetimes of the remaining components from the first sample and second sample as $T_i^{(1),r}$ $i = 1, \ldots n_1 - m$ and $T_j^{(2),r}$ $j = 1, \ldots, n_2 - r + m$ and the randomly ordered values as $T_{m+1:n_1}^{(1)}, \ldots, T_{n_1:n_1}^{(1)}$ and $T_{r-m+1:n_2}^{(2)}, \ldots, T_{n_2:n_2}^{(2)}$ then given $T_{r:n} = x$ and $M = m$, $T_i^{(1),r}$ will be i.i.d with common survival function $\frac{\overline{F}(x+t)}{\overline{F}(x)}$ and $T_j^{(2),r}$ will again be i.i.d with common survival function $\frac{\overline{G}(x+t)}{\overline{G}(x)}$.

Chapter 3

Reliability and System Signature

Reliability is defined as the probability that the system will perform satisfactorily for at least a given period of time under stated conditions. One can define the reliability of a single component, as well as, a system which consists of multiple components.

Reliability evaluation has been done in many areas of engineering such as chemical engineering, electrical engineering, computer engineering and mechanics. It is generally used for maintenance and controlling of engineering systems so it is a vital tool for system engineers. In order to evaluate the reliability of a system correctly one should determine the structure of the system that is, to specify the rules which keep the system functioning and the relationship between the system components. Most of the works that have been done on system reliability have focused on binary system modeling.

In a binary system modeling, the system and all its components may either work or fail. Hence the state of each component and the system itself can be defined as a discrete random variable with two possible outcomes. For series systems all components must function for the system to operate. However for other systems, it may be sufficient for some components to function. These relationship between components and systems are investigated by coherent systems.

Let a system consist of n components. If x_i denotes the state of the ith component in the system. Then

$$x_i = \begin{cases} 1 & \text{if } i\text{th component functions,} \\ 0 & \text{if } i\text{th component fails.} \end{cases} \tag{3.1}$$

for $i = 1, 2, ..., n$ Let ϕ denote the state of system, then it can be defined as

$$\phi(x_1, x_2, ..., x_n) = \begin{cases} 1 & \text{if system functions,} \\ 0 & \text{if system fails.} \end{cases} \tag{3.2}$$

The function $\phi(\vec{x})$, which is called the structure function of system, is a function of states of components.

In real life due to engineering problems some systems may have irrelevant components, in which functioning or failure of these components have no effect to the state of the system.

Definition. Let a system consist of n components. The component $i = 1, 2, \ldots, n$ is said to be irrelevant if and only if

$$\phi(1_i, \vec{x}) = \phi(0_i, \vec{x}) \text{ for all } (\cdot_i, \vec{x}) = (x_1, x_2, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_n).$$

If there exists at least one $\vec{x}$ satisfying $\phi(1_i, \vec{x}) = 1$ and $\phi(0_i, \vec{x}) = 0$ it can be said that component i is relevant.

Using the definition of the structure function we can define coherent systems. For more detailed information on coherent systems one can see [7] and [36].

Definition. A system of components is coherent, if its structure function is increasing and there is no irrelevant component in the system.

In other words a system is coherent if the following conditions are satisfied.

1. $\phi(\mathbf{0}) = 0$ that is system is failed when all components are failed.

2. $\phi(\mathbf{1}) = 1$ that is system is functioning when all components function.

3. $\mathbf{x} < \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y})$ that is improvement of any component does not decrease the performance of the system.

4. For every component i, there exists a component state vector such that the state of component i dictates the state of the system.

Similarly, the reliability of a coherent system consisting of n components can be defined as the probability that the system functions

$$R = P(\phi(\vec{x}) = 1).$$

Reliability of the ith component of this system is defined as the probability that ith component functions

$$P(x_i = 1) = p_i \quad \text{for } i = 1, 2, ..., n.$$

In the literature, different system structures have been defined and their reliability are studied stating different assumptions on components. Series and parallel models are the core of these structures. A series system with n components functions if all components function.On the other hand a parallel system of n components functions if at least one component function. A k-out-of-n:F system which is a generalization of series and parallel system, consists of n components, fails iff at least k of n components fail. Another important system structure that is widely used in the literature is consecutive k-out-of-n:F system. Consecutive systems have been used in modeling in various engineering areas such as telecommunication oil pipeline and vacuum systems in accelerators. It consists of n linearly ordered components such that the system fails iff at least k consecutive components fail. New findings on consecutive k-out-of-n systems can be seen in [15], [16], [17] [45], [61], [62]. An overview of these systems and their generalizations are presented in [19], [36].

In Table 3.1 one can see the structure functions of different coherent systems consisting of n components.

System	*Structure Function*
Series	$\phi(\vec{X}) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, ..., x_n)$
Parallel	$\phi(\vec{x}) = \prod_{i=1}^{n} x_i = \max(x_1, x_2, ..., x_n)$
k-out-of-n:F	$\phi(\vec{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i > n-k \\ 0, & \sum_{i=1}^{n} x_i \leq n-k \end{cases}$
Consecutive k-out-of-n:F	$\phi(\vec{x}) = \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} (1 - x_j))$

Table 3.1 Structure functions of different coherent systems

For more information on the structure of coherent systems one can see [7], [36]. .

3.1 The Signature of Coherent Systems

The reliability of a coherent system is defined as the probability that the system will perform satisfactorily for at least a given period of time t.Let T denotes the lifetime of a coherent system. Then the reliability or survival function of the system is defined by

$$R(t) = P(T > t), t \geq 0$$

Reliability or survival function is one of the most important lifetime characteristic of a system since it gives us the information about how long may the system will continue functioning. Moreover it allows us to evaluate other important lifetime characteristics of the system such as mean time to failure, mean residual life and hazard rate.

Let T_i denote the lifetime of the ith component in a coherent system with the structure function ϕ and lifetime T. Then

$$T = \phi(T_1, T_2, ..., T_n).$$

If we define the binary stochastic process that represents the state of the ith component at time t as follows

$$X_i(t) = \begin{cases} 1 & \text{if } T_i > t \\ 0 & \text{if } T_i \leq t \end{cases}, \quad i = 1, 2, ..., n$$

The survival function $R(t)$ can be investigated by the help of $X_i(t)$s. For example, the survival function of a k-out-of-n:F system can be written as

$$R(t) = P(\sum_{i=1}^{n} X_i(t) > n - k).$$

Another representation for the survival function of coherent systems was given in terms of system signature. In 1985, Samaniego [55] introduced the concept of system signature, which is a very practical tool for representing the lifetime distribution of a coherent system. For a coherent system having lifetime T consisting of n components with independent and identically distributed (i.i.d.) lifetimes $T_1, \ldots, T_n$, the system signature is an n-dimensional vector $\boldsymbol{p}$ whose ith element is given by

$$p_i = P\left\{T = T_{i:n}\right\},$$

where $T_{i:n}$ is the ith order statistic corresponding to the lifetimes $T_1, \ldots, T_n$, $i = 1, \ldots, n$.

$$p_i = \frac{\#\text{ of orderings for which the } i\text{th failure causes system failure}}{n!} \tag{3.3}$$

for $i = 1, ..., n$.The ith element of $\boldsymbol{p}$ can also be computed as

$$p_i = a_{n-i+1}(n) - a_{n-i}(n), \tag{3.4}$$

where

$$a_i(n) = \frac{r_i(n)}{\binom{n}{i}} \tag{3.5}$$

and $r_i(n)$ is the number of path sets of the system with exactly i working components [8]. Using (3.4) and (3.5), we have

$$r_i(n) = \binom{n}{i} \sum_{j=n-i+1}^{n} p_j. \tag{3.6}$$

That is, the signature of a system which can be obtained by computing $r_i(n)$.and $r_i(n)$ can be obtained from system signature. The problem of finding $r_i(n)$ and hence system signature is combinatorial one depending on the structure of a system

The reliability of a coherent system with signature vector $\boldsymbol{p}$ and lifetime T can be represented as a positive mixture of reliability of $T_{1:n}, \ldots, T_{n:n}$ with respective weights $p_1, \ldots, p_n$ as

$$P\{T > t\} = \sum_{i=1}^{n} p_i P\{T_{i:n} > t\}. \tag{3.7}$$

The representation (3.7) was used to compare systems with different structures, evaluation of reliability characteristics and system signature [12], [21], [34], [40], [41], [42], [43], [44], [46], [50], [51], [52], [58] and also ordering properties of coherent systems [53], [63], [64]. For an extensive review of system signature and its applications see the books of Samaniego [56] and Lisnianski and Frenkel [39].

Example 3.1. Let us find the signature of the following consecutive 2-out-of-3:F system. The consecutive 2-out-of-3:F system is a system that consists of $n = 3$ linearly ordered components and fails if and only if at least $k = 2$ consecutive components fail.

We can define the system lifetime T as follows

$$T = \min(\max(T_1, T_2), \max(T_2, T_3))$$

There are totally $3! = 6$ ordering of the component lifetimes which can be given as follows.

Ordering	T
$T_1 < T_2 < T_3$	$T_{2:3}$
$T_1 < T_3 < T_2$	$T_{3:3}$
$T_2 < T_1 < T_3$	$T_{2:3}$
$T_2 < T_3 < T_1$	$T_{2:3}$
$T_3 < T_1 < T_2$	$T_{3:3}$
$T_3 < T_2 < T_1$	$T_{2:3}$

Then we have

$$p_1 = 0$$
$$p_2 = \frac{4}{6}$$
$$p_3 = \frac{2}{6}$$

and the signature $p = (0, \frac{2}{3}, \frac{1}{3})$.

Another representation for the reliability of a coherent system with lifetime T was proposed by Navarro et al [47].

$$P\{T > t\} = \sum_{i=1}^{n} \alpha_i P\{X_{1:i} > t\},$$

where $X_{1:i} = \min(X_1, \ldots, X_i)$ and α_i is the ith element of the minimal signature vector $\boldsymbol{\alpha}$, satisfying $\sum_{i=1}^{n} \alpha_i = 1$, $i = 1, \ldots, n$. Eryilmaz [20] has derived the following formula to compute α_i

$$\alpha_i = \begin{cases} \sum_{j=n-i}^{n-k_{\min}} (-1)^{i+j-n} \binom{j}{n-i} r_{n-j}(n) & \text{if } i \geq k_{\min}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

where $k_{\min}$ is the minimum number of working components required for the functioning of the system.

An alternative concept to the system signature, the survival signature, was introduced by Coolen and Coolen-Maturi [10] and is closely related to the system signature. Let $\Phi(l)$, for $l = 1, \ldots, m$, denote the probability that a system functions given that exactly l of its components function. For coherent systems, $\Phi(l)$ is an increasing function of l, $\Phi(0) = 0$, and $\Phi(m) = 1$. When the failure times of the components in a coherent system are i.i.d.

$$\Phi(l) = \binom{m}{l}^{-1} \sum_{\underline{x} \in S_l} \phi(\underline{x}),$$

where $\underline{x}$ is the state vector, S_l is the set of state vectors which have exactly l functioning components, and ϕ is the structure function of the system. Coolen and Coolen-Maturi [10] showed that the survival signature of a coherent system consisting m components can be written in terms of the system signature of that system:

$$\Phi(l) = \sum_{j=m-l+1}^{m} p_j. \tag{3.9}$$

Chapter 4

Weighted k-out-of-n:G Systems

In order to raise operational availability and productivity or to reduce the loss in industrial systems, it is important to consider process design and reliability. System design for reliability is generally subject to several uncertainties, such as type, working time, failure rate, repair time etc. of the components. If it is difficult to improve the reliability of an individual component in the system, redundancy can be used as an alternative approach in the system design. The k-out-of-n:G system and its variants are widely used in redundant design in order to improve the system reliability. Reliability properties of such systems have been studied widely in the literature [22], [23], [24], [31], [32]. According to the configuration of real life instance, a system's operational availability and productivity depends not only on the availability of the components, but also on their distinct endowments. This type of systems are called weighted k-out-of-n:G systems. In a traditional setup of system reliability problems, components of a system are assumed to have equal weights. However, this assumption is not valid for most of real-life problems. For instance, a factory might have different machines which have different production capacities. In this example, weight of a component is the production capacity of each machine. More real life examples can be found, in heating, cooling and lightning systems. In those systems there can be components with distinct wattage where wattage of a component can be represented as the weights of components. In order to analyze such cases in 1994, Wu and Chen [60] proposed a more general model than $k-$out-of$-n : G$ system and called it as weighted $k-$out-of$-n : G$ system. In this system components may have different positive integer weights and the system works if the total weight of working components is at least a predefined threshold k. This system turns into ordinary $k-$out-of$-n : G$ system if each component has weight unity. The reliability of weighted $k-$out-of$-n : G$ systems can be computed by using recursive formula [9], [18], [33], [37]. There are several studies that have been proposed about the dynamic analysis of weighted $k-$out-of$-n : G$ systems [22] and [57]. Due to the structure of weighted $k-$out-of$-n : G$ systems, components have different reliability which makes computation even harder. Other studies that have been done in this area are [11], [23], [35], [38], [54]. Recent works on weighted k-out-of-$n : G$ systems can be found in [25], [28], [29], [30]. Eryilmaz and Sarikaya [27] considered a special type of weighted $k-$out-of$-n : G$ system which has only two types of components having different weights and reliabilities. One group of components has weight w and common failure time distribution F and the other group has weight w^* and another common failure time distribution G. For this system they have obtained closed form equations for the survival function and mean time to failure (MTTF). This special system arises in real life when factories decide to improve or modify their existing machines. For example, a company may have n_1 machines with common failure time distribution and production capacity w. After a while company may decide to enlarge or improve its production line by adding n_2 new machines with common failure time distribution and production capacity w^* or they can simply replace some of the existing machines with new ones. The underlying cause for the company to take these actions is to improve production capacity, increase reliability and MTTF of the whole system. As well as the advantages there are some downsides of adding or replacing new machines to the system. One and maybe the most important one is cost, which increases as well as the reliability and MTTF. Moreover if the capacities of the new machines (w^*) are greater than the existing ones this will cause an extra increase in the production costs. To overcome this problem, rather than only adding new components to increase reliability and MTTF, companies can use standby components or along with the new components.

Eryilmaz and Sarıkaya [27] derived the following exact reliability of this special weighted $k-$out-of$-n : G$ system without using a recursive formula .

$$P(T_k^{\{n_1,n_2\}} > t) = \sum_{\substack{wi+w^*j\geq k \\ 0\leq i\leq n_1, 0\leq j\leq n_2}}\sum \binom{n_1}{i}\overline{F}(t)^i F(t)^{n_1-i}\binom{n_2}{j}\overline{G}(t)^j G(t)^{n_2-j}, \quad (4.1)$$

where F is the common lifetime distribution of the components of one group and G is the common lifetime distribution of the components of another group.
In this study we will investigate the reliability properties of this special weighted $k-$out-of$-n : G$ equipped with a single cold standby component.

Chapter 5

Standby Systems

A parallel system having n components works successfully even with a single component. However all the components in the system are used simultaneously in the process. Instead of having $n-1$ redundant components another redundancy called standby redundancy can be used in order to increase the reliability of the system. In this case some of the active components may be replaced or additional components may be added as standby components. To do this a sensing and switching mechanism is used to control the operation of the active components. Whenever an active component fails a standby component is put on the operation and becomes active.

There are different types of standby such as hot standby, warm standby and cold standby. Hot standby components have the same failure rate as the active components so they are also called active redundant components. On the contrary cold standby components have zero failure rate which means that they do not fail while they are in standby. Warm standby redundancy is a mixture of these cases. Warm standby components have a failure rate between 0 and failure rate of the active component so warm standby may contain both cold and hot standby cases.

In this study we are solely interested in systems having a single cold standby component and perfect sensing and switching mechanism. In a cold standby system, cold standby component does not fail when it is inactive hence we only need to concentrate on the active components whose failure cause the failure of the system. When the sensing and switching mechanism is perfect, the cold standby component becomes active, as soon as the failure of the active component which causes system failure.

In order to clarify the concept let us consider a system with an active component and a cold standby component. Let T, T_1, T_2 denote respectively the lifetime of the system, lifetime of the active component and lifetime of the cold standby component. Let $F(t)$ and $G(t)$, $t \geq 0$ denote the failure rate distribution of the active and standby component respectively. Denote $f(t)$ as the probability density function of the active component with lifetime T_1. There are two cases for the system to survive until the time t The first one is, active component survives until the time t. The second case is active component fails at time x $(0 \leq x < t)$ and cold standby component is put on operation and survives between time x and t. Hence the reliability of the system is

$$\begin{aligned} P(T > t) &= P(T_1 > t) + \int_0^t P(T_2 > t - x) f(x) dx \\ &= 1 - F(t) + \int_0^t \left(1 - G(t - x)\right) f(x) dx \end{aligned}$$

Chapter 6

Coherent Systems with a Cold Standby

There are different methods in order to increase system reliability. One of them is to equip the system with standby units such as warm, hot and cold. Compared to others, cold standby redundancy can be preferred when switching times are sufficiently short, since cold standby component is inactive which means it does not fail in standby. Van Gemund and Reijns [59] studied k-out-of-n system with a single standby and found an analytical way to compute the mean time to failure of the system. Eryilmaz [23] investigated various mean residual life functions for the same system. Recently, Eryilmaz [24] studied k-out-of-n system equipped with a single warm standby component.
In this study, using system signature, conditioning on the index of the cold standby component and indices of the components failed before cold standby component is put into operation, the reliability of coherent systems having a cold standby component are derived.

In a coherent system with a single cold standby, the index of the standby component as well as the indices of failed components have significant importance which makes the computation of the reliability more difficult. A method for computing the reliability of coherent systems is presented. Moreover, comparison of the reliability and mean time to failure of some systems with and without cold standby component have been illustrated.

Notation

Below the notation that will be used throughout this chapter are provided.

n, number of components in the system;

Y, lifetime of the cold standby component;

T_i, lifetime of the component i, $1 \leq i \leq n$;

$T_{s:n}$, sth smallest among T_i, $1 \leq i \leq n$;

$T_l^{(s)}$, remaining lifetime of the components after $T_{s:n}$ fails: $T_l^{(s)} \stackrel{st}{=} (T_l - T_{s:n} | T_l > T_{s:n})$, $1 \leq l \leq n-s$;

ϕ, structure function of the system;

$T = \phi(T_1, \ldots, T_n)$, lifetime of the system without cold standby component;

T^w, lifetime of the system with a cold standby component;

V_s, discrete random variable representing the index of the cold standby component when $T_{s:n}$ fails: $V_s = c \Leftrightarrow (T_c = T_{s:n} | T = T_{s:n})$, $c = 1, 2, \ldots, n$;

$\mathbf{B}_{s,c} | V_s = c$, a discrete multivariate random variable representing the indices of the failed components given $V_s = c$, $s = 1, \ldots, n$ and $c = 1, \ldots, n$:

$(\mathbf{B}_{s,c} | V_s = c) = (B_1 = b_1, B_2 = b_2, \ldots, B_{s-1} = b_{s-1} | V_s = c) \Leftrightarrow (0_{B_1} = 0_{b_1}, 0_{B_2} = 0_{b_2}, \ldots, 0_{B_{s-1}} = 0_{b_{s-1}} | T_c = T_{s:n}, T = T_{s:n})$ where $\mathbf{0} = (0_{B_1}, 0_{B_2}, \ldots, 0_{B_{s-1}})$ are the components which have failed before $T_{s:n}$;

$\mathbf{R}_{s,c} | V_s = c$, a discrete multivariate random variable representing the indices of the remaining components given $V_s = c$, $s = 1, \ldots, n$ and $c = 1, \ldots, n$: $\mathbf{R}_{s,c} = (R_1 = r_1, R_2 = r_2, \ldots, R_{n-s} = r_{n-s} | V_s = c) \Leftrightarrow (T_{R_1}^{(s)} = T_{r_1}^{(s)}, T_{R_2}^{(s)} = T_{r_2}^{(s)}, \ldots, T_{R_{n-s}}^{(s)} = T_{r_{n-s}}^{(s)} | T_c = T_{s:n}, T = T_{s:n})$.

Consider a binary coherent system with structure function ϕ. Let $T = \phi(T_1, \ldots, T_n)$ denote the lifetime of a coherent system without a cold standby

component and T^w denote the lifetime of the same system with a cold standby component whose lifetime is Y. Moreover, $T_1, \ldots, T_n$ have a common continuous cumulative distribution function (c.d.f); F and Y has a continuous c.d.f G.
Eryilmaz [26] studied on coherent systems equipped with a cold standby component which may put into operation at the time of the first component failure in the system. In this paper, we consider the general case in which standby component may get involved at the time of the sth component failure $s = k_\phi, ..., z_\phi + 1$ where k_ϕ is the minimum number of failed components that cause the system failure whereas z_ϕ is the maximum number of failed components that system can still operate. It is clear that $P(T = T_{s:n}) > 0$ for $s = k_\phi, ..., z_\phi + 1$.
After replacing the standby component with sth failed component which causes the system failure at the same time, the remaining lifetime of the system consisting of $s-1$ failed components ($0's$), $n-s$ functioning components, and a standby component (Y) can be represented as

$$\phi_s(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{s-1}}, Y_{V_s}, T^{(s)}_{R_1}, T^{(s)}_{R_2}, \ldots, T^{(s)}_{R_{n-s}}),$$

When sth failure occurs which causes system failure at the same time, cold standby component gets involved to the system. At this time, there are totally $n - s + 1$ functioning components in the system. The reliability of the remaining lifetime of the system is computed based on these $n-s+1$ functioning components. However, places of the $s - 1$ failed components should be taken into consideration (not their lifetimes since they failed already) in the structure function of the system to calculate the main lifetime random variable T^w.

It is well known that the random variables $T^{(s)}_1, \ldots, T^{(s)}_{n-s}$ are conditionally independent given $T_{s:n} = x$, and

$$P\{T^{(s)}_1 > t_1, \ldots, T^{(s)}_{n-s} > t_{n-s} | T_{s:n} = x\} = \prod_{l=1}^{n-s} \frac{\bar{F}(t_l + x)}{\bar{F}(x)},$$

The main goal is to find the reliability characteristics of T^w, i.e.

$$T^w = T + \sum_{s=k_\phi}^{z_\phi+1} \phi_s(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{s-1}}, Y_{V_s}, T^{(s)}_{R_1}, T^{(s)}_{R_2}, \ldots, T^{(s)}_{R_{n-s}}).$$

Lemma 6.1 *For* $t > 0$ *and* $s = k_\phi, ..., z_\phi + 1$;

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c, T^{(s)}_{r_1}, T^{(s)}_{r_2}, \ldots, T^{(s)}_{r_{n-s}}) > t | X_{s:n} = x\}$$
$$= \frac{1}{\bar{F}^{n-s}(x)} \int_{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},y_c,t_{r_1},t_{r_2},\ldots,t_{r_{n-s}})>t} \cdots \int g(y_c) \prod_{m=1}^{n-s} f(t_{r_m} + x) dt_{r_1} dt_{r_2} \ldots dt_{r_{n-s}} dy_c.$$

Proof. Due to the fact that Y and $T_1, \ldots, T_n$ are independent for $s = k_\phi, ..., z_\phi + 1$

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c, T^{(s)}_{r_1}, T^{(s)}_{r_2}, \ldots, T^{(s)}_{r_{n-s}}) > t | T_{s:n} = x\}$$
$$= \int_{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},y_c,t_{r_1},t_{r_2},\ldots,t_{r_{n-s}})>t} \cdots \int g(y_c) f(t_{r_1}, t_{r_2}, \ldots, t_{r_{n-s}} | t_{s:n} = x) dt_{r_1} dt_{r_2} \ldots dt_{r_{n-s}} dy_c,$$

Since the joint p.d.f. of $T^{(s)}_{r_1}, T^{(s)}_{r_2}, \ldots, T^{(s)}_{r_{n-s}}$ given $T_{s:n} = x$ is

$$f(t_{r_1}, t_{r_2}, \ldots, t_{r_{n-s}} | t_{s:n} = x) = \frac{1}{\bar{F}^{n-s}(x)} \prod_{m=1}^{n-s} f(t_{r_m} + x).$$

The proof is complete. □

Remark. Due to the fact that given $T_{s:n} = x$, the random variables $T^{(s)}_1, \ldots, T^{(s)}_{n-s}$ are independent for $s = k_\phi, ..., z_\phi + 1$. So, the conditional probability given in Lemma 1 is indeed the survival function of the coherent system ϕ_s consisting of $s-1$ failed components, $n-s$ independent component having the same marginal survival function $\frac{\bar{F}(t+x)}{\bar{F}(x)}$ and the v_sth component has the survival function $\bar{G}(t)$. Moreover given $T_{s:n} = x$ if we order the residual lifetime of the remaining $n-s$ components such that

$$T^{(s)}_{1:n-s} \le T^{(s)}_{2:n-s} \le \ldots \le T^{(s)}_{n-s:n-s},$$

The survival function of the kth order statistics of the residual lifetime of the remaining $n-s$ components for $k = 1, 2, \ldots n-s$, can be found as

$$P(T^{(s)}_{k:n-s} > t | T_{s:n} = x) = \sum_{i=0}^{k-1} \binom{n-s}{i} \left(1 - \frac{\bar{F}(t+x)}{\bar{F}(x)}\right)^i \left(\frac{\bar{F}(t+x)}{\bar{F}(x)}\right)^{n-s-i}.$$

Theorem 6.2 *Let* $\mathbf{p}$ *be the signature of a coherent system* $T = \phi(X_1, \ldots, X_n)$ *which has a cold standby component with lifetime distribution* G*. Then*

$$P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} \left(p_s P(T_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \ldots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \ldots, b_{s-1})\right) \times \right.$$
$$\left. \int_0^t P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c, T_{r_1}^{(s)}, T_{r_2}^{(s)}, \ldots, T_{r_{n-s}}^{(s)}) > t - x | T_{s:n} = x\} dF_{s:n}(x) \right).$$

Proof. For a coherent system $P(T = T_{s:n}) > 0$ for $s = k_\phi, \ldots, z_\phi + 1$. Any coherent system operating with n components may fail at the time of sth component failure. If the system failure caused by the failure of the sth component then the standby component gets involved to the system. Therefore the survival function of the coherent system with a standby component can be written as follows

$$\begin{aligned}
P(T^w > t) =& P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, T_{R_1}^{(k_\phi)}, T_{R_2}^{(k_\phi)}, \ldots, T_{R_{n-k_\phi}}^{(k_\phi)}) > t, T = T_{k_\phi:n}\} \\
&+ P(T > t, T > T_{k_\phi:n}) \\
=& p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, T_{R_1}^{(k_\phi)}, T_{R_2}^{(k_\phi)}, \ldots, T_{R_{n-k_\phi}}^{(k_\phi)}) > t | T = T_{k_\phi:n}\} \\
&+ P(T > t, T > T_{k_\phi:n}) \\
=& p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, T_{R_1}^{(k_\phi)}, T_{R_2}^{(k_\phi)}, \ldots, T_{R_{n-k_\phi}}^{(k_\phi)}) > t | T = T_{k_\phi:n}\} + \\
& p_{k_\phi+1} P\{T + \phi_{k_\phi+1}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi}}, Y_{V_{k_\phi+1}}, T_{R_1}^{(k_\phi+1)}, T_{R_2}^{(k_\phi+1)}, \ldots, T_{R_{n-k_\phi-1}}^{(k_\phi+1)}) > t | T = T_{k_\phi+1:n}\} + \\
& P(T > t, T > T_{k_\phi+1:n}) \\
=& p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, T_{R_1}^{(k_\phi)}, T_{R_2}^{(k_\phi)}, \ldots, T_{R_{n-k_\phi}}^{(k_\phi)}) > t | T = T_{k_\phi:n}\} + \\
& p_{k_\phi+1} P\{T + \phi_{k_\phi+1}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{k_\phi}}, Y_{V_{k_\phi+1}}, T_{R_1}^{(k_\phi+1)}, T_{R_2}^{(k_\phi+1)}, \ldots, T_{R_{n-k_\phi-1}}^{(k_\phi+1)}) > t | T = T_{k_\phi+1:n}\} + \cdots + \\
& p_{z_\phi+1} P\{T + \phi_{z_\phi+1}(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{z_\phi}}, Y_{V_{z_\phi+1}}, T_{R_1}^{(z_\phi+1)}, T_{R_2}^{(z_\phi+1)}, \ldots, T_{R_{n-z_\phi-1}}^{(z_\phi+1)}) > t | T = T_{z_\phi+1:n}\} + \\
& P(T > t, T > T_{z_\phi+1:n}),
\end{aligned}$$

It is obvious that $P(T > t, T > T_{z_\phi+1:n}) = 0$.

Now, for $s = k_\phi, \ldots, z_\phi + 1$ consider the conditional probability

$$P\{T+\phi_s(0_{B_1},0_{B_2},\dots,0_{B_{s-1}},Y_{V_s},T^{(s)}_{R_1},T^{(s)}_{R_2},\dots,T^{(s)}_{R_{n-s}})>t|T=T_{s:n}\}$$

$$=\sum_{c=1}^{n}\frac{P\{T_c+\phi_s(0_{B_1},0_{B_2},\dots,0_{B_{s-1}},Y_c,T^{(s)}_{R_1},T^{(s)}_{R_2},\dots,T^{(s)}_{R_{n-s}})>t,T_{s:n}=T_c,T=T_{s:n}\}}{P(T=T_{s:n})}$$

$$=\sum_{c=1}^{n}P\{T_c+\phi_s(0_{B_1},0_{B_2},\dots,0_{B_{s-1}},Y_c,T^{(s)}_{R_1},T^{(s)}_{R_2},\dots,T^{(s)}_{R_{n-s}})>t|T_{s:n}=T_c,T=T_{s:n}\}\times$$

$$P(T_{s:n}=T_c|T=T_{s:n})$$

$$=\frac{\sum\limits_{c=1}^{n}P(V_s=c)\sum\limits_{1\le b_1<\dots<b_{s-1}\le n}P\{T_c+\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t|0_{B_1}=0_{b_1},\dots,0_{B_{s-1}}=0_{b_{s-1}},T_{s:n}=T_c,T=T_{s:n}\}}{P(T_{s:n}=T_c,T=T_{s:n})}$$

$$\times P(0_{B_1}=0_{b_1},\dots,0_{B_{s-1}}=0_{b_{s-1}},T_{s:n}=T_c,T=T_{s:n})$$

$$=\sum_{c=1}^{n}P(V_s=c)\sum_{1\le b_1<\dots<b_{s-1}\le n}P(0_{B_1}=0_{b_1},\dots,0_{B_{s-1}}=0_{b_{s-1}}|T_{s:n}=T_c,T=T_{s:n})\times$$

$$P\{T_c+\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t|0_{B_1}=0_{b_1},\dots,0_{B_{s-1}}=0_{b_{s-1}},T_{s:n}=T_c,T=T_{s:n}\}$$

$$=\sum_{c=1}^{n}P(V_s=c)\sum_{1\le b_1<\dots<b_{s-1}\le n}P\left(\mathbf{B}_{s,c}=(b_1,\dots,b_{s-1})\right)\times$$

$$\int P\{\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t-x|T_{s:n}=x\}dF_{s:n}(x),$$

$$=\sum_{c=1}^{n}P(V_s=c)\sum_{1\le b_1<\dots<b_{s-1}\le n}P\left(\mathbf{B}_{s,c}=(b_1,\dots,b_{s-1})\right)\times$$
$$\left[\int_t^{\infty}dF_{s:n}(x)+\int_0^{t}P\{\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t-x|T_{s:n}=x\}dF_{s:n}(x)\right]$$
$$=\sum_{c=1}^{n}P(V_s=c)\sum_{1\le b_1<\dots<b_{s-1}\le n}P\left(\mathbf{B}_{s,c}=(b_1,\dots,b_{s-1})\right)\times$$
$$\left[\int_0^{t}P\{\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t-x|T_{s:n}=x\}dF_{s:n}(x)+P(T_{s:n}>t)\right]$$

Hence,

$$P(T^w>t)=\sum_{s=k_\phi}^{z_\phi+1}\Bigg(p_sP(T_{s:n}>t)+p_s\sum_{c=1}^{n}P(V_s=c)\sum_{1\le b_1<\dots<b_{s-1}\le n}P\left(\mathbf{B}_{s,c}=(b_1,\dots,b_{s-1})\right)\times$$
$$\int_0^{t}P\{\phi_s(0_{b_1},\dots,0_{b_{s-1}},Y_c,T^{(s)}_{r_1},\dots,T^{(s)}_{r_{n-s}})>t-x|T_{s:n}=x\}dF_{s:n}(x)\Bigg).$$

$\square$

Theorem 6.3 *Consider a coherent system having a signature vector* $\mathbf{p}$*, with a cold standby component having distribution function* G *while other components have common distribution function* F*. Then system reliability can be computed as* *follows*

$$P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} \Bigg(p_s P(T_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \ldots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, ..., b_{s-1})\right) \times$$
$$\int_0^t \left[\bar{G}(t-x) \sum_{k=1}^{n-s} \bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} P(T_{k:n-s}^{(s)} > t - x | T_{s:n} = x) + \bar{G}(t-x) \bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} \right] dF_{s:n}(x) \Bigg),$$

where $\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})}$ is the number of orderings for which kth failure among the $(n-s)$ remaining components and a cold standby component cause the system to fail where cth component (cold standby) assumed to be functioning and components having indices $b_1, b_2, \ldots, b_{s-1}$ have already failed. Moreover $T_{k:n-s}^{(s)}$ is the kth order statistics of the residual lifetime of the remaining $(n-s)$ functioning components.

$$\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} = \frac{\textit{The number of orderings for which the kth failure of the remaining components cause the system to fail}}{n-s!}$$
$$k = 1, \ldots, n-s,$$

and

$$\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = \begin{cases} 1, & \textit{if the failure of the system can be caused by only the failure of the cold standby} \\ 0, & \textit{if the failure of the system can be caused by the remaining components} \end{cases}$$

Proof. When cold standby component is put into operation at time x for the system to survive up to time t, the cold standby component must function between the time x and t since the failure of the cold standby component will lead to system failure with probability 1. Assuming cold standby component functions between the time t and x system failure can be caused by the failure of the remaining components. Given $T_{s:n} = x$ residual lifetime of the remaining $(n-s)$ components are independent and identically distributed. Therefore the survival function of the coherent system $\phi_s(0_{b_1}, \ldots, 0_{b_{s-1}}, Y_c, T_{r_1}^{(s)}, \ldots, T_{r_{n-s}}^{(s)})$ having $s-1$

failed components at places $b_1, b_2, \ldots, b_{s-1}$ and a cold standby component at place c can be computed by its signature function $\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})}$ for $k = 1, \ldots, n-s$ and $\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = 0$. If the failure of the system does not depend on the failure of the remaining components which means system survives until the cold standby component fails in that case $\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} = 0$ for $k = 1, \ldots, n-s$ and $\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = 1$. □

It is known that when both active and standby components have common exponential distribution, the random variables $T_{R_1}^{(s)}, T_{R_2}^{(s)}, \ldots, T_{R_{n-s}}^{(s)}, Y_{V_s}$ are independent and have the same exponential distribution. Therefore, the structure function can be written as

$$\phi_s(0_{B_1}, \ldots, 0_{B_{s-1}}, Y_{V_s}, T_{R_1}^{(s)}, \ldots, T_{R_{n-s}}^{(s)}) \overset{st}{=} \phi_s(0_{B_1}, \ldots, 0_{B_{s-1}}, Y_{V_s}, T_{R_1}, \ldots, T_{R_{n-s}}).$$

Corollary 6.4 *Under the assumption of all components, including the cold standby component, have common exponential distribution $F(x) = 1 - e^{-\lambda x}$, $x > 0$ the reliability of coherent systems with a cold standby component turns into*

$$P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} p_s P(T_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \ldots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \ldots, b_{s-1})\right) \times$$
$$\int_0^t \left[\bar{F}(t-x) \sum_{k=1}^{n-s} \bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} P(T_{k:n-s} > t-x) + \bar{F}(t-x) \bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} \right] dF_{s:n}(x).$$

Example 6.1. Consider the coherent system with lifetime

$$T = \min(T_1, \max(T_2, T_3)).$$

The signature of this system is $p = (\frac{1}{3}, \frac{2}{3}, 0)$. In this system, $k_\phi = z_\phi = 1$. For $s = 1$, there are no failed components ($0's$). $P(V_1 = 1) = 1$, $P(V_1 = 2) = P(V_1 = 3) = 0$ which means only component 1 can be replaced by the cold standby component. The remaining lifetime of the components after the first failure are $X_2^{(1)}$ and $X_3^{(1)}$. $\bar{p}^{1,-}$ can be found as $(0, 1, 0)$ since when the cold standby component functions, the system works until both component 2 and 3 fail. For $s = 2$, all components can be cold standby component with probabilities

$P(V_2 = 1) = \frac{1}{2}$, and $P(V_2 = 2) = P(V_2 = 3) = \frac{1}{4}$. Suppose component 1 is replaced with the cold standby component. Previously failed component can be 2 or 3 (0_2 or 0_3). Moreover, let component 2 (3) replaced by the cold standby component. In this case, previously failed component is 0_3 (0_2).

$\bar{p}^{1,(2)} = \bar{p}^{1,(3)} = \bar{p}^{2,(3)} = \bar{p}^{3,(2)}$ is $(1,0)$ because for each case the failure of the remaining component will lead to system failure.

$s = 1$	c
V_1	1
$\mathbf{B}_{1,c}$	$-$
$\mathbf{R}_{1,c}$	$(2,3)$
$P(V_1 = c)$	1
$\bar{p}^{c,\mathbf{B}_{1,c}}$	$(0,1,0)$

$s = 2$	c			
V_2	1		2	3
$\mathbf{B}_{2,c}$	(2)	(3)	(3)	(2)
$\mathbf{R}_{2,c}$	(3)	(2)	(1)	(1)
$P(V_2 = c)$	$\frac{1}{2}$		$\frac{1}{4}$	$\frac{1}{4}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$

Using Theorem 2

$$
\begin{aligned}
P(T^w > t) = \frac{1}{3}P(T_{1:3} > t) + \frac{2}{3}P(T_{2:3} > t) + \\
\frac{1}{3}\int_0^t \bar{G}(t-x)\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 + 2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right] dF_{1:3}(x) \\
+ \frac{2}{3}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)} dF_{2:3}(x).
\end{aligned}
$$

Example 6.2. Consider the coherent system with lifetime

$$T = \max(\min(T_1, T_2, T_3), \min(T_2, T_3, T_4)).$$

The signature of this system is $p = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ In this system $k_\phi = z_\phi = 1$.

$s=1$	c	
V_1	2	3
$\mathbf{B}_{1,c}$	$-$	$-$
$\mathbf{R}_{1,c}$	$(1,3,4)$	$(1,2,4)$
$P(v_1=c)$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{p}^{c,\mathbf{B}_{1,c}}$	$(\frac{1}{3},\frac{2}{3},0,0)$	$(\frac{1}{3},\frac{2}{3},0,0)$

$s=2$	c					
V_2	1	2		3		4
$\mathbf{B}_{2,c}$	(4)	(1)	(4)	(1)	(4)	(1)
$\mathbf{R}_{2,c}$	$(2,3)$	$(3,4)$	$(1,3)$	$(2,4)$	$(1,2)$	$(2,3)$
$P(V_2=c)$	$\frac{1}{6}$	$\frac{2}{6}$		$\frac{2}{6}$		$\frac{1}{6}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$

$$
\begin{aligned}
P(T^w>t) &= \frac{1}{2}P(T_{1:4}>t)+\frac{1}{2}P(T_{2:4}>t)+\\
&\frac{1}{2}\int_0^t \bar{G}(t-x)\left[\frac{1}{3}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+\frac{2}{3}\left(\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2\right)\right]dF_{1:4}(x)\\
&+\frac{1}{2}\int_0^t \bar{G}(t-x)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 dF_{2:4}(x).
\end{aligned}
$$

Example 6.3. Consider the linear consecutive 3-out-of-5:F system whose lifetime is

$$T=\min(\max(T_1,T_2,T_3),\max(T_2,T_3,T_4),\max(T_3,T_4,T_5)).$$

The signature of this system is $p=(0,0,\frac{3}{10},\frac{1}{2},\frac{2}{10})$. In this system $k_\phi=3$ and $z_\phi=4$.

$s=3$	c								
V_3	1	2		3			4		5
$\mathbf{B}_{3,c}$	$(2,3)$	$(1,3)$	$(3,4)$	$(1,2)$	$(2,4)$	$(4,5)$	$(3,5)$	$(2,3)$	$(3,4)$
$\mathbf{R}_{3,c}$	$(4,5)$	$(4,5)$	$(1,5)$	$(4,5)$	$(1,5)$	$(1,2)$	$(1,2)$	$(1,5)$	$(1,2)$
$P(V_3=c)$	$\frac{1}{9}$	$\frac{2}{9}$		$\frac{3}{9}$			$\frac{2}{9}$		$\frac{1}{9}$
$\bar{p}^{c,\mathbf{B}_{3,c}}$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$

$s=4$	c									
V_4	1	2		3				4		5
$\mathbf{B}_{4,c}$	$(2,3,5)$	$(1,3,4)$	$(1,3,5)$	$(1,2,4)$	$(1,2,5)$	$(1,4,5)$	$(2,4,5)$	$(2,3,5)$	$(1,3,5)$	$(1,3,4)$
$\mathbf{R}_{4,c}$	(4)	(5)	(4)	(5)	(4)	(2)	(1)	(1)	(2)	(2)
$P(V_4=c)$	$\frac{1}{10}$	$\frac{2}{10}$		$\frac{4}{10}$				$\frac{2}{10}$		$\frac{1}{10}$
$\bar{p}^{c,\mathbf{B}_{4,c}}$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(1,0)$

$s=5$	c
V_5	3
$\mathbf{B}_{5,c}$	$(1,2,4,5)$
$\mathbf{R}_{5,c}$	$-$
$P(V_5=c)$	1
$\bar{p}^{c,\mathbf{B}_{5,c}}$	(1)

$$
\begin{aligned}
P(T^w>t) = {} & \frac{3}{10}P(T_{3:5}>t)+\frac{1}{2}P(T_{4:5}>t)+\frac{2}{10}P(T_{5:5}>t)+ \\
& \frac{3}{10}\frac{4}{9}\int_0^t \bar{G}(t-x)\left[\frac{1}{2}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\frac{1}{2}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right]\right]dF_{3:5}(x)+ \\
& \frac{3}{10}\frac{2}{9}\int_0^t \bar{G}(t-x)\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right]dF_{3:5}(x)+\frac{3}{10}\frac{3}{9}\int_0^t \bar{G}(t-x)dF_{3:5}(x)+ \\
& \frac{1}{2}\frac{6}{10}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)}dF_{4:5}(x)+\frac{1}{2}\frac{4}{10}\int_0^t \bar{G}(t-x)dF_{4:5}(x)+ \\
& \frac{2}{10}\int_0^t \bar{G}(t-x)dF_{5:5}(x).
\end{aligned}
$$

Example 6.4. Consider the coherent system with lifetime

$$T=\min(T_1,\max(T_2,T_3),\max(T_3,T_4)).$$

The signature of this system is $p=(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)$ In this system $k_\phi=1$ and $z_\phi=2$.

$s=1$	c
V_1	1
$\mathbf{B}_{1,c}$	$-$
$\mathbf{R}_{1,c}$	$(2,3,4)$
$P(V_1=c)$	1
$\bar{p}^{c,\mathbf{B}_{1,c}}$	$(0,\frac{2}{3},\frac{1}{3},0)$

$s=2$	c						
V_2	1			2	3		4
$\mathbf{B}_{2,c}$	(2)	(3)	(4)	(3)	(2)	(4)	(3)
$\mathbf{R}_{2,c}$	$(3,4)$	$(2,4)$	$(2,3)$	$(1,4)$	$(1,4)$	$(1,2)$	$(1,2)$
$P(V_2=c)$	$\frac{3}{7}$			$\frac{1}{7}$	$\frac{2}{7}$		$\frac{1}{7}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	$(\frac{1}{2},\frac{1}{2},0)$	$(1,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(1,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(1,0,0)$

$s=3$	c	
V_3	1	3
$\mathbf{B}_{3,c}$	$(2,4)$	$(2,4)$
$\mathbf{R}_{3,c}$	(3)	(1)
$P(V_3=c)$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{p}^{c,\mathbf{B}_{3,c}}$	$(1,0)$	$(1,0)$

$$
\begin{aligned}
P(T^w>t) = {} & \frac{1}{4}P(T_{1:4}>t)+\frac{7}{12}P(T_{2:4}>t)+\frac{1}{6}P(T_{3:4}>t)+ \\
& \frac{1}{4}\int_0^t \bar{G}(t-x)\left[\begin{array}{c} \frac{2}{3}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2\right]+ \\ \frac{1}{3}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+3\left(\frac{F(t)-F(x)}{\bar{F}(x)}\right)^2\frac{\bar{F}(t)}{\bar{F}(x)}\right] \end{array}\right] dF_{1:4}(x)+ \\
& \frac{7}{12}\frac{3}{7}\int_0^t \bar{G}(t-x)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 dF_{2:4}(x)+ \\
& \frac{7}{12}\frac{4}{7}\int_0^t \bar{G}(t-x)\left[\frac{1}{2}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\frac{1}{2}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\left(\frac{F(t)-F(x)}{\bar{F}(x)}\right)\frac{\bar{F}(t)}{\bar{F}(x)}\right]\right] dF_{2:4}(x)+ \\
& \frac{1}{6}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)}dF_{3:4}(x)
\end{aligned}
$$

As it can be seen from the examples it is hard to compute $P(T^w>t)$ for systems having complex structures even they have few components. However for some particular systems of order n it can be computed easily.

Example 6.5. Consider the coherent system with lifetime

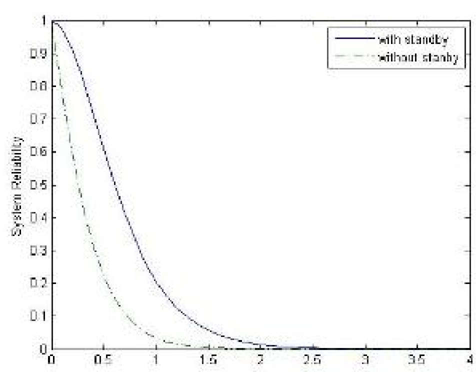
$$T = \min(T_1, \max(T_2, T_3, \dots, T_n)).$$

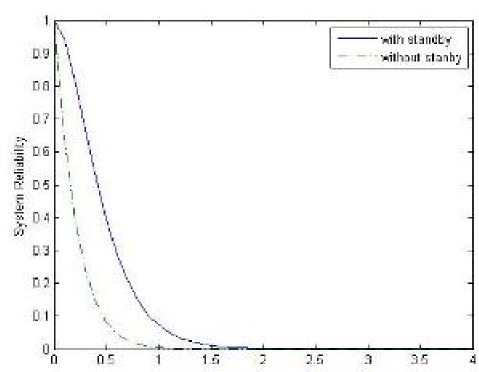
The signature of this system is $p = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{2}{n}, 0)$ In this system $k_\phi = 1$ and $z_\phi = n-2$. For $s = 1, 2, \dots, n-2$, $P(V_s = 1) = 1$ and $P(V_s = c) = 0$, $c = 2, \dots, n$. For $s = n-1$ $P(V_{n-1} = 1) = \frac{1}{2}$ and $P(V_{n-1} = c) = \frac{1}{2(n-1)}$ $c = 2, 3 \dots, n$. Furthermore $\bar{p}^{c, \mathbf{B}_{s,c}} = (\underbrace{0, 0, \dots, 0}_{n-s-1} 1, 0)$ for all s and c. Therefore

$$P(T^w > t) = \sum_{s=1}^{n-1} \left(p_s P(T_{s:n} > t) + p_s \int_0^t \bar{G}(t-x) P(T^{(s)}_{n-s:n-s} > t - x | T_{s:n} = x) dF_{s:n}(x) \right),$$

In general, the computation of $P(T^w > t)$ is not easy even when components have exponential lifetime distributions. In figure 1 one can see the reliability function of the four examples given above with and without a standby when $F(t) = G(t) = 1 - e^{-2t}$, $t > 0$.

Below graphs of reliability functions of the examples 5.1 to 5.4 given respectively.





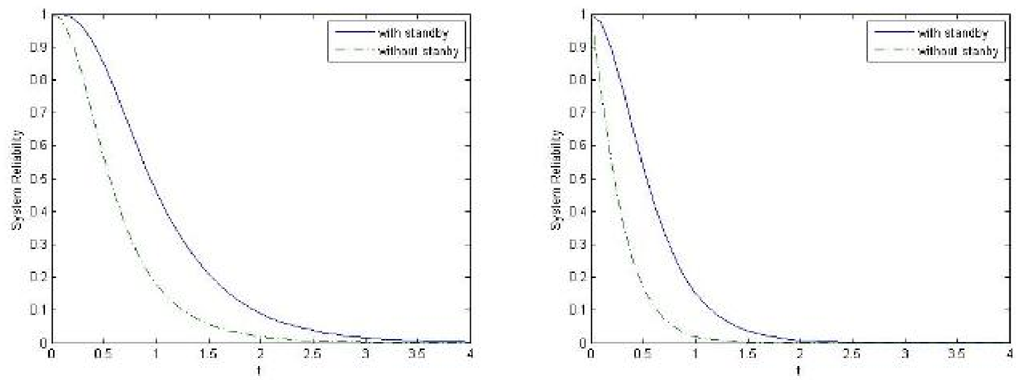


Fig.1 Reliability function of systems with and without standby

In the following table, the mean time to failure of different coherent systems with a standby unit ($E(T^w)$) and without a standby unit ($E(T)$) having independent and identical exponentially distributed components with mean 1, have been computed.

	T	$\mathbf{p}$	$E(T)$	$E(T^w)$
1	$\min(T_1, \max(T_2, T_3))$	$(\frac{1}{3}, \frac{2}{3}, 0)$	0.6667	1.2222
2	$\max(\min(T_1, T_2, T_3), \min(T_2, T_3, T_4))$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	0.4167	0.7917
3	$\min(\max(T_1, T_2, T_3), \max(T_2, T_3, T_4), \max(T_3, T_4, T_5))$	$(0, 0, \frac{3}{10}, \frac{1}{2}, \frac{2}{10})$	1.3333	2.0944
4	$\min(T_1, \max(T_2, T_3), \max(T_2, T_4))$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$	0.5833	1.0625
5	$\min(\max(T_1, T_2), \max(T_2, T_3), \max(T_3, T_4))$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	0.8333	1.3611
6	$\min(\max(T_1, T_2), \max(T_1, T_3), \max(T_1, T_4))$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	1.0833	1.9167
7	$\min(\max(T_1, T_2), \max(T_2, T_3), \max(T_3, T_4), \max(T_4, T_5))$	$(0, \frac{4}{10}, \frac{5}{10}, \frac{1}{10}, 0)$	0.7000	1.1417

Table 1 Mean time to failure of systems with and without standby

It can be seen clearly from the Table 1 that the mean time to failure of coherent systems are nearly doubled by adding a cold standby component which shows the effect of the cold standby to coherent systems.

Chapter 7

Weighted Systems with a Cold Standby

Notation

We will use the following notation throughout this chapter:

n : Number of active components in the system.

$C = \{1, 2, \ldots, n\}$: The index set of the active components in the system.

C_1 : the set of components with weight w

C_2 : the set of components with weight w^*

w_c : the weight of the cold standby component

n_1 : the number of components in C_1

n_2 : the number of components in C_2

k : minimum required weight/capacity for the functioning system

$T_i^{(1)}$: lifetime of the component i, $i \in C_1$

$T_i^{(2)}$: lifetime of the component i, $i \in C_2$

$T_{r:n}$: rth order statistics $r \in C$

M : random variable showing the number of failed components from C_1 at the time of rth failure

$\overline{F}$: survival function of the components in C_1

F : cumulative distribution function of the components in C_1

f : probability density function of the components in C_1

$\overline{G}$: survival function of the components in C_2

G : cumulative distribution function of the components in C_2

g : probability density function of the components in C_2

$\overline{S}$: survival function of the cold standby component

$T_k^{\{n_1,n_2\}}$: lifetime of the system without cold standby

$T_k^{c,\{n_1,n_2\}}$:lifetime of the system with a cold standby

$T_k^{\{n_1,n_2\}}|T_{r:n}, M$: remaining lifetime of the system given $T_{r:n}$ and M

In this study, weighted $k-$out-of$-n$: G systems consisting of two type of components and a cold standby component have been considered. The main difference of this system from regular $k-$out-of$-n$: G systems is that $k-$out-of$-n$: G system fails when $(n-k+1)$th component fails. However weighted $k-$out-of$-n$: G systems may fail upon the failure of the rth component ($r = 1, 2, \ldots, n$) in the system depending on the weights of the components and the threshold k. Another difference is in the lifetime distribution of the components. In this study, the proposed system consists of two types of components. In this setup, when system failure caused by the rth failure (failed component can be either first type or second) $r = 1, 2, \ldots, n$, cold standby component is put into operation and system continues to function with remaining $n-r$ components and a standby component. Therefore reliability calculations of the proposed system is more complex than the reliability calculations of usual $k-$out-of$-n : G$ system with a cold standby.

Now, main assumptions which are used for modeling a weighted $k-$out-of$-n$: G system consisting of two types of components and a cold standby component are given. These assumptions are:

1. The system consists of n independent binary state components and an independent binary state cold standby component.
2. The components are categorized into two groups with respect to their capacity/weights. In addition, there exist a single cold standby component with distinct capacity/weight and reliability
3. The system works if the total weight of the operating components exceeds a predefined threshold.

This modeling and assumptions yield the system reliability calculation which will be given throughout this section.

Eryilmaz and Sarıkaya [27] derived the following equation for the survival function of the system without a standby unit.

$$P(T_k^{\{n_1,n_2\}} > t) = \sum\sum_{\substack{wi+w^*j\geq k \\ 0\leq i\leq n_1, 0\leq j\leq n_2}} \binom{n_1}{i}\overline{F}(t)^i F(t)^{n_1-i}\binom{n_2}{j}\overline{G}(t)^j G(t)^{n_2-j} \qquad (7.1)$$

If the system is equipped with a cold standby component it is obvious that the computation of the system's survival function gets more complicated. Because if there exists a cold standby component, the system has a chance to continue its functioning with the remaining unfailed components and the cold standby component. In order to find the reliability of the system one should consider how many of type 1 and type 2 components are still functioning as well as their remaining lifetimes when the cold standby component becomes active.

Now assume that when the rth failure causes system failure $r = 1, 2, \ldots, n$ at time x $(x < t)$ $(T_{r:n} = x)$ cold standby component starts functioning. In this case, for the system to continue its operation the cold standby component must function between time points t and x, since failure of the cold standby component will lead to system failure. Therefore after the rth failure, the system will have a total of $n - r$ remaining components and a cold standby component. When cold standby component enters the system the total weight of the remaining $n - r$ components should exceed the threshold $k - w_c$. Thus the remaining life of the system can be defined through weighted $(k - w_c)$-out-of-$(n - r) : G$ system. By defining a discrete random variable M, which shows the number of failed type 1 components at the time of rth failure, the reliability of the remaining lifetime can be found. Given $T_{r:n} = x$ and $M = m$, since the residual lifetimes of the remaining components are independent, the reliability of the remaining lifetime of the system consisting of $n_1 - m$ from C_1 and $n_2 - r + m$ components from C_2 can be computed as follows

$$
\begin{aligned}
&P(T_{k-w_c}^{\{n_1-m,n_2-r+m\}} > t - x | T_{r:n} = x, M = m) \\
&= \sum_{\substack{wl+w^* s \geq k-w_c \\ 0 \leq l \leq n_1-m, 0 \leq s \leq n_2-r+m}} \sum \binom{n_1 - m}{l} \left(\frac{\overline{F}(t)}{\overline{F}(x)}\right)^l \left(1 - \frac{\overline{F}(t)}{\overline{F}(x)}\right)^{n_1-m-l} \\
&\qquad \times \binom{n_2 - r + m}{s} \left(\frac{\overline{G}(t)}{\overline{G}(x)}\right)^s \left(1 - \frac{\overline{G}(t)}{\overline{G}(x)}\right)^{n_2-r+m-s}
\end{aligned} \tag{7.2}
$$

In general the reliability of the system with cold standby component can be found as follows

$$
\begin{aligned}
&P(T_k^{c,\{n_1,n_2\}} > t) = P(T_k^{\{n_1,n_2\}} > t) \\
&+ \sum_{r=1}^{n} \sum_{m=\max(0,a)}^{\min(n_1-1,b_1)} \left(\int_0^\infty n_1 f(x) \binom{n_1 - 1}{m} F(x)^m \binom{n_2}{r-1-m} G(x)^{r-1-m} \overline{F}(x)^{n_1-1-m} \overline{G}(x)^{n_2-r+m+1} dx \right. \\
&\times \left. \int_0^t \overline{S}(t-x) P(T_{k-w_c}^{\{n_1-m-1,n_2-r+m+1\}} > t - x | T_{r:n} = x, M = m+1) h_{(r)}(x) dx \right) \\
&+ \sum_{r=1}^{n} \sum_{m=\max(0,a)}^{\min(n_1,b_2)} \left(\int_0^\infty n_2 g(x) \binom{n_1}{m} F(x)^m \binom{n_2 - 1}{r-1-m} G(x)^{r-1-m} \overline{F}(x)^{n_1-m} \overline{G}(x)^{n_2-r+m} dx \right. \\
&\times \left. \int_0^t \overline{S}(t-x) P(T_{k-w_c}^{\{n_1-m,n_2-r+m\}} > t - x | T_{r:n} = x, M = m) h_{(r)}(x) dx \right)
\end{aligned} \tag{7.3}
$$

where $a = \left\lceil \frac{k-n_1w-(n_2+1-r)w^*}{w^*-w} \right\rceil$, $b_1 = \left\lceil \frac{k-(n_1-1)w-(n_2+1-r)w^*}{w^*-w} - 1 \right\rceil$,and $b_2 = \left\lceil \frac{k-n_1w-(n_2-r)w^*}{w^*-w} - 1 \right\rceil$
In equation (5.3) the first term in the summation, $P(T_k^{\{n_1,n_2\}} > t)$, indicates the probability that system survives up to time t without the cold standby component. The second(third) term in the summation is the probability that the system fails before time t due to the failure of type 1(2) components and survived up to time t after the cold standby component becomes active. It should be noted this formula generalizes the formula in Eryilmaz [23] if all components have the same weight of one.

Proof of Equation 5.3. Eryilmaz [25] defined the weight of the component which has the shortest lifetime as $w_{[1]}$. $w_{[1]}$ is a random variable rather than a fixed number and we have the following relation

$$\{w_{[1]} = w_i\} \text{ iff } \{T_{1:n} = T_i\}.$$

In general if $w_{[r]}$ denotes the weight associated with the component which has rth smallest lifetime then

$$\{w_{[r]} = w_i\} \text{ iff } \{T_{r:n} = T_i\}$$

By the law of total probability

$$P(T_k^{\{n_1,n_2\}} = T_{r:n}) = P(T_k^{\{n_1,n_2\}} = T_{r:n}, w_{[r]} = w) + P(T_k^{\{n_1,n_2\}} = T_{r:n}, w_{[r]} = w^*)$$

The event, $(T_k^{\{n_1,n_2\}} = T_{r:n}, w_{[r]} = w)$ implies that there are totally $r-1$ failed components which do not cause system failure and the system fails upon the failure of the component which has the rth smallest lifetime and weight w. The event $(T_k^{\{n_1,n_2\}} = T_{r:n}, w_{[r]} = w^*)$ can be defined similarly.
Without loss of generality let $w^* > w$, and assume that $M = m$ of $r-1$ failed components have weight w. So for the first probability $P(T_k^{\{n_1,n_2\}} = T_{r:n}, w_{[r]} = w)$,

$$(n_1-1)w+n_2w^*-k<mw+(r-1-m)w^*\leq n_1w+n_2w^*-k \text{ and } 0\leq m\leq n_1-1$$
$$\max(0,\left\lceil\frac{k-n_1w-(n_2+1-r)w^*}{w^*-w}\right\rceil\leq m\leq \min(n_1-1,\left\lceil\frac{k-(n_1-1)w-(n_2+1-r)w^*}{w^*-w}-1\right\rceil)$$

Similarly for the second probability $P(T_k^{\{n_1,n_2\}}=T_{r:n},w_{[r]}=w^*)$,

$$\max(0,\left\lceil\frac{k-n_1w-(n_2+1-r)w^*}{w^*-w}\right\rceil\leq m\leq \min(n_1,\left\lceil\frac{k-n_1w-(n_2-r)w^*}{w^*-w}-1\right\rceil) \text{ and } 0\leq m\leq n_1$$

By conditioning on M, we have

$$P(T_k^{\{n_1,n_2\}}=T_{r:n},w_{[r]}=w)=\sum_{m=\max(0,a)}^{\min(n_1-1,b_1)}\int_0^\infty n_1f(x)\binom{n_1-1}{m}F(x)^m\binom{n_2}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-1-m}\overline{G}(x)^{n_2-r+m+1}dx$$

$$P(T_k^{\{n_1,n_2\}}=T_{r:n},w_{[r]}=w^*)=\sum_{m=\max(0,a)}^{\min(n_1,b_2)}\int_0^\infty n_2g(x)\binom{n_1}{m}F(x)^m\binom{n_2-1}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-m}\overline{G}(x)^{n_2-r+m}dx$$

where $a=\left\lceil\frac{k-n_1w-(n_2+1-r)w^*}{w^*-w}\right\rceil$, $b_1=\left\lceil\frac{k-(n_1-1)w-(n_2+1-r)w^*}{w^*-w}-1\right\rceil$ and $b_2=\left\lceil\frac{k-n_1w-(n_2-r)w^*}{w^*-w}-1\right\rceil$.
Therefore the reliability of the system with cold standby is

$$\begin{aligned}
&P(T_k^{c,\{n_1,n_2\}}>t)=P(T_k^{\{n_1,n_2\}}>t)\\
&+\sum_{r=1}^{n}\sum_{m=\max(0,a)}^{\min(n_1-1,b_1)}\Bigg(\int_0^\infty n_1f(x)\binom{n_1-1}{m}F(x)^m\binom{n_2}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-1-m}\overline{G}(x)^{n_2-r+m+1}dx\\
&\times\int_0^t\overline{S}(t-x)P(T_{k-w_c}^{\{n_1-m-1,n_2-r+m+1\}}>t-x|T_{r:n}=x,M=m+1)h_{(r)}(x)dx\Bigg)\\
&+\sum_{r=1}^{n}\sum_{m=\max(0,a)}^{\min(n_1,b_2)}\Bigg(\int_0^\infty n_2g(x)\binom{n_1}{m}F(x)^m\binom{n_2-1}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-m}\overline{G}(x)^{n_2-r+m}dx\\
&\times\int_0^t\overline{S}(t-x)P(T_{k-w_c}^{\{n_1-m,n_2-r+m\}}>t-x|T_{r:n}=x,M=m)h_{(r)}(x)dx\Bigg)
\end{aligned}$$

$\square$

Example 7.1. Let $n = 3, n_1 = 2$, $n_2 = 1, w = 1$ $w^* = 2$, $w_c = 1$ and $k = 3$.

				Remaining Components and total weight		
	Ordering	System failure	M	Type 1	Type 2	RemainingTotal Weight
1	$T_1^{(1)} < T_2^{(1)} < T_1^{(2)}$	$T_3^{\{2,1\}} = T_{2:3}$	2	–	1	2
2	$T_1^{(2)} < T_1^{(1)} < T_2^{(1)}$	$T_3^{\{2,1\}} = T_{1:3}$	0	2	–	2
3	$T_1^{(1)} < T_1^{(2)} < T_2^{(1)}$	$T_3^{\{2,1\}} = T_{2:3}$	1	1	–	1
4	$T_2^{(1)} < T_1^{(1)} < T_1^{(2)}$	$T_3^{\{2,1\}} = T_{2:3}$	2	–	1	2
5	$T_1^{(2)} < T_2^{(1)} < T_1^{(1)}$	$T_3^{\{2,1\}} = T_{1:3}$	0	2	–	2
6	$T_2^{(1)} < T_1^{(2)} < T_1^{(1)}$	$T_3^{\{2,1\}} = T_{2:3}$	1	1	–	1

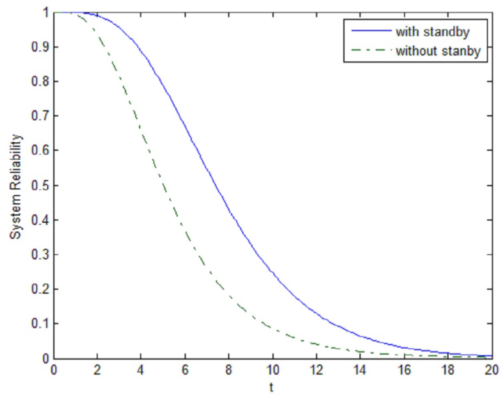
Since the weight of the cold standby component is 1, for the cases $1, 2, 4$ and 5 total weight of the remaining components and cold standby components is 3. Therefore after the system failure in these two cases the system continue to function with the remaining components and cold standby. However in the cases 3 and 6 total weight of the remaining components and cold standby is 2 which is less than k so in this case cold standby component can not prevent system failure. Using equation 2 the reliability of the remaining life of the system for the first and second case are given as

$$P(T_{3-1}^{\{0,1\}} > t - x | T_{2:3} = x, M = 2) = \frac{\overline{G}(t)}{\overline{G}(x)}$$
$$P(T_{3-1}^{\{2,0\}} > t - x | T_{1:3} = x, M = 0) = \left(\frac{\overline{F}(t)}{\overline{F}(x)}\right)^2$$

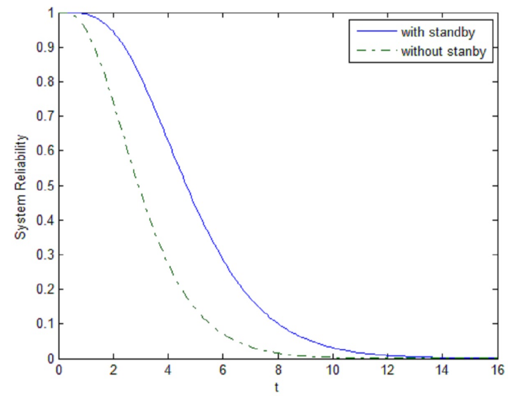
Example 7.2. Let $n = 4, n_1 = 2$, $n_2 = 2, w = 2$ $w^* = 1$, $w_c = 3$ and $k = 5$.

$$
\begin{aligned}
&P(T_5^{c,\{2,2\}} > t) = P(T_5^{\{2,2\}} > t)\\
&+ \int_0^\infty 2f(x)\overline{F}(x)\overline{G}(x)^2 dx \times \int_0^t \overline{S}(t-x) P(T_2^{\{1,2\}} > t-x | T_{1:4} = x, M = 1) h_{(1)}(x) dx\\
&+ \int_0^\infty 2g(x)G(x)\overline{F}(x)^2 dx \int_0^t \overline{S}(t-x) P(T_2^{\{2,0\}} > t-x | T_{2:4} = x, M = 0) h_{(2)}(x) dx\\
&+ \int_0^\infty 4f(x)G(x)\overline{F}(x)\overline{G}(x) dx \int_0^t \overline{S}(t-x) P(T_2^{\{1,1\}} > t-x | T_{2:4} = x, M = 1) h_{(2)}(x) dx\\
&= \overline{F}(t)^2\overline{G}(t)\left(2G(t) + \overline{G}(t)\right)\\
&+ \int_0^\infty 2f(x)\overline{F}(x)\overline{G}(x)^2 dx \int_0^t \overline{S}(t-x) \left(\frac{\overline{F}(t)}{\overline{F}(x)} + \left(1 - \frac{\overline{F}(t)}{\overline{F}(x)}\right)\left(\frac{\overline{G}(t)}{\overline{G}(x)}\right)^2\right) h_{(1)}(x) dx\\
&+ \int_0^\infty 2g(x)G(x)\overline{F}(x)^2 dx \int_0^t \overline{S}(t-x) \left(1 - \left(1 - \frac{\overline{F}(t)}{\overline{F}(x)}\right)^2\right) h_{(2)}(x) dx\\
&+ \int_0^\infty 4f(x)G(x)\overline{F}(x)\overline{G}(x) dx \int_0^t \overline{S}(t-x) \frac{\overline{F}(t)}{\overline{F}(x)} h_{(2)}(x) dx
\end{aligned}
$$

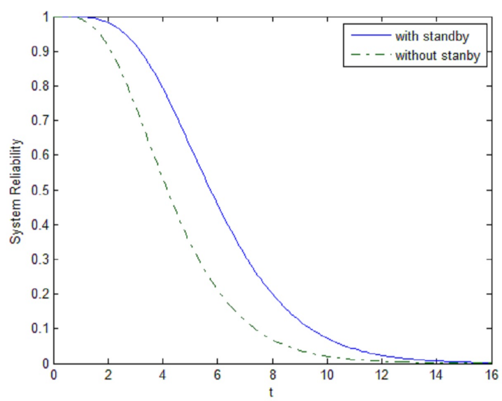
Below reliability function of different weighted k-out-of-n:G systems for $\overline{F}(t) = e^{-0.1t}, \overline{G}(t) = e^{-0.2t}, \overline{S}(t) = e^{-0.3t}$, $w = 1$, $w^* = 2$ and $w_c = 3$ are provided.



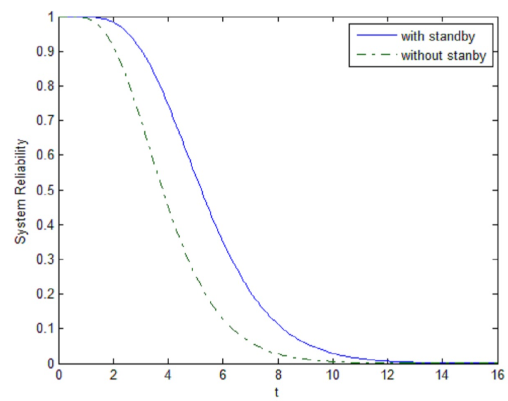
$T_6^{\{3,5\}}$ and $T_6^{c,\{3,5\}}$



$T_9^{\{7,3\}}$ and $T_9^{c,\{7,3\}}$



$T_{12}^{\{5,7\}}$ and $T_{12}^{c,\{5,7\}}$



$T_{13}^{\{9,6\}}$ and $T_{13}^{c,\{9,6\}}$

Figure 1 Reliability function of weighted k-out-of-n:G systems with and without cold standby

The mean time to failure (MTTF) is one of the most important reliability characteristics of the systems. MTTF of a system with lifetime T can be computed with the following equation

$$\mathrm{MTTF} = E\left(T\right) = \int_0^\infty P\left\{T > t\right\} dt.$$

For a weighted k-out-of-n:G system consisting of two types components and a cold standby component, MTTF of the system is

$$E(T_k^{c,\{n_1,n_2\}}) = E(T_k^{\{n_1,n_2\}}) + E(T_{k,w_c}^{\{n_1,n_2\}})$$

where $E(T_k^{\{n_1,n_2\}})$ is the MTTF of the system when the cold standby component is inactive and $E(T_{k,w_c}^{\{n_1,n_2\}})$ is the MTTF of the system when the cold standby component is active. They can be computed from

$$E(T_k^{\{n_1,n_2\}}) = \sum_{\substack{wi+w^*j\geq k \\ 0\leq i\leq n_1, 0\leq j\leq n_2}}\sum \binom{n_1}{i}\binom{n_2}{j}\int_0^\infty \overline{F}(t)^i F(t)^{n_1-i}\overline{G}(t)^j G(t)^{n_2-j}dt$$

and

$$\begin{aligned}E(T_{k,w_c}^{\{n_1,n_2\}}) = &\sum_{r=1}^{n}\Bigg(\sum_{m=\max(0,a)}^{\min(n_1-1,b_1)}\int_0^\infty n_1 f(x)\binom{n_1-1}{m}F(x)^m\binom{n_2}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-1-m}\overline{G}(x)^{n_2-r+m+1}dx \\ &\times\int_0^\infty\int_0^t \overline{S}(t-x)P(T_{k-w_c}^{\{n_1-1-m,n_2-r+m+1\}} > t-x|T_{r:n}=x, M=m+1)h_{(r)}(x)dxdt\Bigg) \\ &+\sum_{r=1}^{n}\Bigg(\sum_{m=\max(0,a)}^{\min(n_1,b_2)}\int_0^\infty n_2 g(x)\binom{n_1}{m}F(x)^m\binom{n_2-1}{r-1-m}G(x)^{r-1-m}\overline{F}(x)^{n_1-m}\overline{G}(x)^{n_2-r+m}dx \\ &\times\int_0^\infty\int_0^t \overline{S}(t-x)P(T_{k-w_c}^{\{n_1-m,n_2-r+m\}} > t-x|T_{r:n}=x, M=m)h_{(r)}(x)dxdt\Bigg)\end{aligned}$$

In Table 5.1 exact and simulated MTTF of weighted k-out-of-n:G system with and without a cold standby are presented for $\overline{F}(t) = e^{-0.2t}, \overline{G}(t) = e^{-0.1t}, \overline{S}(t) = e^{-0.15t}$, $w = 1$, $w^* = 2$ and $w_c = 3$ for different values of n_1, n_2 and k. Monte Carlo estimates of MTTF of weighted k-out-of-n:G system with $(E(T_k^{\{n_1,n_2\}})_{sim})$ and without $(E(T_k^{c,\{n_1,n_2\}})_{sim})$ a cold standby are obtained. All simulation results are based on 50000 repetitions.

n_1	n_2	k	$E(T_k^{\{n_1,n_2\}})$	$E(T_k^{c,\{n_1,n_2\}})$	$E(T_k^{\{n_1,n_2\}})_{sim}$	$E(T_k^{c,\{n_1,n_2\}})_{sim}$
3	5	6	8.2771	11.5271	8.2472	11.5134
6	2	6	4.3765	7.6977	4.3635	7.6937
5	5	7	7.1507	10.5112	7.1565	10.5245
8	2	7	4.1650	6.9918	4.1659	6.9943
7	5	10	4.6792	6.8926	4.6868	6.9090
9	3	10	3.2347	5.2468	3.2366	5.2678

Table 7.1: Simulated and exact MTTF of weighted k-out-of-n:G systems with and without a cold standby

As a special case when all components including the cold standby component are i.i.d and have weight 1 the system turns into ordinary k-out-of-$n : G$ system with a cold standby component. Let $\overline{F} = \overline{G} = \overline{S} = e^{-0.1t}$ and $w = w^* = w_c = 1$.

n	k	$E(T_k^n)$	$E(T_k^{c,n})$
3	2	8.3333	13.3333
5	2	12.8333	17.8333
5	3	7.8333	11.1667
10	3	14.2897	17.6230
10	5	8.4563	10.4563
15	7	8.6823	10.1109

Table 7.2: MTTF of k-out-of-n:G systems with and without a cold standby

The results in Table 5.2 coincides with the ones given in [23].

A way to increase the reliability of the system is to use one of the existing components as a cold standby component. Below the MTTF values of different weighted k-out-of-n:G systems are computed for $\overline{F}(t) = e^{-0.2t}, \overline{G}(t) = e^{-0.1t}, w = 1$ and $w^* = 2$. It can be seen from Table 5.3 that using one of the components

n_1	n_2	k	$E(T_k^{\{n_1,n_2\}})$	$E(T_k^{c,\{n_1-1,n_2\}})$	$E(T_k^{c,\{n_1,n_2-1\}})$
3	5	6	8.2771	8.7195	9.6865
6	2	6	4.3765	4.9060	4.8820
5	5	7	7.1507	7.9209	8.1108
8	2	7	4.1650	4.6059	4.6026
7	5	10	4.6792	5.0657	5.1304
9	3	10	3.2347	3.5014	3.4863

Table 7.3: MTTF of weighted k-out-of-n:G systems with cold standby replacement

as a cold standby component without changing the total number of components in the systems increases the MTTF of the systems dramatically. However the selection of the component type to be used as a cold standby component heavily depends on the number of components in the system $(n_{1,}n_2)$ and the threshold k.

7.1 Optimizing System Configuration of Weighted k-out-of-n:G systems

Weighted k-out-of-$n:G$ systems arise in many real life problems such as logistics, lightning systems, heating and ventilation, and load and capacity problems. Engineers usually tries to decrease system cost subject to several constraints such as reliability, capacity and weight. Improvement can be done in different ways; by using more reliable components, adding redundant components to the system and using standby components. If all these options are available, then engineer faces an optimization problem which can be formulated as a nonlinear mixed integer programming problem. In this paper, for given number of components (n) we decide optimal allocation of two types of components n_1, n_2 and whether to use a cold standby component to minimize system cost by satisfying mean time to failure requirement. Let c_1 and c_2 denote the cost of one element in the first and second group, respectively. Moreover, c_s denotes the cost of cold standby component. If e_0 is the minimum required mean time to failure of the system and y_1 is a binary variable such that

$$y_1 = \begin{cases} 1, & \text{if cold standby having weight } w_c \text{ is used in the system.} \\ 0, & \text{otherwise} \end{cases}$$

then the reliability optimization problem can be formulated as

$$\begin{aligned} &\min n_1c_1 + n_2c_2 + y_1c_s \\ &s.t \\ &E(T_k^{c,\{n_1,n_2\}}) = \Big(E(T_k^{\{n_1,n_2\}}) + y_1E(T_{k,w_c}^{\{n_1,n_2\}})\Big) \geq e_0 \\ &n_1, n_2 \geq 0 \text{ and integers} \\ &y_1 \text{ is binary variable} \end{aligned}$$

Example 7.3. Let $n = 10$ $k = 7$ $w = 1, w^* = 2$, $w_c = 3$, $c_1 = 2$, $c_2 = 5$, $c_s = 6$ $\overline{F}(t) = e^{-0.2t}, \overline{G}(t) = e^{-0.1t}$ and $\overline{S}(t) = e^{-0.15t}$. The minimum required MTTF is $e_0 = 5.5$. Since $c_1 < c_2 < c_s$ it seems reasonable to use component of type 1 as many as possible to minimize the system cost. If the manager decides not to use a cold standby component in order to satisfy the minimum required MTTF $n_1 = 6$ and $n_2 = 4$. Because

$$E(T_7^{\{10,0\}}) = 2.3948$$
$$E(T_7^{\{9,1\}}) = 3.2333$$
$$E(T_7^{\{8,2\}}) = 4.165$$
$$E(T_7^{\{7,3\}}) = 5.1649$$
$$E(T_7^{\{6,4\}}) = 6.1769$$

In this case, total cost of the system is $(6 \times 2) + (5 \times 4) = 32$. Now if the manager uses cold standby component then

$$E(T_7^{c,\{10,0\}}) = 4.7221$$
$$E(T_7^{c,\{9,1\}}) = 5.8364$$

In this case total cost of the system is $(9 \times 2) + (5 \times 1) + 6 = 29$. The cost is minimized by using cold standby component, 9 components of weight $w = 1$ and one component of weight $w^* = 2$.

If the decision maker has several cold standby components with different weights costs and lifetime distributions and if one has to decide whether to use one of them or not, in this case another optimization problem occurs. Let there exist m cold standby components with cost c_{s_j}, weights w_{c_j} and survival functions $\overline{S}_j$, $j = 1, 2, \ldots, m$. Denote y_j as binary variables $j = 1, 2, \ldots, m$ such that

$$y_j = \begin{cases} 1, & \text{if cold standby having weight } w_{c_j} \text{ is used in the system.} \\ 0, & \text{otherwise} \end{cases}$$

then the problem can be formulated as follows

$$\min n_1c_1 + n_2c_2 + \sum_{j=1}^{m} y_j c_{s_j}$$

$$s.t$$

$$E(T_k^{c,\{n_1,n_2\}}) = \left(E(T_k^{\{n_1,n_2\}}) + \sum_{j=1}^{m} y_j E(T_{k,w_{c_j}}^{\{n_1,n_2\}}) \right) \geq e_0$$

$$\sum_{j=1}^{m} y_j \leq 1$$

$n_1, n_2 \geq 0$ and n_1, n_2 are integers

$y_1, y_2, \ldots, y_m$ are binary variables

where

$$\begin{aligned}
E(T_{k,w_{c_j}}^{\{n_1,n_2\}}) = &\sum_{r=1}^{n} \Bigg(\sum_{m=\max(0,a)}^{\min(n_1-1,b)} \int_0^{\infty} n_1 f(x) \binom{n_1-1}{m} F(x)^m \binom{n_2}{r-1-m} G(x)^{r-1-m} \overline{F}(x)^{n_1-1-m} \overline{G}(x)^{n_2-r+m+1} dx \\
&\times \int_0^{\infty}\int_0^{t} \overline{S}_j(t-x) P(T_{k-w_{c_j}}^{\{n_1-1-m,n_2-r+1+m\}} > t-x | T_{r:n} = x, M = m+1) h_{(r)}(x) dx \Bigg) \\
&+ \sum_{r=1}^{n} \Bigg(\sum_{m=\max(0,a)}^{\min(n_1,d)} \int_0^{\infty} n_2 g(x) \binom{n_1}{m} F(x)^m \binom{n_2-1}{r-1-m} G(x)^{r-1-m} \overline{F}(x)^{n_1-m} \overline{G}(x)^{n_2-r+m} dx \\
&\times \int_0^{\infty}\int_0^{t} \overline{S}_j(t-x) P(T_{k-w_{c_j}}^{\{n_1-m,n_2-r+m\}} > t-x | T_{r:n} = x, M = m) h_{(r)}(x) dx \Bigg)
\end{aligned}$$

Example 7.4. Let $n = 8$ $k = 6$ and $m = 3$ which means there exist 3 different cold standby components. Suppose $w = 1, w^* = 2$, $w_{c_1} = 2$, $w_{c_2} = 3$ and $w_{c_3} = 4$, $c_1 = 2$, $c_2 = 5$, $c_{s_1} = 7$ $c_{s_2} = 8$ $c_{s_3} = 11$, $\overline{F}(t) = e^{-0.2t}, \overline{G}(t) = e^{-0.1t}$ and $\overline{S}_1(t) = e^{-0.2t}$, $\overline{S}_2(t) = e^{-0.15t}, \overline{S}_3(t) = e^{-0.25t}$. The minimum required MTTF is $e_0 = 7.5$. If the decision maker conclude not to use a cold standby component in order to satisfy the minimum required MTTF $n_1 = 3$ and $n_2 = 5$. Because

$$E(T_6^{\{8,0\}}) = 2.1726$$
$$E(T_6^{\{7,1\}}) = 3.2011$$
$$E(T_6^{\{6,2\}}) = 4.3765$$
$$E(T_6^{\{5,3\}}) = 5.6277$$
$$E(T_6^{\{4,4\}}) = 6.9358$$
$$E(T_6^{\{3,5\}}) = 8.2771$$

In this case total cost of the system is $(3 \times 2) + (5 \times 5) = 31$. If first cold standby component is used than

$$E(T_6^{c,\{8,0\}}) = 3.8393$$
$$E(T_6^{c,\{7,1\}}) = 5.0860$$
$$E(T_6^{c,\{6,2\}}) = 6.4133$$
$$E(T_6^{c,\{5,3\}}) = 7.8121$$

Total cost of the system is $(5 \times 2) + (3 \times 5) + 7 = 32$. Similarly if second cold standby is used the allocation with minimum cost and satisfying MTTF constraint is $n_1 = 6$, $n_2 = 2$ and total cost is $(6 \times 2) + (2 \times 5) + 8 = 30$. Finally if third standby is used the allocation is again $n_1 = 6$, $n_2 = 2$ and total cost is $(6 \times 2) + (2 \times 5) + 11 = 33$. Hence the cost is minimized using second cold standby component together with 6 components of type 1 and 2 components of type 2.

It can be seen from the examples that even the cost of cold standby component is greater than the usual components, the cold standby component is used in the optimum solutions. However there is no general rule of the optimum solution since the optimum solution depends on the cost of both cold standby and usual components as well as their weights and lifetime distributions.

In most of the works that have been done in reliability theory systems having standby components are assumed to have independent and identical components. In this work, components have been classified in two groups with respect to their weights and lifetime distributions. Moreover, the system is equipped with a single cold standby component which also has a different lifetime distribution function and weight. This assumption makes the problem more realistic but at the same time it is harder to compute the reliability of the system. A nonlinear and complex reliability function makes the system modeling and configuration problem even harder to solve. As a future work one can propose optimization techniques along with heuristic algorithms in order to address this problem. Furthermore one can study the same problem where all components and the cold standby component may have different weights and lifetime distributions.

Chapter 8

Conclusion

In this thesis a method for computing the system reliability of coherent systems with a cold standby component based on system signatures is presented. Even though reliability calculations of coherent systems having complex structures and a cold standby component is difficult, with this proposed method for some important general coherent systems explicit formulas for computing the reliability of those systems can be found. Moreover this study is the first attempt to find a general way to compute the reliability of all coherent systems with a cold standby component. Furthermore in this thesis reliability calculations have been made for systems having independent and non identical components and an independent cold standby component specifically for weighted k-out-of-n systems containing two different type of components and a cold standby component. As a future work, these results can be generalized for system having dependent components or systems having more than two different type of components. Furthermore these findings can be used to find the reliability of systems having more than one cold standby component.

BIBLIOGRAPHY

[1] M. Ahsanullah and V.B. Nevzorov, *Ordered Random Variables*, Nova Science Publishers, Huntington New York, 2001.

[2] M. Ahsanullah and V.B. Nevzorov, *Ordered Statistics. Examples and Excercises*, Nova Science Publishers, Huntington New York, 2005.

[3] B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, *A First Course in Order Statistics*, Wiley, New York, 1992.

[4] M. Asadi, and I. Bayramoglu (Bairamov), On the mean residual life function of the k-out-of-n systems at the system level, IEEE Transactions on Reliability **55** (2006), 314–318.

[5] I. Bairamov and B.C. Arnold, On the residual life lengths of the remaining components in an $n-k+1$-out-of-n system. Statistics & Probability Letters **78** (2008), 945-952.

[6] I. Bairamov and S. Parsi, Order statistics from mixed exchangeable random variables, Journal of Computational and Applied Mathematics **35** (2011), 4629-4638.

[7] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing, Probability Models*, Holt, Rinehart & Winston, 1975.

[8] P. J. Boland, Signatures of indirect majority systems, Journal of Applied Probability **38** (2001), 597–603.

[9] Y. Chen and Q. Yang, Reliability of two-stage weighted k-out-of-n system with components in common, IEEE Transactions on Reliability **54** (2005), 431–440.

[10] F.P.A. Coleen and T. Coleen-Maturi, Generalizing the signature to systems with multiple types of components, In: W. Zamojski et al (Eds.), Complex Systems and Dependability, Springer, Berlin Heidelberg, (2012), 115–130.

[11] L. Cui and M. Xie, On a generalized k-out-of-n system and its reliability, International Journal of Systems Science **36** (2005), 267–274.

[12] G. Da, L. Xia and T. Hu, On computing signatures of k-out-of-n systems consisting of modules, Methodology and Computing in Applied Probability 16 (2014), 223–233.

[13] H.A. David, *Order Statistics*, Wiley, 2nd edn. New York, 1981.

[14] H.A. David and H.N. Nagaraja, *Order Statistics*, Wiley-Interscience, New Jersey, 2003.

[15] S. Eryilmaz, On the lifetime distribution of consecutive k-out-of- n:F system, IEEE Transactions on Reliability **56** (2007), 35–39.

[16] S. Eryilmaz, Lifetime of combined k-out-of-n, and consecutive k_c-out-of-n systems, IEEE Transactions on Reliability **57** (2008), 331–335.

[17] S. Eryilmaz, Reliability properties of consecutive k-out-of-n systems of arbitrarily dependent components, Reliability Engineering and System Safety **94** (2009), 350–356.

[18] S. Eryilmaz and G.Y Tutuncu, Reliability evaluation of linear consecutive-weighted k-out-of-n:F system, Asia Pacific Journal of Operational Research **26** (2009), 805–816.

[19] S. Eryilmaz, Review of recent advances in reliability of consecutive k-out-of-n and related systems, Proceedings of the ImechE, Part O: Journal of Risk and Reliability **224** (2010), 225–237.

[20] S. Eryilmaz, Mixture representations for the reliability of consecutive-k systems, Mathematical and Computer Modelling **51** (2010), 405–412.

[21] S. Eryilmaz, M.V. Koutras and I.S. Triantafyllou, Signature based analysis of m-consecutive-k-out-of-n:F system with exchangeable components, Naval Research Logistics **58** (2011), 344–354.

[22] S. Eryilmaz, Dynamic behavior of k-out-of-n:G systems, Operations Research Letters **39** (2011), 155–159.

[23] S. Eryilmaz, On the mean residual life of a k-out-of-n:G system with a single cold standby component, European Journal of Operational Research **222** (2012), 273–277.

[24] S. Eryilmaz, On reliability of a k-out-of-n system equipped with a single warm standby component, IEEE Transactions on Reliability **62** (2013), 499–503.

[25] S. Eryilmaz, Mean instantaneous performance of a system with weighted components that have arbitrarily distributed lifetimes, Reliability Engineering and System Safety **119** (2013), 290–293.

[26] S. Eryilmaz, A study on reliability of coherent systems equipped with a cold standby component, Metrika **77** (2013), 349–359.

[27] S. Eryilmaz and K. Sarikaya, Modeling and analysis of weighted k-out-of-n:G system consisting of two different types of components. Proceedings of the Institution of Mechanical Engineers, Part O, Journal of Risk and Reliability **228** (2014), 265–271.

[28] S. Eryilmaz and A.R Bozbulut, Computing marginal and joint Birnbaum and Barlow-Proschan importances in weighted k-out-of-n:G systems, Computers & Industrial Engineering **72** (2014a), 255–260.

[29] S. Eryilmaz and A.R Bozbulut, An algorithmic approach for the dynamic reliability analysis of non-repairable multi-state weighted k-out-of-n:G system, Reliability Engineering and System Safety **131** (2014b), 61–65.

[30] S. Eryilmaz, Mean time to failure of weighted k-out-of-n:G systems, Communications in Statistics - Simulation and Computation **44** (2015), 2705–2713.

[31] J. Freixas and M.A. Puente, Consecutive expansions of k-out-of-n systems, Operations Research Letters **37** (2009), 438–442.

[32] S. Gurler and I. Bairamov, Parallel and k-out-of-n:G systems with nonidentical components and their mean residual life functions, Applied Mathematical Modelling **33** (2009), 1116–1125.

[33] Y. Higashiyama, A factored reliability formula for weighted k-out-of-n system, Asia Pacific Journal of Operational Research **18** (2001) 61–66.

[34] S. Kochar, H. Mukerjee and F.J. Samaniego, The signature of a coherent system and its application to comparison among systems, Naval Research Logistics **46** (1999), 507–523.

[35] S. Kochar and M. Xu, On residual lifetimes of k-out-of-n system with nonidentical components, Probability in the Engineering and Informational Sciences **24** (2010), 109–127.

[36] W. Kuo and M. J. Zuo, *Optimal reliability modeling, principles and applications*, New Jersey: John Wiley & Sons, New York, 2003.

[37] X. Li, R. Yan and M.J. Zuo, Evaluating a warm standby system with components having proportional hazard rates, Operations Research Letters **37** (2009), 56–60.

[38] H. Liu, Reliability of a load-sharing k-out-of-n:G system: non-iid components with arbitrary distributions, IEEE Transactions on Reliability **47** (1998), 279–284.

[39] A. Lisnianski and I. Frenkel, *Recent Advances in System Reliability, Signatures, Multistate Systems and Statistical Inference*, Springer, New York, 2011.

[40] J.-L. Marichal and P. Mathonet, Extensions of system signatures to dependent lifetimes: explicit expressions and interpretations, Journal of Multivariate Analysis **102** (2011), 931–936.

[41] J.-L. Marichal, P. Mathonet and T. Waldhauser, On signature-based expressions of system reliability, Journal of Multivariate Analysis **102** (2011), 1410–1416.

[42] J.-L. Marichal and P. Mathonet, Computing system signatures through reliability functions, Statistics & Probability Letters **83** (2013), 710–717.

[43] J.-L. Marichal, P. Mathonet and F. Spizzichino, On modular decompositions of system signatures, Journal of Multivariate Analysis, **134** (2015), 19-32.

[44] J. Navarro, J.M. Ruiz and C.J. Sandoval, A note on comparisons among coherent systems with dependent components using signatures, Statistics and Probability Letters **72** (2005), 179–185.

[45] J. Navarro, and S. Eryilmaz, Mean residual lifetimes of consecutive k-out-of-n:F systems, Journal of Applied Probability **44** (2007), 82–98.

[46] J. Navarro and T. Rychlik, Reliability and expectation bounds for coherent systems with exchangeable components, Journal of Multivariate Analysis **98** (2007), 102–113.

[47] J. Navarro, J.M. Ruiz and C.J. Sandoval, Properties of coherent systems with dependent components, Communications in Statistics – Theory and Methods **36** (2007), 175–191.

[48] J. Navarro and P. J. Hernandez, Mean residual life functions of finite mixtures, order statistics and coherent systems, Metrika **67** (2008b), 277–298.

[49] J. Navarro, A. Guillamon, and M. Ruiz, Generalized mixtures in reliability modelling: applications to the construction of bathtub shaped hazard models and the study of systems, Applied Stochastic Models in Business and Industry **25** (2009), 323–337.

[50] J. Navarro, F.J. Samaniego and N. Balakrishnan, The joint signature of coherent systems with shared components, Journal of Applied Probability **47** (2010), 235–253.

[51] J. Navarro, F. Spizzichino and N. Balakrishnan, Applications of average and projected systems to the study of coherent systems, Journal of Multivariate Analysis **101** (2010), 1471–1482.

[52] J. Navarro, F.J. Samaniego and N. Balakrishnan, Signature-based representations for the reliability of systems with heterogeneous components, Journal of Applied Probability **48** (2011), 856–867.

[53] J. Navarro and R. Rubio, A note on necessary and sufficient conditions for ordering properties of coherent systems with exchangeable components, Naval Research Logistics **58** (2011), 478–489.

[54] J. Navarro, F.J. Samaniego, and N. Balakrishnan, Signature-based representations for the reliability of systems with heterogeneous components, Jornal of Applied Probability **48** (2011), 856–867.

[55] F. J. Samaniego, On closure of the IFR class under formation of coherent systems, IEEE Transactions on Reliability **34** (1985), 69–72.

[56] F. J. Samaniego, *System Signatures and Their Applications in Engineering Reliability*, Springer, New York, 2007.

[57] F. Samaniego and M. Shaked, Systems with weighted components, Statistics & Probability Letters **78** (2007), 815–823.

[58] F.J. Samaniego, N. Balakrishnan and J. Navarro, Dynamic signatures and their use in comparing the reliability of new and used systems, Naval Research Logistics **56** (2009), 577–591.

[59] A.J.C Van Gemund and G.L. Reijns, Reliability analysis of k-out-of-n systems with single cold standby using Pearson distributions, IEEE Transactions on Reliability **61** (2012), 526–532.

[60] J.S Wu and R.J Chen, An algorithm for computing the reliability of weighted k-out-of-n system, IEEE Transactions on Reliability **43** (1994), 327–328.

[61] G. Xiao, Z. Li and T. Li, Dependability for non-Markov consecutive k-out-of-n:F repairable systems by fast simulation, Reliability Engineering and System Safety **92** (2007), 293–299.

[62] W. Y. Yun, G. R. Kim and H. Yamamoto, Economic design of a circular consecutive k-out-of-n:F system with $(k-1)$ step Markov dependence, Reliability Engineering and System Safety **92** (2007), 464–478.

[63] Z. Zhang, Ordering conditional general coherent systems with exchangeable components, Journal of Statistical Planning and Inference **140** (2010), 454–460.

[64] Z. Zhang and X. Li, Some new results on stochastic orders and aging properties of coherent systems, IEEE Transactions on Reliability **59** (2010), 718–724.

VITA

Ceki Franko was born in İzmir, Turkey, on December 27, 1985. He finished his high school education in 60. Yıl Anadolu Lisesi in İzmir. In 2008, he received his bachelor's with first degree from İzmir University of Economics Department of Mathematics. At the same time he completed International Trade and Finance as a double major. After graduating from university he started his master in Applied Statistics in İzmir University of Economics. In 2010 he finished his thesis called residual life lengths of censored data, supervised by Prof. Dr. İsmihan Bayramoğlu. In 2010 he was accepted in Ph. D. programme in Applied Mathematics and Statistics in İzmir University of Economics. He has been awarded both master and PhD scholarship by TÜBİTAK. He worked as a research assistant in İzmir University of Economics Department of Mathematics between the years 2008 and 2015. As 2015 he has been working in American Collegiate Institute as a Math teacher. In 2016 he completed all the requirements for the Doctor of Philosophy degree in the Graduate Program of Applied Mathematics and Statistics at İzmir University of Economics under the supervision of Gözde Yazgı Tütüncü. His research interests include Order Statistics and Reliability Theory.