İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF SCIENCES

CYCLICALLY COMPACT OPERATORS ON KAPLANSKY–HILBERT MODULES

Ph.D. Thesis by

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Department: Mathematics and Computer Science

Programme: Mathematics

Supervisor: Assoc. Prof. Dr. Mert ÇAĞLAR

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KAPLANSKY–HILBERT MODÜLLER ÜZERİNDE DEVRESEL KOMPAKT OPERATÖRLER

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İstanbul, May 2014 Uğur GÖNÜLLÜ

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ÖZET

KAPLANSKY–HILBERT MODÜLLER ÜZERİNDE DEVRESEL KOMPAKT OPERATÖRLER

Uğur GÖNÜLLÜ

Tezin ilk kısmında Kaplansky–Hilbert modülleri üzerindeki devresel kompakt kümeler ve operatörler çalışılmıştır. A. G. Kusraev, Boole-değerli analiz teknikleri kullanarak, devresel kompakt operatörlerin genel bir formunu ispatlamıştır. Tezde, bu genel formun standart kanıtı verilmiştir. Ayrıca, devresel kompakt operatörlerin bazı karakterizasyonları elde edilmiştir. İkinci kısımda Kaplansky–Hilbert modülleri üzerindeki sürekli Λ-lineer operatörlerin Schatten-tipindeki sınıfları çalışılmış ve bunların dualitelerini araştırılmıştır. Öte yandan, Hilbert–Schmidt sınıflarının birer Kaplansky–Hilbert modülü olduğu gösterilmiştir. Son kısımda, Kaplansky–Hilbert modülleri üzerindeki devresel kompakt operatörlerin global özdeğerleri ve bu özdeğerlerin katlılıkları tanımlanmış ve incelenmiştir. Kaplansky–Hilbert modülleri üzerindeki devresel kompakt operatörler için Horn- ve Weyl-tipi eşitsizlikler ve Lidskiĭ iz formülü elde edilmiştir.

Anahtar Kelimeler: Kaplansky–Hilbert modül, devresel kompakt operatör, Schatten-tipi sınıflar, Lidskiĭ iz formülü

University	: İstanbul Kültür University
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SUMMARY

CYCLICALLY COMPACT OPERATORS ON KAPLANSKY–HILBERT MODULES

Uğur GÖNÜLLÜ

The first part of the thesis studies cyclically compact sets and operators on Kaplansky–Hilbert modules. A. G. Kusraev proved a general form of cyclically compact operators in Kaplansky–Hilbert modules using techniques of Boolean-valued analysis. We give a standart proof of this general form. Moreover, we obtain some characterizations of cyclically compact operators. The second part studies the Schatten-type classes of continuous Λ-linear operators on Kaplansky–Hilbert modules and investigates the duality of them. Furthermore, we show that the Hilbert–Schmidt class is a Kaplansky–Hilbert module. In the last part we define and study global eigenvalues of cyclically compact operators on Kaplansky–Hilbert modules and their multiplicities. We obtain Horn- and Weyl-type inequalities and Lidskiĭ trace formula for cyclically compact operators in Kaplansky–Hilbert modules.

Keywords: Kaplansky–Hilbert module, cyclically compact operator, Schatten-type classes, Lidskiĭ trace formula

Chapter 1

Introduction

1.1 State of the Art

The concept of Kaplansky−Hilbert module, or AW^*-module, arose naturally in Kaplansky's study of AW^*-algebras of Type I. Kaplansky−Hilbert module, which is an object like a Hilbert space except that the inner product is not scalar-valued but takes its values in a commutative C^*-algebra Λ which is an order complete vector lattice, was introduced by I. Kaplansky [15]. Such a C^*-algebra is often called a Stone algebra or a commutative AW^*-algebra. I. Kaplansky proved some deep and elegant results for such structures, thereby showing that they have many properties similar to those of the Hilbert spaces. A Kaplansky−Hilbert module X is called λ-homogeneous if X has a basis of cardinality λ. Not every Kaplansky−Hilbert module has a basis, but we can split it into homogeneous parts [15, Theorem 1.]. The concept of strict λ-homogeneity was introduced by A. G. Kusraev and as shown in [20] every Kaplansky−Hilbert module can be splitted into strictly homogeneous parts . In the same paper, Kusraev established functional representations of Kaplansky−Hilbert modules and AW^*-algebras of type I by spaces of continuous vector-functions and strongly continuous operator-functions, respectively. In [42], H. Takemoto gave another representation where each AW^*-module is representable as a continuous field of Hilbert spaces over a Stonean space. By using this representation a variant of the polar decomposition was obtained by H. Takemoto [43], and C. Sunouchi gave another proof [40].

C^*-algebras and von Neumann algebras within Boolean-valued models appeared in the research of G. Takeuti [44, 45]. M. Ozawa started the study of AW^*-modules and algebras by means of Boolean-valued models of set theory [29], in which he gave a negative solution to the I. Kaplansky problem on the unique decomposition of a type I AW^*-algebra into the direct sum of homogeneous bands [33, 34].

The generalization of the concept of Kaplansky−Hilbert module is the Hilbert C^*-module (inner product takes values in a C^*-algebra) which appeared in the papers of W. Paschke and M. Rieffel (see [27]).

The von Neumann−Schatten Classes S_p $(1 \leq p < \infty)$ of linear operators on a Hilbert space H were introduced by von Neumann and Schatten [38]. It turns out that each of these classes is a two-sided ideal in $B(H)$, and consists of compact operators. The space S_p is a Banach space with properties closely analagous to those of the sequence space ℓ_p. The linear spaces $S_p(H)$ and $S_q(H)$ constitute a dual system with respect to the bilinear form $\langle S, T\rangle := \operatorname{tr}(ST)$ where $S \in S_p(H)$, $T \in S_q(H)$ and p is the conjugate index to q. In this sense, $S_p(H)$ can be identified with $S_q(H)'$. In cases $p = 1$ and $p = \infty$, we have $S_1(H)' = B(H)$ and $K(H)' = S_1(H)$, respectively. The latter formulas were obtained by Schatten [36] and Schatten/von Neumann [37]. Though the Banach spaces $S_p(H)$ with $1 \leq p < \infty$ are only semi-classical, they have proved to be quite important. Their main significance, however, stems from the fact that they are even Banach ideals over H.

By a continuous Λ-linear operator T from an AW^*-module X to an AW^*-module Y we mean a mapping of X into Y which is not only linear and continuous as usual, but also a module homomorphism. I. Kaplansky showed that a Λ-linear operator T is continuous if and only if T has an adjoint T^*, and that the set $B_\Lambda(X)$ of all continuous Λ-linear operators in X forms AW^*-algebra of type I in [15]. Every continuous Λ-linear operator is dominated and *bo*-continuos [23]. A continuous Λ-functional on a Kaplansky−Hilbert module X is a continuous Λ-linear operator from X to Λ. Kaplansky also proved that the Riesz Representation Theorem is satisfied on Kaplansky−Hilbert modules [15]. In [49] two versions of a spectral theorem for continuous Λ-linear operators are obtained T on the Kaplansky−Hilbert module X.

In 1936, L. V. Kantorovich introduced the concept of lattice-normed space. These are vector spaces normed by elements of a vector lattice. Every Kaplansky−Hilbert module is a Banach−Kantorovich space which is a decomposable *o*-complete lattice-normed space [23, 7.4.4.].

Cyclically compact sets and operators in lattice-normed spaces were introduced by A. G. Kusraev in [18] and [19], respectively, and a preliminary study of this notions was initiated. Cyclical compactness is the Boolean-valued interpretation of compactness and it also deserves an independent study. For different aspects of cyclical compactness, see [21, 25, 26]. In [22] (see also [23]) a general form of cyclically compact operators in Kaplansky−Hilbert modules, which is similar to the Schmidt representation of compact operators on Hilbert spaces, as well as a variant of the Fredholm alternative for cyclically compact operators, were also given with Boolean-valued techniques. Thus, the natural problem arises to investigate the class of cyclically compact operators in more details. Recently, cyclically compact sets and operators in Banach−Kantorovich spaces over a ring of measurable functions were investigated in [8, 16, 17]. In this vein, the following problems are of importance.

1.2 Statement of the Problem

Introduce and study the Schatten-type classes of continuous linear operators on Kaplansky−Hilbert modules. In particular, we obtain a general form of cyclically compact operators in Kaplansky−Hilbert modules, duality results for the Schatten type classes, and generalized Lidskiĭ trace formula.

1.3 Review of Contents

Chapter 1 of this thesis presents the scope of the study as an introduction.

Chapter 2 contains some background related to theory of Boolean algebras, lattice-normed spaces and AW^*-algebras needed in the sequel.

Chapter 3 deals with Kaplansky−Hilbert modules and cyclically compact operators on them. The first section of the Chapter 3 is an introduction to Kaplansky−Hilbert modules, and the concept of *projection basis* is defined. Section 3.2 is related to cyclically compact sets, and we reprove some characterizations about cyclically compact sets on $C_{\#}(Q, H)$ which were proved for measurable bundles in [16]. In section 3.3 we study operators on Kaplansky−Hilbert modules and define a new notion of *global eigenvalue* of operators, and prove the Polar Decomposition for continuous Λ-linear operators. In the last section of Chapter 3, we prove a general form of cyclically compact operators with standart techniques and give some characterizations about cyclically compact operators on Kaplansky−Hilbert modules and the Rayleigh−Ritz minimax formula. More precisely, we can state the main results of this chapter as follows:

Theorem 1.3.1. *Let T be a cyclically compact operator from X to Y. There exist sequences $(e_k)_{k\in\mathbb{N}}$ in X and $(f_k)_{k\in\mathbb{N}}$ in Y and a sequence $(s_k(T))_{k\in\mathbb{N}}$ of positive elements in Λ such that*

(1) $\langle e_k \mid e_l\rangle = \langle f_k \mid f_l\rangle = 0$ $(k \neq l)$ *and* $[s_k(T)] = \lvert e_k\rvert = \lvert f_k\rvert$ $(k \in \mathbb{N})$*;*

(2) $s_{k+1}(T) \leq s_k(T)$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim_{k\to\infty} s_k(T) = \inf_{k\in\mathbb{N}} s_k(T) = 0$*;*

(3) *there exists a projection π_∞ in $\mathfrak{P}(\Lambda)$ such that $\pi_\infty s_k$ is a weak order-unity in $\pi_\infty\Lambda$ for all $k \in \mathbb{N}$;*

(4) *there exists a partition $(\pi_k)_{k=0}^{\infty}$ of the projection $\pi_\infty^{\perp}$ such that $\pi_0 s_1 = 0$, $\pi_k \leq [s_k]$, and $\pi_k s_{k+1} = 0$, $k \in \mathbb{N}$;*

(5) *for each x the following equality is valid:*

$$\begin{aligned} Tx &= \pi_\infty bo\text{-}\sum_{k=1}^{\infty} s_k(T)\langle x \mid e_k\rangle f_k + bo\text{-}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n} s_k(T)\langle x \mid e_k\rangle f_k \\ &= bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle x \mid e_k\rangle f_k. \end{aligned}$$

Theorem 1.3.2. (The Rayleigh−Ritz minimax formula) *Let T be a cyclically compact operator from X to Y. Then*

$$s_n(T) = \inf\left\{\sup\left\{\lvert Tx\rvert : \lvert x\rvert \leq \mathbf{1},\, x \in J^{\perp}\right\}\right\}$$

where the infmum is taken over all projection orthonormal subset J of X such that $\operatorname{card}(J) < n$, *and the infmum is achieved.*

Theorem 1.3.3. *Let T be in $B_\Lambda(X)$ and Θ denote the set of all finite subsets of the projection basis $\mathscr{E}$. Then the following statements are equivalent:*

(i) *T is a cyclically compact operator on X;*

(ii) *for all projection bases $\mathscr{E}$ in X, the net $\left(\lvert T(I - P_F)\rvert\right)_{F\in\Theta}$ o-converges to 0, where $P_F := \sum_{e\in F}\theta_{e,e}$;*

(iii) *for all projection bases $\mathscr{E}$ in X, $\left(\sup_{e\in F^c}\left\{\lvert Te\rvert\right\}\right)_{F\in\Theta}$ decreases to 0;*

(iv) *for all projection bases $\mathscr{E}$ in X, $\left(\sup_{e\in F^c}\{\lvert\langle Te \mid e\rangle\rvert\}\right)_{F\in\Theta}$ decreases to 0.*

Chapter 4 is devoted to study the Schatten type class of operators. In particular, we investigate the Hilbert−Schmidt class, the trace class and classes $\mathscr{S}_p$, and get duality results for the Schatten-type classes. The main results of this chapter are as follows:

Theorem 1.3.4. *The pair $(\mathscr{S}_2(X,Y), \langle\cdot,\cdot\rangle)$ is a Kaplansky−Hilbert module over Λ and the following equality holds:*

$$\lvert T\rvert \leq \upsilon_2(T) \qquad (T \in \mathscr{S}_2(X,Y))$$

where $\lvert T\rvert$ is exact dominant of T [23, 4.1.1].

Theorem 1.3.5. *If $\varphi : \mathscr{S}_1(Y,X) \to \mathscr{K}(X,Y)^*$ is defined by $\varphi(T)(A) = \mathrm{tr}(TA)$ for all $A \in \mathscr{K}(X,Y)$ and $T \in \mathscr{S}_1(Y,X)$, then φ satisfies the following properties:*

(i) *φ is a bijective Λ-linear operator from $\mathscr{S}_1(Y,X)$ to $\mathscr{K}(X,Y)^*$;*

(ii) $\upsilon_1(T) = \lvert\varphi(T)\rvert$ $(T \in \mathscr{S}_1(Y,X))$.

Theorem 1.3.6. *If $\psi : (B_\Lambda(X,Y), \lvert\cdot\rvert) \to (\mathscr{S}_1(Y,X)^*, \lvert\cdot\rvert_1)$ is defined by $\psi(L)(T) = \mathrm{tr}(TL)$ for all $L \in B_\Lambda(X,Y)$ and $T \in \mathscr{S}_1(Y,X)$. Then ψ satisfies the following properties:*

(i) *ψ is a bijective Λ-linear operator from $B_\Lambda(X,Y)$ to $\mathscr{S}_1(Y,X)^*$;*

(ii) $\lvert L\rvert = \lvert\psi(L)\rvert_1$ $(L \in B_\Lambda(X,Y))$.

Theorem 1.3.7. *Let $1 < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi : (\mathscr{S}_p(X), \upsilon_p(\cdot)) \to (\mathscr{S}_q(X)^*, \lvert\cdot\rvert_q)$ is defined by $\phi(T)(S) = \mathrm{tr}(ST)$ for all $T \in \mathscr{S}_p(X)$ and $S \in \mathscr{S}_q(X)$, then ϕ satisfies the following properties:*

(i) *ϕ is a bijective Λ-linear operator from $\mathscr{S}_p(X)$ to $\mathscr{S}_q(X)^*$;*

(ii) $\upsilon_p(T) = \lvert\phi(T)\rvert_q$ $(T \in \mathscr{S}_p(X))$.

In Chapter 5, we study global eigenvalues of cyclically compact operators and their multiplicities. We prove Horn- and Weyl-type inequalities and Lidskiĭ trace formula for cyclically compact operators. More precisely, the main results of this chapter are as follows:

Theorem 1.3.8. *Let T be a cyclically compact operator on X. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting of global eigenvalues or zeros in Λ with the following properties:*

(1) $|\lambda_k| \leq |T|$, $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim \lambda_k = 0$*;*

(2) *there exists a projection* π_∞ *in* Λ *such that* $\pi_\infty|\lambda_k|$ *is a weak order-unity in* $\pi_\infty\Lambda$ *for all* $k \in \mathbb{N}$*;*

(3) *there exists a partition* (π_k) *of the projection* $\pi_\infty^\perp$ *such that* $\pi_0\lambda_1 = 0$*,* $\pi_k \leq [\lambda_k]$*, and* $\pi_k\lambda_{k+m} = 0$*,* $m, k \in \mathbb{N}$*;*

(4) $\pi\lambda_{k+m} \neq \pi\lambda_k$ *for every nonzero projection* $\pi \leq \pi_\infty + \pi_k$ *and for all* $m, k \in \mathbb{N}$*;*

(5) *every global eigenvalue* λ *of* T *is of the form* $\lambda = \operatorname{mix}_{k\in\mathbb{N}}(p_k\lambda_k)$*, where* $(p_k)_{k\in\mathbb{N}}$ *is a partition of* $[\lambda]$*.*

Theorem 1.3.9. *Let* T *be a cyclically compact operator on* X *and* $(\lambda_k(T))_{k\in\mathbb{N}}$ *be a global eigenvalue sequence of* T *with the multiplicity sequence* $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$*. Then the following properties hold:*

(1) (Weyl-inequality) *if* $(\pi s_k(T))_{k\in\mathbb{N}}$ *is o-summable in* Λ *for some projection* π*, then the following inequality holds*

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(T)|\lambda_k(T)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(T);$$

(2) (Horn-inequality) *Suppose that* T_k *is a cyclically compact operator on* X *for* $1 \leq k \leq K$*. Then*

$$\prod_{i=1}^{N} s_i(T_K \cdots T_1) \leq \prod_{k=1}^{K}\prod_{i=1}^{N} s_i(T_k) \quad (N \in \mathbb{N}).$$

(3) (Lidskiĭ trace formula) *if* $T \in \mathscr{S}_1(X)$*, then the following equality holds*

$$\operatorname{tr}(T) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(T)\lambda_k(T).$$

1.4 Methods Applied

This work uses essentially the methods from the following branches of analysis: Theory of vector lattices, lattice-normed spaces, Kaplansky−Hilbert modules, and AW^*-algebras. In particular, we use intensively the following concepts: order convergence, *bo*-convergence and *bo*-summability, the exact dominant of an operator, spaces with mixed norms, the properties of the vector norm, Λ-valued inner product and cyclically compactness. The main technical tool used in the work is the functional representation of Kaplansky−Hilbert modules and bounded linear operators on them.

1.5 Publications and Reports

The following papers are comprised of the results obtained in the present thesis:

- "Schatten-type classes of operators in Kaplansky−Hilbert modules," in *Studies on Math. Analysis and Diff. Equations*, Vol. 8 (2013), Vladikavkaz.
- "Lidskiĭ trace formula in Kaplansky−Hilbert modules," *Vladikavkaz Math. J.*, forthcoming.
- "The Rayleigh−Ritz minimax formula in Kaplansky−Hilbert modules," submitted.

Besides, parts of the thesis were delivered in the following seminars and symposia:

- *Joint Seminar on Analysis* in Southern Mathematical Institute of the Russian Academy of Sciences, Vladikavkaz, Russia (June 2011).
- *International Conference of Young Scientists*: "Mathematical Analysis and Mathematical Modelling," Fijagdon, Vladikavkaz, Russia (July 25-30, 2011).
- *Joint Seminar on Analysis* in Southern Mathematical Institute of the Russian Academy of Sciences, Vladikavkaz, Russia (September 2013).

Chapter 2

Preliminaries

In this chapter, we set the general background which will be needed in the sequel. For further details one can consult on the books [1, 2, 4, 23], whose terminology is used throughout.

2.1 Boolean Algebras

Let M be a partially ordered set with an order relation $\leq$ (i.e. with a reflexive, antisymmetric and transitive relation $\leq$). A subset A of M is *upward-directed* (*downward-directed*) if, given two elements a, b of A, there is an element c of A such that $a \leq c$ and $b \leq c$ ($c \leq a$ and $c \leq b$). If for x in M $a \leq x$ holds for all a in A, we say that x is an *upper bound* of A. A least element of the set of upper bounds of A is called a *least upper bound* or *supremum* of A and denoted by $\sup A$. Lower bound and infimum are defined similarly. The set of upper bounds for a subset E of M is denoted by u.b.(E). M is called *lattice* if each pair of elements x, y in M has $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. A lattice is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

If a lattice L has the least or greatest element then the former is called the *zero* of L and the latter, the *unity* of L. The zero and unity of L are denoted by $\mathbf{0}_L$

and $\mathbf{1}_L$. If x and x^* satisfy $x \vee x^* = \mathbf{1}_L$ and $x \wedge x^* = \mathbf{0}_L$, then x^* is called a *complement* of x. Elements x, y in L satisfying $x \wedge y = \mathbf{0}_L$ are said to be *disjoint.* If each element in L has at least one complement then we call L a *complemented lattice.*

Definition 2.1.1. A *Boolean algebra* is a distributive complemented lattice with distinct zero and unity.

A Boolean algebra B is called *complete* (*σ-complete*) if every non-empty subset (countable subset) of B has a supremum and an infimum. We say that a subset A of a Boolean algebra B is an *antichain* if all distinct two elements of A is disjoint. An antichain A in B is a *partition of an element* $b \in B$ (and so a *partition of unity* when b is the unity of B) provided that $b = \bigvee A = \sup A$. A subset E of B *minorizes* a subset B_0 of B if to each $0 < b \in B_0$ there is an x in E such that $0 < x \leq b$. We will often use the following theorem.

Theorem 2.1.2. [23, 1.1.6.](Exhaustion Principle). *Let M be a nonempty subset of a Boolean algebra B. Assume given a subset E of B that minorizes the band B_0 of B generated by M. Then some antichain E_0 exists, $E_0 \subset E$, such that* u.b.(E_0) = u.b.(M) *and to each $x \in E_0$ there is an element y in M satisfying $x \leq y$.*

We say that a subset F of a Boolean algebra B is a *filter*, if $x, y \in F$ implies $x \wedge y \in F$, and if $b \in B$, $x \in F$ and $b \geq x$ imply $b \in F$. A filter other than B is *proper.* A maximal element of the inclusion-ordered set of all proper filters on B is an *ultrafilter* on B. Let $U(B)$ stand for the set of all ultrafilters on B, and denote by $U(b)$ the set of ultrafilters containing b. We endow $U(B)$ with the topology with base $\{U(b) : \ b \in B\}$. Clearly, $U(x \wedge y) = U(x) \cap U(y)$ $(x, y \in B)$, i.e., $\{U(b) : \ b \in B\}$ is closed under finite intersections. The topological space $U(B)$ is often referred to as the *Stone space of* B and is denoted by $\mathscr{S}(B)$. Recall that a topological space is called *extremally* (*quasiextremally*) *disconnected* or simply *extremal* (*quasiextremal*) if the closure of an arbitrary open set (open F_σ-set) in it is open or, which is equivalent, the interior of an arbitrary closed set (closed G_δ-set) is closed.

Theorem 2.1.3. [23, 1.2.4.](Ogasawara Theorem). *A Boolean algebra is complete* (*σ-complete*) *if and only if its Stone space is extremal* (*quasiextremal*).

In the sequel, by a *Boolean algebra of projections* in a vector space X we mean a set $\mathscr{B}$ of commuting idempotent linear operators that act in X. Moreover, the

Boolean operations have the following form:

$$\pi \wedge \rho := \pi \circ \rho, \qquad \pi \vee \rho := \pi + \rho - \pi \circ \rho, \qquad \pi^* = I_X - \pi \qquad (\pi, \rho \in \mathscr{B}).$$

and the zero and identity operators in X serve as the zero and unity of the Boolean algebra $\mathscr{B}$.

2.2 Vector Lattices

A real vector space E is said to be an *ordered vector space* whenever it is equipped with an order relation $\leq$ that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

(1) If $x \leq y$, then $x + z \leq y + z$ holds for all $x, y, z \in E$,

(2) If $x \leq y$, then $\lambda x \leq \lambda y$ holds for all $x, y \in E$ and $0 \leq \lambda \in \mathbb{R}$.

An element x in an ordered vector space E is called *positive* whenever $0 \leq x$ holds. The set of all positive elements of E is called the *positive cone* of E and it will be denoted by E_+. A *vector lattice* (*Riesz space*) is an ordered vector space that is also a lattice. A vector lattice is called *Dedekind complete* or *order complete* (in the Russian literature, *K-space*) whenever every nonempty subset bounded above has a supremum. Note also we assume that every vector lattice is Archimedean.

Let u be a positive vector of a vector lattice E. A vector $e \in E$ is said to be a *fragment*, or a *part*, or a *component of* u, or a *unit element with respect to* u whenever $e \wedge (u - e) = 0$. The set of all fragments of u is denoted by $\mathfrak{C}(u)$.

Theorem 2.2.1. [2, Theorem 3.15.] *Let E be a vector lattice and $u \in E_+$. Then $\mathfrak{C}(u)$ is a Boolean algebra consisting precisely of all extreme points of the convex set $[0, u]$. Moreover, in case E is Dedekind complete, $\mathfrak{C}(u)$ is likewise Dedekind complete.*

The disjoint complement $M^\perp$ of a nonempty set $M \subset E$ is defined as

$$M^\perp := \{x \in E : \text{ for all } y \in M,\ x \wedge y = 0\}.$$

A nonempty set $K \subset E$ satifying $K = K^{\perp\perp}$ is called a *band of* E. The set of all bands of E is denoted by $\mathfrak{B}(E)$. Every band of the form $\{x\}^{\perp\perp}$ with $x \in E$ is called *principal*. It is well known that $\mathfrak{B}(E)$ is a complete Boolean algebra with the inclusion-order. The Boolean operation of $\mathfrak{B}(E)$ take the shape:

$$L \wedge K = L \cap K, \qquad L \vee K = (L \cup K)^{\perp\perp}, \qquad L^* = L^{\perp} \qquad (L, K \in \mathfrak{B}(E)).$$

We say that $K \in \mathfrak{B}(E)$ is a *projection band* if $E = K \oplus K^{\perp}$. The projection π onto the band K along the band $K^{\perp}$ is called a *band projection* (or *order projection*). The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \leq \rho \Leftrightarrow \pi \circ \rho = \pi$ is a Boolean algebra. The Boolean operations of $\mathfrak{P}(E)$ take the shape

$$\pi \wedge \rho = \pi \circ \rho, \qquad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \qquad \pi^* = I_E - \pi \qquad (\pi, \rho \in \mathfrak{P}(E)).$$

A vector lattice E is said to have the *projection property* whenever every band in E is a projection band.

Let $(A, \leq)$ be an upward-directed set. We say that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice E *o-converges* to $x \in E$ if there is a net $e_\beta \downarrow 0$ in E and for each β there is $\alpha(\beta)$ with $|x_\alpha - x| \leq e_\beta$ $(\alpha \geq \alpha(\beta))$. We call x the *o-limit* of the net $(x_\alpha)_{\alpha \in A}$ and write $x = o\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(o)} x$. If a net (e_β) in this definition is replaced by a sequence $(\lambda_n e)_{n \in \mathbb{N}}$, where $e \in E_+$ and $(\lambda_n)_{n \in \mathbb{N}}$ is a numerical sequence with $\lim_{n \to \infty} \lambda_n = 0$, then we say that a net $(x_\alpha)_{\alpha \in A}$ *converges relatively uniformly* or more precisely *e-uniformly* to $x \in E$. The elements e and x are called the *regulator of convergence* and the *r-limit* of $(x_\alpha)_{\alpha \in A}$, respectively. The notations $x = r\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(r)} x$ are also frequent. A net $(x_\alpha)_{\alpha \in A}$ is called *o-fundamental* (*r-fundametal with regulator* e) if the net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in A \times A}$ o-converges (respectively, r-converges with regulator e) to zero. A vector lattice is said to be *relatively uniformly complete* if every r-fundamental sequence is r-convergent.

A linear subspace J of a vector lattice is called an *order ideal* or *o-ideal* (or, finally, just an ideal) if the inequality $|x| \leq |y|$ implies $x \in J$ for arbitrary $x \in E$ and $y \in J$. If an ideal J possesses the additional property $J^{\perp\perp} = E$ (or, $J^{\perp} = \{0\}$) then J is referred to as an *order-dense ideal* of E. The o-ideal generated by the element $0 \leq u \in E$ is the set $E(u) := \cup_{n=1}^{\infty} [-nu, nu] = \{x \in E : \ |x| \leq \lambda u, \ \lambda \in \mathbb{R}\}$. If $E(u) = E$ then we say that u is a *strong unity* or *strong order-unity.* If $E(u)^{\perp\perp} = E$ then we say that u is an *order-unity* or *weak order-unity.* A *vector sublattice* is a vector subspace $E_0 \subset E$ such that $x \wedge y, \ x \vee y \in E_0$ for all $x, y \in E_0$. A vector lattice is called *disjointly complete* (*disjointly σ-complete*) if every its order-bounded antichain (countable antichain) has supremum.

A norm $\|\cdot\|$ on a vector lattice is said to be a *lattice norm* whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice equipped with a lattice norm is known as a *normed vector lattice.* If a normed vector lattice is also norm complete, then it is referred to as a *Banach lattice.* A Banach lattice E is called an *abstract M-space* or *AM-space* if $\|x \vee y\| = \|x\| \vee \|y\|$ $(x, y \in E_+)$. If the unit ball of an AM-space E contains a largest element e, then e is a strong order-unity and the unit ball of E coincides with the symmetric order interval $[-e, e]$. In this case E is said to be an *AM-space with unit.*

Theorem 2.2.2. (Kakutani-Bohnenblust and M. Krein-S. Krein). *Let E be an AM-space. Then there exist a compact Q and a family of triples $(t_\alpha, s_\alpha, \lambda_\alpha)_{\alpha \in A}$ with $t_\alpha, s_\alpha \in Q$ and $0 \leq \lambda_\alpha < 1$ such that E is linearly isometric and order isomorphic to the closed sublattice*

$$F := \{x \in C(Q) : \ (\forall \alpha \in A) \ x(t_\alpha) = \lambda_\alpha x(s_\alpha)\}.$$

In particular, every AM-space with unity is linearly isometric and order isomorphic to the space of continuous functions C(Q) on some compact space Q.

Theorem 2.2.3. [23, Theorem 1.5.9.] *For a compact space Q, the following are equivalent:*

(1) *$C(Q)$ is order complete (σ-complete);*

(2) *$C(Q)$ is disjointly complete (σ-complete);*

(3) *Q is extremal (quasiextremal);*

(4) *$C(Q)$ possesses the projection property (principal projection property).*

2.3 Lattice-Normed Spaces

Let X be a vector space and E be a real vector lattice. A mapping $\lvert\!\lvert\cdot\rvert\!\rvert : X \to E_+$ is called a *vector* (E*-valued*) *norm* if it satisfies the following axioms:

(1) $\lvert\!\lvert x\rvert\!\rvert = 0 \Leftrightarrow x = 0$ $(x \in X)$;

(2) $\lvert\!\lvert \lambda x\rvert\!\rvert = |\lambda| \lvert\!\lvert x\rvert\!\rvert$ $\ (\lambda \in \mathbb{R}, x \in X)$;

(3) $\lvert\!\lvert x + y\rvert\!\rvert \leq \lvert\!\lvert x\rvert\!\rvert + \lvert\!\lvert y\rvert\!\rvert$ $(x, y \in X)$.

A vector norm is called *decomposable* or *Kantorovich norm* if

(4) for all $e_1, e_2 \in E_+$ and $x \in X$, from $\lvert\!\lvert x\rvert\!\rvert = e_1 + e_2$, it follows that there exists $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\lvert\!\lvert x_k\rvert\!\rvert = e_k$ $(k = 1, 2)$.

In the case when condition (4) is valid only for disjoint $e_1, e_2 \in E_+$, the norm is said to be *disjointly-decomposable* or, in short, *d-decomposable.*

A triple $(X, \lvert\cdot\rvert, E)$ is a *lattice-normed space* if $\lvert\cdot\rvert$ is an E-valued norm in the vector space X. The space E is called the *norm lattice* of X. If the norm is decomposable, then the space $(X, \lvert\cdot\rvert)$ is called *decomposable.*

If $\lvert x\rvert \wedge \lvert y\rvert = 0$, then we call the elements $x, y \in X$ *disjoint* and write $x \perp y$. As in the case of a vector lattice, a set of the form

$$M^{\perp} := \{x \in X : \text{ for all } y \in M,\ x \perp y\ \}$$

$\varnothing \neq M \subset X$, is called a *band.* The symbol $\mathscr{B}(X)$ denotes the set of all bands ordered by inclusion. We say that $K \in \mathscr{B}(X)$ is a *projection band* if $K \oplus K^{\perp} = X$. The projection $h(\pi)$ onto the band K along the band $K^{\perp}$ is called a *band projection.* A lattice-normed space X is said to have the *projection property* whenever every band in X is a projection band. For uniformity, we often write $\mathfrak{B}(X)$ instead of $\mathscr{B}(X)$. Given $L \subset E$ and $M \subset X$, we let by definition

$$h(L) := \{x \in X : \lvert x\rvert \in L\} \quad \text{and} \quad \lvert M\rvert := \{\lvert x\rvert : x \in M\}.$$

It is clear that $\lvert h(L)\rvert = L \cap \lvert X\rvert$.

Theorem 2.3.1. [23, 2.1.2.(1)] *Suppose that every band of the vector lattice $E_0 := \lvert X\rvert^{\perp\perp}$ contains the norm of some nonzero element. Then $\mathfrak{B}(X)$ is a complete Boolean algebra and the mapping $L \mapsto h(L)$ is an isomorphism of the Boolean algebras $\mathfrak{B}(\lvert X\rvert^{\perp\perp})$ and $\mathfrak{B}(X)$.*

Theorem 2.3.2. [23, 2.1.2.(4)] *Suppose that every band of the vector lattice $E_0 := \lvert X\rvert^{\perp\perp}$ contains the norm of some nonzero element and X is d-decomposable and there exist a band projection π onto $L \in \mathfrak{B}(E_0)$. Then the projection $h(\pi)$ onto the band $K := h(L)$ along the band $K^{\perp}$ exists and, moreover, $\pi\lvert x\rvert = \lvert h(\pi)x\rvert$ for all $x \in X$.*

Theorem 2.3.3. [23, 2.1.3.] *Suppose that $E_0 := \lvert X\rvert^{\perp\perp}$ is a vector lattice with the projection property and the space X is d-decomposable. Then X have the projection property. Moreover, there exists a complete Boolean algebra $\mathscr{B}$ of band projections in X and an isomorphism h from $\mathfrak{P}(E_0)$ onto $\mathscr{B}$ such that $b\lvert x\rvert = \lvert h(b)x\rvert$ $(b \in \mathfrak{P}(E_0),\ x \in X)$.*

We identify the Boolean algebras $\mathfrak{P}(E_0)$ and $\mathscr{P}(X) := \mathscr{B}$ and write $\pi\lvert x\rvert = \lvert \pi x\rvert$ $(\pi \in \mathfrak{P}(E_0),\ x \in X)$.

We say that a net $(x_\alpha)_{\alpha\in A}$ *bo-converges* to an element $x \in X$ and write $x = bo\text{-}\lim x_\alpha$ if there exists a net $e_\gamma \downarrow 0$ in E such that for every γ, there is an $\alpha(\gamma)$ such that $\lvert x_\alpha - x\rvert \le e_\gamma$ for all $\alpha \ge \alpha(\gamma)$. Given an element $e \in E_+$, let the following condition be satisfied: for every number $\epsilon > 0$, there is an index $\alpha(\epsilon)$ such that $\lvert x_\alpha - x\rvert \le \epsilon e$ for all $\alpha \ge \alpha(\epsilon)$. Then we say that $(x_\alpha)_{\alpha\in A}$ *br-converges* to x and write $x = br\text{-}\lim x_\alpha$. A net $(x_\alpha)_{\alpha\in A}$ is said to be *bo-fundamental* (*br-fundamental*) if the net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ *bo*-converges (*br*-converges) to zero. A lattice-normed space is called *bo-complete* (*br-complete*) if every *bo*-fundamental (*br*-fundamental) net in it *bo*-converges (*br*-converges) to an element of the space.

Take a family $(x_\xi)_{\xi\in\Xi}$ and associate with the net $(y_\alpha)_{\alpha\in A}$, where $A := \wp_{fin}(\Xi)$ is the set of all finite subsets of Ξ and $y_\alpha := \sum_{\xi\in\alpha} x_\xi$. If $x := bo\text{-}\lim y_\alpha$ exists then the family (x_ξ) is said to be *bo-summable* and x is its *sum*. It is conventional to write $x = bo\text{-}\sum_{\xi\in\Xi} x_\xi$ in this case.

A set $M \subset X$ is called *norm-bounded* if there exists an $e \in E_+$ such that $\lvert x\rvert \le e$ for all $x \in M$. A space X is called *disjointly complete* or *d-complete* if every norm-bounded set in X of pairwise disjoint elements is *bo*-summable.

Definition 2.3.4. A *Banach–Kantorovich space* is a decomposable *bo*-complete lattice-normed space.

Theorem 2.3.5. [23, 2.2.1.] *Let (X, E) be a Banach–Kantorovich space and $E = \lvert X\rvert^{\perp\perp}$. For every bounded family $(x_\xi)_{\xi\in\Xi}$ in X and every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(X)$, the sum $x = bo\text{-}\sum_{\xi\in\Xi} \pi_\xi x_\xi$ exists. Moreover, x is a unique element in X satisfying the relations $\pi_\xi x = \pi_\xi x_\xi$ $(\xi \in \Xi)$.*

Theorem 2.3.6. [23, 2.2.3.] *A decomposable lattice-normed space is bo-complete if and only if it is disjointly complete and complete with respect to relative uniform convergence.*

Let $(X, \lvert\cdot\rvert, E)$ be a lattice-normed space with E a norm lattice of X and E be a normed lattice. Then we have a norm in X defined by

$$|||x||| := \|\,\lvert x\rvert\,\| \qquad (x \in X).$$

The normed space $(X, |||\cdot|||)$ is called a *space with mixed norm* and $|||\cdot|||$ is called *mixed norm* in X. From inequality $\big|\,\lvert x\rvert - \lvert y\rvert\,\big| \le \lvert x - y\rvert$ and monotonicity of the norm in E, the vector norm $\lvert\cdot\rvert$ is a norm continuous. A *Banach space with mixed norm* is a pair (X, E) in which E is a Banach lattice and X is a *br*-complete lattice-normed space with E-valued norm.

Theorem 2.3.7. [23, 7.1.2.] *Let E be a Banach lattice. Then $(X, |||\cdot|||)$ is a Banach space if and only if the lattice-normed space (X, E) is br-complete.*

2.4 Normed B-Spaces

Let X be a normed space and B be a Boolean algebra. Suppose that $\mathscr{L}(X)$ has a complete Boolean algebra of norm one projections $\mathscr{B}$ which is isomorphic to B. In this event we will identify the Boolean algebras $\mathscr{B}$ and B, writing $B \subset \mathscr{L}(X)$. Say that X is a *normed B-space* if $B \subset \mathscr{L}(X)$ and for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in B the two conditions hold:

(1) If $b_\xi x = 0$ $(\xi \in \Xi)$ for some $x \in X$ then $x = 0$;

(2) If $b_\xi x = b_\xi x_\xi$ $(\xi \in \Xi)$ for $x \in X$ and a family $(x_\xi)_{\xi\in\Xi}$ in X then $\|x\| \leq \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

Conditions (1) and (2) amount to the respective conditions $(1')$ and $(2')$:

$(1')$ To each $x \in X$ there corresponds the greatest projection $b \in B$ such that $bx = 0$;

$(2')$ If x, (x_ξ), and (b_ξ) are the same as in (2) then $\|x\| = \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

From $(2')$ it follows in particular that

$$\left\| \sum_{k=1}^{n} b_k x \right\| = \max_{k=1,\dots,n} \|b_k x\|$$

for $x \in X$ and pairwise disjoint projections $b_1, \dots, b_n$ in B.

Given a partition of unity (b_ξ), we refer to $x \in X$ satisfying the condition $b_\xi x = b_\xi x_\xi$ $(\xi \in \Xi)$ as a *mixing* of (x_ξ) by (b_ξ). If (1) holds then there is a unique mixing x of (x_ξ) by (b_ξ). In these circumstances we naturally call x the *mixing* of (x_ξ) by (b_ξ). Condition (2) may be paraphrased as follows: the unit ball U_X of X is closed under mixing or is *mix-complete.*

A normed B-space X is *B-cyclic* if we may find in X a mixing of each norm-bounded family by any partition of unity in B. Considering what was said above, we may assert that X is a B-cyclic normed space if and only if, given a partition of unity $(b_\xi) \subset B$ and a family $(x_\xi) \subset U_X$, we may find a unique element $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ.

2.5 AW^*-Algebras

We recall some preliminaries concerning complex algebras. Note also that by an algebra we always mean a unital associative algebra. An *involutive algebra* or $*$-*algebra* A is a complex algebra with *involution*, i.e. a mapping $x \to x^*$ $(x \in A)$ satisfying the conditions:

(1) $x^{**} = x$ $(x \in A)$;

(2) $(x + y)^* = x^* + y^*$ $(x, y \in A)$;

(3) $(\lambda x)^* = \bar{\lambda} x^*$ $(\lambda \in \mathbb{C},\ x \in A)$;

(4) $(xy)^* = y^* x^*$ $(x, y \in A)$.

An element x of an involutive algebra A is called *hermitian* if $x^* = x$. The set of all hermitian elements of A is denoted by ReA. An element x of A is called *normal* if $x^*x = xx^*$. A hermitian element p is a *projection* whenever p is an idempotent, i.e. $p^2 = p$. The symbol $\mathfrak{P}(A)$ stands for the set of all projections of an involutive algebra A. Two projections $p, q \in \mathfrak{P}(A)$ are called *orthogonal* if $pq = 0$. A projection p is *central* if $px = xp$ for all $x \in A$. Denote the set of all central projections by $\mathfrak{P}_c(A)$.

A scalar $\lambda \in \mathbb{C}$ is a *spectral value* of x, if $\lambda - x$ is not invertible in A. The set of all spectral values of x is called the *spectrum* of x and denoted by Sp(x). An element x of a $*$-algebra A is called *positive* if x is hermitian and Sp$(x) \subset \mathbb{R}_+$. If x is positive, this is denoted by $x \geq 0$. The set of all positive elements of A is denoted by A_+.

If $(A, *)$ and $(B, *)$ are involutive algebras and $\mathscr{R} : A \to B$ is a multiplicative linear operator or a homomorphism, then $\mathscr{R}$ is called a $*$-*representation* or a $*$-*homomorphism* of A in B whenever $\mathscr{R}(x^*) = \mathscr{R}(x)^*$ for all $x \in A$. If $\mathscr{R}$ is also an isomorphism then $\mathscr{R}$ is a $*$-*isomorphism* of A and B.

A norm $\|\cdot\|$ on an algebra A is *submultiplicative* if

$$\|xy\| \leq \|x\|\|y\| \qquad (x, y \in A).$$

A *Banach algebra* A is an algebra furnished with a submultiplicative norm making A into a Banach space. A C^*-*algebra* is a Banach algebra which is also an involutive algebra and its involution satisfies the condition

$$\|xx^*\| = \|x\|^2 \qquad (x \in A).$$

The spectrum of an element of a Banach algebra is a nonempty compact subset of $\mathbb{C}$ [6, Theorem VII.3.6.]. Let $C(Q)$ denote the C^*-algebra of continuous complex-valued functions on a topological space Q.

Let A be a commutative Banach algebra and Σ denote the set of all nonzero homomorhism of $A \to \mathbb{C}$. Give Σ the relative weak* topology that it has as a subset of closed unit ball of A' [6, Proposition VII.8.4.]. Σ with this topology is called the *maximal ideal space* of A. By the Banach−Alaoglu theorem, the maximal ideal space Σ is a compact Hausdorff space. If $a \in A$, then *Gelfand transform* of a is the function $\hat{a} : \Sigma \to \mathbb{C}$ defined by $\hat{a}(h) = h(a)$. The homomorphism $a \mapsto \hat{a}$ of A into $C(\Sigma)$ is called the *Gelfand transform* of A.

Theorem 2.5.1. [6, Theorem VIII.2.1.] *If A is a commutative C^*-algebra with identity and Σ is its maximal ideal space, then the Gelfand transform $\gamma : A \to C(\Sigma)$ is an isometric $*$-isomorphism of A onto $C(\Sigma)$.*

Let B be an arbitrary C^*-algebra with identity and let a be a normal element of B. So, if $A = C^*(a)$, the C^*-algebra generated by a and unity $\mathbf{1}$, i.e., $C^*(a)$ is the closure of $\{p(a, a^*) : \; p(z, \bar{z}) \text{ is a polynomial in } z \text{ and } \bar{z}\}$, A is commutative.

Proposition 2.5.2. [6, Proposition VIII.2.3.] *If A is a commutative C^*-algebra with maximal ideal space Σ and $a \in A$ such that $A = C^*(a)$, then the map $\tau : \Sigma \to \mathrm{Sp}(a)$ defined by $\tau(h) = h(a)$ is a homeomorphism. If $p(z, \bar{z})$ is a polynomial in z and $\bar{z}$ and $\gamma : A \to C(\Sigma)$ is the Gelfand transform, then $\gamma(p(a, a^*)) = p\tau$.*

If $\tau : \Sigma \to \mathrm{Sp}(a)$ is defined as in the preceding proposition, $\tau^\sharp : C(\mathrm{Sp}(a)) \to C(\Sigma)$ is defined by $\tau^\sharp(f) = f\tau$. Note that $\tau^\sharp$ is a $*$-isomorphism and an isometry, because τ is a homeomorphism.

Theorem 2.5.3. (Spectral Theorem). *Let x be a normal element of a C^*-algebra A. There is a unique isometric $*$-representation $\mathscr{R}_x : C(\mathrm{Sp}(x)) \to A$ such that $x = \mathscr{R}_x(i)$, where i is the identity mapping on $\mathrm{Sp}(x)$.*

The representation $\mathscr{R}_x : C(\mathrm{Sp}(x)) \to A$ is called the *continuous functional calculus* (for a normal element x of A). The element $\mathscr{R}_x(f)$ with $f \in C(\mathrm{Sp}(x))$ is usually denoted by $f(x)$. Note that if $p(z, \bar{z})$ is a polynomial in z and $\bar{z}$, then $\mathscr{R}_x(p(z, \bar{z})) = p(x, x^*)$. In particular, $\mathscr{R}_x(z^n \bar{z}^m) = x^n(x^*)^m$ so that $\mathscr{R}_x(z) = x$ and $\mathscr{R}_x(\bar{z}) = x^*$. Also, $\mathscr{R}_x(\mathbf{1}) = \mathbf{1}$.

Theorem 2.5.4. [6, Theorem VIII.2.7.](Spectral Mapping Theorem). *If A is a C^*-algebra and x is a normal element of A, then for every f in $C(\mathrm{Sp}(x))$,*

$$\mathrm{Sp}(f(x)) = f(\mathrm{Sp}(x)).$$

Theorem 2.5.5. [6, Theorem VIII.3.6.] *If A is a C^*-algebra and $x \in A$, then the following statements are equivalent.*

(1) $x \geq 0$;

(2) $x = b^2$ *for some* $b \in \mathrm{Re}A$;

(3) $x = a^*a$ *for some* $a \in A$;

(4) $x = x^*$ *and* $\|t - x\| \leq t$ *for all* $t \geq \|x\|$;

(5) $x = x^*$ *and* $\|t - x\| \leq t$ *for some* $t \geq \|x\|$.

Form Spectral Theorem and the theorem above, we have for every positive $x \in A$ the square root $\sqrt{x}$ is defined, since $\mathrm{Sp}(x) \subset \mathbb{R}_+$, and for each normal $x \in A$ the modulus can be defined as $|x| = \sqrt{x^*x}$. Note that if $x, y \in A_+$ and $x \leq y$, then $x^\beta \leq y^\beta$ holds for $0 \leq \beta \leq 1$.

Consider an involutive algebra A. Given a nonempty set $M \subset A$,

$$M^\perp := \{y \in A : (\forall x \in M) xy = 0\}$$

and call $M^\perp$ the *right annihilator* of M. Similarly,

$${}^\perp M := \{y \in A : (\forall x \in M) yx = 0\}$$

denotes the *left annihilator* of M. A *Baer $*$-algebra* is involutive algebra A such that for each nonempty $M \subset A$, there is some p in $\mathfrak{P}(A)$ satisfying $M^\perp = pA$. An *AW^*-algebra* is a C^*-algebra that is a Baer $*$-algebra.

Theorem 2.5.6. [4, Theorem 7.1.] *Let A be a commutative C^*-algebra with unity and write $A = C(T)$, T compact. In order that A be an AW^*-algebra, it is necessary and sufficient that T be an extremally disconnected.*

Note that if A is a commutative C^*-algebra with unity, then A is an AW^*-algebra if and only if its maximal ideal space is extremally disconnected. For more information, we refer to [4, 23].

Let Λ be a commutative AW^*-algebra. Then $\mathrm{Re}\Lambda$ is a Dedekind complete vector lattice with strong order-unity $\mathbf{1}$ and $\mathfrak{P}(\Lambda)$ is a complete Boolean algebra. Denote $[\lambda] = \inf\{\pi \in \mathfrak{P}(\Lambda) : \pi\lambda = \lambda\}$, the *support* of λ in Λ. In case $\Lambda = C(Q)$, $[\lambda]$ is the characteristic function of the clopen set $\mathrm{cl}\left(\{q \in Q : \lambda(q) \neq 0\}\right)$. Note that $[\lambda] = \sup\{|f\lambda| \in \mathfrak{P}(\Lambda) : f \in \Lambda\}$ and $[\lambda]$ is the projection of $\mathbf{1}$ onto the band generated by $|\lambda|$. By [4, §3, Propostion 3] and the preceding observations we can deduce the following proposition.

Proposition 2.5.7. *Let Λ be a commutative AW^*-algebra and $\lambda, \delta \in \Lambda$. Then the following properties are holds:*

(1) $[\lambda]\,\lambda = \lambda$;

(2) $\lambda\delta = 0$ *iff* $[\lambda]\,\delta = 0$;

(3) $0 \leq \lambda \leq \delta$ *implies* $[\lambda] \leq [\delta]$;

(4) $\lambda \geq 0$ *and* $\lambda\delta \geq 0$ *implies* $[\lambda]\,\delta \geq 0$.

Chapter 3

Kaplansky−Hilbert Modules

In this chapter, we introduce and study the notion of a Kaplansky−Hilbert module and operators on them. A Kaplansky−Hilbert module is like a Hilbert space except that the field of complex numbers is replaced by an arbitrary commutative C^*-algebra which is order complete vector lattice. Kaplansky−Hilbert modules appeared in the paper [15] of I. Kaplansky. Moreover, we deal with cyclically compact sets and operators in Kaplansky−Hilbert modules which are introduced by A.G. Kusraev and recently, they are studied in [8, 16].

3.1 Kaplansky−Hilbert Modules (AW^*-Modules)

In this section, we will study some facts of Kaplansky−Hilbert modules which can be found [15, 23]. Recall that a commutative C^*-algebra A with unity is called *Stone algebra* if it is a Dedekind complete vector lattice with respect to cone A_+. Other name used for Stone algebras in the literature are commutative AW^*-algebras that were proposed by I. Kaplansky [4, 13, 14, 15, 23]. If Σ is the maximal ideal space of Stone algebra A, then Σ is an extremal compact space with weak* topology [4, Theorem 7.1.] and A is isometric $*$-isomorphic to $C(\Sigma)$ (Theorem 2.5.1), and so $C(\Sigma, \mathbb{R})$, the algebra of continuous real-valued functions on Σ, is a Dedekind complete vector lattice (Theorem 2.2.3). Now suppose that E is an order complete complex AM-space with strong order-unity $\mathbf{1}$. According to the M. Kreĭn−S. Kreĭn−Kakutani Theorem and Theorem 2.2.3 E is linearly isometric and order isomorphic to the space of continuous functions $C(Q)$ on some extremal compact space Q. Therefore, E can be endowed with some multiplication and involution so that E becomes a commutative C^*-algebra with unity $\mathbf{1}$, i. e. a Stone algebra. Observe that an element $e \in E$ is a projection if and only if it is a component of $\mathbf{1}$. Moreover, the isomorphism $E \to C(Q)$ defines a bijection between the set of components of $\mathbf{1}$ and the set of characteristic functions of clopen sets in Q, so that the Boolean algebras $\mathfrak{C}(E) := \mathfrak{C}(\mathbf{1})$ which is the set of all component of $\mathbf{1}$ coincides with the set of all projections $\mathfrak{P}(E)$ and is isomorphic to Clop(Q). Moreover, by [46, Theorem IV.3.7, Corollary IV.8.1., Proposition V.(a)] we have $\mathfrak{C}(E) = \{\pi(\mathbf{1}) : \pi$ band projection on $E\}$ and $\pi(x) = x\pi(\mathbf{1})$ holds for all band projection π and x in E. Given a complete Boolean algebra B there exists a unique (up to $*$-isomorphism) Stone algebra Λ such that B and $\mathfrak{P}(\Lambda)$ are isomorphic. Each of these algebras will be denoted by $\mathscr{S}(B)$. Note that every Stone algebra is also a f-algebra with unity, and $a \perp b$ means $ab = 0$ for $a, b \in \Lambda$.

Let Λ be a Stone algebra and X be a Λ-module. Suppose X equipped with a Λ-*valued inner product*, $\langle \cdot \mid \cdot \rangle : X \times X \to \Lambda$ satisfying the following conditions:

(1) $\langle x \mid x \rangle \geq 0$; $\langle x \mid x \rangle = 0 \Leftrightarrow x = 0$;

(2) $\langle x \mid y \rangle = \langle y \mid x \rangle^*$;

(3) $\langle ax \mid y \rangle = a \langle x \mid y \rangle$;

(4) $\langle x + y \mid z \rangle = \langle x \mid z \rangle + \langle y \mid z \rangle$,

for all x, y, z in X and a in Λ. As in Hilbert spaces, we can introduce the norms in X by the formulas

$$\lvert x \rvert := \sqrt{\langle x \mid x \rangle}, \qquad |||x||| := \| \langle x \mid x \rangle \|^{\frac{1}{2}},$$

for all x in X. Employing the continuous functional calculus [23, Theorem 7.4.2.] we may deduce from the properties (2) and (3) that $\lvert \lambda x \rvert = |\lambda| \lvert x \rvert$ for all λ in Λ and x in X. Since $\lvert x \rvert$ is regarded as a function on some extremal compact space Q, it follows that the Cauchy−Bunyakovskiĭ−Schwarz inequality

$$| \langle x \mid y \rangle | \leq \lvert x \rvert \lvert y \rvert$$

holds. Thus, $\lvert \cdot \rvert$ satisfies the triangle inequality, and $\lvert \cdot \rvert$ is a Λ-valued norm in X. On the other hand, on taking norms in the above inequality, we further get the numerical version of the Cauchy−Bunyakovskiĭ−Schwarz inequality

$$\| \langle x \mid y \rangle \| \leq |||x||| \, |||y||| \, .$$

So, we have that X is a normed space. Moreover, by definitions of $\lvert\cdot\rvert$ and $|||\cdot|||$,

$$|||x||| = \| \lvert x\rvert \|$$

holds for all x in X, since $\|a\| = \|(\sqrt{a})^2\| = \|\sqrt{a}\|^2$ for every positive a in Λ. Therefore, the normed space $(X, |||\cdot|||)$ is a space with mixed norm. On the other hand, it has the following properties:

(1) let x be an arbitrary element in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$ with $e_\xi x = 0$ for all $\xi \in \Xi$, then $x = 0$;

(2) if $e_\xi x = e_\xi x_\xi$ $(\xi \in \Xi)$ for $x \in X$, for a family $(x_\xi)_{\xi\in\Xi}$ in X and a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$, then $|||x||| \le \sup\{|||e_\xi x_\xi||| : \ \xi \in \Xi\}$

since Λ has same properties [13, Lemmas 2.2 and 2.5]. Clearly, it follows from (1) that the element x of (2) is unique, we shall write $x = \operatorname{mix}_{\xi\in\Xi}(e_\xi x_\xi)$. We call X a *C^*-module* over Λ if it is complete with respect to the mixed norm $|||\cdot|||$. We call X a *Kaplansky–Hilbert module* or an *AW^*-module* over Λ if it is a C^*-module over Λ and has the following additional property:

(3) let $(x_\xi)_{\xi\in\Xi}$ be a norm-bounded family in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$; then there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$.

Note that $x = \operatorname{mix}_{\xi\in\Xi}(e_\xi x_\xi)$ means $x = bo\text{-}\sum_{\xi\in\Xi} e_\xi x_\xi$ for all norm-bounded family $(x_\xi)_{\xi\in\Xi} \subset X$ and partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$. On the other hand, for each projection e in $\mathfrak{P}(\Lambda)$ it can be defined a band projection on Λ such that $a \mapsto ea$. Thus, $\mathscr{L}(X)$ has a complete Boolean algebra of norm-one projection B which is isomorphic to $\mathfrak{P}(\Lambda)$, i. e. X is a normed B-space.

Theorem 3.1.1. [23, Theorem 7.4.4.] *Let X be a C^*-module over Λ. Then the following statements are equivalent:*

(i) *X is a Kaplansky–Hilbert module over Λ;*

(ii) *$(X, |||\cdot|||)$ is a B-cyclic Banach space where B is isomorphic to $\mathfrak{P}(\Lambda)$;*

(iii) *$(X, \lvert\cdot\rvert)$ is a Banach–Kantorovich space over Λ.*

Note that the inner product is bo-continuous in each variable. In particular,

$$\left\langle bo\text{-}\sum_{\xi\in\Xi} e_\xi x_\xi \,\middle|\, y \right\rangle = o\text{-}\sum_{\xi\in\Xi} \langle e_\xi x_\xi \mid y\rangle .$$

for every norm-bounded family $(x_\xi)_{\xi\in\Xi}$ in X, and partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$ [15, Lemma 2].

Let X be a Kaplansky$-$Hilbert module over Λ. By an *Kaplansky$-$Hilbert submodule* or an *AW^*-submodule* X_0 we mean X_0 is a submodule in the algebric sense, closed in norm topology and containing all sums of the form *bo*-$\sum_{\xi\in\Xi} e_\xi x_\xi$, where $(x_\xi)_{\xi\in\Xi}$ is a bounded family in X_0 and $(e_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathfrak{P}(\Lambda)$ [15].

We remark that a Kaplansky$-$Hilbert submodule is itself a Kaplansky$-$Hilbert module over Λ, and the intersection of any number of Kaplansky$-$Hilbert submodules is again a Kaplansky$-$Hilbert submodule, and consequently for any subset M there exists the smallest Kaplansky$-$Hilbert submodule containing M; it is called the Kaplansky$-$Hilbert *submodule generated by* M. Moreover, a submodule $X_0 \subset X$ is a Kaplansky$-$Hilbert submodule if and only if it is *bo*-closed [23]. The *orthogonal complement* of M in X is defined as

$$M^\perp := \{x \in X \ : \ (\forall y \in M) \ \langle x \mid y \rangle = 0\}.$$

Then the set $M^\perp$ for any subset M of X is a Kaplansky$-$Hilbert submodule of X [15, Lemma 6], and if X_0 is a Kaplansky$-$Hilbert submodule of X, then $X = X_0 \oplus X_0^\perp$ [15, Theorem 3]. Hence, Kaplansky$-$Hilbert submodule generated by a subset M of X is $M^{\perp\perp}$. A Kaplansky$-$Hilbert module over Λ is called *faithful* if for every $a \in \Lambda$ the condition $(\forall x \in X)\ ax = 0$ implies that $a = 0$. In the sequel we restrict our attention to faithful Kaplansky$-$Hilbert modules over Λ.

Clearly the following identity can be verified in Kaplansky$-$Hilbert modules,

$$4\langle u \mid y\rangle = \sum_{k=0}^{3} i^k \left\langle u + i^k v \mid x + i^k y\right\rangle \tag{3.1}$$

for each $x, y, u, v \in X$, and so the Polarization identity holds:

$$4\langle x \mid y\rangle = \lvert x+y \rvert^2 - \lvert x-y \rvert^2 + i\lvert x+iy \rvert^2 - i\lvert x-iy \rvert^2. \tag{3.2}$$

Let U be a subset of X and $\mathrm{mix}\,(U)$ denote a set of all $x \in X$ such that there exist $(x_\xi)_{\xi\in\Xi}$ in U and arbitrary partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$ with $e_\xi x = e_\xi x_\xi$ $(\xi \in \Xi)$. The set $\mathrm{mix}\,(U)$ is called the *mix-closure* of U. We say that U is *mix-closed* if $U = \mathrm{mix}\,(U)$. Moreover, $\mathrm{mix}_{\xi\in\Xi}\,(e_\xi x_\xi) \in \mathrm{mix}\,(U)$ means that the sum *bo*-$\sum_{\xi\in\Xi} e_\xi x_\xi$ exists in X. In particular, $\mathrm{mix}_{\xi\in\Xi}\,(e_\xi x_\xi) =$ *bo*-$\sum_{\xi\in\Xi} e_\xi x_\xi$. We say that U is *mix-complete* if, for all partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$ and

$(x_\xi)_{\xi\in\Xi} \subset U$, there is $x \in U$ such that $x = \operatorname{mix}_{\xi\in\Xi}(e_\xi x_\xi)$. From (2) and (3), the closed unit ball of X is mix-complete. Clearly, every mix-complete set is mix-closed. Mix-complete means mix-closed whenever the set is bounded.

Lemma 3.1.2. [47, Lemma 2.3.] *Let M be a submodule of X. Then* $\operatorname{cl}(\operatorname{mix}(M))$ *is the Kaplansky–Hilbert submodule generated by M.*

Lemma 3.1.3. *Let x be a nonzero element of a Kaplansky–Hilbert module X over Λ. Then the following statements hold:*

(i) *there are a nonzero $\mu \in \mathfrak{P}(\Lambda)$ and a positive element $a \in \Lambda$ with $a\,|x| = \mu$ and $\mu = [a] \le [\,|x|\,]$;*

(ii) *if $|x| \ge \pi$ is satisfied for some nonzero projection π, then there exists $a \in \Lambda_+$ such that $\pi a = a$ and $a\,|x| = \pi$.*

Proof. (i) According to [15, Lemma 4.] there exists a nonzero $\mu \in \mathfrak{P}(\Lambda)$ and an element $b \in \Lambda$ with $b\,|x| = \mu$. Define $a := \mu b$, and note that $a\,|x| = \mu$ and $a = \mu a$. So, we have $[a] \le \mu$. From $(\mathbf{1} - [a])a = (\mathbf{1} - [\,|x|\,])\,|x| = 0$ we see that $(\mathbf{1} - [a])\mu = (\mathbf{1} - [\,|x|\,])\mu = 0$. Thus, $\mu \le [a]$ and $\mu \le [\,|x|\,]$ hold, and so $\mu = [a] \le [\,|x|\,]$. Moreover, it follows from $a\,|x| \ge 0$ that $a\,[\,|x|\,] \ge 0$. Hence, $a = a\mu = \mu a\,[\,|x|\,] \ge 0$, and the proof of (i) is finished.

(ii) Let $|x| \ge \pi$. By (i) for each $0 < \mu \le \pi$ there exists $g \in \Lambda_+$ such that $\mu \ge \mu' := g\,|x| \in \mathfrak{P}(\Lambda) \setminus \{0\}$, and so $\|\mu' g\| \le 1$ since $g\,|x| \ge g\mu'$. Consider the set

$$S := \left\{(\mu, g) : g \in \Lambda_+ \text{ and } \pi \ge \mu = g\,|x| \in \mathfrak{P}(\Lambda)\right\}.$$

Thus we have $\pi = \sup\{\mu : (\mu, g) \in S\}$. Using Exhaustion Principle [23, 1.1.6.(1)] we get an antichain $(\mu_\alpha)_{\alpha\in A}$ in $\mathfrak{P}(\Lambda)$ and a bounded family $(g_\alpha)_{\alpha\in A}$ in Λ such that $\pi = \sup_{\alpha\in A} \mu_\alpha$ where $(\mu_\alpha, g_\alpha) \in S$ with $\mu_\alpha g_\alpha = g_\alpha$. Define $a = o\text{-}\sum_{\alpha\in A} \mu_\alpha g_\alpha$, and note that $\pi a = a$ and $a\,|x| = \pi$. The proof of the lemma is now complete. □

Definition 3.1.4. Let X be a Kaplansky–Hilbert module over Λ. A subset $\mathscr{E}$ of X is said to be *orthonormal* (*projection orthonormal*) if

(1) $\langle x \mid y\rangle = 0$ for all distinct $x, y \in \mathscr{E}$;

(2) $\langle x \mid x\rangle = \mathbf{1}$ ($\langle x \mid x\rangle \in \mathfrak{P}(\Lambda) \setminus \{0\}$) for every $x \in \mathscr{E}$.

An orthonormal (projection orthonormal) set $\mathscr{E} \subset X$ is a *basis* (*projection basis*) for X provided that

(3) the condition $(\forall e \in \mathscr{E})\ \langle x \mid e \rangle = 0$ imply $x = 0$.

A Kaplansky–Hilbert module is said to be λ-*homogeneous* if λ is a cardinal and X has a basis of cardinality λ. A Kaplansky–Hilbert module is called *homogeneous* if it is λ-homogeneous for some cardinal λ. For $0 \neq b \in \mathfrak{P}(\Lambda)$, denote by $\varkappa(b)$ the least cardinal γ such that a Kaplansky–Hilbert module bX over $b\Lambda$ is γ-homogeneous. If X is homogeneous then $\varkappa(b)$ is defined for all $0 \neq b \in \mathfrak{P}(\Lambda)$. It is convenient to assume that $\varkappa(0) = 0$. We shall say that a Kaplansky–Hilbert module X is *strictly* γ-*homogeneous* if X is homogeneous and $\gamma = \varkappa(b)$ for all nonzero $b \in \mathfrak{P}(\Lambda)$. A Kaplansky–Hilbert module is said to be *strictly homogeneous* if it is strictly λ-homogeneous for some cardinal λ.

Not every Kaplansky–Hilbert module has a basis, but we can split it into strictly homogeneous parts. Thus, every Kaplansky–Hilbert module has a projection basis.

Theorem 3.1.5. [23, 7.4.7.(2)] *Let X be a Kaplansky–Hilbert module over Λ. Then there exists a partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(\Lambda)$ such that $b_\xi X$ is a strictly $\varkappa(b_\xi)$-homogeneous Kaplansky–Hilbert module over $b_\xi \Lambda$.*

Suppose that Q is an extremal compact space. Let $C_\infty(Q, E)$ be the set of cosets of continuous vector-functions u that act from comeager subsets $\mathrm{dom}(u) \subset Q$ into some normed space E. Recall that a set is called *comeager* if its complement is meager. Vector-functions u and v are equivalent if $u(t) = v(t)$ whenever $t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. The set $C_\infty(Q, E)$ is endowed, in a natural way, with the structure of a module over $C_\infty(Q)$. Moreover, the continuous extension of the pointwise norm defines a decomposable vector norm on $C_\infty(Q, E)$ with values in $C_\infty(Q)$. Indeed, given any $z \in C_\infty(Q, E)$, there exists a unique function $x_z \in C_\infty(Q)$ such that $\|u(t)\| = x_z(t)$ $(t \in \mathrm{dom}(u))$ for every representative u of the coset z. Assign $\lvert z \rvert = x_z$ and

$$C_\#(Q, E) := \left\{ z \in C_\infty(Q, E) : \lvert z \rvert \in C(Q) \right\}.$$

If E is a Banach space, then $C_\#(Q, E)$ is a Banach–Kantorovich space [23, 2.3.3.]. Let H be a Hilbert space and $\langle \cdot, \cdot \rangle$ be the inner product of H. Then we can introduce some $C(Q)$-valued inner product in $C_\#(Q, H)$ as follows: Take continuous vector-functions $u : \mathrm{dom}(u) \to H$ and $v : \mathrm{dom}(v) \to H$. The function $q \mapsto \langle u(q), v(q) \rangle$ $(q \in \mathrm{dom}(u) \cap \mathrm{dom}(v))$ is continuous and admits a unique continuation $z \in C(Q)$ to the whole of Q. If x and y are the cosets containing vector-functions u and v then assign $\langle x \mid y \rangle := z$. Clearly, $\langle \cdot \mid \cdot \rangle$ is a $C(Q)$-valued inner product and $\lvert x \rvert = \left\lvert \sqrt{\langle x \mid x \rangle} \right\rvert$ $(x \in C_\#(Q, H))$.

Theorem 3.1.6. [23, 7.4.8.(1)] *Suppose that Q is an extremal compact space, and H is a Hilbert space of dimension λ. The space $C_{\#}(Q, H)$ is a λ-homogeneous Kaplansky–Hilbert module over the algebra $\Lambda := C(Q)$.*

Let $\mathscr{E}$ be a nonempty set and denote by $\ell_2(\mathscr{E}, \Lambda)$ the set of all families $(a_e)_{e\in\mathscr{E}}$ of elements of Stone algebra Λ such that $o\text{-}\sum_{e\in\mathscr{E}} |a_e|^2$ is o-summable in Λ. Define a Λ-valued inner product in $\ell_2(\mathscr{E}, \Lambda)$ as

$$\langle u \mid v\rangle := o\text{-}\sum_{e\in\mathscr{E}} u_e v_e^* \qquad (u, v \in \ell_2(\mathscr{E}, \Lambda))\,.$$

Since $|u_e v_e^*| \leq |u_e|\,|v_e|$, the following inequality [15, Lemma 8.]

$$\left| o\text{-}\sum_{e\in\mathscr{E}} u_e v_e \right|^2 \leq o\text{-}\sum_{e\in\mathscr{E}} |u_e|^2\, o\text{-}\sum_{e\in\mathscr{E}} |v_e|^2$$

implies $\langle u \mid v\rangle$ is well-defined.

Theorem 3.1.7. [23, 7.4.8.(2)] *For any nonempty set $\mathscr{E}$ with $\lambda := |\mathscr{E}|$ the space $\ell_2(\mathscr{E}, \Lambda)$ is a λ-homogeneous Kaplansky–Hilbert module over Λ.*

Corollary 3.1.8. [23, 7.4.8.(3)] *For a Stone algebra Λ and a cardinal number λ there exists a λ-homogeneous Kaplansky–Hilbert modules over Λ.*

Lemma 3.1.9. *Let X be a Kaplansky–Hilbert module over Λ and $\lvert x\rvert \in \mathfrak{P}(\Lambda)$. Then $x = \lvert x\rvert x$ holds.*

Proof. Given $\lvert x\rvert \in \mathfrak{P}(\Lambda)$, we deduce

$$\begin{aligned}\big|x - |x|x\big| &= \big|(\mathbf{1} - |x|)x\big| = (\mathbf{1} - |x|)\,|x| \\ &= |x| - |x|\,|x| = |x| - |x| = 0\end{aligned}$$

where $\mathbf{1}$ is unity of $\mathfrak{P}(\Lambda)$. Thus, $x = \lvert x\rvert x$ holds. ☐

Lemma 3.1.10. (Bessel's inequality). *Let x be a element of Kaplansky–Hilbert module X over Λ and $\{\xi_\alpha \mid \alpha \in A\}$ be a projection orthonormal subset in X. Then $(|\langle x \mid \xi_\alpha\rangle|^2)_{\alpha\in A}$ is o-summable and*

$$o\text{-}\sum_{\alpha\in A} |\langle x \mid \xi_\alpha\rangle|^2 \leq |x|^2.$$

Proof. Let $F \subset A$ be finite set. By using Lemma 3.1.9,

$$\begin{aligned}0 \leq \left| x - \sum_{k\in F} \langle x \mid \xi_k\rangle\, \xi_k \right|^2 &= |x|^2 - 2\sum_{k\in F} |\langle x \mid \xi_k\rangle|^2 + \sum_{k\in F} |\langle x \mid \xi_k\rangle|^2 |\xi_k|^2 \\ &= |x|^2 - 2\sum_{k\in F} |\langle x \mid \xi_k\rangle|^2 + \sum_{k\in F} \big|\big\langle x \mid |\xi_k|\xi_k\big\rangle\big|^2 \\ &= |x|^2 - \sum_{k\in F} |\langle x \mid \xi_k\rangle|^2.\end{aligned}$$

Therefore, we have for all $F \subset A$ finite set

$$\sum_{k\in F} | \langle x \mid \xi_k \rangle |^2 \leq \boldsymbol{|} x \boldsymbol{|}^2.$$

As Λ is Dedekind complete, $(| \langle x \mid \xi_\alpha \rangle |^2)_{\alpha\in A}$ is o-summable and

$$o\text{-}\sum_{\alpha\in A} | \langle x \mid \xi_\alpha \rangle |^2 = \sup \left\{ \sum_{k\in F} | \langle x \mid \xi_k \rangle |^2 : F \subset A \text{ is finite} \right\} \leq \boldsymbol{|} x \boldsymbol{|}^2. \qquad \square$$

In case of Hilbert C^*-modules a variant of Bessel's inequality is proved in [7]. The following Lemmas 3.1.11 and 3.1.12 were proved for orthonormal sets in [15, 7.4.9.(1),(2)].

Lemma 3.1.11. *Let X_0 be the Kaplansky–Hilbert submodule generated by $\mathscr{E}$ which is a projection orthonormal subset in X. If $\{a_e \mid e \in \mathscr{E}\}$ is a family in Λ such that $\{|a_e|^2 \boldsymbol{|} e \boldsymbol{|}\}_{e\in\mathscr{E}}$ is o-summable then there exists an element $x_0 \in X_0$ with*

$$x_0 = bo\text{-}\sum_{e\in\mathscr{E}} a_e e, \quad \boldsymbol{|} x_0 \boldsymbol{|}^2 = o\text{-}\sum_{e\in\mathscr{E}} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}, \quad \langle x_0 \mid e \rangle = a_e \boldsymbol{|} e \boldsymbol{|} \quad (e \in \mathscr{E}).$$

Proof. Let Θ be the set of all finite subsets of $\mathscr{E}$. Given $\theta \in \Theta$, put

$$s_\theta := \sum_{e\in\theta} a_e e, \quad \sigma_\theta := \sum_{e\in\theta} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}, \quad \sigma := o\text{-}\sum_{e\in\mathscr{E}} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}, \quad \delta_\theta := \sigma - \sigma_\theta$$

Take $\theta, \theta_1, \theta_2 \in \Theta$ with $\theta \subset \theta_1 \cap \theta_2$ and denote by θ' and $\theta_1 \triangle \theta_2$ the complement of θ and the symmetric difference of θ_1 and θ_2, respectively. Since the set $\mathscr{E}$ is projection orthonormal, we may write

$$\begin{aligned}\boldsymbol{|} s_{\theta_1} - s_{\theta_2} \boldsymbol{|}^2 &= \left\boldsymbol{|} \sum_{e\in\theta_1\triangle\theta_2} a_e e \right\boldsymbol{|}^2 = \sum_{e\in\theta_1\triangle\theta_2} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}^2 \\ &= \sum_{e\in\theta_1\triangle\theta_2} |a_e|^2 \boldsymbol{|} e \boldsymbol{|} \leq o\text{-}\sum_{e\in\theta'} |a_e|^2 \boldsymbol{|} e \boldsymbol{|} = \sigma - \sigma_\theta = \delta_\theta.\end{aligned}$$

By hypothesis $(\delta_\theta)_{\theta\in\Theta}$ decreases to zero, so that $(s_\theta)_{\theta\in\Theta}$ is bo-fundamental. By X is bo-complete, the bo-limit of $(s_\theta)_{\theta\in\Theta}$ exists in X. Denote

$$x_0 := bo\text{-}\lim_{\theta\in\Theta} s_\theta := bo\text{-}\sum_{e\in\mathscr{E}} a_e e.$$

Now we deduce

$$\langle x_0 \mid e \rangle = o\text{-}\sum_{\zeta\in\mathscr{E}} a_\zeta \langle \zeta \mid e \rangle = a_e \boldsymbol{|} e \boldsymbol{|}^2 = a_e \boldsymbol{|} e \boldsymbol{|}.$$

Moreover, we have

$$\boldsymbol{|} x_0 \boldsymbol{|}^2 = o\text{-}\sum_{e\in\mathscr{E}} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}$$

since $\boldsymbol{|} s_\theta \boldsymbol{|}^2 = \sum_{e\in\theta} |a_e|^2 \boldsymbol{|} e \boldsymbol{|}$. $\square$

Lemma 3.1.12. *Let X_0 be Kaplansky–Hilbert submodule of X with a projection basis $\mathscr{E}$. Then $\mathscr{E}$ and X_0 have the same orthogonal complements in X.*

Proof. The Kaplansky–Hilbert submodule generated by $\mathscr{E}$ is denoted by $Y_0 \subset X_0$. Since $\mathscr{E} \subset X_0$, we have $X_0^{\perp} \subset \mathscr{E}^{\perp}$. Let $x \in \mathscr{E}^{\perp}$ and $y \in X_0$. By Bessel's inequality, $(|\langle y \mid e \rangle|^2)$ is o-summable. By the preceding lemma, there exists $x_0 \in Y_0$ such that

$$x_0 = bo\text{-}\sum_{e \in \mathscr{E}} \langle y \mid e \rangle\, e, \quad \langle x_0 \mid e \rangle = \langle y \mid e \rangle \left| e \right| \quad (e \in \mathscr{E}).$$

Since $\langle y \mid e \rangle \left| e \right| = \langle y \mid e \rangle$, we obtain $\langle x_0 - y \mid e \rangle = 0$ for all $e \in \mathscr{E}$, i. e., $x_0 = y$ (or $X_0 = Y_0$). Moreover, from

$$\langle x \mid y \rangle = \left\langle x \,\middle|\, bo\text{-}\sum_{e \in \mathscr{E}} \langle y \mid e \rangle\, e \right\rangle = o\text{-}\sum_{e \in \mathscr{E}} \langle e \mid y \rangle \langle x \mid e \rangle = 0$$

it follows that $X_0^{\perp}$ and $\mathscr{E}^{\perp}$ are same Kaplansky–Hilbert submodule. □

As an immediate corollary, if X_0 is a Kaplansky–Hilbert submodule of X with a projection basis $\mathscr{E}$ then $X_0 = \mathscr{E}^{\perp\perp}$, i. e., X_0 is Kaplansky–Hilbert submodule generated by $\mathscr{E}$ and

$$x = bo\text{-}\sum_{e \in \mathscr{E}} \langle x \mid e \rangle\, e, \qquad \left| x \right|^2 = o\text{-}\sum_{e \in \mathscr{E}} |\langle x \mid e \rangle|^2$$

hold for all $x \in X_0$ from Lemmas 3.1.11 and 3.1.12. All projection orthonormal subset is a projection basis for Kaplansky–Hilbert submodule generated by it.

Definition 3.1.13. Let $\mathscr{E}$ be a basis for Kaplansky–Hilbert module X over Λ and $x \in X$. We say that the family $\widehat{x} := (\widehat{x}_e)_{e \in \mathscr{E}}$ in $\Lambda^{\mathscr{E}}$, given by the identity $\widehat{x}_e := \langle x \mid e \rangle$, is the *Fourier coefficient family* of x with respect to $\mathscr{E}$ or the *Fourier transform* of x (relative to $\mathscr{E}$).

Observe that by Bessel's inequality, the Fourier coefficient family of x is square o-summable; moreover, the following identities hold

$$x = bo\text{-}\sum_{e \in \mathscr{E}} \widehat{x}_e e, \qquad \left| x \right|^2 = o\text{-}\sum_{e \in \mathscr{E}} |\widehat{x}_e|^2$$

from Lemma 3.1.11.

Proposition 3.1.14. (Riesz–Fisher Isomorphism Theorem) [23, 7.4.10.(4)] *Let X be a homogeneous Kaplansky–Hilbert module over Λ with a basis $\mathscr{E}$. The*

Fourier transform $\mathscr{F} : x \mapsto \widehat{x}$ *(relative to* $\mathscr{E}$*) is an isometric isomorphism of* X *onto* $\ell_2(\mathscr{E}, \Lambda)$*. The inverse Fourier transform, the Fourier summation* $\mathscr{F}^{-1}$: $\ell_2(\mathscr{E}, \Lambda) \to X$*, acts by the rule* $\mathscr{F}^{-1}(\widehat{x}) = bo\text{-}\sum_{e\in\mathscr{E}} x_e e$ *for* $\widehat{x} = (\widehat{x}_e)_{e\in\mathscr{E}} \in \ell_2(\mathscr{E}, \Lambda)$*. Moreover, the Fourier transform preserves inner product or, in other words, for all* $x, y \in X$ *the Parseval identity holds:*

$$\langle x \mid y\rangle = o\text{-}\sum_{e\in\mathscr{E}} \widehat{x}_e \widehat{y}_e^*.$$

Corollary 3.1.15. *Any two* λ*-homogeneous Kaplansky–Hilbert modules over a Stone algebra are isomorphic.*

Proposition 3.1.16. *Let* $\mathscr{E}$ *be a finite subset of* X *with* $\left|e\right| = \left|f\right| \in \mathfrak{P}(\Lambda)$ $(e, f \in \mathscr{E})$*. Suppose that* $\sum_{e\in\mathscr{E}} f_e e = 0$ *implies* $\left|f_e e\right| = 0$ *where* $f_e \in \Lambda$*. If* $\mathscr{E}$ *is a subset of* $\mathscr{F}^{\perp\perp}$*, where* $\mathscr{F}$ *is a projection orthonormal finite subset of* X*, then* $\operatorname{card}(\mathscr{F}) \geq \operatorname{card}(\mathscr{E})$.

Proof. Let $\mathscr{F} = \{y_1, y_2, ..., y_k\}$. The proof is by induction on $n = \operatorname{card}(\mathscr{E})$. For $n = 1$. The validity of the statement is obvious. Assume that the result to be true for some $n \in \mathbb{N}$. Let $\mathscr{E} = \{x_1, x_2, ..., x_n, x_{n+1}\} \subset X$ such that $\left|x_i\right| = \left|x_j\right| \in \mathfrak{P}(\Lambda)$ $(1 \leq i, j \leq n+1)$ and $f_1x_1 + f_2x_2 + \cdots + f_{n+1}x_{n+1} = 0$ implies $\left|f_1x_1\right| = \left|f_2x_2\right| = \cdots = \left|f_{n+1}x_{n+1}\right| = 0$ where $f_i \in \Lambda$ $(1 \leq i \leq n+1)$. Let $\mathscr{E}$ is a subset of $\mathscr{F}^{\perp\perp}$. Then, $x_i = \sum_{j=1}^{k} \langle x_i \mid y_j\rangle\, y_j$ holds for each $i = 1, ..., n+1$. Thus, it follows from $x_{n+1} \neq 0$ that there exists j such that $\langle x_{n+1} \mid y_j\rangle \neq 0$. We can assume $j = k$. By Lemma 3.1.3 (i) there is $g \in \Lambda$ such that $\mu := g\langle x_{n+1} \mid y_k\rangle = \left|g\langle x_{n+1} \mid y_k\rangle\, y_k\right| \in \mathfrak{P}(\Lambda) \setminus \{0\}$. Note that $\mu = \mu\left|x_{n+1}\right|$ since $(\mathbf{1} - \left|x_{n+1}\right|)\langle x_{n+1} \mid y_k\rangle = 0$. We have the following statement by simple calculations

$$\mu x_i - g\langle x_i \mid y_k\rangle\, x_{n+1} = \sum_{j=1}^{k-1} (\mu\langle x_i \mid y_j\rangle - g\langle x_i \mid y_k\rangle\langle x_{n+1} \mid y_j\rangle)\, y_j \quad (1 \leq i \leq n).$$

Moreover, $\left|\mu x_i - g\langle x_i \mid y_k\rangle\, x_{n+1}\right|^2 = \mu + \left|g\langle x_i \mid y_k\rangle\right|^2 \geq \mu$ are satisfied $i = 1, ..., n+1$. By Lemma 3.1.3 (ii) there is $g_i \in \Lambda_+$ such that $\mu g_i = g_i$ and $\mu = g_i\left|\mu x_i - g\langle x_i \mid y_k\rangle\, x_{n+1}\right|$. Define $z_i := g_i(\mu x_i - g\langle x_i \mid y_k\rangle\, x_{n+1})$, and note that $z_i \in \{y_1, y_2, ..., y_{k-1}\}^{\perp\perp}$ and $\left|z_i\right| = \mu$ $(1 \leq i \leq n)$. Assume that $\lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n = 0$ holds for some $\lambda_i \in \Lambda$ $(1 \leq i \leq n)$. Then we have

$$\begin{aligned} 0 = \lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n &= \sum_{i=1}^{n} \lambda_i g_i(\mu x_i - g\langle x_i \mid y_k\rangle\, x_{n+1}) \\ &= \sum_{i=1}^{n} \lambda_i g_i \mu x_i - \left(\sum_{i=1}^{n} g\lambda_i g_i\langle x_i \mid y_k\rangle\right) x_{n+1}, \end{aligned}$$

and so $\left|\lambda_i z_i\right| = 0$ since $\left|\lambda_i g_i\right| = \left|\lambda_i g_i \mu x_i\right| = 0$. Therefore, from our induction hypothesis we get $n \leq k-1$, and so $n+1 \leq k$. □

Corollary 3.1.17. *Let $\mathscr{E}$ and $\mathscr{F}$ be projection orthonormal finite subsets of X. If $\pi := \min\left\{\left|x\right| : x \in \mathscr{E}\right\} \neq 0$ and $\operatorname{card}(\mathscr{F}) < \operatorname{card}(\mathscr{E})$, then there exists $x \in \mathscr{E}^{\perp\perp} \cap \mathscr{F}^{\perp}$ with $\left|x\right| = \pi$.*

Proof. Let $\mathscr{E} := \{x_1, x_2, ..., x_n\}$ and $\mathscr{F} := \{y_1, y_2, ..., y_k\}$. Assume by way of contradiction that our claim is false. Then $\mu\left(\mathscr{E}^{\perp\perp} \cap \mathscr{F}^{\perp}\right) = \{0\}$ holds for some $0 < \mu \leq \pi$. For $1 \leq i \leq n$ define $z_i := \sum_{y \in \mathscr{F}} \langle x_i \mid y\rangle\, y$, and note that $pz_i \neq 0$ are satisfied for all $p \in \mathfrak{P}(\Lambda)$ with $0 < p \leq \mu$. Thus, there exist $g_i \in \Lambda$ and $p \in \mathfrak{P}(\Lambda)$ such that $0 < p = g_i\left|z_i\right| \leq \mu$, $(1 \leq i \leq n)$. Suppose that $\lambda_1 g_1 z_1 + \lambda_2 g_2 z_2 + \cdots + \lambda_n g_n z_n = 0$ holds for some $\lambda_i \in \Lambda$ $(1 \leq i \leq n)$. Therefore, we have $\lambda_1 g_1 x_1 + \lambda_2 g_2 x_2 + \cdots + \lambda_n g_n x_n \in \mathscr{F}^{\perp}$. By assumption, $\mu\left(\lambda_1 g_1 x_1 + \lambda_2 g_2 x_2 + \cdots + \lambda_n g_n x_n\right) = 0$. Since $\mathscr{E}$ is projection orthonormal set we have $\left|\mu\lambda_1 g_1 x_1\right| = \left|\mu\lambda_2 g_2 x_2\right| = \cdots = \left|\mu\lambda_n g_n x_n\right| = 0$. So, it follows from $\left|\mu\lambda_i g_i x_i\right|^2 = \left|\mu\lambda_i g_i z_i\right|^2 + \left|\mu\lambda_i g_i (x_i - z_i)\right|^2$ that $\left|\lambda_i g_i z_i\right| = 0$ for $1 \leq i \leq n$. Thus, $k < n$, contradicting Proposition 3.1.16. □

Now we recall the notion of C^*-sum, for details see [4]. Let $(A_\xi)_{\xi \in \Xi}$ be a family of (commutative) AW^*-algebras. If

$$A := \sum_{\xi \in \Xi}^{\oplus} A_\xi := \left\{ a = (a_\xi)_{\xi \in \Xi} \in \prod_{\xi \in \Xi} A_\xi : \sup_{\xi \in \Xi} \{\|a_\xi\|\} < \infty \right\}$$

is equipped with the coordinatewise $*$-algebra operations, and the norm $\|a\| := \sup_{\xi \in \Xi}\{\|a_\xi\|\}$, then A is an (commutative) AW^*-algebra and $\mathfrak{P}(A) = \prod_{\xi \in \Xi} \mathfrak{P}(A_\xi)$ and $\mathfrak{P}_c(A) = \prod_{\xi \in \Xi} \mathfrak{P}_c(A_\xi)$ ([4, Proposition 10.1]).

The notion of C^*-sum can be given for Kaplansky−Hilbert modules. Let Y_ξ be a Kaplansky−Hilbert module over A_ξ. Then

$$Y := \sum_{\xi \in \Xi}^{\oplus} Y_\xi := \left\{ x = (x_\xi)_{\xi \in \Xi} \in \prod_{\xi \in \Xi} Y_\xi : \sup_{\xi \in \Xi} \{|||x_\xi|||\} < \infty \right\}$$

equipped with the coordinatewise module operations over A and the inner product $\langle x \mid y\rangle := (\langle x_\xi \mid y_\xi\rangle)_{\xi \in \Xi}$, is a Kaplansky−Hilbert module over A. In particular, $\left|x\right| = \left(\left|x_\xi\right|\right)_{\xi \in \Xi}$ and $|||x||| = \sup_{\xi \in \Xi}\{|||x_\xi|||\}$ are satisfied for all $x = (x_\xi)_{\xi \in \Xi}$ in Y.

The following result on functional representation of Kaplansky−Hilbert modules is one of the main tools of our investigation. We refer for the definition of γ-stable to [23, 7.4.11.].

Theorem 3.1.18. [23, Theorem 7.4.12.] *To each Kaplansky–Hilbert module X over Λ there is a family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals, such that Q_γ is γ-stable for all $\gamma \in \Gamma$ and the following unitary equivalence holds:*

$$X \simeq \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}\left(Q_\gamma, \ell_2(\gamma)\right).$$

If some family $(P_\delta)_{\delta\in\Delta}$ of extremal compact spaces satisfies the above conditions then $\Gamma = \Delta$ and P_γ is homeomorphic with Q_γ for all $\gamma \in \Gamma$.

The unitary equivalence means that there are an isomorphism

$$\Psi : X \to \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}\left(Q_\gamma, \ell_2(\gamma)\right) \tag{3.3}$$

and a $*$-isomorphism

$$\Phi : \Lambda \to \sum_{\gamma\in\Gamma}^{\oplus} C(Q_\gamma) \tag{3.4}$$

(also Φ is an isometry [6, VIII.4.8. Teorem]) such that

$$\Phi \langle x \mid y\rangle = \langle \Psi(x) \mid \Psi(y)\rangle \quad (x, y \in X).$$

So, for every $x, y \in X$, $a \in \Lambda$ and $\pi \in \mathfrak{P}(\Lambda)$ the following hold:

(i) $\Psi(ax) = \Phi(a)\Psi(x)$;

(ii) $\Phi\left|x\right| = \left|\Psi(x)\right|$ and $|||x||| = |||\Psi(x)||| = \sup_{\gamma\in\Gamma}\{\|(\Psi(x))_\gamma\|\}$.

From (ii), Φ is the exact dominant $\left|\Psi\right|$ of Ψ. Moreover, it follows from Φ is a bijective positive operator that Φ is o-continuous, and so Ψ is bo-continuous.

3.2 Cyclically Compact Sets in Kaplansky–Hilbert Modules

Now we turn our attention to the study of cyclically compact sets in Kaplansky–Hilbert modules. In this section, X will denote a Kaplansky–Hilbert module over Λ

Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}_\sigma(B)$ the set of sequences $\nu : \mathbb{N} \to B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}_\sigma(B)$, the formula $\nu_1 \ll \nu_2$ abbreviates the following assertion:

$$\text{if } m, n \in \mathbb{N} \text{ and } \nu_1(m) \wedge \nu_2(n) \neq 0_B, \text{ then } m < n.$$

Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \to K$, and a partition $\nu \in \mathrm{Prt}_\sigma(B)$, put $s_\nu := \mathrm{mix}_{n\in\mathbb{N}}\,(\nu(n)s(n))$. A *cyclic subsequence* of $s : \mathbb{N} \to K$ is any sequence of the form $(s_{\nu_k})_{k\in\mathbb{N}}$, where $(\nu_k)_{k\in\mathbb{N}} \subset \mathrm{Prt}_\sigma(B)$ and $(\forall k \in \mathbb{N})$ $\nu_k \ll \nu_{k+1}$. A subset $C \subset X$ is said to be *cyclically compact* if C is mix-complete and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set. Moreover, every cyclically compact set is bounded in X.

Observe that if for some $n \in \mathbb{N}$ $\nu_1 \ll \nu_2$ and $\nu_1(k) = 0$ for all $k < n$, then $\nu_2(k) = 0$ for all $k \le n$. Thus, if $(\nu_k)_{k\in\mathbb{N}} \subset \mathrm{Prt}_\sigma(B)$ and $(\forall k \in \mathbb{N})$ $\nu_k \ll \nu_{k+1}$, then $\nu_k(i) = 0$ for all $i < k$.

Lemma 3.2.1. *Let K be a relatively cyclically compact subset in X. Then* $\mathrm{cl}\,(\mathrm{mix}\,(K))$ *is a cyclically compact subset of X.*

Proof. It is enough to show that $\mathrm{cl}\,(\mathrm{mix}\,(K))$ is mix-complete. To this end, let $(x_\alpha)_{\alpha\in A}$ be a family in $\mathrm{cl}\,(\mathrm{mix}\,(K))$ and $(\pi_\alpha)_{\alpha\in A}$ be a partition of unity in $\mathfrak{P}(\Lambda)$. Define $x := \mathrm{mix}_{\alpha\in A}\,(\pi_\alpha x_\alpha)$. For each $n \in \mathbb{N}$ there is a family $(y_\alpha)_{\alpha\in A}$ in $\mathrm{mix}\,(K)$ such that $|||x_\alpha - y_\alpha||| \le 1/n$. Since $\mathrm{mix}\,(K)$ is mix-complete, $y_n := \mathrm{mix}_{\alpha\in A}\,(\pi_\alpha y_\alpha) \in \mathrm{mix}\,(K)$. Using the following inequality

$$\left| x - y_n \right| = \left| bo\text{-}\sum_{\alpha\in A} \pi_\alpha x_\alpha - bo\text{-}\sum_{\alpha\in A} \pi_\alpha y_\alpha \right| = o\text{-}\sum_{\alpha\in A} \pi_\alpha \left| x_\alpha - y_\alpha \right| \le \frac{1}{n}\mathbf{1}$$

we obtain $x \in \mathrm{cl}\,(\mathrm{mix}\,(K))$, as desired. □

The following lemma is a corollary of [23, 2.2.9.(1)]. A set V is called *finitely cyclic* if for every $n \in \mathbb{N}$, $\{x_1, x_2, ..., x_n\} \subset V$ and a partition of unity $\{p_1, p_2, ..., p_n\} \subset \mathfrak{P}(\Lambda)$, $\sum_{i=1}^n p_i x_i \in V$.

Lemma 3.2.2. *Let V be a finitely cyclic subset of X and $x \in X$. Then there exists a net $(v_\alpha)_{\alpha\in A}$ in V such that the net $\left(\left|x - v_\alpha\right|\right)_{\alpha\in A}$ decreases and*

$$\left\{ \left|x - v_\alpha\right| : \alpha \in A \right\} = \left\{ \left|x - v\right| : v \in V \right\}.$$

In particular, $o\text{-}\lim_{\alpha\in A} \left|x - v_\alpha\right| = \inf_{v\in V} \left|x - v\right|$.

Lemma 3.2.3. *Let K be a cyclically compact subset of X and $x \in X$. Then there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in K that converges (in norm) to some w in K and satisfies*

$$\left| \left|x - w_n\right| - r \right| \le \frac{1}{n}\mathbf{1} \text{ and } \left|x - w\right| = r$$

where $r = \inf \left\{ \left|x - k\right| : k \in K \right\}$.

Proof. By Lemma 3.2.2, we have a net $(v_\alpha)_{\alpha\in A}$ in K such that o-$\lim |x - v_\alpha| = r$. So, using [23, Theorem 8.1.8.] for each $n \in \mathbb{N}$ we obtain a partition of unity $(\pi_\alpha^n)_{\alpha\in A}$ in $\mathfrak{P}(\Lambda)$ such that

$$\pi_\alpha^n \left| |x - v_\alpha| - r \right| \leq \frac{1}{n}\mathbf{1}$$

holds for all $\alpha \in A$. Since K is bounded, the net $\left(|x - v_\alpha|\right)_{\alpha\in A}$ is also bounded. Thus, we have $k_n \in K$ with

$$\left| |x - k_n| - r \right| \leq \frac{1}{n}\mathbf{1}$$

where $k_n = bo\text{-}\sum_{\alpha\in A} \pi_\alpha^n v_\alpha = \operatorname{mix}_{\alpha\in A}(\pi_\alpha^n v_\alpha)$. Since K is cyclically compact subset, there is a cyclic subsequence $(k_{\nu_n})_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ in K that is norm-convergent to some w in K. Therefore, if we define $w_n := k_{\nu_n}$ then it follows from

$$\left| |x - w_n| - r \right| \leq \frac{1}{n}\mathbf{1}$$

that $|x - w| = r$, as desired. □

The following Lemmas 3.2.4 and 3.2.5 and Propositions 3.2.8 and 3.4.7 are proved in [8, 16] for measurable bundles.

Proposition 3.2.4. *Let K be a cyclically compact subset of $C_\#(Q, H)$. Then for each $q \in Q$, $K(q)$ is a closed set in H where $K(q) := \{x(q) : \widetilde{x} \in K, q \in \operatorname{dom}(x)\}$.*

Proof. Fix $q \in Q$ and let x_q be an element of $\operatorname{cl}(K(q))$. So, we have

$$0 = \inf\left\{\|x_q - k(q)\| : \widetilde{k} \in K, q \in \operatorname{dom}(k)\right\}.$$

By $x : t \mapsto x_q$ $(t \in Q)$ and Lemma 3.2.3, we obtain $\widetilde{v} \in K$ such that

$$|\widetilde{x} - \widetilde{v}| = \inf\left\{|\widetilde{x} - \widetilde{k}| : \widetilde{k} \in K\right\}.$$

Thus, from $|\widetilde{x} - \widetilde{v}|(t) \leq |\widetilde{x} - \widetilde{k}|(t)$ for every $\widetilde{k} \in K$ and $t \in Q$ we have $|\widetilde{x} - \widetilde{v}|(q) = 0$, and so we can assume $q \in \operatorname{dom}(v)$. Therefore, it follows from $|\widetilde{x} - \widetilde{v}|(q) = \|x_q - v(q)\|$ that $x_q = v(q)$, i. e., $K(q)$ is a closed set in H. □

Lemma 3.2.5. *Let K be a relatively cyclically compact subset of $C_\#(Q, H)$. Then there exists a comeager set $Q_0 \subset Q$ such that $K(q)$ is precompact in H for all $q \in Q_0$.*

Proof. Fix n in $\mathbb{N}$. According to [23, Theorem 8.5.2.] there exist a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)$ and a sequence $(\theta_k)_{k\in\mathbb{N}}$ of finite subsets $\theta_k :=$

$\{\widetilde{x}_{k,1}, \widetilde{x}_{k,2}, ..., \widetilde{x}_{k,l(k)}\} \subset K$ such that for every $\widetilde{x} \in K$ there exists a partition of unity $\{\rho_{k,1}, \rho_{k,2}, ..., \rho_{k,l(k)}\}$ in $\mathfrak{P}(\Lambda)$ with

$$\left\| \pi_k \left| \widetilde{x} - \sum_{i=1}^{l(k)} \rho_{k,i} \widetilde{x}_{k,i} \right| \right\| \leq \frac{1}{n}.$$

Define

$$Q_n := \left(\bigcup_{k \in \mathbb{N}} U_k \right) \bigcap \left(\bigcap_{k \in \mathbb{N}} \bigcap_{i=1}^{l(k)} \operatorname{dom}(x_{k,i}) \right)$$

where U_k is a clopen set in Q corresponding to the projection π_k in $\mathfrak{P}(\Lambda)$. So, if we define a comeager set $Q_0 := \bigcap_{n \in \mathbb{N}} Q_n$, then it is not difficult to see that Q_0 is a comeager set and for every q in Q_0 there is k_0 in $\mathbb{N}$ with $q \in U_{k_0}$ and

$$K(q) \subset \bigcup_{i=1}^{l(k_0)} \operatorname{cl}\left(B_{\frac{1}{n}}(x_{k_0,i}(q)) \right)$$

where $B_{\frac{1}{n}}(x_{k_0,i}(q))$ is the open ball centered at $x_{k_0,i}(q)$ with radius $\frac{1}{n}$. Thus, $K(q)$ is precompact in H for all $q \in Q_0$. □

A useful consequence of the preceding results is the following.

Corollary 3.2.6. *Let K be a cyclically compact subset of $C_\#(Q, H)$. Then there exists a comeager set $Q_0 \subset Q$ such that $K(q)$ is compact in H for all $q \in Q_0$.*

The following lemma was proved in [16] for measurable bundles.

Lemma 3.2.7. *Let K be a mix-complete subset of X and ε be a positive real number. Then there exist a subset $\theta := \{x_n : n \in \mathbb{N}\}$ of K and an increasing sequence $(\pi_n)_{n \in \mathbb{N}} \subset \mathfrak{P}(\Lambda)$ such that the following statements hold:*

(i) *for every $x \in K$ and $n \in \mathbb{N}$ there exists $x' \in \operatorname{mix}(\theta_n)$ such that*

$$\pi_n \left| x - x' \right| \leq \varepsilon \mathbf{1}$$

where $\theta_n := \{x_1, ..., x_n\}$;

(ii) $\pi_i^\perp \left| x_j - x_i \right| \geq \varepsilon \pi_i^\perp$, $(i < j)$.

Proof. The sequences can be constructed by induction as follows. Given $n \in \mathbb{N}$. Suppose that $\theta_n = \{x_1, x_2, ..., x_n\} \subset K$ and $\{\pi_1, \pi_2, ..., \pi_{n-1}\} \subset \mathfrak{P}(\Lambda)$ with $\pi_{j-1} \leq \pi_j$ and the following statements hold:

(a) for every $x \in K$ and $k < n$ there exists some $x' \in \operatorname{mix}(\theta_k)$ such that $\pi_k \left| x - x' \right| \leq \varepsilon \mathbf{1}$;

(b) $\pi_i^{\perp} \left| x_j - x_i \right| \geq \varepsilon \pi_i^{\perp}$, $(i < j < n)$.

Now we will deduce the existence of $x_{n+1} \in K$ and $\pi_n \in \mathfrak{P}(\Lambda)$. If $\pi_{n-1} = \mathbf{1}$, then take $x_{n+1} := x_n$ and $\pi_n := \pi_{n-1}$. Assume that $\pi_{n-1} \neq \mathbf{1}$. Consider the set

$$A_n := \left\{ \pi \in \mathfrak{P}(\Lambda) : (\forall x \in K)(\exists\, x' \in \operatorname{mix}(\theta_n))\, \pi \left| x - x' \right| \leq \varepsilon \mathbf{1} \right\},$$

then we will prove the following statements hold:

(1) $\pi_n := \bigvee A_n \in A_n$

(2) There exists $x_{n+1} \in K$ such that $\pi_i^{\perp} \left| x_{n+1} - x_i \right| \geq \varepsilon \pi_i^{\perp}$ for $i \leq n$.

Firstly, we will show that A_n is a band in $\mathfrak{P}(\Lambda)$. For this, by definition of A_n, $\pi \in A_n$ and $\mu \leq \pi$ implies $\mu \in A_1$. If $\pi, \mu \in A_n$, then there exists $y_1, y_2 \in \operatorname{mix}(\theta_n)$ such that

$$\pi \left| x - y_1 \right| \leq \varepsilon \mathbf{1} \quad \text{and} \quad (\mu \wedge \pi^{\perp}) \left| x - y_2 \right| \leq \varepsilon \mathbf{1}.$$

If we define

$$x' := \pi y_1 + (\mu \wedge \pi^{\perp}) y_2 + (\pi \vee \mu)^{\perp} y_1,$$

then $x' \in \operatorname{mix}(\theta_n)$ and $(\pi \vee \mu) \left| x - x' \right| \leq \varepsilon \mathbf{1}$. So, A_n is an ideal in $\mathfrak{P}(\Lambda)$. Using the Exhaustion Principle [23, 1.1.6.(1)] there exists an antichain $(\nu_\alpha)_{\alpha \in A}$ in A_n such that $\sup_{\alpha \in A} \nu_\alpha = \pi_n$. So, for each $\alpha \in A$ we have $y_\alpha \in \operatorname{mix}(\theta_n)$ such that $\nu_\alpha \left| x - y_\alpha \right| \leq \varepsilon \mathbf{1}$. Since $\operatorname{mix}(\theta_n)$ is mix-complete,

$$x' := \pi_n^{\perp} x_1 + bo\text{-}\sum_{\alpha \in A} \nu_\alpha y_\alpha$$

is an element of $\operatorname{mix}(\theta_n)$, and so we have

$$\begin{aligned} \pi_n \left| x - x' \right| &= \pi_n \left| bo\text{-}\sum_{\alpha \in A} \nu_\alpha (x - y_\alpha) \right| = o\text{-}\sum_{\alpha \in A} \nu_\alpha \left| x - y_\alpha \right| \\ &= \sup_{\alpha \in A} \nu_\alpha \left| x - y_\alpha \right| \leq \varepsilon \mathbf{1}. \end{aligned}$$

Thus, A_n is a band in $\mathfrak{P}(\Lambda)$, and this proves (1).

For the proof of (2), we define

$$C_n = \left\{ \pi \in \mathfrak{P}(\Lambda) : (\exists\, x \in K) \pi \left| x - x_i \right| > \varepsilon \pi,\ i = 1, 2, ..., n \right\},$$

then clearly, $C_n \cup \{\mathbf{0}\}$ is an ideal in $\mathfrak{P}(\Lambda)$ and $\bigvee C_n \leq \pi_n^{\perp}$. Since $0 < \pi \leq \pi_n^{\perp}$ implies $\pi \notin A_n$, we have $x_\pi \in K$ such that $\pi \left| x_\pi - y \right| \nleq \varepsilon \mathbf{1}$ holds for all $y \in \operatorname{mix}(\theta_n)$. Now we define

$$D_i := \left\{ \mu \in \mathfrak{P}(\Lambda) : \mu \leq \pi \text{ and } \mu \left| x_\pi - x_i \right| > \varepsilon \mu \right\}$$

for $i = 1, 2, ..., n$. Clearly, $D_i \neq \varnothing$ and $D_i \cup \{\mathbf{0}\}$ is an ideal in $\mathfrak{P}(\Lambda)$. Moreover,

$$\mu_i \left| x_\pi - x_i \right| \geq \varepsilon \mu_i \quad \text{and} \quad \pi \mu_i^{\perp} \left| x_\pi - x_i \right| \leq \varepsilon \mathbf{1}$$

where $\mu_i := \bigvee D_i$ $(i = 1, 2, ..., n)$. If $\bigwedge_{i=1}^n \mu_i = 0$, then there exists an antichain $\{\lambda_i : i = 1, 2, ..., n\}$ such that $0 \leq \lambda_i \leq \mu_i^{\perp}$ and $\bigvee_{i=1}^n \lambda_i = \bigvee_{i=1}^n \mu_i^{\perp} = \mathbf{1}$. Since $\{\pi\lambda_i : i = 1, 2, ..., n\} \cup \{\pi^{\perp}\}$ is a partition of unity,

$$x' = \sum_{i=1}^{n} \pi \lambda_i x_i + \pi^{\perp} x_1 \in \operatorname{mix}(\theta_n) \quad \text{and} \quad \pi \left| x_\pi - x' \right| \leq \varepsilon \mathbf{1}$$

are satisfied. This is a contradiction, and thus $\bigwedge_{i=1}^n \mu_i \neq 0$. So, there exists $\mu \in \cap_{i=1}^n D_i$ such that $0 < \mu \leq \bigwedge_{i=1}^n \mu_i \leq \mu_i$ and $\mu \left| x_\pi - x_i \right| > \varepsilon \mu$ holds for $i = 1, 2, ..., n$, i. e., $\mu \in C_n$. This implies $\pi_n^{\perp} = \bigvee C_n$. Again, by the Exhaustion Principle, we obtain an antichain $(\nu_\gamma)_{\gamma \in \Gamma} \subset C_n$ and $(x_\gamma)_{\gamma \in \Gamma} \subset K$ such that $\pi_n^{\perp} = \bigvee_{\gamma \in \Gamma} \nu_\gamma$ and $\nu_\gamma \left| x_\gamma - x_i \right| > \varepsilon \nu_\gamma$ hold for $i = 1, 2, ..., n$. If we define

$$x_{n+1} := bo\text{-}\sum_{\gamma \in \Gamma} \nu_\gamma x_\gamma + \sum_{i=2}^{n} (\pi_i \wedge \pi_{i-1}^{\perp}) x_i + \pi_1 x_1,$$

then $x_{n+1} \in K$ and $\pi_i^{\perp} \left| x_{n+1} - x_i \right| \geq \pi_i^{\perp} \varepsilon$ holds for $i = 1, 2, ..., n$, and the proof of the lemma is finished. □

Proposition 3.2.8. *Let K be a mix-complete subset of $C_\#(Q, H)$ and $Q_0 \subset Q$ be a comeager subset. If K_q is a compact set in H for $q \in Q_0$ and*

$$K \subset \{\widetilde{x} \in C_\#(Q, H) : x(q) \in K_q \ (\forall q \in \operatorname{dom}(x) \cap Q_0)\},$$

then K is relatively cyclically compact subset of $C_\#(Q, H)$.

Proof. According to [23, Theorem 8.5.2.], it is enough show that the supremum of sequence $(\pi_n)_{n \in \mathbb{N}}$, which is constructed in Lemma 3.2.7, equals to the unity $\mathbf{1}$ of $\mathfrak{P}(\Lambda)$. Assume that $\bigvee_{n \in \mathbb{N}} \pi_n \neq \mathbf{1}$. Then for all $i, j \in \mathbb{N}$ the following inequality

$$\rho \left| \widetilde{x}_i - \widetilde{x}_j \right| \geq \varepsilon \rho$$

holds where $\rho^{\perp} := \bigvee_{n \in \mathbb{N}} \pi_n$. Since $Q_1 := (\bigcap_{n \in \mathbb{N}} \operatorname{dom}(x_n)) \cap Q_0$ is a comeager set in Q, we have some $q \in Q_1 \cap U_\rho$ where U_ρ is clopen set in Q corresponding to the element ρ in $\mathfrak{P}(\Lambda)$. Therefore, K_q is not compact in H since $\|x_i(q) - x_j(q)\| \geq \varepsilon$ $(i, j \in \mathbb{N})$. This is a contradiction. Hence, K is relatively cyclically compact. □

As a corollary, we see that the closed unit ball of a Stone algebra is a cyclically compact set.

3.3 Operators on Kaplansky−Hilbert Modules

Let X, Y be Kaplansky−Hilbert modules over Λ. Let $B_\Lambda(X,Y)$ denote the set of all continuous Λ-linear operators from X into Y. For brevity, $B_\Lambda(X,X)$ will be denoted by $B_\Lambda(X)$. We call a Λ-linear operator $T^* : Y \to X$ the *adjoint* of Λ-linear operator $T : X \to Y$ if $\langle Tx \mid y\rangle = \langle x \mid T^*y\rangle$ for all x and y. I. Kaplansky showed that a Λ-linear operator $T : X \to Y$ is continuous if and only if T has an adjoint [15, Theorem 6]. Moreover, he also showed that $B_\Lambda(X)$ is an AW^*-algebra of type I with center isomorphic to Λ [15, Theorem 7].

Let T in $B_\Lambda(X)$, and A be a subset of X. If $T(A) \subset A$, then $T^*(A^\perp) \subset A^\perp$. From this we observe, if A is a subset of X and $T(A) \subset A$, then Kaplansky−Hilbert submodule generated by A is T-invariant since $A^{\perp\perp}$ is the submodule generated by A.

Proposition 3.3.1. [23, 7.5.7 (1)] *Let X and Y be Kaplansky−Hilbert modules over Λ, and T in $B_\Lambda(X,Y)$. Then T is dominated and bo-continuous. In addition, the kernel of T is a Kaplansky−Hilbert submodule of X.*

Proposition 3.3.2. *Let T be a bijection in $B_\Lambda(X,Y)$. Then T^{-1} is an element of $B_\Lambda(Y,X)$.*

Proof. Using the Banach's Isomorphism Theorem, we have $T^{-1} \in L(Y,X)$ where $L(Y,X)$ is the set of all continuous linear operator from Y into X. Let $x \in X$, $f \in \Lambda$. As T is onto, there are $y_1, y_2 \in X$ such that $Ty_1 = fx$ and $Ty_2 = x$. As T is one to one, we have $y_1 = fy_2$. Thus, it follows from

$$T^{-1}(fx) = T^{-1}Ty_1 = y_1 = fy_2 = f(T^{-1}Ty_2) = fT^{-1}x$$

that $T^{-1} \in B_\Lambda(Y,X)$. □

Corollary 3.3.3. *Suppose that $T \in B_\Lambda(X)$, then T has the same spectrums as an element of $L(X)$ and as an element of $B_\Lambda(X)$, that is,* $\mathrm{Sp}(T) := \mathrm{Sp}_{L(X)}(T) = \mathrm{Sp}_{B_\Lambda(X)}(T)$ *holds.*

Proposition 3.3.4. *Let T be a continuous Λ-linear operator on X. If for some positive $a \in \Lambda$ the following inequality holds*

$$a\left|Tx\right| \geq \left|x\right| \quad (x \in X),$$

then $T(X)$ *is a Kaplansky–Hilbert submodule of* X *where* $T(X)$ *denotes the range of* T.

Proof. Clearly, $T(X)$ is submodule of X. Now we must show that $T(X)$ is bo-closed in X. For this, take a net $(Tx_\alpha)_{\alpha\in A} \subset T(X)$ with $bo\text{-}\lim Tx_\alpha = x$ in X. This implies that $(Tx_\alpha)_{\alpha\in A}$ is bo-fundamental net in X. From $a\left|Tx\right| \geq \left|x\right|$, $(x_\alpha)_{\alpha\in A}$ is a bo-fundamental net in X. So, there exists $y \in X$ such that $y = bo\text{-}\lim x_\alpha$. By Proposition 3.3.1, $Ty = x$ holds, i. e., $T(X)$ is a Kaplansky–Hilbert submodule of X. □

As in Hilbert space case, the following identities are valid for each continuous Λ-linear operator T from X to Y and $u, v, x, y \in X$

$$\operatorname{Ker} T = (T^*(Y))^{\perp} \text{ and } \operatorname{cl}(\operatorname{mix}(T^*(Y))) = (\operatorname{Ker} T)^{\perp}.$$

By identity (3.1), for every $x \in X$, $\langle Tx \mid x\rangle = 0$ implies $T = 0$. Since $B_\Lambda(X)$ is a C^*-algebra, for every positive T in $B_\Lambda(X)$ there is a unique positive $T^{1/2}$ in $B_\Lambda(X)$ such that $T = \left(T^{1/2}\right)^2$. So, for any element T in $B_\Lambda(X, Y)$, we can define absolute value of T by $|T| := (T^*T)^{1/2}$. Another proof of the following proposition is given in [28, Proposition 2.1.3].

Proposition 3.3.5. *Let* T *in* $B_\Lambda(X)$. *Then the following statements are equivalent:*

(i) T *is a positive element of* $B_\Lambda(X)$*;*

(ii) $\langle Tx \mid x\rangle$ *is positive element of* Λ *for all* $x \in X$*;*

(iii) $\langle Tx \mid x\rangle$ *is positive element of* Λ *for all* $x \in X$ *with* $\left|x\right| \in \mathfrak{P}(\Lambda)$*;*

(iv) $\langle Tx \mid x\rangle$ *is positive element of* Λ *for all* $x \in X$ *with* $\left|x\right| = \mathbf{1}$.

Proof. The implications (iii) $\Rightarrow$ (iv) : and (ii) $\Rightarrow$ (iii) : are obvious and (iv) $\Rightarrow$ (iii) : follows from the faithfulness of X.

(i) $\Rightarrow$ (ii) : Let T be a positive element of $B_\Lambda(X)$. Then there exists $A \in B_\Lambda(X)$ such that $T = A^*A$. This implies that $\langle Tx \mid x\rangle = \left|Ax\right|^2 \geq 0$.

(ii) $\Rightarrow$ (i) : Assume that $\langle Tx \mid x\rangle$ is positive in Λ for all $x \in X$. Using the identity (3.1), we obtain $T = T^*$. Let λ be a negative number. So,

$$\begin{aligned}\left|(T - \lambda I)x\right|^2 &= \left|Tx\right|^2 - 2\lambda \langle Tx \mid x\rangle + \lambda^2 \left|x\right|^2 \\ &\geq -2\lambda \langle Tx \mid x\rangle + \lambda^2 \left|x\right|^2 \geq \lambda^2 \left|x\right|^2\end{aligned}$$

since $\lambda < 0$ and $\langle Tx \mid x\rangle \geq 0$. This implies that $\mathrm{Ker}(T-\lambda I) = \{0\}$ and $(T - \lambda I)(X)$ is a Kaplansky−Hilbert submodule of X. Moreover, from $\mathrm{Ker}(T-\lambda I) = ((T^* - \lambda I)(X))^{\perp}$ and $T = T^*$, we have $(T-\lambda I)(X) = X$. Hence, it follows from Proposition 3.3.2 that $T - \lambda I$ is invertible in $B_\Lambda(X)$, i. e., $\lambda \notin \mathrm{Sp}(T)$. Therefore, T is a positive element of $B_\Lambda(X)$.

(iii) ⇒ (ii) : Given x in X. Using the Exhaustion Principle and Lemma 3.1.3 (i) there exist a partition $(\pi_\alpha)_{\alpha\in A}$ of $\big[\big|x\big|\big]$ and a family $(a_\alpha)_{\alpha\in A}$ in Λ_+ such that $a_\alpha \big|x\big| = \pi_\alpha$ and $\pi_\alpha = [a_\alpha]$. From $a_\alpha^2 \langle Tx, x\rangle = \langle T(a_\alpha x), a_\alpha x\rangle \geq 0$ it follows that $\pi_\alpha \langle Tx, x\rangle \geq 0$ holds for all $\alpha \in A$. Therefore, $\langle Tx, x\rangle = \big[\big|x\big|\big] \langle Tx, x\rangle \geq 0$ holds, and the proof is finished. □

An element V in $B_\Lambda(X,Y)$ is said to be a *partial isometry* if $\big|Vx\big| = \big|x\big|$ for $x \in (\mathrm{Ker}\, V)^{\perp}$.

Lemma 3.3.6. *If V is a partial isometry, V^*V is a projection on* $(\mathrm{Ker}\, V)^{\perp}$.

Proof. For all $x \in (\mathrm{Ker}\, V)^{\perp}$ and $y \in \mathrm{Ker}\, V$,

$$\langle V^*Vx \mid x\rangle = \langle Vx \mid Vx\rangle = \big|Vx\big|^2 = \big|x\big|^2 = \langle x \mid x\rangle$$

and

$$\langle V^*Vy \mid y\rangle = \langle Vy \mid Vy\rangle = 0.$$

Thus, it follows from $\langle V^*V(x+y) - x \mid x+y\rangle = 0$ that $V^*V(x+y) = x$, i. e., V^*V is a projection on $(\mathrm{Ker}\, V)^{\perp}$. □

Now we shall prove the polar decomposition for operators on Kaplansky−Hilbert module case. A variant of following lemma is proved in [43, Theorem 5.5].

Lemma 3.3.7. (Polar Decomposition) *Let T be a continuous Λ-linear operator from X to Y. Then T has a polar decomposition $T = V|T|$, where V is a partial isometry for which*

$$\mathrm{Ker}\, V = \mathrm{Ker}\, T \quad \text{and} \quad V(X) = \mathrm{cl}\,(\mathrm{mix}\,(T(X)))\,.$$

Moreover, $V^*T = |T|$, and if $T = UP$, where $P \geq 0$ and U is a partial isometry with $\mathrm{Ker}\, U = \mathrm{Ker}\, P$*, then $P = |T|$ and $U = V$.*

Proof. If $x \in X$, $\big||T|x\big|^2 = \langle |T|x \mid |T|x\rangle = \langle |T|^2 x \mid x\rangle = \langle T^*Tx \mid x\rangle = \langle Tx \mid Tx\rangle = \big|Tx\big|^2$. Hence, the map

$$V : |T|(X) \to T(X),\ |T|x \mapsto Tx,$$

is a well-defined surjection and $\left|Vx\right| = \left|x\right|$ for $x \in |T|(X)$. From [47, Lemma 2.4.], V can be extended to a unique map $\mathrm{cl}\,(\mathrm{mix}\,(|T|(X)))$ onto $\mathrm{cl}\,(\mathrm{mix}\,(T(X)))$, and so $\left|Vx\right| = \left|x\right|$ holds for all $x \in \mathrm{cl}\,(\mathrm{mix}\,(|T|(X)))$. Extend V to all of X by letting to be 0 on $(\mathrm{cl}\,(\mathrm{mix}\,(|T|(X))))^{\perp} = (|T|(X))^{\perp} = \mathrm{Ker}\,|T|$. Thus, V is a partial isometry and $\mathrm{Ker}\,V = \mathrm{Ker}\,|T| = \mathrm{Ker}\,T$. Now the following equality

$$\langle V^*Tx \mid |T|y\rangle = \langle Tx \mid Ty\rangle = \langle T^*Tx \mid y\rangle = \langle |T|x \mid |T|y\rangle$$

implies $\langle V^*Tx \mid z\rangle = \langle |T|x \mid z\rangle$ for all $z \in \mathrm{cl}\,(\mathrm{mix}\,(|T|(X)))$, and therefore for all z in X. So, we have $V^*T = |T|$.

To show uniqueness, note that $T^*T = PU^*UP$. By Lemma 3.3.6, $E := U^*U$ is the projection onto $(\mathrm{Ker}\,U)^{\perp} = (\mathrm{Ker}\,P)^{\perp} = \mathrm{cl}\,(\mathrm{mix}\,(P(X)))$. Thus, $T^*T = P^2$, so that $P = |T|$. For $x \in X$,

$$V|T|x = Tx = UPx = U|T|x,$$

that is, V and U agree on $|T|(X)$, and hence $V = U$. □

By Proposition 3.3.1 and [23, 4.1.2.] all T in $B_\Lambda(X,Y)$ has the exact dominant $\left|T\right|$. Since $x \perp y$ means $\left|y\right| x = 0$, it follows from $\left|y\right| \left|Tx\right| = \left|T(\left|y\right| x)\right|$ that T is band preserving. Thus, from [23, 5.1.8.(1),(2)] $\left|T\right|$ is an element of $\mathrm{Orth}(\Lambda)$, and so $\left|T\right| \in \Lambda$ and by [23, 4.1.8. and 4.1.11.] we have

$$\left|T\right| = \sup\left\{\left|Tx\right| : \left|x\right| \leq \mathbf{1}\right\} = \sup\left\{\left|Tx\right| : \left|x\right| = \mathbf{1}\right\}.$$

In particular, $\left|\lambda T\right| = |\lambda| \left|T\right|$ holds for all $\lambda \in \Lambda$.

Proposition 3.3.8. $(B_\Lambda(X,Y), \left|\cdot\right|, \Lambda)$ *is a Banach−Kantorovich space and admits a compatible module structure over* Λ. *In addition, the mixed norm is equal to the operator norm in* $B_\Lambda(X,Y)$, *i. e.,* $\left\| \left|T\right| \right\| = |||T|||$ $(T \in B_\Lambda(X,Y))$.

Proof. Clearly, we see that $(B_\Lambda(X,Y), \left|\cdot\right|, \Lambda)$ is a lattice-normed space. Since $\left|\lambda T\right| = |\lambda| \left|T\right|$ holds for all $\lambda \in \Lambda$ the norm $\left|\cdot\right|$ is decomposable. Indeed, assume that $\left|T\right| = e_1 + e_2$ is satisfied for some $e_1, e_2 \in \Lambda_+$. Then since $0 \leq e_1 \leq \left|T\right|$ there exists an orthomorpism S on Λ such that $S\left|T\right| = e_1$ and $0 \leq S \leq I$ [2, Theorems 2.49]. So, it follows from $\mathrm{Orth}(\Lambda) = \Lambda$ that there is g in Λ such that $g\left|T\right| = e_1$ and $0 \leq g \leq \mathbf{1}$ [2, Theorems 2.62]. Therefore, $e_2 = \left|T\right| - e_1 = \left|T\right| - g\left|T\right| = (\mathbf{1}-g)\left|T\right|$ and $T = gT + (\mathbf{1}-g)T$, i. e., $B_\Lambda(X,Y))$ is a decomposable lattice-normed space. From Proposition 3.3.1 the space of dominated operators $M(X,Y)$ which is Banach−Kantorovich space over $L^{\sim}(\Lambda)$ [23, 4.2.6] contains $B_\Lambda(X,Y)$. So, assume that $(T_\alpha)_{\alpha\in A}$ *bo*-converges to T with $(T_\alpha)_{\alpha\in A} \subset B_\Lambda(X,Y)$

and $T \in M(X,Y)$. Clearly, T is a Λ-linear. Since every positive operator in $L^{\sim}(\Lambda)$ is norm-continuous [2, Theorem 4.3] T is also norm-continuous. Therefore, $B_\Lambda(X,Y)$ is a Banach$-$Kantorovich space. On the other hand $B_\Lambda(X,Y)$ admits a compatible module structure over Λ from [23, 2.1.8.(3)]. Now we will show that $\| \lvert T \rvert \| = |||T||| \ (T \in B_\Lambda(X,Y))$. Obviously $\lvert x \rvert \leq \mathbf{1}$ means $|||x||| \leq 1$ for all $x \in X$. Let $\lvert x \rvert \leq \mathbf{1}$. So, from $|||Tx||| = \| \lvert Tx \rvert \| \leq \| \lvert T \rvert \lvert x \rvert \| \leq \| \lvert T \rvert \|$ we have $\| \lvert T \rvert \| \geq |||T|||$. Let $|||x||| \leq 1$. Then it follows from $\lvert Tx \rvert \leq |||Tx\,|||\,\mathbf{1} \leq |||T\,|||\,\mathbf{1}$ that $\| \lvert T \rvert \| \leq |||T|||$, and the proof is finished. □

Definition 3.3.9. Let T be an operator on X. A scalar $\lambda \in \Lambda$ is said to be an *eigenvalue* if there exists nonzero $x \in X$ such that $Tx = \lambda x$. A nonzero eigenvalue λ is called a *global eigenvalue* if for any nonzero projection $\pi \in \Lambda$ with $\pi \leq [\lambda]$ there exists a nonzero $x \in \pi X$ such that $Tx = \lambda x$.

Proposition 3.3.10. *Let T be a continuous Λ-linear operator on X and λ be a nonzero scalar. Then the following statements are equivalent:*

(i) *The scalar $\lambda \in \Lambda$ is a global eigenvalue of T.*

(ii) *There is $x \in X$ such that $Tx = \lambda x$ and $\left[\lvert x \rvert \right] \geq [\lambda]$.*

(iii) *There is $x \in X$ such that $Tx = \lambda x$ and $\lvert x \rvert \in \mathfrak{P}(\Lambda)$ with $\lvert x \rvert \geq [\lambda]$*

Proof. (iii) $\Rightarrow$ (ii) : Obvious.

(ii) $\Rightarrow$ (i) : If there is $x \in X$ with $Tx = \lambda x$ and $\left[\lvert x \rvert \right] \geq [\lambda]$, then $\pi x \neq 0$ and $T(\pi x) = \lambda \pi x$ hold for all nonzero projection π with $\pi \leq [\lambda]$. Thus, λ is a global eigenvalue of T.

(i) $\Rightarrow$ (iii) : Consider the set

$$C := \left\{ (\lvert x \rvert, x) : \lvert x \rvert \in \mathfrak{P}(\Lambda),\, 0 < \lvert x \rvert \leq [\lambda]\,,\, Tx = \lambda x \right\}.$$

From the definition of global eigenvalue and Lemma 3.1.3 (i), C is a non-empty set. We claim that $[\lambda] = \sup \left\{ \lvert x \rvert : (\lvert x \rvert, x) \in C \right\}$ holds. Indeed, assume that $\mu := \sup \left\{ \lvert x \rvert : (\lvert x \rvert, x) \in C \right\} \neq [\lambda]$. Then using Lemma 3.1.3 (i) there is $\lvert x_0 \rvert \in \mathfrak{P}(\Lambda)$ with $Tx_0 = \lambda x_0$ and $0 < \lvert x_0 \rvert \leq \mu^{\perp} \wedge [\lambda]$. This is a contradiction, and so $[\lambda] = \sup \left\{ \lvert x \rvert : (\lvert x \rvert, x) \in C \right\}$. Using the Exhaustion Principle [23, 1.1.6.(1)] there exists an antichain $(\mu_\alpha)_{\alpha \in A}$ in $\mathfrak{P}(\Lambda)$ such that $\sup_{\alpha \in A} \mu_\alpha = [\lambda]$ and for each $\alpha \in A$ there is $(\lvert x_\alpha \rvert, x_\alpha) \in C$ with $\mu_\alpha \leq \lvert x_\alpha \rvert$. Hence, we get $x := bo\text{-}\sum_{\alpha \in A} \mu_\alpha x_\alpha$ with $\lvert x \rvert = [\lambda]$ and $Tx = \lambda x$, and the proof is finished. □

Corollary 3.3.11. *To each global eigenvalue λ of T, $|\lambda| \leq \lvert T \rvert$.*

Corollary 3.3.12. *Let $\lambda, \mu \in \Lambda$ be global eigenvalues of T. If for some $x \in X$, $Tx = \lambda x = \mu x$, then $\left[\left|x\right|\right]\lambda = \left[\left|x\right|\right]\mu$.*

Corollary 3.3.13. *Let $(\lambda_\alpha)_{\alpha\in A}$ be a bounded family of eigenvalues of T and $(p_\alpha)_{\alpha\in A}$ be a partition of unity. Then $\operatorname{mix}_{\alpha\in A}(p_\alpha\lambda_\alpha)$ is an eigenvalue of T. Moreover, if $(\lambda_\alpha)_{\alpha\in A}$ consists of global eigenvalues of T, then $\operatorname{mix}_{\alpha\in A}(p_\alpha\lambda_\alpha)$ is a global eigenvalue of T.*

Proposition 3.3.14. *Let T be a positive operator on X. Then*

$$\left|T\right| = \sup\left\{\langle Tx \mid x\rangle : \left|x\right| = \mathbf{1}\right\}.$$

Proof. Consider the set $D := \left\{\langle Tx \mid x\rangle : \left|x\right| = \mathbf{1}\right\}$. From $\left|Tx\right| \le \left|T\right|\left|x\right|$ $(x \in X)$ $\left|T\right|$ is an upper bound of D. Clearly, $\left|T\right| \ge \langle Tx \mid x\rangle$ holds for each $\left|x\right| \in \mathfrak{P}(\Lambda)$. Assume that λ is an upper bound of D with $\lambda \le \left|T\right|$. Since $\lambda \ge \langle Ty \mid y\rangle$ holds for all $y \in X$ with $\left|y\right| = \mathbf{1}$ we get that $\lambda I - T$ is positive from Proposition 3.3.5. Thus, if $x \in X$,

$$\langle(\lambda I - T)(Tx) \mid Tx\rangle \ge 0 \text{ and } \langle T(\lambda I - T)x \mid (\lambda I - T)x\rangle \ge 0.$$

Adding, $\langle(\lambda T - T^2)(x) \mid \lambda x\rangle \ge 0$, so that $\lambda^2\langle Tx \mid x\rangle \ge \lambda\langle T^2x \mid x\rangle = \lambda\left|Tx\right|^2$. Thus, $\lambda^2\left|T\right| \ge \lambda\left|T\right|^2$, and so we have $\lambda \ge [\lambda]\left|T\right|$. Moreover, for every $\left|x\right| = \mathbf{1}$ $\pi\langle Tx \mid x\rangle = 0$ holds where $\pi := \mathbf{1} - [\lambda]$. Therefore, $\pi T^{1/2}x = 0$ holds for every $\left|x\right| = \mathbf{1}$, hence $\pi Tx = 0$. This implies $\pi\left|T\right| = 0$, and so $\left|T\right|$ is the supremum of D. □

Now, we recall important definitions that are used in functional representation of type I AW^*-algebras. For details see [23].

Suppose that Q is some extremal compact space, H is a Hilbert space, and $B(H)$ is the space of bounded linear endomorphisms of H. Denote by $\mathfrak{C}(Q, B(H))$ the set of all operator-functions $u : \operatorname{dom}(u) \to B(H)$, defined on the comeager sets $\operatorname{dom}(u) \subset Q$ and continuous in the strong operator topology.

If $u \in \mathfrak{C}(Q, B(H))$ and $h \in H$, then the vector-function $uh : q \mapsto u(q)h$ $(q \in \operatorname{dom}(u))$ is continuous thus determining a unique element $\widetilde{uh} \in C_\infty(Q, H)$ from the condition $uh \in \widetilde{uh}$. Introduce an equivalence on $\mathfrak{C}(Q, B(H))$ by putting $u \sim v$ if and only if u and v agree on $\operatorname{dom}(u) \cap \operatorname{dom}(v)$. If $\widetilde{u}$ is a coset of the operator-function $u : \operatorname{dom}(u) \to B(H)$ then $\widetilde{u}h := \widetilde{uh}$ $(h \in H)$ by definition.

Denote by $SC_\infty(Q, B(H))$ the set of all cosets $\widetilde{u}$ such that $u \in \mathfrak{C}(Q, B(H))$ and the set $\left\{\left|\widetilde{u}h\right| : \|h\| \le 1\right\}$ is bounded in $C_\infty(Q)$.

Since $\left|\widetilde{u}h\right|$ agrees with the function $q \mapsto \|u(q)h\|$ $(q \in \operatorname{dom}(u))$ on some comeager set, the inclusion $\widetilde{u} \in SC_\infty(Q, B(H))$ means that the function $q \mapsto \|u(q)\|$ $(q \in \operatorname{dom}(u))$ is continuous on a comeager set. Hence, there are an element $\left|\widetilde{u}\right| \in C_\infty(Q)$ and a comeager set $Q_0 \subset Q$ satisfing $\left|\widetilde{u}\right|(q) = \|u(q)\|$ $(q \in Q_0)$. Moreover, $\left|\widetilde{u}\right| = \sup\left\{\left|\widetilde{u}h\right| : \|h\| \leq 1\right\}$, where the supremum is taken over $C_\infty(Q)$. Also, $SC_\infty(Q, B(H))$ can be equipped with $*$-algebra and a unitary $C_\infty(Q)$-module.

If $\widetilde{u} \in SC_\infty(Q, B(H))$ and the element $\widetilde{x} \in C_\infty(Q, H)$ is determined by a continuous vector-function $x : \operatorname{dom}(x) \to H$ then we may define $\widetilde{u}\widetilde{x} := \widetilde{ux} \in C_\infty(Q, H)$, with $ux : q \mapsto u(q)x(q)$ $(q \in \operatorname{dom}(u) \cap \operatorname{dom}(x))$; since the last function is continuous on a comeager set. We also have

$$\left|\widetilde{u}x\right| \leq \left|\widetilde{u}\right| \left|x\right| \quad (x \in C_\infty(Q, H)).$$

It follows in particular that

$$\left|\widetilde{u}\right| = \sup\left\{\left|\widetilde{u}x\right| : x \in C_\infty(Q, H),\ \left|x\right| \leq \mathbf{1}\right\}.$$

Denote the operator $x \mapsto \widetilde{u}x$ by $S_{\widetilde{u}}$. We now introduce the following normed $*$-algebra,

$$SC_\#(Q, B(H)) := \left\{v \in SC_\infty(Q, B(H)) : \left|v\right| \in C(Q)\right\},$$

$$\|v\| = \left\|\left|v\right|\right\|_\infty \quad (v \in SC_\#(Q, B(H))).$$

Theorem 3.3.15. [23, Theorem 7.5.10.] *To each operator $U \in \operatorname{End}(C_\#(Q, H))$ there is a unique element $u \in SC_\#(Q, B(H))$ satisfying $U = S_u$. The mapping $U \mapsto u$ is a $*$-B-isomorphism of $\operatorname{End}(C_\#(Q, H))$ onto $A := SC_\#(Q, B(H))$. In particular, A is a λ-homogeneous algebra. Moreover, if Q is a λ-stable compact space then A is a strictly λ-homogeneous AW^*-algebra, with $\lambda = \dim(H)$.*

In preceding theorem, $\operatorname{End}(C_\#(Q, H))$ denotes $B_{C(Q)}(C_\#(Q, H))$.

Observe that if $U = S_{\widetilde{u}}$ is a positive operator, then $u(q)$ is also positive operator for all $q \in \operatorname{dom}(u)$.

Let the family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals satisfy functional representation of X as in Theorem 3.1.18. On the other hand, there exists an isometrically $*$-isomorphism

$$\mathscr{P} : B_A\left(\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))\right) \to \sum_{\gamma\in\Gamma}^{\oplus} \operatorname{End}\left(C_\#(Q_\gamma, \ell_2(\gamma))\right) \tag{3.5}$$

satisfying $Tx = (T_\gamma x_\gamma)_{\gamma\in\Gamma}$ for all $x = (x_\gamma)_{\gamma\in\Gamma}$ and $\mathscr{P}(T) = (T_\gamma)_{\gamma\in\Gamma}$ where $A := \sum_{\gamma\in\Gamma}^{\oplus} C(Q_\gamma)$. Using these facts, we obtain an isometrically $*$-isomorphism

$$\mathscr{R} : B_\Lambda(X) \to \sum_{\gamma\in\Gamma}^{\oplus} \mathrm{End}\left(C_\#\left(Q_\gamma, \ell_2(\gamma)\right)\right) \tag{3.6}$$

with $\Psi(Tx) = (\mathscr{P}^{-1}\mathscr{R}(T))\Psi(x)$ hold for all $T \in B_\Lambda(X)$ and $x \in X$. So, $\mathscr{R}(T) := (T_\gamma)_{\gamma\in\Gamma}$ implies $\Psi(Tx) = (T_\gamma x_\gamma)_{\gamma\in\Gamma}$ for $T \in B_\Lambda(X)$ and $x \in X$ with $\Psi(x) := (x_\gamma)_{\gamma\in\Gamma}$.

The following result on functional representation of type I AW^*-algebras is one of the main tools of our investigation.

Theorem 3.3.16. [23, Theorem 7.5.12.] *To every type I AW^*-algebra A there exists a family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ such that the following conditions are met:*

(1) Γ *is a set of cardinals and Q_γ is γ-stable for each $\gamma \in \Gamma$;*

(2) *There is a $*$-B-isomorphism:*

$$A \simeq \sum_{\gamma\in\Gamma}^{\oplus} SC_\#\left(Q_\gamma, B(\ell_2(\gamma))\right).$$

This family is unique up to congruence.

3.4 Cyclically Compact Operators on Kaplansky–Hilbert Modules

In this section a special class of operators called cyclically compact will be studied. Some properties of these operators have been investigated in [8, 23]. Our result for the main object of our interest which is cyclically compact operator is a standart proof of [23, Theorem 8.5.6.]. An operator $T \in B_\Lambda(X, Y)$ is called *cyclically compact* if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact in Y. The set of all cyclically compact operators is denoted by $\mathscr{K}(X, Y)$.

Theorem 3.4.1. [23, Theorem 8.5.6.] *Let T in $\mathscr{K}(X, Y)$ be a cyclically compact operator from a Kaplansky–Hilbert module X to a Kaplansky–Hilbert module Y. There are orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y, and a family $(\mu_k)_{k\in\mathbb{N}}$ in Λ such that the following hold:*

(1) $\mu_{k+1} \le \mu_k$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim_{k\to\infty} \mu_k = 0$;

(2) *there exists a projection* π_∞ *in* Λ *such that* $\pi_\infty \mu_k$ *is a weak order-unity in* $\pi_\infty \Lambda$ *for all* $k \in \mathbb{N}$;

(3) *there exists a partition* $(\pi_k)_{k=0}^{\infty}$ *of the projection* $\pi_\infty^\perp$ *such that* $\pi_0 \mu_1 = 0$, $\pi_k \le [\mu_k]$, *and* $\pi_k \mu_{k+1} = 0$, $k \in \mathbb{N}$;

(4) *the representation*

$$T = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\sharp \otimes f_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k e_k^\sharp \otimes f_k$$

is valid.

Corollary 3.4.2. *Let* T *in* $\mathscr{K}(X,Y)$. *Then the following statements hold:*

(1) $T = \Big(\pi_\infty + o\text{-}\sum_{n=0}^{\infty} \pi_n\Big)\Big(bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\sharp \otimes f_k\Big) = bo\text{-}\sum_{k\in\mathbb{N}} \mu_k e_k^\sharp \otimes f_k$

(2) $\big|Tx\big| \ge \mu_n \big|x\big|$ *holds for all* $x \in X_0$ *where* X_0 *is Kaplansky–Hilbert submodule generated by* $\{e_1, e_2, ..., e_n\}$,

(3) $T^* = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k f_k^\sharp \otimes e_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k f_k^\sharp \otimes e_k$,

(4) $T^*T = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k^2 e_k^\sharp \otimes e_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k^2 e_k^\sharp \otimes e_k$,

(5) $|T| = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\sharp \otimes e_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k e_k^\sharp \otimes e_k$.

Proposition 3.4.3. *Let* T *be a nonzero positive cyclically compact operator on* X. *Then* $\big|T\big|$ *is a global eigenvalue of* T. *In particular, there exists* $y \in X$ *such that* $Ty = \big|T\big| y$ *and* $\big|y\big| = \mathbf{1}$.

Proof. Let $\big|x\big| = \mathbf{1}$. By Lemma 3.3.14 and

$$\begin{aligned} 0 \le \big|Tx - \big|T\big|x\big|^2 &= \big|Tx\big|^2 - 2\big|T\big| \langle Tx \mid x\rangle + \big|T\big|^2 \\ &\le 2\big|T\big|(\big|T\big| - \langle Tx \mid x\rangle), \end{aligned}$$

we obtain $\inf\big\{\big|Tx - \big|T\big|x\big| : \big|x\big| = \mathbf{1}\big\} = 0$, and therefore using [23, 2.2.9.(1) and 8.1.8.(3)], it is easy to observe that there exists $(x_n)_{n\in\mathbb{N}}$ with $\big|x_n\big| = \mathbf{1}$ such that for each $n \in \mathbb{N}$

$$\big|Tx_n - \big|T\big|x_n\big| \le \frac{1}{n}\mathbf{1}.$$

As T is cyclically compact, there is a cyclic subsequence $(Tx_{\nu_n})_{n\in\mathbb{N}}$ of $(Tx_n)_{n\in\mathbb{N}}$ which is norm-convergent to some w, and since the following inequality is valid:

$$\big|Tx_{\nu_n} - \big|T\big|x_{\nu_n}\big| \le \frac{1}{n}\mathbf{1}$$

for every n, $(\lvert T\rvert x_{\nu_n})_{n\in\mathbb{N}}$ is norm-convergent to w and $\lvert T\rvert = \lvert w\rvert$. So, we have $Tw = \lvert T\rvert w$, i. e., $\lvert T\rvert$ is a global eigenvalue of T. If $[\lvert w\rvert] \neq \mathbf{1}$, then from Proposition 3.3.10 and $[\lvert T\rvert]^{\perp} Tx = 0$ $(x \in X)$ we can find y with $Ty = \lvert T\rvert y$ and $\lvert y\rvert = \mathbf{1}$. □

According to Theorem 3.3.15 for each operator $U \in \mathrm{End}(C_{\#}(Q,H))$ there is a unique element $\widetilde{u} \in SC_{\#}(Q,B(H))$ satisfying $U\widetilde{x} = S_{\widetilde{u}}\widetilde{x} := \widetilde{u}\widetilde{x} := \widetilde{ux}$ $(\widetilde{x} \in C_{\#}(Q,H))$ where $ux : q \mapsto u(q)x(q)$ $(q \in \mathrm{dom}(u) \cap \mathrm{dom}(x))$. Clearly, if U is a positive operator, then $u(q)$ is also positive operator for all $q \in \mathrm{dom}(u)$. The closed unit balls of X and H will be denoted by X^1 and H^1, respectively.

Lemma 3.4.4. *Let X_0 be a Kaplansky–Hilbert submodule of $C_{\#}(Q,H)$. Then $X_0(q) := \{x(q) : \widetilde{x} \in X_0,\ q \in \mathrm{dom}(x)\}$ is a closed subspace of H for all $q \in Q$. In particular, if πX_0 is n-homogenouos over $\pi C(Q)$ for some non zero projection π, then there is a comeager subset Q_0 of Q such that the dimensional of $X_0(q)$ equals to n for all $q \in Q_0 \cap A_\pi$ where A_π is clopen set in Q corresponding to the projection π.*

Proof. Clearly, $X_0(q)$ is subspace of H. Therefore, we will show that $X_0(q)$ is closed in H. Assume that $(\widetilde{x}_n)_{n\in\mathbb{N}} \subset X_0$ and $(x_n(q))_{n\in\mathbb{N}} \subset X_0(q)$ converges to h in H. By [15, Theorem 3.], corresponding to the continuous function $y : t \to h$ $(t \in Q)$, there exist $\widetilde{y}_1 \in X_0$ and $\widetilde{y}_2 \in X_0^{\perp}$ such that $\widetilde{y} = \widetilde{y}_1 + \widetilde{y}_2$. Since $\langle \widetilde{y}_2 \mid \widetilde{x}_n\rangle = 0$ holds for each n, we have

$$\begin{aligned} |\langle h, x_n(q)\rangle| &= |\langle y(q), x_n(q)\rangle| = |\langle \widetilde{y} \mid \widetilde{x}_n\rangle|(q) = |\langle \widetilde{y}_1 \mid \widetilde{x}_n\rangle|(q) \\ &\leq \lvert\widetilde{y}_1\rvert(q)\lvert\widetilde{x}_n\rvert(q) = \lvert\widetilde{y}_1\rvert(q)\,\|x_n(q)\|, \end{aligned}$$

and so taking limits with respect to n, we get $\|h\| \leq \lvert\widetilde{y}_1\rvert(q)$. Moreover, using the inequality

$$\|h\|^2 = \lvert\widetilde{y}\rvert^2(q) = \lvert\widetilde{y}_1\rvert^2(q) + \lvert\widetilde{y}_2\rvert^2(q) \geq \lvert\widetilde{y}_1\rvert^2(q)$$

we see that $\|h\| = \lvert\widetilde{y}_1\rvert(q)$ and $\lvert\widetilde{y}_2\rvert(q) = 0$. Thus, we can assume that $q \in \mathrm{dom}(y_1) \cap \mathrm{dom}(y_2)$. Then $y_2(q) = 0$ and $y_1(q) = h$, i. e., $X_0(q)$ is the closed subspace of H. Let $\{\pi\widetilde{e}_1, \pi\widetilde{e}_2, ..., \pi\widetilde{e}_n\}$ be a basis for πX_0 with $\widetilde{e}_i \in C_{\#}(Q,H)$. Since $\mathrm{dom}(e_i)$ is a comeager in Q,

$$Q_0 := \bigcap_{i=1}^{n} \mathrm{dom}(e_i)$$

is also comeager in Q. Thus, we obtain $\{e_1(q), e_2(q), ..., e_n(q)\}$ is an orthonormal set in $X_0(q)$ for all $q \in Q_0 \cap A_\pi$. Let $x(q)$ be an alement of $X_0(q)$ for some

$q \in Q_0 \cap A_\pi$ with $\widetilde{x} \in X_0$. Thus, it follows from

$$x(q) = \sum_{i=1}^{n} \langle \widetilde{x} \mid \widetilde{e}_i \rangle (q) e_i(q)$$

that $\{e_1(q), e_2(q), ..., e_n(q)\}$ is the basis of $X_0(q)$ for all $q \in Q_0 \cap A_\pi$, as desired. □

Corollary 3.4.5. *Let X_0 be a Kaplansky–Hilbert submodule of $C_\# (Q, H)$ and P be a projection on $C_\# (Q, H)$ with range X_0. If the unique function corresponding to P is denoted by $\widetilde{u} \in SC_\# (Q, B(H))$, then $u(q)$ is a projection with range $X_0(q)$ for all $q \in \mathrm{dom}(u)$. In particular, $(X_0(q))^\perp = X_0^\perp(q)$ $(q \in \mathrm{dom}(u))$.*

Proof. From Lemma 3.4.4 $X_0(q)$ is a closed subspace of H $(q \in Q)$. Let $q \in \mathrm{dom}(u)$ and $h \in H$. Clearly, we see that $u(q)$ is a projection on H and $X_0(q) \subset u(q)(H)$ holds. Moreover, $\widetilde{uz}(q) = u(q)h \in X_0(q)$ holds for the continuous function $z : t \mapsto h$ $(t \in Q)$ since $\widetilde{uz} \in X_0$ and $q \in \mathrm{dom}(uz)$. Therefore, $u(q)$ is a projection on H with range $X_0(q)$ for all $q \in \mathrm{dom}(u)$. Hence, $I - u(q)$ is a projection onto $X_0^\perp(q)$ $(q \in \mathrm{dom}(u))$ since $I - \widetilde{u}$ is projection onto $X_0^\perp$. So, $(X_0(q))^\perp = X_0^\perp(q)$ holds for all $q \in \mathrm{dom}(u)$. □

Proposition 3.4.6. *Let T be a positive compact operator on H and $(s_n(T))$ be the singular number of T. If the set $\{h_i : i = 1, ..., n\}$ which satifies $Th_i = s_i(T)h_i$ and $\|h_i\| = 1$, is linearly independent, then $s_{n+1}(T) = \|TP_n\|$ where P_n is the projection with range $\{h_1, ..., h_n\}^\perp$.*

Proof. From [9, Theorem 15.7.1], we see that $s_{n+1}(T) \leq \|TP_n\|$. Choose $l \in \mathbb{N}$ minimal with respect to the property $s_{n-l+1}(T) \neq s_{n+1}(T)$. On the other hand we can write $T = \sum_{k\in\mathbb{N}} s_k(T) \langle \cdot, e_k \rangle e_k$ where (e_n) is an orthonormal sequence in H. Let $h \in H$. Since $P_n h \in \{e_1, ..., e_{n-l+1}\}^\perp$, it follows from

$$\begin{aligned} \|TP_n h\|^2 &= \sum_{k\in\mathbb{N}} s_k(T)^2 | \langle P_n h, e_k \rangle |^2 = \sum_{k=n-l+2}^{\infty} s_k(T)^2 | \langle P_n h, e_k \rangle |^2 \\ &\leq s_{n-l+2}(T)^2 \sum_{k=n-l+2}^{\infty} | \langle P_n h, e_k \rangle |^2 \leq s_{n-l+2}(T)^2 \| P_n h\|^2 \\ &\leq s_{n-l+2}(T)^2 \|h\|^2 \end{aligned}$$

that $\|TP_n\| \leq s_{n-l+2}(T) = s_{n+1}(T)$. Thus, we have the desired equality. □

Proposition 3.4.7. *Let $U = S_{\widetilde{u}}$ be in $\mathrm{End}\,(C_\# (Q, H))$. Then the following statements are equivalent:*

(i) *$U = S_{\widetilde{u}}$ is a cyclically compact operator on $C_\# (Q, H)$;*

(ii) *there is a comeager subset Q_0 of Q such that $u(q)$ is compact operator on H for all $q \in Q_0$.*

Proof. (i) $\Rightarrow$ (ii) : Let U be a cyclically compact operator on $C_{\#}(Q, H)$. Then $K := U(X^1)$ is relatively cyclically compact subset. By Lemma 3.2.5, there exists a comeager set $Q_1 \subset Q$ such that $K(q)$ is precompact in H for all $q \in Q_1$. Clearly, $u(q)(H^1) \subset K(q)$ holds for each $q \in Q_0 := Q_1 \cap \mathrm{dom}(u)$. Thus, $u(q)$ is compact operator on H for all $q \in Q_0$.

(ii) $\Rightarrow$ (i) : Clearly, $U(X^1)$ is a mix-complete subset. Moreover,

$$U(X^1) \subset \{\widetilde{x} \in C_{\#}(Q, H) : (\forall q \in \mathrm{dom}(x) \cap Q_0)\, x(q) \in K_q\},$$

where $K_q := u(q)(H^1)$ for $q \in Q_0$. Using Proposition 3.2.8 we obtain U is a cyclically compact operator on $C_{\#}(Q, H)$. □

Theorem 3.4.8. *Let $U = S_{\widetilde{u}}$ be a positive cyclically compact operator on $C_{\#}(Q, H)$. There exist a sequence $(\widetilde{e}_k)_{k\in\mathbb{N}}$ in $C_{\#}(Q, H)$ and a sequence $(s_k)_{k\in\mathbb{N}}$ in $C(Q)$ such that the following statements hold:*

(1) *$\langle \widetilde{e}_k \mid \widetilde{e}_l \rangle = 0$ $(k \neq l)$, and $\left| \widetilde{e}_k \right| = \mathbf{1}$ whenever $\widetilde{e}_k \neq 0$;*

(2) *$0 \leq s_{k+1} \leq s_k$ $(k \in \mathbb{N})$ and o-$\lim s_k = \inf_{k\in\mathbb{N}} s_k = 0$;*

(3) *there exists a projection π_∞ in $\mathfrak{C}(\mathbf{1})$ such that $\pi_\infty s_k$ is a weak order-unity in $\pi_\infty C(Q)$ for all $k \in \mathbb{N}$;*

(4) *there exists a partition $(\pi_k)_{k=0}^{\infty}$ of the projection $\pi_\infty^\perp$ such that $\pi_0 s_1 = 0$, $\pi_k \leq [s_k]$, and $\pi_k s_{k+1} = 0$, $k \in \mathbb{N}$;*

(5) *for each $\widetilde{x} \in C_{\#}(Q, H)$ the following equality is valid*

$$\begin{aligned} U\widetilde{x} &= \pi_\infty bo\text{-}\sum_{k=1}^{\infty} s_k \langle \widetilde{x} \mid \widetilde{e}_k \rangle \widetilde{e}_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} s_k \langle \widetilde{x} \mid \widetilde{e}_k \rangle \widetilde{e}_k \\ &= bo\text{-}\sum_{k\in\mathbb{N}} s_k \langle \widetilde{x} \mid \widetilde{e}_k \rangle \widetilde{e}_k. \end{aligned}$$

Proof. If $U = 0$, we can take $\widetilde{e}_n = 0$, $s_n = 0$, $\pi_0 = \mathbf{1}$ and $\pi_k = 0$ $(k \in \mathbb{N} \cup \{\infty\})$. Assume that $U \neq 0$. The proof is by induction. According to Corollary 3.4.5 and Propositions 3.4.3 and 3.4.7, there exists a comeager subset Q_1 such that for every $q \in Q_1$ the following statements hold:

(i) $u(q)$ is a positive compact operator on H,

(ii) there is $\left|\widetilde{e}_1\right| = \mathbf{1}$ such that $u(q)e_1(q) = s_1(q)e_1(q)$, i. e., $U\widetilde{e}_1 = s_1\widetilde{e}_1$ where $s_1 := \left|U\right|$,

(iii) there is a projection $P_1 = S_{\widetilde{u}_1}$ with range $\{\widetilde{e}_1\}^\perp$ such that $u_1(q)$ is a projection with range $\{e_1(q)\}^\perp$,

(iv) $\left|U\right|(q) = \|u(q)\|$, and so $s_1(q) = s_{1,q}$ where $(s_{k,q})_{k\in\mathbb{N}}$ is the singular number of $u(q)$.

Now if $\widetilde{y} \in \{\widetilde{e}_1\}^\perp$ and $\widetilde{z} \in \{\widetilde{e}_1\}^{\perp\perp}$, then

$$\langle U\widetilde{y} \mid \widetilde{e}_1\rangle = \langle \widetilde{y} \mid U\widetilde{e}_1\rangle = s_1 \langle \widetilde{y} \mid \widetilde{e}_1\rangle = 0,$$

i. e., U leaves $\{\widetilde{e}_1\}^\perp$ invariant. Thus,

$$\langle U\widetilde{z} \mid \widetilde{y}\rangle = \langle \widetilde{z} \mid U\widetilde{y}\rangle = 0$$

implies that U leaves $\{\widetilde{e}_1\}^{\perp\perp}$ invariant, and so $P_1U = UP_1$ holds. If $UP_1 = 0$, we can take $\widetilde{e}_n = 0$, $s_n = 0$ and $\pi_n = 0$ $(n > 1)$, $\pi_\infty = 0$, $\pi_0 = [s_1]^\perp$ and $\pi_1 = [s_1]$. Suppose that s_k, $\widetilde{e}_k$, $P_k = S_{\widetilde{u}_k}$ and Q_k are constructed and $UP_k \neq 0$ for all $k < n$. So, since UP_n is positive cyclically compact, again using to Corollary 3.4.5 and Propositions 3.4.3, 3.4.6 and 3.4.7, there exists a comeager subset $Q_{n+1} \subset Q_n$ such that for every $q \in Q_{n+1}$ the following statements hold:

(i) $u(q)u_n(q)$ is positive compact operator on H,

(ii) there exists $\widetilde{e}_{n+1} \in \{\widetilde{e}_1, ..., \widetilde{e}_n\}^\perp$ such that $\left|\widetilde{e}_{n+1}\right| = \mathbf{1}$ and $U\widetilde{e}_{n+1} = s_{n+1}\widetilde{e}_{n+1}$ where $s_{n+1} := \left|UP_n\right|$,

(iii) there is a projection $P_{n+1} = S_{\widetilde{u}_{n+1}}$ with range $\{\widetilde{e}_1, ..., \widetilde{e}_{n+1}\}^\perp$ such that $u_{n+1}(q)$ is a projection with range $\{e_1(q), ..., e_{n+1}(q)\}^\perp$,

(iv) $s_{n+1}(q) = \left|UP_n\right|(q) = \|u(q)u_n(q)\| = s_{n+1,q}$.

Clearly, we can see that $\{\widetilde{e}_1, ..., \widetilde{e}_{n+1}\}^\perp$ and $\{\widetilde{e}_1, ..., \widetilde{e}_{n+1}\}^{\perp\perp}$ are invariant under U. If $UP_{n+1} = 0$ we can take $\widetilde{e}_k = 0$, $s_k = 0$ and $\pi_k = 0$ $(k > n + 1)$, $\pi_\infty = 0$, $\pi_0 = [s_1]^\perp$ and $\pi_i = [s_i]\wedge[s_{i+1}]^\perp$ $(i = 1, ..., n)$ and $\pi_{n+1} = [s_{n+1}]$. If for each $k \in \mathbb{N}$, $UP_k \neq 0$, then by induction, we have a sequence $(\widetilde{e}_k)_{k\in\mathbb{N}}$ in $C_\#(Q, H)$, a sequence $(s_k)_{k\in\mathbb{N}}$ of positive functions, a sequence $(P_k = S_{\widetilde{u}_k})$ of projections with range $\{\widetilde{e}_1, ..., \widetilde{e}_k\}^\perp$ and a decreasing comeager-set-sequence (Q_k). Thus, $\pi_\infty = \bigwedge_{k\in\mathbb{N}} [s_k]$, $\pi_0 = [s_1]^\perp$ and $\pi_k = [s_k] \wedge [s_{k+1}]^\perp$ $(k \in \mathbb{N})$ implies (1), (3) and (4). If we define $Q_0 := \bigcap_{k\in\mathbb{N}} Q_k$, then $s_k(q) \downarrow 0$ for all $q \in Q_0$ implies (2). Moreover, if $\langle \widetilde{x} \mid \widetilde{e}_k\rangle = 0$

holds for each k, then it follows from $\left|U\widetilde{x}\right| = \left|UP_k\widetilde{x}\right| \leq \left|UP_k\right| \left|\widetilde{x}\right| = s_{k+1}\left|\widetilde{x}\right|$ that $U\widetilde{x} = 0$. Assume that $\mathscr{E}$, which contains $(\widetilde{e}_k)_{k\in\mathbb{N}}$, is a projection basis. Then for all $\widetilde{x} \in C_{\#}(Q, H)$

$$U\widetilde{x} = U\left(bo\text{-}\sum_{\widetilde{e}\in\mathscr{E}} \langle\widetilde{x} \mid \widetilde{e}\rangle\,\widetilde{e}\right) = bo\text{-}\sum_{\widetilde{e}\in\mathscr{E}} \langle\widetilde{x} \mid \widetilde{e}\rangle\,U\widetilde{e}$$
$$= bo\text{-}\sum_{k\in\mathbb{N}} \langle\widetilde{x} \mid \widetilde{e}_k\rangle\,U\widetilde{e}_k = bo\text{-}\sum_{k\in\mathbb{N}} s_k\,\langle\widetilde{x} \mid \widetilde{e}_k\rangle\,\widetilde{e}_k$$

implies (5). □

Note that the cardinality of e_k satisfying $\left|e_k\right| = \mathbf{1}$ is related with the dimension of H.

Let the family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals satisfy functional representation of X as in Theorem 3.1.18. Suppose that $(f_\alpha)_{\alpha\in I}$ is a net in $\sum^{\oplus}_{\gamma\in\Gamma} C(Q_\gamma)$ with $f_\alpha = (f_{\alpha,\gamma})_{\gamma\in\Gamma}$. Clearly, we see that the net $(f_\alpha)_{\alpha\in I}$ is decreasing and $\inf_{\alpha\in I} f_\alpha = 0$ iff the net $(f_{\alpha,\gamma})_{\alpha\in I}$ is decreasing and $\inf_{\alpha\in I} f_{\alpha,\gamma} = 0$ for all $\gamma \in \Gamma$. Therefore, if $(e_\alpha)_{\alpha\in I}$ is a bounded net in $\sum^{\oplus}_{\gamma\in\Gamma} C(Q_\gamma)$ with $e_\alpha = (e_{\alpha,\gamma})_{\gamma\in\Gamma}$, then $(e_\alpha)_{\alpha\in I}$ o-converges to $e = (e_\gamma)_{\gamma\in\Gamma}$ in $\sum^{\oplus}_{\gamma\in\Gamma} C(Q_\gamma)$ iff $(e_{\alpha,\gamma})_{\alpha\in I}$ o-converges to e_γ in $C(Q_\gamma)$ and $\sup_{\gamma\in\Gamma}\|e_\gamma\| < \infty$ for all $\gamma \in \Gamma$. Moreover, $[a] = ([a_\gamma])_{\gamma\in\Gamma}$ is satisfied for all $a = (a_\gamma)_{\gamma\in\Gamma} \in \sum^{\oplus}_{\gamma\in\Gamma} C(Q_\gamma)$.

Let $\upsilon = (\upsilon(k))_{k\in\mathbb{N}}$ be a sequence in $\sum^{\oplus}_{\gamma\in\Gamma} C(Q_\gamma)$ with $\upsilon(k) = (\upsilon_\gamma(k))_{\gamma\in\Gamma}$. Then $\upsilon = (\upsilon(k))_{k\in\mathbb{N}}$ is a partition of unity in $\prod_{\gamma\in\Gamma} \mathfrak{C}(\mathbf{1}_\gamma)$ iff $\upsilon^\gamma := (\upsilon_\gamma(k))_{k\in\mathbb{N}}$ is a partition of unity in $\mathfrak{C}(\mathbf{1}_\gamma)$ for all $\gamma \in \Gamma$. Moreover, if $\upsilon = (\upsilon(k))_{k\in\mathbb{N}}$ and $\mu = (\mu(k))_{k\in\mathbb{N}}$ are partitions of unity in $\prod_{\gamma\in\Gamma} \mathfrak{C}(\mathbf{1}_\gamma)$, then $\upsilon \ll \mu$ iff $\upsilon^\gamma \ll \mu^\gamma$ for all $\gamma \in \Gamma$. Given $\xi \in \Gamma$, denote by $h_\xi := (\delta_{\gamma\xi}\mathbf{1}_\gamma)_{\gamma\in\Gamma}$ the element of $\prod_{\gamma\in\Gamma} \mathfrak{C}(\mathbf{1}_\gamma)$ with $\mathbf{1}_\xi$ the ξth place and 0's elsewhere.

Let Ψ, $\mathscr{P}$ and $\mathscr{R}$ be as (3.3), (3.5) and (3.6), respectively, Then T is a cyclically compact operator on X iff $\mathscr{P}^{-1}\mathscr{R}(T) = \Psi T \Psi^{-1}$ is a cyclically compact operator on $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$.

Proposition 3.4.9. *Let T be an operator on $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$ with $\mathscr{P}(T) = (T_\gamma)_{\gamma\in\Gamma}$. Then T is a cyclically compact operator on $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$ iff T_γ is a cyclically compact operator on $C_{\#}(Q_\gamma, \ell_2(\gamma))$ for all $\gamma \in \Gamma$.*

Proof. Suppose that T is a cyclically compact operator on $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$ and $(x_{k,\xi})_{k\in\mathbb{N}}$ is a bounded sequence in $C_{\#}(Q_\xi, \ell_2(\xi))$ for some $\xi \in \Gamma$. If x_k is an element of $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$ with $x_{k,\xi}$ the ξth place and 0's elsewhere, it is clear that $|||x_k||| = \|x_{k,\xi}\|$ and $(x_k)_{k\in\mathbb{N}}$ is also bounded in $\sum^{\oplus}_{\gamma\in\Gamma} C_{\#}(Q_\gamma, \ell_2(\gamma))$. As T is cyclically compact operator, there is a cyclic subsequence $(Tx_{\nu_k})_{k\in\mathbb{N}}$ of $(Tx_k)_{k\in\mathbb{N}}$ which is norm-convergent to some x. Since $h_\xi x_k = x_k$, we have $h_\xi x_{\nu_k} = x_{\nu_k}$ and $h_\xi x = x$. Thus, the following equality holds,

$$|||Tx_{\nu_k} - x||| = |||h_\xi(Tx_{\nu_k} - x)||| = \|T_\xi x_{\nu_k^\xi,\xi} - x_\xi\|$$

since $x_{\nu_k^\xi,\xi} = \operatorname{mix}_{n\in\mathbb{N}}(\nu_{k,\xi}(n)x_{\xi,n})$ with $\nu_k^\xi := (\nu_{k,\xi}(n))_{n\in\mathbb{N}}$, and so we get a cyclic subsequence $(T_\xi x_{\nu_k^\xi,\xi})_{k\in\mathbb{N}}$ of $(T_\xi x_{k,\xi})_{k\in\mathbb{N}}$ which is norm-convergent to x_ξ. Therefore, T_ξ is a cyclically compact operator on $C_\#(Q_\xi, \ell_2(\xi))$.

Conversely, assume that T_γ is a cyclically compact on $C_\#(Q_\gamma, \ell_2(\gamma))$ for all $\gamma \in \Gamma$ and $(x_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$ with $x_k = (x_{k,\gamma})_{\gamma\in\Gamma}$. Since $(x_{k,\gamma})_{k\in\mathbb{N}}$ is a bounded sequence and T_γ is cyclically compact, there is a cyclic subsequence $(T_\gamma x_{\nu_{k,\gamma}})_{k\in\mathbb{N}}$ of $(T_\gamma x_{k,\gamma})_{k\in\mathbb{N}}$ which is norm-convergent to some x_γ. Moreover, we can assume

$$\|T_\gamma x_{\nu_{k,\gamma}} - x_\gamma\| \le \frac{1}{k}$$

for all $\gamma \in \Gamma$ and $k \in \mathbb{N}$. Clearly, we see that $x := (x_\gamma)_{\gamma\in\Gamma}$ in $\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$ and

$$|||Tx_{\nu_k} - x||| = \sup_{\gamma\in\Gamma} \|T_\gamma x_{\nu_{k,\gamma}} - x_\gamma\| \le \frac{1}{k}$$

since $x_{\nu_k} = \left(x_{\nu_{k,\gamma}}\right)_{\gamma\in\Gamma} = (\operatorname{mix}_{n\in\mathbb{N}}(\nu_{k,\gamma}(n)x_{n,\gamma}))_{\gamma\in\Gamma}$ where $\nu_n = (\nu_n(k))_{k\in\mathbb{N}}$ and $\nu_n(k) = (\nu_{n,\gamma}(k))_{\gamma\in\Gamma}$. So, T is a cyclically compact operator on $\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$.

□

Let T be a positive operator on $\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$. This means that T_γ is a positive operator on $C_\#(Q_\gamma, \ell_2(\gamma))$ for all $\gamma \in \Gamma$. If T is a positive cyclically compact operator on $\sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$, then according to Theorem 3.4.8 and Proposition 3.4.9 we can define $s_k(T) := (s_k(T_\gamma))_{\gamma\in\Gamma}$, $e_k := (\widetilde{e}_{k,\gamma})_{\gamma\in\Gamma}$ and $\pi_k := (\pi_{k,\gamma})_{\gamma\in\Gamma}$ $(k \in \mathbb{N} \cup \{0, \infty\})$ where $(s_k(T_\gamma))_{k\in\mathbb{N}}$, $(\pi_{k,\gamma})$ and $(\widetilde{e}_{k,\gamma})_{k\in\mathbb{N}}$ satisfy the representation of cyclically compact operator T_γ as in Theorem 3.4.8. Therefore, from the functional representation of X we have the following theorem.

Proposition 3.4.10. *Let T be a positive cyclically compact operator on X. There exist a sequence $(e_k)_{k\in\mathbb{N}}$ in X and a sequence $(s_k(T))_{k\in\mathbb{N}}$ of positive elements in Λ such that*

(1) $\langle e_k \mid e_l \rangle = 0$ $(k \ne l)$ *and* $[s_k(T)] \le \left| e_k \right| \in \mathfrak{P}(\Lambda)$*;*

(2) $s_{k+1}(T) \le s_k(T)$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim s_k(T) = \inf_{k\in\mathbb{N}} s_k(T) = 0$*;*

(3) *there exists a projection* π_∞ *in* $\mathfrak{P}(\Lambda)$ *such that* $\pi_\infty s_k(T)$ *is a weak order-unity in* $\pi_\infty\Lambda$ *for all* $k \in \mathbb{N}$*;*

(4) *there exists a partition* $(\pi_k)_{k=0}^\infty$ *of the projection* $\pi_\infty^\perp$ *such that* $\pi_0 s_1(T) = 0$, $\pi_k \leq [s_k(T)]$*, and* $\pi_k s_{k+1}(T) = 0$, $k \in \mathbb{N}$*;*

(5) *for each* x *the following equality is valid*

$$Tx = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} s_k(T)\langle x \mid e_k\rangle e_k + bo\text{-}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n} s_k(T)\langle x \mid e_k\rangle e_k$$
$$= bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle x \mid e_k\rangle e_k.$$

Using the Polar Decomposition for T and the preceding theorem, we obtain Theorem 3.4.1 as follows.

Theorem 3.4.11. *Let* T *be a cyclically compact operator from* X *to* Y*. There exist sequences* $(e_k)_{k\in\mathbb{N}}$ *in* X *and* $(f_k)_{k\in\mathbb{N}}$ *in* Y *and a sequence* $(s_k(T))_{k\in\mathbb{N}}$ *of positive elements in* Λ *such that*

(1) $\langle e_k \mid e_l\rangle = \langle f_k \mid f_l\rangle = 0$ $(k \neq l)$ *and* $[s_k(T)] = \lvert e_k\rvert = \lvert f_k\rvert$ $(k \in \mathbb{N})$

(2) $s_{k+1}(T) \leq s_k(T)$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim s_k(T) = \inf_{k\in\mathbb{N}} s_k(T) = 0$*;*

(3) *there exists a projection* π_∞ *in* $\mathfrak{P}(\Lambda)$ *such that* $\pi_\infty s_k(T)$ *is a weak order-unity in* $\pi_\infty\Lambda$ *for all* $k \in \mathbb{N}$*;*

(4) *there exists a partition* $(\pi_k)_{k=0}^\infty$ *of the projection* $\pi_\infty^\perp$ *such that* $\pi_0 s_1(T) = 0$, $\pi_k \leq [s_k(T)]$*, and* $\pi_k s_{k+1}(T) = 0$, $k \in \mathbb{N}$*;*

(5) *for each* x *the following equality is valid*

$$Tx = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} s_k(T)\langle x \mid e_k\rangle f_k + bo\text{-}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n} s_k(T)\langle x \mid e_k\rangle f_k$$
$$= bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle x \mid e_k\rangle f_k.$$

Proof. Using the Polar Decomposition for T we obtain a partial isometry V such that $T = V|T|$ and $|T| = V^*T$. Since $|T|$ is a positive cyclically compact operator on X, there exist decreasing null sequence $(s_k(T))_{k\in\mathbb{N}}$ in Λ and family $(e_k)_{k\in\mathbb{N}}$ in X which satisfy the properties of Proposition 3.4.10. Thus, we have

$$T = V|T| = V\left(bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle\cdot \mid e_k\rangle e_k\right) = bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle\cdot \mid e_k\rangle V(e_k).$$

From $s_k(T)e_k \in |T|(X)$ and $[s_k(T)]^{\perp} e_k \in \operatorname{Ker}|T|$, we see that $\left|Vs_k(T)e_k\right| = \left|s_k(T)e_k\right| = s_k(T)$ and $[s_k(T)]^{\perp} Ve_k = V[s_k(T)]^{\perp} e_k = 0$. The former means $[s_k(T)]\left|Ve_k\right| = [s_k(T)]$, and so this and the latter imply that $\left|Ve_k\right| = [s_k(T)]$ $(k \in \mathbb{N})$. Since V^*V is the projection onto $(\operatorname{Ker}|T|)^{\perp} = \operatorname{cl}(\operatorname{mix}(|T|(X)))$, we get

$$s_n(T)\langle Ve_n \mid Ve_m\rangle = \langle V^*Vs_n(T)e_n \mid e_m\rangle = \langle s_n(T)e_n \mid e_m\rangle = 0 \quad (n \neq m),$$

and so $\langle Ve_n \mid Ve_m\rangle = [s_n(T)]\langle Ve_n \mid Ve_m\rangle = 0$. Define $f_k := Ve_k$, and the proof is finished. □

Theorem 3.4.12. (The Rayleigh−Ritz minimax formula) *Let T be a cyclically compact operator from X to Y. Then*

$$s_n(T) = \inf\left\{\sup\left\{\left|Tx\right| : \left|x\right| \leq \mathbf{1},\, x \in J^{\perp}\right\}\right\}$$

where the infmum is taken over all projection orthonormal subset J of X such that $\operatorname{card}(J) < n$*, and the infmum is achieved.*

Proof. Let $\alpha_n := \inf\left\{\sup\left\{\left|Tx\right| : \left|x\right| \leq \mathbf{1},\, x \in J^{\perp}\right\} : \operatorname{card}(J) < n\right\}$ where J is a projection orthonormal subset of X. If $\mathscr{E}_{n-1} := \{e_1, e_2, ..., e_{n-1}\}$, then $\left|Tx\right| \leq s_n(T)$ are satisfied for each $x \in \mathscr{E}_{n-1}^{\perp}$ with $\left|x\right| \leq \mathbf{1}$, and so $\alpha_n \leq s_n(T)$. Suppose that J is a projection orthonormal subset of X with $\operatorname{card}(J) < n$. If $x \in \mathscr{E}_n^{\perp\perp}$, then $\left|Tx\right| \geq s_n(T)\left|x\right|$. From Corollary 3.1.17, there exists $x \in \mathscr{E}_n^{\perp\perp} \cap J^{\perp}$ with $\left|x\right| = \left|e_n\right|$. Thus, $\left|Tx\right| \geq s_n(T)\left|x\right| = s_n(T)$, and so $\alpha_n \geq s_n(T)$. Finally, the infimum is achieved on $\mathscr{E}_{n-1}^{\perp}$. □

Let $x \in X$, $y \in Y$. Define the operator $\theta_{x,y} : X \to Y$ by the formula

$$\theta_{x,y}(z) := \langle z \mid x\rangle\, y \qquad (z \in X).$$

An operator of the this form is called an *elementary operator* [27]. We denote the Λ-linear span of the set of all elementary operators by $\mathscr{E}(X, Y)$. Clearly, $\mathscr{E}(X, Y) \subset B_{\Lambda}(X, Y)$ and the following equalities are satisfied;

(i) $(\theta_{x,y})^* = \theta_{y,x}$;

(ii) $\theta_{x,y}\theta_{v,u} = \theta_{\langle x|u\rangle v,y} = \theta_{v,\langle u|x\rangle y}$ $(u \in X,\, v \in Y)$;

(iii) $T\theta_{x,y} = \theta_{x,Ty}$ $(T \in B_{\Lambda}(Y, Z))$;

(iv) $\theta_{x,y}S = \theta_{S^*x,y}$ $(S \in B_{\Lambda}(Z, X))$.

Clearly, $S \in B_{\Lambda}(Z, X)$, $T \in B_{\Lambda}(Y, Z)$ and $L \in \mathscr{E}(X, Y)$ implies $TL \in \mathscr{E}(X, Z)$ and $LS \in \mathscr{E}(Z, Y)$.

Lemma 3.4.13. *Let X and Y be Kaplansky–Hilbert modules over Λ. Then the following hold:*

(i) $\mathscr{E}(X,Y) \subset \mathscr{K}(X,Y)$ *;*

(ii) $\mu_n = 0$ *holds for* $n \geq 2$ *where* $(\mu_n)_{n\in\mathbb{N}}$ *satisfy representation of* $\theta_{x,y}$ *as in Theorem 3.4.1;*

(iii) $\left|\theta_{x,y}\right| = \left|x\right| \left|y\right|$ $(x \in X, y \in Y)$*;*

(iv) *for each* $T \in \mathscr{K}(X,Y)$ *there exists a sequence* $(T_k)_{k\in\mathbb{N}}$ *in* $\mathscr{E}(X,Y)$ *which* $\left(\left|T - T_k\right|\right)_{k\in\mathbb{N}}$ *decreases to* 0.

Proof. To show (i), it suffices to show that $\theta_{x,y} \in \mathscr{K}(X,Y)$ for all $x \in X$ and $y \in Y$. As a corollary of Proposition 3.2.8 the closed unit ball of Λ is cyclically compact set, so $\theta_{x,y} \in \mathscr{K}(X,Y)$. From (i), $\theta_{x,y}$ has a representation

$$\theta_{x,y} = bo\text{-}\sum_{n\in\mathbb{N}} \mu_n \langle \cdot \mid e_n \rangle f_n$$

as in Theorem 3.4.1. Since $\theta_{x,y}(e_n) = \mu_n f_n = \langle e_n \mid x \rangle y$ $(n \in \mathbb{N})$, we have $\mu_n = \mu_n \left|f_n\right| = |\langle e_n \mid x \rangle| \left|y\right|$ and

$$\begin{aligned} 0 &= |\langle \mu_1 f_1 \mid \mu_k f_k \rangle| = |\langle \langle e_1 \mid x \rangle y \mid \langle e_k \mid x \rangle y \rangle| \\ &= |\langle e_1 \mid x \rangle| \left|y\right| |\langle e_k \mid x \rangle| \left|y\right| \\ &= \mu_1 \mu_k. \end{aligned}$$

is satisfied for $k \neq 1$. Thus, it follows from $\mu_1 \geq \mu_n$ that $\mu_n = 0$ for $n \geq 2$. This implies (ii). Now for any $z \in X$, it follows from

$$\left|\theta_{x,y}(z)\right| = \left|\langle z \mid x \rangle y\right| = |\langle z \mid x \rangle| \left|y\right| \leq \left|x\right| \left|y\right| \left|z\right|$$

that $\left|\theta_{x,y}\right| \leq \left|x\right| \left|y\right|$. This and $\theta_{x,y}(x) = \left|x\right|^2 y$ imply (iii). From Theorem 3.4.1, for all $T \in \mathscr{K}(X,Y)$, we can define $T_n \in \mathscr{E}(X,Y)$ as

$$T_n := \sum_{k=1}^{n} \mu_k \langle \cdot \mid e_k \rangle f_k$$

for every $n \in \mathbb{N}$. Using the following inequality for each $x \in X$

$$\begin{aligned} \left|Tx - T_{n-1}x\right|^2 &= o\text{-}\sum_{k\in\mathbb{N}_n} \mu_k^2 |\langle x \mid e_k \rangle|^2 \leq \mu_{n+1}^2 o\text{-}\sum_{k\in\mathbb{N}_n} |\langle x \mid e_k \rangle|^2 \\ &\leq \mu_n^2 \left|x\right|^2 \end{aligned}$$

where $\mathbb{N}_n = \{k \in \mathbb{N} : k \geq n\}$ $(n \in \mathbb{N})$, we get $\left|T - T_{n-1}\right| \leq \mu_n$. Thus, $\left(\left|T - T_k\right|\right)_{k\in\mathbb{N}}$ decreases to 0 from Theorem 3.4.1 (1). □

Theorem 3.4.14. *Let T be in $B_\Lambda(X)$ and Θ denote the set of all finite subsets of projection basis $\mathscr{E}$. Then the following statements are equivalent.*

(i) *T is a cyclically compact operator on X;*

(ii) *for all projection basis $\mathscr{E}$ in X, the net $\left(\left|T(I-P_F)\right|\right)_{F\in\Theta}$ o-converges to 0 where $P_F := \sum_{e\in F}\theta_{e,e}$;*

(iii) *for all projection basis $\mathscr{E}$ in X $\left(\sup_{e\in F^c}\left\{\left|Te\right|\right\}\right)_{F\in\Theta}$ decreases to 0;*

(iv) *for all projection basis $\mathscr{E}$ in X $(\sup_{e\in F^c}\{|\langle Te\mid e\rangle|\})_{F\in\Theta}$ decreases to 0.*

Proof. (i) $\Rightarrow$ (ii) : Define $T_n := \sum_{k=1}^{n}\mu_k\theta_{e_k,f_k}$ as in above proof. So, the following inequality holds

$$\begin{aligned}\left|\theta_{e_k,f_k}(I-P_F)x\right|^2 &= \left|\langle (I-P_F)x\mid e_k\rangle f_k\right|^2\\ &= \langle\langle (I-P_F)x\mid e_k\rangle f_k \mid \langle (I-P_F)x\mid e_k\rangle f_k\rangle\\ &= \left|f_k\right|^2|\langle (I-P_F)x\mid e_k\rangle|^2 = \left|f_k\right|^2|\langle x\mid (I-P_F)e_k\rangle|^2\\ &\le \left|f_k\right|^2\left|x\right|^2\left|(I-P_F)e_k\right|^2.\end{aligned}$$

for every $x\in X$ from which it follows that

$$\left|\theta_{e_k,f_k}(I-P_F)\right| \le \left|f_k\right|\left|(I-P_F)e_k\right| \quad (k\in\mathbb{N}).$$

Thus, since $((I-P_F)e_k)_{F\in\Theta}$ *bo*-converges to 0 for each $k\in\mathbb{N}$ (Lemma 3.1.11), the same is true for $(\theta_{e_k,f_k}(I-P_F))_{F\in\Theta}$, hence, for $(T_n(I-P_F))_{F\in\Theta}$. Therefore, result now follows from $\left|T-T_n\right|\xrightarrow{(o)}0$.

(ii) $\Rightarrow$ (iii) : Let $F\in\Theta$ and $e\in\mathscr{E}$. Then, it follows from

$$\left|Te\right| \le \left|TP_Fe\right| + \left|T(I-P_F)\right|\left|e\right| \le \left|TP_Fe\right| + \left|T(I-P_F)\right|$$

that $\left|Te\right| \le \left|T(I-P_F)\right|$ $(e\in F^c)$. This implies (iii).

(iii) $\Rightarrow$ (iv) : The proof follows from $|\langle Te\mid e\rangle| \le \left|Te\right|\left|e\right| \le \left|Te\right|$ for all $e\in\mathscr{E}$.

(iv) $\Rightarrow$ (i) : Given a positive integer n and a nonzero projection $\pi\in\mathfrak{P}(\Lambda)$, consider the class S_π of all projection orthonormal sets in X for which

$$\pi|\langle Te\mid e\rangle| \ge \frac{1}{4n}\pi \quad (e\in A)$$

(we allow the empty set as one possible choice of A). By (iii), πA is a finite set for each A in S_π. The inclusion ordered set S_π clearly obeys the hypotheses of Kuratowski$-$Zorn Lemma. Therefore, there is a maximal element $A_0\in S_\pi$.

So, there exists a projection $\mu \in \mathfrak{P}(\Lambda)$ with $0 < \mu \leq \pi$ and $|\langle Tx \mid x\rangle| \leq \frac{1}{4n}\mu$ whenever $x \in \mu A_0^{\perp}$ and $\lvert x\rvert \in \mathfrak{P}(\Lambda)$; for otherwise this contradicts the maximality of A_0. By the Exhaustion Principle there exist a partition of unity (π_α) and the family of finite projection orthonormal sets (A_α) such that $|\langle Tx \mid x\rangle| \leq \frac{1}{4n}\pi_\alpha$ whenever $x \in \pi_\alpha A_\alpha^{\perp}$ and $\lvert x\rvert \in \mathfrak{P}(\Lambda)$. Therefore, it follows from identity (3.1) that

$$|\langle Tu \mid v\rangle| \leq \frac{1}{n}\pi_\alpha$$

for all $u, u \in \pi_\alpha A_\alpha^{\perp}$, $\lvert u\rvert \leq \mathbf{1}$ and $\lvert v\rvert \leq \mathbf{1}$. By taking $u = (I - P_\alpha)x$ and $v = (I-P_\alpha)y$, where P_α is the projection from X to $A_\alpha^{\perp\perp}$, i. e., $P_\alpha := \sum_{e\in A_\alpha}\theta_{e,e} \in \mathscr{E}(X)$ we deduce that

$$\pi_\alpha|\langle (I - P_\alpha)T(I - P_\alpha)x \mid y\rangle| \leq \frac{1}{n}\pi_\alpha$$

whenever $x, y \in X$, $\lvert x\rvert \leq \mathbf{1}$ and $\lvert y\rvert \leq \mathbf{1}$. Thus, $\pi_\alpha\lvert(I - P_\alpha)T(I - P_\alpha)\rvert \leq \frac{1}{n}\pi_\alpha$. The operator $F_{\alpha,n} := P_\alpha T + TP_\alpha - P_\alpha TP_\alpha$ is in $\mathscr{E}(X)$ and $\lvert\pi_\alpha T - \pi_\alpha F_{\alpha,n}\rvert \leq \frac{1}{n}\pi_\alpha$. Since $\mathscr{K}(X)$ is a Banach−Kantorovich space there exists a cyclically compact operator $F_n = bo\text{-}\sum_\alpha \pi_\alpha F_{\alpha,n}$ with

$$\lvert T - F_n\rvert \leq \frac{1}{n}\mathbf{1}.$$

Again using $\mathscr{K}(X)$ is a Banach−Kantorovich space we obtain (iii) implies (i). □

Observe that if T is a positive cyclically compact operator, then $T^{1/2}$ is also.

Chapter 4

The Schatten Type Classes of Operators in Kaplansky−Hilbert Modules

In this chapter, we generalize the Schatten−von Neumann classes of operators on a Hilbert space which were introduced by von Neumann and Schatten [38]. There are two particularly important classes, the trace class and the Hilbert−Schmidt class. We investigate the classes $\mathscr{S}_p$ and get duality results for the Schatten-type classes. Throughout this chapter, the letters X and Y denote Kaplansky−Hilbert modules over Λ, and orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y and family $(\mu_k)_{k\in\mathbb{N}}$ in Λ will stand for the representation of the cyclically compact operator T as given in Theorem 3.4.1.

4.1 The Hilbert−Schmidt Class

In this section, we generalize Hilbert−Schmidt operators on a Hilbert space and study several properties of Hilbert−Schmidt operators on a Kaplansky−Hilbert modules. Some equivalent characterizations are given and we show that the Hilbert−Schmidt class is a Kaplansky−Hilbert module.

Proposition 4.1.1. *Let T be an element of $\mathscr{K}(X,Y)$. The following conditions are equivalent:*

(i) *for every projection basis $\mathscr{E}$ in X the family $\left(|Te|^2\right)_{e\in\mathscr{E}}$ is o-summable;*

(ii) *for some projection basis $\mathscr{E}$ in X the family $\left(|Te|^2\right)_{e\in\mathscr{E}}$ is o-summable;*

(iii) *$(\mu_k^2)_{k\in\mathbb{N}}$ is o-summable.*

In particular, if the family $\left(|Te|^2\right)_{e\in\mathscr{E}}$ is o-summable in Λ for some projection basis $\mathscr{E}$ of X, then the sum equals to o-$\sum_{k\in\mathbb{N}}\mu_k^2$.

Proof. Let $(\xi_i)_{i\in I}$ and $(\zeta_j)_{j\in J}$ be projection bases for X and Y, respectively. By Lemma 3.1.14 we have

$$\begin{aligned} o\text{-}\sum_{i\in I}|T\xi_i|^2 &= o\text{-}\sum_{i\in I}\left(o\text{-}\sum_{j\in J}|\langle T\xi_i \mid \zeta_j\rangle|^2\right) \\ &= o\text{-}\sum_{i\in I}\left(o\text{-}\sum_{j\in J}|\langle \xi_i \mid T^*\zeta_j\rangle|^2\right) \\ &= o\text{-}\sum_{j\in J}\left(o\text{-}\sum_{i\in I}|\langle T^*\zeta_j \mid \xi_i\rangle|^2\right) \\ &= o\text{-}\sum_{j\in J}|T^*\zeta_j|^2. \end{aligned}$$

Thus, the sum does not depend on the choice of projection basis. Assume that a projection basis $\mathscr{E}$ contains $\{e_n : n\in\mathbb{N}\}$. So, it follows from $Te_n = \mu_n f_n$ and $Te = 0$ $(e\in\mathscr{E}\setminus\{e_n : n\in\mathbb{N}\})$ that

$$o\text{-}\sum_{e\in\mathscr{E}}|Te|^2 = o\text{-}\sum_{k\in\mathbb{N}}|Te_n|^2 = o\text{-}\sum_{k\in\mathbb{N}}\mu_n^2.$$

Therefore, the equivalence of (i), (ii) and (iii) are obtained. □

Definition 4.1.2. The *Hilbert–Schmidt* class $\mathscr{S}_2(X,Y)$ consists of cyclically compact operators T such that $(\mu_k^2)_{k\in\mathbb{N}}$ is o-summable in Λ. Put

$$v_2(T) := \left(o\text{-}\sum_{k\in\mathbb{N}}\mu_k^2\right)^{1/2}.$$

The operators of the class $\mathscr{S}_2(X,Y)$ are called *Hilbert–Schmidt operators.*

Using the proposition above, $\mathscr{E}(X,Y)\subset\mathscr{S}_2(X,Y)$ and $v_2(\theta_{x,y}) = |x|\,|y|$ hold for all $x\in X$ and $y\in Y$. Note that $T\in\mathscr{S}_2(X,Y)$ implies $T^*\in\mathscr{S}_2(Y,X)$.

Proposition 4.1.3. *Let S, T be in $\mathscr{S}_2(X,Y)$. Then the family $(|\langle Se \mid Te\rangle|)_{e\in\mathscr{E}}$ is o-summable for all projection basis $\mathscr{E}$ in X. In particular, the sum o-$\sum_{e\in\mathscr{E}} \langle Se \mid Te\rangle$ is same for all projection basis $\mathscr{E}$ of X. If we define*

$$\langle S, T\rangle := o\text{-}\sum_{e\in\mathscr{E}} \langle Se \mid Te\rangle\,,$$

then $\langle\cdot,\cdot\rangle$ is Λ-valued inner product on $\mathscr{S}_2(X,Y)$ for which $\langle T,T\rangle = v_2^2(T)$ for all $T \in \mathscr{S}_2(X,Y)$.

Proof. Let F be a finite subset of $\mathscr{E}$. From

$$\left(\sum_{f\in F} |\langle Sf \mid Tf\rangle|\right)^2 \leq \left(\sum_{f\in F} |\!|Sf|\!|\,|\!|Tf|\!|\right)^2 \leq \sum_{f\in F} |\!|Sf|\!|^2 \sum_{f\in F} |\!|Tf|\!|^2$$

[15, Lemma 8] $(|\langle Se \mid Te\rangle|)_{e\in\mathscr{E}}$ is o-summable. It follows from the polarization identity (3.2) and Proposition 4.1.1 that the sum o-$\sum_{e\in\mathscr{E}} \langle Se \mid Te\rangle$ is same for all projection basis $\mathscr{E}$ of X. Clearly, $\langle S, T\rangle$ is Λ-valued inner product on $\mathscr{S}_2(X,Y)$. $\square$

Theorem 4.1.4. *The pair $(\mathscr{S}_2(X,Y), \langle\cdot,\cdot\rangle)$ is a Kaplansky–Hilbert module over Λ and the following equality holds:*

$$|\!|T|\!| \leq v_2(T) \qquad (T \in \mathscr{S}_2(X,Y))$$

where $|\!|T|\!|$ is exact dominant of T [23, 4.1.1].

Proof. By Proposition 4.1.1, we see that $\mathscr{S}_2(X,Y)$ is a submodule of $B_\Lambda(X,Y)$ and $v_2(\lambda T) = |\lambda|\, v_2(T)$ holds for all $\lambda \in \Lambda$. Moreover, $(\mathscr{S}_2(X,Y), v_2(\cdot), \Lambda)$ is a decomposable lattice-normed space. Indeed, from Proposition 4.1.1 it is a lattice-normed space. We will prove that it is decomposable, and for this we assume that $v_2(T) = a_1 + a_2$ is satisfied for some $a_1, a_2 \in \Lambda_+$. Then, since $0 \leq a_1 \leq v_2(T)$ there exists an orthomorphism L on Λ such that $Lv_2(T) = a_1$ and $0 \leq L \leq I$ [2, Theorem 2.49]. So, it follows from $\mathrm{Orth}(\Lambda) = \Lambda$ that there is g in Λ such that $gv_2(T) = a_1$ and $0 \leq g \leq \mathbf{1}$ [2, Theorem 2.62]. Therefore, $T = gT + (\mathbf{1} - g)T$, $v_2(gT) = a_1$ and $v_2((\mathbf{1} - g)T) = a_2$, i. e., $(\mathscr{S}_2(X,Y), v_2(\cdot), \Lambda)$ is a decomposable lattice-normed space. On the other hand, by Proposition 4.1.3 $\mathscr{S}_2(X,Y)$ has an Λ-inner product such that $v_2(T) = \langle T, T\rangle^{1/2}$. From [23, 7.4.4.], it suffices to show that $(\mathscr{S}_2(X,Y), v_2(\cdot), \Lambda)$ is bo-complete. Firstly, we will prove the inequality $|\!|T|\!| \leq v_2(T)$ $(T \in \mathscr{S}_2(X,Y))$. Given $x \in X$, and the representation of T we have;

$$\begin{aligned}\left|Tx\right|^2 &= o\text{-}\sum_{n\in\mathbb{N}}\mu_n^2|\langle x\mid e_n\rangle|^2\left|f_n\right|^2 = o\text{-}\sum_{n\in\mathbb{N}}\mu_n^2|\langle x\mid e_n\rangle|^2\\ &\le o\text{-}\sum_{n\in\mathbb{N}}\mu_n^2\left|e_n\right|^2\left|x\right|^2 = \left|x\right|^2 o\text{-}\sum_{n\in\mathbb{N}}\mu_n^2\\ &= v_2(T)^2\left|x\right|^2\end{aligned}$$

from which the desired inequality follows. Now let $(T_\alpha)_{\alpha\in A}$ be *bo*-fundamental net in $\mathscr{S}_2(X,Y)$. So, there exists $T\in\mathscr{K}(X,Y)$ such that $(T_\alpha)_{\alpha\in A}$ *bo*-converges to T in $\mathscr{K}(X,Y)$ since $\mathscr{K}(X,Y)$ is a Banach−Kantorovich space. Without loss of generality we can assume that there exists $h\in\Lambda$ such that $v_2(T_\alpha - T_\beta)\le h$ holds for all $\alpha,\beta\in A$. Let $\mathscr{E}$ be a projection basis of X and $\alpha\in A$. Then, we have

$$\sum_{e\in F}\left|(T_\alpha - T)e\right|^2 = o\text{-}\lim_{\beta\in A}\sum_{e\in F}\left|(T_\alpha - T_\beta)e\right|^2 \le \sup_{\beta\ge\alpha} v_2^2(T_\alpha - T_\beta)\le h^2$$

where F is a finite subset of $\mathscr{E}$, and so $T_\alpha - T\in\mathscr{S}_2(X,Y)$ and $v_2(T_\alpha - T)\le \sup_{\beta\ge\alpha} v_2(T_\alpha - T_\beta)\le h$. Thus, $T_\alpha = (T_\alpha - T) + T$ implies $T\in\mathscr{S}_2(X,Y)$. Moreover, from $\inf\{\sup\{v_2(T_\alpha - T_\beta) : \beta\ge\alpha\} : \alpha\in A\} = 0$, we have $(T_\alpha)_{\alpha\in A}$ *bo*-converges to T in $\mathscr{S}_2(X,Y)$. □

4.2 The Trace Class

In this section, we generalize theory of the trace class operators on a Hilbert spaces to operators on Kaplansky−Hilbert modules and study several properties of the trace class operators on Kaplansky−Hilbert modules.

Proposition 4.2.1. *Let T be a positive cyclically compact operator in $B_\Lambda(X)$. The following statements are equivalent:*

(i) *for every projection basis $\mathscr{E}$ in X the family $(\langle Te\mid e\rangle)_{e\in\mathscr{E}}$ is o-summable in Λ;*

(ii) *for some projection basis $\mathscr{E}$ in X the family $(\langle Te\mid e\rangle)_{e\in\mathscr{E}}$ is o-summable in Λ;*

(iii) *$(\mu_k)_{k\in\mathbb{N}}$ is o-summable in Λ.*

In particular, if the family $(\langle Te\mid e\rangle)_{e\in\mathscr{E}}$ is o-summable for some projection basis $\mathscr{E}$ of X, then the sum equals to $o\text{-}\sum_{k\in\mathbb{N}}\mu_k$.

Proof. Let T be a positive cyclically compact operator in $B_\Lambda(X)$. Define

$$S := bo\text{-}\sum_{k\in\mathbb{N}} \mu_k^{1/2}\theta_{e_k,e_k},$$

and note that $T = S^2$ and S is a positive cyclically compact operator on X.

(i) ⇒ (ii) : Obvious.

(ii) ⇒ (iii) : Let $\mathscr{E}$ satisfy the properties of (ii). Since $\langle Te \mid e\rangle = \langle Se \mid Se\rangle = \lvert Se\rvert^2$ holds for every $e \in \mathscr{E}$, we obtain $S \in \mathscr{S}_2(X)$. The result follows from Proposition 4.1.1.

(iii) ⇒ (i) : Let $(\mu_k)_{k\in\mathbb{N}}$ be o-summable and $\mathscr{E}$ be a projection basis of X. Then $S \in \mathscr{S}_2(X)$. Again, from Proposition 4.1.1 and $\langle Te \mid e\rangle = \langle Se \mid Se\rangle = \lvert Se\rvert^2$ $(e \in \mathscr{E})$ (i) follows.

Assume that $\mathscr{E}$ is a projection basis containing $\{e_n : n \in \mathbb{N}\}$. From $\langle Te_k \mid e_k\rangle = \langle \mu_k e_k \mid e_k\rangle = \mu_k \lvert e_k\rvert = \mu_k$ $(k \in \mathbb{N})$, we have

$$o\text{-}\sum_{e\in\mathscr{E}} \langle Te \mid e\rangle = o\text{-}\sum_{k\in\mathbb{N}} \langle Te_k \mid e_k\rangle = o\text{-}\sum_{k\in\mathbb{N}} \mu_k.$$ □

Proposition 4.2.2. *Let T be an element of $\mathscr{K}(X,Y)$. The following conditions are equivalent:*

(i) $(\mu_n)_{n\in\mathbb{N}}$ *is o-summable in* Λ*;*

(ii) *there exist families $(x_i)_{i\in I}$ in X and $(y_i)_{i\in I}$ in Y such that $\left(\lvert x_i\rvert\,\lvert y_i\rvert\right)_{i\in I}$ is o-summable in Λ and*

$$Tx = bo\text{-}\sum_{i\in I} \langle x \mid x_i\rangle\, y_i \quad (x \in X).$$

In particular, if there exist projection orthonormal families $(x_i)_{i\in I}$ in X, $(y_i)_{i\in I}$ in Y and a posivite elements family $(\alpha_i)_{i\in I}$ such that families $(\alpha_i x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy (ii), then $o\text{-}\sum_{k\in\mathbb{N}} \mu_k = o\text{-}\sum_{i\in I} \alpha_i \lvert x_i\rvert\,\lvert y_i\rvert$.

Proof. (i) ⇒ (ii) : Let $(\mu_n)_{n\in\mathbb{N}}$ be o-summable in Λ. Then if we take $x_n := \mu_n e_n$ and $y_n := f_n$, then (ii) is satisfied.

(ii) ⇒ (i) : For all $k \in \mathbb{N}$ we obtain

$$\begin{aligned}
\sum_{n=1}^{k} \mu_n = \sum_{n=1}^{k} \langle Te_n \mid f_n \rangle &= \sum_{n=1}^{k} \left\langle bo\text{-}\sum_{i\in I} \langle e_n \mid x_i \rangle \, y_i \,\middle|\, f_n \right\rangle \\
&= \sum_{n=1}^{k} \left(o\text{-}\sum_{i\in I} \langle e_n \mid x_i \rangle \langle y_i \mid f_n \rangle \right) \\
&\leq o\text{-}\sum_{i\in I} \left(\left(\sum_{n=1}^{k} | \langle e_n \mid x_i \rangle |^2 \right)^{1/2} \left(\sum_{n=1}^{k} | \langle y_n \mid f_i \rangle |^2 \right)^{1/2} \right) \\
&\leq o\text{-}\sum_{i\in I} \lvert x_i \rvert \, \lvert y_i \rvert .
\end{aligned}$$

This implies (i).

Let $(x_i)_{i\in I} \subset X$ and $(y_i)_{i\in I} \subset Y$ be projection orthonormal families and a posivite elements family $(\alpha_i)_{i\in I}$ such that families $(\alpha_i x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy (ii). Then as above inequality, we have for all finite subset F of I

$$\begin{aligned}
\sum_{i\in F} \alpha_i \lvert x_i \rvert \, \lvert y_i \rvert = \sum_{i\in F} \langle Tx_i \mid y_i \rangle &= \sum_{i\in F} \left\langle bo\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle x_i \mid e_k \rangle \, f_k \,\middle|\, y_i \right\rangle \\
&= \sum_{i\in F} \left(o\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle x_i \mid e_k \rangle \langle f_k \mid y_i \rangle \right) \\
&\leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k \left(\sum_{i\in F} | \langle x_i \mid e_k \rangle || \langle f_k \mid y_i \rangle | \right) \\
&\leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k \left(\left(\sum_{i\in F} | \langle x_i \mid e_k \rangle |^2 \right)^{1/2} \left(\sum_{i\in F} | \langle f_k \mid y_i \rangle |^2 \right)^{1/2} \right) \\
&\leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k \lvert e_k \rvert \, \lvert f_k \rvert = o\text{-}\sum_{k\in\mathbb{N}} \mu_k .
\end{aligned}$$

Thus, we obtain $o\text{-}\sum_{k\in\mathbb{N}} \mu_k = o\text{-}\sum_{i\in I} \alpha_i \lvert x_i \rvert \, \lvert y_i \rvert$. □

Definition 4.2.3. The *trace class* $\mathscr{S}_1(X,Y)$ consists of cyclically compact operators T such that $(\mu_k)_{k\in\mathbb{N}}$ is o-summable in Λ. We put

$$v_1(T) := o\text{-}\sum_{k\in\mathbb{N}} \mu_k .$$

The operators of class $\mathscr{S}_1(X,Y)$ are called *trace class* operators.

The proposition above yields that have $\mathscr{E}(X,Y) \subset \mathscr{S}_1(X,Y)$ and the following which shows that $v_1(T)$ is well-defined.

Corollary 4.2.4. *Let $T \in \mathscr{S}_1(X,Y)$ and $\lambda \in \Lambda$. Then $v_1(\lambda T) = |\lambda|\, v_1(T)$ and*

$$v_1(T) = \inf\left\{ o\text{-}\sum_{i\in I} \lvert x_i \rvert \, \lvert y_i \rvert \in \Lambda : (x_i)_{i\in I} \subset X,\ (y_i)_{i\in I} \subset Y \right\}$$

where $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy condition (ii) of Proposition 4.2.2.

Proposition 4.2.5. *$(\mathscr{S}_1(X,Y), v_1(\cdot))$ is a Banach−Kantorovich space and the following equality holds,*

$$\lvert T \rvert \leq v_1(T) \qquad (T \in \mathscr{S}_1(X,Y))$$

where $\lvert T \rvert$ is exact dominant of T, [23, 4.1.1].

Proof. By Proposition 4.2.2 and Corollary 4.2.4, $\mathscr{S}_1(X,Y)$ is a submodule of $B_\Lambda(X,Y)$ and, respectively, $(\mathscr{S}_1(X,Y), v_1(\cdot), \Lambda)$ is a lattice-normed space and $v_1(\lambda T) = |\lambda|\, v_1(T)$ for all $\lambda \in \Lambda$. Thus, $(\mathscr{S}_1(X,Y), v_1(\cdot), \Lambda)$ is a decomposable lattice-normed space (for decomposablity see the proof of Theorem 4.1.4). Now we will prove the inequality $\lvert T \rvert \leq v_1(T)$ $(T \in \mathscr{S}_1(X,Y))$. From representation of T we have for $x \in X$

$$\begin{aligned} \lvert Tx \rvert &= \left\lvert bo\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle x \mid e_k \rangle f_k \right\rvert \leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k |\langle x \mid e_k \rangle| \, \lvert f_k \rvert \\ &\leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k \lvert e_k \rvert \, \lvert x \rvert \, \lvert f_k \rvert = \lvert x \rvert\, o\text{-}\sum_{k\in\mathbb{N}} \mu_k \\ &= v_1(T) \lvert x \rvert \end{aligned}$$

from which the desired inequality follows. Moreover, to show $\mathscr{S}_1(X,Y)$ is a Banach−Kantorovich space, it suffices to show that it is *bo*-complete. Assume that $(T_\alpha)_{\alpha\in A}$ is a *bo*-fundamental net in $\mathscr{S}_1(X,Y)$. Then there exists $T \in \mathscr{K}(X,Y)$ such that $(T_\alpha)_{\alpha\in A}$ *bo*-converges to T in $\mathscr{K}(X,Y)$ since $\mathscr{K}(X,Y)$ is Banach−Kantorovich space. We can assume that there exists $g \in \Lambda$ such that $v_1(T_\alpha - T_\beta) \leq g$ holds for all $\alpha, \beta \in A$. By Proposition 4.2.2 and Theorem 3.4.1 there exist families $(x_i)_{i\in I}$, (ξ_n) in X and $(y_i)_{i\in I}$, (ζ_n) in Y and family (ν_n) in Λ_+ such that

$$(T_\alpha - T_\beta)x = bo\text{-}\sum_{i\in I} \langle x \mid x_i \rangle\, y_i \text{ and } (T_\alpha - T)x = bo\text{-}\sum_{n\in\mathbb{N}} \nu_n \langle x \mid \xi_n \rangle\, \zeta_n$$

and $\left(\left|x_i\right| \left|y_i\right|\right)_{i\in I}$ is o-summable. Thus,

$$\begin{aligned}\sum_{n=1}^{k} |\langle (T_\alpha - T_\beta)\xi_n \mid \zeta_n\rangle| &= \sum_{n=1}^{k} \left| o\text{-}\sum_{i\in I} \langle \xi_n \mid x_i\rangle \langle y_i \mid \zeta_n\rangle \right| \\ &\leq o\text{-}\sum_{i\in I} \left(\left(\sum_{n=1}^{k} |\langle \xi_n \mid x_i\rangle|^2 \right)^{1/2} \left(\sum_{n=1}^{k} |\langle y_n \mid \zeta_i\rangle|^2 \right)^{1/2} \right) \\ &\leq o\text{-}\sum_{i\in I} \left|x_i\right| \left|y_i\right|.\end{aligned}$$

from which

$$\sum_{n=1}^{k} |\langle (T_\alpha - T_\beta)\xi_n \mid \zeta_n\rangle| \leq v_1(T_\alpha - T_\beta)$$

holds for each $\alpha, \beta \in A$. Fix $\alpha \in A$. Using the inequality

$$\begin{aligned}\sum_{n=1}^{k} \nu_n = \sum_{n=1}^{k} \langle (T_\alpha - T)\xi_n \mid \zeta_n\rangle &= o\text{-}\lim_{\beta\in A} \sum_{n=1}^{k} |\langle (T_\alpha - T_\beta)\xi_n \mid \zeta_n\rangle| \\ &\leq \sup_{\beta\geq\alpha} v_1(T_\alpha - T_\beta) \leq g\end{aligned}$$

we see $T \in \mathscr{S}_1(X,Y)$ and $v_1(T_\alpha - T) \leq \sup_{\beta\geq\alpha} v_1(T_\alpha - T_\beta)$. Therefore $(T_\alpha)_{\alpha\in A}$ bo-converges to T in $\mathscr{S}_1(X,Y)$ since $\inf\{\sup\{v_1(T_\alpha - T_\beta) : \beta \geq \alpha\} : \alpha \in A\} = 0$. □

This, together with [23, 2.1.8.(3)], yields the following.

Corollary 4.2.6. *$\mathscr{S}_1(X,Y)$ admits a compatible module structure over Λ.*

Note that $(\mathscr{S}_1(X,Y), \Lambda)$ is a Banach space with mixed norm which is defined by $|||T|||_1 := \|v_1(T)\|$ $(T \in \mathscr{S}_1(X,Y))$.

Lemma 4.2.7. *Let $T \in \mathscr{S}_1(X)$. Then the net $(|\langle Te \mid e\rangle|)_{e\in\mathscr{E}}$ is o-summable for all projection basis $\mathscr{E}$, and the sum $o\text{-}\sum_{e\in\mathscr{E}} \langle Te \mid e\rangle$ is the same for all projection basis $\mathscr{E}$ of X.*

Proof. There exist positive cyclically compact operator R_1 and cyclically compact operator R_2 in $\mathscr{S}_2(X)$ such that $T = R_1R_2$ and $\langle Te \mid e\rangle = \langle R_2e \mid R_1e\rangle$ hold for every $e \in \mathscr{E}$, namely,

$$R_1 := bo\text{-}\sum_{k\in\mathbb{N}} \mu_k^{1/2}\theta_{f_k,f_k} \text{ and } R_2 := bo\text{-}\sum_{k\in\mathbb{N}} \mu_k^{1/2}\theta_{e_k,f_k}.$$

The net $(|\langle Te \mid e\rangle|)_{e\in\mathscr{E}}$ is o-summable in Λ by Proposition 4.1.3, and the sum $o\text{-}\sum_{e\in\mathscr{E}} \langle Te \mid e\rangle$ is the same for all projection basis of X. □

We will now utilize the preceding proposition to define a trace for operators in the trace class.

Definition 4.2.8. For $T \in \mathscr{S}_1(X)$ define the *trace* of T by

$$\operatorname{tr}(T) := o\text{-}\sum_{e \in \mathscr{E}} \langle Te \mid e \rangle$$

where $\mathscr{E}$ is a projection basis of X.

Note that $v_1(T) = \operatorname{tr}(T)$ is satisfied for every positive operator T in $\mathscr{S}_1(X)$.

Lemma 4.2.9. *Let $T \in \mathscr{S}_1(X)$. If $Tx = bo\text{-}\sum_{i \in I} \langle x \mid x_i \rangle y_i$ where $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ satisfy the condition (ii) of Proposition 4.2.2, then we have*

$$\operatorname{tr}(T) = o\text{-}\sum_{i \in I} \langle y_i \mid x_i \rangle .$$

Proof. Let $\mathscr{E}$ be a projection basis of X. First, observe that

$$\left(o\text{-}\sum_{e \in \mathscr{E}} |\langle e \mid x_i \rangle| \, |\langle y_i \mid e \rangle| \right)^2 \leq o\text{-}\sum_{e \in \mathscr{E}} |\langle e \mid x_i \rangle|^2 \, o\text{-}\sum_{e \in \mathscr{E}} |\langle y_i \mid e \rangle|^2 \leq \left| y_i \right|^2 \left| x_i \right|^2$$

for each $i \in I$. Hence we have

$$\begin{aligned} \operatorname{tr}(T) = o\text{-}\sum_{e \in \mathscr{E}} \langle Te \mid e \rangle &= o\text{-}\sum_{e \in \mathscr{E}} \left(o\text{-}\sum_{i \in I} \langle e \mid x_i \rangle \langle y_i \mid e \rangle \right) \\ &= o\text{-}\sum_{i \in I} \left(o\text{-}\sum_{e \in \mathscr{E}} \langle e \mid x_i \rangle \langle y_i \mid e \rangle \right) = o\text{-}\sum_{i \in I} \langle y_i \mid x_i \rangle \end{aligned}$$

from which the desired result follows. □

Lemma 4.2.10. *The following statements hold:*

(i) $\operatorname{tr} : \mathscr{S}_1(X) \to \Lambda$ *is a dominated, bo-continuous Λ-linear operator and*

$$|\operatorname{tr}(T)| \leq v_1(T) \quad (T \in \mathscr{S}_1(X)).$$

In particular, $\left|\operatorname{tr}\right| = \mathbf{1}$ and tr is $|||\cdot|||_1$-continuous Λ-linear operator and is a band preserving operator;

(ii) $\operatorname{tr}(T^*) = \operatorname{tr}(T)^*$ $(T \in \mathscr{S}_1(X))$;

(iii) $\operatorname{tr}(TL) = \operatorname{tr}(LT)$ *whenever* $TL, LT \in \mathscr{S}_1(X)$ $(T \in \mathscr{K}(X)$ *and* $L \in B_\Lambda(X))$. *In particular, the following holds*

$$\operatorname{tr}(TL) = \operatorname{tr}(LT) = o\text{-}\sum_{k \in \mathbb{N}} \mu_k \langle Lf_k \mid e_k \rangle ;$$

(iv) *if* $T \in \mathscr{S}_1(Y,X)$ *and* $L \in B_\Lambda(X,Y)$*, then* $TL \in \mathscr{S}_1(X)$*,* $LT \in \mathscr{S}_1(Y)$ *and* $|\mathrm{tr}(TL)| \leq v_1(T) \left| L \right|$*;*

(v) $S^*T \in \mathscr{S}_1(X)$ *is satisfied for all* $S,T \in \mathscr{S}_2(X,Y)$*.*

Proof. (i) tr is a Λ-linear operator by Lemma 4.2.9. Moreover, utilising

$$|\mathrm{tr}(T)| = \left| o\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle f_k \mid e_k \rangle \right| \leq o\text{-}\sum_{k\in\mathbb{N}} \mu_k = v_1(T)$$

tr is bo-continuous and subdominated [23, 4.1.10.] and hence dominated, by virtue of [23, 4.1.11.(1)]. Since $v_1(T) = \mathrm{tr}(T)$ is satisfied for every positive operator T in $\mathscr{S}_1(X)$ and $\mathrm{Orth}(\Lambda) = \Lambda$, we have $\left|\mathrm{tr}\right| = \mathbf{1}$, and hence it is a band preserving operator [23, 5.1.8.(1)].

(ii) Let $\mathscr{E}$ be a projection basis of X. The proof follows from Corollary 3.4.2 and the following equality

$$\mathrm{tr}(T^*) = o\text{-}\sum_{e\in\mathscr{E}} \langle T^*e \mid e \rangle = o\text{-}\sum_{e\in\mathscr{E}} \langle e \mid Te \rangle = \mathrm{tr}(T)^*.$$

(iii) We use the representation of T to obtain

$$LT = bo\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle \cdot \mid e_k \rangle Lf_k \text{ and } (TL)^* = L^*T^* = bo\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle \cdot \mid f_k \rangle L^*e_k.$$

This implies that

$$\begin{aligned}\mathrm{tr}(LT) &= o\text{-}\sum_{n\in\mathbb{N}} \mu_n \langle Lf_n \mid e_n \rangle = \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n \langle L^*e_n \mid f_n \rangle \right)^* \\ &= \mathrm{tr}((TL)^*)^* = \mathrm{tr}(TL).\end{aligned}$$

(iv) Assume that $Tx = bo\text{-}\sum_{i\in I} \langle x \mid x_i \rangle y_i$ where $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the condition (ii) of Proposition 4.2.2. Then $TL = bo\text{-}\sum_{i\in I} \theta_{L^*x_i, y_i}$ holds. Moreover, since $(L^*x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the condition (ii) of Proposition 4.2.2. we have $TL \in \mathscr{S}_1(X)$. Using Lemma 4.2.9, we obtain

$$\begin{aligned}|\mathrm{tr}(TL)| &= \left| o\text{-}\sum_{i\in I} \langle y_i \mid L^*x_i \rangle \right| = \left| o\text{-}\sum_{i\in I} \langle Ly_i \mid x_i \rangle \right| \\ &\leq o\text{-}\sum_{i\in I} \left|Ly_i\right| \left|x_i\right| \leq \left|L\right| o\text{-}\sum_{i\in I} \left|y_i\right| \left|x_i\right|.\end{aligned}$$

The result now follows from Corollary 4.2.4.

(v) The representation of T yields

$$S^*Tx = bo\text{-}\sum_{k\in\mathbb{N}} \mu_k \langle x \mid e_k \rangle S^*f_k,$$

and so for all $n \in \mathbb{N}$ the following inequality holds

$$\sum_{k=1}^{n} \mu_k \left|e_k\right| \left|S^* f_k\right| = \sum_{k=1}^{n} \mu_k \left|S^* f_k\right| \\ \leq \left(\sum_{k=1}^{n} \mu_k^2\right)^{1/2} \left(\sum_{k=1}^{n} \left|S^* f_k\right|^2\right)^{1/2}.$$

That $S^*T \in \mathscr{S}_1(X)$ follows from Propositions 4.1.1 and 4.2.2. □

Let $(\mathscr{X}, |\cdot|, \Lambda)$ be a Banach−Kantorovich space. From [23, 2.1.8.(3)] $\mathscr{X}$ admits a compatible module structure over $\mathrm{Orth}(\Lambda) = \Lambda$. Moreover, every Λ-linear operator f from $\mathscr{X}$ to Λ is band preserving. Indeed, assume that $|x| \perp |y|$. This means that $|x|\,|y| = 0$. So, $|y|\,|f(x)| = \big||y| f(x)\big| = \big|f(|y| x)\big| = 0$ implies f is band preserving. Therefore, every dominated Λ-linear operator f from $\mathscr{X}$ to Λ is *bo*-continuous [23, 5.1.8.(1)]

Proposition 4.2.11. *Let $(\mathscr{X}, |\cdot|, \Lambda)$ be a Banach−Kantorovich space. If $f : \mathscr{X} \to \Lambda$ is a Λ-linear operator, then the following statements are equivalent:*

(i) *f is dominated;*

(ii) *f is mixed norm $|||\cdot|||$-continuous.*

Proof. (i) ⇒ (ii) : By [23, 5.1.8.(1)] the exact dominant $|f|$ is in $\mathrm{Orth}(\Lambda) = \Lambda$. Since $|f|$ is norm-continuous (ii) follows from $|||fx||| \leq \big\| |f|\,|x| \big\|$.

(ii) ⇒ (i) : Let $\lambda \in \Lambda_+$. For all $|x| \leq \lambda$

$$|fx| \leq |||fx\,|||\, \mathbf{1} \leq |||f\,|||\, |||x\,|||\, \mathbf{1} \leq |||f\,|||\, \|\lambda\|\, \mathbf{1}$$

implies that $\{|fx| : |x| \leq \lambda\}$ is bounded. Therefore, f is subdominated, and so it is dominated [23, 4.1.11]. □

Denote by $\mathscr{X}^*$ the set of all Λ-linear operator $\eta : \mathscr{X} \to \Lambda$ such that there exists $c \in \Lambda$ with $|\eta(x)| \leq c|x|$ $(x \in \mathscr{X})$. By the proposition above $\mathscr{X}^*$ consists of all $|||\cdot|||$-continuous Λ-linear operators $\eta : \mathscr{X} \to \Lambda$.

Theorem 4.2.12. *If $\varphi : \mathscr{S}_1(Y, X) \to \mathscr{K}(X, Y)^*$ is defined by $\varphi(T)(A) = \mathrm{tr}(TA)$ for all $A \in \mathscr{K}(X, Y)$ and $T \in \mathscr{S}_1(Y, X)$, then φ satisfies the following properties:*

(i) *φ is a bijective Λ-linear operator from $\mathscr{S}_1(Y, X)$ to $\mathscr{K}(X, Y)^*$;*

(ii) $v_1(T) = |\varphi(T)|$ $(T \in \mathscr{S}_1(Y, X))$.

Proof. By Lemma 4.2.10 (i) and (iv), φ is a well-defined dominated Λ-linear operator, and $\lvert\varphi(T)\rvert \leq v_1(T)$ for all $T \in \mathscr{S}_1(Y,X)$. Given $\phi \in \mathscr{K}(X,Y)^*$. From Theorem 4.1.4 $\phi_{|\mathscr{S}_2(X,Y)}$ is in $\mathscr{S}_2(X,Y)^*$ and there exists a unique $S \in \mathscr{S}_2(X,Y)$ such that $\phi_{|\mathscr{S}_2(X,Y)} = \langle \cdot, S\rangle$ since $\mathscr{S}_2(X,Y)$ is Kaplansky−Hilbert module [23, 7.5.7.(2)]. Thus, for all $A \in \mathscr{S}_2(X,Y)$, we obtain $\phi_{|\mathscr{S}_2(X,Y)}(A) = \langle A, S\rangle = \operatorname{tr}(S^*A)$ since $S^*A \in \mathscr{S}_1(X)$. Let $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$, and $(\lambda_k)_{k\in\mathbb{N}}$ be the representation of S^* as in Theorem 3.4.1. Define $P_m := \sum_{k=1}^{m} \theta_{y_k,x_k}$ $(m \in \mathbb{N})$, and note that $\lvert P_m\rvert \leq \mathbf{1}$. Thus, the following inequality

$$\lvert\phi\rvert = \lvert\phi\rvert\mathbf{1} \geq \lvert\phi\rvert \, \lvert P_m\rvert \geq |\phi(P_m)| = |\operatorname{tr}(S^*P_m)| = \sum_{k=1}^{m} \lambda_k$$

implies that $S^* \in \mathscr{S}_1(Y,X)$. For all $A \in \mathscr{K}(X,Y)$ there is $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{E}(X,Y) \subset \mathscr{S}_2(X,Y)$ such that $\lvert A - A_n\rvert \xrightarrow{(o)} 0$. It follows from the *bo*-continuity of $\varphi(S^*)$ that $\varphi(S^*)(A_n) = \phi(A_n)$ implies $\varphi(S^*)(A) = \phi(A)$. Thus, φ is onto and $\lvert\varphi(S^*)\rvert \geq v_1(S^*)$, and the proof is finished. □

A variant of the following lemma is proved in [49, Proposition 1.3].

Lemma 4.2.13. *If the mapping $\sigma : X \times Y \to \Lambda$ satisfies the following properties*

(i) $\sigma(\lambda x_1 + \mu x_2, y) = \lambda\sigma(x_1, y) + \mu\sigma(x_2, y)$ *for each* $x_1, x_2 \in X$, $y \in Y$ *and each* $\lambda, \mu \in \Lambda$;

(ii) $\sigma(x, \lambda y_1 + \mu y_2) = \lambda^*\sigma(x, y_1) + \mu^*\sigma(x, y_2)$ *for each* $x \in X$, *each* $y_1, y_2 \in Y$ *and each* $\lambda, \mu \in \Lambda$;

(iii) *There exists some* $\lambda \in \Lambda_+$ *such that* $|\sigma(x,y)| \leq \lambda\lvert x\rvert \, \lvert y\rvert$ *holds for each* $x \in X$ *and each* $y \in Y$,

then there exists a unique $A \in B_\Lambda(X,Y)$ *such that* $\lvert A\rvert \leq \lambda$ *and* $\sigma(x,y) = \langle Ax \mid y\rangle$ *hold for each* $x \in X$ *and each* $y \in Y$.

Proof. Given $x_0 \in X$, we define $A_{x_0}(y) := \sigma(x_0, y)^*$. Then $A_{x_0} \in B_\Lambda(Y, \Lambda)$. This implies that there exists a unique $y_0 \in Y$ such that $A_{x_0}(y) = \langle y \mid y_0\rangle$ holds for all $y \in Y$ [15, Theorem 5.]. Therefore we can define $Ax_0 := y_0$. Thus, we see that A is Λ-linear operator from X to Y and $\sigma(x,y) = \langle Ax \mid y\rangle$ is satisfied for every $x \in X$, $y \in Y$. By the property (iii) we have

$$|\langle Ax \mid Ax\rangle| = \lvert Ax\rvert^2 \leq \lambda\lvert x\rvert \, \lvert Ax\rvert,$$

and so $\lvert A\rvert \leq \lambda$. □

Theorem 4.2.14. *If $\psi : (B_\Lambda(X,Y), \lvert\cdot\rvert) \to (\mathscr{S}_1(Y,X)^*, \lvert\cdot\rvert_1)$ is defined by $\psi(L)(T) = \operatorname{tr}(TL)$ for all $L \in B_\Lambda(X,Y)$ and $T \in \mathscr{S}_1(Y,X)$. Then ψ satisfies the following properties:*

(i) *ψ is a bijective Λ-linear operator from $B_\Lambda(X,Y)$ to $\mathscr{S}_1(Y,X)^*$;*

(ii) *$\lvert L\rvert = \lvert\psi(L)\rvert_1$ $(L \in B_\Lambda(X,Y))$.*

Proof. By Lemma 4.2.10 (i) and (iv), ψ is a well-defined, dominated and Λ-linear operator, and $\lvert\psi(L)\rvert_1 \leq \lvert L\rvert$ holds for all $L \in B_\Lambda(X,Y)$. Let τ be in $\mathscr{S}_1(Y,X)^*$. For every $x \in X$ and $y \in Y$ it follows from $\theta_{y,x} \in \mathscr{K}(X,Y)$ and Proposition 4.2.2 that $\theta_{y,x} \in \mathscr{S}_1(X,Y)$. Define $\sigma : X \times Y \to \Lambda$ by $\sigma(x,y) := \tau(\theta_{y,x})$, whence

$$|\sigma(x,y)| = |\tau(\theta_{y,x})| \leq \lvert\tau\rvert_1 v_1(\theta_{y,x}) \leq \lvert\tau\rvert_1 \lvert x\rvert\,\lvert y\rvert.$$

Therefore there exists $A \in B_\Lambda(X,Y)$ such that $\sigma(x,y) = \langle Ax \mid y\rangle$. This implies

$$\psi(A)(\theta_{y,x}) = \operatorname{tr}(A\theta_{y,x}) = \operatorname{tr}(\theta_{y,Ax}) = \langle Ax \mid y\rangle = \tau(\theta_{y,x})$$

and

$$\lvert Ax\rvert^2 = \langle Ax \mid Ax\rangle = |\tau(\theta_{Ax,x})| \leq \lvert\tau\rvert_1 v_1(\theta_{Ax,x}) \leq \lvert\tau\rvert_1 \lvert Ax\rvert\,\lvert x\rvert.$$

Thus, $\lvert A\rvert \leq \lvert\tau\rvert_1$ and for all $T \in \mathscr{S}_1(Y,X)$ we obtain $\psi(A)(T) = \tau(T)$, i. e., $\psi(A) = \tau$. Therefore $\lvert\psi(L)\rvert_1 \geq \lvert L\rvert$ holds for all $L \in B_\Lambda(X,Y)$. So, (i) and (ii) are satisfied. □

4.3 Classes $\mathscr{S}_p$

This section is concerned with certain classes $\mathscr{S}_p$ $(1 \leq p < \infty)$ of cyclically compact operators on Kaplansky−Hilbert modules. It turns out that each of these classes becomes a Banach−Kantorovich space when provided with a suitable vector norm.

Proposition 4.3.1. *Let T be a positive cyclically compact operator on X and $1 \leq p < \infty$. Then the following are equivalent:*

(i) *$(\mu_k^p)_{k\in\mathbb{N}}$ is o-summable in Λ;*

(ii) *$(\langle Te \mid e\rangle^p)_{e\in\mathscr{E}}$ is o-summable in Λ for all projection orthonormal subsets $\mathscr{E}$ of X.*

In this case,

$$\left(o\text{-}\sum_{k\in\mathbb{N}}\mu_k^p\right)^{\frac{1}{p}} = \max\left\{\left(o\text{-}\sum_{e\in\mathscr{E}}\langle Te \mid e\rangle^p\right)^{\frac{1}{p}} \in \Lambda : \mathscr{E}\subset X\right\}$$

where $\mathscr{E}$ is a projection orthonormal subset of X.

Proof. (ii) $\Rightarrow$ (i) : Follows directly from the equality $\langle Te_k \mid f_k\rangle = \mu_k$ and the fact that $\{e_k : k\in\mathbb{N}\}$ is a projection orthonormal subset of X.

(i) $\Rightarrow$ (ii) : Assume that $\mathscr{E}$ is a projection orthonormal subset of X. If $p=1$, the proof is finished by Proposition 4.2.1. Assume that $1<p<\infty$ and q is conjugate index to p. Then we have

$$\begin{aligned}
\langle Te \mid e\rangle &= \left\langle bo\text{-}\sum_{n\in\mathbb{N}}\mu_n\langle e\mid e_n\rangle e_n \,\middle|\, e\right\rangle = o\text{-}\sum_{n\in\mathbb{N}}\mu_n|\langle e\mid e_n\rangle|^2 \\
&= o\text{-}\sum_{n\in\mathbb{N}}\mu_n|\langle e\mid e_n\rangle|^{2/p}|\langle e\mid e_n\rangle|^{2/q} \\
&\leq \left(o\text{-}\sum_{n\in\mathbb{N}}\left(\mu_n|\langle e\mid e_n\rangle|^{2/p}\right)^p\right)^{1/p}\left(o\text{-}\sum_{n\in\mathbb{N}}\left(|\langle e\mid e_n\rangle|^{2/q}\right)^q\right)^{1/q} \\
&\leq \left(o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p|\langle e\mid e_n\rangle|^2\right)^{1/p}\left|e\right|^{2/q} \leq \left(o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p|\langle e\mid e_n\rangle|^2\right)^{1/p}
\end{aligned}$$

for all $e\in\mathscr{E}$. This implies

$$\begin{aligned}
\sum_{e\in F}\langle Te\mid e\rangle^p &\leq \sum_{e\in F}\left(o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p|\langle e\mid e_n\rangle|^2\right) = o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p\left(\sum_{e\in F}|\langle e\mid e_n\rangle|^2\right) \\
&\leq o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p\left|e_n\right|^2 = o\text{-}\sum_{n\in\mathbb{N}}\mu_n^p,
\end{aligned}$$

where F is a finite subset of $\mathscr{E}$. This proves the proposition. ☐

Let T be in $\mathscr{K}(X,Y)$. Using the Polar Decomposition, there exists a partial isometry $U_T\in B_\Lambda(X,Y)$ such that $\left|U_T\right|\leq\mathbf{1}$, that $T=U_T|T|$, and that $|T|=U_T^*T$ with

$$|T|x = bo\text{-}\sum_{k\in\mathbb{N}}\mu_k\langle x\mid e_k\rangle e_k \text{ and } U_Tx = bo\text{-}\sum_{k\in\mathbb{N}}\langle x\mid e_k\rangle f_k,$$

where $|T|=(T^*T)^{1/2}$, and therefore $|T|\in\mathscr{K}(X,Y)$.

Definition 4.3.2. Let $1\leq p<\infty$. The set of all cyclically compact operators T such that $(\mu_k^p)_{k\in\mathbb{N}}$ is o-summable in Λ will be denoted by $\mathscr{S}_p(X,Y)$. We put

$$\upsilon_p(T) := \left(o\text{-}\sum_{k\in\mathbb{N}}\mu_k^p\right)^{\frac{1}{p}}.$$

Observe that the inclusion $\mathscr{S}_p(X,Y) \subset \mathscr{S}_r(X,Y)$ holds for $1 \leq p \leq r$.

Corollary 4.3.3. *Let T be in $\mathscr{K}(X,Y)$ and $1 \leq p < \infty$. Then the following are equivalent:*

(i) *T is in $\mathscr{S}_p(X,Y)$;*

(ii) *$U_T\,|T|\,U_T^*$ is a positive cyclically compact operator in $\mathscr{S}_p(Y)$;*

(iii) *$|T|$ is a positive cyclically compact operator in $\mathscr{S}_p(X)$;*

(iv) *$(\langle |T|\,e \mid e\rangle^p)_{e\in\mathscr{E}}$ is o-summable for all projection orthonormal subsets $\mathscr{E}$ of X.*

In this case, $v_p(T) = v_p(|T|) = v_p(U_T\,|T|\,U_T^*)$ and

$$v_p(T) = \max\left\{\left(o\text{-}\sum_{e\in\mathscr{E}} \langle |T|\,e \mid e\rangle^p\right)^{\frac{1}{p}} : \mathscr{E} \subset X\right\}$$

where $\mathscr{E}$ is a projection orthonormal subset of X.

Proof. Clearly, (i), (ii) and (iii) are equivalent. From the preceding proposition (iii) and (iv) are equivalent. □

Every positive operator A in $B_\Lambda(X)$ has a unique positive operator $A^{1/2}$ in $B_\Lambda(X)$ such that $A = \left(A^{1/2}\right)^2$. From the Cauchy−Bunyakovskiĭ−Schwarz inequality and this fact we deduce the Generalized Schwarz's inequality,

$$|\,\langle Ax \mid y\rangle\,|^2 \leq \langle Ax \mid x\rangle\,\langle Ay \mid y\rangle$$

where $x, y \in X$ and A is a positive operator in $B_\Lambda(X)$.

Proposition 4.3.4. *Let T be a cyclically compact operator from X to Y and $1 \leq p < \infty$. Then the following statements are equivalent:*

(i) *T is in $\mathscr{S}_p(X,Y)$;*

(ii) *$(|\,\langle Te_\alpha \mid f_\alpha\rangle\,|^p)_{\alpha\in\mathscr{A}}$ is o-summable for all projection orthonormal subsets $(e_\alpha)_{\alpha\in\mathscr{A}}$ and $(f_\alpha)_{\alpha\in\mathscr{A}}$ in X and Y, respectively.*

In this case, for all $T \in \mathscr{S}_p(X,Y)$, the following equality holds

$$v_p(T) = \max\left\{\left(o\text{-}\sum_{\alpha\in\mathscr{A}} |\,\langle Te_\alpha \mid f_\alpha\rangle\,|^p\right)^{\frac{1}{p}} \in \Lambda : (e_\alpha)_{\alpha\in\mathscr{A}} \subset X,\ (f_\alpha)_{\alpha\in\mathscr{A}} \subset Y\right\}$$

where $(e_\alpha)_{\alpha\in\mathscr{A}}$ and $(f_\alpha)_{\alpha\in\mathscr{A}}$ are projection orthonormal subsets of X and Y, respectively.

Proof. (ii) $\Rightarrow$ (i) : (i) follows from $\langle Te_k \mid f_k\rangle = \mu_k$, and $(e_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ projection orthonormal subsets. The following equality also holds:

$$v_p(T) = \left(o\text{-}\sum_{k\in\mathbb{N}} \mu_k^p\right)^{\frac{1}{p}} = \left(o\text{-}\sum_{k\in\mathbb{N}} \langle Te_k \mid f_k\rangle^p\right)^{\frac{1}{p}}.$$

(i) $\Rightarrow$ (ii) : Let $(e_\alpha)_{\alpha\in\mathscr{A}}$ and $(f_\alpha)_{\alpha\in\mathscr{A}}$ be projection orthonormal subsets of X and Y, respectively. By Generalized Schwarz's inequality,

$$\begin{aligned}
|\langle Te_\alpha \mid f_\alpha\rangle| &= |\langle U_T |T| e_\alpha \mid f_\alpha\rangle| = |\langle |T| e_\alpha \mid U_T^* f_\alpha\rangle| \\
&\le |\langle |T| e_\alpha \mid e_\alpha\rangle|^{1/2} |\langle |T| U_T^* f_\alpha \mid U_T^* f_\alpha\rangle|^{1/2} \\
&= |\langle |T| e_\alpha \mid e_\alpha\rangle|^{1/2} |\langle U_T |T| U_T^* f_\alpha \mid f_\alpha\rangle|^{1/2}
\end{aligned}$$

holds for all $\alpha \in \mathscr{A}$. So, we use Corollary 4.3.3 to obtain

$$\begin{aligned}
\sum_{\alpha\in\mathscr{F}} |\langle Te_\alpha \mid f_\alpha\rangle|^p &\le \sum_{\alpha\in\mathscr{F}} |\langle |T| e_\alpha \mid e_\alpha\rangle|^{p/2} |\langle U_T |T| U_T^* f_\alpha \mid f_\alpha\rangle|^{p/2} \\
&\le \left(\sum_{\alpha\in\mathscr{F}} \left(\langle |T| e_\alpha \mid e_\alpha\rangle^{\frac{p}{2}}\right)^2\right)^{\frac{1}{2}} \left(\sum_{\alpha\in\mathscr{F}} \left(\langle U_T |T| U_T^* f_\alpha \mid f_\alpha\rangle^{\frac{p}{2}}\right)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{\alpha\in\mathscr{F}} \langle |T| e_\alpha \mid e_\alpha\rangle^p\right)^{\frac{1}{2}} \left(\sum_{\alpha\in\mathscr{F}} \langle U_T |T| U_T^* f_\alpha \mid f_\alpha\rangle^p\right)^{\frac{1}{2}} \\
&\le v_p(|T|)^{\frac{p}{2}} v_p(U_T |T| U_T^*)^{\frac{p}{2}} = v_p(T)^p
\end{aligned}$$

where $\mathscr{F}$ is a finite subset of $\mathscr{A}$. Thus, (ii) is satisfied. $\square$

Proposition 4.3.5. *Let T be in $\mathscr{K}(X,Y)$ and $1 \le p \le 2$. Then $T \in \mathscr{S}_p(X,Y)$ if and only if $\left(\left|Te\right|^p\right)_{e\in\mathscr{E}}$ is o-summable for some projection basis $\mathscr{E}$ of X. In this case, for $T \in \mathscr{S}_p(X,Y)$, the following equality holds*

$$v_p(T) = \min\left\{\left(o\text{-}\sum_{e\in\mathscr{E}} \left|Te\right|^p\right)^{\frac{1}{p}} \in \Lambda : \mathscr{E} \subset X\right\}$$

where $\left(\left|Te\right|^p\right)_{e\in\mathscr{E}}$ is o-summable for projection basis $\mathscr{E}$.

Proof. There exists a projection basis $\mathscr{E}$ containing $\{e_k : k \in \mathbb{N}\}$. Thus, $Te = 0$ and $Te_k = \mu_k f_k$ hold for all $e \in \mathscr{E} \setminus \{e_k : k \in \mathbb{N}\}$ and $k \in \mathbb{N}$, respectively. If $T \in \mathscr{S}_p(X,Y)$, then it follows from $\left|Te_k\right| = \mu_k$ that $\left(\left|Te\right|^p\right)_{e\in\mathscr{E}}$ is o-summable for the projection basis $\mathscr{E}$ of X and we also have the following equality

$$v_p(T) = \left(o\text{-}\sum_{e\in\mathscr{E}} \left|Te\right|^p\right)^{\frac{1}{p}}.$$

Conversely, suppose that $\left(\left|Te\right|^p\right)_{e\in\mathscr{E}}$ is o-summable for some projection basis $\mathscr{E}$ of X. Using Hölder's inequality with conjugate exponents $2/p$ and $2/(2-p)$, we have

$$\begin{aligned}
\sum_{n\in F}\mu_n^p &= \sum_{n\in F}\mu_n^p\left(o\text{-}\sum_{e\in\mathscr{E}}|\langle e \mid e_n\rangle|^2\right) = o\text{-}\sum_{e\in\mathscr{E}}\left(\sum_{n\in F}\mu_n^p|\langle e \mid e_n\rangle|^2\right)\\
&= o\text{-}\sum_{e\in\mathscr{E}}\left(\sum_{n\in F}\mu_n^p|\langle e \mid e_n\rangle|^p|\langle e \mid e_n\rangle|^{2-p}\right)\\
&\le o\text{-}\sum_{e\in\mathscr{E}}\left(\left(\sum_{n\in F}\mu_n^2|\langle e \mid e_n\rangle|^2\right)^{p/2}\left(\sum_{n\in F}|\langle e \mid e_n\rangle|^2\right)^{1-p/2}\right)\\
&\le o\text{-}\sum_{e\in\mathscr{E}}\left|Te\right|^p\left|e\right|^{2-p} \le o\text{-}\sum_{e\in\mathscr{E}}\left|Te\right|^p
\end{aligned}$$

where F is a finite subset of $\mathbb{N}$. Thus, we have the desired result. □

Proposition 4.3.6. *Let T be in $\mathscr{K}(X,Y)$ and $2 \le p < \infty$. Then T is in $\mathscr{S}_p(X,Y)$ if and only if $\left(\left|Te\right|^p\right)_{e\in\mathscr{E}}$ is o-summable for all projection orthonormal subset $\mathscr{E}$ of X. In this case, for $T \in \mathscr{S}_p(X,Y)$, the following equality holds*

$$v_p(T) = \max\left\{\left(o\text{-}\sum_{e\in\mathscr{E}}\left|Te\right|^p\right)^{\frac{1}{p}} \in \Lambda : \mathscr{E} \subset X\right\}$$

where $\mathscr{E}$ is a projection orthonormal subset of X.

Proof. (ii) ⇒ (i) : $\{e_k : k \in \mathbb{N}\}$ is a projection orthonormal set and $\left|Te_k\right| = \mu_k$ holds for all $k \in \mathbb{N}$. This implies (i), and the following equality holds:

$$v_p(T) = \left(o\text{-}\sum_{k\in\mathbb{N}}\left|Te_k\right|^p\right)^{\frac{1}{p}}.$$

(i) ⇒ (ii) : Let T be in $\mathscr{S}_p(X,Y)$ and $\mathscr{E}$ be a projection orthonormal subsets of X. Using Hölder's inequality with conjugate exponents $p/2$ and $p/(p-2)$, we have

$$\begin{aligned}
\sum_{e\in\mathscr{F}} \lVert Te\rVert^p &= \sum_{e\in\mathscr{F}} \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^2 |\langle e \mid e_n\rangle|^2 \right)^{p/2} \\
&= \sum_{e\in\mathscr{F}} \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^2 |\langle e \mid e_n\rangle|^{4/p} |\langle e \mid e_n\rangle|^{2-4/p} \right)^{p/2} \\
&\leq \sum_{e\in\mathscr{F}} \left(\left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p |\langle e \mid e_n\rangle|^2 \right) \left(o\text{-}\sum_{n\in\mathbb{N}} |\langle e \mid e_n\rangle|^2 \right)^{(p-2)/2} \right) \\
&= \sum_{e\in\mathscr{F}} \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p |\langle e \mid e_n\rangle|^2 \right) \lVert e\rVert^{p-2} \leq \sum_{e\in\mathscr{F}} o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p |\langle e \mid e_n\rangle|^2 \\
&= o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p \left(\sum_{e\in\mathscr{F}} |\langle e \mid e_n\rangle|^2 \right) \leq o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p \lVert e_n\rVert^2 \\
&= o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p.
\end{aligned}$$

where $\mathscr{F}$ is a finite subset of $\mathscr{E}$. Clearly, we obtain the desired result. □

Proposition 4.3.7. *Let $1 \leq p < \infty$. Then the following inequality holds:*

$$\lVert T\rVert \leq v_p(T) \qquad (T \in \mathscr{S}_p(X,Y))$$

where $\lVert T\rVert$ is the exact dominant of T.

Proof. Let T be in $\mathscr{S}_p(X,Y)$. If $1 \leq p \leq 2$, then using representation of T and Hölder's inequality with q the conjugate index to p, we have

$$\begin{aligned}
\left\lVert \sum_{n\in F} \mu_n \langle x \mid e_n\rangle f_n \right\rVert &\leq \sum_{n\in F} \mu_n |\langle x \mid e_n\rangle| \lVert f_n\rVert = \sum_{n\in F} \mu_n |\langle x \mid e_n\rangle|^{1-2/q} |\langle x \mid e_n\rangle|^{2/q} \\
&\leq \left(\sum_{n\in F} \mu_n^p |\langle x \mid e_n\rangle|^{(1-2/q)p} \right)^{1/p} \left(\sum_{n\in F} |\langle x \mid e_n\rangle|^2 \right)^{1/q} \\
&\leq \left(\sum_{n\in F} \mu_n^p \lVert x\rVert^{(1-2/q)p} \right)^{1/p} \lVert x\rVert^{2/q} \\
&= \lVert x\rVert^{1-2/q} \left(\sum_{n\in F} \mu_n^p \right)^{1/p} \lVert x\rVert^{2/q} \\
&\leq v_p(T) \lVert x\rVert
\end{aligned}$$

where F is a finite subset of $\mathbb{N}$ and $x \in X$. Therefore, $\lVert Tx\rVert \leq v_p(T) \lVert x\rVert$ is satisfied for all $x \in X$. If $2 \leq p$, then using Hölder's inequality with conjugate exponents

$p/2$ and $p/(p-2)$, we have

$$\begin{aligned}
\left|Tx\right|^2 &= o\text{-}\sum_{n\in\mathbb{N}} \mu_n^2 |\langle x \mid e_n\rangle|^2 = o\text{-}\sum_{n\in\mathbb{N}} \mu_n^2 |\langle x \mid e_n\rangle|^{4/p} |\langle x \mid e_n\rangle|^{2-4/p} \\
&\leq \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p |\langle x \mid e_n\rangle|^2\right)^{2/p} \left(o\text{-}\sum_{n\in\mathbb{N}} |\langle x \mid e_n\rangle|^2\right)^{(p-2)/p} \\
&\leq \left|x\right|^{4/p} \left(o\text{-}\sum_{n\in\mathbb{N}} \mu_n^p\right)^{2/p} \left|x\right|^{2(p-2)/p} \\
&= v_p(T)^2 \left|x\right|^2.
\end{aligned}$$

Finally, $\left|Tx\right| \leq v_p(T)\left|x\right|$ is satisfied for every $x \in X$ with $1 \leq p < \infty$, and so $\left|T\right| \leq v_p(T)$. □

Proposition 4.3.8. *Let $1 \leq p < \infty$. $(\mathscr{S}_p(X,Y), v_p(\cdot), \Lambda)$ is a Banach–Kantorovich space and the following properties hold:*

(i) *if $T \in \mathscr{S}_p(X,Y)$, then $T^* \in \mathscr{S}_p(Y,X)$ and $v_p(T) = v_p(T^*)$;*

(ii) *if $L \in B_\Lambda(Y,Z)$ and $S \in B_\Lambda(W,X)$, then $LT \in \mathscr{S}_p(X,Z)$ and $TS \in \mathscr{S}_p(W,Y)$ hold for all $T \in \mathscr{S}_p(X,Y)$. Moreover,*

$$v_p(LT) \leq \left|L\right| v_p(T) \quad \text{and} \quad v_p(TS) \leq \left|S\right| v_p(T).$$

Proof. By Proposition 4.3.4 we have (i) and $\mathscr{S}_p(X,Y)$ is a submodule of $B_\Lambda(X,Y)$. Moreover, $v_p(\lambda T) = |\lambda|\, v_p(T)$ holds for all $T \in \mathscr{S}_p(X,Y)$ and $\lambda \in \Lambda$, and so $(\mathscr{S}_p(X,Y), v_p(\cdot), \Lambda)$ is a decomposable lattice-normed space (see the proof of Theorem 4.1.4). Let $L \in B_\Lambda(Y,Z)$, $S \in B_\Lambda(W,X)$ and $T \in \mathscr{S}_p(X,Y)$. By (i) and Propositions 4.3.5 and 4.3.6 we obtain $LT \in \mathscr{S}_p(X,Z)$, $TS \in \mathscr{S}_p(W,Y)$. So, it follows from $\left|S\right| = \left|S^*\right|$ that

$$v_p(LT) \leq \left|L\right| v_p(T) \quad \text{and} \quad v_p(TS) \leq \left|S\right| v_p(T).$$

Now we will show $\mathscr{S}_p(X,Y)$ is a Banach–Kantorovich space.To this end, let $(T_\alpha)_{\alpha\in A}$ be *bo*-fundamental net in $\mathscr{S}_p(X,Y)$. Using Proposition 4.3.7 and $\mathscr{K}(X,Y)$ Banach–Kantorovich space, there exists $T \in \mathscr{K}(X,Y)$ such that $(T_\alpha)_{\alpha\in A}$ *bo*-converges to T in $\mathscr{K}(X,Y)$. We can assume $v_p(T_\alpha - T_\beta) \leq g$ for some $g \in \Lambda$ and all $\alpha, \beta \in A$. Fix $\alpha \in A$. Since $T_\alpha - T \in \mathscr{K}(X,Y)$ there exist orthonormal families $(\xi_n)_{n\in\mathbb{N}}$ in X and $(\zeta_n)_{n\in\mathbb{N}}$ in Y, and family $(\lambda_n)_{n\in\mathbb{N}}$ verifying representation of $T_\alpha - T$ as in Theorem 3.4.1. Thus, we use Proposition 4.3.4 to obtain

$$\begin{aligned}
\sum_{n=1}^{k} \lambda_n^p = \sum_{n=1}^{k} |\langle (T_\alpha - T)\xi_n \mid \zeta_n\rangle|^p &= o\text{-}\lim_{\beta\in A} \sum_{n=1}^{k} |\langle (T_\alpha - T_\beta)\xi_n \mid \zeta_n\rangle|^p \\
&\leq \sup_{\beta\geq\alpha} v_p(T_\alpha - T_\beta)^p \leq g^p,
\end{aligned}$$

and so $v_p(T_\alpha - T) \leq \sup_{\beta \geq \alpha} v_p(T_\alpha - T_\beta)$, i. e., $T_\alpha - T \in \mathscr{S}_p(X,Y)$. So, it follows from $T = T_\alpha - (T_\alpha - T)$ that $T \in \mathscr{S}_p(X,Y)$. On the other hand, the net $(T_\alpha)_{\alpha \in A}$ *bo*-converges to T in $\mathscr{S}_p(X,Y)$ since $\inf \{\sup \{v_1(T_\alpha - T_\beta) : \beta \geq \alpha\} : \alpha \in A\} = 0$. Therefore, $\mathscr{S}_p(X,Y)$ is a Banach−Kantorovich space. □

From [24, Lemmas 5.1. and 7.1.] we can compute the mix-norm $|||T|||_p := \|v_p(T)\|$ $(T \in \mathscr{S}_p(X,Y))$ for $1 \leq p < \infty$ as follows:

$$|||T|||_p = \left\| \left(o\text{-}\sum_{k \in \mathbb{N}} \mu_k^p \right)^{\frac{1}{p}} \right\| = \sup_{l \in \mathbb{N}} \inf_{(\pi_k) \in \mathrm{Prt}_\sigma} \sup_{k \in \mathbb{N}} \left(\sum_{n=1}^{l} \|\pi_k \mu_n\|^p \right)^{1/p}$$

where Prt_σ is the set of sequences $\pi : \mathbb{N} \to \mathfrak{P}(\Lambda)$ which are partitions of unity in $\mathfrak{P}(\Lambda)$

Lemma 4.3.9. *Let $1 \leq p < \infty$. Then $\mathscr{E}(X,Y) \subset \mathscr{S}_p(X,Y)$ and for each $T \in \mathscr{S}_p(X,Y)$ there exists a sequence $(T_k)_{k \in \mathbb{N}}$ in $\mathscr{E}(X,Y)$ which $(v_p(T - T_k))_{k \in \mathbb{N}}$ o-converges to* 0. *In particular $v_p(\theta_{x,y}) = \left|x\right| \left|y\right|$ $(x \in X, y \in Y)$.*

Proof. By Lemma 3.4.13 (i) $\theta_{x,y} \in \mathscr{K}(X,Y)$ for all $x \in X$ and $y \in Y$. Let $(e_\alpha)_{\alpha \in \mathscr{A}}$ and $(f_\alpha)_{\alpha \in \mathscr{A}}$ be projection orthonormal subsets in X and Y, respectively. From $|\langle e_\alpha \mid x \rangle| \leq \left|e_\alpha\right| \left|x\right| \leq \left|x\right|$ and $|\langle f_\alpha \mid y \rangle| \leq \left|y\right|$ $(\alpha \in \mathscr{A})$, we have

$$\begin{aligned}
\sum_{\alpha \in F} |\langle \theta_{x,y} e_\alpha \mid f_\alpha \rangle|^p &= \sum_{\alpha \in F} |\langle e_\alpha \mid x \rangle|^p \, |\langle y \mid f_\alpha \rangle|^p \\
&\leq \left(\sum_{\alpha \in F} |\langle e_\alpha \mid x \rangle|^{2p} \right)^{1/2} \left(\sum_{\alpha \in F} |\langle y \mid f_\alpha \rangle|^{2p} \right)^{1/2} \\
&\leq \left(\sum_{\alpha \in F} |\langle e_\alpha \mid x \rangle|^2 |\langle e_\alpha \mid x \rangle|^{2p-2} \right)^{1/2} \left(\sum_{\alpha \in F} |\langle y \mid f_\alpha \rangle|^2 |\langle y \mid f_\alpha \rangle|^{2p-2} \right)^{1/2} \\
&\leq \left(\left|x\right|^{2p-2} \sum_{\alpha \in F} |\langle e_\alpha \mid x \rangle|^2 \right)^{1/2} \left(\left|y\right|^{2p-2} \sum_{\alpha \in F} |\langle y \mid f_\alpha \rangle|^2 \right)^{1/2} \\
&\leq \left(\left|x\right|^{2p-2} \left|x\right|^2 \right)^{1/2} \left(\left|y\right|^{2p-2} \left|y\right|^2 \right)^{1/2} \\
&= \left|x\right|^p \left|y\right|^p
\end{aligned}$$

where F is a finite subset of $\mathscr{A}$. From Proposition 4.3.4 we see that $\theta_{x,y} \in \mathscr{S}_p(X,Y)$ and $v_p(\theta_{x,y}) \leq \left|x\right| \left|y\right|$. Thus, $\mathscr{E}(X,Y) \subset \mathscr{S}_p(X,Y)$ by the preceding proposition. By Lemma 3.4.13 (iii) and Proposition 4.3.7 we have $v_p(\theta_{x,y}) = \left|x\right| \left|y\right|$. We will show finally that for each $T \in \mathscr{S}_p(X,Y)$ there exists a sequence $(T_k)_{k \in \mathbb{N}}$ in $\mathscr{E}(X,Y)$ which $(v_p(T - T_k))_{k \in \mathbb{N}}$ *o*-converges to 0. Using represention of T we can define T_m in $\mathscr{E}(X,Y)$ by for

$$T_m := \sum_{k=1}^{m} \mu_k \theta_{e_k, f_k}$$

for $m \in \mathbb{N}$. Then

$$T - T_m = bo\text{-}\sum_{k \in \mathbb{N}_{m+1}} \mu_k \theta_{e_k, f_k}$$

where $\mathbb{N}_{m+1} = \{k \in \mathbb{N} : k \geq m+1\}$ and we obtain $v_p(T - T_m)^p = bo\text{-}\sum_{k \in \mathbb{N}_{m+1}} \mu_k^p$. Thus, $(v_p(T - T_k))_{k \in \mathbb{N}}$ o-converges to 0. □

Lemma 4.3.10. *Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $S \in \mathscr{S}_p(X, Y)$ and $T \in \mathscr{S}_q(Y, X)$, then $TS \in \mathscr{S}_1(X)$ and $v_1(TS) \leq v_q(T) v_p(S)$.*

Proof. We may assume that $p \geq 2$ since otherwise $q > 2$ holds and then the proof is similar. So, we use Proposition 4.3.6 and

$$TSx = o\text{-}\sum_{n \in \mathbb{N}} \mu_n \langle x \mid S^* e_n \rangle f_n$$

to obtain

$$\begin{aligned} \sum_{n \in F} \left|\mu_n f_n\right| \left|S^* e_n\right| &\leq \left(\sum_{n \in F} \left|\mu_n f_n\right|^q\right)^{\frac{1}{q}} \left(\sum_{n \in F} \left|S^* e_n\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(o\text{-}\sum_{n \in \mathbb{N}} \mu_n^q\right)^{\frac{1}{q}} v_p(S^*) = v_q(T) v_p(S) \end{aligned}$$

where F is a finite subset of $\mathbb{N}$. The proof follows immediately from Proposition 4.2.2 and Corollary 4.2.4. □

Together with Lemma 4.2.10 (iii), this yields the following corollary.

Corollary 4.3.11. *For each $T \in \mathscr{S}_p(X, Y)$ and $S \in \mathscr{S}_q(Y, X)$, $tr(TS) = tr(ST)$.*

Theorem 4.3.12. *Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi : (\mathscr{S}_p(X), v_p(\cdot)) \to (\mathscr{S}_q(X)^*, \left|\cdot\right|_q)$ is defined by $\phi(T)(S) = \mathrm{tr}(ST)$ for all $T \in \mathscr{S}_p(X)$ and $S \in \mathscr{S}_q(X)$, then ϕ satisfies the following properties:*

(i) *ϕ is a bijective Λ-linear operator from $\mathscr{S}_p(X)$ to $\mathscr{S}_q(X)^*$;*

(ii) *$v_p(T) = \left|\phi(T)\right|_q$ $(T \in \mathscr{S}_p(X))$.*

Proof. By Lemma 4.2.10 (i) and Lemma 4.3.10, ϕ is a well-defined dominated Λ-linear operator, and $v_p(T) \geq \left|\phi(T)\right|_q$ is satisfied for all $T \in \mathscr{S}_p(X)$. Given η in $\mathscr{S}_q(X)^*$. From Lemma 4.3.9, we have $\theta_{y,x} \in \mathscr{S}_q(X)$ for all $x, y \in X$. So, if $\sigma : X \times X \to \Lambda$ is defined by $\sigma(x, y) := \eta(\theta_{y,x})$, then by Lemma 4.2.13 there exists $T \in B_\Lambda(X)$ such that $\sigma(x, y) = \langle Tx \mid y \rangle$ since

$$|\sigma(x, y)| = |\eta(\theta_{y,x})| \leq \left|\eta\right|_q v_q(\theta_{y,x}) \leq \left|\eta\right|_q \left|x\right| \left|y\right|.$$

Thus, using Lemma 4.2.9 we get

$$\operatorname{tr}(\theta_{y,x}T) = \operatorname{tr}(\theta_{T^*y,x}) = \langle x \mid T^*y \rangle = \langle Tx \mid y \rangle = \eta(\theta_{y,x}).$$

Since tr is Λ-linear operator, $\operatorname{tr}(ST) = \eta(S)$ is satisfied for every $S \in \mathscr{E}(X)$. If $T \in \mathscr{K}(X)$, then we must show $T \in \mathscr{S}_p(X)$. Define

$$T_n := \sum_{k=1}^{n} \mu_k^{p-1} \theta_{f_k,e_k},$$

and note that $T_n \in \mathscr{E}(X) \subset \mathscr{S}_q(X)$ and

$$v_q(T_n) = \left(\sum_{k=1}^{n} \mu_k^{(p-1)q} \right)^{1/q} = \left(\sum_{k=1}^{n} \mu_k^{p} \right)^{1/q}.$$

Therefore, we have

$$\sum_{k=1}^{n} \mu_k^{p} = \operatorname{tr}(T_nT) = \eta(T_n) \leq \left\| \eta \right\|_q v_q(T_n) = \left\| \eta \right\|_q \left(\sum_{k=1}^{n} \mu_k^{p} \right)^{1/q}.$$

This implies that $(\sum_{k=1}^{n} \mu_k^{p})^{1/p} \leq \left\| \eta \right\|_q$, and so we have $T \in \mathscr{S}_p(X)$ and $v_p(T) \leq \left\| \eta \right\|_q$. Given $S \in \mathscr{S}_q(X)$. Then there exists a sequence $(S_k)_{k\in\mathbb{N}}$ in $\mathscr{E}(X)$ which $(v_q(S - S_k))_{k\in\mathbb{N}}$ o-converges to 0. Thus, it follows from $|\operatorname{tr}(ST - S_kT)| \leq v_1(ST - S_kT) \leq v_q(S - S_k)v_p(T)$ that $\operatorname{tr}(ST) = \eta(S)$, i. e., $\phi(T) = \eta$. This implies (i) and (ii). To finish the proof we will show that $T \in \mathscr{K}(X)$. Let $\mathscr{E}$ be a projection basis of X. If $\mathscr{F}$ is a finite subset of $\mathscr{E}$, then $P_{\mathscr{F}} := \sum_{e\in\mathscr{F}} \theta_{e,e}$ is an element of $\mathscr{E}(X)$. So, we form the Λ-linear functional from $\mathscr{S}_q(X)$ to Λ

$$\eta_{\mathscr{F}}(A) := \eta(P_{\mathscr{F}}AP_{\mathscr{F}}) \qquad (A \in \mathscr{S}_q(X)).$$

From Proposition 4.3.8 and $\left\| P_{\mathscr{F}} \right\| \leq \mathbf{1}$, we have

$$|\eta_{\mathscr{F}}(A)| = |\eta(P_{\mathscr{F}}AP_{\mathscr{F}})| \leq \left\| \eta \right\|_q v_q(A),$$

and so the Λ-linear functional $\eta_{\mathscr{F}}$ is dominated. It follows from $P_{\mathscr{F}}TP_{\mathscr{F}} \in \mathscr{E}(X)$, and belongs to $\mathscr{S}_p(X)$ that

$$\eta_{\mathscr{F}}(A) = \eta(P_{\mathscr{F}}AP_{\mathscr{F}}) = \operatorname{tr}(P_{\mathscr{F}}AP_{\mathscr{F}}T) = \operatorname{tr}(AP_{\mathscr{F}}TP_{\mathscr{F}}) = \phi(P_{\mathscr{F}}TP_{\mathscr{F}})(A),$$

i. e., $\eta_{\mathscr{F}} = \phi(P_{\mathscr{F}}TP_{\mathscr{F}})$. From the preceding discussion, we have

$$\left\| \eta \right\|_q \geq \left\| \eta_{\mathscr{F}} \right\|_q = \left\| \phi(P_{\mathscr{F}}TP_{\mathscr{F}}) \right\|_q = v_p(P_{\mathscr{F}}TP_{\mathscr{F}})$$

and using Proposition 4.3.4 the following inequality is satisfied

$$\begin{aligned}
\left|\eta\right|_q^p \geq v_p(P_{\mathscr{F}} T P_{\mathscr{F}})^p &\geq \sum_{e\in\mathscr{E}} |\left\langle P_{\mathscr{F}} T P_{\mathscr{F}} e \mid e\right\rangle|^p \\
&= \sum_{e\in\mathscr{F}} |\left\langle P_{\mathscr{F}} T e \mid e\right\rangle|^p = \sum_{e\in\mathscr{F}} |\left\langle T e \mid P_{\mathscr{F}} e\right\rangle|^p \\
&= \sum_{e\in\mathscr{F}} |\left\langle T e \mid e\right\rangle|^p.
\end{aligned}$$

Therefore, $\inf\{\sup\{|\langle Te \mid e\rangle| : e \in F^c\} : F \in \Theta\} = 0$ holds for every projection basis $\mathscr{E}$ of X where Θ is the set of all finite subsets of $\mathscr{E}$. This and Theorem 3.4.14 completes the proof. □

Chapter 5

Global Eigenvalues of Cyclically Compact Operators on Kaplansky−Hilbert Modules

In this chapter, we study the global eigenvalues of cyclically compact operators on Kaplansky−Hilbert modules and give variants of Weyl- and Horn-type inequalities and Lidskiĭ trace formula. Throughout this chapter, X and Y will denote Kaplansky−Hilbert modules over Λ, and Q and H will denote an extremally disconnected compact space and a Hilbert space, respectively.

5.1 The Multiplicity of Global Eigenvalues

In this section, we define the multiplicity of global eigenvalues of cyclically compact operators on X which is an element of the universally complete vector lattice $(\mathrm{Re}\Lambda)^{\infty}$, which in turn is the universal completion of $\mathrm{Re}\Lambda$. Let λ be an eigenvalue of $T \in B_{\Lambda}(X, Y)$. The set $N_{\lambda} := \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\,(T - \lambda I)^n$ is called the *generalized eigenspace*, corresponding to the eigenvalue λ. We start with the following lemma which gives a relation between the generalized eigenspace N_{λ} and $\mathrm{Ker}\,(T - \lambda I)^n$ $(n \in \mathbb{N})$.

Lemma 5.1.1. *Let T be a cyclically compact operator on X and λ be a global* *eigenvalue of T. If π is a projection with $0 < \pi \le [\lambda]$, then there exist a projection μ with $0 < \mu \le \pi$ and $n \in \mathbb{N}$ such that $\mu \mathrm{Ker}\,(T-\lambda I)^n = \mu \mathrm{Ker}\,(T-\lambda I)^{n+1}$, i. e., $\mu N_\lambda = \mu \mathrm{Ker}\,(T-\lambda I)^n$.*

Proof. Firstly, note that for any $0 < \nu \le [\lambda]$ and $n \in \mathbb{N}$,

$$\nu \mathrm{Ker}\,(T-\lambda I)^n = \nu \mathrm{Ker}\,(T-\lambda I)^{n+1}$$

which means that

$$\nu \left((\mathrm{Ker}\,(T-\lambda I)^n)^{\perp} \cap \mathrm{Ker}\,(T-\lambda I)^{n+1} \right) = \{0\}\,.$$

Assume by way of contradiction that the lemma is false. Given $n \in \mathbb{N}$ and $y \in \pi \mathrm{Ker}\,(T-\lambda I)^n$ with $\left| y \right| \in \mathfrak{P}(\Lambda)$. Then there exists

$$z \in \pi \left((\mathrm{Ker}\,(T-\lambda I)^n)^{\perp} \cap \mathrm{Ker}\,(T-\lambda I)^{n+1} \right)$$

with $\left| y \right| = \left| z \right|$. To see this, if Ψ is a set consisting of the pair (μ, x) such that

$$x \in \mu \left((\mathrm{Ker}\,(T-\lambda I)^n)^{\perp} \cap \mathrm{Ker}\,(T-\lambda I)^{n+1} \right) \setminus \{0\}\,,$$

$0 < \mu \le \left| y \right|$ and $\mu = \left| x \right| \in \mathfrak{P}(\Lambda)$, then it follows from the assumption that $\left| y \right| = \sup \{\mu :\, (\mu, x) \in \Psi\}$. By the Exhaustion Principle, we can deduce z, as desired. So, a sequence $(x_n)_{n \in \mathbb{N}}$ can be constructed such that $\pi = \left| x_n \right|$ and

$$x_n \in \pi \left((\mathrm{Ker}\,(T-\lambda I)^n)^{\perp} \cap \mathrm{Ker}\,(T-\lambda I)^{n+1} \right).$$

Therefore, it follows from

$$(T-\lambda I)^n \left((T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m \right) = 0 \quad (m < n)$$

that $(T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m \in \mathrm{Ker}\,(T-\lambda I)^n$, and so

$$\begin{aligned} \left| Tx_n - Tx_m \right|^2 &= \left| \lambda x_n + ((T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m) \right|^2 \\ &\ge \left| \lambda x_n \right|^2 + \left| (T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m \right|^2 \\ &\ge |\lambda|^2 \left| x_n \right| = \pi |\lambda|^2 \ne 0 \end{aligned}$$

which contradicts cyclically compactness of T, and the proof is finished. □

Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. Define $\rho_N(\lambda) := \sup \left\{ \pi \in \mathfrak{P}(\Lambda) : \pi N_\lambda = \pi \mathrm{Ker}\,(T-\lambda I)^N,\ \pi \le [\lambda] \right\}$ for each $N \in \mathbb{N}$. Using the lemma above, we immediately have the following corollary.

Corollary 5.1.2. *Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. The following conditions are satisfied:*

(1) $\rho_N(\lambda) \leq \rho_{N+1}(\lambda)$.

(2) $\rho_N(\lambda)\mathrm{Ker}\,(T-\lambda I)^N = \rho_N(\lambda)\mathrm{Ker}\,(T-\lambda I)^{N+1}$.

(3) $\rho_N(\lambda)N_\lambda = \rho_N(\lambda)\mathrm{Ker}\,(T-\lambda I)^N$.

(4) $[\lambda] = \bigvee_{N\in\mathbb{N}} \rho_N(\lambda) = \sup\{\rho_N(\lambda) : N \in \mathbb{N}\}$.

(5) $\rho_N(\lambda)N_\lambda$ *is a Kaplansky–Hilbert module over* $\rho_N(\lambda)\Lambda$.

According to Theorem 3.1.5, for each $N \in \mathbb{N}$, there exists a partition of $\rho_N(\lambda)$, $(b_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$ such that $b_\xi N_\lambda$ is a strictly $\varkappa(b_\xi)$-homogeneous Kaplansky–Hilbert module over $b_\xi\Lambda$. Since T is cyclically compact, $\varkappa(b_\xi)$ must be a finite number. From [23, 7.4.7.(1) Theorem], we can assume that $\Xi = \mathbb{N}$ and $\varkappa(\tau_{\lambda,N}(n)) = n$ where $\tau_{\lambda,N}(n) := b_n$. So, there is a unique sequence $(\tau_{\lambda,l})_{l\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)^{\mathbb{N}}$ such that $\tau_{\lambda,l} := (\tau_{\lambda,l}(n))_{n\in\mathbb{N}}$ is a partition of $\rho_l(\lambda)$ and $\tau_{\lambda,l}(n)N_\lambda = \tau_{\lambda,l}(n)\mathrm{Ker}\,(T-\lambda I)^l$ is a strictly n-homogeneous Kaplansky–Hilbert module over $\tau_{\lambda,l}(n)\Lambda$. Moreover, $\tau_{\lambda,l}(n) \leq \tau_{\lambda,l+1}(n)$ and $\tau_{\lambda,l}(n) \wedge \tau_{\lambda,k}(m) = 0$ are satisfied for all $k, l, m, n \in \mathbb{N}$ with $n \neq m$. So, $(\tau_\lambda(n))_{n\in\mathbb{N}}$ is a partition of $[\lambda]$ where $\tau_\lambda(n) := \sup\{\tau_{\lambda,l}(n) : l \in \mathbb{N}\}$.

Definition 5.1.3. Let λ be a global eigenvalue of T. We call

$$\overline{\tau}_\lambda := o\text{-}\sum_{n\in\mathbb{N}} n\tau_\lambda(n) = o\text{-}\sum_{n\in\mathbb{N}} n \sup_{l\in\mathbb{N}}\{\tau_{\lambda,l}(n)\} = \sup_{n,l\in\mathbb{N}}\{n\tau_{\lambda,l}(n)\} \in (\mathrm{Re}\Lambda)^\infty$$

the *multiplicity* of global eigenvalue λ of T.

5.2 Global Eigenvalues of Cyclically Compact Operators on $C_\#(Q,H)$

In this section, we give some characterizations about the global eigenvalues of cyclically compact operators on $C_\#(Q,H)$ and prove that there exists a sequence consisting of global eigenvalues satisfying the corresponding properties. Throughout this section, A_λ will denote the clopen set corresponding to the projection $[\lambda]$ $(\lambda \in \Lambda)$.

Proposition 5.2.1. *Let* $U = S_{\widetilde{u}}$ *be a cyclically compact operator on* $C_\#(Q,H)$ *and a nonzero* $\lambda \in C(Q)$*. If for some meager* A_0 *of* Q*,* $\lambda(q)$ *is a nonzero eigenvalue of* $u(q)$ *for all* $q \in A_\lambda \setminus A_0$*, then* λ *is a global eigenvalue of* U*.*

Proof. Let q be element of $A_\lambda \setminus A_0$. Consider the set

$$B_q := \left\{ U\widetilde{x} - \lambda\widetilde{x} : u(q)x(q) = \lambda(q)x(q), \ \left|\widetilde{x}\right| = [\lambda] \right\}.$$

Since $\lambda(q)$ is an eigenvalue of $u(q)$, B_q is a non-empty finitely cyclic subset. If we define

$$r := \inf \left\{ \left|U\widetilde{x} - \lambda\widetilde{x}\right| : U\widetilde{x} - \lambda\widetilde{x} \in B_q \right\},$$

then it follows from $\left|U\widetilde{x} - \lambda\widetilde{x}\right|(q) = \|u(q)x(q) - \lambda(q)x(q)\| = 0$ that $r(q) = 0$. Moreover, given $q' \in A_\lambda \setminus A_0$ with $q \neq q'$. Then there is $\widetilde{x} \in C_\#(Q, H)$ such that $U\widetilde{x} - \lambda\widetilde{x} \in B_q \cap B_{q'}$. Thus, from $\left|U\widetilde{x} - \lambda\widetilde{x}\right|(q') = 0$, we have $r(q') = 0$, i. e., $r = 0$. Using [23, 2.2.9.(1) and 8.1.8.(3)], there exists $(\widetilde{x}_n)_{n\in\mathbb{N}}$ with $\left|\widetilde{x}_n\right| = [\lambda]$ such that for each $n \in \mathbb{N}$

$$\left|U\widetilde{x}_n - \lambda\widetilde{x}_n\right| \leq \frac{1}{n}\mathbf{1}.$$

As U is cyclically compact, there is a cyclic subsequence $(U\widetilde{x}_{\nu_n})_{n\in\mathbb{N}}$ of $(U\widetilde{x}_n)_{n\in\mathbb{N}}$ which is norm-convergent to some $\widetilde{y}$, and since the following is valid

$$\left|U\widetilde{x}_{\nu_n} - \lambda\widetilde{x}_{\nu_n}\right| \leq \frac{1}{n}\mathbf{1}$$

for every n, we have $(\lambda\widetilde{x}_{\nu_n})_{n\in\mathbb{N}}$ is norm-convergent to $\widetilde{y}$. Therefore, $U\widetilde{y} = \lambda\widetilde{y}$ and $\left|\widetilde{y}\right| = |\lambda|$ are satisfied. From Proposition 3.3.10, λ is a global eigenvalue of U. □

Lemma 5.2.2. *Let $U = S_{\widetilde{u}}$ be in $\operatorname{End}(C_\#(Q, H))$ and the function λ be a global eigenvalue of U. Then there is a meager subset B_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus B_0$.*

Proof. By Proposition 3.3.10 $U\widetilde{x} = \lambda\widetilde{x}$ is satisfied for some $\widetilde{x} \in C_\#(Q, H)$ with $\left|\widetilde{x}\right| = [\lambda]$. Thus, $u(q)x(q) = \lambda(q)x(q)$ holds for all $q \in Q_0 := \operatorname{dom}(u) \cap \operatorname{dom}(x)$ which is comeager. Define $V_\lambda := \{q \in Q : \lambda(q) \neq 0\}$, and so it is open in Q. It follows from $A_\lambda = \operatorname{cl}(V_\lambda)$ that $A_\lambda \setminus V_\lambda$ is a nowhere dense set in Q. Therefore, $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in V_\lambda \cap Q_0$ since $A_\lambda = \left\{q \in Q : \left|x\right|(q) \neq 0\right\}$. Define $B_0 := Q_0^c \cup (A_\lambda \setminus V_\lambda)$ and note that B_0 is a meager set and $A_\lambda \setminus B_0 = V_\lambda \cap Q_0$. This completes the proof of the lemma. □

Note that in a topological space $K \setminus \operatorname{int}(K)$ is a nowhere dense set for every closed set K. We denote by π_V the projection corresponding to clopen set V in Q.

Lemma 5.2.3. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q, H)$ and λ be a global eigenvalue of U. Then there is a meager subset A_0 such that*

$$\operatorname{Ker}(U - \lambda I)(q) := (\operatorname{Ker}(U - \lambda I))(q) = \operatorname{Ker}(u(q) - \lambda(q)I)$$

hold for all $q \in A_\lambda \setminus A_0$.

Proof. Clearly, $q \in \text{dom}(u)$ implies $\text{Ker}\,(U - \lambda I)(q) \subset \text{Ker}\,(u(q) - \lambda(q)I)$. As U is cyclically compact operator, there exists a partition of $[\lambda]$, $(b_k)_{k\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)$ such that $b_n\text{Ker}\,(U - \lambda I)$ is a strictly n-homogeneous Kaplansky−Hilbert module over $b_n C(Q)$. Fix $k \in \mathbb{N}$. Let $\{\widetilde{e}_i : i = 1, ..., k\}$ be a basis for $b_k\text{Ker}\,(U - \lambda I)$. By Lemma 3.4.4, we have a meager set A_k such that $\{e_i(q) : i = 1, ..., k\}$ is a basis of $\text{Ker}\,(U - \lambda I)(q)$ for all $q \in V_k \setminus A_k$ where V_k is clopen set corresponding to the projection b_k. From the lemma above we obtain a meager subset B_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus B_0$. Now we will show that the set

$$C_k := \{q \in V_k \setminus (A_k \cup B_0) : \text{Ker}\,(U - \lambda I)(q) \neq \text{Ker}\,(u(q) - \lambda(q)I)\}$$

is meager. Note that $\lambda(q) \neq 0$ $(q \in C_k)$. Define $\Delta := \Phi \cap C_k$ where $\Phi := \text{int}(\text{cl}(C_k))$, and note that $\Phi = \text{cl}(\Delta)$, and $C_k \setminus \Phi = C_k \setminus \Delta$ is a nowhere dense set in Q. On the other hand, for every $q \in \Delta$ there exists $h_q \in \text{Ker}\,(u(q) - \lambda(q)I) \cap \text{Ker}\,(U - \lambda I)(q)^\perp$ with $\|h_q\| = 1$. Since $u - \lambda I : \text{dom}(u) \to B(H)$ is continuous in the strong operator topology there exists a clopen set $U_{q,n} \subset \Phi$ for each $n \in \mathbb{N}$ and $q \in \Delta$ such that

$$\|(u(w) - \lambda(w)I)h_q - (u(q) - \lambda(q)I)h_q\| = \|u(w)h_q - \lambda(w)h_q\| \leq \frac{1}{n}$$

and

$$\left|\left\langle \widetilde{e}_i \mid \widetilde{h}_q \right\rangle(w) - \left\langle \widetilde{e}_i \mid \widetilde{h}_q \right\rangle(q)\right| = \left|\left\langle \widetilde{e}_i \mid \widetilde{h}_q \right\rangle(w)\right| \leq \frac{1}{n} \quad (i = 1, ..., k)$$

for $w \in U_{q,n} \cap \text{dom}(u)$ where $h_q : t \mapsto h_q$ $(t \in Q)$. Thus, it follows from $U_{q,n} = \text{cl}\,(U_{q,n} \cap \text{dom}(u))$ that

$$\pi_{q,n}\left|U\widetilde{h}_q - \lambda\widetilde{h}_q\right| \leq \frac{1}{n}\pi_{q,n} \text{ and } \pi_{q,n}\left|\left\langle \widetilde{e}_i \mid \widetilde{h}_q \right\rangle\right| \leq \frac{1}{n}\pi_{q,n} \quad (i = 1, ..., k)$$

where $\pi_{q,n} := \pi_{U_{q,n}}$. From $\Phi = \text{cl}(\Delta)$ we get $\pi = \bigvee_{q\in\Delta}\pi_{q,n}$ where $\pi := \pi_\Phi$. In view of the Exhaustion Principle, there exists an antichain (μ_α) such that $\pi = \bigvee \mu_\alpha$ and for every α there is $q \in \Delta$ with $\mu_\alpha \leq \pi_{q,n}$, and denote $q_\alpha := q$. If we define $\widetilde{x}_n = bo\text{-}\sum \mu_\alpha \widetilde{h}_{q_\alpha}$ where $h_{q_\alpha} : t \mapsto h_{q_\alpha}$, then we have $\left|\widetilde{x}_n\right| = \pi$ and

$$\left|U\widetilde{x}_n - \lambda\widetilde{x}_n\right| \leq \frac{1}{n}\pi \text{ and } \pi\left|\langle \widetilde{e}_i \mid \widetilde{x}_n\rangle\right| \leq \frac{1}{n}\pi \quad (i = 1, ..., k).$$

Since U is cyclically compact operator, there is a cyclic subsequence $(U\widetilde{x}_{\nu_n})_{n\in\mathbb{N}}$ of $(U\widetilde{x}_n)_{n\in\mathbb{N}}$ which is norm-convergent to some $\widetilde{x}$. So, $(\lambda\widetilde{x}_{\nu_n})_{n\in\mathbb{N}}$ is also norm-convergent to $\widetilde{x}$. This implies $U\widetilde{x} = \lambda\widetilde{x}$ and $\pi\left|\lambda\right| = \left|\widetilde{x}\right|$ and $\pi\left|\langle \widetilde{e}_i \mid \widetilde{x}\rangle\right| = 0$ $(i = 1, ..., k)$. Thus, $x(q) \in \text{Ker}\,(U - \lambda I)(q) \subset \text{Ker}\,(u(q) - \lambda(q)I)$ and $\langle e_i(q), x(q)\rangle = 0$

$(i = 1, ..., k)$ are satisfied for all $q \in \Delta \cap \mathrm{dom}(x)$, and so $x(q) = 0$. This and $\pi\left|\lambda\right| = \left|\widetilde{x}\right|$ imply $\lambda(q) = 0$ for all $q \in \Delta \cap \mathrm{dom}(x)$, and so $\Delta \subset \mathrm{dom}(x)^c$, i. e., Δ is meager. Hence, it follows from $C_k = (C_k \setminus \Delta) \cup \Delta$ that C_k is meager. Thus, $B_k := A_k \cup B_0 \cup C_k$ is meager for each $k \in \mathbb{N}$. So, $\mathrm{Ker}\,(U - \lambda I)(q) = \mathrm{Ker}\,(u(q) - \lambda(q) I)$ holds for all $q \in A_\lambda \setminus A_0$ where $A_0 = \left(A_\lambda \setminus \left(\bigcup_{k\in\mathbb{N}} V_k\right)\right) \cup \left(\bigcup_{k\in\mathbb{N}} B_k\right)$ is meager, as desired. □

Corollary 5.2.4. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_{\#}(Q, H)$ and λ be a global eigenvalue of U. Then there exists a meager set B_0 such that for all $q \in A_\lambda \setminus B_0$ the following statements hold:*

(1) *$\lambda(q)$ is a nonzero eigenvalue of compact operator $u(q)$;*

(2) *$\left(\mathrm{Ker}\,(U - \lambda I)^k\right)(q) = \mathrm{Ker}\,(u(q) - \lambda(q) I)^k$ $(k \in \mathbb{N})$;*

(3) *$N_\lambda(q) = N_{\lambda(q)}$ where $N_{\lambda(q)}$ is the generalized eigenspace, corresponding to the eigenvalue $\lambda(q)$;*

(4) *$\overline{\tau}_\lambda(q) = m(\lambda(q))$ where $m(\lambda(q))$ is the algebraic multiplicity of $\lambda(q)$.*

Proof. For all $q \in Q$ the following equality is satisfied

$$N_\lambda(q) = \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\,(U - \lambda I)^n(q).$$

Moreover, for every $n \in \mathbb{N}$ there exists a cyclically compact operator U_n such that $(U - \lambda I)^n = U_n - \lambda^n I$ where λ^n is a global eigenvalue of U_n. Using this and Lemma 5.2.3 we obtain a meager subset A_n such that $\mathrm{Ker}\,(U - \lambda I)^n(q) = \mathrm{Ker}\,(u(q) - \lambda(q) I)^n$ holds for all $q \in A_\lambda \setminus A_n$. Define $A_0 := \cup_{n\in\mathbb{N}} A_n$ and we deduce that A_0 is meager and

$$N_\lambda(q) = \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\,(u(q) - \lambda(q) I)^n \quad (q \in A_\lambda \setminus A_0).$$

This means that

$$N_\lambda(q) = N_{\lambda(q)} \quad (q \in A_\lambda \setminus A_0),$$

and so $\dim(N_\lambda(q)) = \dim\left(N_{\lambda(q)}\right) = m(\lambda(q))$. On the other hand, from definition of $\overline{\tau}_\lambda$ and Lemma 3.4.4, there exists a meager set C_0 such that $\overline{\tau}_\lambda(q) = \dim(N_\lambda(q))$ holds for every $q \in A_\lambda \setminus C_0$. Therefore, if we define $B_0 := A_0 \cup C_0$, then the proof is finished. □

Denote by $\mathrm{Sp}^*(u(q)) := \mathrm{Sp}(u(q)) \setminus \{0\}$ the set of nonzero elements of the spectrum of $u(q)$.

Lemma 5.2.5. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_{\#}(Q,H)$ and $\lambda_q \in \mathrm{Sp}^*(u(q))$ for all $q \in A_u \subset Q$. If A_u is not meager in Q, then there are a global eigenvalue λ of U and a comeager set Q_0 that satisfy the following conditions:*

(1) $[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ *where π_N is the projection corresponding to clopen set* $U_N := \mathrm{int}\,(\mathrm{cl}\,(A_N))$ *with*

$$A_N := \left\{ q \in A_u : |\lambda_q| \geq \frac{1}{N} \right\};$$

(2) $\pi_N|\lambda| \geq \frac{1}{N}\pi_N$ $(N \in \mathbb{N})$*;*

(3) *if q is in $Q_0 \cap A_N$ for some $N \in \mathbb{N}$, then $|\lambda(q)| \geq \frac{1}{N}$;*

(4) $\lambda(q) \in \mathrm{Sp}^*(u(q))$ *whenever $q \in Q_0$ and $\lambda(q) \neq 0$;*

(5) $q \notin A_u$ *whenever $q \in Q_0$ and $\lambda(q) = 0$.*

Proof. Without loss of generality we may assume that $u(q)$ is compact operator on H for all $q \in \mathrm{dom}(u)$ by Proposition 3.4.7. There exists an eigenvector h_q of $u(q)$ corresponding to λ_q with $\|h_q\| = 1$ for all $q \in A_u$. Since $A_u = \bigcup_{N\in\mathbb{N}} A_N$ is not meager, $U_{N_0} \neq \varnothing$ for some N_0, i. e., $\pi_{N_0} \neq 0$. Since u is continuous in the strongly operator topology, for every $n \in \mathbb{N}$ and $q \in U_N \cap A_N$ we have a clopen set $U_{q,n,N} \subset U_N$ such that

$$\|u(w)h_q - u(q)h_q\| = \|u(w)h_q - \lambda_q h_q\| \leq \frac{1}{n}$$

for each $w \in U_{q,n,N} \cap \mathrm{dom}(u)$. Moreover, it follows from $U_N = \mathrm{cl}(U_N \cap A_N)$ that $\pi_N = \bigvee_{q\in U_N\cap A_N} \pi_{q,n,N}$ where $\pi_{q,n,N} := \pi_{U_{q,n,N}}$. In view of the Exhaustion Principle, there exists an antichain (μ_α) such that $\pi_N = \bigvee \mu_\alpha$ and for every α there is $q \in U_N \cap A_N$ with $\mu_\alpha \leq \pi_{q,n,N}$, and denote $q_\alpha := q$. If we define $\lambda_n^N := o\text{-}\sum \lambda_{q_\alpha}\mu_\alpha$ and $x_n^N = bo\text{-}\sum \mu_\alpha z_\alpha$ where $z_\alpha : t \mapsto h_{q_\alpha}$ $(t \in Q)$, then $\left|x_n^N\right| = \left[\lambda_n^N\right] = \pi_N$ and $\left|\lambda_n^N\right| \geq \frac{1}{N}\pi_N$. Since $(\lambda_n^N)_{n\in\mathbb{N}}$ is a bounded sequence in $C(Q)$, there exists a cyclical subsequence $\left(\lambda_{\nu_n}^N\right)_{n\in\mathbb{N}}$ of $(\lambda_n^N)_{n\in\mathbb{N}}$ which is norm-convergent to some λ_N, and so $|\lambda_N| \geq \frac{1}{N}\pi_N$ and $[\lambda_N] = \pi_N$ are satisfied. On the other hand, we have

$$\left|Ux_n^N - \lambda_n^N x_n^N\right| \leq \frac{1}{n}\pi_N,$$

and so

$$\left|Ux_{\nu_n}^N - \lambda_{\nu_n}^N x_{\nu_n}^N\right| \leq \frac{1}{n}\pi_N.$$

As U is cyclically compact operator, there is a cyclic subsequence $\left(Ux^N_{\eta_n}\right)_{n\in\mathbb{N}}$ of $\left(Ux^N_{\nu_n}\right)_{n\in\mathbb{N}}$ which is norm-convergent to some x_N, and since the following is valid for every n

$$\left|Ux^N_{\eta_n} - \lambda^N_{\eta_n} x^N_{\eta_n}\right| \leq \frac{1}{n}\pi_N,$$

$\left(\lambda^N_{\eta_n} x^N_{\eta_n}\right)_{n\in\mathbb{N}}$ is norm-convergent to x_N. Thus, we have $Ux_N = \lambda_N x_N$ and $\left|\lambda_N\right| = \left|x_N\right|$, and so λ_N is a global eigenvalue of U for $N \geq N_0$. Define $x := \pi_1 x_1 + bo\text{-}\sum_{N\in\mathbb{N}}(\pi_{N+1} - \pi_N)x_{N+1}$ and $\lambda := \pi_1\lambda_1 + o\text{-}\sum_{N\in\mathbb{N}}(\pi_{N+1} - \pi_N)\lambda_{N+1}$, and note that $\left|x\right| = |\lambda|$ and $[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ and $\pi_N|\lambda| \geq \frac{1}{N}\pi_N$ and λ is a global eigenvalue of U since $\pi_{N_0} > 0$. This implies (1) and (2). From Proposition 5.2.2, there exists a meager set A_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus A_0$ where A_λ is the clopen set corresponding to the projection $[\lambda]$. If we define

$$Q_0^c := \left(\bigcup_{N\in\mathbb{N}} (A_N \setminus U_N)\right) \cup \operatorname{dom}(u)^c \cup A_0,$$

then Q_0 is a comeager set, and (3), (4) and (5) are satisfied. □

Lemma 5.2.6. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q,H)$ and let Σ be a finite subset of $C(Q)$ and the set*

$$A_u \subset \{q \in \operatorname{dom}(u) : \operatorname{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\} \neq \varnothing\}$$

be not meager in Q. If λ_q is in $\operatorname{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$ for each $q \in A_u$, then there is a global eigenvalue λ of U and a comeager set Q_0 that satisfy the following conditions:

(1) $[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ *where π_N is the projection corresponding to clopen set $U_N := \operatorname{int}(\operatorname{cl}(A_N))$ with*

$$A_N := \{q \in A_u : (\forall \sigma \in \Sigma) |\sigma(q) - \lambda_q| \geq 1/N \text{ and } |\lambda_q| \geq 1/N\};$$

(2) $\pi_N|\lambda| \geq \frac{1}{N}\pi_N$ *and* $\pi_N|\sigma - \lambda| \geq \frac{1}{2N}\pi_N$ $(N \in \mathbb{N}, \sigma \in \Sigma)$;

(3) *if q is in $A_N \cap Q_0$, then $|\lambda(q)| \geq \frac{1}{N}$ and $|\sigma(q) - \lambda(q)| \geq \frac{1}{2N}$ hold for each $\sigma \in \Sigma$;*

(4) *if $\lambda(q) \neq 0$ holds for some $q \in Q_0$, then $\lambda(q) \in \operatorname{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$;*

(5) *if $\lambda(q) = 0$ holds for some $q \in Q_0$, then $q \notin A_u$.*

Proof. The proof is similar to the proof of Lemma 5.2.5. $U_{N_0} \neq \varnothing$ for some N_0 since $A_u = \bigcup_{N\in\mathbb{N}} A_N$ is not meager. Let h_q be an eigenvector of $u(q)$ corresponding to λ_q with $\|h_q\| = 1$ for every $q \in A_u$. For every $N, n \in \mathbb{N}$ and $q \in U_N \cap A_N$ we can find clopen set $U_{q,n,N} \subset U_N$ such that

$$\|u(w)h_q - \lambda_q h_q\| \leq \frac{1}{n} \text{ and } |\sigma(w) - \lambda_q| \geq \frac{1}{2N} \quad (\sigma \in \Sigma)$$

for all $w \in U_{q,n,N} \cap \mathrm{dom}(u)$. As in the proof of Lemma 5.2.5, we can find a global eigenvalue λ_N of U such that $[\lambda_N] = \pi_N$ and

$$|\lambda_N| \geq \frac{1}{N}\pi_N \text{ and } \pi_N|\sigma - \lambda_N| \geq \frac{1}{2N}\pi_N \quad (\sigma \in \Sigma).$$

Therefore, if we define

$$\lambda := \pi_1\lambda_1 + o\text{-}\sum_{N\in\mathbb{N}} (\pi_{N+1} - \pi_N)\lambda_{N+1},$$

then $[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ and λ is a global eigenvalue of U since $\pi_{N_0} > 0$. This implies (1) and (2). From Proposition 5.2.2, there exists a meager set A_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus A_0$. If we define

$$Q_0^c := \left(\bigcup_{N\in\mathbb{N}} (A_N \setminus U_N)\right) \cup \mathrm{dom}(u)^c \cup \left(A_\lambda \setminus \bigcup_{N\in\mathbb{N}} U_N\right) \cup A_0,$$

then Q_0 is a comeager set, and (3), (4) and (5) are satisfied. □

Proposition 5.2.7. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q, H)$. Then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$ and a comeager set Q_0 that satisfy the following conditions:*

(1) $\mathrm{Sp}^*(u(q)) = \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}$ $(q \in Q_0)$*;*

(2) $\lambda_{n+1} = 0$ *whenever* $\lambda_n = 0$*;*

(3) $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$*;*

(4) $\lambda_n(q) \neq \lambda_k(q)$ *whenever* $\lambda_n(q) \neq 0$ *or* $\lambda_k(q) \neq 0$ *for* $n \neq k$ $(q \in Q_0)$*.*

Proof. Without loss of generality we may assume that $u(q)$ is a compact operator on H $(q \in \mathrm{dom}(u))$. We shall construct by induction a sequence (λ_n) consisting of global eigenvalues or zeros, and a decreasing comeager set sequence (Q_n) such that:

(i) For $q \in Q_n$, $\lambda_n(q) \in \mathrm{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, ..., n-1\}$ whenever $\lambda_n(q) \neq 0$.

(ii) For $q \in Q_n$, $\mathrm{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, ..., n-1\} = \varnothing$ whenever $\lambda_n(q) = 0$.

(iii) $\mathrm{Sp}^*(u(q)) = \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}$ is satisfied for all $q \in Q_0 := \bigcap Q_n$.

For $n = 1$. If

$$A_{u_1} := \{q \in \mathrm{dom}(u) : \mathrm{Sp}^*(u(q)) \neq \varnothing\}$$

is meager, then we take $\lambda_n := 0$ and $Q_n := Q \setminus A_{u_1} = \{q \in \mathrm{dom}(u) : \mathrm{Sp}^*(u(q)) = \varnothing\}$ for every n. If not, we can choose a family $(\lambda_q)_{q \in A_{u_1}}$ so that $\lambda_q \in \mathrm{Sp}^*(u(q))$ and $|\lambda_q|$ is the maximum element of $|\mathrm{Sp}^*(u(q))|$ for all $q \in A_{u_1}$. Thus, from Lemma 5.2.5 we get a global eigenvalue λ_1 and there exists comeager set Q_1 such that (i) and (ii) are satisfied. Now suppose that the elements λ_n and Q_n are already constructed as above. If the set

$$A_{u_{n+1}} := \{q \in Q_n : \mathrm{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, ..., n\} \neq \varnothing\}$$

is meager, then we take $\lambda_{k+1} := 0$ and

$$Q_{k+1} := \{q \in Q_k : \mathrm{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, ..., k\} = \varnothing\} = Q_n \cap \left(A_{u_{n+1}}\right)^c$$

for every $k \geq n$. If not, we can choose a family $(\lambda_q)_{q \in A_{u_{n+1}}}$ so that $\lambda_q \in \mathrm{Sp}^*(u(q))$ and $|\lambda_q|$ is the maximum element of $|\mathrm{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, ..., n\}|$ for all $q \in A_{u_{n+1}}$. Thus, it follows from Lemma 5.2.6 that we get a global eigenvalue λ_{n+1} and there exists comeager set $Q_{n+1} \subset Q_n$ such that (i) and (ii) are satisfied. From (i), $\{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\} \subset \mathrm{Sp}^*(u(q))$ holds for every n. Assume $\mathrm{Sp}^*(u(q)) \neq \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}$ for some $q \in Q_0$. Thus, there exists $\mu_q \in \mathrm{Sp}^*(u(q)) \setminus \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}$ such that $|\mu_q|$ is the maximum of $|\mathrm{Sp}^*(u(q)) \setminus \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}|$. Moreover, there is $N \in \mathbb{N}$ such that

$$|\mu_q| \geq \frac{1}{N} \text{ and } |\lambda_n(q) - \mu_q| \geq \frac{1}{N} \quad (n \in \mathbb{N}).$$

Thus, there exists $K \in \mathbb{N}$ such that $q \in A_{u_k}$ and $|\mu_q|$ is the maximum of $|\mathrm{Sp}^*(u(q)) \setminus \{\lambda_n(q) : n = 1, ..., k-1\}|$ for $k \geq K$. It follows from Lemma 5.2.6 (3) that

$$|\lambda_k(q)| \geq \frac{1}{N} \quad (K \leq k)$$

which contradicts $\lambda_k(q)$ converging to zero, and so (iii) holds. Clearly, (i), (ii) and (iii) complete the proof of the proposition. □

It is well known that if $(\lambda_\alpha)_{\alpha \in A}$ is a net in $C(Q)$, then $\inf_{\alpha \in A} \lambda_\alpha = 0$ in $C(Q)$ iff there exists some comeager set Q_0 in Q such that $\inf_{\alpha \in A} \lambda_\alpha(q) = 0$ in $\mathbb{R}$ for all $q \in Q_0$. Thus, if $f_\alpha \overset{(o)}{\to} 0$ in $C(Q)$, then $f_\alpha(q) \to 0$ holds on some comeager set.

Conversely, if $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $C(Q)$ and $f_n(q) \to 0$ is satisfied on a comeager set, then $(f_n)_{n\in\mathbb{N}}$ o-converges to 0. Note that $\left|U\right|(q) = \|u(q)\|$ holds on a comeager subset of Q for each $U = S_{\widetilde{u}}$ in $\mathrm{End}\,(C_{\#}\,(Q,H))$.

In what follows, the phrase "consisting of global eigenvalues or zeros," refers to the fact that λ_k is a global eigenvalue whenever $\lambda_k \neq 0$.

Theorem 5.2.8. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_{\#}\,(Q,H)$. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting of global eigenvalues or zeros in $C(Q)$ with the following properties:*

(1) $|\lambda_k| \leq \left|U\right|$, $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim \lambda_k = 0$;

(2) *there exists a projection π_∞ in $C(Q)$ such that $\pi_\infty|\lambda_k|$ is a weak order-unity in $\pi_\infty C(Q)$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition $(\pi_k)_{k\in\mathbb{N}}$ of the projection $\pi_\infty^\perp$ such that $\pi_0\lambda_1 = 0$, $\pi_k \leq [\lambda_k]$, and $\pi_k\lambda_{k+m} = 0$, $m, k \in \mathbb{N}$;*

(4) *$\pi\lambda_{k+m} \neq \pi\lambda_k$ for every nonzero projection $\pi \leq \pi_\infty + \pi_k$ and for all $m, k \in \mathbb{N}$;*

(5) *every global eigenvalue λ of U is of the form $\lambda = \mathrm{mix}_{k\in\mathbb{N}}\,(p_k\lambda_k)$, where $(p_k)_{k\in\mathbb{N}}$ is a partition of $[\lambda]$.*

Proof. By Proposition 5.2.7, we have a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting global eigenvalues or zeros. Define $\pi_\infty := \bigwedge_{k\in\mathbb{N}} [\lambda_k]$ and $\pi_0 := [\lambda_1]^\perp$ and $\pi_k := [\lambda_k] \wedge [\lambda_{k+1}]^\perp$ $(k \in \mathbb{N})$, and so (2), (3) and (4) hold. Since $\mathrm{Sp}^*(u(q)) = \{\lambda_n(q) : \lambda_n(q) \neq 0\}$ and $|\lambda_n(q)| \leq \|u(q)\|$ hold on a comeager set we have $|\lambda_n| \leq \left|U\right|$ and $\lim_{k\to\infty} \lambda_k(q) = 0$ on a comeager set, and so we obtain $o\text{-}\lim \lambda_k = 0$, i. e., (1) holds. Let λ be a global eigenvalue of U. From Lemma 5.2.2 and Proposition 5.2.7, there exists some meager set A_0 such that $A_\lambda \setminus A_0 = \bigcup_{k\in\mathbb{N}} A_k$ where

$$A_k := \{q \in A_\lambda \setminus A_0 : \lambda(q) = \lambda_k(q)\} \quad (k \in \mathbb{N}).$$

Since $A_k \setminus \mathrm{int}\,(\mathrm{cl}\,(A_k))$ is nowhere dense, $[\lambda] = \bigvee_{k\in\mathbb{N}} \mu_k$ and $\mu_k\lambda = \mu_k\lambda_k$ where μ_k denotes the projection corresponding clopen set $\mathrm{int}\,(\mathrm{cl}\,(A_k))$. Thus, there exists a partition $(p_k)_{k\in\mathbb{N}}$ of $[\lambda]$ such that $\lambda = \mathrm{mix}_{k\in\mathbb{N}}\,(p_k\lambda_k)$, and the proof is finished. □

Let $(\lambda_k)_{k\in\mathbb{N}}$ be as above theorem. Note that the statements of the Proposition 5.2.7 is satisfied by $(\lambda_k)_{k\in\mathbb{N}}$.

Let the family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals satisfy functional representation of X as in Theorem 3.1.18. Let T be a cyclically compact operator on $\sum_{\gamma\in\Gamma}^{\oplus} C_{\#}(Q_\gamma, \ell_2(\gamma))$. From Theorem 3.4.9, T_γ is a cyclically compact operator on $C_{\#}(Q_\gamma, \ell_2(\gamma))$ for all $\gamma \in \Gamma$ where $\mathscr{P}(T) = (T_\gamma)_{\gamma\in\Gamma}$. So, we have a sequence $(\lambda_k(T_\gamma))_{k\in\mathbb{N}}$ satisfing the statements of the Theorem 5.2.8 for each $\gamma \in \Gamma$. Define $\lambda_k(T) := (\lambda_k(T_\gamma))_{\gamma\in\Gamma}$, and note that $\lambda_k(T)$ is a global eigenvalue of T or zero for each $k \in \mathbb{N}$. On the other hand, given a global eigenvalue $\lambda = (\lambda_\gamma)_{\gamma\in\Gamma}$ of T, the function λ_γ is a global eigenvalue of T_γ whenever $\lambda_\gamma \neq 0$. Let $p_k = (p_{k,\gamma})_{\gamma\in\Gamma}$ be a projection for each $k \in \mathbb{N}$. Then a partition $(p_k)_{k\in\mathbb{N}}$ of $[\lambda]$ means that $(p_{k,\gamma})_{k\in\mathbb{N}}$ is a partition of $[\lambda_\gamma]$ for every $\gamma \in \Gamma$ since $[\lambda] = ([\lambda_\gamma])_{\gamma\in\Gamma}$. Using the representation of Kaplansky−Hilbert modules we can also generalize the theorem above as following.

Theorem 5.2.9. *Let T be a cyclically compact operator on X. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting of global eigenvalues or zeros in Λ with the following properties:*

(1) $|\lambda_k| \leq |\!|T|\!|$, $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim \lambda_k = 0$*;*

(2) *there exists a projection π_∞ in Λ such that $\pi_\infty|\lambda_k|$ is a weak order-unity in $\pi_\infty\Lambda$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition (π_k) of the projection $\pi_\infty^\perp$ such that $\pi_0\lambda_1 = 0$, $\pi_k \leq [\lambda_k]$, and $\pi_k\lambda_{k+m} = 0$, $m, k \in \mathbb{N}$;*

(4) *$\pi\lambda_{k+m} \neq \pi\lambda_k$ for every nonzero projection $\pi \leq \pi_\infty + \pi_k$ and for all $m, k \in \mathbb{N}$;*

(5) *every global eigenvalue λ of T is of the form $\lambda = \mathrm{mix}_{k\in\mathbb{N}}(p_k\lambda_k)$, where $(p_k)_{k\in\mathbb{N}}$ is a partition of $[\lambda]$.*

Let $(\lambda_k)_{k\in\mathbb{N}}$ be as above theorem. If $\lambda_k \neq 0$, then by definition of $(\rho_l(\lambda_k))_{l\in\mathbb{N}}$ and $(\tau_{\lambda_k,l}(n))_{n\in\mathbb{N}}$ there exists a unique sequence $\tau_{k,l}(n) := \tau_{\lambda_k,l}(n) \wedge \rho_{l-1}^\perp(\lambda_k)$ such that $(\tau_{k,l}(n))_{l,n\in\mathbb{N}}$ is a partition of $[\lambda_k]$ for every $k \in \mathbb{N}$, and $\tau_{k,l}(n) \neq 0$ implies $\tau_{k,l}(n)\lambda_k$ is a global eigenvalue of T of multiplicity n, i. e., $\tau_{k,l}(n)N_{\lambda_k} = \tau_{k,l}(n)\mathrm{Ker}\,(T - \lambda_k I)^l$ is n-homogenuous. If $\lambda_k = 0$, take $\tau_{k,l}(n) := 0$ and $\overline{\tau}_{\lambda_k} := 0$ for all $k, l, n \in \mathbb{N}$.

Definition 5.2.10. The sequence $(\lambda_k(T))_{k\in\mathbb{N}}$, where $\lambda_k(T) := \lambda_k$ is given by the above theorem, is called a *global eigenvalue sequence* of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$ where $\overline{\tau}_k(T) := \overline{\tau}_{\lambda_k}$.

5.3 Weyl and Horn Inequalities and Lidskiĭ Trace Formula

In this section, we give a variant of Weyl- and Horn-type inequalities and Lidskiĭ trace formula for cyclically compact operator on X. Throughout this section, sequences $(\widetilde{e}_k)_{k\in\mathbb{N}}$ and $(\widetilde{f}_k)_{k\in\mathbb{N}}$ in $C_\#(Q,H)$ and positive functions sequence $(s_k(U))_{k\in\mathbb{N}}$ in $C(Q)$ will satisfy the statements of Proposition 3.4.11 for cyclically compact operator U on $C_\#(Q,H)$.

Lemma 5.3.1. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q,H)$. The following statements are satisfied on a comeager subset of Q:*

(1) *the numbers $s_k(U)(q)$ are the singular numbers of compact operator $u(q)$ and*

$$u(q)h = \sum_{k=1}^{\infty} s_k(U)(q)\,\langle h, e_k(q)\rangle\, f_k(q);$$

(2) $\operatorname{tr}(U)(q) = \operatorname{tr}(u(q))$ *and* $v_1(U)(q) = v_1(u(q))$ *whenever* $U \in \mathscr{S}_1(C_\#(Q,H))$.

Proof. We may assume that $u(q)$ is compact operator on H ($q \in \operatorname{dom}(u)$) by Proposition 3.4.7. Let $\widetilde{x} \in C_\#(Q,H)$ and $n \in \mathbb{N}$. Since $(s_k(U))_{k\in\mathbb{N}}$ is a decreasing sequence we have the following inequality

$$\begin{aligned}
\left|U\widetilde{x} - \sum_{k=1}^{n-1} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle\,\widetilde{f}_k\right|^2 &= \left| bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle\,\widetilde{f}_k \right|^2 \\
&= \left\langle bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle \widetilde{f}_k \,\middle|\, bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle \widetilde{f}_k \right\rangle \\
&= o\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)^2\,|\langle \widetilde{x} \mid \widetilde{e}_k\rangle|^2\,\left|\widetilde{f}_k\right| \\
&\le s_n(U)^2 \left(o\text{-}\sum_{k\in\mathbb{N}_n} |\langle \widetilde{x} \mid \widetilde{e}_k\rangle|^2 \right) \\
&\le s_n(U)^2 \left|x\right|^2
\end{aligned}$$

where $\mathbb{N}_n := \{k \in \mathbb{N} : k \ge n\}$. From $\inf_{k\in\mathbb{N}} s_k(U) = 0$, there is a comeager set Q_1 in Q with $\inf_{k\in\mathbb{N}} s_k(U)(q) = 0$ for all $q \in Q_1$. Define

$$Q_0 := Q_1 \cap \operatorname{dom}(u) \cap \left(\bigcap_{k\in\mathbb{N}} \operatorname{dom}(e_k) \right) \cap \left(\bigcap_{k\in\mathbb{N}} \operatorname{dom}(f_k) \right),$$

and note that Q_0 is a comeager set in Q. Moreover, $\{e_k(q) : k \in \mathbb{N}\} \setminus \{0\}$ and $\{f_k(q) : k \in \mathbb{N}\} \setminus \{0\}$ are orthonormal sets in H for all $q \in Q_0$. Given $h \in H$.

Define the function $z : t \mapsto h$ $(t \in Q)$, and note that $\left|\widetilde{z}\right|(t) = \|h\|$ $(t \in Q)$ and $\operatorname{dom}(z) = Q$. Therefore, for each $q \in Q_0$ the following inequality holds

$$\left\| u(q)h - \sum_{k=1}^{n} s_k(U)(q) \langle h, e_k(q) \rangle f_k(q) \right\| = \left| U\widetilde{z} - \sum_{k=1}^{n} s_k(U) \langle \widetilde{z} \mid \widetilde{e}_k \rangle \widetilde{f}_k \right| (q)$$

$$\leq s_{n+1}(U)(q) \left|\widetilde{z}\right|(q) = s_{n+1}(U)(q) \|h\| .$$

Thus, we deduce that for each $q \in Q_0$

$$u(q)h = \sum_{k=1}^{\infty} s_k(U)(q) \langle h, e_k(q) \rangle f_k(q),$$

and so the numbers $s_k(U)(q)$ are the singular numbers of the compact operator $u(q)$ by the Rayleigh$-$Ritz minimax formula [9, Theorem 15.7.1]. Now assume that $U \in \mathscr{S}_1 (C_\# (Q, H))$. Using Proposition 4.2.2 we obtain that $(s_k(U))_{k\in\mathbb{N}}$ is o-summable in $C(Q)$ and $v_1(U) = o\text{-}\sum_{k\in\mathbb{N}} s_k(U)$. Moreover, by Lemma 4.2.9 we have

$$\operatorname{tr}(U) = o\text{-}\sum_{k\in\mathbb{N}} s_k(U) \left\langle \widetilde{f}_k \mid \widetilde{e}_k \right\rangle .$$

Thus, there is a comeager set $Q_2 \subset Q_0$ such that for all $q \in Q_2$

$$\operatorname{tr}(U)(q) = \sum_{k\in\mathbb{N}} s_k(U)(q) \langle f_k(q), e_k(q) \rangle \quad \text{and} \quad v_1(U)(q) = \sum_{k\in\mathbb{N}} s_k(U)(q).$$

So, $u(q) \in \mathscr{S}_1 (H)$. Again, by Proposition 4.2.2 and Lemma 4.2.9 $\operatorname{tr}(U)(q) = \operatorname{tr}(u(q))$ and $v_1(U)(q) = v_1(u(q))$ hold on comeager set Q_2. This completes the proof of the lemma. □

Theorem 5.3.2. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\# (Q, H)$ and $(\lambda_k(U))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of U with the multiplicity sequence $(\overline{\tau}_k(U))_{k\in\mathbb{N}}$. The following statements hold:*

(1) (Weyl-inequality) *if $(\pi s_k(U))_{k\in\mathbb{N}}$ is o-summable in $C(Q)$ for some projection π, then the following inequality holds*

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(U)|\lambda_k(U)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(U);$$

(2) (Horn-inequality) *Suppose that $U_k = S_{\widetilde{u}_k}$ is a cyclically compact operator on $C_\# (Q, H)$ for $1 \leq k \leq K$. Then*

$$\prod_{i=1}^{N} s_i(U_K \cdots U_1) \leq \prod_{k=1}^{K} \prod_{i=1}^{N} s_i(U_k) \quad (N \in \mathbb{N}).$$

(3) (Lidskiĭ trace formula) *if* $U = S_{\widetilde{u}} \in \mathscr{S}_1(C_\#(Q,H))$, *then the following equality holds*

$$\operatorname{tr}(U) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(U)\lambda_k(U).$$

Proof. Let $(\lambda_k(U))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of U with the multiplicity sequence $(\overline{\tau}_k(U))_{k\in\mathbb{N}}$. From Corollary 5.2.4, Proposition 5.2.7 and Lemma 5.3.1 there exists a comeager set Q_0 such that for each $q \in Q_0$, the following statements hold:

(i) the numbers $s_k(U)(q)$ are the singular numbers of compact operator $u(q)$ and

$$u(q)h = \sum_{k=1}^{\infty} s_k(U)(q)\langle h, e_k(q)\rangle f_k(q);$$

(ii) $\operatorname{tr}(U)(q) = \operatorname{tr}(u(q))$ and $v_1(U)(q) = v_1(u(q))$ if $U \in \mathscr{S}_1(C_\#(Q,H))$;

(iii) $\operatorname{Sp}^*(u(q)) = \{\lambda_n(U)(q) : \lambda_n(U)(q) \neq 0\}$;

(iv) $\lambda_n(U)(q) \neq \lambda_m(U)(q)$ if $\lambda_n(U)(q) \neq 0$ or $\lambda_m(U)(q) \neq 0$ for $n \neq m$;

(v) if $\lambda_k(U)(q) \neq 0$, then $\overline{\tau}_k(U)(q) = m(\lambda_k(U)(q)) \in \mathbb{N}$ where $m(\lambda_k(U)(q))$ is the algebraic multiplicity of $\lambda_k(U)(q)$.

Moreover, $s_k(U)(q) \neq 0$ implies that $\|e_k(q)\| = \|f_k(q)\| = 1$.

(1) Let $(\pi s_k(U))_{k\in\mathbb{N}}$ be o-summable for some projection π. Using (i), (iii), (iv), (v) and Weyl's inequality for compact operator $u(q)$ we get that

$$\sum_{k=1}^{\infty} \overline{\tau}_k(U)(q)|\lambda_k(U)(q)| = \sum_{k=1}^{\infty} m(\lambda_k(U)(q))|\lambda_k(U)(q)| \leq \sum_{k=1}^{\infty} s_k(U)(q)$$

holds on a comeager set Q_0. This implies

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(U)|\lambda_k(U)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(U)$$

since $\sum_{k=1}^{\infty} \pi(q)s_k(U)(q)$ is finite for each $q \in Q$.

(2) $U_K \cdots U_1 = S_{\widetilde{u}_K} \cdots S_{\widetilde{u}_1} = S_{\widetilde{u}_K \cdots \widetilde{u}_1}$ and $(u_K \cdots u_1)(q) = u_K(q) \cdots u_1(q)$ are satisfied. So, using (i) there exits a comeager set Q_0 such that for each $q \in Q_0$ the numbers $s_k(U_K \cdots U_1)(q)$ and $s_k(U_k)(q)$ are the singular numbers of compact operators $u_K(q) \cdots u_1(q)$ and $u_k(q)$ $(1 \leq k \leq K)$, respectively. Therefore, from Horn's inequality for compact operators $u_k(q)$ with $1 \leq k \leq K$ we get that

$$\prod_{i=1}^{N} s_i(U_K \cdots U_1)(q) \leq \prod_{k=1}^{K}\prod_{i=1}^{N} s_i(U_k)(q)$$

holds for all $q \in Q_0$. Thus, we have the desired inequality.

(3) Let $U = S_{\widetilde{u}} \in \mathscr{S}_1(C_\#(Q, H))$. Then $(s_k(U))_{k\in\mathbb{N}}$ is o-summable in $C(Q)$ and $\upsilon_1(U) = o\text{-}\sum_{k\in\mathbb{N}} s_k(U)$. Using (i), (ii), (iii), (iv), (v) and Lidskiĭ trace formula for compact operator $u(q)$ we obtain that

$$\operatorname{tr}(U)(q) = \operatorname{tr}(u(q)) = \sum_{k=1}^{\infty} \overline{\tau}_k(U)(q)\lambda_k(U)(q)$$

is absolutely convergent on the comeager set Q_0, and so

$$\operatorname{tr}(U)(q) = \operatorname{tr}(u(q)) = \sum_{k\in\mathbb{N}} \overline{\tau}_k(U)(q)\lambda_k(U)(q)$$

holds on Q_0. From (1) we see that $(\overline{\tau}_k(U)\lambda_k(U))_{k\in\mathbb{N}}$ is o-summable in $C(Q)$, and so we have

$$\operatorname{tr}(U) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(U)\lambda_k(U),$$

as desired. □

Let $(A_\xi)_{\xi\in\Xi}$ be a family of commutative AW^*-algebras and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $A := \sum_{\xi\in\Xi}^{\oplus} A_\xi$ with $x_k := (x_{k,\xi})_{\xi\in\Xi}$ and $(\sum_{k\in\alpha} x_k)_{\alpha\in I}$ be a bounded family with $I = \mathscr{P}_{fin}(\mathbb{N})$. Then $(x_k)_{k\in\mathbb{N}}$ is o-summable in A if and only if $(x_{k,\xi})_{k\in\mathbb{N}}$ is o-summable in A_ξ for every $\xi \in \Xi$. In particular, $o\text{-}\sum_{k\in\mathbb{N}} x_k = \left(o\text{-}\sum_{k\in\mathbb{N}} x_{k,\xi}\right)_{\xi\in\Xi}$. Therefore, we have the following lemma.

Lemma 5.3.3. *Let T be a cyclically compact operator on $X := \sum_{\gamma\in\Gamma}^{\oplus} C_\#(Q_\gamma, \ell_2(\gamma))$ and $1 \le p < \infty$. Then T is in $\mathscr{S}_p(X)$ if and only if T_γ is in $\mathscr{S}_p(C_\#(Q_\gamma, \ell_2(\gamma)))$ for all $\gamma \in \Gamma$ and $\sup_{\gamma\in\Gamma} \|\upsilon_p(T_\gamma)\| < \infty$ where $\mathscr{P}(T) = (T_\gamma)_{\gamma\in\Gamma}$. In particular, if $T \in \mathscr{S}_1(C_\#(Q_\gamma, \ell_2(\gamma)))$, then $\operatorname{tr}(T) = (\operatorname{tr}(T_\gamma))_{\gamma\in\Gamma}$.*

Proof. Suppose that T in $\mathscr{S}_p(X)$. Let $(e_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, and $(s_k(T))_{k\in\mathbb{N}}$ satisfy the condition of Proposition 3.4.11 for T. Then $\upsilon_p(T)^p = o\text{-}\sum_{k\in\mathbb{N}} s_k(T)^p$. If $e_k = (e_k(\gamma))_{\gamma\in\Gamma}$, $f_k = (f_k(\gamma))_{\gamma\in\Gamma}$ and $s_k(T) = (s_k(T)(\gamma))_{\gamma\in\Gamma}$, then $(e_k(\gamma))_{k\in\mathbb{N}}$, $(f_k(\gamma))_{k\in\mathbb{N}}$, and $(s_k(T)(\gamma))_{k\in\mathbb{N}}$ satisfy the condition of Proposition 3.4.11 for T_γ. Therefore, $\upsilon_p(T)(\gamma)^p = o\text{-}\sum_{k\in\mathbb{N}} s_k(T)(\gamma)^p = \upsilon_p(T_\gamma)^p$, i. e., T_γ in $\mathscr{S}_p(C_\#(Q_\gamma, \ell_2(\gamma)))$.

Conversely, assume that T_γ is in $\mathscr{S}_p(C_\#(Q_\gamma, \ell_2(\gamma)))$ for each $\gamma \in \Gamma$ and $\sup_{\gamma\in\Gamma} \|\upsilon_p(T_\gamma)\| < \infty$. Let $(e_k(\gamma))_{k\in\mathbb{N}}$, $(f_k(\gamma))_{k\in\mathbb{N}}$, and $(s_k(T_\gamma))_{k\in\mathbb{N}}$ satisfy the condition of Proposition 3.4.11 for T_γ. Then $\upsilon_p(T_\gamma)^p = o\text{-}\sum_{k\in\mathbb{N}} s_k(T_\gamma)^p$. Denote by $e_k := (e_k(\gamma))_{\gamma\in\Gamma}$, $f_k = (f_k(\gamma))_{\gamma\in\Gamma}$ and $s_k(T) = (s_k(T_\gamma))_{\gamma\in\Gamma}$. Thus, $(e_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, and $(s_k(T))_{k\in\mathbb{N}}$ satisfy the condition of Proposition 3.4.11 for T. Since $\sup_{\gamma\in\Gamma} \|\upsilon_p(T_\gamma)\| < \infty$ we have T in $\mathscr{S}_p(X)$.

Let $T \in \mathscr{S}_1\left(C_\#\left(Q_\gamma, \ell_2(\gamma)\right)\right)$ and $\mathscr{E}$ be a projection orthonormal subset of X. Thus, $\mathscr{E}_\gamma \setminus \{0\}$ is a projection orthonormal subset of $C_\#\left(Q_\gamma, \ell_2(\gamma)\right)$ where

$$\mathscr{E}_\gamma := \{e_\gamma : e = (e_\gamma)_{\gamma\in\Gamma} \in \mathscr{E}\}$$

for each $\gamma \in \Gamma$. The proof follows immediately from $\langle Te \mid e\rangle = (\langle T_\gamma e_\gamma \mid e_\gamma\rangle)_{\gamma\in\Gamma}$ $(e = (e_\gamma)_{\gamma\in\Gamma} \in \mathscr{E})$. □

Note that if $T = (T_\gamma)_{\gamma\in\Gamma}$ and $L = (L_\gamma)_{\gamma\in\Gamma}$ are operators on X, then $TL = (T_\gamma L_\gamma)_{\gamma\in\Gamma}$.

Let the family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals satisfy functional representation of X as in Theorem 3.1.18. Given a cyclically compact operator T on $\sum_{\gamma\in\Gamma}^{\oplus} C_\#\left(Q_\gamma, \ell_2(\gamma)\right)$. By Theorem 3.4.9, this means that T_γ is a cyclically compact operator on $C_\#\left(Q_\gamma, \ell_2(\gamma)\right)$ for all $\gamma \in \Gamma$ where $\mathscr{P}(T) = (T_\gamma)_{\gamma\in\Gamma}$. Assume that $Y_\gamma \subset C_\#\left(Q_\gamma, \ell_2(\gamma)\right)$ is a Kaplansky−Hilbert submodule over $C(Q_\gamma)$. Denote

$$\sum_{\gamma\in\Gamma}^{\oplus} Y_\gamma := \left\{x = (x_\gamma)_{\gamma\in\Gamma} \in X : (\forall\gamma \in \Gamma)\ x_\gamma \in Y_\gamma\right\}.$$

Then $Y := \sum_{\gamma\in\Gamma}^{\oplus} Y_\gamma$ is n-homogenuous over $\pi\Lambda$ with projection $\pi = (\pi_\gamma)_{\gamma\in\Gamma}$ if and only if Y_γ is n-homogenuous over $\pi_\gamma C(Q_\gamma)$ for each $\gamma \in \Gamma$. Let $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. Then for each $\gamma \in \Gamma$, $(\lambda_k(T_\gamma))_{k\in\mathbb{N}}$ is also a global eigenvalue sequence of T_γ where $\lambda_k(T) = (\lambda_k(T_\gamma))_{\gamma\in\Gamma}$. Let $(\overline{\tau}_k(T_\gamma))_{k\in\mathbb{N}}$ be the multiplicity sequence corresponding to global eigenvalue sequence $(\lambda_k(T_\gamma))_{k\in\mathbb{N}}$. Moreover, the following holds

$$\pi \mathrm{Ker}\,(T - \lambda_k(T)I)^l = \sum_{\gamma\in\Gamma}^{\oplus} \pi_\gamma \mathrm{Ker}\,(T_\gamma - \lambda_k(T_\gamma)I)^l$$

for each $\pi = (\pi_\gamma)_{\gamma\in\Gamma} \in \mathfrak{P}(\Lambda)$ and $k, l \in \mathbb{N}$. Thus, $\rho_l(\lambda_k(T)) = (\rho_l(\lambda_k(T_\gamma)))_{\gamma\in\Gamma}$ and $\tau_{\lambda_k(T),l}(n) = \left(\tau_{\lambda_k(T_\gamma),l}(n)\right)_{\gamma\in\Gamma}$ are satisfied for all $k, l, n \in \mathbb{N}$, and so we have $\tau_{\lambda_k(T),l}(n)\overline{\tau}_k(T) = \left(\tau_{\lambda_k(T_\gamma),l}(n)\overline{\tau}_k(T_\gamma)\right)_{\gamma\in\Gamma}$ since $\tau_{\lambda_k(T),l}(n)\overline{\tau}_k(T) = n\tau_{\lambda_k(T),l}(n)$ and $\tau_{\lambda_k(T_\gamma),l}(n)\overline{\tau}_k(T_\gamma) = n\tau_{\lambda_k(T_\gamma),l}(n)$. Using the representation of Kaplansky−Hilbert modules we can also generalize the theorem above as follows.

Theorem 5.3.4. *Let T be a cyclically compact operator on X and $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. Then the following properties hold:*

(1) (Weyl-inequality) *if* $(\pi s_k(T))_{k\in\mathbb{N}}$ *is o-summable in* Λ *for some projection* π, *then the following inequality holds*

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(T)|\lambda_k(T)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(T);$$

(2) (Horn-inequality) *Suppose that* T_k *is a cyclically compact operator on* X *for* $1 \leq k \leq K$. *Then*

$$\prod_{i=1}^{N} s_i(T_K \cdots T_1) \leq \prod_{k=1}^{K}\prod_{i=1}^{N} s_i(T_k) \quad (N \in \mathbb{N}).$$

(3) (Lidskiĭ trace formula) *if* $T \in \mathscr{S}_1(X)$, *then the following equality holds*

$$\mathrm{tr}(T) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(T)\lambda_k(T).$$

Proof. Let $(\pi s_k(T))_{k\in\mathbb{N}}$ be o-summable sequence in Λ for the projection π. Then $\pi\overline{\tau}_k(T)\lambda_k(T) \in \Lambda$ $(k \in \mathbb{N})$. Indeed, $\tau_{\lambda_k(T),l}(n)\pi\overline{\tau}_k(T)|\lambda_k(T)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(T)$ holds for each $k, n \in \mathbb{N}$ from Theorem 5.3.2 and $\tau_{\lambda_k(T),l}(n)\overline{\tau}_k(T)|\lambda_k(T)| = \big(\tau_{\lambda_k(T_\gamma),l}(n)\overline{\tau}_k(T_\gamma)|\lambda_k(T_\gamma)|\big)_{\gamma\in\Gamma}$. Since Λ is B-cyclic we have $\pi\overline{\tau}_k(T)\lambda_k(T) \in \Lambda$. Thus, the proof of the theorem is immediately from Theorem 5.3.2 and the preceding lemma. □

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