İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF SCIENCE

MULTI-NORMS

M.Sc. THESIS by
MEHMET SELÇUK TÜRER

Programme: Mathematics and Computer Science
Science Programme: Mathematics and Computer Science

Thesis Supervisor: Assoc. Prof. Dr. Mert ÇAĞLAR

JUNE 2010

İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF SCIENCE

MULTI-NORMS

M.Sc. THESIS by
MEHMET SELÇUK TÜRER
(0809041043)

Date of submission: 14 June 2010
Date of deffence examination: 24 June 2010

Thesis Supervisor: Assoc. Prof. Dr. Mert ÇAĞLAR

JUNE 2010

İSTANBUL KÜLTÜR ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

ÇOKLU-NORMLAR

YÜKSEK LİSANS TEZİ
MEHMET SELÇUK TÜRER
(0809041043)

Tezin Enstitüye verildiği tarih: 14 Haziran 2010
Tezin savunulduğu tarih: 24 Haziran 2010

Tez danışmanı: Doç. Dr. Mert ÇAĞLAR

HAZİRAN 2010

"Multi-norms," a dissertation prepared by Mehmet Selçuk TÜRER in partial fulfillment of the requirements for the degree Master of Science, has been accepted by the commitee in charge:

Assoc. Prof. Dr. Mert ÇAĞLAR (Supervisor) ________________

Assist. Prof. Dr. R. Tunç MISIRLIOĞLU ________________

Assoc. Prof. Dr. Erhan ÇALIŞKAN (YTÜ) ________________

____

Date

Abstract

MULTI-NORMS

TÜRER, Mehmet Selçuk
M.Sc., Department of Mathematics and Computer Science
Supervisor: Assoc. Prof. Dr. Mert ÇAĞLAR

June 2010, 37 pages

The present work deals with the so-called "multi-normed spaces," developed by H. G. Dales and M. E. Polyakov. The main goal of the thesis is to study an open problem given by H. G. Dales, about direct sum decompositions within the context of Banach lattices. We present an approach to the solution of it for the case of Banach lattice $L^p(\mathbb{I})$, where $\mathbb{I}$ is the closed unit interval.

Keywords: Banach lattice, multi-norm, multi-normed space.

ÖZET

ÇOKLU-NORMLAR

TÜRER, Mehmet Selçuk
Yüksek Lisans, Matematik-Bilgisayar Bölümü
Tez Danışmanı: Doç. Dr. Mert ÇAĞLAR

Haziran 2010, 37 sayfa

Eldeki çalışmada H. G. Dales ve M. E. Polyakov tarafından geliştirilen "çok-normlu uzaylar" ile ilgilenilmektedir. Bu tezin asıl amacı, Banach örgülerinin direkt toplam ayrışımları hakkında H. G. Dales tarafından verilen açık bir problem üzerine çalışmaktır. $\mathbb{I}$ kapalı birim aralık olmak üzere, $L^p(\mathbb{I})$ Banach örgüsü için problemin çözümü verilmiştir.

Anahtar Kelimeler: Banach örgüsü, çoklu-norm, çok-normlu uzay.

To my family

Acknowledgments

I would like to express my sincere gratitude to my supervisor Mert ÇAĞLAR for introducing me to the multi-norm theory and for his numerous suggestions and improvements. I also would like to thank R. Tunç MISIRLIOĞLU who has always provided to come into being new ideas in my mind.

I am grateful to my friends Uğur GÖNÜLLÜ and Mesut YILDIRIM for their supports.

TABLE OF CONTENTS

ABSTRACT iv

ÖZET v

ACKNOWLEDGMENTS vii

TABLE OF CONTENTS viii

CHAPTER

1 INTRODUCTION 1

2 THE AXIOMS AND THEIR CONSEQUENCES 3

2.1 Preliminaries 3

2.2 The Axioms 12

3 EXAMPLES OF MULTI-NORMS 19

3.1 The minimum multi norm 19

3.2 The maximum multi-norm 21

3.3 Specific elementary examples 22

4 MULTI-NORMS ON BANACH SPACES 24

4.1 Banach lattices and multi-norms 24

4.2 Operator spaces and multi-norms 26

5 Orthogonality 28
5.1 Terminology 28
5.2 Orthogonal Decompositions 29
5.3 A problem on direct sum decompositions 31

references 35

Vita 37

CHAPTER 1

Introduction

The notion of a multi-normed space was introduced by H. G. Dales and M. E. Polyakov. They generalized the normed linear space E to a 'multi-normed space,' and constructed a new theory, namely 'multi-norm theory'. It is a similar generalization of a Banach algebra to a 'multi-Banach algebra'. The motivation therein was to answer some problems of amenability. This notion has also a natural counterpart in the theory of operator spaces. A multi-normed space can also be seen as arising as an operator sequence space, which is developed in detail by Effros and Ruan (see **[7]** and the references therein).

In **[12]**, 'type-p multi-normed spaces' was defined by Paul Ramsden. This generalizes the construction of Dales and Polyakov. In **[5]** Dales and Moslehian investigate some properties of 'multi-bounded' mappings on multi-normed spaces. Moreover, they prove a generalized Hyers−Ulam−Rassias stability theorem associated to the Cauchy additive equation for mappings from linear spaces into multi-normed spaces. In **[11]**, Moslehian and Srivastava investigate the Hyers−Ulam stability of the Jensen functional equation for mappings from linear spaces into multi-normed spaces. They establish an assymptotic behavior of the Jensen equation in the framework of multi-normed spaces.

The present thesis consists of 5 chapters. In Chapter 2, we begin by reminding some standard notions, and give multi-norm axioms and immediate consequences of them.

Chapter 3 deals with the most obvious multi-norms examples, namely minimum and maximum multi-norms, and we give some specific examples.

In Chapter 4, we deal with the multi-norms on Banach spaces. We give a relation between Banach lattices and multi-norms, and operator spaces and multi-norms.

Finally, Chapter 5, which is the core of our study, is devoted to a solution of a problem on the direct sum decompositions within the context of complex Banach lattices which is given in **[6, 15]**. For the Banach lattices $C(\Omega)$ and ℓ^p, the solutions are given by H. G.

Dales (see **[6]**), and in this context, we give a solution of the problem for the Banach lattice $L^p(\mathbb{I})$, where $\mathbb{I}$ is the closed unit interval.

CHAPTER 2

THE AXIOMS AND THEIR CONSEQUENCES

2.1 Preliminaries

This chapter is devoted to construct the multi-norms and multi-normed spaces. We will closely follow [**6**] throughout.

Notations

The sets $\mathbb{N}$ and $\mathbb{Z}$ denote the natural numbers and the integers, respectively. The real field is $\mathbb{R}$. Moreover, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ and $\mathbb{R}^+ = [0, \infty)$; the unit interval $[0, 1]$ in $\mathbb{R}$ is denoted by $\mathbb{I}$. The complex field is $\mathbb{C}$; the open unit disc in $\mathbb{C}$ is $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and its closure is $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. We write $\mathbb{T}$ for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in $\mathbb{C}$.

For each $n \in \mathbb{N}$, we denote by $\mathbb{N}_n$ and $\mathbb{Z}_n^+$ the sets $\{1, \ldots, n\}$ and $\{0, 1, \ldots, n\}$, respectively. Also, we denote by $\mathfrak{S}_n$ the group of permutations on n symbols; we write $\mathfrak{S}_{\mathbb{N}}$ for the group of all permutations of $\mathbb{N}$.

Let E be a linear space (always taken to be over the complex field $\mathbb{C}$, unless otherwise stated). The dimension of E and the linear subspace spanned by a subset S of E are denoted by $\dim E$ and $\operatorname{span} S$, respectively.

Let F and G be linear subspaces of a linear space E. Set $F+G = \{x+y : x \in F, y \in G\}$; if further $F \cap G = \{0\}$ and $F + G = E$ then, $E = F \oplus G$.

For each $n \in \mathbb{N}$, and for a linear space E, the direct sum of n copies of the linear space E is $E^n := E \oplus \cdots \oplus E$, so E^n consist of n-tuples $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in E$. The linear operations on E^n are defined coordinatewise. For each $x \in E$, *the constant sequence with value* x is the sequence $(x) = (x, \ldots, x) \in E^n$.

Some classical spaces

Let S be a non-empty set. The space $\mathbb{C}^S$ is the linear space of functions from S to $\mathbb{C}$; $\mathbb{C}^S$ is an algebra for the pointwise operations. For functions $f, g \in \mathbb{C}^S$, we define $(f \vee g)(x) := f(x) \vee g(x)$ for each $x \in E$. The functions $f \wedge g, |f|, \exp f$, etc. are defined similarly.

For $n \in \mathbb{N}$, set
$$\delta_n = (\delta_{m,n} : m \in \mathbb{N}) \in \mathbb{C}^{\mathbb{N}}$$
where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ if $m \neq n$, and set
$$e_n = \delta_1 + \ldots + \delta_n = (\underbrace{1, 1, \ldots, 1}_{n-terms}, 0, 0, \ldots).$$

Define
$$c_{00} := \mathrm{span}\{\delta_n : n \in \mathbb{N}\} \subset \mathbb{C}^{\mathbb{N}},$$
and, for $1 \leq p < \infty$, set
$$\ell^p := \Big\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{i=1}^{\infty} |\alpha_i|^p < \infty\Big\}$$
so that ℓ^p is a Banach space for the norm given by
$$\|(\alpha_i)\| = \Big(\sum_{i=1}^{\infty} |\alpha_i|^p\Big)^{1/p} \qquad ((\alpha_i) \in \ell^p).$$
Further, set
$$\ell^\infty := \Big\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \|(\alpha_i)\|_\infty = \sup_{i \in \mathbb{N}} |\alpha_i| < \infty\Big\},$$
so that $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space. The space
$$c_0 = \{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \lim_{i \to \infty} \alpha_i = 0\}$$
of *null sequences* is a closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$. It is well known that c_{00} is a dense linear subspace of each ℓ^p and of c_0, and $\{\delta_n : n \in \mathbb{N}\}$ is a Schauder basis for each of these spaces; it is called the *standart basis.* One can check that $\|\delta_n\| = 1$ for each $n \in \mathbb{N}$, where $\|\cdot\|$ is calculated in any of the spaces ℓ^p for $p \geq 1$ or c_0.

Similarly, we regard $\{\delta_1, \ldots, \delta_n\}$ as the standard basis of $\mathbb{C}^n$ for $n \in \mathbb{N}$.

The real-valued versions of these spaces will be denoted by $\ell^p_{\mathbb{R}}, \ell^\infty_{\mathbb{R}}$, and $c_{0,\mathbb{R}}$.

For $1 < p < \infty$, and for q, which is *conjugate index* to p, satisfying the condition that $\frac{1}{p} + \frac{1}{q} = 1$ we have; $c_0' = \ell^1, (\ell^1)' = \ell^\infty$, and $(\ell^p)' = \ell^q$. We regard 1 and ∞ as being conjugate index to each other.

For each $n \in \mathbb{N}$, the n-dimensional versions of the above spaces are denoted by ℓ^p_n for $p \geq 1$ and by ℓ^∞_n.

Let A be a non-empty index set, and let $\{(E_\alpha, \|\cdot\|_\alpha) : \alpha \in A\}$ be a family of normed spaces. Then we shall consider the spaces,

$$\ell^\infty(E_\alpha) = \{\{x_\alpha : \alpha \in A\} : \|(x_\alpha)\| = \sup_\alpha \|x_\alpha\|_\alpha < \infty\}$$

and for $1 \leq p < \infty$,

$$\ell^p(E_\alpha) = \left\{\{x_\alpha : \alpha \in A\} : \|(x_\alpha)\| = \Big(\sum_\alpha \|x_\alpha\|^p_\alpha\Big)^{1/p} < \infty\right\}.$$

It is straightforward to check that $\ell^\infty(E_\alpha)$ and $\ell^p(E_\alpha)$ are normed spaces; further, they are Banach spaces if each E_α is.

The space $\mathcal{L}(E, F)$ and some special operators

Let E and F be linear spaces. The space $\mathcal{L}(E, F)$ consists of linear operators from E to F. We write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$; the identity operator on E is I_E, so that $\mathcal{L}(E)$ is a unital algebra with respect to the composition of operators.

Let $E_1, \ldots, E_n$ and F be linear spaces. Then the space of n-linear maps from $E_1 \times \cdots \times E_n$ to F is denoted by $\mathcal{L}(E_1, \ldots, E_n; F)$.

Let E be a linear space, and let $n \in \mathbb{N}$. For $\sigma \in \mathfrak{S}_n$, let

$$A_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \quad (x = (x_1, \ldots, x_n) \in E^n)$$

so that $A_\sigma \in \mathcal{L}(E^n)$. For $\alpha = (\alpha_i) \in \mathbb{C}^n$, set

$$M_\alpha(x) = (\alpha_i x_i) \quad (x = (x_1, \ldots, x_n) \in E^n),$$

so that $M_\alpha \in \mathcal{L}(E^n)$. The operator A_σ is said to be a *permutation operator* and the operator M_α is said to be a *multiplication operator*.

Let E be a linear space, and let S be a subset of $\mathbb{N}_n$. For each $x = (x_i) \in E^n$, we set

$$P_S(x) = (y_i), \quad \textit{where } y_i = x_i \ (i \in S) \textit{ and } y_i = 0 \ (i \notin S),$$

$$Q_S(x) = (y_i), \quad \textit{where } y_i = x_i \ (i \notin S) \textit{ and } y_i = 0 \ (i \in S).$$

Thus P_S is the *projection onto* S and Q_S is the *projection onto the complement of* S. Clearly P_S and Q_S are idempotents in the algebra $\mathcal{L}(E^n)$, and $P_S + Q_S = I_{E^n}$. Also, for $x = (x_1, \ldots, x_n) \in E^n$, we set

$$P_i(x) = (0, \ldots, 0, x_i, 0, \ldots, 0) \text{ and } Q_i(x) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n),$$

so that $P_i = P_{\{i\}}$ and $Q_i = Q_{\{i\}}$.

A closed subspace F of a Banach space E is called *complemented* if there is a continuous projection P of E onto F, and λ-*complemented* for $\lambda \geq 1$ if there is a projection P of E onto F with $\|P\| \leq \lambda$.

Ordered vector spaces

Let E be a real vector space. The space E is said to be an *ordered vector space* if it is equipped with an order relation $\geq$ which is competible with the algebraic structure of E.

A *Riesz space* or a *vector lattice* is an ordered vector space E, which satisfy the property that for each pair of vectors $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ both exist in E. It is conventional to write $x \vee y$ and $x \wedge y$ for the supremum and the infimum of the set $\{x, y\}$, respectively.

For $x \in E$, we set

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x).$$

In a Riesz space, two elements x and y is said to be *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$. Two subsets A and B is said to be disjoint, written $A \perp B$, if $a \perp b$ holds for each $a \in A$ and for each $b \in B$.

Let E be a Riesz space and let A be a subset of E, the *disjoint complement* A^d is defined by

$$A^d := \{x \in E : x \perp y \text{ for each } y \in A\}.$$

One can check that $A \cap A^d = \{0\}$.

Let $E = E_1 \oplus \cdots \oplus E_n$ be a direct sum decomposition of E. We say this decomposition is *orthogonal* if $E_i \perp E_j$ whenever $i, j \in \mathbb{N}_n$ and $i \neq j$, and we write

$$E = E_1 \perp \cdots \perp E_n.$$

Let E be a Riesz space and let $x, y \in E$ with $x \leq y$. The *order interval* $[x, y]$ is a subset of E is defined by

$$[x, y] := \{z \in E : x \leq z \leq y\}.$$

A subset A of E is said to be *bounded below* (or *bounded above*) if there exist $z \in E$ such that $z \leq x$ (or $x \leq z$) for each $x \in A$. The subset A is said to be *order bounded* if it is bounded from below and from above.

A net $(x_\alpha : \alpha \in A)$ is order bounded if the set $\{x_\alpha : \alpha \in A\}$ is. A net is *increasing* (*decreasing*) if $x_\alpha \leq x_\beta$ $(x_\alpha \geq x_\beta)$ whenever $\alpha \leq \beta$ in A. We say the net (x_α) increases to $x \in E$ (in symbols $x_\alpha \uparrow x$) if (x_α) is an increasing net in E and $x = sup\{x_\alpha : \alpha \in A\}$. The net (x_α) decreases to $x \in E$ (in symbols $x_\alpha \downarrow x$) defined similarly.

A net (x_α) of a Riesz space E is said to be *order convergent* to x (in symbols $x_\alpha \xrightarrow{o} x$) if there exist a net (y_α) with the same index set such that $y_\alpha \downarrow 0$ and such that $|x_\alpha - x| \leq y_\alpha$. A subset A of E is *order closed* if $(x_\alpha) \subset A$ with $x_\alpha \xrightarrow{o} x$ imply $x \in A$.

A Riesz space E is *Dedekind complete* if every nonempty subset bounded from above has a supremum.

A subset A of a Riesz space E is said to be *solid* if $|y| \leq |x|$ and $x \in A$ imply $y \in A$. If A is a solid vector subspace of E then we say A is an *ideal* of E. An order closed ideal is referred to be a *band*. A band B in E is said to be *projection band* if $E = B \oplus B^d$.

There is a useful condition which ensures an ideal necessarily to be a band.

Theorem 2.1.1 ([3, Theorem 3.6]). *Let A and B be two ideals in a Riesz space E such that $E = A \oplus B$. Then A and B are both bands satisfying $A = B^d$ and $B = A^d$.* □

The following theorem is due to F.Riesz.

Theorem 2.1.2 ([3, Theorem 3.8]). *If B is a band in a Dedekind complete Riesz space E, then $E = B \oplus B^d$ holds.* □

Linear topological spaces

Let S be a subset of a linear space E, then for each $\lambda \in \mathbb{R}$ we let

$$\lambda S := \{\lambda s : s \in S\}.$$

A nonempty subset S of E is said to be

(1) *convex*, whenever $x, y \in S$ and $0 \le \lambda < 1$ imply $\lambda x + (1-\lambda)y \in S$;

(2) *circled*, whenever $x \in S$ and $|\lambda| \le 1$ imply $\lambda x \in S$;

(3) *absorbing*, if for each $x \in E$ there exist some $\lambda > 0$ satisfying $x \in \lambda S$.

S is called *absolutely convex* if it is convex and circled. Equivalently, S is absolutely convex if $\alpha x + \beta y \in S$ whenever $x, y \in S$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \le 1$.

The *convex hull* co S is the smallest (with respect to inclusion) convex set that includes S. The set co S consists of all convex combinations of S, i.e.,

$$\text{co } S := \Big\{ \sum_{i=1}^{n} \lambda_i x_i \ : \ x_i \in S, \ \lambda_i \ge 0, \ \textit{and} \ \sum_{i=1}^{n} \lambda_i = 1 \Big\},$$

and its closure is $\overline{\text{co}}\ S$.

For an absolutely convex and absorbing subset K of E, the $Minkowski\ functional$ p_K of K, defined by

$$p_K(x) = \inf\{t > 0 : x \in tK\} \quad (x \in E),$$

is a seminorm on E; and p_K is a norm if and only if

$$\bigcap \{(1/n)K : n \in \mathbb{N}\} = 0.$$

Throughout, a compact topological space is supposed to be Hausdorff.

Let Ω be a non-empty, compact space. Then $C(\Omega)$ is the space of all complex-valued, continuous functions on Ω, and $C_{\mathbb{R}}(\Omega)$ is the real subspace of real-valued functions in $C(\Omega)$.

Suppose that $(E, \|\cdot\|)$ is a normed space. We denote by $E_{[r]}$ the closed ball in E with centre 0 and radius $r \ge 0$.

Now let $x, y \in E_{[1]}$ and let $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. Then we have,

$$\|\alpha x + \beta y\| \leq \|\alpha x\| + \|\beta y\| = |\alpha|\|x\| + |\beta|\|y\| \leq |\alpha| + |\beta| \leq 1,$$

and for $x \in E$, take $\lambda = \frac{1}{\|x\|}$, then $\lambda x = \frac{x}{\|x\|} \in E_{[1]}$. Hence, we observe that $E_{[1]}$ is an absolutely convex, absorbing, and closed neigbourhood of 0. Also, the unit sphere of E denoted by S_E, so that

$$S_E = \{x \in E : \|x\| = 1\}.$$

Banach lattices

Let E be a Riesz space. A norm $\|\cdot\|$ on E is said to be a *lattice norm* if $\|x\| \leq \|y\|$ in $\mathbb{R}^+$ whenever $x, y \in E$ with $|x| \leq |y|$. A *normed Riesz space* is a Riesz space which is equipped with a lattice norm. A *Banach lattice* is a normed Riesz space which is a Banach space with respect to the lattice norm.

Let E be a normed Riesz space, we have

$$\|x\| = \|\,|x|\,\| \quad \text{and} \quad \|x^+ - y^+\| \leq \|x - y\| \quad \text{and} \quad \|\,|x| - |y|\,\| \leq \|x - y\|,$$

for all $x, y \in E$.

Let $(E, \|\cdot\|)$ be a Banach lattice, and let $E = E_1 \perp \cdots \perp E_n$. We have

$$\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\| \quad \text{(by [10] , Theorem 1.1.1.)}$$

for $x_j \in E_j$ and $j \in \mathbb{N}_n$.

A lattice norm $\|\cdot\|$ is said to be *order continuous* if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. If the condition holds for sequences, then the norm $\|\cdot\|$ is said to be *σ-order continuous*.

A Banach lattice $(E, \|\cdot\|)$ is said to be;

(1) An *AL-space* if $\|x + y\| = \|x\| + \|y\|$ whenever $x, y \in E^+$ with $x \wedge y = 0$.

(2) An *AM-space* if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ whenever $x \wedge y = 0$ in E.

The most important example of AL-spaces is $L^1(\Omega)$, where Ω is a measure space, and the most important example of AM-spaces is $C(\Omega)$, where Ω is a compact Hausdorff space.

The space $\mathcal{B}(E,F)$ and the dual notion

Let E an F be normed spaces. The normed space $\mathcal{B}(E,F)$ (with respect to the operator norm) consist of all bounded linear operators from E to F; $\mathcal{B}(E,F)$ is a Banach space whenever F is. If $T \in \mathcal{B}(E,F)$, then the operator norm denoted by $\|T\|$.

We write again $\mathcal{B}(E)$ for $\mathcal{B}(E,E)$, so that $\mathcal{B}(E)$ is a unital normed algebra.

If $T \in \mathcal{B}(E,F)$, then the *dual* T' of T is defined by the equation

$$\langle x, T'\lambda\rangle = \langle Tx, \lambda\rangle \quad \text{for } x \in E \text{ and for } \lambda \in F';$$

so that $T' \in \mathcal{B}(F', E')$ and $\|T\| = \|T'\|$.

The dual space of a normed space $(E, \|\cdot\|)$ is E'; the action $\lambda \in E'$ on $x \in E$ gives the complex number $\langle x, \lambda\rangle$; we shall denote the dual norm on E' by $\|\cdot\|'$. The second dual space of E is denoted by E'', and the action $\phi \in E''$ on $\lambda \in E'$ gives $\langle \phi, \lambda\rangle$ in our notation; we shall denote the dual norm on E'' by $\|\cdot\|''$. The canonical embedding $\imath : E \to E''$ is defined by the equation

$$\langle \imath(x), \lambda\rangle = \langle x, \lambda\rangle \quad \text{for } x \in E \text{ and for } \lambda \in E';$$

so that $\imath$ is an isometry. In fact, we usually identify x with $\imath(x)$ and write $\|\cdot\|$ for the second dual norm on E''. The weak topology on E is denoted by $\sigma(E, E')$, the weak-* topology on E'' is $\sigma(E'', E')$; of course, by the Banach-Alaoglu theorem, $E_{[1]}$ is $\sigma(E'', E')$-dense in $E''_{[1]}$.

Matrices and matrix norms

Let E be a linear space, for each $m, n \in \mathbb{N}$, the linear space of all $m \times n$ matrices with coefficients in E is denoted by $\mathbb{M}_{m,n}(E)$; we write $\mathbb{M}_n(E)$ for $\mathbb{M}_{n,n}(E)$. We write $\mathbb{M}_{m,n}$ and $\mathbb{M}_n$ for $\mathbb{M}_{m,n}(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$, respectively. If $v \in \mathbb{M}_m(E)$ and $w \in \mathbb{M}_n(E)$, then $v \oplus w$ is the matrix in $\mathbb{M}_{m+n}(E)$ of the form

$$\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}.$$

Let $x = (x_{ij}) \in \mathbb{M}_{m,n}(E)$. The *transpose* of x is the matrix

$$x^t = (x_{ji}) \in \mathbb{M}_{n,m}(E).$$

Let E be a linear space and let $m, n \in \mathbb{N}$. Each element $a \in \mathbb{M}_{m,n}$ defines an element of $\mathcal{L}(E^n, E^m)$ by matrix multiplication.

Let $m, n \in \mathbb{N}$. We identify $\mathbb{M}_{m,n}$ with the Banach space $\mathcal{B}(\ell_n^\infty, \ell_m^\infty)$, so that $(\mathbb{M}_{m,n}, \|\cdot\|)$ is a Banach space. Where

$$\|a\| = \max\left\{\sum_{j=1}^{n} |a_{ij}| : i \in \mathbb{N}_m\right\} \quad (a = (a_{ij}) \in \mathbb{M}_{m,n}). \tag{2.1}$$

More generally, for $p, q \in [1, \infty]$, we can also identify $\mathbb{M}_{m,n}$ with $\mathcal{B}(\ell_n^p, \ell_m^q)$. Similarly we denote the norm of $a \in \mathbb{M}_{m,n}$ by,

$$\|a : \ell_n^p \to \ell_m^q\| = \max\left\{\sum_{i=1}^{n} |a_{ij}| : j \in \mathbb{N}_n\right\} \tag{2.2}$$

Let $p_1, p_2 \in [1, \infty]$, and take q_1, q_2 to be the conjugate indices to p_1 and p_2, respectively. For each $a \in \mathbb{M}_{m,n}$, we have $a^t = a'$ and

$$\|a : \ell_n^{p_1} \to \ell_m^{p_2}\| = \|a^t : \ell_n^{q_2} \to \ell_m^{q_1}\|.$$

The norm $|||\cdot|||$

Let $(E, \|\cdot\|)$ be a normed space, and let $k \in \mathbb{N}$. Let $|||\cdot|||$ be any norm on E^k such that

$$|||x||| \geq \max\{\|x_i\| : i \in \mathbb{N}_k\} \quad (x = (x_i) \in E^k) \tag{2.3}$$

and

$$|||(0, \ldots, 0, x_i, 0, \ldots, 0)||| = \|x_i\| \quad (x_i \in E, i \in \mathbb{N}_k) \tag{2.4}$$

For each $\lambda_1, \ldots, \lambda_k \in E'$, set $\lambda = (\lambda_1, \ldots, \lambda_k) \in (E')^k$, and define λ on E^k by

$$\langle x, \lambda \rangle = \sum_{i=1}^{k} \langle x_i, \lambda_i \rangle \quad (x = (x_1, \ldots, x_k) \in E^k).$$

Then λ is a linear functional on E^k, and

$$|\langle x, \lambda \rangle| \leq \left(\sum_{i=1}^{k} \|\lambda_i\|\right) \max\{\|x_i\| : i \in \mathbb{N}_k\} \leq \left(\sum_{i=1}^{k} \|\lambda_i\|\right) |||x|||$$

for each $x \in E^k$. Thus $\lambda \in (E^k, |||\cdot|||)'$ with $\|\lambda\| \leq \sum_{i=1}^{k} \|\lambda_i\|$. Further, each element in $(E^k, |||\cdot|||)'$ arises in this way. Thus we may regard $(E')^k$ as a Banach space for the norm $|||\cdot|||'$.

2.2 The Axioms

Definition 2.2.1. *Let $(E, \|\cdot\|)$ be a normed space, and let $n \in \mathbb{N}$. A multi-norm of level n on $\{E^k : k \in \mathbb{N}\}$ is a sequence*

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following Axioms (A1)-(A4) *are satisfied for each $k \in \mathbb{N}_n$ with $k \geq 2$.*

(A1) *for each $\sigma \in \mathfrak{S}_k$ and $x \in E^k$, we have*

$$\|A_\sigma(x)\|_k = \|x\|_k \ ;$$

(A2) *for each $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $x \in E^k$, we have*

$$\|M_\alpha(x)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|x\|_k \quad (\alpha = (\alpha_1, \ldots, \alpha_k));$$

(A3) *for each $x_1, \ldots, x_{k-1} \in E$, we have*

$$\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \ ;$$

(A4) *for each $x_1, \ldots, x_{k-1} \in E$, we have*

$$\|(x_1, \ldots, x_{k-2}, x_{k-1}, x_{k-1})\|_k = \|((x_1, \ldots, x_{k-2}, x_{k-1})\|_{k-1} \ .$$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ is a multi-normed space of level n.

A multi-norm on $\{E^k \ : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a multi-norm of level n for each $n \in \mathbb{N}$. In this case, we say that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.

The follows observations from the axioms are immediate; the Axiom (A1) says that A_σ preserves the norm, hence it is an isometry on $(E^k, \|\cdot\|_k)$ whenever $\sigma \in \mathfrak{S}$, the Axiom

(A2) says that the multiplication operator M_α is a bounded linear operator on $(E^k, \|\cdot\|_k)$ whenever $\alpha \in \overline{\mathbb{D}}^k$, and further

$$\|M_\alpha\| = \max_{i\in\mathbb{N}_k} |\alpha_i| \quad (\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n).$$

The above definition has a dual version.

Definition 2.2.2. *Let $(E, \|\cdot\|)$be a normed space. A dual multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence*

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the Axioms (A1)-(A3) *and the following modified form of Axiom* (A4) *are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:*

(B4) *for each $x_1, \ldots, x_{k-1} \in E$, we have*

$$\|(x_1, \ldots, x_{k-2}, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-2}, 2x_{k-1})\|_{k-1}.$$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a dual multi-normed space.

After this definitions there are two questions: Are the Axioms (A1)-(A4) independent? For a normed space $(E, \|\cdot\|)$, can the Axioms (A4) and (B4) be both satisfied?

For the first question we have a positive answer. There are examples in **[6]** to show that this is indeed the case. For the second question, suppose $(E, \|\cdot\|)$ is a normed space and the Axioms (A4) and (B4) satisfied for $k = 2$. For each $x \in E$ we have

$$\|x\| = \|(x, x)\| = 2\|x\|$$

hence $x = 0$. Thus a dual multi-normed space is not a multi-normed space unless $E = \{0\}$.

Now we pay attention to the elementary but useful consequences of the axioms.

First, suppose $(E, \|\cdot\|)$ is a normed space, $n \in \mathbb{N}$ with $n \geq 2$, and the sequence $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a norm sequence on $\{E^k : k \in \mathbb{N}_n\}$ such that Axioms (A1)-(A3) satisfied. Thus the consequences apply for both dual multi-normed spaces and multi-normed spaces.

Lemma 2.2.3 ([6, Lemma 2.10])**.** *Let $k \in \mathbb{N}_{n-1}$ and $x_1, \ldots, x_{k+1} \in E$. Then*

$$\|(x_1, \ldots, x_k)\|_k \leq \|(x_1, \ldots, x_k, x_{k+1})\|_{k+1}.$$

Proof. We have

$$\begin{aligned}\|(x_1,\dots,x_k)\|_k &= \|(x_1,\dots,x_k,0)\|_{k+1} \quad \text{by (A3)}\\ &\le \|(x_1,\dots,x_k,x_{k+1})\|_{k+1} \quad \text{by (A2).}\end{aligned}$$

□

Lemma 2.2.4 ([6, Lemma 2.11])**.** *Let $m, k \in \mathbb{N}$ with $m + k \le n$, and let*

$$x_1,\dots,x_m,y_1,\dots,y_k \in E.$$

Then

$$\|(x_1,\dots,x_m,y_1,\dots,y_k)\|_{m+k} \le \|(x_1,\dots,x_m)\|_m + \|(y_1,\dots,y_k)\|_k.$$

□

Lemma 2.2.5 ([6, Lemma 2.12])**.** *Let $x_1,\dots,x_k \in E$, and let $x = (x_{k-1}+x_k)/2$. For each $k \in \mathbb{N}_n$ with $k \ge 2$ we have*

$$\|(x_1,\dots,x_{k-2},x,x)\|_k \le \|(x_1,\dots,x_{k-1},x_k)\|_k.$$

Proof. Note that

$$2(x_1,\dots,x_{k-2},x,x) = (x_1,\dots,x_{k-1},x_k) + (x_1,\dots,x_k,x_{k-1}),$$

so

$$2\|(x_1,\dots,x_{k-2},x,x)\|_k \le \|(x_1,\dots,x_{k-1},x_k)\|_k + \|(x_1,\dots,x_k,x_{k-1})\|_k,$$

then the result follows from Axiom (A1). □

Lemma 2.2.6 ([6, Lemma 2.13])**.** *Let $k \in \mathbb{N}_n$ and $x_1,\dots,x_k \in E$. Then*

$$\max_{i\in\mathbb{N}_k}\|x_i\| \le \|(x_1,\dots,x_k)\|_k \le \sum_{i=1}^{k}\|x_i\| \le k\max_{i\in\mathbb{N}_k}\|x_i\|.$$

Proof. For each $i \in \mathbb{N}_k$ by (A2) and (A3) we have $\|x_i\| \le \|(x_1,\dots,x_k)\|_k$ and hence

$$\max_{i\in\mathbb{N}_k}\|x_i\| \le \|(x_1,\dots,x_k)\|_k.$$

The second inequality follows from Axiom (A3) and the above Lemma 2.2.4. □

We observe from the above lemma that any two norm sequences on $\{E^k : k \in \mathbb{N}_n\}$ which satisfy the Axioms (A1)-(A3) define the same topology on the space E^k. This topology is the product topology.

Corollary 2.2.7 ([6, Corollary 2.14])**.** *Let $(E, \|\cdot\|)$ be a Banach space, and let the sequence $(\|\cdot\|_k : k \in \mathbb{N}_n)$ be a norm sequence on $\{E^k : k \in \mathbb{N}_n\}$ which satisfies* (A1)-(A3)*. Then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}_n$.*

Proof. Let $k \in \mathbb{N}_n$ and let $((x_{i,1}, \ldots, x_{i,k}) : i \in \mathbb{N})$ is a Cauchy sequence in $(E^k, \|\cdot\|_k)$. By Axioms (A2) and (A3) $(x_{i,j} : i \in \mathbb{N})$ is a Cauchy sequence in $(E, \|\cdot\|)$ for each $j \in \mathbb{N}_k$. Since $(E, \|\cdot\|)$ is a Banach space, $(x_{i,j} : i \in \mathbb{N})$ converges in $(E, \|\cdot\|)$. Say x_j to the convergence point for each $j \in \mathbb{N}_n$. Then, by Lemma 2.2.6 we have

$$\|(x_{i,1} - x_1, \ldots, x_{i,k} - x_k)\|_k \leq \sum_{j=1}^{k} \|x_{i,j} - x_j\| \to 0 \quad as \quad i \to \infty.$$

Hence, $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}_n$. □

This corollary is a motivation for the following definition.

Definition 2.2.8. *Let $(E, \|\cdot\|)$ be a Banach space and let $(\|\cdot\|_k : k \in \mathbb{N})$ be a (dual) multi-norm on $\{E^k : k \in \mathbb{N}\}$ then $((E^k, \|\cdot\|_k : k \in \mathbb{N}))$ is a* (dual) multi-Banach space*.*

We now give another lemmas which are satisfied for a multi-normed space.

Lemma 2.2.9. *Let $k \in \mathbb{N}_n$ and $x \in E$. Then $\|(x, \ldots, x)\|_k = \|x\|$*

Proof. This is just Axiom (A4). □

Lemma 2.2.10 ([6, Lemma 2.17])**.** *Let $m, n \in \mathbb{N}_k$, and let $x_1, \ldots, x_m, y_1, \ldots, y_n \in E$ be such that $\{x_1, \ldots, x_m\} \subseteq \{y_1, \ldots, y_n\}$. Then*

$$\|x_1, \ldots, x_m\|_m \leq \|y_1, \ldots, y_n\|_n.$$

□

Now we give some theorems which justify the term dual multi-normed space.

Let $(E, \|\cdot\|)$ be a normed space, let $k \in \mathbb{N}$, and let $\|\cdot\|_k$ be any norm on the space E^k. The dual norm on the space $(E')^k$ is denoted by $\|\cdot\|'_k$, we have

$$\|(\lambda_1, \ldots, \lambda_k)\|'_k = \sup\left\{\left|\sum_{j=1}^{k}\langle x_j, \lambda_j\rangle\right| : \|(x_1, \ldots, x_k)\|_k \leq 1\right\}$$

for $\lambda_1, \ldots, \lambda_k \in E'$.

Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space or dual multi-normed space. It follows from Lemma 2.2.6 and Axiom (A3) that each norm $\|\cdot\|_k$ satisfies (2.3) and (2.4) (with $\|\cdot\|_k$ for $|||\cdot|||$), and so $(E^k, \|\cdot\|_k)'$ is linearly homeomorphic to $(E')^k$ (with the product topology from E'). Thus we have defined a sequence $(\|\cdot\|'_k : k \in \mathbb{N})$ such that $\|\cdot\|'_k$ is a norm on $(E')^k$ for each $k \in \mathbb{N}$. Clearly $\|\lambda\|'_1 = \|\lambda\|'$ for each $\lambda \in E'$.

Theorem 2.2.11 ([6, Theorem 2.28])**.** *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then*
$(((E')^k, \|\cdot\|'_k) : k \in \mathbb{N})$ is a dual multi-Banach space.

Proof. By the definition of the dual norm on the space $(E')^k$, one can easily see that Axioms (A1) and (A2) satisfied. Now let $k \geq 2$ and $\lambda_1, \ldots, \lambda_{k-1} \in E'$. For each $x_1, \ldots, x_k \in E$, we have $\|(x_1, \ldots, x_{k-1})\|_{k-1} \leq \|(x_1, \ldots, x_{k-1}, x_k)\|_k$, and so

$$\begin{aligned}\|(\lambda_1, \ldots, \lambda_{k-1})\|'_{k-1} &= \sup\left\{\left|\sum_{j=1}^{k-1}\langle x_j, \lambda_j\rangle\right| : \|(x_1, \ldots, x_{k-1})\|_{k-1} \leq 1\right\} \\ &\leq \sup\left\{\left|\sum_{j=1}^{k-1}\langle x_j, \lambda_j\rangle + \langle x_k, 0\rangle\right| : \|(x_1, \ldots, x_k)\|_k \leq 1\right\} \\ &= \|(\lambda_1, \ldots, \lambda_{k-1}, 0)\|'_k\end{aligned}$$

and we have $\|(\lambda_1, \ldots, \lambda_{k-1}, 0)\|'_k \leq \|(\lambda_1, \ldots, \lambda_{k-1})\|'_{k-1}$ (by definition),
thus $(\|\cdot\|'_k : k \in \mathbb{N})$ satisfies (A3).

Fix $\lambda_1, \ldots, \lambda_{k-1} \in E'$, and set

$$A = \|(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}, \lambda_{k-1})\|'_k, \qquad B = \|(\lambda_1, \ldots, \lambda_{k-2}, 2\lambda_{k-1})\|'_{k-1}.$$

Take $\epsilon > 0$

First choose $(x_1, \ldots, x_k) \in (E^k, \|\cdot\|_k)_{[1]}$ with

$$\left|\sum_{j=1}^{k-2}\langle x_j, \lambda_j\rangle + \langle x_{k-1}, \lambda_{k-1}\rangle + \langle x_k, \lambda_{k-1}\rangle\right| > A - \epsilon$$

Set $x = (x_{k-1} + x_k)/2$, so that, by Lemma 2.2.5 and (A4), we have $(x_1, \ldots, x_{k-2}, x) \in (E^{k-1}, \|\cdot\|_{k-1})_{[1]}$, and hence

$$\begin{aligned} B &\geq \Big| \sum_{j-1}^{k-2} \langle x_j, \lambda_j \rangle + \langle x, 2\lambda_{k-1} \rangle \Big| \\ &= \Big| \sum_{j-1}^{k-2} \langle x_j, \lambda_j \rangle + \langle x, \lambda_{k-1} \rangle + \langle x, \lambda_{k-1} \rangle \Big| > A - \epsilon. \end{aligned}$$

Second, choose $(x_1, \ldots, x_{k-1}) \in (E^{k-1}, \|\cdot\|_{k-1})_{[1]}$ with

$$\Big| \sum_{j=1}^{k-2} \langle x_j, \lambda_j \rangle + \langle x_{k-1}, 2\lambda_{k-1} \rangle \Big| > B - \epsilon.$$

Then $(x_1, \ldots, x_{k-1}, x_{k-1}) \in (E^k, \|\cdot\|_k)_{[1]}$ by (A4), and so

$$A \geq \Big| \sum_{j=1}^{k-2} \langle x_j, \lambda_j \rangle + \langle x_{k-1}, 2\lambda_{k-1} \rangle \Big| > B - \epsilon.$$

It follows that $A = B$, and so the sequence $(\|\cdot\|_k' : k \in \mathbb{N})$ satisfies Axiom (B4). Thus $(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$ is a dual multi-Banach space. □

In the light of the above theorem, the following definition is reasonable.

Definition 2.2.12. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi normed space. Then $(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$ is the dual multi-Banach space to $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$.*

We have a dual version of the Theorem 2.2.11.

Theorem 2.2.13 ([6, Theorem 2.30]). *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a dual-multi normed space. Then $(((E')^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.*

The proof is similar to Theorem 2.2.11. □

Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then, for each $k \in \mathbb{N}$, the norm on (E'') which is the dual norm to $\|\cdot\|_k'$ on $(E')^k$ is temporarly denoted by $\|\cdot\|_k''$. It is clear from Theorems 2.2.11 and 2.2.13 that $(((E'')^k, \|\cdot\|_k'') : k \in \mathbb{N})$ is a multi-Banach space. Of course the embedding of each space $(E^k, \|\cdot\|_k)$ into $((E'')^k, \|\cdot\|_k'')$ is an isometry of normed spaces, and so we can write $\|\cdot\|_k$ for $\|\cdot\|_k''$ on $(E^k)''$. Thus we have the following conclusion.

Theorem 2.2.14 ([6, Theorem 2.31]). *Let* $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ *be a multi-normed space. Then the multi-normed space* $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ *is a multi-normed subspace of the multi-Banach space* $(((E'')^k, \|\cdot\|_k) : k \in \mathbb{N})$ □

There is an equivalent condition for the multi-norm axioms. But we firstly give a preliminary notion.

Let $m, n \in \mathbb{N}$ and let $a = (a_{ij}) \in \mathbb{M}_{m,n}$. We say $a \in \mathbb{M}_{m,n}$ is *row-special* matrix if there exist at most one non-zero term in each row.

Theorem 2.2.15 ([6, Theorem 2.33]). *Let* $(E, \|\cdot\|)$ *be a normed space, let* $N \in \mathbb{N}$, *and let* $(\|\cdot\|_n : n \in \mathbb{N}_N)$ *be a sequence of norms on the spaces* $E, \ldots, E^N$, *respectively, such that* $\|x\|_1 = \|x\|(x \in E)$. *Then the following are equivalent:*

(a) $(\|\cdot\|_n : n \in \mathbb{N}_n)$ *is a multi-norm of level* N *on the family* $\{E^n : n \in \mathbb{N}_N\}$;

(b) $\|a.x\|_m \leq \|a\|\|x\|_n$ *for each row-special matrix* $a \in \mathbb{M}_{m,n}$, *each* $x \in E^n$, *and each* $m, n \in \mathbb{N}_n$;

(c) $\|a.x\|_m \leq \|a\|\|x\|_n$ *for each matrix* $a \in \mathbb{M}_{m,n}$, *each* $x \in E^n$, *and each* $m, n \in \mathbb{N}_n$.

□

CHAPTER 3

Examples of multi-norms

In this chapter, we give some examples to the multi-normed spaces. These examples are the most important examples for an arbitrary normed space E; namely the minimum and the maximum multi-norm.

3.1 The minimum multi norm

Definition 3.1.1 ([6, Definition 3.1]). *Let $(E, \|\cdot\|)$ be a normed space. For $k \in \mathbb{N}$, define $\|\cdot\|_k$ on E^k by*

$$\|(x_1, \dots, x_k)\|_k = \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \dots, x_k \in E).$$

One can easily see that each $\|\cdot\|_k$ is a norm on E^k, and for each $n \in \mathbb{N}$, the sequence $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a multi-norm of level n. Thus $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ is a multi-normed space of level n.

More generally, let $n \in \mathbb{N}$ and let $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ be a multi normed space of level n on $\{E^k : k \in \mathbb{N}_n\}$. For $m > n$, define

$$\|(x_1, \dots, x_m)\|_m = \max\{\|(y_1, \dots, y_n)\|_n : y_1, \dots, y_n \in \{x_1, \dots, x_m\}\} \quad (x_1, \dots, x_n \in E).$$

Then $((E^m, \|\cdot\|_m) : m \in \mathbb{N})$ is a multi-normed space. Thus a multi-norm of level n can be extended to a multi-norm, in an obvious sense.

The norm sequence $(\|\cdot\|_n : n \in \mathbb{N})$, which is defined above, is said to be minimum multi-norm. The terminology minimum is justified by Lemma 2.2.6.

It is immediate that for an arbitary normed space E, there is indeed a multi-norm on the family $\{E^n : n \in \mathbb{N}\}$, this is minimum multi-norm.

Let us consider the minimum multi-norm of level n on the family $\{\mathbb{C}^k : k \in \mathbb{N}_n\}$. For $k \in \mathbb{N}_n$ we have, by Axiom (A4), $\|(1,\dots,1)\|_k = 1$. And for $(\alpha_1,\dots,\alpha_k) \in \mathbb{C}^k$ we have by (A2)

$$\|(\alpha_1,\dots,\alpha_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|)\|(1,\dots,1)\|_k = \max_{i\in\mathbb{N}_k} |\alpha_i|.$$

And keeping in mind the Lemma 2.2.6 we have the following conclusion.

Lemma 3.1.2 ([6, Lemma 3.3]). *Let $n \in \mathbb{N}$. Then the minimum multi-norm of level n is the unique multi-norm of level n on $\{\mathbb{C}^k : k \in \mathbb{N}_k\}$.* □

Definition 3.1.3. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. For $n \in \mathbb{N}$, set*

$$\varphi_n(E) = \sup\{\|(x_1,\dots,x_n)\|_n : x_1,\dots,x_n \in E_{[1]}\}.$$

The multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm if there exist $C > 0$ with $\varphi_n(E) \leq C$ for $n \in \mathbb{N}$.

One can easily see that $(\varphi_n(E) : n \in \mathbb{N})$ is an increasing sequence for each multi-normed space $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$. To see this we keep in mind the definiton of function $\varphi_n(E)$ and we use Lemma 2.2.6, then we obtain

$$1 \leq \varphi_n(E) \leq n \quad (n \in \mathbb{N})$$

and from Lemma 2.2.4 that

$$\varphi_{m+n}(E) \leq \varphi_m(E) + \varphi_n(E) \quad (m, n \in \mathbb{N}).$$

Moreover, $(\|\cdot\|_n : n \in \mathbb{N})$ is the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$ if and oly if $\varphi_n(E) = 1$ for each $n \in \mathbb{N}$.

We now that for a finite-dimensional space E that all norms on E are equivalent. There is a simple manner for multi-normed spaces as follows.

Proposition 3.1.4 ([6, Proposition 3.5]). *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space such that E is finite-dimensional. Then $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm.* □

3.2 The maximum multi-norm

Definition 3.2.1. *Let* $(E, \|\cdot\|)$ *be a normed space, let* $n \in \mathbb{N}$*, and let*

$$(|||\cdot|||_k : k \in \mathbb{N}_n)$$

be a multi-norm of level n *on* $\{E^k : k \in \mathbb{N}_n\}$*. Then* $(|||\cdot|||_k : k \in \mathbb{N}_k)$ *is the* maximum multi-norm of level n *if*

$$\|(x_1, \ldots, x_k)\|_k \leq |||(x_1, \ldots, x_k)|||_k \quad (x_1, \ldots, x_k \in E, k \in \mathbb{N}_n)$$

whenever $(\|\cdot\|_k : k \in \mathbb{N}_n)$ *is a multi-norm of level* n *on* $\{E^k : k \in \mathbb{N}_n\}$*.*

The maximum multi-norm on $\{E^n : n \in \mathbb{N}\}$ defined to be similar.

Let $(E, \|\cdot\|)$ be a normed space and let $n \in \mathbb{N}$. Consider the family of multi-norms $\{((\|\cdot\|_k^\alpha : k \in \mathbb{N}_n) : \alpha \in A)\}$ on the family $\{E^k : k \in \mathbb{N}_n\}$. Then A is non-empty since there is indeed a multi-norm, namely minimum multi-norm, on the family $\{E^k : k \in \mathbb{N}_k\}$. Then set,

$$|||(x_1, \ldots, x_k)|||_k = \sup_{\alpha \in A} \|(x_1, \ldots, x_k)\|_k^\alpha \quad (x_1, \ldots, x_k \in E).$$

Using Lemma 2.2.6 we see that the supremum is finite in each case and one can see that $(|||\cdot|||_k : k \in \mathbb{N}_n)$ is a multi-norm of level n on the family $\{E^k : k \in \mathbb{N}_n\}$, and hence $(|||\cdot|||_k : k \in \mathbb{N}_n)$ is the maximum multi-norm on $\{E^k : k \in \mathbb{N}_n\}$.

Definition 3.2.2. *Let* $(E, \|\cdot\|)$ *be a normed space.We write*

$$(|||\cdot|||_k^{max} : k \in \mathbb{N})$$

for the maximum multi-norm on $\{E^k : k \in \mathbb{N}\}$*. For* $n \in \mathbb{N}$*, set*

$$\varphi_n^{max}(E) = \sup\{|||(x_1, \ldots, x_n)|||_n^{max} : x_1, \ldots, x_n \in E_{[1]}\}.$$

The sequence $(\varphi_n^{max}(E) : n \in \mathbb{N})$ is intrinsic to the normed space $(E, \|\cdot\|)$. It is interesting to calculate the maximum multi-norm and this sequence for arbitrary normed space E, but we do not mention any more about it in this thesis. One can find considerably remark about this topic in [**6**].

3.3 Specific elementary examples

In this section, we shall give some specific examples of multi-normed spaces.

Example 3.3.1 ([6, Example 3.45]). Let E be one of the spaces ℓ^p (for $p \geq 1$) or c_0, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be the multi-norm on $\{E^n : n \in \mathbb{N}\}$. Let $n \in \mathbb{N}$ and let (α_i) be fixed element of $\mathbb{C}^n$. Set $x_i = \alpha_i \delta_i (i \in \mathbb{N}_n)$, so that $\{x_i : i \in \mathbb{N}_n\} \subset E$. Then

$$\|(x_1, \ldots, x_n)\|_n = \max\{|\alpha_1|, \ldots, |\alpha_n|\} \quad (n \in \mathbb{N}). \tag{3.1}$$

Thus $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ contains $\ell^\infty(\mathbb{N}_n)$ as a closed subspace. □

Example 3.3.2 ([6, Example 3.46]). Let $\Omega = (\Omega, \mu)$ be a measure space, and take p, q with $1 \leq p \leq q < \infty$. We consider the Banach space $E = L^p(\Omega)$, with the norm

$$\|f\| = \left(\int_\Omega |f|^p\right)^{1/p} = \left(\int_\Omega |f|^p d\mu\right)^{1/p} \quad (f \in E).$$

For a measurable subset X of Ω, we write r_X for the seminorm on E specified by

$$r_X(f) = \left(\int_X |f|^p\right)^{1/p} \quad (f \in E)$$

where we suppress in the notation the dependence on p.

Now take $n \in \mathbb{N}$. For each partition $\mathbf{X} = \{X_1, \ldots, X_n\}$ of Ω into measurable subsets and $f_1, \ldots, f_n \in E$, set

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) = (r_{X_1}(f_1)^q + \ldots + r_{X_n}(f_n)^q)^{1/p},$$

so that $r_{\mathbf{X}}$ is a seminorm on E^n and

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) \leq (\|f_1\|^q + \ldots + \|f_n\|^q)^{1/q} \quad (f_1, \ldots, f_n \in E).$$

The norm on E^n is defined to be

$$\|(f_1, \ldots, f_n)\|_n = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \ldots, f_n)) \quad (f_1, \ldots, f_n \in E). \tag{3.2}$$

In the case where $f_1, \ldots, f_n \in E$ have disjoint support, we have

$$\|(f_1, \ldots, f_n)\|_n = (\|f_1\|^q + \ldots + \|f_n\|^q)^{1/q}; \tag{3.3}$$

if, further, we have $p = q$, then

$$\|(f_1, \ldots, f_n)\|_n = \|f_1 + \ldots + f_n\|. \tag{3.4}$$

One can see that the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$. We may consider this multi-norm as a function of q when q belongs to the interval $[p, \infty)$.

For the special case $p = q$ there is an equivalent way of definition $\|f_1, \ldots, f_n\|_n$ for elements $f_1, \ldots, f_n \in E$. Indeed, set $f = |f_1| \vee \ldots \vee |f_n|$, so that

$$f(x) = \max\{|f_1(x)|, \ldots, |f_n(x)|\} \quad (x \in \Omega).$$

Then we see that

$$\|(f_1, \ldots, f_n)\|_n = \Big(\int_\Omega f^p\Big)^{1/p} = \Big(\int_\Omega (|f_1| \vee \ldots \vee |f_n|)^p\Big)^{1/p}. \tag{3.5}$$

In particular, for the case $E = \ell^p$ and for $p = q$, we have

$$\|(f_1, \ldots, f_n)\|_n = \Big(\sum_{j=1}^{\infty} (|f_1(j)| \vee \ldots \vee |f_n(j)|)^p\Big)^{1/p}. \tag{3.6}$$

Definition 3.3.3. *Let Ω be a measure space and take p,q with $1 \le p \le q < \infty$. Set $E = L^p(\Omega)$, as above. Then the standard (p, q)-*multi-norm *on $\{E^n : n \in \mathbb{N}\}$ is the multi-norm described above.*

Proposition 3.3.4 ([6, Proposition 3.48])**.** *Let Ω be a measure space, and set $E = L^1(\Omega)$. Then the standard $(1, 1)$-multi-norm and the maximum multi-norm on $\{E^n : n \in \mathbb{N}\}$ are equal.*

□

Proposition 3.3.5 ([6, Proposition 3.49])**.** *Let $p > 1$. The standard (p, p)-multi-norm and the maximum multi-norm on $\{(\ell^p)^n : n \in \mathbb{N}\}$ are not equal.*

□

CHAPTER 4

Multi-norms on Banach spaces

4.1 Banach lattices and multi-norms

In this section we give definitions of a multi-norm and a dual-multi norm which is connected with a Banach lattice.

Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ set

$$\|(x_1, \ldots, x_n)\|_n = \||x_1| \vee \cdots \vee |x_n|\| \quad (x_1, \ldots, x_n \in E)$$

then one can see that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$ and the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies the Axioms (A1)-(A4) and hence $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$, so $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-Banach space.

Definition 4.1.1. *Let $(E, \|\cdot\|)$ be a Banach lattice. The above multi-norm is the* lattice multi-norm *on $\{E^n : n \in \mathbb{N}\}$.*

Let again $(E, \|\cdot\|)$ be a Banach lattice, and $n \in \mathbb{N}$, set

$$\|(x_1, \ldots, x_n)\|_n = \||x_1| + \cdots + |x_n|\| \quad (x_1, \ldots, x_n \in E)$$

then $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$ and the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies the Axioms (A1)-(A3) and (B4) and hence $(\|\cdot\|_n : n \in \mathbb{N})$ is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$, so $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a dual multi-Banach space.

Definition 4.1.2. *Let $(E, \|\cdot\|)$ be a Banach lattice. The above multi-norm is the* dual lattice multi-norm *on $\{E^n : n \in \mathbb{N}\}$.*

Proposition 4.1.3 ([6, Proposition 3.78])**.** *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the dual of the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the dual lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$.*

Proof. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ and let $n \in \mathbb{N}$, take $\lambda_1, \dots, \lambda_n \in E'$, and write $\lambda = |\lambda_1| + \cdots + |\lambda_n| \in E'$

Suppose that $x_1, \dots, x_n \in E$, with $\|(x_1, \dots, x_n)\|_n \le 1$, and set $x = |x_1| \vee \cdots \vee |x_n|$, so that $\|x\| \le 1$. Using $|\langle z, \lambda\rangle| \le \langle |z|, |\lambda|\rangle$, we see that

$$|\langle x_1, \dots, x_n, \lambda_1, \dots, \lambda_n\rangle| \le \sum_{j=1}^{n} \langle |x_j|, |\lambda_j|\rangle \le \langle x, \lambda\rangle \le \|\lambda\|$$

and hence $\|(\lambda_1 \dots, \lambda_n)\|'_n \le \|\lambda\|$.

For each $\epsilon > 0$ there exist $x \in E_{[1]}$ with $|\langle x, \lambda\rangle| > \|\lambda\| - \epsilon$. We have

$$\|\lambda\| - \epsilon \le \sum_{j=1}^{n} \langle |x|, |\lambda_j|\rangle = |\langle (x, \dots, x), (\lambda_1, \dots, \lambda_n)\rangle|.$$

But $\|(x, \dots, x)\|_n = \|x\| \le 1$, and so $\|\lambda\| - \epsilon \le \|(\lambda_1, \dots, \lambda_n)\|'_n$. □

Proposition 4.1.4 ([6, Proposition 3.79])**.** *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the dual of the dual multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the lattice multi norm on $\{(E')^n : n \in \mathbb{N}\}$.*

Proof. Similar to the above. □

Corollary 4.1.5 ([6, Corollary 3.80])**.** *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the second dual of the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$.* □

There are two useful examples of lattice multi-norms.

Example 4.1.6 ([6, Example 3.81])**.** Let Ω be a measure space, take $p \ge 1$, and let $E = L^p(\Omega)$ which is Banach lattice. The corresponding lattice multi-norm $((E^n, \|\cdot\|))$ is given by

$$\|(f_1, \dots, f_n)\|_n = \Big(\int_\Omega (|f_1| \vee \ldots \vee |f_n|)^p\Big)^{1/p}.$$

By equation(3.5), this is exactly the standard (p, p)-multi-norm on $\{E^n : n \in \mathbb{N}\}$. □

Example 4.1.7 ([6, Example 3,82])**.** Let Ω be compact space, so that the Banach space $(C(\Omega), \|\cdot\|_\infty)$ is a Banach lattice. Then the corresponding lattice multi-norm on the family $\{(C(\Omega))^n : n \in \mathbb{N}\}$ is just the minimum multi-norm.

More generally, for a Banach lattice which is an AM-space, the lattice multi-norm is just the minimum multi-norm. □

Proposition 4.1.8. *Let* $(E, \|\cdot\|)$ *be a Banach lattice, and suppose that* $E = E_1 \perp \cdots \perp E_n$, *where* $n \in \mathbb{N}$. *Then*

$$\|(x_1, \dots, x_n)\|_n = \||x_1| + \cdots + |x_n|\| = \|x_1 + \cdots + x_n\|$$

whenever $x_j \in E_j$ *for* $j \in \mathbb{N}_n$. □

4.2 Operator spaces and multi-norms

In this section we give a relation between operator spaces and multi-norms. We take as our reference for operator spaces the monograph **[7]**.

Concrete and abstract operator spaces

A *concrete function space* on a set S is defined to be a linear subspace E of $\ell_\infty(S)$. All normed spaces arise in this fashion. Let E be any normed space and $x \in E$. By the Hahn-Banach theorem there is a linear functional $f \in E'$ with $\|f\| = 1$ for which $|f(x)| = \|x\|$. Thus if $S = E'_{[1]}$, then an isometry $\Phi : E \to \ell_\infty(S)$ may be defined by letting $\Phi(v)(f) = f(v)$ for $f \in S$. Thus one can say that any normed space E is isometric to a function space.

Given a normed space E and $n \in \mathbb{N}$, the space $\ell_\infty^n(E)$, as usual, consists of n-tuples $x = (x_1, \dots, x_n) \in E^n$ together with the norm

$$\|x\|_\infty = \max\{\|x_j\| : j \in \mathbb{N}_n\}. \tag{4.1}$$

If E is represented as a function space $E \subset \ell_\infty(S)$, then this norm is also determined by the inclusion

$$E^n \subseteq \ell_\infty(S \times n)$$

where n stands for the set $\mathbb{N}_n$, thus $S \times n$ is a disjoint union of n copies of the set S.

Definition 4.2.1. *Let* H *be a Hilbert space. A* concrete operator space V *on* H *is defined to be replacing* $\ell_\infty(S)$ *by* $\mathcal{B}(H)$ *to be a linear subspace of* $\mathcal{B}(H)$.

The natural inclusion $\mathbb{M}_n(V) \subseteq \mathcal{B}(H)$ determines a norm $\|\cdot\|_n$ on $\mathbb{M}_n(V)$. A *matrix norm* $\|\cdot\|$ on a linear space V defined to be an assigment of a norm $\|\cdot\|_n$ on the matrix space $\mathbb{M}_n(V)$ for each $n \in \mathbb{N}$.

Definition 4.2.2. *An* abstract operator space *is such a linear space V and sequence*

$$(\|\cdot\|_n : n \in \mathbb{N})$$

of matrix norms such that:

(M1) $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$

(M2) $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$

for all $v \in \mathbb{M}_m(V)$, for all $w \in \mathbb{M}_n(V)$ and $\alpha \in \mathbb{M}_{n,m}, \beta \in \mathbb{M}_{m,n}$.

The relation between multi-norms and operator spaces

Let V be a linear space and let $k \in \mathbb{N}$ fixed. For $n \in \mathbb{N}$ consider the set of matrices

$$\{T_j = \left(x_{r,s}^{(j)}\right) \in \mathbb{M}_k(v) : j \in \mathbb{N}_n\} \subset \mathbb{M}_k(V).$$

Choose not necessarily distinct $i_1, \ldots, i_k \in \mathbb{N}_n$, and consider the matrix,

$$T_{i_1,\ldots,i_k} = \begin{bmatrix} x_{1,1}^{(i_1)} & \cdots & x_{1,k}^{(i_k)} \\ \vdots & \ddots & \vdots \\ x_{k,1}^{(i_1)} & \cdots & x_{k,k}^{(i_k)} \end{bmatrix} \in \mathbb{M}_k(V).$$

Finally define

$$|||(T_1, \ldots, T_n)|||_n = \max_{i_1,\ldots,i_k} \{\|T_{i_1,\ldots,i_k}\|_k\}.$$

Note that in the case $k = 1$ we have the minumum multi-norm on the family $\{V^n : n \in \mathbb{N}\}$.

The following proposition states that there is a connection between the operator spaces and multi-norms.

Proposition 4.2.3 ([6, Proposition 3.91])**.** *Let V be a linear space, which generates an operator space. Fix $k \in \mathbb{N}$. If $E = \mathbb{M}_k(V)$, then the above defined sequence $(|||\cdot|||_n : n \in \mathbb{N})$ is a multi-norm on the family $\{E^n : n \in \mathbb{N}\}$.* □

CHAPTER 5

ORTHOGONALITY

In this chapter we will give a problem on direct sum decompositions, but we should give necessary materials before the problem. These materials are included in [**6**].

5.1 Terminology

Definition 5.1.1. *Suppose* $(E, \|\cdot\|)$ *be a normed space, and we can consider a family* $\mathcal{K} = \{\{E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}\} : \alpha \in A\}$*, where* A *is an index set,* $n_\alpha \in \mathbb{N}$ *for* $\alpha \in A$*, and*

$$E = E_{1,\alpha} \oplus \cdots \oplus E_{n_\alpha,\alpha}$$

is a direct sum decomposition of E *for each* $\alpha \in A$*. The family* $\mathcal{K}$ *is* closed *provided that the following conditions are satisfied:*

(C1) $\{E_{\sigma(1),\alpha}, \dots, E_{\sigma(n_\alpha),\alpha}\} \in \mathcal{K}$ *when* $\{E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}\} \in \mathcal{K}$ *and* $\sigma \in \mathfrak{S}_{n_\alpha}$*;*

(C2) $\{E_{1,\alpha} \oplus E_{2,\alpha}, E_{3,\alpha}, \dots, E_{n_\alpha,\alpha}\} \in \mathcal{K}$ *when* $\{E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}\} \in \mathcal{K}$ *and* $n_\alpha \geq 2$*;*

(C3) $\mathcal{K}$ *contains all trivial direct sum decompositions.*

Definition 5.1.2. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, let* $k \in \mathbb{N}$*, and let* $\{E_1, \dots, E_k\}$ *be a family of closed linear subspaces of* E*. Then* $\{E_1, \dots, E_k\}$ *is an* orhogonal family *in* E *if, for each partition* $\{S_1, \dots, S_j\}$ *of* $\mathbb{N}_k$ *into non-empty sets, we have*

$$\|(x_1, \dots, x_k)\|_k = \|(y_1, \dots, y_j)\|_j \quad (x_1 \in E_1, \dots, x_k \in E_k)$$

where $y_i = \sum\{x_r : r \in S_i\} \quad (i \in \mathbb{N}_j)$*. A set* $\{x_1, \dots, x_k\}$ *of elements of* E *is* orthogonal *if the family of sets* $\{\mathbb{C}x_1, \dots, \mathbb{C}x_k\}$ *is an orthogonal family.*

Definition 5.1.3. *Suppose* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, let* $k \in \mathbb{N}$, *and let* $E_1, \ldots, E_k$ *be closed linear subspaces in* E *such that* $E = E_1 \oplus \cdots \oplus E_k$ *is a direct sum decomposition. Then the decomposition is an* orthogonal decomposition *of* E *if* $\{E_1, \ldots, E_k\}$ *is an orthogonal family.*

Definition 5.1.4. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, and let* $\mathcal{K} = \{\{E_{1,\alpha}, \ldots, E_{n_\alpha,\alpha}\} : \alpha \in A\}$ *be a closed family of orthogonal decompositions of* E. *Then the multi-normed space is* orthogonal with respect to $\mathcal{K}$ *if*

$$\|(x_1, \ldots, x_n)\|_n = \sup_{\alpha \in A}\{\|(P_{1,\alpha}x_1, \ldots, P_{n_\alpha,\alpha}x_n)\|_n : n_\alpha = n\} \tag{5.1}$$

for each $n \in \mathbb{N}$ *and* $x_1, \ldots, x_n \in E$.

5.2 Orthogonal Decompositions

Theorem 5.2.1 ([6, Theorem 7.34])**.** *Let* $E = C(\Omega)$ *and let* $(\|\cdot\|_n : n \in \mathbb{N})$ *be the lattice multi-norm on the family* $\{E^n : n \in \mathbb{N}\}$. *For* $k \in \mathbb{N}$, $\{E_1, \ldots, E_k\}$ *is an orthogonal decomposition of* E, *with respect to the lattice multi-norm if and only if* $E_i = C(\Omega_i) \quad (i \in \mathbb{N}_k)$, *where* $\{\Omega_1, \ldots, \Omega_k\}$ *is a partition of* Ω *into closed subsets.*

Proof. Let $E = C(\Omega)$ and let $E = E_1 \oplus E_2$ be an orthogonal decomposition of E. Thus, it will be considered just the case where $k = 2$. Let P_1 and P_2 be projections onto E_1 and E_2, respectively.

Now let $f \in E$. Set $f_i = P_i f \quad (i = 1, 2)$, so that $f = f_1 + f_2$. Suppose that $\|f\|_\infty = 1$. Since $\{E_1, E_2\}$ is an orthogonal family we get

$$1 = \|f\|_\infty = \|f_1 + f_2\| = \|(f_1, f_2)\|_2 = \max\{\|f_1\|_\infty, \|f_2\|_\infty\} \quad \text{(By Example 4.1.7).}$$

Then without loss of generality, we may assume that $\|f_1\|_\infty = 1$. Choose $y \in \Omega$ with $|f(y)| = 1$. Assume towards a contradiction that $f_2(y) \neq 0$. Then there exist $\alpha \in \mathbb{T}$ with $|(f_1 + \alpha f_2)(y)| > 1$. But

$$\|f_1 + \alpha f_2\|_\infty = \|(f_1, \alpha f_2)\|_2 = \max\{\|f_1\|_\infty, \|\alpha f_2\|_\infty\} = 1,$$

which is a contradiction. Then we obtain that there exist $y \in \Omega$ such that $\|f\|_\infty = |f(y)|$ with $(P_1 f \cdot P_2 f)(y) = 0$.

Next, let $f \in C(\Omega)$, and let $x \in \Omega$ be such that $|f(x)| = \|f\|_\infty$. Then $(P_1 f . P_2 f)(x) = 0$ To see this, assume towards for a contradiction that $|(P_1 f \cdot P_2 f)(x)| = \eta > 0$, and let V be a neighbourhood of x in Ω such that

$$|(P_1 f \cdot P_2 f)(z)| > \eta/2 \quad \text{for } z \in V. \tag{5.2}$$

There exist $g \in C(\Omega)$ with $g(\Omega) \subset \mathbb{I}$, with $g(z) = 1$ for $z \in \Omega \setminus V$, and with $g(x) = 0$. Set

$$h = \frac{f}{1 + \epsilon g}$$

where $\epsilon > 0$ is such that $\|P_1 f \cdot P_2 f - P_1 h \cdot P_2 h\|_\infty < \eta/2$; such a choise of ϵ is possible. Then $h \in C(\Omega)$ and $|h(x)| = \|h\|_\infty = 1$. By above, there exist $y \in \Omega$ with $|h(y)| = 1$ and $(P_1 h \cdot P_2 h)(y) = 0$, and then $|(P_1 f \cdot P_2 f)(y)| < \eta/2$. We have

$$|h(z)| \leq \frac{1}{1+\epsilon} < 1 \quad (z \in \Omega \backslash V),$$

and so $y \in V$. This contradicts (5.2), and so $(P_1 f \cdot P_2 f)(x) = 0$. Then, there are clopen subsets Ω_1 and Ω_2 of Ω such that $P_1(\mathbf{1}) = \chi_{\Omega_1}$ and $P_2(\mathbf{1}) = \chi_{\Omega_2}$.

Now take $g \in E_2$ with $\|g\|_\infty = 1$. Then $\|\chi_{\Omega_1} + g\|_\infty = 1$, and so, $g \mid \Omega_1 = 0$. Thus $E_2 \subset C(\Omega_2)$. Similarly, $E_1 \subset C(\Omega_1)$. Since $C(\Omega) = E_1 + E_2$, it follows that $E_1 = C(\Omega_1)$ and $E_2 = C(\Omega_2)$. □

Example 5.2.2 ([6, Example 7.35])**.** Let $E = \ell^p$, take p, q with $1 \leq p \leq q < \infty$, and let $\{E^n : n \in \mathbb{N}\}$ have the standard (p,q)-multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$, which is defined as in Definition 3.3.3 by the Equation (3.6).

Let $\{S_1, \ldots, S_k\}$ be a partition of $\mathbb{N}$ (with each S_j non-empty), and let $E_j = \ell^p(S_j)$ for $j \in \mathbb{N}_k$, regarding each E_j as a closed subspace of E. Then $E = E_1 \oplus \cdots \oplus E_k$, $\{E_1, \ldots, E_k\}$ is an orthogonal decomposition of E with respect to the standard (p,q)-multi-norm if and only if $q = p$, and the only possible non-trivial orthogonal decompositions of E have the above form.

To see the first claim, let $\{E_1, \ldots, E_k\}$ be an orthogonal decomposition of E with respect to the standard $(p,q)-$multi-norm, take $n_j \in S_j$ and $x_j = \delta_{n_j}$ for $j \in \mathbb{N}_k$. Then $\|(x_1, \ldots, x_k)\|_k = k^{1/q}$ and $\|x_1 + \cdots + x_k\| = k^{1/p}$, so $p = q$. The sufficient condition follows from Equation (3.4).

For the second claim, let $E = E_1 \oplus E_2$ is a non-trivial orthogonal decomposition of E.

For $k \in \mathbb{N}$, there exist $x_1 = (x_{1,i}) \in E_1$ and $x_2 = (x_{2,i}) \in E_2$ with $\delta_k = x_1 + x_2$ and

$$\|(x_1, zx_2)\|_2 = \|x_1 + zx_2\| \quad (z \in \mathbb{C}).$$

Take $\alpha = x_{1,k} \in \mathbb{C}$ and $y = (y_i) \in E$ with $y_k = 0$ such that $x_1 = \alpha\delta_k + y$, and so $x_2 = (1-\alpha)\delta_k - y$. Suppose that $\alpha \neq 1$, and set

$$\beta = \alpha/(1-\alpha) \quad \text{and} \quad r = \max\{|\beta|, 1\}.$$

For each $z \in \mathbb{C}$ with $|z| \geq r$, we have $|zx_{2,j}| \geq |x_{1,j}|$ for $j \in \mathbb{N}$, so $\|x_1, zx_2\|_2 = |z|\|x_2\|$. Thus $|z|\|x_2\| = \|x_1 + zx_2\|$ so,

$$|z|\|(1-\alpha)\delta_k - y\| = \|\delta_k(\alpha + z(1-\alpha)) + y(1-z)\|.$$

Then by evaluating norms and taking $p-$th power of equation we have,

$$|z|^p(|1-\alpha|^p + \|y\|^p) = |\alpha + z(1-\alpha)|^p + \|y\|^p|1-z|^p.$$

Then the function

$$f(\omega) = \|y\|^p|1-\omega|^p + |1-\alpha|^p|1+\beta\omega|^p$$

is constant on a neigbourhood of 0 in $\mathbb{C}$. Take $\omega = t\zeta$, where $t > 0$ is sufficiently small an $\zeta \in \mathbb{T}$ is such that $\Re\zeta \leq 0$ and $\Re(\beta\zeta) \geq 0$. Then $|1-\omega| > 1$ and $|1+\beta\omega| \geq 1$. So $\|y\| = 0$, and so $y = 0$, and then $\alpha = 0$. Similarly, if $\alpha = 1$, we again conclude that $y = 0$. Thus either $x_1 = \delta_k$ and $x_2 = 0$ or $x_1 = 0$ and $x_2 = \delta_k$. Hence, for each $k \in \mathbb{N}$, $\delta_k \in E_1$ or $\delta_k \in E_2$ as desired. □

5.3 A problem on direct sum decompositions

In **[6]**, which is the central axis of our present work, H.G. Dales and M.E. Polyakov studied Banach lattices as important special examples which are given in the previous section. As pointed out by A.W. Wickstead in **[15]**, multi-norm theory would be simplified if the following problem has an affirmative answer.

Problem 5.3.1. *Let E be a Banach lattice with an algebraic direct sum decomposition $E = E_1 \oplus E_2$ and with the property that*

$$\||x_1| \vee |x_2|\| = \|x_1 + x_2\|$$

for all $x_1 \in E_1$ and $x_2 \in E_2$. Is $E_1 \perp E_2$?

It is known **[15]** that for real scalars this fails: a simple example can be seen by taking $E = \mathbb{R}^2$ with the supremum norm, $E_1 = \{(x, x) : x \in \mathbb{R}\}$, and $E_2 = \{(y, -y) : y \in \mathbb{R}\}$.

For special Banach lattices there is an affirmative answer in **[6]**. For $E = C(\Omega)$ the answer is given by combining Example 4.1.7 and Theorem 5.2.1 and for the case $E = \ell^p$ the answer can be obtained via Example 4.1.6 and Example 5.2.2.

The main goal of this thesis is to go through the above-mentioned problem for the case $E = L^p(\mathbb{I})$. In this regard, our main observation is the subject matter of the following example.

Example 5.3.2. Let $E = L^p(\mathbb{I})$ with $1 \leq p \leq q < \infty$ and let $\{E^n : n \in \mathbb{N}\}$ have the standart (p, q)-multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$.

Let $k \in \mathbb{N}$, and let $S_1, \ldots, S_k$ be measurable subsets of $\mathbb{I}$ with $\lambda(S_j) \neq 0$ for $j = 1, \ldots, k$, $\bigcup_{j=1}^{k} S_j = \mathbb{I}$ and $\lambda(S_m \cap S_n) = 0$ for $m, n \in \mathbb{N}_k$ with $m \neq n$. And define,

$$E_j := \{f \in E : \operatorname{supp} f \subseteq S_j\}.$$

Then we have E_j is a band for each $j \in \mathbb{N}_k$, and $E = E_1 \oplus \cdots \oplus E_k$. Further, since E is Dedekind complete, we have by Theorem 2.1.2, E_j is a projection band for each $j \in \mathbb{N}_k$.

We first claim that $\{E_1, \ldots, E_k\}$ is an orthogonal decomposition of E with respect to the standard (p, q)-multi-norm if and only if $p = q$. To see this, first suppose $p = q$ then the fact that the decomposition is orthogonal follows from equation (3.4).

Now fix $p \geq 1$ and $q \in [p, \infty)$, then take $f_j = m_j \chi_{S_j}$ where $m_j = \left(\frac{1}{\lambda(S_j)}\right)^{1/p}$ for $j \in \mathbb{N}_k$. We have

$$\begin{aligned}
\|(f_1, \ldots, f_k)\|_k &= \left(\|f_1\|^q + \ldots + \|f_n\|^q\right)^{1/q} \quad \text{(by equation 3.3)} \\
&= \left(\sum_{j=1}^{k} \left(m_j^p \lambda(S_j)\right)^q\right)^{1/q} \\
&= k^{1/q}
\end{aligned}$$

Further, we have

$$\begin{aligned}\|f_1 + \ldots + f_k\| &= \Big(\int_{[0,1]} |m_1\chi_{S_1} + \ldots + m_k\chi_{S_k}|^p\Big)^{1/p} \\ &= \Big(\int_{[0,1]} |m_1\chi_{S_1}|^p + \ldots + \int_{[0,1]} |m_k\chi_{S_k}|^p\Big)^{1/p} \quad \text{(Since } \lambda(S_m \cap S_n) = 0\text{)} \\ &= k^{1/p}\end{aligned}$$

thus $p = q$ as desired.

We next claim that the only possible non-trivial orthogonal decomposition of E have the above form.

Indeed, suppose that $E = E_1 \oplus E_2$ is a non-trivial orthogonal decomposition of E.

Let $f_1 \in E_1$. Suppose $|f| \leq |f_1|$ for $f \in E$. Then we have $\operatorname{supp} f \subseteq \operatorname{supp} f_1$. Define,

$$E_1^{\text{supp}} := \{x \in [0,1] : x \in \operatorname{supp} f \quad \text{for some} \quad f \in E_1\}$$

and

$$E_2^{\text{supp}} := \{x \in [0,1] : x \in \operatorname{supp} f \quad \text{for some} \quad f \in E_2\}.$$

Consider the constant function $\mathbf{1} \in E$, let P_1 and P_2 be the projections on E_1 and E_2, respectively. We have $P_1(\mathbf{1})(\mathbb{I}) \subseteq E_1^{\text{supp}}$ and $P_2(\mathbf{1})(\mathbb{I}) \subseteq E_2^{\text{supp}}$, since $E = E_1 \oplus E_2$, we have $E_1^{\text{supp}} \cup E_2^{\text{supp}} = \mathbb{I}$.

Further, let $x \in E_1^{\text{supp}} \cap E_2^{\text{supp}}$ then for all $f_i \in E_i$ we have $|f_i|(x) > 0$ for $i = 1, 2$. Since $E = E_1 \oplus E_2$ is a non-trivial orthogonal decomposition of E, we have $|f_1| \wedge |f_2| = 0$, and so $\lambda(E_1^{\text{supp}} \cap E_2^{\text{supp}}) = 0$.

Since $\operatorname{supp} f \subseteq \operatorname{supp} f_1 \subseteq E_1^{\text{supp}}$, there exists $g \in E_1$ such that $\operatorname{supp} f = \operatorname{supp} g$. Thus $f \in E_1$, so E_1(similarly E_2) is an ideal of E. So, E_1 and E_2 are bands by Theorem 2.1.1.

But we know that all bands of E have of the form above (see **[10]**, p. 263). This gives the second claim.

Finally we conclude that, when $E = L^p(\mathbb{I})$ has the standard $(p,q)-$multi-norm, if $p \neq q$ then there are no non-trivial orthogonal decomposition of E, and if $p = q$ then the only non-trivial orthogonal decompositions of E are

$$E = E_1 \oplus \cdots \oplus E_k,$$

where $\{E_1, \ldots, E_k\}$ as above. Thus regarding $E = L^p(\mathbb{I})$ as a Banach lattice, we have

$$E = E_1 \perp \cdots \perp E_k.$$

If $E = L^p(\mathbb{I})$ has the standard $(p,p)-$multi-norm, and if $\mathcal{K}$ be the family of all orthogonal decompositions of E, then each member of $\mathcal{K}$ has the above form. Clearly the multi-normed space is orthogonal with respect to the family $\mathcal{K}$. □

REFERENCES

[1] Y.A. Abramovich and C.D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, Graduate Studies in Mathematics, Volume 50, Providence, Rhode Island, 2002.

[2] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Graduate Text in Mathematics, Volume 233, Springer, New York, 2006.

[3] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985.

[4] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, Volume 24, Clarendon Press, Oxford, 2001.

[5] H.G. Dales and M. S. Moslehian, "Stability of mappings on multi-normed spaces," *Glasg. Math. J.* **49** (2007), p. 321-332.

[6] H. G. Dales and M. E. Polyakov, *Multi-norms*, Preprint, available at *http://www.amsta.leeds.ac.uk/∼pmt6hgd/preprints/AMSmultinorms.pdf.*

[7] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs, Volume 23, Clarendon Press, Oxford, 2000.

[8] W.A. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland Publishing Company, Amsterdam, 1971.

[9] W.A. Luxemburg and A. C. Zaanen, *Riesz Spaces II*, North-Holland Publishing Company, Amsterdam, 1982.

[10] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, 1991.

[11] M. S. Moslehian and H. M. Srivastava, "Jensens's functional equation in multi-normed spaces," *Taiwanese J. Math.* **14** (2010), no. 2, p.453-462.

[12] P. Ramsden, *Homological properties of semigroup algebras,* Ph. D. Thesis, University of Leeds, 2008.

[13] Inder K. Rana, *An Introduction to Measure and Integration*, American Mathematical Society, Graduate Studies in Mathematics, Volume 45, Narosa Publishing House, 2002, 2nd ed.

[14] W. Rudin, *Real and Complex Analysis*, McGraw Hill, New York, 1987, 3rd ed.

[15] A. W. Wickstead, "Open problems," (Compiled by A.W. Wickstead), *Positivity* **13** (2009), no. 1, 299-306.

[16] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Studies in Advanced Mathematics, Volume 25, Cambridge University Press, 1991.

Vita

Mehmet Selçuk Türer was born in Sakarya, on 5 May 1984. He started to primary school in Sakarya, in 1984. He graduated from Sakarya Ataürk Lisesi, in 2001. He started his undergraduate studies at the Department of Mathematics, Istanbul University, Istanbul, in 2001 and took his Bachelor's degree in 2008. His graduate studies started at the Department of Mathematics and Computer Science, Istanbul Kültür University, in 2008.