İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF SCIENCE

INTEGRAL OF KRIVINE EXTENSIONS AND ORTHOGONALITY AND HERMITIAN PROJECTIONS ON COMPLEX BANACH LATTICES

Ph. D. THESİS

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Department : Mathematics and Computer Science

Programme : Mathematics

Thesis Supervisor : Prof. Dr. Mert ÇAĞLAR

AUGUST 2016

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KRIVINE GENİŞLEMELERİNİN İNTEGRALLERİ VE KOMPLEKS BANACH ÖRGÜLERİ ÜZERİNDE ORTOGONALLİK VE HERMİTSEL PROJEKSİYONLAR

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Abstract

INTEGRAL OF KRIVINE EXTENSIONS AND ORTHOGONALITY AND HERMITIAN PROJECTIONS ON COMPLEX BANACH LATTICES

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Ph.D., Department of Mathematics and Computer Science
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The present work consists of two main parts. In the first part, Krivine extensions of positively homogeneous functions are considered, and it is shown that the Bochner integral of the Krivine extension of a positively homogeneous function coincides with the Krivine extension of its integral.

In the second part, order and hermitian projections on complex Banach lattices are taken into consideration. Introducing the propery (d), it is shown that both types of projections coincide for complex Banach lattices having property (d). Examples of Banach lattices satisfying property (d) are provided and, as a by-product of this, a direct solution to Dales' problem for such Banach lattices is given.

Keywords: Banach lattice, orthogonality, hermitian projections.

ÖZET

KRIVINE GENİŞLEMELERİNİN İNTEGRALLERİ VE KOMPLEKS BANACH ÖRGÜLERİ ÜZERİNDE ORTOGONALLİK VE HERMİTSEL PROJEKSİYONLAR

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Eldeki bu çalışma iki kısımdan oluşmaktadır. Birinci kısımda, pozitif sayılar için homojen fonksiyonların Krivine genişlemeleri ele alınmıştır ve pozitif sayılar için homojen bir fonksiyonun Krivine genişlemesinin Bochner integrali ile bu fonksiyonun integralinin Krivine genişlemesinin aynı olduğu gösterilmiştir.

İkinci kısımda, kompleks Banach örgüleri üzerinde sıra ve hermitsel izdüşümler ele alınmıştır. (d) özelliği tanımlanarak, bu özelliği haiz kompleks Banach örgüleri üzerinde sıra ve hermitsel izdüşümlerin aynı olduğu gösterilmiştir. Bu özelliği haiz Banach örgüleri örneklendirilmiş ve bir uygulama olarak, bu Banach örgüleri için Dales'in problemine direkt bir çözüm verilmiştir.

Anahtar Kelimeler: Banach örgüsü, ortogonallik, hermitsel izdüşümler.

-*Yegâne düşmanınız cehalet olsun ve yalnızca cehalete karşı kazandığınız zaferlere sevinin.*

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Mehmet Selçuk TÜRER

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CHAPTER 1

INTRODUCTION

A natural and fundamental structural problem in Banach lattices is the one stated by Dales in [10], which asks to whether a direct sum decomposition $E \oplus F$ of a Banach lattice X satisfying the norm equality

$$||x + y|| = || \ |x| \vee |y| \ ||, \tag{1.1}$$

for each $x \in E$ and each $y \in F$, is a band decomposition, i.e., $|x| \ \wedge |y| = 0$. The importance of this problem is that an affirmative answer to the problem would simplify the extensive and deep theory of multi-normed spaces introduced by Dales and Polyakov in [10]. As pointed out in [10], the question has a negative answer for real Banach lattices. On the contrary, for complex Banach lattices an affirmative answer has been given by Kalton in [14]. Kalton solved this structural problem making heavy use of hermitian projections. He first showed that for a projection P, induced by the decomposition $E \oplus F$ of a complex Banach lattice X, if $E \oplus F$ satisfies (1.1), then P is an hermitian projection. After that, by using the commutative properties of central operators, (see [1]), and the machinery of Krivine calculus, he showed that P is an order projection. For each $\epsilon > 0$, he proved the inequality

$$\alpha + \beta \leq (1 + \epsilon)(\alpha \vee \beta) + \frac{5}{\epsilon}(\psi(\alpha, \beta) - \alpha \vee \beta), \quad \alpha, \beta \geq 0,$$

where

$$\psi(\alpha, \beta) = \int_0^{2\pi} |\alpha + e^{i\theta}\beta| \ \frac{d\theta}{2\pi}, \quad \alpha, \beta \in \mathbb{C},$$

and extended this inequality to X, by using Krivine calculus.

This thesis deals with two main problems originated from the above mentioned facts and it consists of three parts. In **Chapter 2**, we introduce the fundamental theory of ordered vector spaces and give all the necessary definitions and the results that will be used throughout the thesis. We illustrate the concrete vector lattices, the families of sequence spaces, and function spaces within the context of Banach lattices. To distinguish between real and complex ones, properties of complex Banach lattices are explained in detail in this part as well.

Chapter 3 is devoted to Krivine extension and orthogonality in Banach spaces. We start this chapter by mentioning $C(K)$-representations of vector lattices. Then, as an application of such representations, we introduce the Krivine extension of a positively homogeneous and continuous function. We consider integrals of such functions that are necessarily in the form of of Bochner integrals. Moreover, orthogonality in the sense of Birkhoff-James in normed vector spaces is recovered and for an operator T defined on a complex Banach space the numerical range of T and hermitian operators on Banach spaces are examined.

The main results of the present thesis are given in **Chapter 4**, which consists of two sections. In the first section, we deal with Krivine extension of a positively homogeneous function. We show that the Bochner integral of the Krivine extension of a positively homogeneous function coincides with the Krivine extension of its integral.

The second section of **Chapter 4** deals with the relation between order projections and hermitian projections on a complex Banach lattice. We know that each order projection is an hermitian projection, but the converse implication need not be true. We define a structural property, property (d), on complex Banach lattices and show that each hermitian projection is an order projection on a complex Banach lattice satisfying property (d). We give examples of complex Banach lattices satisfying property (d), and on the contrary, we show that $L_1[0,1]_{\mathbb{C}}$ does not satisfy property (d). We also show that each reflexive complex Banach lattice satisfies the property (d). As an application, we give a direct solution to Dales's problem for complex Banach lattices satisfying property (d).

CHAPTER 2

PRELIMINARIES

This chapter covers basic definitions, fundamental conclusions and concrete examples of vector lattices. In the first section, we give basic definitions, and well known facts without proofs. The second section deals with concrete examples of vector lattices and the structure of its ideals and bands. Finally in the third section, we focus on complex Banach lattices. For more details and the proofs of well known facts, we refer to [1], [3], [4], [17], [18], and [19].

2.1 Ordered vector spaces

Basic definitions

An **ordered vector space** is a real vector space X equipped with an order relation which is compatible with the algebraic structure of X.

We say that $x \in X$ is **positive** if $x \geq 0$ holds. The set of all positive vectors of X is called the **positive cone** of X and denoted by X^+. Thus,

$$X^+ = \{x \in X \mid x \geq 0\}.$$

We say that the positive cone X^+ is **generated** if $X = X^+ - X^+$. One can observe that the positive cone satisfies the following:

(1) $X^+ + X^+ \subseteq X^+$.

(2) $\lambda X^+ \subseteq X^+$ for each $\lambda \in \mathbb{R}^+$.

(3) $X^+ \cap (-X^+) = \{0\}$.

These observations allow us to extend the definition of the positive cone. In vector space X, any subset C satisfying the properties (1)-(3) can be named as a cone. Moreover, if C is a cone of X, then the relation $x \geq y$ if $x - y \in C$ defines an order relation on X and makes X an ordered vector space. Note that the positive cone of X with respect to these order relation is precisely C.

For an ordered vector space X and for each $x, y \in X$, if the set $\{x, y\}$ has supremum and infimum in X, then X is called **Riesz space** or a **vector lattice**. We denote

$$x \vee y := \sup\{x, y\} \qquad \text{and} \qquad x \wedge y := \inf\{x, y\}.$$

Let $x \in X$. We define the **positive part**, the **negative part**, and the **absolute value** of x,

$$x^+ := x \vee 0 \qquad x^- := (-x) \vee 0 \qquad |x| := x \vee (-x),$$

respectively.

It easy to see that $x = x^+ - x^-$, $|x| = x^+ + x^-$ and $x^+ \wedge x^- = 0$.

The operations that maps $x \in X$ to $x^+, x^-, |x|, x \vee y$ and $x \wedge y$ for a fixed $y \in X$ is called **lattice operations** of X.

Let Y be a vector subspace of X. It is clear from the above expressions that if Y is closed under a lattice operation, then it is closed under all the lattice operations. In this case, Y is said to be a **sublattice** of X. One can observe that Y is a vector lattice with the order inherited from X. On the other hand, X can contain a vector subspace Z, which is a vector lattice with its own order. In this case, Z is called a **lattice subspace** of X. Note that every sublattice of X is a lattice subspace but the converse needs not be true.

A norm on a vector lattice X is said to be **lattice norm** if $|x| \leq |y|$ implies $||x|| \leq ||y||$ for each $x, y \in X$. A **normed lattice** is a vector lattice which is also a normed space equipped with a lattice norm. If, in addition, the norm is complete, then it is called a **Banach lattice**. We know that lattice operations in a normed lattice are continuous.

If $x, y \in X$ with $x \leq y$, then the set $[x, y]$, defined by

$$[x, y] : \{z \in X \mid x \leq z \leq y\},$$

is called an **order interval**.

A subset A of a vector lattice X is called **bounded above** if there exists $y \in X$ such that $x \leq y$ for each $x \in A$. Boundedness from below is defined in a very similar way. A subset A of X is said to be **order bounded** if it is bounded both from below and above. Note that the interval $[x, y]$ is order bounded for each $x, y \in X$ with $x \leq y$. It is clear that A is order bounded if and only if it is included in an order interval.

A subset A of an ordered vector space is called **directed upwards** if for each pair a, b of the elements of A there exists some $c \in A$ such that $a \leq c$ and $b \leq c$. Directed downward set is defined analogously. A set is said to be **directed** if it is directed both upward and downward. A **net** $\{x_\alpha\}$ in a vector lattice X is a function defined from some directed set to X. For $\alpha, \beta \in A$, if $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$, then the net $\{x_\alpha\}$ is called **increasing** and denoted by $x_\alpha \uparrow$. The notation $x_\alpha \uparrow x$ stands for $x_\alpha \uparrow$ and $\sup\{x_\alpha\} = x$. The notations $x_\alpha \downarrow$ and $x_\alpha \downarrow x$ are anologous.

A net $\{x_\alpha\}$ in a vector lattice X is said to be **order convergent** to an element of $x \in X$ and denoted by $x_\alpha \xrightarrow{o} x$, if there exists another net $\{y_\alpha\}$, with the same indices, such that $|x_\alpha - x| \leq y_\alpha \downarrow 0$. The element x is said to be **order limit** of the net $\{x_\alpha\}$. It is well known that each net in a vector lattice has at most one order limit. A subset A of X is said to be **order closed** if it includes all of its order limit points, i.e., if $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$, then $x \in A$. The **σ-order closed** subsets are defined analogously.

A vector lattice X is said to be **Archimedean** if the set $\{nx\}$ is unbounded above in X for each $x \in X^+$, or equivalently $\frac{1}{n}x \downarrow 0$ holds for each $x \in X^+$. It is well-kwon that every normed lattices are Archimedean. We will be fine to study with Archimedean vector lattices. Throughout this thesis all vector lattices will be regarded as Archimedean, unless otherwise stated.

A vector lattice X is said to be **Dedekind complete** if every nonempty bounded above subset of X has a supremum. It turns out that a vector lattice is Dedekind complete if and only if the inequality $0 \leq x_\alpha \uparrow \leq x$ implies the existence of $\sup\{x_\alpha\}$.

A subset A of a vector lattice X is said to be **order dense** if for every $0 < x \in X$ there exists $a \in A$ such that $0 < a \leq x$.

In a vector lattice X, the elements x and y are called disjoint if $|x| \wedge |y| = 0$ and denoted by $x \perp y$. Note that disjointness is a symmetric relation by definition. If

$x, y \in X$, then we have by [4, Theorem 1.7]

$$\begin{aligned}\big||x+y|-|x-y|\big| &= 2\big(|x+y| \vee |x-y|\big) - \big(|x+y|+|x-y|\big)\\ &= 2\big(|x|+|y|\big) - 2\big(|x| \vee |y|\big)\\ &= 2\big(|x| \wedge |y|\big).\end{aligned}$$

So that, x and y are disjoint if and only if $|x+y| = |x-y|$ if and only if $|x+y| = |x|+|y|$. Two subsets A and B of a vector lattice are called disjoint if a and b are disjoint for each $a \in A$ and each $b \in B$, and denoted by $A \perp B$.

For a non-empty subset A of a vector lattice X, the set

$$A^d := \{x \in X \mid x \perp y \text{ for each } y \in A\},$$

is called the **disjoint complement** of A.

A subset A of a vector lattice X is called **solid** if $|x| \leq |y|$ and $y \in A$ imply $x \in A$. If A is both vector subspace of X and solid, then it is called an **ideal**. An order closed ideal is said to be a **band**.

For each subset A of a vector lattice X, there exists a smallest ideal E_A containing A, and it is called the **the ideal generated by** A. One can observe that, the set

$$E = \{x \in X \mid \text{ there exists } x_1, \dots . x_n \in A \text{ and } \lambda \geq 0 \text{ such that } |x| \leq \lambda \sum_{i=1}^{n} |x_i|\},$$

is an ideal containing A and included by any ideal that contains A. Hence, it is the ideal generated by A. In particular, if A is a singleton $\{a\}$, then the ideal generated by $A = \{a\}$ is called a **principal ideal**, and denoted by E_a. It should be clear that

$$E_a = \{x \in X \mid \text{ there exists } \lambda \geq 0 \text{ such that } |x| \leq \lambda |a|\}.$$

For a subset A of a vector lattice X, the smallest band containing A is called the band generated by A and denoted by B_A. In particular, if A is a singleton $\{a\}$, then the band generated by $A = \{a\}$ is called a **principal band**, and denoted by B_a.

An element $0 < u \in X$ is said to be **order unit** if $E_u = X$, or equivalently for every $x \in X$ there exists some $\lambda \in \mathbb{R}$ such that $|x| \leq \lambda u$.

A band B in a vector lattice X is called **projectoion band** if $B \oplus B^d = X$. Each projection band B defines a natural projection P_B on X with range B and kernel B^d.

Indeed, each $x \in X$ could be written as $x = x_1 + x_2$ where $x_1 \in B$ and $x_2 \in B^d$. If we define $P_B(x) = x_1$, then it should be clear that $\text{Range}P_B = B$ and $\text{Ker}P_B = B^d$. Any projection of this kind is called an **order projection** (or a **band projection**).

An element u in a vector lattice is said to be a **projection element** if the principle band B_u is a projection band.

A vector lattice X is said to have the **projection property** if each band $B \subseteq X$ is a projection band. If each principle band $B \subseteq X$, then X is said to have the **principle projection property**.

A linear map between two vector spaces is called an **operator**. A **positive operator** is an operator $T : X \to Y$ between two ordered spaces that maps positive elements of X to positive elements of Y, i.e., $Tx \geq 0$ for each $x \geq 0$, or equivalently $T(X^+) \subseteq Y^+$. If T is a positive operator, then $x \leq y$ implies $Tx \leq Ty$ for $x, y \in X$.

For vector lattices X and Y, the vector space of all operators from X to Y will be denoted by $\mathcal{L}(X, Y)$. It should be clear that it is an ordered vector space under the ordering $S \leq T$ if $Sx \leq Tx$ holds for each $x \in X^+$.

For an operator $T : X \to Y$ between two vector lattices if the supremum $T \vee (-T)$ exists in $\mathcal{L}(X, Y)$, then it is called the **modulus of** T, and denoted by $|T|$. If this is the case, then

$$|Tx| \leq |T|(|x|)$$

holds for each $x \in X$.

An operator $T : X \to Y$ is called **order bounded** if it maps order bounded subsets of X to order bounded subsets of Y. The vector space of all order bounded operators from X to Y is denoted by $\mathcal{L}_b(X, Y)$. For brevity, $\mathcal{L}(X, X)$ and $\mathcal{L}_b(X, X)$ will be denoted by $\mathcal{L}(X)$ and $\mathcal{L}_b(X)$, respectively.

The ordered vector space $\mathcal{L}_b(X, \mathbb{R})$ is called the **order dual of** X, and will be denoted by $X^\sim$.

For a non-empty subset A of a vector lattice X, the **annihilator of** A is defined by

$$A^\circ = \{T \in \mathcal{L}_b(X, Y) \mid Ta = 0 \text{ for each } a \in A\}.$$

Similarly, for a non-empty subset E of $\mathcal{L}_b(X, Y)$ the **inverse annihilator of** E

is defined by

$$^{\circ}E = \{x \in X \mid Tx = 0 \text{ for each } T \in E\}.$$

An operator $P : X \to Y$ between two vector lattices is said to be:

(1) **band preserving** if $P(B) \subseteq B$ for each band $B \subseteq X$.

(2) an **orthomorphism** if $|x| \wedge |y| = 0$ implies $|Px| \wedge |y| = 0$.

(3) a **lattice homomorphism** if $x \wedge y = 0$ implies $Px \wedge Py = 0$.

(4) a **lattice isomorphism** if P is a one-to-one lattice homomorphism.

Fundamental conclusions

We now give some useful identities and some conclusions without proofs which we use in this thesis. For the proofs of the identities and the conclusions we refer to [1], [3], [4], [17], [18], and [19].

The following result is not directly used in this thesis, but it is a corner stone of the theory of vector lattices.

Theorem 2.1 (The Riesz Decomposition Property)**.** Let x be any element of a vector lattice X, and let

$$|x| \leq |y_1 + y_2 + \ldots + y_n|$$

for some $y_1, y_2, \ldots, y_n \in X$. Then there exists $x_1, x_2, \ldots, x_n \in X$ such that

$$x = x_1 + x_2 + \ldots + x_n \text{ and that } |x_i| \leq |y_i| \text{ for each } i = 1, \ldots, n.$$

In particular, if x is positive vector, then the elements x_i can be chosen to be positive.

In the main part of this thesis, we deal with disjointness properties of elements of a vector lattice, which can be found in the following.

Theorem 2.2 (Disjointness Properties)**.** Let x, y and z be elements of a vector lattice and let $\alpha, \beta \in \mathbb{R}$. Then we have

(1) If $x \perp y$ and $x \perp z$, then $x \perp (\alpha y + \beta z)$.

(2) $x \perp y$ if and only if $|x+y| = |x-y|$.

(3) If $x \perp y$, then

$$|x+y| = |x-y| = |x| + |y| = \big||x| - |y|\big| = |x| \vee |y|.$$

We now focus on the subsets of a vector lattice. The following are about disjoint complements, ideals, bands, and projection bands of a vector lattice.

Lemma 2.3. An ideal E of a vector lattice X is a band if and only if $0 \leq x_\alpha \uparrow x$ and $\{x_\alpha\} \subseteq X$ imply $x \in A$.

Theorem 2.4. If E is an ideal of a vector lattice X, then;

(1) E is order dense in X if and only if $A^d = \{0\}$.

(2) $E \oplus E^d$ is order dense in X.

(3) E is order dense in E^{dd}.

Theorem 2.5. In a vector lattice X the band generated by a non-empty subset A is precisely A^{dd}. Moreover, a subset A of X is band if and only if $A = A^{dd}$.

Theorem 2.6. Let E and F be two ideals in a vector lattice X such that $X = E \oplus F$. Then E and F are both bands satisfying $E = F^d$ and $F = E^d$

Theorem 2.7. Let X be vector lattice. Then;

(1) a band B of X is a projection band if and only if for each $x \in X$ the set

$$\{y \in B \mid 0 \leq y \leq x\}$$

has a supremum in X. In this case we have

$$P_B(x) = \sup\{y \in B \mid 0 \leq y \leq x\}.$$

(2) a principal band B_a of X is a projection band if and only if for each $x \in X$ the set

$$\{y \in B_a \mid 0 \leq y \leq x\}$$

has a supremum in X. In this case we have

$$P_a(x) = \sup\{y \in B \mid 0 \leq y \leq x\}.$$

Theorem 2.8 (F. Riesz)**.** Each band in a Dedekind complete vector lattice is a projection band.

Theorem 2.9. Let $T : X \to X$ be an operator on a vector lattice. Then the following are equivalent.

(1) T is an order projection.

(2) T is a projection and $0 \leq T \leq I$.

(3) T and $I - T$ have disjoint ranges.

We close this section with results that will be used in the main part of this thesis.

Theorem 2.10. Each reflexive Banach lattice is Dedekind complete.

Lemma 2.11. Let X and Y be vector lattices with Y Dedekind complete.

(1) If A is an ideal of X, then A° is a band of $\mathcal{L}_b(X, Y)$.

(2) If E is an ideal of $\mathcal{L}_b(X, Y)$, then ${}^\circ E$ is an ideal of X.

2.2 Ideals, bands, and sublattices of concrete vector lattices

In this section, we give some examples of concrete vector lattices. We discuss the examples of vector lattices in two main parts. First, we mention about sequence spaces and characterize sublattices of such spaces. Then, we give examples of fundamental function spaces and characterize ideals and bands of such spaces.

Sequence spaces

Example 2.12. Let $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$ for some $n \geq 1$. We define a partial order on $\mathbb{R}^n$ by ordering vectors coordinatewise, i.e., $x \leq y$ if and only if $x_i \leq y_i$ for each $1 \leq i \leq n$. This order makes $\mathbb{R}^n$ a Dedekind complete and Archimedean vector lattice. But this order is not the unique order that makes $\mathbb{R}^n$ a vector lattice. We can define an order on $\mathbb{R}^2$ by $x \leq y$ if and only if $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$. This order is called **lexicographic order**. For $n > 2$ the lexicographic order can be defined analogously. With lexicographic order the vector space R^n becomes a *non-Archimedean* vector lattice.

One can assume that $\mathbb{R}^n$ is an n-dimensional sequence space. We may also put coordinatewise ordering on infinite dimensional sequence spaces.

Example 2.13. Let $R^{\mathbb{N}} = \{(x_n) \mid \{x_n\} \subseteq \mathbb{R}\}$. Then it is a Dedekind complete and Archimedean vector lattice with coordinatewise ordering.

Example 2.14. Let $c_o = \{(x_n) \mid \{x_n\} \subseteq \mathbb{R} \text{ and } \lim\limits_{n\to\infty} x_n = 0\}$. If we put coordinatewise ordering on the vector space c_0, then it becomes an Archimedean vector lattice but it is not Dedekind complete. Moreover, it is a Banach lattice under the norm $||x|| = \sup_n |x_n|$.

Example 2.15. Let $\ell_p = \{(x_n) \mid \{x_n\} \subseteq \mathbb{R} \text{ and } \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$, for some $1 \leq p < \infty$. The vector space ℓ_p is a Dedekind complete and Archimedean vector lattice with coordinatewise ordering. Under the norm $||x|| = \big(\sum\limits_{n=1}^{\infty} |x_n|^p\big)^{\frac{1}{p}}$, it is a Banach lattice as well.

One can observe that if X is a sequence space and if $x = (x_n), y = (y_n) \in X$, then

$$x \vee y = (x_n \vee y_n), \quad x \wedge y = (x_n \wedge y_n), \quad x^+ = (x_n^+), \text{ and } |x| = (|x_n|).$$

The following result characterizes the sublattices of some sequence spaces.

Theorem 2.16. Suppose that $X = \mathbb{R}^{\mathbb{N}}$, or $X = c_0$, or $X = \ell_p$ for $1 \leq p < \infty$. Then every closed sublattice of X is of the form a subspace that is generated by a (finite or infinite) disjoint positive sequence $\{x^{(n)}\} \subseteq X$.

Example 2.17. Let $\ell_\infty = \{(x_n) \mid \sup\limits_n x_n < \infty\}$. If we put coordinatewise ordering on the vector space ℓ_∞, then it becomes a Dedekind complete and Archimedean vector lattice. Moreover, under the norm $||x||_\infty = \sup_n |x_n|$, it is a Banach lattice.

Function spaces

Example 2.18. Let Ω be any non-empty set and let $\mathbb{R}^\Omega = \{f \mid f : \Omega \to \mathbb{R}\}$. The order relation defined by $f \leq g$ if and only if $f(t) \leq g(t)$ for each $t \in \Omega$ is called pointwise ordering, and makes $\mathbb{R}^\Omega$ a vector lattice. It should be clear that

$$(f \vee g)(t) = f(t) \vee g(t) \;\; (f \wedge g)(t) = f(t) \wedge g(t), \;\; f^+(t) = (f(t))^+ \;\; \text{and } |f|(t) = |f(t)|,$$

for each $f, g \in \mathbb{R}^\Omega$.

Example 2.19. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let

$$L_0(\mu) = \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable}\}.$$

This space is a vector lattice under the pointwise ordering. Now, for $1 \leq p < \infty$, let $L_p(\mu)$ denote the linear subspace of $L_0(\mu)$ such that $\int |f|^p$ is finite. It should be clear that $L_p(\mu)$ is a vector lattice under the pointwise ordering. Moreover, under the norm $||f|| = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$, it is a Banach lattice. Similarly, the subspace $L_\infty(\mu)$ of $L_0(\mu)$ consisting essentially bounded functions is a Banach lattice under the norm $||f||_\infty = ess \sup_{t \in \Omega} |f(t)|$.

Theorem 2.20. Let $X = L_p(\Omega, \mathcal{F}, \mu)$ for some $1 \leq p < \infty$ and a finite measure μ. Then the closed sublettices of X containing the constant function $\mathbf{1}$ are of the form $L_p(\Omega, \mathcal{G}, \mu)$, where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$.

For a measurable subset A of Ω, suppose that $\mathcal{F}_A = \{b \in \mathcal{F} \mid B \subseteq A\}$ and that μ_A is the measure obtained by restricting μ to $\mathcal{F}_A$. The bands of $L_p(\Omega, \mathcal{F}, \mu)$ are characterized as follows:

Theorem 2.21. Let $X = L_p(\Omega, \mathcal{F}, \mu)$ where μ is a σ-finite measure and $1 \leq p < \infty$. Then the closed ideals and bands of X are of the form $L_p(\Omega, \mathcal{F}_A, \mu_A)$ for some $A \in \mathcal{F}$.

Note that each band in $L_p(\mu)$ is a projection band since $L_p(\mu)$ is Dedekind complete.

Example 2.22. Let Ω be a topological Hausdorff space. Then the space of all continuous functions defined from Ω to $\mathbb{R}$, i.e,

$$C(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is continuous}\},$$

is an Archimedean vector lattice under the pointwise ordering. If, in addition, Ω is compact, then it is a Banach lattice under the so called **sup-norm** $||f|| = \sup_{t \in \Omega} |f(t)|$.

Remark 2.23. In the literature the letter K is used in general instead of Ω, when the underlying topological space is compact Hausdorff space. Throughout this thesis, the term $C(K)$-spaces will stand for spaces of continuos functions on a compact Hausdorff space K.

On the other hand, $C(K)$-spaces are not Dedekind complete in general. Recall that a topological space is Stonean if the closure of every open subset is open. See the following:

Theorem 2.24. Let K be a compact Hausdorff space. Then, $C(K)$ is Dedekind complete if and only if K is Stonean.

Theorem 2.25. Every closed sublattice of a $C(K)$-space containing constant function $\mathbf{1}$ is a $C(K)$-space again.

For a subset A of K the set I_A stands for the set of functions that vanishes on A, i.e.,

$$I_A = \{f \in C(K) \mid f \text{ vanishes on } A\}.$$

Theorem 2.26. A subset A in $C(K)$ is a band if and only if $A = I_{\overline{U}}$ for some open set U in K.

Recall that a band in $C(K)$ needs not to be a projection band, unless K is Stonean. See the following:

Theorem 2.27. The only projection bands in $C[0,1]$ are $\{0\}$ and $C[0,1]$.

2.3 Complex Banach lattices

This section deals with complex vector lattices, especially complex Banach lattices. Recall that a complex Banach lattice is noting but the complexification of a real Banach lattice. We start with a real Banach lattice, and do complexification to obtain a complex Banach lattice. For more details and proofs, we refer to [1] and [18].

Let X be a real vector lattice. Then the complex vector space

$$X_{\mathbb{C}} = X \oplus iX = \{x + iy \mid x, y \in X\},$$

is called the **complexification** of X, and its vector space operations are defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2) \text{ and}$$
$$(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\beta x + \alpha y).$$

Note that X sits in $X_{\mathbb{C}}$, since $X = X + i\{0\}$.

For an element $z \in X_{\mathbb{C}}$, the **modulus** of z is defined by the formula

$$|z| = \sup_{\theta \in \mathbb{R}} [x \cos\theta + y \sin\theta] = \sqrt{x^2 + y^2}.$$

(See the definition of $\sqrt{x^2 + y^2}$ in Chapter 3.1.)

If X is also a normed lattice, then we can extend the norm of X to a norm on $X_{\mathbb{C}}$ by

$$||z||_{\mathbb{C}} = ||\ |z|\ || = ||\sqrt{x^2 + y^2}||,$$

for each $z = x + iy \in X_{\mathbb{C}}$. It should be clear that $||x|| = ||x + i0||$ for each $x \in X$ and that $|z_1| \le |z_2|$ implies $||z_1|| \le ||z_2||$ for each $z_1, z_2 \in X_{\mathbb{C}}$. Moreover, if $z = x + iy$, then

$$\frac{1}{2}\big(||x|| + ||y||\big) \le ||z||_{\mathbb{C}} \le ||x|| + ||y||.$$

This observations give rise for the following definition.

Definition 2.28. Any complex Banach space of the form $X_{\mathbb{C}} = X \oplus iX$, where X is a real Banach lattice, is called a **complex Banach lattice**.

We give now some useful identities and inequalities which are true in a complex Banach lattice.

Theorem 2.29. Let $X_{\mathbb{C}}$ be a complex Banach lattice.

(1) If $z_1, z_2 \in X_{\mathbb{C}}$ satisfy $|z_1| \perp |z_2|$, then

$$|z_1 - z_2| = |z_1 + z_2| = |z_1| + |z_2| = \big||z_1| - |z_2|\big| = \sup\{|z_1|, |z_2|\}.$$

(2) If $z = x + iy$ and $z' = |x| + i|y|$ then $|z| = |z'|$.

(3) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$, then $|z_1| \leq |z_2|$.

Let A be a subset of a complex Banach lattice $X_{\mathbb{C}}$. Then the set $A \cap X$ is called **real part of** A and denoted by $A_{\mathbb{R}}$.

Now, we can take a look of the structure of ideals in a complex vector lattice.

Theorem 2.30. Let E be an ideal in $X_{\mathbb{C}}$. Then $E_{\mathbb{R}}$ is an ideal in X, and $E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}$. Conversely, if F is an ideal in X, then $F \oplus iF$ is an ideal in $X_{\mathbb{C}}$, and $(F \oplus iF)_{\mathbb{R}} = F$.

Bands in complex vector lattices are defined via bands in real vector lattices. An ideal E is said to be band in $X_{\mathbb{C}}$ if $E_{\mathbb{R}}$ is a band in X. A band B is said to be a projection band in $X_{\mathbb{C}}$ if $B_{\mathbb{R}}$ is a projection band in X.

For a non-empty subset A of a complex vector lattice $X_{\mathbb{C}}$, the disjoint complement A^d of A is defined similarly as in the real case by

$$A^d = \{z \in X_{\mathbb{C}} \mid |z| \perp |a| \text{ for each } a \in A\}.$$

Unlike the real case, A^d is a band in $X_{\mathbb{C}}$ for each $A \subseteq X_{\mathbb{C}}$.

An operator $T : X \to Y$ between two real vector lattices can be extended to an operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ via the formula

$$T_{\mathbb{C}}(x + iy) = Tx + iTy.$$

Lemma 2.31. If $T : X \to Y$ is a bounded operator, then $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is also bounded and satisfies $||T|| = ||T_{\mathbb{C}}||$.

On the other hand, each operator in $\mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ can be identified with an operator $\mathcal{T} = T + iS \in \mathcal{L}(X,Y) \oplus i\mathcal{L}(X,Y)$, where the action of the operator $T + iS$ on $X_{\mathbb{C}}$ is

$$(T + iS)(x + iy) = (Tx - Sy) + i(Sx - Ty).$$

For a linear functional $f : X_{\mathbb{C}} \to \mathbb{C}$ on a complex vector lattice $X_{\mathbb{C}}$, there exist a unique linear functional $g : X_{\mathbb{R}} \to \mathbb{R}$ such that

$$f(x) = g(x) - ig(ix)$$

holds for each $x \in X_{\mathbb{C}}$, and $||f|| = ||g||$. This functional is called the **real part of** f, and denoted by Ref. Thus, we have the following:

Theorem 2.32. If $X_{\mathbb{C}}$ is a complex vector lattice, then $(X_{\mathbb{C}})^* = (X^*)_{\mathbb{C}}$.

CHAPTER 3

Krivine extensions and orthogonality

3.1 $C(K)$ representations and Krivine's extension

In this section, we will outline $C(K)$ representations of the ideals of a vector lattice, and as an application of such representations we will give Krivine's extension of a positively homogeneous function. For detailed information about Krivine's calculus we refer to [16, p. 40-42].

Let X be a vector lattice. For each $a \in X^+$ the formula

$$||x||_a = \inf\{\lambda \geq 0 \mid |x| \leq \lambda a\},$$

defines a norm on E_a, i.e., the ideal generated by $a \in X^+$. If the normed space $(E_a, ||\cdot||_a)$ is complete for every $a \in X^+$, then X is said to be **uniformly complete**. It is known that every σ-order complete space and every Banach lattice is uniformly complete.

Suppose that X is a Banach lattice and that $u \in X^+$ is an order unit, i.e., $E_u = X$, or equivalently for every $x \in X$, there exists λ such that $|x| \leq \lambda u$. Then $||\cdot||_u$ is a lattice norm on X. Moreover, it is equivalent to the original norm of X.

Theorem 3.1. Let X be a uniformly complete vector lattice. Then for every $a \in X^+$, the space $(E_a, ||\cdot||_a)$ is lattice isometric to a $C(K)$-space, with a corresponding the constant function $\mathbf{1}$ on K.

The above theorem has many applications. We now present one of these applications.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **positively homogeneous** if

$$f(\lambda t_1, \ldots, \lambda t_n) = \lambda f(t_1, \ldots, t_n),$$

for each $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and each $\lambda \geq 0$. The set of all positively homogeneous and continuous functions of n variables is denoted by H_n. It should be clear that H_n is a vector lattice under pointwise operations. Moreover, H_n is a normed vector lattice under the norm

$$||f||_{H^n} = \sup\{f(t_1, \ldots, t_n) \mid |t_1| \vee \ldots \vee |t_n| = 1\}.$$

Now let X be a uniformly complete vector lattice and let $x_1, x_2 \in X$. It should be clear that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(t_1, t_2) = \sqrt{t_1^2 + t_2^2}$$

is positively homogeneous and continuous on $\mathbb{R}^2$. By using $C(K)$ representation, we can give a meaning to the expression $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Indeed, put $a = |x_1| \vee |x_2|$ and note that $x_1, x_2 \in E_a$. We know that E_a is lattice isometric to a $C(K)$ space for a compact Hausdorff K. Thus we can see the elements x_1, x_2 of X as continuous functions $x_1(t), x_2(t)$ with $|x_i(t)| \leq \mathbf{1}$ on K. Now define

$$f(x_1, x_2)(t) = f(x_1(t), x_2(t)) = \sqrt{x_1^2(t) + x_2^2(t)},$$

for each $t \in K$. It is obvious that $f(x_1, x_2)(t)$ is a continuous function on K. Thus we can uniquely extend the function $f(t_1, t_2) = \sqrt{t_1^2 + t_2^2}$, which is defined on $\mathbb{R}^2$, to a function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ defined on X^2.

The above technic can be applied for every positively homogeneous and continuous functions defined on real or complex numbers.

Theorem 3.2. Let X be a uniformly complete vector lattice and let $x_1, \ldots, x_n \in X$. If $a = |x_1| \vee \ldots \vee |x_n|$, then there exist a unique lattice homomorphism $L : H_n \to E_a$ such that $Lf = f(x_1, \ldots, x_n)$ for each $f \in H_n$.

For a vector lattice X, the function $f(x_1, \ldots, x_n) : X^n \to X$ is called the **Krivine extension** of the positively homogeneous function $f(t_1, \ldots, t_n) : \mathbb{R}^n \to \mathbb{R}$.

By Krivine extension, the vector expressions, such as $\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$, can be well defined for any positive real p, and any elements $x_1, x_2, \ldots, x_n$ of a uniformly complete vector lattice.

3.2 Bochner integral

This section is devoted to cover the definition of Bohner integral. We also give some basic conclusions without proofs. For more details and proofs, we refer to [9].

Throughout this section $(\Omega, \mathcal{F})$ will be a measurable space, and X will be a Banach space.

The σ-algebra of Borel subsets of X, i.e., the σ-algebra generated by the open subsets of X, will be denoted by $\mathcal{B}(X)$. A function $f : \Omega \to X$ is said to be **Borel measurable** if it is $(\mathcal{F}, \mathcal{B}(X))$-measurable, i.e., the inverse image of each element of $\mathcal{B}(X)$ is an element of $\mathcal{F}$. We call that f is **strongly measurable** if it has a separable range and Borel measurable. The function f is said to be **simple** if it takes only finitely many values. It should be clear that a simple function is Borel measurable if and only if it is strongly measurable.

Proposition 3.3. If $f : \Omega \to X$ is Borel measurable then the function $x \mapsto ||f(x)||$ is $\mathcal{F}$-measurable.

Proposition 3.4. The pointwise limits of Borel measurable (strongly measurable) functions are Borel measurable (strongly measurable).

Proposition 3.5. If $f : \Omega \to X$ is strongly measurable then there exists a sequence $\{f_n\}$ of strongly measurable simple functions such that f is the pointwise limit of f_n and that $||f_n(x)|| \leq ||f(x)||$ holds for each $x \in \Omega$

The immediate consequence of the above propositions is that a function f is strongly measurable if and only if it is pointwise limit of a sequence of strongly (or Borel) measurable functions. On the other hand, one can conclude from the above propositions that the set of all strongly measurable functions forms a vector space.

Now we are able to define the integral of the function which takes its values in a Banach space. After this point $(\Omega, \mathcal{F}, \mu)$ will be a measure space.

A function $f : \Omega \to X$ is said to be **Bochner integrable** if it is strongly measurable and the function $x \mapsto ||f(x)||$ is integrable. Its integral is defined as follows:

First suppose that f is a simple and Bochner integrable function. Let $x_1, \ldots, x_n$ be the values of f and let $X_i = f^{-1}(\{x_i\})$ for $i = 1, \ldots, n$. Then one can conclude that

$\mu(X_i)$ is finite, since the function $x \mapsto ||f(x)||$ is integrable. So that the expression $\sum_{i=1}^{n} x_i\mu(X_i)$ is well-defined. We define the integral of f to be this sum and denote by $\int f d\mu$. Thus

$$\int f d\mu = \sum_{i=1}^{n} x_i\mu(X_i).$$

It should be clear that $|| \int f d\mu|| \leq \int ||f|| d\mu$.

Now suppose that f, g are simple and Bochner integrable functions and that a, b are scalars. It is easy to see that $af + bg$ is also simple and Bochner integrable function, and

$$\int af + bg d\mu = a \int f d\mu + b \int g d\mu.$$

Finally, suppose that f is an arbitrary Bochner integrable function. We can pick a sequence $\{f_n\}$ of simple and Bochner integrable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ hold for each $x \in \Omega$ and such that the function $x \mapsto \sup_n ||f_n(x)||$ is integrable. Then the Bochner integral of f is defined to be the limit of the sequence $\int f_n d\mu$. Thus,

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Of course, the integral is independent from the choice of the sequence $\{f_n\}$.

We now give some basic properties of the Bochner integral.

Proposition 3.6. If f is Bochner integrable then $|| \int f d\mu|| \leq \int ||f|| d\mu$.

Proposition 3.7. If f and g are integrable and a, b are scalars then $af + bg$ is also Bocher integrable and

$$\int af + bg d\mu = a \int f d\mu + b \int g d\mu.$$

3.3 Orthogonality in normed vector spaces

This section will cover the notion of orthogonality in normed vector spaces. In fact, there are different concepts of orthogonality in normed vector spaces in the literature. We will agree with G. Birkhoff and R. J. James, and for detailed information we refer to [6] and [13].

Throughout this section X will stand for a normed vector space.

An element x of X is said to be **orthogonal** to an element y if $||x|| \leq ||x + ky||$ holds for each scalar k, and denoted by $x \perp_J y$. An element x is said to be orthogonal to a subset A if $x \perp_J a$ holds for each $a \in A$. It is clear that if an inner product defined on X, then $x \perp_J y$ if and only if $< x, y >= 0$. Thus, the definition of orthogonality in a normed vector spaces is an extension of the notion of inner product space sense orthogonality.

This definition is an anology of a well-known fact : If two lines intersects at a point p, then they are orthogonal if and only if for any point $x_0 \neq p$ of the first line, the distance between x_0 and p is less than or equal to the distance between x_0 and q, where q is an arbitrary point of the second line. Since this definition comes in a very natural way, it has many advantages, e.g., it can be related to other concepts, such as hyperplanes, strict convexity, weak compactness, and especially linear functionals. In this thesis, we mainly use the relation between orthogonality and linear functionals.

Recall that a proper subspace H of X is an **hyperplane** if it is not properly contained in a proper subspace M of X. If H is an hyperplane then $x + H$ is called a **flat** for each $x \in X - \{0\}$.

We now give some basic observations on orthogonality.

(1) If an element of X is orthogonal to itself then it should be zero.

(2) Orthogonality is homogeneous but is neither symmetric nor additive.

(3) For any elements, $x, y \in X$ there exists a scalar a such that $x \perp_J ax + y$.

(4) If a sequence $\{y_n\}$ converges to y and $x \perp_J y_n$ holds for each n, then $x \perp_J y$.

(5) For any $x \in X$ the set $H_x; = \{y \in X \mid x \perp_J\}$ is closed.

Theorem 3.8. If $x \in X$ is orthogonal to a subset A of X, then there exists a linear functional f such that $f(x) = ||f|| \cdot ||x||$ and $f(a) = 0$ holds for each $a \in A$, and there exists a hyperplane H such that $x \perp_J H$ with $A \subseteq H$.

From the above theorem, we can observe the following:

Corollary 3.9. Any element of a normed vector space is orthogonal to some hyperplane.

Corollary 3.10. For $x, y \in X$, $x \perp_J ax + y$ if and only if there exists a linear functional f such that $|f(x)| = ||f|| \cdot ||x||$ and $a = -\frac{f(y)}{f(x)}$. In particular, if $x \perp_J ax+y$, then $|a| \leq \frac{||y||}{||x||}$.

As we said before, orthogonality is not symmetric, i.e., $x \perp_J y$ does not imply $y \perp_J x$, in general. Thus the above corollary does not guarantee the existence of a scalar b such that $bx + y \perp_j x$.

Theorem 3.11. For any elements, $x, y \in X$ there exists a scalar a such that $ax+y \perp_J x$. Moreover, if $ax + y \perp_J x$ and $bx + y \perp_J x$, then $cx + y \perp_J x$ for each scalar c between a and b.

3.4 Numerical range and hermitian operators on Banach spaces

In this section, we recall the definition of the numerical range and hermitian operators in Banach spaces. We will also outline the characterizations of hermitian operators that we will use. For details, we refer to [7].

Recall that for an operator $T : X \to X$ on a Hilbert space X, the numerical range $W(T)$ is defined by

$$W(T) = \{< Tx, x > \ \mid ||x|| = 1\}.$$

The definition of the numerical range is introduced by Toeplitz in 1918, and it becomes an important tool in Hilbert spaces. However, in this section we do not deal with Hilbert spaces. One can see details in [12].

Now, let X be a complex Banach space and let X^* be the dual of X. The **unit sphere** of X is the set

$$S(X) = \{x \in X \mid ||x|| = 1\}.$$

The 2-tuple (x, x^*) is called a **primary state** if both $(x, x^*) \in S(X) \times S(X^*)$ and $x^*(x) = 1$ hold. The set of all primary states on X will be denoted by $\Pi(X)$.

For an operator $T : X \to X$ on a complex Banach space X, the **numerical range** $V(T)$ of T is defined by

$$V(T) = \{x^*(Tx) \mid (x, x^*) \in \Pi(X)\}.$$

Note that if X is an Hilbert space, then (x, x^*) is a primary state if and only if $x^* =< x', \cdot >$ where $x' \in S(X)$. Thus the numerical range $V(T)$ of an operator T is nothing but the classical numerical range $W(T)$ defined in terms of the inner product of X. Furthermore, if X is a Banach space with a smooth unit ball (see [8]), then $V(T)$ also coincides $W(T)$.

Now we are able to give the definition of an hermitian operator on a complex Banach space.

For a complex Banach space X, an operator T on X is said to be an **hermitian operator** if its numerical range $V(T)$ is real, i.e., $V(T) \subseteq \mathbb{R}$.

Recall that an operator P is called **projection** if $P^2 = P$. A projection P on X is said to be an **hermitian projection** if it is an hermitian operator. A decomposition $E \oplus F$ of X is said to be **hermitian decomposition** if the induced projection $P : X \to E$ is an hermitian projection.

For an operator T on a complex Banach space X, the exponential function is defined by

$$\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

There is a characterization of hermitian operators in terms of the exponential function as follows:

Lemma 3.12. Let X be a complex Banach space. An operator T on X is an hermitian operator if and only if $\exp(i\theta T)$ is an isometry for each $\theta \in \mathbb{R}$.

Corollary 3.13. Let $E \oplus F$ be a direct sum decomposition of a complex Banach space X. Then, $E \oplus F$ is an hermitian decomposition if and only if

$$||x + y|| = ||x + e^{i\theta} y||$$

for each $x \in E$, each $y \in F$, and each $\theta \in \mathbb{R}$.

CHAPTER 4

Main results

4.1 Integral of Krivine extensions

Let $f : \mathbb{K} \to \mathbb{K}$ be a positively homogeneous function, where $\mathbb{K}$ is the scalar field, i.e., $\mathbb{R}$ or $\mathbb{C}$, and suppose that X is a (complex) Banach lattice. We know that f can be uniquely extended to X. We will denote the Krivine extension of f by $\widetilde{f}$. If f is integrable with respect to a parameter θ, then integral of f is also a positively homogeneous function, thus it can be extended to X as well. On the other hand, we may also calculate the Bochner integral of $\widetilde{f}$. At this point, it is natural to ask that if the Krivine extension of the integral of f coincides with the Bochner integral of $\widetilde{f}$.

In this section, we will give an answer to this question for a special case. We first prove that if a positively homogeneous function is continuous with respect to a parameter θ, then the Krivine extension of integral of the function f and the Bochner integral of $\widetilde{f}$ are coincide.

Theorem 4.1. Let $f(s_1, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be a continuous function of parameters $s_1, \ldots, s_n \in \mathbb{R}$ and $\theta \in [0,1]$ such that for each $\theta \in [0,1]$ the function $f_\theta(s_1, \ldots, s_n) := f(s_1, \ldots, s_n, \theta)$ is positively homegenuous. Define $F(s_1, \ldots, s_n) = \int_0^1 f_\theta(s_1, \ldots, s_n) d\theta$. Then for each Banach lattice X, the Krivine extension $\widetilde{F}$ of F is the Bochner integral of the Krivine extension $\widetilde{f}$ of f,i.e.,

$$\widetilde{F}(x_1, x_2, \ldots, x_n) = \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta.$$

Proof. By induction, it suffices to prove the lemma for the function $f(s, t, \theta) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$. Let H_2 be the set of all continuous positively homogenuous functions of two variables. For $h \in H_2$ we define

$$||h||_{H_2} = \sup_{S^2_\infty} |h| = \sup\{|h(s,t)| \mid |s| \vee |t| = 1\}.$$

For a Banach lattice X, let $\widetilde{h} : X^2 \to X$ be the Krivine extension of h. Then

$$||\widetilde{h}(x,y)|| \leq ||h||_{H_2}.\Big|\Big||x| \vee |y|\Big|\Big|,$$

(see, [16]). For each n, s and t, let $f^{(n)}(s,t,\theta)$ be the standard "left end-point step-function approximation" of $f(s,t,\theta)$, that is, $f^{(n)}(s,t,\theta) = f(s,t,\frac{k}{n})$ whenever $\theta \in [\frac{k}{n}, \frac{k}{n+1}]$ as $k = 0, 1, \ldots, n-1$. Put $f_\theta^{(n)}(s,t) = f^{(n)}(s,t,\theta)$; clearly, for $\theta \in [\frac{k}{n}, \frac{k}{n+1}]$, we have $f_\theta^{(n)} = f_{\frac{k}{n}}^{(n)}$. In particular, $f_\theta^{(n)}$ is positively homogenuous. Fix $\epsilon > 0$. Since $S_\infty^2 \times [0,1]$ is compact, f is uniformly continuous there. Hence, there exist n such that for each $(s,t) \in S_\infty^2$ and all $\theta \in [\frac{k}{n}, \frac{k}{n+1}]$ we have $|f(s,t,\theta) - f(s,t,\frac{k}{n})| < \epsilon$. It follows that for each $(s,t) \in S_\infty^2$ and all $\theta \in [0,1]$, we have

$$|f_\theta(s,t) - f_\theta^{(n)}(s,t)| = |f(s,t,\theta) - f^{(n)}(s,t,\theta)| < \epsilon,$$

so that $||f_\theta^{(n)} - f_\theta||_{H_2} < \epsilon$ for each $\theta \in [0,1]$. Fix $x, y \in X$. We may assume without loss of generality that $\Big|\Big||x| \vee |y|\Big|\Big| = 1$. For each $\theta \in [0,1]$, let $\widetilde{f_\theta}$ and $\widetilde{f}_\theta^{(n)}$ be the Krivine extensions of f_θ and $f_\theta^{(n)}$. By properties of Bochner integral , we have

$$\begin{aligned}\Big|\Big|\int_0^1 \widetilde{f}_\theta^{(n)}(x,y)d\theta - \int_0^1 \widetilde{f}_\theta(x,y)d\theta\Big|\Big| &\leq \int_0^1 ||\widetilde{f}_\theta^{(n)}(x,y) - \widetilde{f}_\theta(x,y)||d\theta \\ &\leq \int_0^1 ||f_\theta^{(n)} - f_\theta||_{H_2}.\Big|\Big||x| \vee |y|\Big|\Big|d\theta < \epsilon.\end{aligned}$$

For $s, t \in \mathbb{R}$, define

$$F^{(n)}(s,t) = \int_0^1 f(s,t,\theta)d\theta = \frac{1}{n}\sum_{k=1}^{n-1} f(s,t,\frac{k}{n}) = \frac{1}{n}\sum_{k=1}^{n-1} f_{\frac{k}{n}}(s,t).$$

It follows that $F^{(n)}$ is positively homogenuous. For each $s, t \in S_\infty$, we have

$$|F(s,t) - F^{(n)}(s,t)| \leq \int_0^1 \big|f(s,t,\theta) - f^{(n)}(s,t,\theta)\big|d\theta < \epsilon,$$

so that $||F^{(n)} - F||_{H_2} < \epsilon$. It follows that $\Big|\Big|\widetilde{F}^{(n)}(x,y) - \widetilde{F}(x,y)\Big|\Big| < \epsilon$. For each $\theta \in [\frac{k}{n}, \frac{k}{n+1}]$, it follows from $f_\theta^{(n)} = f_{\frac{k}{n}}^{(n)}$ in H_2 that $\widetilde{f}_\theta^{(n)} = \widetilde{f}_{\frac{k}{n}}^{(n)}$. In particular $\widetilde{f}_\theta^{(n)}(x,y) = \widetilde{f}_{\frac{k}{n}}^{(n)}(x,y)$. This means that the function $\theta \in [0,1] \to \widetilde{f}_\theta^{(n)}(x,y)$ is a simple function and its Bochner integral

$$\int_0^1 \widetilde{f}_\theta^{(n)}(x,y)d\theta = \frac{1}{n}\sum_{k=1}^{n-1} \widetilde{f}_{\frac{k}{n}}(x,y) = \widetilde{F}^{(n)}(x,y).$$

The last inequality follows from the definition of $F^{(n)}$. Combining the inequalities, we get

$$\left|\left|\widetilde{F}(x,y) - \int_0^1 \widetilde{f}_\theta(x,y)d\theta\right|\right| < 2\epsilon.$$

□

Next step will be to give an affirmative answer for integrable functions. To this end, we first do an approximation.

Lemma 4.2. Let $f(s_1, s_2, \ldots, s_n, \theta) : [0, 2\pi]^n \times [0, 1] \to \mathbb{R}$ be an integrable function of parameter $\theta \in [0, 1]$ and continuous function of parameters $s_1, s_2, \ldots, s_n \in [0, 2\pi]$ uniformly on θ. Then for each $\epsilon > 0$ there exists a function

$$g(s_1, s_2, \ldots, s_n, \theta) : [0, 2\pi]^n \times [0, 1] \to \mathbb{R},$$

which is continuous of parameters $s_1, s_2, \ldots, s_n \in [0, 2\pi]$ and $\theta \in [0, 1]$ such that $||f(s_1, s_2, \ldots, s_n, .) - g(s_1, s_2, \ldots, s_n, .)||_1 < \epsilon$.

Proof. Let $f(s, \theta)$ be a real-valued function which is integrable of parameter $\theta \in [0, 1]$, and continuous of parameter $s \in [0, 2\pi]$ uniformly on θ, i.e., for each $\epsilon > 0$ there exist $\delta > 0$ such that for each $\theta \in [0, 1]$ we have $|f(s_1, \theta) - f(s_2, \theta)| < \epsilon$ whenever $|s_1 - s_2| < \delta$. Fix $\epsilon > 0$. Since $f(s, .) \in L_1[0, 1]$ for each $s \in [0, 2\pi]$, there exist $g(s, .) \in C[0, 1]$ such that $||f(s, .) - g(s, .)||_1 < \epsilon$. Now, let $\delta > 0$ such that $|f(s_1, \theta) - f(s_2, \theta)| < \epsilon$ whenever $|s_1 - s_2| < \delta$, for each $\theta \in [0, 1]$. Pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, put $s_k = k\frac{2\pi}{n}$ for $k = 0, 1, \ldots, n$. Note that, for each $s \in [0, 2\pi]$ there exist $\lambda \in [0, 1]$ such that $s = \lambda s_k + (1 - \lambda)s_{k+1}$ for some $k = 0, 1, \ldots, n - 1$. Define

$$f_n(s, \theta) := \lambda g(s_k, \theta) + (1 - \lambda)g(s_{k+1}, \theta)$$

if $s = \lambda s_k + (1-\lambda)s_{k+1}$. Then $f_n(s, \theta)$ is a continuous function of parameters s and θ. Let $s \in [0, 2\pi]$. Since $s_k \le s \le s_{k+1}$ for some $k = 0, 1, \ldots, n-1$, we have $|s - s_k| < \delta$. So that $|f(s, \theta) - f(s_k, \theta)| < \epsilon$ for each $\theta \in [0, 1]$. Therefore

$$||f(s, .) - f(s_k, .)||_1 < \epsilon.$$

On the other hand $||f(s_k, .) - f_n(s_k, .)||_1 < \epsilon$ since $f^n(s_k, .) = g(s_k, .)$. Also by the linearity of f_n, for each $\theta \in [0, 1]$, we have

$$|f_n(s_k, \theta) - f_n(s, \theta)| \le |f_n(s_k, \theta) - f_n(s_{k+1}, \theta)|,$$

so that

$$\begin{aligned}||f_n(s_k,.)-f_n(s,.)||_1 &\le ||f_n(s_k,.)-f_n(s_{k+1},.)||_1 = ||g(s_k,.)-g(s_{k+1},.)||_1\\ &\le ||g(s_k,.)-f(s_k,.)||_1 + ||f(s_k,.)-f(s_{k+1},.)||_1\\ &+ ||f(s_{k+1},.)-g(s_{k+1},.)||_1 < 3\epsilon.\end{aligned}$$

Thus

$$||f_n(s_k,.)-f_n(s,.)||_1 < 3\epsilon,$$

therefore,

$$||f(s,.)-f_n(s,.)||_1 \le ||f(s,.)-f(s_k,.)||_1+||f(s_k,.)-f_n(s_k,.)||_1+||f_n(s_k,.)-f_n(s,.)||_1 < 5\epsilon.$$

Now let $f(s,t,\theta)$ be a Real-valued function which is integrable of parameter $\theta \in [0,1]$, and continuous of parameters $s,t \in [0,2\pi]$ uniformly on θ. Fix $\epsilon > 0$. Since the function $f(s,t,\theta)$ is continuous of parameters $s,t \in [0,2\pi]$ uniformly on $\theta \in [0,1]$, there exists $\delta > 0$ such that for each $\theta \in [0,1]$, we have

$$|f(s_1,t_1,\theta)-f(s_2,t_2,\theta)| < \epsilon,$$

whenever $\max\{|s_1-s_2|,|t_1-t_2|\} < \delta$. Now, pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and put $t_k = \frac{2\pi k}{n}$ for $k = 0,1,\ldots,n$. We know from the previous case that for each $k = 0,1,\ldots,n$ there exists a function $f_n^k(s,t_k,\theta)$ which is continuous of parameters $s \in [0,2\pi]$ and $\theta \in [0,1]$ such that $||f(s,t_k,.)-f_n^k(s,t_k,.)||_1 < \epsilon$. We also know that, for each $t \in [0,2\pi]$ there exists $\lambda \in [0,1]$ such that $t = \lambda t_k + (1-\lambda)t_{k+1}$ for some $k = 0,1,\ldots,n-1$. Define

$$f_n(s,t,\theta) := \lambda f_n^k(s,t_k,\theta) + (1-\lambda) f_n^{k+1}(s,t_{k+1},\theta)$$

if $t = \lambda t_k + (1-\lambda)t_{k+1}$. Then $f_n(s,t,\theta)$ is a continuous function of parameters s,t and θ. Let $s,t \in [0,2\pi]$. Since $t_k \le t \le t_{k+1}$ for some $k = 0,1,\ldots,n-1$ we have $|t-t_k| < \delta$. So that $|f(s,t,\theta)-f(s,t_k,\theta)| < \epsilon$ for each $\theta \in [0,1]$. Therefore

$$||f(s,t,.)-f(s,t_k,.)||_1 < \epsilon.$$

On the other hand,

$$\begin{aligned}||f_n(s,t_k,.)-f_n(s,t_{k+1},.)||_1 &\le ||f_n(s,t_k,.)-f(s,t_k,.)||_1 + ||f(s,t_k,.)-f(s,t_{k+1},.)||_1\\ &+ ||f(s,t_{k+1},.)-f_n(s,t_{k+1},.)||_1 < 3\epsilon,\end{aligned}$$

so that

$$||f_n(s,t_k,.) - f_n(s,t,.)||_1 \leq ||f_n(s,t_k,.) - \lambda f_n(s,t_k,.) - (1-\lambda)f_n(s,t_{k+1},.)||_1$$
$$= (1-\lambda)||f_n(s,t_k,.) - f_n(s,t_{k+1},.)||_1 < 3(1-\lambda)\epsilon.$$

Therefore

$$||f(s,t,.) - f_n(s,t,.)||_1 \leq ||f(s,t,.) - f(s,t_k,.)||_1 + ||f(s,t_k,.) - f_n(s,t_k,.)||_1$$
$$+ ||f_n(s,t_k,.) - f_n(s,t,.)||_1 < (2 + 3(1-\lambda))\epsilon.$$

The conclusion follows by induction. □

We know that positively homogeneous and continuous functions on $\mathbb{R}^n$ can be characterized as continuous functions on the unit sphere of $\mathbb{R}^n$, which we will denote by S^{n-1}. We use this fact to extend our approximation.

Lemma 4.3. Let $f(s_1, s_2, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be an integrable function of parameter $\theta \in [0,1]$ and positively homegenuous and continuous function of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ uniformly on θ. Then for each $\epsilon > 0$ there exists a function $g(s_1, s_2, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ which is continuous of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ and $\theta \in [0,1]$ and positively homogenuous of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ such that $||f(s_1, s_2, \ldots, s_n, .) - g(s_1, s_2, \ldots, s_n, .)||_1 < \epsilon$.

Proof. Let f_0 denote the restiriction of f on $S^{n-1} \times [0,1]$. Since f is an integrable function of parameter $\theta \in [0,1]$ and positively homegenuous and continuous function of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ uniformly on θ, we have that f_0 is an integrable function of parameter $\theta \in [0,1]$ and continuous function of parameters $s_1, s_2, \ldots, s_{n-1} \in [0, 2\pi]$ uniformly on θ. By Lemma 4.2, we know that for each $\epsilon > 0$ there exists a function $g_0(s_1, s_2, \ldots, s_{n-1}, \theta)$ which is continuous of parameters $s_1, s_2, \ldots, s_n \in [0, 2\pi]$ and $\theta \in [0,1]$ such that $||f_0(s_1, s_2, \ldots, s_{n-1}, .) - g_0(s_1, s_2, \ldots, s_{n-1}, .)||_1 < \epsilon$. Since positively homegenuous functions are chracterized by the continuous functions on the unit sphere, we have that there exist a function $g(s_1, s_2, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ which uniquely extends g_0, and satisfies the desired properties. □

We now ready to give an affirmative answer for an integrable function.

Theorem 4.4. Let $f(s_1, s_2, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be an integrable function of parameter $\theta \in [0,1]$ and positively homegenuous and continuous function of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ uniformly on θ and let

$$F(s_1, s_2, \ldots, s_n) = \int_0^1 f(s_1, s_2, \ldots, s_n, \theta) d\theta.$$

Then for each Banach lattice X, the Krivine extension $\widetilde{F}$ of F is the Bochner integral of the Krivine extension $\widetilde{f}$ of f,i.e.,

$$\widetilde{F}(x_1, x_2, \ldots, x_n) = \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta.$$

Proof. Let X be a Banach lattice. By the previous lemma, for each $\epsilon > 0$ we can pick a function $g(s_1, s_2, \ldots, s_n, \theta) : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ which is continuous of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ and $\theta \in [0,1]$ and positively homogenuous of parameters $s_1, s_2, \ldots, s_n \in \mathbb{R}$ such that $||f(s_1, s_2, \ldots, s_n, .) - g(s_1, s_2, \ldots, s_n, .)||_1 < \epsilon$. Define $G(s_1, s_2, \ldots, s_n) := \int_0^1 g(s_1, s_2, \ldots, s_n, \theta) d\theta$. Then we have

$$|F - G| = \Big| \int_0^1 f d\theta - \int_0^1 g d\theta \Big| \leq \int_0^1 |f - g| d\theta = ||f - g||_1 < \epsilon.$$

Now, let " $\widetilde{.}$ " stands for the Krivine extension of " . " to the Banach lattice X. So by Krivine extension we have

$$||\widetilde{F} - \widetilde{G}|| = || \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta - \int_0^1 \widetilde{g}(x_1, x_2, \ldots, x_n, \theta) d\theta || < \epsilon.$$

We also know from Lemma 4.1 that

$$\widetilde{G}(x_1, x_2, \ldots, x_n) = \int_0^1 \widetilde{g}(x_1, x_2, \ldots, x_n, \theta) d\theta,$$

where the right handside of the equality understood as a Bochner integral. Therefore

$$\begin{aligned} &||F(x_1, x_2, \ldots, x_n) - \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta|| \\ &\quad \leq ||\widetilde{F} - \widetilde{G}|| + ||\widetilde{G} - \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta|| \\ &\quad = ||\widetilde{F} - \widetilde{G}|| + || \int_0^1 \widetilde{g}(x_1, x_2, \ldots, x_n, \theta) d\theta - \int_0^1 \widetilde{f}(x_1, x_2, \ldots, x_n, \theta) d\theta || < 2\epsilon. \end{aligned}$$

□

4.2 Orthogonality and Hermitian projections on complex Banach lattices

Hermitian projections on complex Banach lattices

Suppose that X is a complex Banach space. A bounded linear operator $T : X \to X$ is said to be **hermitian** if the numerical range of T is real, e.i., $V(T) \subseteq \mathbb{R}$. A projection P on X is an **hermitian projection** if it is an hermitian operator. A direct sum decomposition $E \oplus F$ of X is an **hermitian decomposition** if the induced projection $P : X \to E$ is an hermitian projection.

We know that a decomposition $E \oplus F$ of X is hermitian if and only if

$$||x + e^{i\theta} y|| = ||x + y||, \quad x \in E, \ y \in F, \ \theta \in \mathbb{R}.$$

Now, let X be a complex Banach lattice and let $P : X \to X$ be an order projection. Then $E = \text{Range } P$ is a band in X and $X = E \oplus E^d$ is a band decomposition of X. Note that for every $x \in E$, and every $y \in E^d$, we have

$$|x + y| = |x| + |y| = |x| \vee |y| + |x| \wedge |y| = |x| \vee |y|,$$

thus $||x + y|| = || \, |x| \vee |y| \, ||$. We deduce from this equality that

$$||x + e^{i\theta} y|| = || \, |x| \vee |e^{i\theta} y| \, || = || \, |x| \vee |y| \, || = ||x + y||,$$

for each $\theta \in \mathbb{R}$. Thus, $P : X \to E$ is an hermitian projection. With this in hand, it is natural to ask if every hermitian projection is an order projection on a complex Banach lattice. As pointed out in [14], this is not the case generally. Indeed, let $X = \ell_2^2$, and $E = \{(\alpha, \alpha) \ : \ \alpha \in \mathbb{C}\}$ and $F = \{(\beta, -\beta) \ : \ \beta \in \mathbb{C}\}$, so that $X = E \oplus F$. Let $P : X \to E$ be the projection on E. For $x = (\alpha, \alpha) \in E$ and $y = (\beta, -\beta) \in F$, we have

$$||x + e^{i\theta} y||^2 = 2(|\alpha|^2 + |\beta|^2) = ||x + y||^2, \quad \theta \in [0, 2\pi),$$

and so P is an hermitian projection. However it is not an order projection.

Orthogonality in Banach Spaces

In [13], James introduced the notion of orthogonality in normed vector spaces. In a normed vector space X, an element x is said to be **orthogonal** to an element y if and only if $||x|| \leq ||x + \alpha y||$ for scalars α, or equivalently there exists a functional $f \in X^*$ such that $|f(x)| = ||x||$ and $f(y) = 0$. This definition extends the notion of orthogonality in Hilbert spaces to normed linear spaces.

We know that an operator T on a Hilbert space H is hermitian if and only if $\langle x, y \rangle = 0$ for $x \in$ Range T, $y \in$ Ker T. Thus, orthogonality is coherent with hermitian projections in Hilbert spaces. It is natural to ask that if this coherence stands in Banach spaces.

Lemma 4.5. Let X be a complex Banach space. If P is an hermitian projection on X, then,

$$||x|| \leq ||x + \alpha y||,$$

for each $x \in$ Range P, each $y \in$ Ker P, and each $\alpha \in \mathbb{C}$.

Proof. Let $x \in$ Range P, $y \in$ Ker P. Then there exists $x^* \in X^*$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$, By the Hahn-Banach Theorem. For every $\alpha \in \mathbb{C}$, one has

$$||x|| = x^*(x) = x^*P(x) = x^*P(x + \alpha y) \leq ||x^*P||\ ||x + \alpha y||.$$

On the other hand, $||x^*P|| \leq ||x^*||\ ||P|| = 1$ and

$$1 = \frac{x^*(x)}{||x||} = x^*P(\frac{x}{||x||}),$$

thus $||x^*P|| = 1$. Hence,

$$||x|| \leq ||x + \alpha y|| \quad \alpha \in \mathbb{C}.$$

□

The converse of the Lemma 4.5 is not true in general, in that, there are some non-hermitian projections satisfying $\|x\| \leq \|x + \alpha y\|$, where x is in the range, y is in the kernel and α is a scalar.

Example 4.6. Let $X = (\ell_\infty)_{\mathbb{C}}$, and let (e_i) be the standard basis of X. Put $f_2 = e_1 + e_2$ and observe that $(e_1, f_2, e_3, \ldots)$ is also a basis of X. Now, define

$$P : X \to X, \quad P(x = z_1 e_1 + z_2 f_2 + x_3 e_3 + \ldots) = z_2 f_2 = (z_2, z_2, 0, 0, \ldots).$$

Then P is a projection on X with Ran $P = \{(z, z, 0, 0, \ldots) \in \ell_\infty \mid z \in \mathbb{C}\}$ and Ker $P = \{(z_1, 0, z_3, z_4, \ldots) \in \ell_\infty \mid z_i \in \mathbb{C}\}$. Now let $x = (z, z, 0, 0, \ldots) \in$ Ran P and let $y = (z_1, 0, z_3, z_4, \ldots) \in$ Ker P. For each $\alpha \in \mathbb{C}$, we have

$$\|x\| = \|(z, z, 0, 0, \ldots)\| = |z| \leq \sup\{|z + \alpha z_1|, |z|, |\alpha z_3|, \ldots\} = \|x + \alpha y\|.$$

But P is not hermitian since for $x = (1, 1, 0, 0, \ldots) \in$ Ran P, and $y = (1, 0, 0, 0, \ldots) \in$ Ker P,

$$\|x + y\| = 2 \neq \sqrt{2} = |1 + e^{i\pi}| = \|x + e^{i\pi} y\|.$$

Now, let X be a (complex) Banach lattice. We will now check the relation between the orthogonality and the lattice structure of X. We start by checking if one of the notions orthogonality and disjointness implies the other one for arbitrary elements x and y of X.

Lemma 4.7. Let X be a complex Banach lattice and let $x, y \in X$ with $|x| \wedge |y| = 0$. Then for each $\alpha \in \mathbb{C}$, we have

$$||x|| \leq ||x + \alpha y||.$$

Proof. Let $|x| \wedge |y| = 0$. Then, $|x| \wedge |\alpha y| = 0$ for each $\alpha \in \mathbb{C}$. Thus

$$|x + \alpha y| = |x| + |\alpha y| \geq |x|,$$

hence $||x|| \leq ||x + \alpha y||$. □

In general, orthogonality does not imply disjointness.

Example 4.8. Let $X = (c_0)_\mathbb{C}$, and let $x = (1, 1, 0, 0, \ldots),\ y = (1, 0, 0, \ldots)$. Then for each $\alpha \in \mathbb{C}$, we have

$$||x + \alpha y|| = ||(1 + \alpha, 1, 0, 0, \ldots)|| = \max\{|1 + \alpha|, 1\} \geq 1 = ||x||.$$

However, $x \wedge y \neq 0$.

It is natural to ask under which conditions orthogonality implies disjointness. This is first discussed in [20, Theorem 1].

Theorem 4.9. Let X be a smooth Banach lattice. Then the following are equivalent:

(i) X is strictly monotone;

(ii) For $x, y \in X^+$, if $||x|| \leq ||x + \alpha y||$ then $x \wedge y = 0$.

On the other hand, the relation between orthogonality and disjointness can also be analyzed by comparing complemented subspaces of X. Let $X = E \oplus F$ be a direct sum decomposition. By Lemma 4.7, we know that if $E \oplus F$ is a band decomposition, then it is an orthogonal decomposition as well. But it is not trivial to see if the converse implication holds. We introduce the following definition.

Definition 4.10. A direct sum decomposition $E \oplus F$ of a complex Banach lattice X is said to satisfy property (d) if

$$||x|| \leq ||x + \alpha y|| \text{ implies } |x| \wedge |y| = 0, \quad x \in E,\ y \in F,\ \alpha \in \mathbb{C}.$$

A complex Banach lattice X is said to satisfy property (d) if every direct sum decomposition $E \oplus F$ of X satisfies the property (d).

Example 4.11. Let X be one of the spaces $(\ell_p)_{\mathbb{C}}, 1 \leq p < \infty$, or $(c_0)_{\mathbb{C}}$, then X satisfies the property (d). Indeed, by Pełczyński's theorem, we know that every infinite-dimensional complemented subspace of X is isomorphic to X. Thus, if $X = E \oplus F$, then it is a band decomposition. For more details, we refer to [8, Chapter 5].

The above example shows that if X is one of the spaces $(\ell_p)_{\mathbb{C}}, 1 \leq p < \infty$ or $(c_0)_{\mathbb{C}}$, then every direct decomposition of X is orthogonal.

We have shown by Example 4.8 that if $x, y \in (c_0)_{\mathbb{C}}$, then the orthogonality of the elements x and y does not imply the disjointness in general. But, thanks to Pełczyński's theorem, direct sum decompositions of $(c_0)_{\mathbb{C}}$ are disjoint.

Example 4.12. Let $X = (\ell_2^2)_{\mathbb{C}}$ and let $E = \{(\alpha, \alpha) \mid \alpha \in \mathbb{C}\},\ F = \{(\beta, -\beta) \mid \beta \in \mathbb{C}\}$. Then $X = E \oplus F$ and for $x = (\alpha, \alpha) \in E,\ y = (\beta, -\beta) \in F$, we have

$$\|x + \theta y\|^2 = 2|\alpha|^2 + 2|\theta|^2|\beta|^2 \geq 2|\alpha|^2 = \|x\|,\ \theta \in \mathbb{C},$$

but $|x| \wedge |y| \neq 0$. Hence, ℓ_2^2 does not satisfy property (d).

Remark 4.13. Note that $(\ell_2^2)_{\mathbb{C}}$ is smooth and strictly monotone but does not satisfy property (d). Also note that the complemented subspace F which is given in the previous example, does not contain any positive element of $(\ell_2^2)_{\mathbb{C}}$.

Theorem 4.14. Every infinite-dimensional reflexive complex Banach lattice satisfies property (d).

Proof. Let X be a reflexive complex Banach lattice and let $X = E \oplus F$ be a direct sum decomposition of X with

$$||x|| \leq ||x + \alpha y||,$$

for each $x \in E$, each $y \in F$ and each $\alpha \in \mathbb{C}$. This is equivalent to, by [13, Theorem 2.1], the fact that for each $x \in E$ there exists $x^* \in X'$ such that $x^*(x) = 1$ and $x^*(y) = 0$ for each $y \in F$. Define

$$E' = \{x^* \in X' \mid \exists x \in E \text{ such that } x^*(x) = 1 \text{ and } x^*(y) = 0 \ \ \forall y \in F\}.$$

We claim that E' is an ideal in $X' = X^{\sim}$, where $X^{\sim}$ is the order dual of X. Indeed, if $a^*, b^* \in E'$, then by Hahn-Banach Theorem there exists $x'' \in X''$ such that $(\alpha a^* + \beta b^*)(x'') = 1$ for each scalars α, β. But $x'' \in X$, since X is reflexive. Put $x'' = x + y$ where $x \in E, y \in F$. Then,

$$1 = (\alpha a^* + \beta b^*)(x'') = (\alpha a^* + \beta b^*)(x + y) = (\alpha a^* + \beta b^*)(x).$$

Hence $\alpha x^* + \beta y^* \in E'$. On the other hand if $|a^*| \leq |b^*|$ with $b^* \in E'$, then, by Hahn-Banach Theorem again, there exists $a, b \in X$ such that $a^*(a) = 1 = b^*(b)$. Put $a = x_a + y_a, b = x_b + y_b$ where $x_a, x_b \in E, y_a, y_b \in F$. Then $b^*(y_b) = b^*(y_a) = 0$ since $b^* \in E'$. So, $a^*(y_a) = 0$ since $|a^*| \leq |b^*|$. Therefore,

$$1 = a^*(a) = a^*(x_a + y_a) = a^*(x_a),$$

thus $a^* \in E'$.

Now let $^{\circ}E'$ be the inverse annihilator of E', i.e.,

$$^{\circ}E' = \{a \in X \mid x^*(a) = 0 \ \ \forall x^* \in E'\}.$$

Then $F = {}^{\circ}E'$. Indeed, if $y \in F$, then $x^*(y) = 0$ for each $x^* \in E'$. Thus $y \in {}^{\circ}E'$. Conversely, if $a \in {}^{\circ}E'$ with $a = x_a + y_a$ where $x_a \in E$ and $y_a \in F$, then there exists $x_a^* \in E'$ such that $x_a^*(x_a) = ||x_a||$. On the other hand, $x_a^*(a) = 0$ and $x_a^*(y_a) = 0$ since $a, y_a \in {}^{\circ}E'$. So,

$$||x_a|| = x_a^*(x_a) = x_a^*(x_a + y_a) = x_a^*(a) = 0,$$

thus $x_a = 0$ and therefore $a \in F$.

Since E' is an ideal of X', its inverse annihilator F is an ideal of X, which means $E \oplus F$ is a band decomposition of X. $\square$

Lemma 4.15. The complex Banach lattice $L_1[0,1]_{\mathbb{C}}$ does not satisfy property (d).

Proof. Let $E = \{f \in L_1[0,1]_{\mathbb{C}} \mid f = c \text{ for some } c \in \mathbb{C}\}$. It is easy to observe that E is a complemented subspace of $L_1[0,1]_{\mathbb{C}}$ and its complement is

$$F = \{g \in L_1[0,1]_{\mathbb{C}} \mid \int g = 0\}.$$

Indeed, if $(h_n)_{n=0}^{\infty}$ is the Haar system and P is the projection on h_0, then $E =$Range P and $F =$Ker P.

On the other hand, for each $f \in E$, each $g \in F$, and each $\alpha \in \mathbb{C}$, we have

$$||f|| = \int |f| \leq \int |f + \alpha g| = ||f + \alpha g||.$$

Thus $L_1[0,1]_{\mathbb{C}} = E \oplus F$ is an orthogonal decomposition, but clearly it is not a band decomposition. $\square$

Applications

In [10], Dales asked if a direct sum decomposition $E \oplus F$ of a complex Banach lattice X, satisfying

$$||x + y|| = ||\ |x| \vee |y|\ || \tag{4.1}$$

for $x \in E,\ y \in F$ is a band decomposition. For real Banach lattices, he gave a counter example: Let $X = \ell_1^2$ and let $\{e_1, e_2\}$ be the canonical basis of X. Let $E = \text{span}\{e_1 + e_2\}$ and $F = \text{span}\{e_1 - e_2\}$. Then, for each $a, b \in \mathbb{R}$, we have

$$||a(e_1+e_2)+b(e_1-e_2)|| = |a+b|+|a-b| = \max(2|a|, 2|b|) = ||\ |a||e_1+e_2|\vee|b||e_1-e_2|\ ||.$$

Thus, (3.1) holds, but $E \oplus F$ is not a band decomposition.

In [14], Kalton solved the problem for complex Banach lattices. He proved, by heavily using hermitian projections, that in a complex Banach lattice X, every direct sum decomposition $E \oplus F$ of X satisfiying (4.1) is a band decomposition of X.

For complex Banach lattices satisfying the property (d), we will give a direct solution to Dales's problem. First, we characterize hermitian projections.

Theorem 4.16. Let X be a complex Banach lattice, and let P be an hermitian projection on X. Then, P is an order projection if and only if the decomposition $X = \text{Range } P \oplus \text{Ker } P$ satisfies property (d).

Proof. Let $x \in \text{Range } P,\ y \in \text{Ker } P$. Since P is an hermitian projection, by Lemma 4.5, we have $\|x\| \le \|x + \alpha y\|$ for $\alpha \in \mathbb{C}$.

Now, if P is an order projection on X, then $X = \text{Range } P \oplus \text{Ker } P$ is a band decomposition of X, i.e., $|x| \wedge |y| = 0$. Hence Range $P \oplus$ Ker P satisfies property (d).

Conversely, if Range $P \oplus$ Ker P satisfies property (d), then $|x| \wedge |y| = 0$ since $\|x\| \le \|x + \alpha y\|$ for $\alpha \in \mathbb{C}$. Thus, P is an order projection. □

Corollary 4.17. Let X be a complex Banach lattice. If X satisfies property (d), then every hermitian projection on X is an order projection on X.

Proof. If $P : X \to X$ is an hermitian projection then, the decomposition Ran$P \oplus$ Ker$P$ satisfies property (d) since X does. Hence P is an order projection by Theorem 4.16. □

Now we are able to give a direct solution to Dales's problem for complex Banach lattices satisfying property (d).

Corollary 4.18. Let X be a complex Banach lattice with property (d), and let $E \oplus F$ be a direct sum decomposition of X. If $||x+y|| = ||\ |x| \vee |y|\ ||$ for each $x \in E,\ y \in F$, then $E \oplus F$ is a band decomposition.

Proof. Let P be the induced projection on E. Then for each $\alpha \in C$ with $|\alpha| = 1$,

$$||x+y|| = ||\ |x| \vee |y|\ || = ||\ |x| \vee |\alpha y|\ || = ||x + \alpha y||.$$

Thus P is an hermitian projection. Hence P is a band projection since X satisfies the property (d). So that $E \oplus F$ is a band decomposition. □

We do not know if $C_{\mathbb{C}}[0,1]$ satisfies property (d), but we know about the bands in $C_{\mathbb{C}}[0,1]$. It is well-known that the only projection bands on the Banach lattice $C_{\mathbb{C}}[0,1]$ are $\{0\}$ and $C_{\mathbb{C}}[0,1]$. So, we have the following:

Lemma 4.19. $C_{\mathbb{C}}[0,1]$ admits no non-trivial hermitian projections.

Proof. Let P be an hermitian projection on $C_{\mathbb{C}}[0,1]$, and let x be an element of $C_{\mathbb{C}}[0,1]$ such that $||x|| = 1$, and that $x(t) = 1$ for some $t \in [0,1]$. One can observe that if $x^* = \delta_t$, where δ_t is the Kronecker delta, then (x, x^*) is a primary state, i.e., $||x|| = 1 = ||x^*||$ and $x^*(x) = 1$. Hence, $x^*(Px) = P^*x^*(x)$ is real. We claim that this forces P^*x^* to be a multiple of x^*. By way of contradiction, suppose that there exists an open interval (a,b) such that $t \notin [a,b]$ and that P^*x^* is not zero on (a,b). Now pick some $x_0 \in C_{\mathbb{C}}[0,1]$ with $x_0(t) = 1$, $x_0(a) = 0 = x_0(b)$, $||x_0|| = 1$, and $\mu(x_0) \neq 0$, where μ is the restriction of the measure P^*x^* on (a,b). Now put $x_1 = \alpha x_0$ and observe that (x_1, x^*) is a primary state and $P^*x^*(x_1)$ is not real for a suitable $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

This implies that, P is a multiplication operator since P^*x^* is a multiple of x^*. Thus, P is a band projection. □

Corollary 4.20. Let $C_{\mathbb{C}}[0,1] = E \oplus F$ be a direct sum decomposition with

$$||x+y|| = ||\ |x| \vee |y|\ ||$$

for each $x \in E,\ y \in F$. Then either E or F is $C_{\mathbb{C}}[0,1]$.

Proof. Let P be the induced projection on E. Then it is easy to see that P is an hermitian projection. So, by Lemma 4.19, either E or F is $C_{\mathbb{C}}[0,1]$. $\square$

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Vita

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