ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

ON THE RICCI SOLITONS WITH PARALLEL VECTOR FIELDS

M.Sc. THESIS

Merve ATASEVER

Department of Mathematical Engineering

Mathematical Engineering Programme

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Merve ATASEVER, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509161292 successfully defended the thesis entitled "ON THE RICCI SOLITONS WITH PARALLEL VECTOR FIELDS", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family

FOREWORD

This thesis came into being as a result of intense study which has been last for two years. It would not have been possible without the help and understanding of a few people. First of all, I sincerely thank to my advisor Prof. Dr. Sezgin Altay Demirbağ for her guidance throughout the research process.

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December 2018 Merve ATASEVER

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SYMBOLS

TM : Tangent bundle of M
$\boldsymbol{T^{(r,s)}M}$: Bundle of (r,s)-tensors on M
$\boldsymbol{T_pM}$: Tangent space at point $p \in M$
$\boldsymbol{\chi(M)}$: The space of vector fields on manifold M
$\boldsymbol{C^\infty(M;R)}$: The space of smooth functions from M to $\mathbb{R}$
g : Riemannian metric
$\boldsymbol{\nabla}$: Connection related to metric g
R : Riemann curvature tensor
Ric : Ricci tensor
r : scalar curvature
$\boldsymbol{\Gamma}$: Christoffel symbol
[.,.] : Lie bracket
W : Weyl tensor
B : Bach tensor
C : Cotton tensor
d : Differential operator
div : Divergence
grad : Gradient
$\boldsymbol{\Delta}$: Laplace operator
Hessf : Hessian tensor of f
$\boldsymbol{\delta}$: Kronecker symbol
$\mathscr{L}_{\xi}$: Lie derivative in the direction of vector field $\boldsymbol{\xi}$
$\otimes$: Tensor product
K : Sectional curvature
$\boldsymbol{\partial}$: Partial derivative operator

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YÖK watermark

ON THE RICCI SOLITONS WITH PARALLEL VECTOR FIELDS

SUMMARY

The Riemannian geometry has been an outstanding branch of mathematics due to its importance in understanding many geometrical structures. In the last century, some scientists have introduced new concepts to describe more complex structures. The new improvements helped Perelman to solve Poincaré conjecture by using the Ricci flow. There are still lots of unknowns in the associated topics such as gradient Einstein-type manifolds, quasi-Einstein manifolds, and Ricci solitons which are examples for the most significant ones. To contribute to the field, we have tried to find some special structures as instances for the Ricci solitons.

In this thesis, we started with the introduction chapter which is composed of simple answers for how these new objects emerge, why we need them, and what is the geometric meaning behind. The subjects which range from Einstein manifolds to Ricci solitons have taken in hand according to the historical developments. Some mathematicians who had contributed to the field mentioned throughout the chapter. At the end, the aim of this study which try to relate some famous geometric concepts is given.

In the second chapter, we have dealt with the basic knowledge on the Riemannian geometry. Firstly, the definition of differentiable manifolds that are smooth, locally Euclidean spaces is given. Then, some topological properties which belong to these manifolds are mentioned. It is known that the Riemannian metric g, a distance function, defined on differentiable manifolds to be able to examine curvatures. Moreover, the Riemann curvature tensor which is obtained from the metric g gives another significant instrument,the Ricci tensor, by contraction. The tools like these to study the Riemannian geometry are described in detailed. Later on, we built up all the work on this knowledge.

In the third chapter, we entered the main subjects of the thesis. Einstein manifolds and quasi-Einstein manifolds are defined with respect to the Ricci tensor. Afterwards, gradient Einstein-type manifolds and their classifications are introduced in the existence of some vector field X on the manifold. Several examples and couples of recent theorems are given in the following. Also, the trace-free Weyl tensor, and related Cotton, Bach, Schouten tensors are mentioned to consider the concept of curvature in different ways. The relations between these tensors are observed through some important lemmas and theorems.

In the next chapter, Hamilton's Ricci flow and Ricci solitons which are very popular topics because of the Poincaré conjecture are taken in hand. The equation for the gradient Ricci soliton is analyzed for the next chapter. Additionally, the cumulative knowledge in the literature up to now has been shared with the help of theorems and examples.

In the last chapter, we have discussed the results of our study, and related them with the recent works of other colleagues. It has been searched for the Ricci soliton structure after admitting parallel vector field on various types of Einstein manifolds. Then, this case is checked out if it fits to the well-known theorems of the topics. The outcomes of the research are noteworthy in a way that the studied structure sets an example for the recent findings on the topic.

RİCCİ SOLİTONLARI VE PARALEL VEKTÖR ALANLARI

ÖZET

Geometri esas olarak evrenin küçük bir noktadan dev bir kara deliğe kadar matematiksel yorumudur. Objeleri, doğrular, eğriler ve açılar cinsinden ifade edip, analiz yapmaya yardımcıdır. Riemann geometrisi, daha kompleks yapıları anlamadaki öneminden ötürü matematiğin göze çarpan bir dalı olmuştur. Geometriye getirdiği yeni kavramlar sayesinde yüksek boyutlu eğimli yüzeylere sahip uzaylar üzerinde çalışmak daha kolay hale gelmiştir.

Lokal olarak Öklid uzaylarına benzeyen, türetilebilir Riemann manifoldları bu alanda çalışırken kullandığımız en temel yapılardır. Bu objeler daha kompleks uzayları, iyi bildiğimiz, nispeten basit Öklid uzayları yardımıyla anlamamızı sağlar. Sadece matematikte değil, klasik mekanik ve genel görelilik gibi fiziğin birçok alanında da kullanılan bir konsepttir.

Geometriciler, bir manifold üzerindeki en iyi Riemann yapısını bulmaya çalışmışlar ve sonrasında en iyi yapının sabit eğrilikli manifoldlarda bulunduğu gösterilmiştir. Bir manifold üzerindeki eğriliği hesaplamak için iki temel araç kullanılmıştır; Riemann eğrilik tensörü ve Ricci tensörü. Geçtiğimiz yüzyılda, bazı bilim insanları bu araçlar yardımıyla daha kompleks yapıları tasvir edebilmek için yeni kavramlar tanımlamışlardır. Bu yeni kavramlar Perelman'ın bir asır boyu cevapsız kalmış olan ünlü Poincaré sanısını Ricci akışı yardımıyla çözmesine yardımcı olmuştur.

Gradiyent Einstein tipi manifoldlar, yarı Einstein manifoldları, ve Ricci solitonları gibi arkasında derin fiziksel anlamlar barındıran yapılar hakkında hala bilinmeyen birçok şey bulunmaktadır. Bu alanda katkılı olabilmek adına, biz Ricci solitonlara örnek olabilecek bazı özel yapıları gradiyent Einstein tipi manifoldlar üzerinde araştırdık.

Tezin giriş bölümünde, bu oluşumların nereden çıktığını, niye böyle tanımlamalara ihtiyaç duyduğumuzu ve arkalarında yatan geometrik yorumu cevaplamaya çalıştık. Konuları Einstein manifoldlarından başlayarak Ricci solitonlarına kadar tarihi ilerleyişine göre ele aldık. Bölüm içinde bu alanlara katkıda bulunmuş bazı değerli matematikçilerden de bahsettik. Son olarak, çalışmanın amacı, bazı çokça bilinen geometrik kavramları ilişkilendirmek olarak verildi.

İkinci bölümde, Riemann geometrisi üzerine genel bilgiler ile ilgilendik. Öncelikle, bölgesel olarak Öklid uzaylarına benzeyen, türetilebilir manifoldların tanımını verdik. Sonrasında, bu manifoldların sahip olduğu topolojik özelliklerden bahsettik. Kovaryant türev, katlı çarpım gibi ilerleyen bölümlerde sıkça kullanacağımız işlemlerin tanımlarını verdik. Daha sonra, türetilebilir manifoldlar üzerindeki eğimleri incelemek için, uzaklık fonksiyonu diyebileceğimiz "g" Riemann metriğinin tanımlı olduğu gösterdik. Dahası, bu metrikten elde edilen Riemann eğrilik tensörünün daraltılması ile başka önemli bir araç olan Ricci tensörü elde ediliyor. Bu alandaki Einstein manifoldları gibi birçok manifold Ricci tensörünün yapısına

göre çeşitlendiriliyor. Bu açıdan çalışmamızın temel taşını oluşturuyor. Riemann geometrisini çalışmak için gerekli diğer benzer araçlar da tezin ilgili kısmında ayrıntılı olarak işlenmiştir. Bu temel bilgiler baz alınarak ilerleyen bölümlerdeki yapılar oluşturulmuştur.

Üçüncü bölümde, tezin temel konularına giriş yapılmıştır. Biliyoruz ki, Ricci tensörünün metrik tensörüyle orantılı olduğu manifoldlara Einstein manifoldları deniliyor. Bu manifoldlar, uzay-zaman düzleminde bir kütle tarafında yaratılan çekim gücünü açıklamaya çalışan Einstein'ın ünlü alan denklemleriyle yakından ilişkilidir. Bu açıdan birçok matematikçi ve fizikçinin ilgisini çekmektedir. Einstein manifoldları üzerindeki en değerli çalışmalardan biri M. C. Chaki ve R. K. Maity tarafından yürütülmüştür. 2000 yılında, Einstein manifoldlarının genelleştirilmiş konsepti olan yarı Einstein manifoldlarını, bir sonraki sene de genelleştirilmiş yarı Einstein manifoldlarını tanıtmışlardır. Genel göreliliğin anlaşılması ve modellenmesi bu bağlamda kolaylaşmıştır. Einstein manifoldları ve yarı Einstein manifoldları, ilgili Ricci tensörlerinin yapısına göre tanımlanmıştır. Sonrasında, manifoldları üzerlerinde belli koşulları sağlayan bir X vektör alanına sahip olmaları haline göre gradiyent Einstein tipi manifoldlar ve onların sınıflandırılması olarak çeşitlendirilmiştir. Çalışmanın devamında, çeşitli örnekler ve yakın zamanda ispatlanmış teoremler verilmiştir. Ayrıca, eğrilik kavramını farklı açılardan değerlendirmek için Weyl tensörü ve onunla ilişkili Cotton, Bach, Schouten tensörlerinden bahsedilmiştir. Bu tensörlerin birbiriyle ilişkisi bazı önemli önsav ve teoremler aracılığıyla gözlemlenmiştir.

Bir sonraki bölümde, ünlü Poincaré sanısının çözümündeki rollerinden dolayı popülerliği artmış Ricci akışı ve Ricci solitonları ele alınmıştır. Uzun yıllar çözülememiş sanı şunu iddia etmekteydi; her basit bağıntılı, kapalı, 3 boyutlu manifold ile 3-küre arasında bir homeomorfizma vardır. Sonrasında, bu sanının daha genel hali olan Thurton'ın geometrikleştirme sanısı her 3 boyutlu kompakt manifoldu sınıflandırmayla ilgiydi. Bu problemlerin çözümüyle ilgili en büyük adım 1982'de Ricci akışını literatüre kazandıran Hamilton tarafından atılmıştır. Geçtiğimiz yıllarda, Perelman Ricci akışını kullanarak Poincaré sanısını (artık teorem) ispatlamıştır. Yeni bir kavram olan Ricci solitonları bu şekilde ortaya çıkmıştır. Ricci solitonları, Ricci akış denkleminin kendi kendine benzer çözümleridir. Bu bölümde gradiyent Ricci solitonunu veren denklem ilerleyen bölümlerde kullanılmak üzere analiz edilmiştir. Ek olarak, bu alan ile ilgili şimdiye kadar yapılmış literatürdeki çalışmalar taranmış, teorem ve örnekler yardımıyla paylaşılmıştır.

Son bölümde, kendi çalışmamız üzerine yoğunlaşıp, sonuçlarımızı güncel çalışmalarla ilişkilendirdik. Literatüre baktığımızda, Einstein tipi yapılara sahip Riemann manifoldlarının Ricci soliton örneği bulmak için araştırıldığını görüyoruz. Genelleştirilmiş Einstein manifoldlarından Ricci solitonlarına geçişte, Ricci tensörü, Hessian tensörü, ve tensör çarpımından oluşan m-Bakry-Emery-Ricci tensörünün kullanıldığı görülmektedir. Bu tezde, çeşitli Einstein tipi manifoldlarda paralel vektör alanı tanımlandığında Ricci soliton yapısı elde edilip edilemeyeceği araştırılmıştır. Üzerinde paralel vektör alanı tanımlanmış bir gradiyent Einstein tipi manifoldun, sabit skaler eğrilikli Ricci soliton ve yaklaşık yarı Einstein yapılarına sahip olduğu gözlemlenmiştir. Sonucunda da, bu yapının konuyla ilgili bilinen temel teoremlere uyumlu olup olmadığı kontrol edilmiştir. H. D. Cao ve Q. Chen'in çalışmaları yardımıyla, boyutu $n \geq 5$ olan manifoldların bazı koşullar altında harmonik Weyl tensörüne sahip olduğu ve Z. Hu, D. Li ve S. Zhai'nin çalışmalarıyla ilişkilendirildiğinde bir aralık ile $(n-1)$ boyutlu bir Einstein manifoldunun katlı çarpımına isometrik olduğu görülmüştür. Son olarak da, bu yapıya örnek olarak 3 boyutlu, Bach düz yapıya sahip bir manifold verilmiştir.

1. INTRODUCTION

Geometry is essentially a mathematical interpretation of the universe (including the higher dimensional forms) in the scale of a point to a black hole. It analyzes objects or structures in terms of lines, angles, and surfaces. Up to three dimensional space, the concept is much easier because it is the space that we are living in. It is clear that the study of any geometrical object is more concrete in a sense that we can see or imagine a structure. On the other hand, it gets harder to imagine as the dimension of the space increases. If we work in an Euclidean space, it is still respectively easy because many scientists have worked in the Euclidean spaces throughout the history, which is associated with a very basic distance formula. When it comes to the curvature complex spaces, the Riemannian geometry helps us understand the more complicated structures.

In the Riemannian geometry, we define the notion of manifolds which locally look like Euclidean spaces. Hence, the complex spaces can be expressed by using well-known simpler spaces with the help of this concept. Moreover, subsets of a manifold which are locally Euclidean are called *charts*. If these charts are compatible on the manifold, then computations like differentiability and integrability are defined on the differentiable (smooth) manifold as well. Smooth manifolds with a Riemann metric g (distance function) are said to be *Riemannian manifolds* which is denoted by (M^n, g) with dimension n. Riemannian manifolds play an important role in both mathematics and physics, especially in classical mechanics and Einstein's general relativity. [1]

Geometers started to think about what would be the best Riemannian structure on a manifold in time. They have found that the manifolds of constant curvature are actually the best Riemannian structures. In this thesis, we will describe the most significant tools; the Riemann curvature tensor and the Ricci tensor to measure the corresponding curvatures.

One of the attractive topics in the Riemannian geometry is the notion of Einstein manifolds. On a Riemannian manifold, if the Ricci tensor is proportional to the metric tensor,i.e. there exists a function λ such that $Ric = \lambda g$, then we called the manifold *Einstein*. These manifolds are very related to the Einstein's field equations which try to explain the gravitational effects produced by a mass in space-time. [2]

In the following years, the concept of Einstein manifolds has attracted considerable attention of several mathematicians and physicists. One of the most noteworthy study had been carried out by M. C. Chaki and R. K. Maity. They introduced the notion of quasi-Einstein manifolds as a generalization of Einstein manifolds in 2000. [3] It was seen that quasi-Einstein manifolds model the space-time continuum which composed of perfect fluid matter and satisfies the Einstein's field equations. One year later, the generalized quasi-Einstein manifolds was presented with regard of cases of different fluid density. General relativity has been modelled and understood better in this context. [4]

On the other side, there was famous unsolved problem of Poincaré conjecture in mathematical aspect. The conjecture (theorem now) was about if every simply connected, closed 3-dimensional manifold is homeomorphic to the 3-sphere. Besides that, Thurton proposed a more general question of the classification of every compact 3-dimensional manifold, which is known as the geometrization conjecture. [5] The biggest step to solve these problems came from R. S. Hamilton in 1982. [6] He introduced the Ricci flow in his famed article but the Ricci flow tends to create singularities which causes the flow to stop. Perelman achieved to get away from the singularities by his surgery method and solved the Poincare conjecture after a century.

The new structure,Ricci solitons, emerged from the Ricci flow during the progress. Ricci solitons illustrate the formation of singularities in the Ricci flow and fit as self-similar solutions. [7] When we look at the literature, we see that the Riemannian manifolds which have Einstein-like structure have been searched to find examples for Ricci solitons. The m-Bakry-Emery-Ricci tensor is used in the transition from the concept of generalized quasi-Einstein manifolds to the Ricci solitons. With the cumulative knowledge on the topics and the same purpose, we have researched that in which conditions the Ricci soliton structure can be observed on special type of Einstein manifolds in this thesis. We have mentioned the main properties of these structures and given some theorems which have proven recently.

2. BASIC CONCEPTS

In this chapter, some basic definitions and significant theorems related to Riemannian geometry are given. Resources which are numbered [8], [9], [10], [11], [12], and [13] are used for this purpose.

2.1 Differential Geometry and Riemannian Manifolds

The notion of differentiable manifolds is studied not only in differential geometry but also several areas of mathematics and physics. Although it takes some time to digest the theory, the geometry of manifolds have helped scientists to understand the complex curvature surfaces in a better way. That's why, elementary definitions are good materials to start with.

Every metric space gives rise to a topological space naturally, so the manifold (M, g) may be seen as a topological space. We will be referring to some properties of a topological spaces, such as compactness, connectedness, and etc.

Definition 2.1.1 Let X be a set. A *topology* on X is a collection T of subsets of X satisfying

- T contains $\emptyset$ and X,
- T is closed under arbitrary unions, i.e. if $U_i \in T$ for $i \in I$ then $\cup_{i\in I} U_i \in T$,
- T is closed under finite intersections, i.e. if $U_1, U_2 \in T$ then $U_1 \cap U_2 \in T$.

Definition 2.1.2 A topological space M is called

- *Hausdorff* if for all distinct points $x, y \in M$ there exist disjoint open subsets N_x, N_y such that $x \in N_x$ and $y \in N_y$.
- *compact* if every open cover of M has a finite subcover.
- *disconnected* if there exists non-empty open subsets U and V of M such that $U \cap V = \emptyset$ and $M = U \cup V$. If M is not disconnected, then it is *connected*.

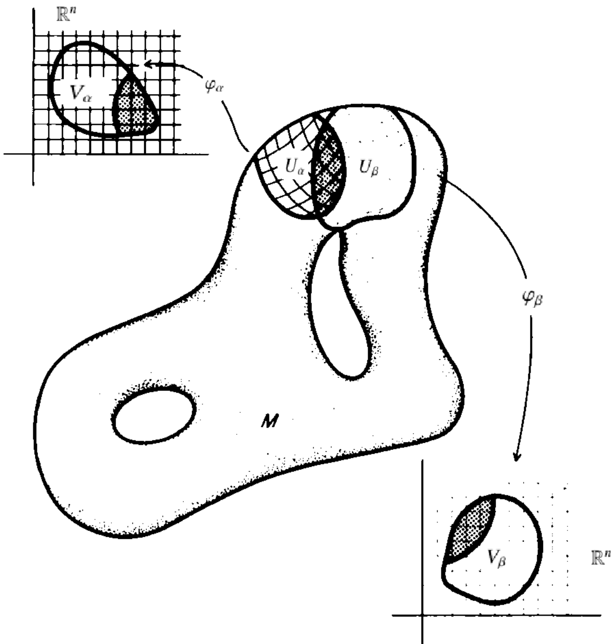

Figure 2.1 : Coordinate Charts

Definition 2.1.3 We call a Hausdorff topological space M as an *n-dimensional manifold* with the properties below:

1. M is locally Euclidean of dimension n,

2. M has a countable basis of open sets.

By saying "locally Euclidean", we intent to say that for each point $p \in M$ there is a neighborhood $U_p \subseteq M$ which is homeomorphic to an open subset of Euclidean space of dimension n.
In other words $\forall p \in M \quad \exists (U_p, \varphi_p = (x_p^1, x_p^2, ..., x_p^n))$ such that $\varphi_p : U_p \subseteq M \longrightarrow V_p \subseteq \mathbb{R}^n$ where φ_p is bijective and continuous function with continuous inverse function ${\varphi_p}^{-1}$. We call the pair (U_p, φ_p) a coordinate chart of the manifold M. A collection of charts whose domains cover M is called an atlas where any two maps $(\varphi_\alpha, \varphi_\beta)$ of it overlap smoothly.

Definition 2.1.4 Given two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), we say that the maps $(\varphi_\alpha, \varphi_\beta)$ are C^∞*-compatible (or overlap smoothly)* if the mappings $(\varphi_\alpha \circ {\varphi_\beta}^{-1})$ and $(\varphi_\beta \circ {\varphi_\alpha}^{-1})$ of the open subsets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ of $\mathbb{R}^n$ are smooth. Hence, these mappings are diffeomorphisms that are maps between manifolds with differentiable inverses. It is drawn in Figure 2.1.

Definition 2.1.5 Assume that M is an n-dimensional topological manifold. If there exists an atlas on M, then the manifold is said to be *differentiable manifold*.

We will suppose that all the manifolds are differentiable (smooth) manifolds in the rest of the thesis. The property of differentiability on manifolds allows us to mention about some notions like integration, vector fields, tangent fields, and etc. On smooth surfaces (i.e surfaces without corners or edges), there exist a normal vector N_p and tangent plane T_pM for each point $p \in M$, which varies continuously as we move from a point to another. For example, the surface of a cube is homeomorphic to S^2 but the cube has no tangent plane or normal vector at the corners and the edges. Hence, even if they are equivalent spaces, working calculus on a 2-sphere is easier.

We have described the curved spaces that will be the base for this study up to here. Now, it is time to move on a new notion to measure lengths on the manifold.

Definition 2.1.6 Let M be a smooth manifold and T_pM be a tangent space at a point $p \in M$. Define an inner product $g : T_pM \times T_pM \longrightarrow \mathbb{R}$ for any point p on the manifold with the following properties:

1. $g(u,u) = 0$ if and only if $u = 0$.
2. $g(u,u) \geq 0$ for all $u \in T_pM$.
3. $g(u,v) = g(v,u)$ for all $u, v \in T_pM$.

Then, such smoothly chosen g_{ij} is called *Riemannian metric*. In particular, *Riemannian manifold* (M,g) is a smooth manifold which is furnished with a Riemannian metric g. For a coordinate system $(x^1, x^2, ..., x^n)$, the metric is calculated as $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$. In an orthonormal basis, we have $g_{ij} = \delta_{ij}$.

Theorem 2.1.1 [8] Every smooth manifold carries a Riemannian metric.

Proof: Let (U_β, φ_β) be coordinate charts for a smooth manifold M which is covered by the union of U_β. For each β, consider the Reimannian metric g_β in U_β whose local expression $((g_\beta)_{ij})$ is the identity matrix. Let ρ_β be a smooth partition of unity of M subordinate to the covering U_β, and define

$$g = \sum_\beta \rho_\beta g_\beta.$$

Since the family of supports of the ρ_β is locally finite, the above sum is locally finite, and hence g is well-defined and smooth. Moreover, it is bilinear and symmetric at each point. Because of ρ_β being nonnegative for all and $\sum_\beta \rho_\beta = 1$, it follows that g is positive definite and a Riemannian metric in M as well. (The Einstein summation convention will be used throughout the thesis.)

Definition 2.1.7 Consider the Riemannian manifolds (M_1^q, g) and (M_2^{n-q}, g') with corresponding charts (ϕ_1, x^α) and (ϕ_2, x^α). We call $M_1 \times_f M_2$ *the warped product* of the manifold $M_1 \times M_2$ with the metric $g_\star = g \times_f g'$ such that

$$g \times_f g' = \pi_1^* g + (f \circ \pi_1)^2 \pi_2^* g' \tag{2.1}$$

where π_i $(i = 1,2)$ are natural projections and f is a pozitive smooth function on M_1. Here, if the function f is constant, it is called Riemannian product.

Now, we know how to find the distance between two points on a manifold with the help of metric tensor g_{ij}. It is required to introduce some new objects which will be used for tensor differentiation later.

2.2 Covariant Derivative And Some Special Vector Fields

Definition 2.2.1 The functions

$$\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial x^i}(g_{jk}) + \frac{\partial}{\partial x^j}(g_{ki}) - \frac{\partial}{\partial x^k}(g_{ij})]$$

are called *Christoffel symbols of first kind*. The following notation is also used in some books:

$$\Gamma_{ijk} = \frac{1}{2}(g_{jki} + g_{kij} - g_{ijk})$$

The important property of the Christoffel symbols of first kind is the symmetry in the first two indices, i.e. $\Gamma_{ijk} = \Gamma_{jik}$. Furthermore, the equation $\Gamma^i_{jk} = g^{ir}\Gamma_{jkr}$ gives *the Christoffel symbols of second kind* where (g^{ir}) is the inverse matrix of (g_{ir}). (Since the metric g is a linear transformation indeed, the corresponding matrix is nonsingular.) We also have the symmetry in the lower indices in this case, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Definition 2.2.2 Let X and Y be smooth vector fields on a manifold M. We define *the Lie Bracket* or *the commutator* of X and Y as

$$[X,Y] = XY - YX,$$

that is, for a smooth function $f : M \longrightarrow \mathbb{R}$,

$$[X,Y](f) = X(Yf) - Y(Xf)$$

with the properties:

1. $[X,Y] = -[Y,X]$.
2. $[X_1 + X_2, Y] = [X_1,Y] + [X_2,Y] \quad and \quad [X, Y_1 + Y_2] = [X,Y_1] + [X,Y_2]$.
3. For any smooth functions $a,b : M \longrightarrow \mathbb{R}$

$$[aX, bY] = ab[X,Y] + a(Xb)Y - b(Ya)X.$$

4. Jacobi Identity: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Definition 2.2.3 Let M be a C^∞ n-manifold, $X,Y,Z \in \chi(M)$, and $f \in C^\infty(M)$, then we define *the affine connection* ∇ as follows:

$$\nabla : \chi(M) \times \chi(M) \longrightarrow \chi(M)$$
$$(X,Y) \longrightarrow \nabla(X,Y) = \nabla_X Y$$

where $\nabla_X Y$ is called *the covariant derivative* of Y in the direction of X with the below properties:

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.
2. $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$.
3. $\nabla_{(fX)} Y = f \nabla_X Y$.
4. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

To make this operator clear, let $e_1, e_2, ..., e_n$ be a C^∞ base field about a point $m \in M$, and let $X_m = \sum_1^n a_i(m)(e_i)_m$ and $Y = \sum_1^n b_j e_j$ on the domain of the base field (intersected with the domain of Y). Then

$$(\nabla_X Y)_m = [\nabla_X(\sum b_j e_j)]_m = \sum_j [(X_m b_j)(e_j)_m + b_j(m) \sum_i a_i(m)(\nabla_{e_i} e_j)_m]$$

where $\nabla_{e_i} e_j = \sum_{k=1}^n \Gamma_{ij}^k e_k$.

Definition 2.2.4 Let (x^i) be a coordinate system. The covariant derivative of a tensor $T = T^{i_1 i_2 \dots i_p}_{j_1 j_2 \cdots j_q}$ with respect to x^k is the following tensor:

$$T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q, k} = \frac{\partial T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}}{\partial x^k} + \Gamma^{i_1}_{tk} T^{t i_2 \dots i_p}_{j_1 j_2 \dots j_q} + \Gamma^{i_2}_{tk} T^{i_1 t \dots i_p}_{j_1 j_2 \dots j_q} + \dots + \Gamma^{i_p}_{tk} T^{i_1 i_2 \dots t}_{j_1 j_2 \dots j_q}$$
$$- \Gamma^{t}_{j_1 k} T^{i_1 i_2 \dots i_p}_{t j_2 \dots j_q} - \Gamma^{t}_{j_2 k} T^{i_1 i_2 \dots i_p}_{j_1 t \dots j_q} - \dots - \Gamma^{t}_{j_q k} T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots t}.$$

In particular, the covariant derivative of the covariant and contravariant vectors are

$$T_{i,k} = (\frac{\partial T_i}{\partial x^k} - \Gamma^t_{ik} T_t) \quad and \quad T^i_{,k} = (\frac{\partial T^i}{\partial x^k} + \Gamma^i_{tk} T^t).$$

Notice that the covariant derivative will be induced to classical partial derivative if all components of (g_{ij}) are constant, which in turn all the Christoffel symbols being zero.

The covariant derivative is one of the substantial tools to analyze the curves and vector fields on a manifold. Let σ be a curve on a smooth manifold M with a tangent field T, then a C^∞-vector field Y is said to be *parallel* along σ if and only if $\nabla_T Y = 0$ on σ. Furthermore,for the curve σ being a *geodesic* the necessary and sufficient conditions is that $\nabla_T T = 0$ on σ. There is one more way to measure a change in a tensor field from a point to another point, which is Lie derivative. The concept of the Lie derivatives is very useful in differential geometry and physics because they help to describe the invariants. For instance, the change of a function under a flow can be measured simply with the help of this concept.

Definition 2.2.5 *The Lie derivative* of a metric tensor g with respect to the vector field X is given by

$$\mathscr{L}_X g_{ij} = X_{i,j} + X_{j,i}$$

where $X_{i,j}$ is a covariant derivative.

Definition 2.2.6 We define *conformal vector field* ξ on a Riemannian manifold (M^n, g) if it satisfies

$$\mathscr{L}_\xi g = 2\Omega g$$

for a smooth function Ω on M. This function is known as *conformal factor*. For constant Ω, we call the vector field ξ *homothetic*. For identically zero Ω, the vector field ξ is said to be *killing*.

Definition 2.2.7 If the vector field ξ satisfy the following condition

$$\nabla_X \xi = \Omega X$$

for all X and a smooth function Ω, then it is called *closed conformal vector field*. Additionally, the closed conformal vector field is *parallel* if Ω vanishes.

2.3 Riemann Curvature Tensor And Ricci Tensor

Definition 2.3.1 Let ∇ be an affine connection on a smooth manifold (M,g) as mentioned above. If the connection satisfy two additional properties:

1. $\nabla_X Y - \nabla_Y X = [X,Y]$ (torsion-free/symmetric connection)

2. $X(g(Y,Z)) = X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (metric compatibility)

where $X,Y,Z \in \chi(M)$, then this unique connection is called *Riemannian connection* or *Levi-Civita connection*.

An interesting question emerges from this point. If we apply the symmetric connection to a covariant vector (V_i) twice with respect to x^j and x^k, does the order of differentiation matter?

Definition 2.3.2 Let M be a smooth manifold with the Levi-Civita connection ∇, and (V_i) be an arbitrary covariant vector. Then,

$$V_{i,jk} - V_{i,kj} = R^l_{ijk} V_l \quad (Ricci\ identity)$$

where $R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^r_{ij}\Gamma^l_{rk}$ is called *Riemann curvature tensor of second order*. We define the Riemann curvature tensor or Riemann tensor of type (1,3) on the manifold M as follows:

$$R : \chi(M) \times \chi(M) \times \chi(M) \longrightarrow \chi(M)$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Here, $R_{ijkl} = g_{ir} R^r_{jkl}$ or $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ is called *Riemann tensor of first kind*, which is $(0,4)$-type tensor. Note that if $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ are coordinate vector fields then $[X,Y] = 0$, and hence the Riemann tensor becomes

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z.$$

The Riemann tensor possesses the following properties:

1. $R_{ijkl} = R_{klij}$;

2. $R^l_{ijk} = -R^l_{ikj}$;

3. $R_{ijkl} = -R_{jikl}$ and $R_{ijkl} = -R_{ijlk}$;

4. $R^l_{ikj} + R^l_{kji} + R^l_{jik} = 0$ (First Bianchi Identity);

5. $R^l_{mij,k} + R^l_{mki,j} + R^l_{mjk,i} = 0$ (Second Bianchi Identity).

As well-known, the Riemann curvature tensor is composed of the Ricci tensor and the Weyl tensor. While the Ricci tensor is the trace of the Riemann tensor, the Weyl tensor is the traceless part of the Riemann tensor. We will give a definition for the Weyl tensor in a later chapter.

Definition 2.3.3 A contraction of the Riemann curvature tensor gives a significant tool in the relativity and the Riemannian geometry;

$$R_{ij} = R^k_{ijk}$$

where R_{ij} is called *the Ricci tensor* which is symmetric, and (0,2)-type tensor.

Definition 2.3.4 *The scalar curvature* r can be calculated from the Ricci tensor as below:

$$r = R^i_i = g^{is} R_{si}.$$

In Riemannian geometry, the main purpose of the Ricci tensor is to describe the growth rate along geodesics. For example, it tells how much the volume changes due to the local curvature in 3-dimensional spaces. Additionally, if all the components of the Ricci tensor is zero on a smooth manifold M, then M is called *Ricci-flat*.

Lemma 2.3.1 [2] (Schur's Identity) The Ricci tensor and the corresponding scalar curvature satisfy the equation below;

$$r_k = 2R_{tk,t}. \tag{2.2}$$

Proof: To begin with, the second Bianchi identity leads

$$R_{ijtk,s} + R_{ijks,t} + R_{ijst,k} = 0. \tag{2.3}$$

When we take the trace with respect to i and s, it is found that

$$R_{ijtk,i} + R_{ijki,t} + R_{ijit,k} = 0. \tag{2.4}$$

It is known that covariant derivative commutes with tracing, hence

$$R_{ijtk,i} = R_{jt,k} - R_{jk,t}. \tag{2.5}$$

Contracting this with respect to j and t gives

$$R_{ik,i} = -R_{tk,t} + r_k \tag{2.6}$$

$$\Longrightarrow 2R_{tk,t} = r_k. \tag{2.7}$$

which is known as Schur's identity. Here, r_k denotes the covariant derivative of the scalar curvature r.

3. GRADIENT EINSTEIN-TYPE MANIFOLDS

In this chapter, we will give an intuitive introduction for Einstein manifolds and some special type of quasi-Einstein manifolds. The discussion of the Einstein manifolds developed from the interpretation of the general relativity in the context of Riemannian geometry. Einstein needed some tool to describe the general relativity which is independent of the coordinate systems chosen. In today's approach, the physicists treat space-time as a 4-dimensional Riemannian manifold and consider the quasi-Einstein manifolds as solutions of the fields equations which are associated to the curvature of the space-time (gravity) as again a manifold.

3.1 Quasi-Einstein Manifolds

Definition 3.1.1 For (M,g) be a Riemannian manifold and $X,Y \in \chi(M)$, if there exists a function $\lambda : M \longrightarrow \mathbb{R}$ such that

$$Ric(X,Y) = \lambda g(X,Y)$$

then, M is said to be *Einstein manifold*. [9]

Definition 3.1.2 [3] A non-flat Riemannian manifold (M^n,g) is said to be a *quasi-Einstein manifold* if

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y) \quad \forall X,Y \in \chi(M) \tag{3.1}$$

where a and b are some scalar functions such that $b \neq 0$, and A is a non-zero 1-form satisfying

$$A(X) = g(X,\xi) \quad \forall X \in \chi(M) \quad and \quad g(\xi,\xi) = A(\xi) = 1$$

for the associated unit tangent vector ξ which is said to be *the generator of the manifold*. We use the notation $(QE)_n$ for this kind manifolds.

Definition 3.1.3 [14] A non-flat Riemannian manifold (M^n,g) is called a *nearly quasi-Einstein manifold* if

$$Ric(X,Y) = ag(X,Y) + bE(X,Y) \quad \forall X,Y \in \chi(M) \tag{3.2}$$

for some scalar functions $a, b \neq 0$, where E is a non-zero (0,2)-type symmetric tensor. We use the notation $N(QE)_n$ for this kind manifolds.

3.2 Gradient Einstein-Type Manifolds

In this thesis, it has been searched that under which conditions we can associate Einstein-type manifolds with Ricci solitons. Because of this purpose, Einstein-type manifolds has importance in the rest of the study.

Definition 3.2.1 It is said that a connected Riemannian manifold (M^n, g) of dimension $n \geq 3$ is an *Einstein-type manifold* if there exists a smooth vector field $X \in \chi(M)$ and $\lambda \in C^\infty(M)$ such that

$$\alpha Ric + \frac{\beta}{2}\mathscr{L}_X g + \mu X^\flat \otimes X^\flat = (\rho r + \lambda)g \tag{3.3}$$

for some constants $\alpha, \beta, \rho, \mu \in \mathbb{R}$, with $(\alpha, \beta, \mu) \neq (0,0,0)$. Here $X^\flat$ denotes the 1-form metrically dual to X. If $X = \nabla f$ for some $f \in C^\infty(M)$, the manifold (M^n, g) is said to be a *gradient Einstein-type manifold*, so the equation (3.3) becomes

$$\alpha Ric + \beta Hess(f) + \mu df \otimes df = (\rho r + \lambda)g \tag{3.4}$$

for some $\alpha, \beta, \rho, \mu \in \mathbb{R}$. Here $Hess$ stands for the Hessian. Catino introduced the notion of gradient Einstein-type manifolds and added that these manifolds are *nondegenerate* if $\beta \neq 0$ and $\beta^2 \neq (n-2)\alpha\mu$, otherwise $(for \quad \beta \neq 0 \quad and \quad \beta^2 = (n-2)\alpha\mu)$ manifolds are called *degenerate* gradient Einstein-type manifolds. [15]

Remark that the manifold (M, g) becomes Einstein manifold obviously for constant f. Moreover, the manifolds which have Einstein-type structure can be classified according to the parameters and possibly the function λ as [15] :

1. Einstein manifolds: $(\alpha, \beta, \mu, \rho) = (1, 0, 0, \frac{1}{m})$, $\quad \lambda = 0$ (or equivalently for $m \geq 3$, $\quad \rho = 0$ and $\lambda = \frac{r}{m}$);

2. Ricci solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0)$, $\quad \lambda \in \mathbb{R}$;

3. Ricci almost solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0)$, $\quad \lambda \in C^\infty(M)$;

4. Yamabe solitons: $(\alpha, \beta, \mu, \rho) = (0, 1, 0, 1)$, $\quad \lambda \in \mathbb{R}$;

5. Quasi-Einstein manifolds: $(\alpha, \beta, \mu, \rho) = (1, 1, -\frac{1}{k}, 0)$, $\quad k \neq 0$, $\quad \lambda \in \mathbb{R}$;

6. ρ-Einstein manifolds: $(\alpha,\beta,\mu,\rho)=(1,1,0,\rho), \quad \rho\neq 0, \quad \lambda\in\mathbb{R}$.

Definition 3.2.2 [16] The expression

$$W_{ijkl}=R_{ijkl}-\frac{1}{n-2}(R_{ik}g_{jl}-R_{il}g_{jk}+R_{jl}g_{ik}-R_{jk}g_{il}) \\ +\frac{R}{(n-1)(n-2)}(g_{ik}g_{jl}-g_{il}g_{jk}) \tag{3.5}$$

defines *the Weyl tensor* which is the traceless part of the Riemann tensor and invariant under conformal transformations. There is another geometric interpretations of curvature which is highly related to the conformal curvature tensor, ie. Weyl tensor

$$C_{ijk}=R_{ij,k}-R_{ik,j}-\frac{1}{2(n-1)}(r_{,k}g_{ij}-r_{,j}g_{ik}) \tag{3.6}$$

is called *the Cotton tensor* which can be expressed

$$C_{ijk}=A_{jk,i}-A_{ik,j} \tag{3.7}$$

and

$$W_{ijkl}=R_{ijkl}-\frac{1}{n-2}(g_{ik}A_{jl}-g_{il}A_{jk}+g_{jl}A_{ik}-g_{jk}A_{il}) \tag{3.8}$$

as well. Here,

$$A_{ij}=R_{ij}-\frac{r}{2(n-1)}g_{ij} \tag{3.9}$$

is *the Schouten tensor*. ([17], [18], and [19])

Lemma 3.2.1 [20] The Weyl and the Cotton tensors defined above has the following relation for the dimension $n\geq 4$;

$$\nabla^l W_{ijkl}=-\frac{n-3}{n-2}C_{ijk}. \tag{3.10}$$

Proof: Let's start with the equation

$$\nabla^l A_{jl}=\nabla^l R_{jl}-\nabla^l\frac{r}{2(n-1)}g_{jl}=\frac{1}{2}\nabla_j r-\frac{1}{2(n-1)}\nabla_j r=\frac{(n-2)}{2(n-1)}\nabla_j r \tag{3.11}$$

that is found by the Schur's identity. On the other side, the divergence of the Weyl given by

$$g^{ls}\nabla_s W_{ijkl}=g^{ls}\nabla_s R_{ijkl}-\frac{1}{n-2}(g_{ik}g^{ls}\nabla_s A_{jl}-g_{il}g^{ls}\nabla_s A_{jk}+g_{jl}g^{ls}\nabla_s A_{ik} \\ -g_{jk}g^{ls}\nabla_s A_{il}). \tag{3.12}$$

If we use the contracted 2^{nd} Bianchi identity, we get

$$g^{ls}\nabla_s W_{ijkl} = \nabla_j R_{ik} - \nabla_i R_{jk} - \frac{1}{n-2}(g_{ik}\nabla^l A_{jl} - \nabla_i A_{jk} + \nabla_j A_{ik} - g_{jk}\nabla^l A_{il}). \quad (3.13)$$

Combine this with the equation (3.7) to obtain

$$g^{ls}\nabla_s W_{ijkl} = \nabla_j R_{ik} - \nabla_i R_{jk} - \frac{1}{n-2}(g_{ik}\nabla^l A_{jl} - g_{jk}\nabla^l A_{il} - C_{ijk}). \quad (3.14)$$

When we use the equation (3.8), we have

$$g^{ls}\nabla_s W_{ijkl} = \nabla_i(-R_{jk} + \frac{r}{2(n-1)}g_{jk}) - \nabla_j(-R_{ik} + \frac{r}{2(n-1)}g_{ik}) + \frac{1}{n-2}C_{ijk} \quad (3.15)$$

$$= -\nabla_i A_{jk} + \nabla_j A_{ik} + \frac{1}{n-2}C_{ijk} \quad (3.16)$$

by the definition of the Schouten tensor. Then,

$$g^{ls}\nabla_s W_{ijkl} = -\frac{(n-3)}{(n-2)}C_{ijk} = \nabla^l W_{ijkl} \quad (3.17)$$

follows.

We say the manifold (M^n, g) has harmonic Conformal curvature tensor if the divergence of W vanishes. Furthermore, we say the manifold (M, g) is *locally conformally flat* if $C_{ijk} = 0$ for $n = 3$ or $W_{ijkl} = 0$ for $n \geq 4$. It is obvious from the above equation that in case of the Weyl tensor is zero, the Cotton tensor vanishes too in dimension $n > 3$.

Definition 3.2.3 [20] *The Bach tensor B* is composed of the Weyl and the Cotton tensor as below:

$$B_{ij} = \frac{1}{(n-2)}(C_{jik,k} + R_{kt}W_{ikjt}) \quad or \quad (3.18)$$

$$B_{ij} = \frac{1}{(n-3)}W_{ikjt,tk} + \frac{1}{(n-2)}R_{kt}W_{ikjt} \quad for \quad n \geq 4. \quad (3.19)$$

Note that if the manifold (M, g) is either conformally flat or Einstein manifold, then the Bach tensor vanishes. In particular, we have an practical identity $B_{ij,j} = \frac{n-4}{(n-2)^2}R_{kt}C_{kti}$, which will be used in a theorem later.

Definition 3.2.4 [20] We also define the tensor D as follows:

$$D_{ijk} = \frac{1}{(n-2)}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) \quad (3.20)$$
$$- \frac{r}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f)$$

where f is the potential function in the equation (3.20).

Theorem 3.2.1 [20] Let the manifold (M,g) of dimension $n \geq 3$ be a gradient Einstein-type manifold and β is non-zero. Then the integrability conditions which are stated below hold:

$$\alpha C_{ijk} + \beta f_t W_{tijk} = [\beta - \frac{(n-2)\alpha\mu}{\beta}]D_{ijk}, \tag{3.21}$$

$$\alpha B_{ij} = \frac{1}{n-2}[\beta - \frac{(n-2)\alpha\mu}{\beta}D_{ijk,k}] + \beta(\frac{n-3}{n-2})f_t C_{jit} - \mu f_t f_k W_{itjk}. \tag{3.22}$$

Proof: We start by rewriting the equation (3.4) as

$$\alpha R_{ij} + \beta f_{ij} + \mu f_i f_j = (\rho r + \lambda)\delta_{ij}. \tag{3.23}$$

Taking the covariant derivative of this equation gives

$$\alpha R_{ij,k} + \beta f_{ij,k} + \mu(f_{ik} f_j + f_i f_{jk}) = (\rho r_k + \lambda_k)\delta_{ij}. \tag{3.24}$$

If we skew-symmetrize with respect to $j \leftrightarrow k$ and use the identity $f_{ijk} = f_{ikj} + f_t R_{tijk}$ [15], we get

$$\alpha(R_{ij,k} - R_{ik,j}) + \beta f_t R_{tijk} + \mu(f_{ik} f_j - f_k f_{ij}) = \rho(r_k \delta_{ij} - r_j \delta_{ik}) + (\lambda_k \delta_{ij} - \lambda_j \delta_{ik}) \tag{3.25}$$

To obtain the following equation, we multiply the equation (3.25) by g^{ij} and use the Schur's identity $r_k = 2R_{tk,t}$:

$$[\alpha - 2\rho(n-1)]r_k = -2\beta f_t R_{tk} + 2(n-1)\lambda_k - 2\mu(f_t f_{tk} - \Delta f f_k). \tag{3.26}$$

Now pulling the term f_{ij} from the equation (3.23) and taking the trace gives

$$f_{ij} = \frac{1}{\beta}[-\alpha R_{ij} - \mu f_i f_j + (\rho r + \lambda)g_{ij}]. \tag{3.27}$$

$$\Delta f = \frac{1}{\beta}[(n\rho - \alpha)r + n\lambda - \mu|\nabla f|^2]. \tag{3.28}$$

When putting these two relations into the equation (3.26), we obtain

$$[\alpha - 2\rho(n-1)]r_k = 2(-\beta + \frac{\alpha\mu}{\beta})f_t R_{tk} + 2(n-1)\lambda_k - 2\frac{\mu}{\beta}[\alpha - \rho(n-1)]r f_k + 2\frac{\mu}{\beta}(n-1)\lambda f_k. \tag{3.29}$$

From the definitions of Weyl and D_{ijk} tensors, we deduce that

$$f_t R_{tijk} = f_t W_{tijk} - D_{ijk} - \frac{1}{n-1}(f_t R_{tk}\delta_{ij} - f_t R_{tj}\delta_{ik}). \tag{3.30}$$

Inserting the equations (3.6), (3.29), and (3.30) into (3.25) we get the equation (3.27).

For the second part of the proof, let's take the divergence of the equation (3.21):

$$\alpha C_{ijk,k} - \beta f_{tk} W_{itjk} - \beta (\frac{n-3}{n-2}) f_t C_{jit} = [\beta - \frac{(n-2)\alpha\mu}{\beta}] D_{ijk,k}. \tag{3.31}$$

Here we use the definition of the Bach tensor, the equation (3.27), and the symmetries of the Weyl tensor to get the desired result. (The Weyl tensor possesses the same symmetries with the Riemann tensor.)

3.3 (m,ρ)-Quasi Einstein Manifolds

We are now familiar with the concept of the Einstein manifolds and their several generalization. In this part of the study, we will go little bit deeper and work on a very special type of manifolds.

Definition 3.3.1 [17] The tensor given below, which is composed of the Ricci tensor, Hessian, and the tensor product parts as:

$$Ric_f^m = Ric + Hessf - \frac{1}{m} df \otimes df; \quad 0 < m \leqslant \infty \tag{3.32}$$

is called *the m-Bakry-Emery Ricci tensor* where f is a smooth function and m is a positive integer. It has a similar structure with the equation (3.4), but different constants. As it seen easily, when f is constant, the m-Bakry-Emery-Ricci tensor becomes the usual Ricci tensor.

Definition 3.3.2 [17] A Riemannian manifold (M^n, g) with a potential function f on M is said to be *m-generalized quasi Einstein manifold* if

$$Ric + Hessf - \frac{1}{m} df \otimes df = \lambda g \tag{3.33}$$

where $m \in \mathbb{N}$ and $\lambda \in C^\infty(M)$. It has the following characterizations:

- If $\lambda \in \mathbb{R}$, the manifold becomes *m-quasi Einstein manifold.*
- For $m = \infty$, the manifold reduces to a gradient Ricci soliton which will be defined in the next chapter.
- For constant potential function f, the manifold becomes Einstein. We call this incident *rigidity*.

Definition 3.3.3 [21] A Riemannian manifold (M^n, g) with a potential function f on M is said to be (m,ρ)-*quasi Einstein manifold* if

$$Ric + Hessf - \frac{1}{m} df \otimes df = (\rho r + \lambda) g \tag{3.34}$$

where $m \in \mathbb{N},\ \rho, \lambda \in \mathbb{R}$ and r denotes the scalar curvature.

Theorem 3.3.1 [22] For an (m,ρ)-quasi Einstein manifold with admitted parallel vector field ϕ, we have the following form for the Ricci tensor:

$$Ric = (\rho r + \lambda)[g - U^\flat \otimes U^\flat] \tag{3.35}$$

where $U^\flat$ is a 1-form associated with the unit vector field in the direction of ϕ.

The proof will be given in the last chapter. Let's assume that the theorem holds for now.

Theorem 3.3.2 [22] Let (M^n, g) be an (m,ρ)-quasi Einstein manifold with admitted gradient parallel vector field ϕ. Then, there exists an isomorphism between the manifold M and the direct product $M^* \times \mathbb{R}$ where M^* is an complete Einstein manifold with dimension $(n-1)$.

Proof: It is well-known fact from Tashiro's theorem [23] that such a manifold M is the direct product $M^* \times I$ of an $(n-1)$ dimensional Riemannian manifold M^* with a straight line I. Hence, we can decompose the metric g as $g = g_{M^*} + (dt)^2$ where g_{M^*} is the metric on M^* and $Ric = Ric_{M^*} + Ric_I$. If we choose $U = \partial_t$, then by the previous theorem, we have

$$Ric = (\rho r + \lambda)[g - \partial_t^\flat \otimes \partial_t^\flat] \tag{3.36}$$

which implies $Ric_I = 0$, so we get

$$Ric_{M^*}(X,Y) = (\rho r + \lambda) g_{M^*}(X,Y) \quad for\ all\ X,Y \in \chi(M^*). \tag{3.37}$$

It can be seen that M^* is an Einstein manifold with constant scalar curvature, so the proof is completed.

In the following, we will give an example for (m,ρ)-quasi Einstein manifolds of dimension $n = 3$.

Example 3.3.1 Consider a Riemannian manifold M with $(n=3)$, which has local coordinates (t,x,y), frame fields $\partial_t, \partial_x, \partial_y$, and the Rimennian metric tensor

$$g_h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & h(t) \end{pmatrix} \tag{3.38}$$

for some function h depending only on t. [22] Then, we have

$$\nabla_{\partial_t}\partial_y = \frac{1}{2}h_t\partial_t, \qquad \nabla_{\partial_y}\partial_y = \frac{1}{2}hh_t\partial_t - \frac{1}{2}h_t\partial_y \tag{3.39}$$

$$R(\partial_t,\partial_y)\partial_t = -\frac{1}{2}h_{tt}\partial_t, \qquad R(\partial_t,\partial_y)\partial_y = -\frac{1}{2}hh_{tt}\partial_t + \frac{1}{2}h_{tt}\partial_y \tag{3.40}$$

where the corresponding Ricci tensor is found as

$$Ric = \begin{pmatrix} 0 & 0 & \frac{1}{2}h_{tt} \\ 0 & 0 & 0 \\ \frac{1}{2}h_{tt} & 0 & \frac{1}{2}hh_{tt} \end{pmatrix} \tag{3.41}$$

whose scalar curvature is $r = h_{tt}$. Additionally, the manifold (M, g_h) should satisfy the eq.(3.34) so that it has the (m,ρ)-quasi Einstein structure. Using the components of the Ricci tensor and the metric g in the equation (3.34) yields

$$u_i u_j = \begin{cases} 1 & i=j=2 \\ 0 & otherwise \end{cases} \tag{3.42}$$

which implies that we have a solution when the potential function f depends only on x. Hence, we obtain the following reduced system of differential equations:

$$f_{xx} - \frac{1}{m}(f_x)^2 = \rho h'' + \lambda, \qquad \frac{1}{2}h'' = \rho h'' + \lambda \tag{3.43}$$

where "’" denotes the partial derivative with respect to t. These can be solved as follows:

$$f(x) = c_2 - m\ln(\cos(\frac{\sqrt{k}(c_1+x)}{\sqrt{m}})) \; or \; f(x) = c_2 - m\ln(\sin(\frac{\sqrt{k}(c_1+x)}{\sqrt{m}})) \; for \; k>0 \tag{3.44}$$

and the potential function will be in the same form with the hyperbolic functions instead of cos & sin functions for $k<0$ where $h(t) = kt^2 + lt + p$ for some constants $k, l, p, c_1, c_2 \in R$.

4. RICCI SOLITONS

4.1 Ricci Flow

The primary purpose of this section is an introduction for the Ricci solitons to answer the questions such as where these structures emerge from and what is the physical interpretation of this kind of geometric forms.

In the beginning of the 20^{th} century, French mathematician Henri Poincaré claimed that every simply connected, closed 3-dimensional manifolds are homeomorphic to the $\mathbb{S}^3$ in a sense that 3-sphere is the only 3-dimensional space where each loop is able to shrunk to a point continuously. The conjecture may seem easy but it had stayed unsolved until the next century. After Poincaré, Thurston stated the geometrization conjecture which is about the complete classification of the compact 3-dimensional manifolds. Hence, it could give an answer for Poincaré conjecture as well. In 1982, Hamilton introduced the concept of Ricci flow which was used to solve geometrization conjecture by Perelman later.

Definition 4.1.1 Let (M, g_0) be a Riemannian manifold. The following partial differential equation is called *the Ricci flow* which evolves the metric tensor:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2Ric(g(t)), \\ g(0) &= g_0. \end{aligned} \tag{4.1}$$

The Ricci flow can be considered as heat-type equation in which it averages out the curvature as homogeneous heat radiation. [24] To gain some intuition about the geometry of it, let's consider the mean curvature flow. Take a closed 2-dimensional surface in the 3-dimensional space and draw a tangent plane to each point: Now move each point in the perpendicular direction to the tangent plane at that point. Additionally, let the distance of the movement be proportional to the curvature of the point and the inside volume be fixed. Then, the surface will get rounder and be a sphere eventually as in the Figure 4.1. (If we do not take the inside volume fixed, then

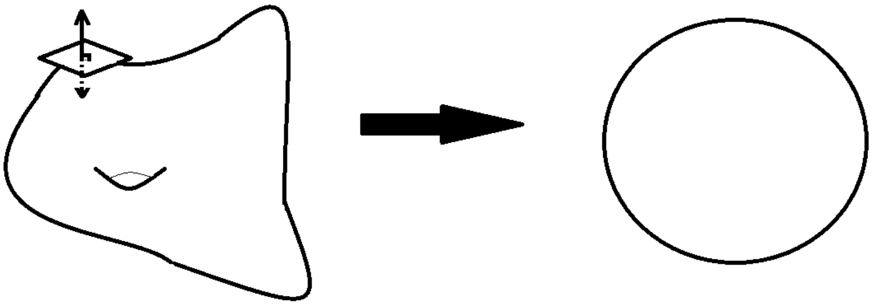

Figure 4.1 : Mean Curvature Flow

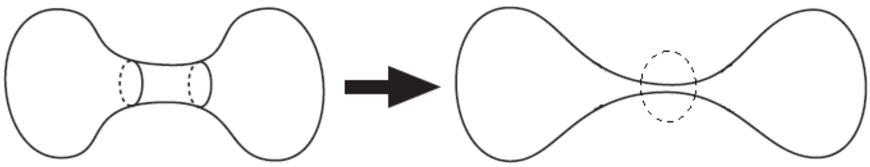

Figure 4.2 : Singularity Points

it will shrink to a point at the end.) The problem occurs if we take an hour glass-like surface initially. After some time, we will end up with some point where the curvature blows up, which is drawn in the Figure 4.2. In the case of the Ricci flow, the metric g evolves in time proportional to the something including the (Ricci) curvature. It is likely that the flow will create singularities as we tried to illustrate in the case of the mean curvature flow. There is a structure which corresponds to self-similar solutions of the Ricci flow and models the formation of singularities in the flow will be defined the next.

4.2 Gradient Ricci Solitons

Definition 4.2.1 A complete Riemannian manifold (M,g) is said to be a *Ricci soliton* if the following equation

$$Ric+\frac{1}{2}\mathscr{L}_{\xi}g=\lambda g \tag{4.2}$$

holds for some smooth vector field ξ and a constant λ. A Ricci soliton is said to be

- *steady* if $\lambda=0$,
- *shrinking* if $\lambda>0$,
- *expanding* if $\lambda<0$.

Furthermore, we have *gradient Ricci soliton* if the potential field ξ is the gradient of some smooth function f on M. In this case, the eq.(4.2) becomes

$$Ric + \nabla^2 f = \lambda g \tag{4.3}$$

where $\nabla^2 f$ stands for the Hessian of f. Notice that we get an Einstein manifold when f is constant and say that a soliton is trivial. Gradient Ricci solitons are significant part of the studies on the Ricci flow since they correspond to the solutions which evolve along symmetries of the flow. Besides from that, the Ricci solitons do not need to be compact. In particular, the compact Ricci solitons are known as the fixed points of the Ricci flow. [25]

Lemma 4.2.1 [26] Take a complete gradient Ricci soliton (M, g) and let f be the corresponding potential function. Then we have

$$r + |\nabla f|^2 - 2\lambda f = C \tag{4.4}$$

for some constant C.

Proof: A gradient Ricci soliton must satisfy the equation (4.3). If we consider the covariant derivative of the equation (4.3) and skew-symmetrize with respect to $k \leftrightarrow j$, then we get

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl} \nabla_l f = 0. \tag{4.5}$$

Multiplying by g^{jk} and recall the contracted second Bianchi identity $\nabla_j R_{ij} = \frac{1}{2} \nabla_i r$ gives

$$\nabla_i r = 2 R_{ij} \nabla_j f. \tag{4.6}$$

Adding the same terms in the both sides does not change the equality, so

$$\nabla_i (r + |\nabla f|^2 - 2\lambda f) = 2(R_{ij} + \nabla_i \nabla_j f - \lambda g_{ij}) \nabla_j f = 0. \tag{4.7}$$

Therefore,

$$r + |\nabla f|^2 - 2\lambda f = C \tag{4.8}$$

for some constant C.

Remark: It is possible to rescale the metric and shift the function f by a constant for shrinking Ricci solitons so that

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij} \text{ and } r + |\nabla f|^2 - f = 0. \tag{4.9}$$

Proposition 4.2.1 [26] [27] Any compact steady or expanding gradient Ricci soliton must be Einstein.

Proof: We are going to show the statement for the expanding gradient Ricci solitons. The steady case is similar. For a compact shrinking gradient Ricci soliton (M, g) with some $\lambda < 0$, if we take the trace of the soliton equation, we obtain

$$r + \Delta f = n\lambda. \tag{4.10}$$

By using the lemma 4.2.1, we find

$$\Delta f - |\nabla f|^2 = -2\lambda f + C_0 \tag{4.11}$$

for some constant C_0. Using the maximum principle, we see

$$-2\lambda f|_{max} + C_0 \leq 0$$
$$-2\lambda f|_{min} + C_0 \geq 0$$

which implies $f|_{max} = f|_{min}$, so f is constant. Thus, the soliton is Einstein.

It can be said that there are no compact gradient steady or expanding Ricci solitons in the dimensions of $n = 2, 3$ except for those of constant curvature.

Lemma 4.2.2 [20] Let (M^n, g) be a Riemannian manifold. Then, the equation $C_{kij,k} = 0$ holds for the Cotton tensor.

Proof: Let's start with taking the covariant derivative of the Cotton tensor:

$$C_{ijk,t} = R_{ij,kt} - R_{ik,jt} - \frac{1}{2(n-1)}(r_{,kt}\delta_{ij} - r_{,jt}\delta_{ik}). \tag{4.12}$$

We also know that [20]

$$R_{ik,jk} = R_{ik,kj} + R_{tijk}R_{tk} + R_{tkjk}R_{ti} = \frac{1}{2} r_{ij} - R_{tk}R_{itjk} + R_{it}R_{tj} \tag{4.13}$$

from the Schur's identity. Then, the divergence of the Cotton tensor can be expressed as

$$C_{ijk,k} = R_{ij,kk} - \frac{n-2}{2(n-1)} r_{,ij} + R_{tk}R_{itjk} - R_{it}R_{tj} - \frac{1}{2(n-1)} \Delta r \delta_{ij} \tag{4.14}$$

which shows that $C_{ijk,k} = C_{jik,k}$. We also have the next equation from the equation (3.15);

$$W_{ijkt} + W_{itjk} + W_{iktj} = 0 \tag{4.15}$$

which is known as the first Bianchi identity for the Weyl tensor, that implies

$$C_{ijk} + C_{jki} + C_{kij} = 0. \tag{4.16}$$

If we look at the covariant derivative of the equation (4.16), we get

$$C_{kij,k} = 0. \tag{4.17}$$

Proposition 4.2.2 [19] Let (M^n, g) be a gradient Ricci soliton with potential function f. Then the Weyl, Cotton, Bach, and D tensors satisfy the following equations:

$$C_{ijk} + f_t W_{tijk} = D_{ijk}, \tag{4.18}$$

$$(n-2)B_{ij} - \left(\frac{n-3}{n-2}\right) f_t C_{jit} = D_{ijk,k}, \tag{4.19}$$

$$R_{kt}C_{kti} = (n-2)D_{itk,tk}, \tag{4.20}$$

$$\frac{1}{2}|C|^2 + R_{kt}C_{kti,i} = (n-2)D_{itk,tki}. \tag{4.21}$$

Proof: We will prove the first two equations. Let's start with taking the covariant derivative of the equation (4.3);

$$R_{ij,k} + f_{ijk} = 0. \tag{4.22}$$

If we skew-symmetrize this with respect to $k \leftrightarrow j$ and use the commutation relation $f_{ijk} - f_{ikj} = f_t R_{tijk}$ and the formula for the Weyl tensor, we get

$$R_{ij,k} - R_{ik,j} = f_t R_{tikj} \tag{4.23}$$

$$= f_t[W_{tikj} + \frac{1}{n-2}(R_{tk}g_{ij} - R_{tj}g_{ik} + R_{ij}g_{tk} - R_{ik}g_{tj}) \tag{4.24}$$

$$- \frac{r}{(n-1)(n-2)}(g_{tk}g_{ij} - g_{tj}g_{ik})]$$

$$= -f_t W_{tijk} + \frac{1}{n-2}(f_t R_{tk}g_{ij} - f_t R_{tj}g_{ik} + f_k R_{ij} - f_j R_{ik}) \tag{4.25}$$

$$- \frac{r}{(n-1)(n-2)}(f_k g_{ij} - f_j g_{ik})$$

Now when we use the definitions of the Cotton tensor and the D tensor, the equation (4.18) follows easily. To prove the next equation, take the divergence of the equation (4.18);

$$C_{ijk,k} + f_{tk}W_{tijk} + f_t W_{tijk,k} = D_{ijk,k}. \tag{4.26}$$

Put this into the definition of the Bach tensor to get

$$(n-2)B_{ij} = D_{jik,k} - f_{tk}W_{tjik} - f_t W_{tjik,k} + R_{kt}W_{ikjt}. \tag{4.27}$$

Using the soliton equation, the relation formula for the Weyl and the Cotton tensor, and the fact that the Weyl tensor is trace-free gives the equation (4.19) as desired.

Note that these four equations we just proved are known as integrability conditions for the gradient Ricci solitons.

4.3 Examples For Ricci Solitons

For the examples throughout the section, we have considered the resources [7] and [28].

Example 4.3.1 (Hamilton's Cigar Soliton) A complete noncompact steady soliton on $\mathbb{R}^2$ with the given metric and the potential function by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$
$$f = -log(1 + x^2 + y^2)$$

The Cigar Soliton has positive (Gaussian) curvature and linear volume growth. Also, it is asymptotic to a cylinder of finite perimeter at ∞. It is also known as Witten's black hole in general relativity.

Example 4.3.2 (Byrant Soliton) It is an analog of Cigar soliton for higher dimensions $(n \geq 3)$, which is rotationally symmetric and have positive sectional curvature. Also, it is asymptotic to a paraboloid in a different way from the Cigar case.

Example 4.3.3 (Gaussian Soliton) It is $(\mathbb{R}^n, g_0)$ with the flat Euclidean metric which has both shrinking and expanding gradient Ricci solitons, called the Gaussian shrinker or expander.

- A gradient shrinker with potential function $f = \frac{|x|^2}{4}$;

$$Ric + \nabla^2 f = \frac{1}{2} g_0.$$

- A gradient expander with potential function $f = -\frac{|x|^2}{4}$;

$$Ric + \nabla^2 f = -\frac{1}{2} g_0.$$

Example 4.3.4 (Warped Products) It is constructed by using doubly warped product and multiple warped product to produce noncompact gradient steady solitons by Ivey and Dancer-Wang.

5. THE STRUCTURE OF RICCI SOLITONS ON SOME SPECIAL KIND OF EINSTEIN MANIFOLDS

In the previous chapters, we have mentioned about the concepts of the Riemannian geometry, special type of Einstein manifolds, Ricci solitons, and their importance in differential geometry and physics. In this last chapter, we will explain about what we have worked on, how our work contribute to the related fields even to a little extent, and what are the ongoing problems and the current studies.

In this master's thesis, it has been searched for the Ricci soliton structure after admitting different kinds of vector fields on various types of Einstein manifolds.

5.1 The Gradient Einstein-Type Manifold With a Parallel Vector Field

Let (M^n, g) be a gradient Einstein-type manifold. Then it has a potential function f satisfying the equation

$$\alpha R_{ij} + \beta f_{ij} + \mu f_i f_j = (\rho r + \lambda) g_{ij} \tag{5.1}$$

which is given in the earlier chapter. Now, take a parallel vector field ϕ given on the manifold M. Since $\nabla_X \phi = 0$ for all $X \in \chi(M)$, we have $\phi_{i,j} = 0$. If we apply the Ricci identity, we also get

$$\phi_{i,jk} - \phi_{i,kj} = \phi_h R^h_{ijk} = 0 \implies \phi^h R_{hijk} = 0 \implies \phi^h R_{hk} = 0. \tag{5.2}$$

Define a new function A such that

$$A = \phi^i f_i \implies A_j = \phi^i f_{ij} \tag{5.3}$$

$$\implies A_{,jk} = \phi^i f_{ijk} = Hess(\phi(f)). \tag{5.4}$$

From the above equation, it is seen that $A_{,jk} - A_{,kj} = 0$ since the Hessian is symmetric. When we turn back to the equation (5.1) and multiply the the equation by ϕ^i, we obtain

$$\alpha \phi^i R_{ij} + \beta \phi^i f_{ij} + \mu \phi^i f_i f_j = (\rho r + \lambda) \phi^i g_{ij}. \tag{5.5}$$

Using the equations (5.3) and (5.2) in the equation (5.5) gives

$$\beta A_j + \mu A f_j = (\rho r + \lambda)\phi_j. \tag{5.6}$$

If we take the covariant derivative with respect to k, we find

$$\beta A_{,jk} + \mu A_k f_j + \mu A f_{jk} = \rho r_k \phi_j. \tag{5.7}$$

Skew-symmetrize the equation with respect to $k \leftrightarrow j$ to get

$$\mu(A_k f_j - A_j f_k) = \rho(r_k \phi_j - r_j \phi_k). \tag{5.8}$$

From the equation (5.6), we have

$$\mu(A_k f_j - A_j f_k) = \mu[\frac{1}{\beta}(-\mu A f_k + (\rho r + \lambda)\phi_k) f_j - \frac{1}{\beta}(-\mu A f_j + (\rho r + \lambda)\phi_j) f_k] \tag{5.9}$$

When we arrange the above equation, we get

$$\mu(A_k f_j - A_j f_k) = \frac{\mu(\rho r + \lambda)}{\beta}(\phi_k f_j - \phi_j f_k). \tag{5.10}$$

If we combine the equations (5.8) and (5.10), we see

$$[\frac{\mu(\rho r + \lambda)}{\beta} f_j + \rho r_j]\phi_k - [\frac{\mu(\rho r + \lambda)}{\beta} f_k + \rho r_k]\phi_j = 0. \tag{5.11}$$

Leave this equation here and multiply the equation (5.8) by ϕ^k to get

$$\mu \phi^k A_k f_j - \mu A_j \phi^k f_k = \rho \phi^k r_k \phi_j - \rho r_j \phi^k \phi_k. \tag{5.12}$$

Let $\mu \phi^k A_k = \psi$ and $\phi^k r_k = \gamma$, then the equation becomes

$$\rho r_j \|\phi\|^2 = \mu A A_j + \rho \gamma \phi_j - \psi f_j. \tag{5.13}$$

Combining this equation with the equation (5.6) leads us the following equation;

$$\rho r_j \|\phi\|^2 = (-\frac{\mu^2 A^2}{\beta} - \psi) f_j + (\frac{\mu A(\rho r + \lambda)}{\beta} + \rho \gamma)\phi_j. \tag{5.14}$$

Pull ρr_j and ρr_k from the above equation and put them into the equation (5.11), then we have

$$[\frac{\mu(\rho r + \lambda)}{\beta} - \frac{1}{\|\phi\|^2}(\frac{\mu^2 A^2}{\beta} + \psi)] f_j \phi_k = [\frac{\mu(\rho r + \lambda)}{\beta} - \frac{1}{\|\phi\|^2}(\frac{\mu^2 A^2}{\beta} + \psi)] f_k \phi_j. \tag{5.15}$$

From the above equation, a solution naturally comes that the derived vector fields from f and ϕ are collinear, i.e. $f_{,j} = \varepsilon\phi_{,j}$ for some smooth function ε. As a consequence of this

$$A = \varepsilon\|\phi\|^2 \implies \varepsilon_{,j} = \frac{A_{,j}}{\|\phi\|^2} \tag{5.16}$$

comes out. On the other hand, the next equation comes into being from the previous equation and the equation (5.6);

$$f_{jk} = \varepsilon_k\phi_j = \frac{1}{\|\phi\|^2\beta}[(\rho r+\lambda)\phi_k - \mu A f_k]\phi_j, \tag{5.17}$$

$$= \frac{1}{\|\phi\|^2\beta\varepsilon}[\frac{(\rho r+\lambda)}{\varepsilon} - \mu A]f_k f_j. \tag{5.18}$$

Before continuing, let's check for the incident that is

$$\frac{(\rho r+\lambda)}{\varepsilon} = \mu A. \tag{5.19}$$

This indicates f_{ij}, A_j, and ε_j are all zero with regard of the equations (5.16) and (5.17), ie. the functions A and ε are constant. Then, the main equation becomes

$$\alpha R_{ij} + \mu f_i f_j = (\rho r+\lambda)g_{ij}. \tag{5.20}$$

And hence

$$\alpha R_{ij} + \mu\varepsilon^2\phi_i\phi_j = (\rho r+\lambda)g_{ij}. \tag{5.21}$$

which indicates that

$$R_{ij} = \frac{(\rho r+\lambda)}{\alpha}(g_{ij} - u_i u_j) \implies r = \frac{(\rho r+\lambda)}{\alpha}(n-1) \tag{5.22}$$

for $u_i = \frac{\phi_i}{\|\phi\|}$. Now, assume that $\frac{(\rho r+\lambda)}{\varepsilon} \neq \mu A$. Putting the equation (5.18) into the equation (5.1) gives

$$\alpha R_{ij} + [\frac{(\rho r+\lambda)}{\|\phi\|^2\varepsilon^2}]f_i f_j = (\rho r+\lambda)g_{ij}. \tag{5.23}$$

We have $f_i = \varepsilon\phi_i = \varepsilon u_i\|\phi\|$, and hence

$$R_{ij} = \frac{(\rho r+\lambda)}{\alpha}(g_{ij} - u_i u_j). \tag{5.24}$$

Take the trace of the equation to obtain

$$r = \frac{(\rho r+\lambda)}{\alpha}(n-1) \implies r = \frac{\lambda(n-1)}{\alpha - \rho(n-1)} \equiv constant. \tag{5.25}$$

which is similar with the other case. Note that $\alpha \neq \rho(n-1)$ here. It can be seen the Ricci soliton structure for

$$G_{ij} = \frac{(\rho r + \lambda)}{\alpha} u_i u_j, \tag{5.26}$$

then the equation (5.1) becomes

$$Ric + G = \lambda^* g \tag{5.27}$$

where $G = \frac{1}{2}\mathscr{L}_G g = \frac{\beta}{\alpha} f_{,ij} + \frac{\mu}{\alpha} f_{,i} f_{,j}$ and $\lambda^* = \frac{(\rho r+\lambda)}{\alpha} = \frac{r}{(n-1)}$ as desired. It can be also seen the nearly quasi-Einstein manifold structure for

$$E_{ij} = u_i u_j, \tag{5.28}$$

then the equation (5.1) becomes

$$R_{ij} = a g_{ij} + b E_{ij} \tag{5.29}$$

where

$$a = \frac{r}{n-1} \quad and \quad b = -\frac{r}{n-1} \tag{5.30}$$

by the equations (5.22) and (5.24). Hence, we have the following theorems.

Theorem 5.1.1 A gradient Einstein-type manifold M with admitted a parallel vector field ϕ has nearly quasi-Einstein manifold. Moreover, its associated scalars are constant such that the sum is equal to zero.

From the equations (5.26) and (5.27), we get the following theorem.

Theorem 5.1.2 A gradient Einstein-type manifold M with admitted a parallel vector field ϕ has Ricci soliton.

On the other hand, if we take $\frac{(\rho r+\lambda)}{\alpha} = \frac{1}{2}$, then we get a shrinking gradient Ricci soliton corresponding to a self-similar solution $g_{ij}(t)$ for the Ricci flow in which

$$g_{ij}(t) = (1-t)\varphi_t^*(g_{ij}) \quad t < 1 \tag{5.31}$$

where φ_t are the 1-parameter family of diffeomorphisms generated by $\frac{\nabla f}{(1-t)}$ according to the study of Cao. [29]

Theorem 5.1.3 A gradient Einstein-type manifold (M^n, g) with admitted a parallel vector field ϕ has vanishing Cotton tensor for $n \geq 3$.

Proof: The Ricci tensor of the manifold M is parallel with regard of the equation (5.24). If we use this fact and the scalar curvature r being constant in the definition of the Cotton tensor, it is easily seen that $C_{ijk} = 0$ for $n \geq 3$.

Theorem 5.1.4 A gradient Einstein-type manifold (M^n, g) with admitted a parallel vector field ϕ has vanishing D_{ijk} tensor for $n \geq 3$ where

$$D_{ijk} = \frac{1}{(n-2)}(R_{jk}f_i - R_{ik}f_j) + \frac{1}{(n-1)(n-2)}(R_{il}f^l g_{jk} - R_{jl}f^l g_{ik}) - \frac{r}{(n-1)(n-2)}(g_{jk}f_i - g_{ik}f_j).$$

Proof: If we use the equations (5.24) and (5.25) in the above equation, we get

$$\begin{aligned} D_{ijk} = &\frac{1}{(n-2)}[f_i \frac{r}{(n-1)}(g_{jk} - u_j u_k) - f_j \frac{r}{(n-1)}(g_{ik} - u_i u_k)] \\ &+ \frac{1}{(n-1)(n-2)} f^l [\frac{r}{(n-1)}(g_{li} - u_i u_l) g_{jk} - \frac{r}{(n-1)}(g_{lj} - u_l u_j) g_{ik}] \\ &- \frac{r}{(n-1)(n-2)}(f_i g_{jk} - f_j g_{ik}) \end{aligned} \tag{5.32}$$

which leads us

$$D_{ijk} = 0. \tag{5.33}$$

since $f_i = \varepsilon u_i \|\phi\|$.

We could show the same result by using the below relation

$$D_{ijk} = \frac{1}{n-2}(A_{jk}\nabla_i f - A_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}E_{il} - g_{ik}E_{jl})\nabla_l f \tag{5.34}$$

where Einstein tensor $E_{ij} = R_{ij} - \frac{r}{2}g_{ij}$. [30] This structure has importance in the field because it also verifies couples of other theorems recently proved. In H. D. Cao and Q. Chen's relevant work [30], they showed that (M^n, g) has the Cotton tensor $C_{ijk} = 0$ for $n \geq 5$ if it is a complete gradient Ricci soliton with tensor $D_{ijk} = 0$, which is supported by our case. Additionally, it is compatible with the theorem (5.1.3). The relation $\nabla^l W_{ijkl} = -\frac{n-3}{n-2} C_{ijk}$ and the theorem (5.1.3) implies the next theorem.

Theorem 5.1.5 A gradient Einstein-type manifold M with admitted a parallel vector field ϕ has conharmonic curvature tensor.

Theorem 5.1.6 If (M^4, g, f) is a complete gradient Einstein-type manifold with admitted a parallel vector field and positive scalar curvature, then the manifold is locally conformally flat.

Proof: Cao and Chen also showed that complete gradient Ricci solitons with $D_{ijk} = 0$ are locally conformally flat for $n = 4$. [30] With regard of this fact and the equations (5.24), (5.25), (5.27), and (5.33) the result is seen clearly.

Theorem 5.1.7 If (M^n, g, f) is a complete gradient Einstein-type manifold with admitted a parallel vector field, then the manifold is Bach flat.

Proof: We know that [19]

$$(n-2)B_{ij} - \left(\frac{n-3}{n-2}\right) f_t C_{jit} = D_{ijk,k}. \tag{5.35}$$

Hence from the theorems (5.1.3) and (5.1.4), the result follows.

Thus, we can say that the manifold in the next example is Bach flat.

Example 5.1.1 Consider a Riemannian manifold M with $(n = 3)$, which has local coordinates (t, x, y), frame fields $\partial_t, \partial_x, \partial_y$, and the Rimennian metric tensor

$$g_h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & h(t) \end{pmatrix} \tag{5.36}$$

for some function h depending only on t. [22] Then, we have

$$\nabla_{\partial_t}\partial_y = \frac{1}{2}h_t\partial_t, \qquad \nabla_{\partial_y}\partial_y = \frac{1}{2}hh_t\partial_t - \frac{1}{2}h_t\partial_y \tag{5.37}$$

$$R(\partial_t, \partial_y)\partial_t = -\frac{1}{2}h_{tt}\partial_t, \qquad R(\partial_t, \partial_y)\partial_y = -\frac{1}{2}hh_{tt}\partial_t + \frac{1}{2}h_{tt}\partial_y \tag{5.38}$$

where the corresponding Ricci tensor is found as

$$Ric = \begin{pmatrix} 0 & 0 & \frac{1}{2}h_{tt} \\ 0 & 0 & 0 \\ \frac{1}{2}h_{tt} & 0 & \frac{1}{2}hh_{tt} \end{pmatrix} \tag{5.39}$$

whose scalar curvature is $r = h_{tt}$. Additionally, we also have

$$u_i u_j = \begin{cases} 1 & i = j = 2 \\ 0 & otherwise \end{cases} \tag{5.40}$$

as well, so the potential function f will be depending only on x. However, the manifold M should satisfy the equation for gradient Einstein-type manifolds this time. It brings us the following differential equation system;

$$\frac{\beta}{\alpha}f_{xx} + \frac{\mu}{\alpha}(f_x)^2 = \frac{h_{tt}}{2} = \frac{r}{(n-1)} \tag{5.41}$$

$$\frac{h_{tt}}{2} = \frac{(\rho h_{tt} + \lambda)}{\alpha} \tag{5.42}$$

where $h(t) = a_1t^2 + a_2t + a_3$ for some constants $a_1, a_2, a_3 \in \mathbb{R}$, $r = 2a_1$, and $\frac{r}{(n-1)} = \frac{(\rho r+\lambda)}{\alpha}$. This system can be solved as;

$$f(x) = \begin{cases} f(x) = \frac{\ln(|e^{2\sqrt{k_1k_2}-2\sqrt{k_1x}}+1|)+\sqrt{k_1x}}{\frac{\mu}{\beta}} + k_3 & -\sqrt{k_1} \leq \frac{\mu}{\beta} f_x \leq \sqrt{k_1}, \\ f(x) = \frac{\ln(|e^{2\sqrt{k_1k_2}-2\sqrt{k_1x}}-1|)+\sqrt{k_1x}}{\frac{\mu}{\beta}} + k_3 & |\frac{\mu}{\beta} f_x| > \sqrt{k_1}, \end{cases} \tag{5.43}$$

where $k_1 = \frac{\mu}{\beta^2}(2a_1\rho + \lambda) = \frac{\mu}{\beta^2}(\rho r + \lambda)$, $k_1, k_2 > 0$, and for some constant $k_3 \in \mathbb{R}$.

Theorem 5.1.8 Let M be the 3-dimensional complete gradient Einstein-type manifold with admitted a parallel vector field and local coordinates (t,x,y) endowed with the metric $g_h = 2dtdy + (dx)^2 + h(t)(dy)^2$ where f is the potential function depending only on x given by the equation (5.43). Then the manifold M is a Ricci soliton where λ^* is constant for $Ric + G = \lambda^* g$ as in the equation (5.27).

Special Case: In the equation (5.1), if we take constants $(\alpha, \beta, \mu) = (1, 1, -\frac{1}{m})$, the manifold becomes (m, ρ)-quasi Einstein manifold. Applying the same process gives a gradient Ricci soliton with the following results:

$$R_{ij} = (\rho r + \lambda)(g_{ij} - u_iu_j) = \frac{r}{(n-1)}(g_{ij} - u_iu_j) \tag{5.44}$$

$$r = (\rho r + \lambda)(n-1) \tag{5.45}$$

$$G_{ij} = (\rho r + \lambda)u_iu_j \tag{5.46}$$

Remark that the expression of Ricci tensor with the scalar curvature r is just like in the general case. Hence, the same results hold for (m, ρ)-quasi Einstein manifolds as well.

Theorem 5.1.9 Let (M^n, g) be a complete $(n \geq 3)$ nontrivial gradient Einstein-type manifold with admitted a parallel vector field ϕ. Then the manifold M is isometric to the direct product of an interval and a $(n-1)$ dimensional Einstein manifold.

Proof: It is easily seen that the Ricci solitons we obtained earlier possess parallel Ricci tensor with regard of the equations (5.24) and (5.25). Then, the result is coming from the work of Z. Hu, D. Li and S. Zhai in their paper. [31] Also, we know that (m, ρ)-quasi Einstein manifolds are a special case of gradient Einstein-type manifolds. The corresponding theorem for (m, ρ)-quasi Einstein manifolds is proved in [22].

[32], [33], [34], [35], [36], [37], [38], [39], and [40] are the additional resources that have been checked out in the process.

6. CONCLUSIONS AND RECOMMENDATIONS

The Riemannian geometry is valuable part of overall science due to the fact that it has developed the perception of surfaces and geometrical structures. Einstein manifolds and Ricci solitons are respectively new topics of this concept. These two notions have serious physical meanings behind. Thus, they have become intriguing in the context to look for the relation between them.

In this thesis, it has been searched for the Ricci soliton structure after admitting different kinds of vector fields on some special types of Einstein manifolds. With this purpose, we have considered parallel vector fields. We have obtained some interesting outcomes from the research. First one is the manifold hosts both Ricci soliton and nearly quasi-Einstein manifold structures in this case. Secondly, the manifold has vanishing Cotton tensor, so does conharmonic curvature tensor. This situation help us come to the conclusion about the Weyl and Bach tensors as well. According to the last observation, the gradient Einstein-type manifolds with associated parallel vector field is isomorphic to the warped product of an interval and $(n-1)$ dimensional Einstein manifold.

For future studies, there may also some interesting findings for the conformal vector fields on various types of Einstein manifolds. Moreover, this case may be related to Yamabe solitons.

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