İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF SCIENCES

TABLEAUX APPROACHES FOR REGION BASED THEORIES OF SPACE

Ph.D. Thesis by
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Departmant: MATHEMATICS AND COMPUTER SCIENCE
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İSTANBUL KÜLTÜR UNIVERSITY
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İstanbul, September 2016 Zafer Özdemir

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ÖZET

UZAYIN BÖLGEYE DAYALI TEORİLERİ İÇİN TABLO YAKLAŞIMI

Zafer ÖZDEMİR

Uzayın bölgeye dayalı teorilerini için tablo yaklaşımını incelediğimiz bu tez beş bölümden oluşmaktadır. Tezin ilk kısmında problemin tanımı ve uygulanan yöntemler verilmiştir. Tezin ikinci ve üçüncü kısmında uzayın bölgeye dayalı teorisinin sentaks, semantik ve aksiyomatik özellikleri tanıtılarak uzayın bölgeye dayalı teorisi için tablo kuralları verilmiş ve tablo yönteminin temel kavramları olan; başlangıç tablosu, açık ve kapalı tablo, dal, düğüm kavramları ifade edilerek tablo kurallarının uygulamaları örneklerle ayrıntılı olarak açıklanmıştır. Tablo kurallarının sonlanma ve sağlamlık teoremleri ve kanıtları verilmiştir. Ayrıca tablo kuralları için tamlık teoreminin kanıtında kullanılan sistematik tablo inşa yöntemi, doğruluk lemması ve tamlık teoremleri ve ispatıları verilmiştir. Ek olarak, uzayın bölgeye dayalı teorisinin modellerini genişleterek simetrik, yansımalı ve geçişmeli olması durumunda tablo kuralları tanımlanarak, sonlanma, sağlamlık ve tamlık teoremlerinin kanıtında kullanılan yardımcı teoremler ve bu teoremlerin kanıtları ayrıntılı olarak verilmiştir. Tezin dördüncü kısmında, bağıntılı mantıkların bir genişlemesi olan evrensel modalite içeren bağıntılı mantıkların; sentaks, semantik ve aksiyomatik özellikleri tanıtılarak, tablo kuralları verilmiş ve uygulamaları örneklerle ayrıntılı olarak açıklanmıştır. Ardından tablo kurallarının sonlanma ve sağlamlık teoremleri ve kanıtları verilmiştir. Ayrıca tablo kuralları için

doğruluk lemması ve tamlık teoreminin kanıtları verilmiştir. Beşinci bölümde, bağıntılı mantıkların farklı bir semantiği olan reel sayı aralıkları üzerindeki yorumu üzerine çalışılmış ve bu bağlamda; sentaks, semantik ve aksiyomatik özellikleri tanıtılarak, tablo kuralları verilmiş ve tablo kurallarının uygulamaları örneklerle ayrıntılı olarak açıklanmıştır. Ek olarak, tablo kurallarının sonlanma ve sağlamlık teoremleri kanıtları ile verilmiştir. Ayrıca tablo kuralları için doğruluk lemması ve tamlık teoreminin kanıtı verilmiştir.

Anahtar Kelimeler: Uzayın bölgeye dayalı teorisi, kontak mantıklar ve karar verme yöntemleri, hesaplanabilirlik.

University	**: İstanbul Kültür University**
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SUMMARY

TABLEAUX APPROACHES FOR REGION BASED THEORIES OF SPACE

Zafer ÖZDEMİR

In this study, we examined tableaux approaches for region based theories of space. This thesis consists of five chapters. In the first chapter is devoted to statement of the problem, review of contents, methods applied. In Chapter 2 and Chapter 3, we introduce region based theories of space. In Chapter 2, we introduce historical background, syntax and semantics, axiomatization of the theory. In semantics subsection, we present relational semantics and topological semantics. Following sections continue with applications and definability result in region based theories of space. In Chapter 3, we study tableau approaches for region based theories of space, we give basic definitions about tableau approaches, we prove termination, soundness and completeness theorems. In last section of Chapter 3, we give tableaux rules for variants of region based theories of space. In its subsections, we prove soundness and completeness theorem for variants. In Chapter 4, we presents generalized contact logics. The first section of Chapter 4 consists of syntax-semantics, definability and axiomatizationn. The chapter continue with general tableau approaches for generalized contact logics. Section 4.4 and 4.3, consist of soundness-completeness theorems and proofs. In the end of Chapter 4, we give tableau rules for variants. And then we give soundness-completeness theorem and proofs. Chapter 6 is devoted to the study of interval semantics of contact logics and its tableau

approaches. In particular, we give syntax and semantics. After that we present tableau rules for contact logics interpreted over intervals. The last two sections are about soundness and completeness theorems. We give proofs of these theorems.

Keywords: Region based theories, contact logics, tableaux-based decision procedures, computability.

Chapter 1

INTRODUCTION

1.1 Statement of the Problem

We introduce and study the general tableaux approaches for region based theories of space. In particular, we give sound and complete tableaux-based decision procedures for contact logics and its variants. Developing such tableaux-based decision procedures, we obtain new decidability/complexity results.

1.2 Review of Contents

Chapter 1 of this thesis presents the scope of the study as an introduction. Chapter 2 contains some background related to region based theories of space. The first section of Chapter 2 start with historical background of the theory. Section 2.2 and Section 2.3 deals with syntax, semantics and axiomatization. Section 2.4 is about applications in artificial intelligence and geographical information systems. Section 2.5 is about some definability results for region based theories of space.

Chapter 3 is about tableaux approaches for region based theories of space. First section of Chapter 3 is about basic definitions. Section 3.2 is about soundness and termination theorems. In section 3.3, we present some theorems and proofs which are related to proof of completeness theorem. In the end of the section we present completeness theorem and its proof. Following sections we extend the theory with symmetric, reflexive, serial and dense models. We prove soundness and completeness theorems for variants of contact logics.

In Chapter 4, we presents generalized contact logics. The first section of Chapter 4 consists of syntax-semantics, definability and axiomatization.

In Chapter 4 continue with tableau approaches for generalized contact logics. Sections 4.6 and 4.7 consist of soundness-completeness theorems and proofs. In the end of Chapter 4, we give tableau rules for variants, and we give soundness-completeness theorems and proofs for variants of generalized contact logics.

Chapter 5 is devoted to the study of interval semantics of contact logics and its tableau approaches. In particular, we give syntax and semantics. After that we present tableau rules for contact logics interpreted over intervals. The last two sections are about soundness and completeness theorems. We give proofs of these theorems.

1.3 Methods Applied

This work uses essentially the methods from the following branches of mathematical logics: region based theories, tableaux-based decision procedures, computability. In particular, we use intensively the following concepts: tableau approaches, termination of tableau approaches, saturated tableaux, termination theorem, soundness theorem, truth lemma and completeness theorem . The main technical tool used in the work is semantic tableau approach.

Chapter 2

REGION BASED THEORIES OF SPACES

In this chapter, we set general background about region based theories of space which also known as contact logic. It is modal logics that has been recently considered in order to obtain decidable fragments of the region-based theories of space introduced by De Laguna [19] and Whitehead [64].

2.1 History

The region-based theory of space is an alternative to the point-based theory of space. The region-based theory of space contains a notion such as region corresponding to the intuitive notion of spatial body and some basic relations between regions such as the mereological relations of part-of or overlap and the topological relation of connection (recently called contact relation). This does not mean that points should be disregarded; they must be defined and one of the main aims of the theory is to show an equivalence between the pointless and point-based approaches. Good analogs of regions in the point-based approach are regular closed (or regular open) sets in some topological spaces. Then two regular closed regions are in a contact if they have a non-empty intersection and the part-of relation is identified with the set-theoretical inclusion. Remark that regular sets (open or closed) form a Boolean algebra; this suggests that is considered region-based theories of space as special Boolean algebras with some additional relations, also called contact algebras. So topology is one of the main sources of (point-based) models of continuous region-based theories of space. The origin of region-based theory of space goes back to de Laguna [19] and Whitehead [64]. Early papers related to this approach include Tarski [58], Leonard and Goodman [33], Grzegorczyk [32], Clarke [16]. Gerla give a good survey of the pointless approach to the theory of space in [31]. Recent results in this field, concentrated on the correspondence with the classical, point-based approach, are Roeper [52], Mormann [48], Pratt and Schoop [49], Vakarelov et al. [60, 61], Duntsch and Winter [23], Dimov and Vakarelov [21] (see also the survey by Vakarelov [59]). One of the significant systems of region-based theory of space is the Region Connection Calculus (RCC) which is introduced by Randel et al. [50]. An axiomatization of RCC based on Boolean algebras was given by Stell [50]. In recent years, the RCC system is in the center of the so called Qualitative Spatial Reasoning (QSR), an intensive research area in artificial intelligent and information theory. A survey of the research in QSR was given by Cohn and Hazarika in [17]. In connection with this some authors pointed out that topological models do not fit well with some discrete models of space. Galton [29] proposed a model consisting of a set of elements called cells and a binary relation R between cells called adjacency relation. The regions in the adjacency spaces are arbitrary sets of cells and two regions A and B are in a contact if there exists cells $x \in A$ and $y \in B$ such that xRy [4]. A pointless approach to such a discrete region-based theory of space was given by Duntsch and Vakarelov [24] and Li and Ying [47] proposed a discrete generalization of RCC denoted as GRCC.

2.2 Syntax and Semantics

In this section, we present syntax and semantics for contact logics. We give the relational semantics which is based on Kripke models and also the topological semantics which is based on topological spaces. For further details one can consult on [4], whose terminology is used throughout.

2.2.1 Syntax

Let BV be a countably infinite set of Boolean variables (with members denoted by p, q, etc). The set of all Boolean terms based on BV (with members denoted by a, b, etc) is defined as follows:

- $a := p \mid 0 \mid -a \mid (a \cup b)$.

The Boolean constructs 1 and $\cap$ are defined as usual. The set of all formulas based on BV (with members denoted by ϕ, ψ, etc) is defined as follows:

- $\phi := a \leq b \mid C(a, b) \mid \bot \mid \neg\phi \mid (\phi \vee \psi)$.

The other Boolean constructs for formulas ($\top$, $\wedge$, etc) are defined as usual. Given Boolean terms a and b, let $a \not\leq b$ be $\neg\ a \leq b$, $a \equiv b$ be $(a \leq b \wedge b \leq a)$ and $a \not\equiv b$ be $\neg\ a \equiv b$. The intuitive readings of the formula $a \leq b$ and $C(a, b)$ are: "region a is part-of region b" and "region a is in contact with region b". As usual, we will follow the standard rules for omission of the parentheses. Examples of formulas are: $a \not\equiv 0 \to C(a, a)$ ("if a is nonempty then a is in contact with itself") and $C(a, b) \to C(b, a)$ ("if a is in contact with b then b is in contact with a").

2.2.2 Relational Semantics

A frame is a pair $\mathcal{F} = (W, R)$ where W is a nonempty set and $R \subseteq W \times W$. The elements of W are called points and, naturally, regions are sets of points, i.e. elements of 2^W. A model based on $\mathcal{F}$ is a 3-tuple $\mathcal{M} = (W, R, V)$ where V: $BV \longrightarrow 2^W$. The function V is the valuation function of the model. It associates a region of $\mathcal{M}$ to each Boolean variable of the language. It can be extended into the function $\bar{V}$ defined as follows on the set of all Boolean terms:

$\bar{V}(p) = V(p)$,

$\bar{V}(0) = \emptyset$,

$\bar{V}(-a) = W \setminus \bar{V}(a)$,

$\bar{V}(a \cup b) = \bar{V}(a) \cup \bar{V}(b)$.

Now, we have everything that is needed to define the satisfiability relation between a model $\mathcal{M} = (W, R, V)$ and a formula ϕ, in symbols $\mathcal{M} \models \phi$, as follows:

$\mathcal{M} \models a \leq b$ iff $\bar{V}(a) \subseteq \bar{V}(b)$,

$\mathcal{M} \models C(a, b)$ iff there exists $x, y \in W$ such that $x \in \bar{V}(a)$, $y \in \bar{V}(b)$ and xRy,

$\mathcal{M} \not\models \bot$,

$\mathcal{M} \models \neg\phi$ iff not $\mathcal{M} \models \phi$,

$\mathcal{M} \models \phi \vee \psi$ iff $\mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$.

We shall say that ϕ is satisfiable in a frame $\mathcal{F}$ iff $\mathcal{M} \models \phi$ for some $\mathcal{M}$ based on $\mathcal{F}$. ϕ is said to be valid in $\mathcal{F}$ iff $\mathcal{M} \models \phi$ for every $\mathcal{M}$ based on $\mathcal{F}$. Let $\mathcal{C}$ be a class of frames. We shall say that ϕ is satisfiable in $\mathcal{C}$ iff $\mathcal{M} \models \phi$ for some $\mathcal{M}$ based on a frame in $\mathcal{C}$. ϕ is said to be valid in $\mathcal{C}$ iff $\mathcal{M} \models \phi$ for every $\mathcal{M}$ based on a frame in $\mathcal{C}$. Satisfiability and validity with respect to a class of models are similarly defined. Examples of formulas valid in the class of all models are: $C(a,b) \rightarrow a \not\equiv 0 \wedge b \not\equiv 0$, $C(a, b \cup c) \leftrightarrow C(a,b) \vee C(a,c)$ and $C(a \cup b, c) \leftrightarrow C(a,c) \vee C(b,c)$. The formulas $a \not\equiv 0 \rightarrow C(a,a)$ and $C(a,b) \rightarrow C(b,a)$ are valid in, respectively, the class of all reflexive models and the class of all symmetric models.

2.2.3 Topological semantics

Let X be a topological space and let $RC(X)$ be the contact algebra of regular closed sets of X. We will use $RC(X)$ as a topological semantics for the language of contact logic in the following way. By a valuation of Boolean variables in a topological space X we mean any function v which assigns to any Boolean variable p a regular closed set $V(p)$. Then V is extended to all Boolean terms as follows:

$\bar{V}(p) = V(p)$,

$\bar{V}(0) = \emptyset$,

$\bar{V}(-a) = Cl(X \setminus \bar{V}(a))$,

$\bar{V}(a \cup b) = \bar{V}(a) \cup \bar{V}(b)$.

The pair $M = (X, V)$ is called a (topological) model. Then formulas are interpreted in such models as follows:

$M \models C(a,b)$ iff $\bar{V}(a) \cap \bar{V}(b) = \emptyset$,

$M \models a \leq b$ iff $\bar{V}(a) \subseteq \bar{V}(b)$,

$\mathcal{M} \not\models \perp$,

$\mathcal{M} \models \neg\phi$ iff not $\mathcal{M} \models \phi$,

$\mathcal{M} \models \phi \vee \psi$ iff $\mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$.

We say that a formula is true in the space X if it is true in all models over X.

2.3 Axiomatization

In this section an axiomatic system for contact logic will be given. This will be a Hilbert-style axiomatic system consisting of a set of axioms and a set of inference rules.

Axioms for contact logic L_{Con}.

(I). A complete set of axiom schemes for the classical propositional logic (or all formulas which are tautologies of the classical propositional logic).

(II). A set of axiom schemes for Boolean algebra in terms of the part-of $\leq$ (a, b, c are arbitrary Boolean terms):

$a \leq a$,

$(a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c)$,

$(0 \leq a)$,

$(a \leq 1)$,

$(c \leq (a \sqcap b)) \Leftrightarrow (c \leq a) \wedge (c \leq b)$,

$((a \sqcup b) \leq c) \Leftrightarrow (a \leq c) \wedge (b \leq c)$,

$(a \sqcap (b \sqcup c)) \leq (a \sqcap b) \sqcup (a \sqcap c)$,

$(c \sqcap a \leq 0) \Leftrightarrow (c \leq -a)$,

$(- - a \leq a)$.

(III). A set of axiom schemes for the contact C:

(C1) $C(a, b) \rightarrow (a \not\equiv 0) \wedge (b \not\equiv 0)$,

(C2) $C(a, b) \wedge (a \leq a') \wedge (b \leq b') \rightarrow C(a', b')$,

(C3) $C(a, (b \sqcup c)) \rightarrow C(a, b) \vee C(a, c), C((b \sqcup c), a) \rightarrow C(b, a) \vee C(c, a)$.

Rule of inference. Modus ponens:

From ϕ and $\phi \rightarrow \psi$, infer ψ, symbolically:

$$\text{(MP)} \quad \frac{\phi, \phi \rightarrow \psi}{\psi}$$

The notion of proof in L_{Con} is the standard one. All provable formulas will be called theorems of L_{Con}. It can be proved easily that the set of the theorems of L_{Con} is closed under the rule of substitution: if $\alpha(p_1, ..., p_n)$ is a theorem of L_{Con} and $p_1, ..., p_n$ is a sequence of different Boolean variables then for any boolean terms $b_1, ..., b_n$, the formula $\alpha(b_1, ..., b_n)$ is a theorem of L_{Con}. An equivalent formulation of the axiomatic system L_{Con} can be obtained as follows: consider the rule of substitution as a separate rule and instead of considering axiom schemes for the Boolean part and for the contact C consider for them concrete axioms in a sense that the terms a, b, c in their formulation are different fixed Boolean variables.

2.4 Applications In Artificial Intelligence And Geographical Information Systems

The need for formal languages to express and reason about spatial concepts is of crucial importance in many areas of artificial intelligence and geographic information systems. In this respect, the RCC system is in the center of the so called Qualitative Spatial Reasoning (QSR). Research in QSR is motivated by a wide range of possible application areas including Geographic Information System (GIS), robotic navigation, high level vision, spatial propositional semantics of natural languages, engineering design, common-sense reasoning about physical systems and specifying visual language syntax and semantics. There are numerous other application areas including qualitative document-structure recognition [63], biology (e.g. [56, 18]) and domains where space is used as a metaphor (e.g. [44, 51])

2.5 Some Definability Results

The definition of modal definability of a class of frames by a formula is the same as the definition of global modal definability in ordinary modal logic. Namely, we say that the class Σ of frames is modally definable by the formula ϕ if for every frame F we have: $F \in \Sigma$ iff $F \models \phi$. If the class Σ is defined by a first-order formula F then we say that F is modally definable by ϕ or that F is a first-order equivalent of ϕ.

The following lemma is a kind of modal definability statement.

Lemma 2.5.1. *Let $F = (W, R)$ be a frame and p, q be Boolean variables. Then the following equivalences are true:*

(i) $R \neq \emptyset$ *iff* $F \models C(1, 1)$ *(non-emptiness of R),*

(ii) $(\forall x \in W)(\exists y \in W)(xRy)$ *iff* $F \models p \not\equiv 0 \Rightarrow C(p,1)$ *(right seriality of R),*

(iii) $(\forall y \in W)(\exists x \in W)(xRy)$ *iff* $F \models p \not\equiv 0 \Rightarrow C(1,p)$ *(left seriality of R),*

(iv) $(\forall x \in W)(\exists y \in W)(xRy \text{ or } yRx)$ *iff* $F \models p \not\equiv 0 \Rightarrow C(1,p) \vee C(p,1)$ *(weak seriality of R),*

(v) $(\forall x \in W)(xRx)$ *iff* $F \models (p \not\equiv 0 \Rightarrow C(p,p))$ *(reflexivity of R),*

(vi) $(\forall x, y \in W)(xRy \rightarrow yRx)$ *iff* $F \models (C(p,q) \Rightarrow C(q,p))$ *(symmetry of R),*

(vii) $(\forall x, y \in W)(xRy \Leftrightarrow x = y)$ *iff* $F \models (C(p,q) \Leftrightarrow p \sqcap q \not\equiv 0)$ *(contact coincides with overlap),*

(viii) $(\forall x, y \in W)(xRy)$ *iff* $F \models p = 0 \wedge q \not\equiv 0 \Rightarrow C(p,q)$ *(universality of R).*

Let us note that the first-order conditions from $(i), (iii), (iv)$ and $(viii)$ are not modally definable in the ordinary modal language. On the other hand, there are examples of first-order conditions definable in the ordinary modal logic which are not modally definable in the present language. Such a condition is for instance transitivity of R.

The language of contact logics can be seen as a first-order language without quantifiers. See also [21, 23]. Nevertheless, we call it *modal* because most concepts, tools and techniques typical of ordinary modal languages can be applied to it: filtration method, canonical model construction, etc. For instance, with respect to modal definability, a Sahlqvist-like Correspondence Theorem can be obtained for contact logics [2]. It happens that some elementary properties that are definable in the ordinary language of modal logic are not definable in contact logics and, on the contrary, some second-order properties that are definable in contact logics are not definable in the ordinary modal language. Concerning the satisfiability problem, an interesting result for the contact logics is the following [4]: the satisfiability problem with respect to the class of all Kripke models or with respect to the class of all reflexive and symmetric Kripke models is NP-complete. These definability and computability results show that the language of contact logics can be sometimes more expressive than the corresponding language of modal logic whereas the satisfiability problem can be, in some cases, easier to decide.

Chapter 3

TABLEAUX APPROACHES FOR REGION BASED THEORIES OF SPACES

In this chapter we give sound and complete tableaux- based decision procedures for contact logics and some variants of these logics. Although the language of contact logics has been based on the connectives of part-of $\leq$ and contact C, we will base the tableau rules on the connectives of equality $\equiv$ and contact C. Seeing that $\equiv$ and $\leq$ are easily interdefinable ($a \equiv b = (a \leq b \wedge b \leq a)$ and $a \leq b = (a \cup b) \equiv b$), this change in the choice of the primitives is harmless: the results concerning the decidability and complexity of the problem of deciding the satisfiability of formulas with respect to such or such class of models will be the same in a $\leq$-base language or in an $\equiv$-based one.

3.1 Tableaux Approaches

The language of contact logics is based on two types of expressions: Boolean terms and formulas. For these reasons, tableau nodes will be labeled by the following types of expressions:

(i) formulas;

(ii) expressions of the form $x : a$;

(iii) expressions the form $x\triangle y$;

where x, y are symbols and a is a Boolean term. Given a formula ϕ, its initial tableau is the labeled tree consisting of exactly one node (called root) labeled with ϕ. The tableau rules are given in two parts: Boolean-rules (This page) and formula-rules (next page).

Union Rule

$$\frac{x : (a \cup b)}{x : a \quad \Big| \quad x : b}$$

Intersection Rule

$$\frac{x : -(a \cup b)}{\begin{array}{c} x : -a \\ x : -b \end{array}}$$

Negation Rule

$$\frac{x : - - a}{x : a}$$

Rules are applied in a standard way by extending branches of constructed trees. For example, given a current tree t, a branch β in t and a node n in β labeled with a formula $C(a, b)$, applying the C rule to n consists in successively adding to the end of β three new nodes respectively labeled with $x : a$, $y : b$ and $x\triangle y$ where x, y are new symbols.

Disjunction Rule

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

Conjunction Rule

$$\frac{\neg(\phi \vee \psi)}{\begin{array}{c}\neg\phi \\ \neg\psi\end{array}}$$

Negation Rule

$$\frac{\neg\neg\phi}{\phi}$$

C Rule

$$\frac{C(a,b)}{\begin{array}{c} x : a \\ y : b \\ x \triangle y \end{array}} \quad (x \text{ and } y \text{ are new in the branch})$$

$\neg C$ Rule

$$\frac{\begin{array}{c} x \triangle y \\ \neg C(a,b) \end{array}}{x : -a \mid y : -b} \quad (x \text{ and } y \text{ already occur in the branch})$$

$\equiv$ Rule

$$\frac{a \equiv b}{\begin{array}{c|c} x : a & x : -a \\ x : b & x : -b \end{array}} \quad (x \text{ already occurs in the branch})$$

$\not\equiv$ Rule

$$\frac{a \not\equiv b}{\begin{array}{c|c} x : a & x : -a \\ x : -b & x : b \end{array}} \quad (x \text{ is new in the branch})$$

Definition 3.1.1. A branch is said to be closed if and only if one of the following conditions holds:

(i) it contains a node labeled with $x : 0$;

(ii) it contains two nodes respectively labeled with $x : a$, $x : -a$;

(iii) it contains a node labeled with $\perp$.

Definition 3.1.2. A tableau is closed when all its branches are closed.

Let us consider the contact formula $C(a,b) \wedge \neg C(b,a)$ and show how the rules apply. The tableau obtained for this formula by applying our rules has two open branches.

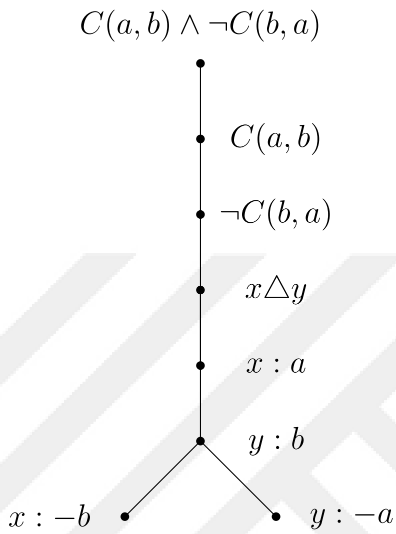

Figure 3.1 Open tableau

Let us consider the contact formula $(a \cup b) \equiv 0 \wedge C(a,b)$ and show how the rules apply. The tableau obtained for this formula by applying our rules has two closed branches.

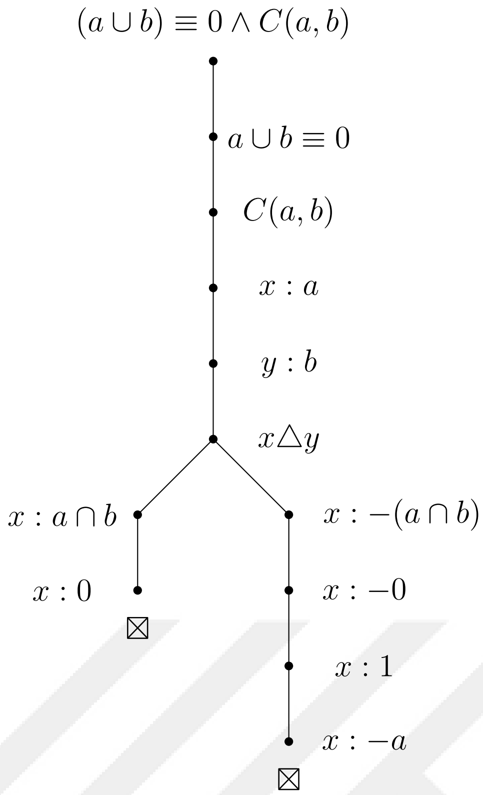

Figure 3.2 Closed tableau

In order to prove that the tableaux of satisfiable formulas cannot be closed, we introduce the concept of interpretability of a branch in a model.

Definition 3.1.3. Let $M = (W, R, V)$ be a model. Let β be a branch in a tableau and W' be the set of all variables occurring in β. The branch β is said to be interpretable in M if there exists a function $f : W' \to W$ such that:

(i) if ϕ occurs in β, then $M \models \phi$,

(ii) if $x \triangle y$ occurs in β, then $f(x) R f(y)$,

(iii) if $x : a$ occurs in β, then $f(x) \in \bar{V}(a)$.

These conditions are called compatibility conditions for f.

Definition 3.1.4. Let t be a semantic tableau and M be a model. The semantic tableau t is said to be interpretable in M if and only if there exist a branch in t which is interpretable in M.

Obviously, interpretable branches and interpretable tableaux are open.

3.2 Soundness

In this section, we show the soundness of the tableau rules for contact logics.

Proposition 3.2.1. *Let $M = (W, R, V)$ be a model and ϕ be a formula. If $M \models \phi$, then every semantic tableau computed from the initial tableau of ϕ is interpretable in M.*

Proof. Suppose $M \models \phi$. Since the initial tableau of ϕ consists of a single node labeled with ϕ, therefore the initial tableau of ϕ is interpretable in M. The fact that the tableau rules preserve the interpretability property in M follows from the strict similarity between the relational Kripke semantics of contact logics and the tableau rules presented at Pages 4 and 6.

□

Now, rule show the termination of our tableau procedure.

Theorem 3.2.2. *Let ϕ be a contact formula. After a finite number of steps from the initial tableau of ϕ, no tableau rules can be applied.*

Proof. Let t be a tableau computed from the initial tableau for ϕ. We have to show that after a finite number of steps, no rule can be applied. Suppose the contrary. Thus there exists an infinite sequence $t_0, t_1, \ldots$ of tableaux such that $t_0 = t$ and t_{n+1}, is obtained from t_n by applying a rule. Remark that each node occurring in these tableaux has at most two successors. As a result, from the sequence $t_0, t_1, \ldots$, we can extract an infinite branch β. The branch β contains information of the form: ψ, $x \triangle y$ and $x : a$, where ψ is a sub-formula or the negation of a subformula of ϕ and a is a sub-term or the complement of a subterm of ϕ. Note that the symbol $x, y, \ldots$ occurring in β have been introduced by two specific rules: the C rule and the $\not\equiv$ rule. Remark also that the application of these two rules are triggered by the occurrence of atomic formula of the form $C(a, b)$ and $a \not\equiv b$. These atomic formulas being sub-formulas or negations of sub-formulas of ϕ, the above two specific rules will be applied finitely many time only. This means that, all in all, the infinite sequence $t_0, t_1, \ldots$ does not exist. □

3.3 Completeness

In this section, we prove the completeness of the tableau method, i.e. for all valid formulas ϕ, after finitely many steps one can obtain a closed tableau from the initial tableau of $\neg\phi$. In this respect, the concept of saturation is essential.

Definition 3.3.1. A branch β in some tableau is called saturated, if the following conditions holds for all nodes $n \in \beta$:

- if n is labeled with $\neg\neg\phi$, then β contains a node labeled with ϕ,
- if n is labeled with $\phi \vee \psi$, then β contains a node labeled with ϕ or ψ,
- if n is labeled with $\neg(\phi \vee \psi)$, then β contains nodes labeled with $\neg\phi$ and $\neg\psi$,
- if n is labeled with $C(a, b)$, then β contains nodes labeled with $x\triangle y$, $x : a$ and $y : b$,
- if n is labeled with $\neg C(a, b)$ and β contains a node labeled with $x\triangle y$, then β contains a node labeled with $x : -a$ or $y : -b$,
- if n is labeled with $a \equiv b$ and β contains the symbol x, then β contains nodes labeled with $x : a$ and $x : b$, or β contains nodes labeled with $x : -a$ and $x : -b$,
- if n is labeled with $a \not\equiv b$, then for some symbol x either β contains nodes labeled with $x : a$ and $x : -b$, or β contains nodes labeled with $x : -a$ and $x : b$,
- if n is labeled with $x : - - a$, then β contains a node labeled with $x : a$,
- if n is labeled with $x : a \cup b$, then β contains a node labeled with $x : a$ or $x : b$,
- if n is labeled with $x : -(a \cup b)$, then β contains nodes labeled with $x : -a$ and $x : -b$.

Definition 3.3.2. A tableau is said to be saturated if all its branches are saturated.

Definition 3.3.3. Let t be a saturated tableau and β be a branch in t. A triple $M = (W, R, V)$ is called a saturated model for β, if

(*i*) W is the set of all variables occurring in β,

(*ii*) R is the binary relation on W defined by xRy iff β contains the information $x\triangle y$,

(iii) $V(p)$ is the set of all $x \in W$ such that β contains the information $x : p$.

The following lemma is crucial for proving the completeness of our method.

Lemma 3.3.4. *Let t be a saturated tableau, β be a branch in t and $M = (W, R, V)$ be a saturated model for β. We have the following:*

(i) *If β contains $x : a$, then $x \in \bar{V}(a)$,*

(ii) *If β contains a contact formula ϕ, then $M \models \phi$.*

Proof. (i) The proof is done by induction on a. The base case follows from the definition of V. The induction steps are left to reader.
(ii) The proof is by done induction on ϕ. We only consider the case $C(a, b)$, the other cases being left to the reader. Suppose β contains contact formula $C(a, b)$. Since β is saturated, β contains nodes labeled with $x \triangle y$, $x : a$ and $y : b$. By item (i), we have $x \in \bar{V}(a)$ and $y \in \bar{V}(b)$. Since β contains information $x \triangle y$, then xRy. Therefore $M \models C(a, b)$. □

Now, we are ready to prove the completeness of our method.

Theorem 3.3.5. *Let ϕ be a formula and t a saturated tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules. If ϕ is valid in the class of all models then t is closed.*

Proof. Suppose t is open. Thus, t contains an open branch β. Since t is saturated, therefore β is saturated too. Let $M = (W, R, V)$ be the saturated model for β. By the truth lemma, we have $M \models \neg\phi$, contradicting the validity of ϕ. □

Obviously, the formula $a \not\equiv 0$ is satisfiable iff the Boolean term a is consistent in Boolean Logic. For this reason, the problem of determining if a given formula is satisfiable is NP-hard. A careful analysis of the tableau rules immediately leads us to the conclusion that the depth of a tableau computed from a given formula ϕ is linear in the number of symbols in ϕ. Since tableaux are finitely branching, for this reason, together with Proposition 3.2.1 and Theorem 3.3.2 (Termination), Theorem 3.3.5 (Completeness) allows us to conclude that the problem of determining if a given formula is satisfiable in the class of all models is NP-complete. This complexity result has already been discussed in [4] where it was obtained by means of a more complicate argument based on the filtration method.

3.4 Variants

In this section, we extend our systems with adding new tableau rules which are Ser (for serial models), Ref (for reflexive models), Ser (for serial models) and Den (for dense models). We give sound and complete tableaux based decision procedure for some variants of contact logics.

3.4.1 Symmetric models

A model $M = (W, R, V)$ is symmetric if for all $x, y \in W$, xRy if and only if yRx. Let us consider the class of all symmetric models of contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Sym)} \qquad \frac{x \triangle y}{y \triangle x} \; (x \text{ and } y \text{ already occur in the branch})$$

Obviously, (Sym) preserves the property of interpretability in symmetric models. Thus, if ϕ is satisfiable in a symmetric model, then all semantic tableaux computed from the initial tableau of ϕ are open. Seeing that there is no need to apply (Sym) twice to the same pair (x, y), termination of our extended tableau system easily follows. Thus, after a finite number of steps from an initial tableau, no rules can be applied. Finally, within the context of our extended system, the model defined in the previous section is clearly symmetric. The proof of the Truth Lemma being repeated as such, we therefore obtain the following theorem:

Theorem 3.4.1. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules augmented with (Sym). If ϕ is valid in the class of all symmetric models then t is closed.*

Again, a careful analysis of the tableau rules augmented with (Sym) immediately leads us to the conclusion that the depth of a tableau computed from a given formula ϕ is linear in the number of symbols in ϕ. Consequently, the problem of determining if a given formula is satisfiable in the class of all symmetric models is NP-complete.

3.4.2 Reflexive models

A model $M = (W, R, V)$ is reflexive if for all $x \in W$, xRx. Let us consider the class of all reflexive models of contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Ref)} \qquad \frac{\cdot}{x \triangle x} \qquad (x \text{ already occurs in the branch})$$

The same line of reasoning as the one considered in the case of the class of all models would easily allow us to obtain the soundness, termination, completeness of the extended tableau system.

3.4.3 Serial models

A model $M = (W, R, V)$ is serial if for all $x \in W$, there exist $y \in W$ such that xRy. Let us consider the class of all serial models of contact logics. Apparently, the seriality property is quite innocent. Nevertheless, the satisfiability problem for the restriction of contact logic to the class of all serial models seems to be more difficult than expected. Since we are interested in the class of all serial models, the following rule is of some importance

$$\text{(Ser)} \qquad \frac{\cdot}{x \triangle y} \qquad \begin{array}{l}(x \text{ already occurs in the branch} \\ \text{and } y \text{ is new in the branch})\end{array}$$

Obviously, the application of (Ser) preserves the property of interpretability in serial models. The main problem with (Ser) clearly concerns the termination property. Nevertheless, by means of a strategy, we can obtain a terminating tableaux-based decision procedure. To define our strategy, we need the following preliminary definitions.

Definition 3.4.2. Let β be a branch and x be a symbol occurring in β. We say that x is *successor-free*, if there is no symbol y such that $x \triangle y$ occurs in β.

Definition 3.4.3. Let β be a branch and $term(x, \beta)$ be the set of all terms which is associated with x. x is said to be *twin-free*, if there is no symbol y occurring in β, $x \neq y$, such that $term(x, \beta) = term(y, \beta)$ and y has a successor.

Recall from the example presented at Page 13 that the contact formula $C(a, b) \wedge \neg C(b, a)$ has an open tableau. According to our definition, the symbols x and y are twin-free in both branches. On the other hand, x is not successor-free: x

has a successor, namely y, in both branches. Besides, y has no successor in both branches. Thus, y is a successor-free symbol.

Our strategy is the following:

(i) Apply the formula rules and term rules as much as possible,

(ii) Choose a successor-free and twin-free symbol x already existing in the branch. Apply the (Ser) rule to x and go to (i) otherwise, go to (iii),

(iii) Halt.

Now, let us show how our strategy will terminate. Remark first that in any branch β of a tableau constructed from ϕ's initial tableau, $term(x, \beta)$ contains, for each symbol x occuring in β, only subterms or complements of subterms of ϕ. There exists finitely many subterms of ϕ. Consequently, at some point of the computation, in each branch β of the constructed tree, each β successor-free symbol is not β twin-free. Thus, our strategy terminate. Finally, we now prove the completeness of our strategy.

Definition 3.4.4. Suppose β is an open branch (if there is at least one) in the tableau obtained by means of our strategy from the initial tableau of ϕ. Let $M = (W, R, V)$ be the structure defined as follows:

- W is the set of all symbols x in β that are not successor-free,
- for all $x, y \in W$, xRy if and only if $x \triangle y$ occurs in β or there is a successor-free symbol z in β such that $term(y, \beta) = term(z, \beta)$ and $x \triangle z$ occurs in β (we will say that y is a *twin* of z).
- $V(p)$ is the set of all $x \in W$ such that β contains the information $x : p$.

Obviously, M is serial.

Lemma 3.4.5. *Let t be an open tableau obtained with our strategy, β be an open branch in t and $M = (W, R, V)$ be the model defined as above. The following conditions hold: (i) If β contains the expression $x : a$, then $x \in \bar{V}(a)$; (ii) If β contains the contact formula φ, then $M \models \varphi$.*

Proof. (i) is proved by induction on a. As for (ii), we just consider the case of formula $C(a, b)$. Suppose β contains $C(a, b)$, we show that $M \models C(a, b)$. Since C rule is applied, β contains the following expressions: $x \triangle y$, $x : a$ and $y : b$. Then,

$x \in W$. Since $x : a$ occurs in β, therefore by item (i), $x \in \bar{V}(a)$. As for y, there are two cases to consider:

Case 1: y has a successor. Then, $y \in W$ too. Since $y : b$ occurs in β, therefore by item (i), $y \in \bar{V}(b)$.

Case 2: y has no successor. Hence, $y \notin W$. Let z be a twin of y in β. By definition of the twin-free property, z has a successor, $z \in W$ and xRz. Moreover, $z : b$ belongs to the branch β. By item (i), $z \in \bar{V}(b)$. □

Theorem 3.4.6. *Let ϕ be a formula and t a saturated tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Ser). If ϕ is valid in the class of all serial models then t is closed.*

Proof. Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Ser). If ϕ is valid in the class of all serial models then t is closed. □

Concerning the computational complexity of deciding the satisfiability of formulas with respect to the class of all serial models, obviously, the number N of pairwise distinct sets $term(\cdot, \beta)$ associated to symbols in the branch β of a tableau computed from ϕ is exponential in the size of ϕ. Thus, we immediately obtain from the tableau approach for the class of all serial models the following result: the computational complexity of deciding the satisfiability of formulas is in $NEXPTIME$. Nevertheless, in a branch β, during a computation, there is no need, when all formulas of the form $C(a, b)$ and $a \not\equiv b$ have been taken into account, and we know that the number of such formulas is linearly bounded in the size of ϕ, to keep in memory all expressions of the form $x : a$ occurring in β. All we have to do is:

1. when all formulas of the form $C(a, b)$ and $a \not\equiv b$ have been taken into account, each time (Ser) is applied, apply the $\neg C$ and the $\equiv$ rules as much as possible, then eliminate from the branch the symbol x that has triggered the execution of (Ser);

2. count the number of times the rule (Ser) have been applied;

3. once this number is greater than N, stop.

This would give us an improved strategy that can be implemented in polynomial space. Consequently, the computational complexity of deciding the satisfiability of formulas with respect to the class of all serial models is in $PSPACE$. This complexity result is new; it cannot be easily obtained by means of an argument based on the filtration method.

3.4.4 Dense models

A model $\mathcal{M} = (W, R, V)$ is said to be dense if for all $x, y \in W$, if xRy then there exists $z \in W$ such that xRz and zRy. Let us consider the class of all dense models of contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Den)} \qquad \frac{x \triangle y}{\begin{array}{c} x \triangle z \\ z \triangle y \end{array}} \qquad \begin{array}{l}(x \text{ and } y \text{ already occur in the branch} \\ \text{and } z \text{ is new in the branch})\end{array}$$

Obviously, (Den) is sound with respect to dense models, i.e. it preserves the interpretability property of tableaux. Nevertheless, as in the case of (Ser), it may lead to infinite computations. Thus, we have to define a strategy that will guarantee the soundness, the completeness and the termination of our tableaux-based system.

Definition 3.4.7. Let β be a branch and the pair (x, y) be a symbol occurring in β. We will say the pair (x, y) is intermediate-free if $x\Delta y$ occurs in β and for all symbols z in β, either $x\Delta z$ does not occur in β, or $z\Delta y$ does not occur in β.

Definition 3.4.8. Let β be a branch, x be a symbol occurring in β. let $term(x, \beta) = \{a : x : a \text{ occurs in } \beta\}$. We will say the pair (x, y) of symbols occurring in β is said to be twin-free if $x\Delta y$ occurs in β and for all symbols z_1, z_2, z_3 occurring in β, if $term(x, \beta) = term(z_1, \beta)$ and $term(y, \beta) = term(z_3, \beta)$ then either $z_1\Delta z_2$ does not occur in β, or $z_1\Delta z_3$ does not occur in β, or $z_2\Delta z_3$ does not occur in β.

Our strategy is the following:

(i) Apply the formula rules and the term rules as much as possible,

(ii) Choose an intermediate-free twin-free pair (x, y) of symbols occurring in a branch β; Apply the rule (Den) to (x, y) and go to (i) otherwise go to (iii),

(iii) Halt.

In order to show that our strategy terminate, it suffices to follow an argument similar to the one developed in the previous section. Let us be more precise.

Firstly, remark that in any branch β of a tableau constructed from the initial formula ϕ and for any x occurring in β, $term(x, \beta)$ only contains subterms or negation of subterms from ϕ. Seeing that there exists finitely many subterms from ϕ, at some point of the computation, each intermediate-free pair (x, y) of symbols occurring in a branch β is not twin-free. Therefore, our strategy terminates. Secondly, let us prove the soundness and the completeness of our tableau system extended with (Den) and following the above strategy. Obviously, every tableau constructed, by following the above strategy, from the initial tableau of a formula ϕ satisfiable in a dense model will be open. Conversely, suppose β is an open branch obtained, by means of our strategy, at the end of the tableau computation from an initial formula ϕ. Let W be the set of all x, y, etc occurring in β. As expected, we define on W the valuation V such that for all Boolean variables p, $V(p) = \{x \in W\colon\ x : p$ occurs in $\beta\}$. Now, for the accessibility relation R on W, it is defined as follows: for all $x, y \in W$, xRy iff $x\Delta y$ occurs in β. Defining $\mathcal{M} = (W, R, V)$, as in the cases of the previous classes of models that we have considered we obtain the following Truth Lemma which proof is similar to the proof of Theorem 3.3.2.

Lemma 3.4.9. *Let t be an open tableau obtained with our strategy, β be an open branch in t and $M = (W, R, V)$ be the model defined as above. The following conditions hold: (i) If $x : a$ occurs in Δ then $x \in \bar{V}(a)$. (ii) If ψ occurs in β then $\mathcal{M} \models \psi$.*

Since the considered tableau has been computed from the initial tableaux of ϕ, the branch β contains the formula ϕ and, by item (ii) of the above Truth Lemma, $\mathcal{M} \models \phi$. The main drawback is that R might be not dense. Suppose R is not dense. In order to prove that ϕ can be satisfied in a dense model, we have to transform $\mathcal{M}$ into a modally equivalent dense model $\mathcal{M}'$. Clearly, the accessibility relation R can be seen as the set of all pairs (x, y) in $W \times W$ such that $x\Delta y$ occurs in β. Since R is not dense, therefore R, as a subset of $W \times W$, is nonempty. A pair (x, y) in R is said to be an R-defect if there is no $z \in W$ such that xRz and zRy. By definition of R, seeing that β is a branch in a tableau that has been constructed by following the strategy described in the previous page, each R-defect (x, y) can be associated to a triple $\delta(x, y) = (z_1, z_2, z_3)$ of elements of W such that $term(x, \beta) = term(z_1, \beta)$, $term(y, \beta) = term(z_3, \beta)$ and $z_1\Delta z_2$, $z_1\Delta z_3$ and $z_2\Delta z_3$ occur in β. Let $R_0, R_1, \ldots$ be the sequence of binary relation on W and $\delta_0, \delta_1, \ldots$ be the sequence of functions defined as follows.

1. Let $R_0 = R$ and $\delta_0 = \delta$. By definition, for each R_0-defect (x, y), the triple

$\delta_0(x, y) = (z_1, z_2, z_3)$ of elements of W is such that $term(x, \beta) = term(z_1, \beta)$, $term(y, \beta) = term(z_3, \beta)$ and $z_1 R_0 z_2$, $z_1 R_0 z_3$ and $z_2 R_0 z_3$.

2. Suppose for some $i \in \mathbb{N}$ we have already defined a binary relation R_i on W and a function δ_i associating, for each R_i-defect (x, y), a triple $\delta_i(x, y) = (z_1, z_2, z_3)$ of elements of W such that $term(x, \beta) = term(z_1, \beta)$, $term(y, \beta) = term(z_3, \beta)$ and $z_1 R_i z_2$, $z_1 R_i z_3$ and $z_2 R_i z_3$.

3. If R_i has no defect, we define $R_{i+1} = R_i$. Otherwise, let (x, y) be an R_i-defect. Let z_1, z_2, z_3 be elements of W such that $\delta_i(x, y) = (z_1, z_2, z_3)$. We define $R_{i+1} = R_i \cup \{(x, z_2), (z_2, y)\}$.

For all $i \in \mathbb{N}$, let $\mathcal{M}_i = (W, R_i, V)$. Remark that, according to step 3 above, the only differences between $\mathcal{M}_{i+1}$ and $\mathcal{M}_i$ are the possibly new links (x, z_2) and (z_2, y). Since, according to step 3, the elements z_1, z_2, z_3 of W are such that $\delta_i(x, y) = (z_1, z_2, z_3)$, then $term(x, \beta) = term(z_1, \beta)$, $term(y, \beta) = term(z_3, \beta)$, $z_1 R_i z_2$ and $z_2 R_i z_3$. Thus, for all formula ψ based on Boolean variables occurring in ϕ, $\mathcal{M}_{i+1} \models \psi$ iff $\mathcal{M}_i \models \psi$. Since $\mathcal{M} \models \phi$, therefore for all $i \in \mathbb{N}$, $\mathcal{M}_i \models \phi$. Since W is finite, there exists $i \in \mathbb{N}$ such that R_i has no defect. Hence, R_i is a dense relation on W. Let $\mathcal{M}' = (W, R_i, V)$. By the above discussion, $\mathcal{M}' \models \phi$. Hence, ϕ can be satisfied in a dense model and we obtain the following result.

Theorem 3.4.10. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules augmented with (Den) and following the above strategy. If ϕ is valid in the class of all dense models then t is closed.*

Proof. Let ϕ be a formula and t be a tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Den). If ϕ is valid in the class of all dense models then t is closed. □

Thus, the problem of the satisfiability of formulas with respect to the class of all dense models is decidable. Unfortunately, we do not know its exact computational complexity. Note that this decidability result is new; it does not seem that it can be easily obtained by means of an argument based on the filtration method, seeing that the filtration construction does not preserve the elementary property of density.

Chapter 4

TABLEAUX APPROACHES FOR GENERALIZED CONTACT LOGICS

In this chapter we give syntax, semantics and axiomatization for generalized contact logics with a restricted form of universal modality which is a family of modal logics with restricted form of universal modality. The language of these family of modal logics allows the expression of Boolean combinations of formulas of the form $[U]\phi$ where ϕ is a formula of the ordinary language of modal logic. By means of a tableaux-based approach, we provide decision procedures for their satisfiability problem.

4.1 Syntax and Semantics

4.1.1 Syntax

Our language $L_r(\Box, \exists)$ is defined using a countable set BV of Boolean variables (with typical members noted p, q, r, etc). We inductively define the set $t(BV)$ of terms (with typical members noted A, B, C, etc) as follows:

$A ::= p \mid 1 \mid -A \mid (A \cap B) \mid \Box A.$

The other Boolean constructs for terms are defined as usual: 0 for -1, $(A \cup B)$ for $-(-A \cap -B)$ and $(A \to B)$ for $-(A \cap -B)$. We obtain the term $\Diamond A$ as an abbreviation: $\Diamond A$ for $-\Box - A$. We inductively define the set $f(BV)$ of formulas (with typical members noted ϕ, ψ, κ, etc) as follows:

$\phi ::= \exists A \mid \top \mid \neg\phi \mid (\phi \wedge \psi)$.

The other Boolean constructs for formulas are defined as usual: $\bot$ for $\neg\top$, $\neg(\phi \vee \psi)$ for $\neg(\neg\phi \wedge \neg\psi)$ and $(\phi \to \psi)$ for $\neg(\phi \wedge \neg\psi)$. We obtain the formulas $\forall A$, $A \leq B$ and $A \equiv B$ as abbreviations for $\neg\exists - A$, $\neg\exists(A \cap -B)$ and $(\neg\exists(A \cap -B) \wedge \neg\exists(-A \cap B))$. The notion of sub-term and the notion of sub-formula are standard. We adopt the standard rules for omission of the parentheses. For all sets x of terms, let $\Box x$ be the set of all terms A such that $\Box A \in x$. For all sets S of formulas, let $\forall S$ be the set of all terms A such that $\forall A \in S$. If A is a term then $BV(A)$ will denote the set of all Boolean variables occurring in A whereas if ϕ is a formula then $BV(\phi)$ will denote the set of all Boolean variables occurring in ϕ. For all $BV' \subseteq BV$, $t(BV')$ will denote the set of all $A \in t(BV)$ such that $BV(A) \subset BV'$ whereas $f(BV')$ will denote the set of all $\phi \in f(BV)$ such that $BV(\phi) \subseteq BV'$.

4.1.2 Semantics

A frame is an ordered pair $F = (W, R)$ where W is a non-empty set of possible worlds (with typical members noted x, y, z, etc) and R is a binary relation on W. For all $x \in W$, let $R(x)$ be the set of all $y \in W$ such that xRy. A valuation based on F is a function V assigning to each Boolean variable p a subset $V(p)$ of W. V induces a function $(\cdot)^V$ assigning to each term A a subset $(A)^V$ of W such that

- $(p)^V = V(p)$,
- $(1)^V = W$,
- $(-A)^V = W \setminus (A)^V$,
- $(A \cap B)^V = (A)^V \cap (B)^V$,
- $(\Box A)^V = \{x : R(x) \subseteq (A)^V\}$.

As a result,

- $(0)^V = \emptyset$,

- $(A \cup B)^V = (A)^V \cup (B)^V$,

- $(A \to B)^V = \{x : x \in (A)^V$ only if $x \in (B)^V\}$.

Moreover, $(\Diamond A)^V = \{x : R(x) \cap (A)^V \neq \emptyset\}$. A model is an ordered triple $M = (W, R, V)$ where $F = (W, R)$ is a frame and V is a valuation based on F. The satisfiability of a formula ϕ in M, in symbols $M \models \phi$, is defined as follows:

- $M \models \exists A$ iff $(A)^V \neq \emptyset$,

- $M \models \top$,

- $M \models \neg\phi$ iff $M \models \phi$,

- $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.

As a result,

- $M \not\models \bot$,

- $M \models \phi \vee \psi$ iff $M \models \phi$ or $M \models \psi$,

- $M \models \phi \to \psi$ iff $M \models \phi$ only if $M \models \psi$.

Moreover, $M \models \forall A$ iff $(A)^V = W$, $M \models A \leq B$ iff $(A)^V \subseteq (B)^V$ and $M \models A \equiv B$ iff $(A)^V = (B)^V$. Let F be a frame. A formula ϕ is said to be valid in F, in symbols $F \models \phi$, iff for all models M based on $F, M \models \phi$. Let C be a class of frames. A formula ϕ is said to be valid in C, in symbols $C \models \phi$, iff for all frames F in C, $F \models \phi$.

4.2 Definability

Formulas can be used to define classes of frames. For all classes C of frames and for all formulas ϕ, let $C(C, \phi)$ be the class of all frames F in C such that $F \models \phi$. A class C' of frames is said to be modally definable with respect to a class C of frames iff there exists a formula ϕ such that $C' = C(C, \phi)$. For instance, with respect to C_{all}, the class of all reflexive frames and the class of all symmetric frames are modally defined by, respectively, the formulas $\forall(\Box p \to p)$ and $\forall(p \to \Box \diamond p)$. Let $L(\Box)$ be the ordinary language of modal logics.

Proposition 4.2.1. *Let C, C be classes of frames. Let A be a term. With respect to C, the two following conditions are equivalent:*

1. C is modally definable by A (considered as a formula in $L(\Box)$),

2. C is modally definable by $\forall A$ (considered as a formula in $L_r(\Box, \exists)$).

At first sight, the capacity of $L_r(\Box, \exists)$ to modally define classes of frames seem to be equal to the corresponding capacity of $L(\Box)$. It is not true: with respect to C_{all}, there exists classes of frames that are modally definable in $L_r(\Box, \exists)$ but that are not modally definable in $L(\Box)$. Witness, the class of all connected frames and the class of all non-two-colourable frames that are modally defined in $L_r(\Box, \exists)$ by, respectively, the formulas $\forall(p \to \Box p) \to (\exists p \to \forall p)$ and $\forall(p \cup \Box p) \to \exists(p \cap \diamond p)$ but that are not modally definable in $L(\Box)$. Let $L(\Box, \exists)$ be the ordinary language of modal logics enriched with the universal modality.

Proposition 4.2.2. *Let C, C' be classes of frames. With respect to C, if C' is modally definable in $L_r(\Box, \exists)$ then C is modally definable in $L(\Box, \exists)$.*

Proposition 4.2.3. *Let C, C' be classes of frames. With respect to C, if C is modally definable in $L(\Box, \exists)$ then C is modally definable in $L_r(\Box, \exists)$.*

Hence, $L_r(\Box, \exists)$ and $L(\Box, \exists)$ have the same expressive power. At this point, an interesting question arises: is there a class of frames that is modally defined in $L(\Box, \exists)$ in an exponentially more succinct way than it is modally defined in $L_r(\Box, \exists)$? We leave open this question. Finally, there is also the problem of deciding the modal definability of a given elementary class of frames. Within the context of $L(\Box)$, Chagrova's Theorem says that this problem is undecidable. See [10] for details. The proof of Chagrova's Theorem is based on the undecidability of a variant of the halting problem concerning Minsky machines. Within the context of $L_r(\Box, \exists)$, it cannot be easily repeated for demonstrating that the problem of deciding the modal definability of a given elementary class of frames is undecidable too, as we will do in the following

Proposition 4.2.4. *Within the context of $L_r(\Box, \exists)$, the problem of deciding the modal definability of a given elementary class of frames is undecidable.*

4.3 Axiomatization

In our setting, a normal logic is a set of formulas containing the following axioms and closed under the rule of modus ponens and the rule of uniform substitution: (TAU) tautologies, (CON) congruence axioms:

– $A \equiv B \to -A \equiv -B$,

– $A \equiv B \wedge C \equiv D \to A \sqcap C \equiv B \sqcap D$,

– $A \equiv B \to \Box A \equiv \Box B$

(BOO) nondegenerate Boolean algebras axioms:

– $(A \sqcap B) \sqcap C \equiv A \cap (B \sqcap C)$,

– $A \sqcap B \equiv B \sqcap A$,

– $A \sqcap A \equiv A$,

– etc,

(MOD) modal algebras axioms:

– $\Box(A \sqcap B) \equiv \Box A \sqcap \Box B$,

– $\Box 1 \equiv 1$.

Let L_{min} be the minimal normal logic.

Lemma 4.3.1. *The following formulas are in L:*

1. $\forall(A \Rightarrow B) \to (\forall A \Rightarrow \forall B)$,
2. $\forall(-A \Rightarrow (A \Rightarrow 0))$,
3. $\forall(0 \Rightarrow A)$,
4. $\forall(A \cap B \Rightarrow A)$,
5. $\forall(A \cap B \Rightarrow B)$,
6. $\forall(A \Rightarrow (B \Rightarrow A \cap B))$,
7. $\forall A \to \forall(B \Rightarrow A)$,
8. $\forall((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
9. $\forall(A \Rightarrow (B \Rightarrow A))$,
10. $\forall(A \Rightarrow A)$,

11. $\forall(A \Rightarrow 0) \Rightarrow -A)$,

12. $\forall A \to \forall \Box A$,

13. $\forall(\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$,

14. $\forall(\Box - -A \Rightarrow \Box A)$,

15. $\neg\forall 0$,

16. $\neg\exists A \to \forall - A$,

17. $\exists A \to \neg\forall - A$.

We shall say that a set S of formulas is an L-theory iff S contains L and S is closed under the rule of modus ponens, i.e. for all formulas ϕ, ψ, if $\phi \to \psi \in S$ and $\phi \in S$ then $\psi \in S$. Let us be clear that the set of all L-theories is a partially ordered set with respect to set inclusion. The least element is L and the greatest element is the set of all formulas. Let S be an L-theory. The following lemmas some well-known facts about theories and maximal theories.

Lemma 4.3.2. *The following conditions are equivalent:*

1. *S is the set of all formulas,*
2. *there exists a formula ϕ such that $\phi \in S$ and $\neg\phi \in S$,*
3. *$\bot \in S$.*

S is said to be consistent iff for all formulas ϕ, $\phi \in S$ or $\neg\phi \in S$. We shall say that S is maximal iff for all formulas ϕ, $\phi \in S$ or $\neg\phi \in S$. Suppose S is maximal and consistent.

Lemma 4.3.3. *The following conditions are equivalent:*

1. *$\top \in S$,*
2. *$\neg\phi \in S$ iff $\phi \in S$,*
3. *$\phi \wedge \psi \in S$ iff $\phi \in S$ and $\psi \in S$.*

A set x of terms is said to be an S-theory iff x contains $\forall S$ and x is closed under the rule of modus ponens, i.e. for all terms A, B, if $A \to B \in x$ and $A \in x$ then $B \in x$. We will use x, y, z, etc, for S-theories. Let us be clear that the set of all S-theories is a partially ordered set with respect to set inclusion. The least element is $\forall S$ (use item 1 of Lemma 4.3.2) and the greatest element is the set of all terms.

Lemma 4.3.4. *Let x be an S-theory. The following conditions are equivalent:*

1. *x is the set of all terms,*
2. *there exists a term A such that $A \in x$ and $-A \in x$,*
3. *$0 \in x$.*

An S-theory x is said to be consistent iff for all terms A, $A \in x$ or $-A \in x$. We shall say that an S-theory x is maximal iff for all terms A, $A \in x$ or $-A \in x$.

Lemma 4.3.5. *Let x be a maximal consistent S-theory.*

1. *$1 \in x$,*
2. *$-A \in x$ iff $A \notin x$,*
3. *$A \cap B \in x$ iff $A \in x$ and $B \in x$.*

Let x be an S-theory. For all terms A, let $x + A$ be the set of all terms B such that $A \Rightarrow B \in x$.

Lemma 4.3.6. *Let x be an S-theory. For all terms A, $x + A$ is an S-theory containing x and A. Moreover, $x + A$ is consistent iff $-A \in x$.*

Lemma 4.3.7. *Let x be a consistent S-theory. For all terms A, $x+A$ is consistent or $x + -A$ is consistent.*

Three lemmas support the technique of the canonical model for L: the Lindenbaum's lemma, the diamond lemma and the truth lemma. The next lemma duplicates the Lindenbaum's lemma in $L(\Box)$. There is no change in the proof.

Lemma 4.3.8. *Let x be a consistent S-theory. There exists a maximal consistent S-theory y such that $x \subseteq y$.*

Lemma 4.3.9. *Let x be an S-theory. $\Box x$ is an S-theory.*

The next lemma duplicates the diamond lemma in $L(\Box)$. There is no change in the proof.

Lemma 4.3.10. *Let x be an S-theory. Let A be a term. If $\Box A \in x$ then there exists a maximal consistent S-theory y such that $\Box x \subseteq y$ and $A \in y$.*

The canonical model for S is the ordered triple $M_S = (W_S, R_S, V_S)$ where W_S is the set of all maximal consistent S-theories, R_S is the binary relation on W_S such that xR_Sy iff $\Box x \subseteq y$ and V_S is the function assigning to each Boolean variable p the subset $V_S(p)$ of W_S such that $x \in V_S(p)$ iff $p \in x$. The pair $F_S = (W_S, R_S)$ is called the canonical frame for S. Remark that W_S is nonempty (use item 15 of Lemma 1 and Lemma 8). The next lemma duplicates the truth lemma in modal logic. There is no change in the proof.

Lemma 4.3.11. *Let A be a term. Let ϕ be a formula. For all $x \in W_S$,*

– $x \in (A)^{V_S}$ *iff* $A \in x$,

– $M_S \models \phi$ *iff* $\phi \in S$.

Proposition 4.3.12. *For all formulas ϕ, if ϕ is valid in the class of all frames then $\phi \in L_{min}$.*

Proof. By Lemma 4.3.11. □

4.4 Tableaux Approaches

In this section we present the tableau rules for generalized contact logics with a restricted universal modality. To prove the termination theorem, we present a strategy for termination of tableau rules. We also prove that some lemmas and theorems which are related to proofs of soundness and completeness theorems. In section 6.4, we introduce the tableau rules for variants of generalized contact logic with a restricted universal modality. We should add the following rules to our systems: Sym (for symmetric models), Ref (for reflexive models) and Den (for dense models). We give strategies for termination of tableau rules of the variants and give proofs of termination, soundness and completeness theorems. First we star with tableau rules for generalized contact logics.

Tableau rules for formulas:

Conjunction Rule $$\frac{\phi \wedge \psi}{\begin{matrix}\phi \\ \psi\end{matrix}}$$

Disjunction Rule $$\frac{\neg(\phi \wedge \psi)}{\neg\phi \mid \neg\psi}$$

Negation Rule $$\frac{\neg\neg\phi}{\phi}$$

$\exists$ Rule $$\frac{\exists A}{x : A}$$ (x is new in the branch)

$\neg\exists A$ Rule $$\frac{\neg\exists A}{x : -A}$$ (x already exists in the branch)

Moreover, the language is based on two types of expressions: terms and formulas. For these reasons, tableau nodes will be labeled by the following types of expressions:

(i) formulas;

(ii) expressions of the form $x : A$;

(iii) expressions the form $x \triangle y$;

where x, y are symbols and A is a Boolean term. Given a formula ϕ, its initial tableau is the labeled tree consisting of exactly one node (called root) labelled with ϕ.

Box Rule

$$\frac{x : \Box A \quad x \triangle y}{y : A} \quad (y \text{ already occurs in the branch})$$

Negation Box *Rule*

$$\frac{x : -\Box A}{x \triangle y \quad y : -A} \quad (y \text{ is new in the branch})$$

Intersection Rule

$$\frac{x : A \cap B}{x : A \quad x : B}$$

Union Rule

$$\frac{x : -(A \cap B)}{x : -A \mid x : -B}$$

Negation Rule

$$\frac{x : - - A}{x : A}$$

The tableau rules are given in two parts: formula-rules (Page 41) and Boolean-rules (this page). Rules are applied in a standard way by extending branches of constructed trees. For example, given a current tree t, a branch β in t and a node n in β labeled with a formula $\exists A$, applying the $\exists$ rule to n consists in adding to the end of β a new node labeled with $x : A$ where x is new symbol.

Definition 4.4.1. A branch is said to be closed if and only if one of the following conditions holds:

(*i*) it contains a node labeled with $x : 0$;

(*ii*) it contains two nodes respectively labeled with $x : A$, $x : -A$;

(*iii*) it contains a node labeled with $\perp$.

A tableau is closed when all its branches are closed. Let us consider the formula $\exists - \Box(A \cap B) \wedge \neg\exists - \Box(A \cap B)$ and let us see how the rules apply. (Figure *Open tableau* in next page). The tableau obtained for this formula by applying our rules has two closed branches.

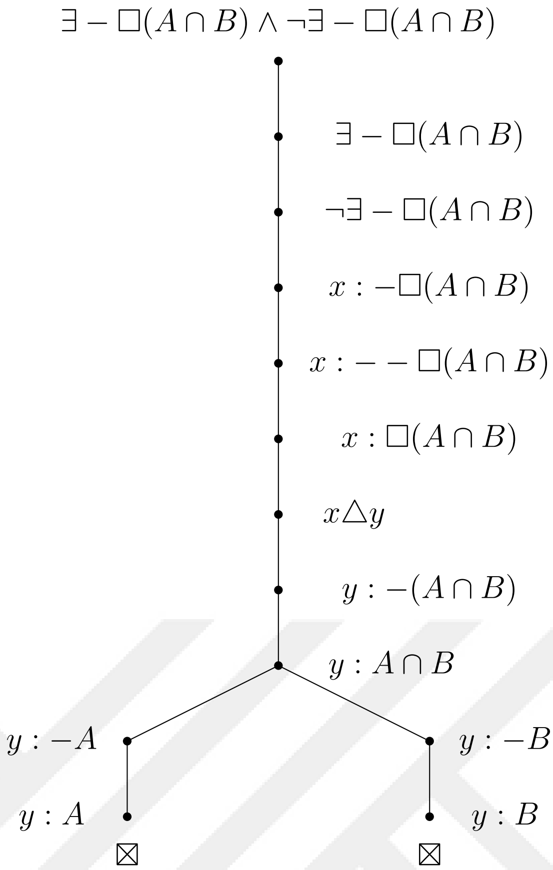

Figure 4.1 Closed tableau

Let us consider the formula another $\exists - (A \cup B) \wedge \neg\exists(A \cup B)$ and show how the rules apply. The tableau obtained for this formula by applying our rules has two open branches.

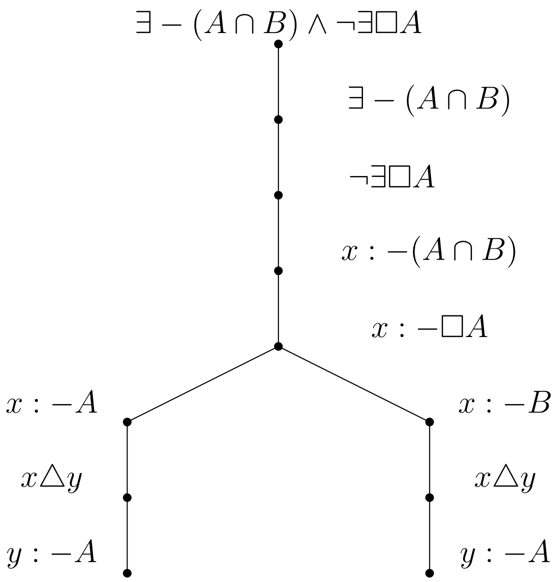

Figure 4.2 Open tableau

In order to prove that the tableaux of satisfiable formulas cannot be closed, we introduce the concept of interpretability of a branch in a model.

Definition 4.4.2. Let $M = (W, R, V)$ be a model. Let β be a branch in a tableau and W' be the set of all variables occurring in β. The branch β is said to be interpretable in M if there exists a function $f : W' \to W$ such that:

(i) if ϕ occurs in β, then $M \models \phi$,

(ii) if $x \triangle y$ occurs in β, then $f(x)Rf(y)$,

(iii) if $x : A$ occurs in β, then $f(x) \in \bar{V}(A)$.

Let t be a tableau and M be a model. The tableau t is said to be interpretable in M if and only if there exist a branch in t which is interpretable in M. Obviously, interpretable branches and, then, interpretable tableaux are open.

4.5 Soundness

In this section, we show the soundness of the tableau rules for modal logics with a restricted universal modality.

Proposition 4.5.1. *Let $M = (W, R, V)$ be a model and ϕ be a formula. If $M \models \phi$, then every semantic tableau computed from the initial tableau of ϕ is interpretable in M.*

Proof. Suppose ϕ is satisfiable. We want to show that all semantic tableaux computed from the initial tableau of *phi* are open. Since ϕ is satisfiable, there exists a model $M = (W, R, V)$ such that $M \models \phi$. Hence, it suffices to prove that the tableau rules preserve the interpretability property in M.
We only consider here the case of the $\exists A$ rule. Let β be a branch in tableau t and suppose β is interpretable in M. Assume that β contains a node n labeled with the formula $\exists A$ and apply the $\exists$ rule to n with respect to β. Then, one obtains from t a new tableau t' by extending β into a new branch β' obtained after adding a node n_1 to the end of β' labeled with $x' : A$ where x' is new symbol in the branch β'. Let W_1 be the set of symbols occurring in β and W_2 be the set of symbols occurring in β'. Obviously, $W_2 = W_1\{x'\}$. Since β is interpretable in M, there exists a function $f_1 : W_1 \to W$ satisfying the compatibility conditions for β with respect to M. In particular, $\exists A$ occurs in β. Consequently, there

exists $x \in W$ such that, $x \in V(A)$. Let $f_2 : W_2 \rightarrow W$ be the such that $f_{2|W_1} = f_1$ and $f_2(x') = x$. We want to show that the function $f_2 : W_2' \rightarrow W$ satisfies the compatibility conditions for β' with respect to M.

Suppose $w : B$ occurs in β'. We want to show that $f_2(w) \in V(B)$. There are two cases:

Case 1: $w : B$ occurs in β, then $f_1(w) \in V(B)$. We know that $f_{2|W_1} = f_1$, so $f_2(w) \in V(B)$.

Case 2: $w : B$ does not occur in β, then $w = x', B = A$. So $f_2(w) = f_2(x)$. Therefore $f_2(w) \in V(B)$.

Since, f_2 satisfies the compatibility conditions for β', then the $\exists$ rule preserve the interpretibility. The other tableaux rules are left to the reader. $\square$

The previous proposition shows that the tableau rules for modal logics with a restricted universal modality preserves property of interpretability in general models. As for the termination property, we need to define the following preliminary definitions.

Definition 4.5.2. Let β be a branch of some tableau and x be a symbol in β. Let $(A_1, ..., A_n)$ be a list of all modal terms A such that $x : -\Box A$ is in β. We will say that x is successor-free in β if and only if there exists $i \in \{1, .., n\}$ such that for all y in β, if $x \triangle y$ is in β then $y : -A_i$ is not β.

Definition 4.5.3. We will say that x is twin-free in β iff for all y in β, if $terms(x, \beta) = terms(y, \beta)$ then for all $u_1, ..., u_n$ in β, if $t\Delta u_1$ is in $\beta, ..., t\Delta u_n$ is in β then either $u_1 : -A_1$ is not in $\beta, ...,$ or $u_n : -A_n$ is not in β.

The strategy is the following:

(i) Apply all rules (except the $-\Box$ rule) as much as possible.

(ii) Apply the $-\Box$ rule to $x : -\Box A$ in β, when x is successor-free in β and twin-free in β.

(iii) Halt.

Now, let us show how the termination strategy will terminate. Note that for any symbol occurring in branches of a tableau constructed from ϕ's initial tableau, $term(x; \beta)$ contains only sub-terms or complements of sub-terms of ϕ. There exists finitely many sub-terms of ϕ. Consequently, at some point of the computation, in each branch of the constructed tree, each successor-free symbol is not twin-free. Thus, the strategy terminates.

4.6 Completeness

In this section, we prove the completeness of the tableau method, i.e. for all valid formulas ϕ, after finitely many steps one can obtain a closed tableau from the initial tableau of $\neg\phi$.

Definition 4.6.1. Let t be a tableau obtained after applying our strategy as much as possible. Let β be a branch in t. Suppose β is open. Let M be the model defined as follows:

(i) W is the set of all non successor-free x in β,

(ii) R is the binary relation on W defined by xRy iff either $x\triangle y$ occurs in β or there exists a symbol z in β such that z is successor free in β, $term(y,\beta) = term(z,\beta)$ and $x\triangle z$ occurs in β.

(iii) $V(p)$ is the set of all $x \in W$ such that β contains the information $x : p$.

The following lemma is crucial for proving the completeness of our method.

Lemma 4.6.2. *Let t be a tableau, β be a branch in t and $M = (W, R, V)$ be a model for β. We have the following:*

(i) *If β contains $x : A$ and $x \in W$, then $x \in V(A)$,*

(ii) *If β contains a formula ϕ, then $M \models \phi$.*

Proof. (i) The proof is done by induction on A. We consider the case $-\Box A$ and $\Box A$, the other cases being left to the reader. Let us start with $-\Box A$. Let $x \in W$. Thus, x is not successor free with respect to some Boolean term. Suppose $''x : -\Box A'' \in \beta$. We want to show that $x \in V(-\Box A)$. Since x is in W, therefore x is not successor-free. Since $x : -\Box A$ is in β, therefore, for some y in $\beta, x\Delta y$ is in β and $y : -A$ is in β. Now, we have to consider two cases. In the first case, y is in W. Since $y : -A$ is in β, therefore by induction hypothesis, we have that y is in $V(-A)$. Moreover, since $x\Delta y$ is in β, therefore xRy. Consequently, x is in $V(-\Box A)$. In the second case, y is not in W. Thus, y is successor-free. Hence, according to our strategy, y is not twin-free. Let $B_1, ..., B_n$ be a list of the terms B such that $y : -\Box B$ is in β. Since y is not twin-free, therefore for some $t, u_1, ...u_n in\beta, term(y, \beta) = term(t, \beta), t\Delta u_1$ is in $\beta, ..., t\Delta u_N$ is in $\beta, u_1 : -B_1$ is in $\beta, ..., u_n : -B_n$ is in β. It follows that t is not successor-free and, as a result, t is in W. Since $y : -A$ is in β and $term(y, \beta) = term(t, \beta)$, hence $t : -A$ is in β and, by induction hypothesis, t is in $V(-A)$. Since t is in W, $x\Delta y$ is in β, y is successor-free in β and $term(y, \beta) = term(t, \beta)$, therefore xRt. Since t is in $V(-A)$, therefore x is in $V(-\Box A)$.

Let us prove the case $\Box A$. Let $x \in W$. Thus, x is not successor free. Suppose $x : \Box A \in \beta$. We want to show that $x \in V(\Box A)$. Let $y \in W$ such that xRy. We have to show that $y : A$ is in β. We have to consider two cases. In the first case, $x\Delta y$ is in β. Since $x : \Box A$ is in β, therefore by the Box rule, $y : A$ is in β. In the second case, let z in β be such that z is successor-free in β, $term(y, \beta) = term(z, \beta)$ and $x\Delta z$ in β. Since $x : \Box a$ is in β, therefore by the Box rule we have that $z : A$ is in β. Since $term(y, \beta) = term(z, \beta)$, therefore $y : A$ is in β.

(ii) The proof is by done induction on ϕ. We consider the cases $\exists A$ and $\forall A$, the other cases being left to the reader. Suppose β contains a formula $\exists A$. The rule $\exists$ is applied, $\exists y \in \beta$ such that $x : A$ occurs in β. By item (i), we have $x \in V(A)$. Therefore $M \models \exists A$.

Let us now prove the case $\forall$, Suppose β contains a formula $\forall A$. When the rule $\forall$ is applied, for all $x \in \beta$, $x : A$ occurs in β. By item (i), we have $x \in V(A)$. Therefore $M \models \forall A$. □

Now, we are ready to prove the completeness of our method.

Theorem 4.6.3. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules and our strategy as much as possible. If ϕ is valid in the class of all models then t is closed.*

Proof. Suppose t is open. Thus, t contains an open branch β. Let $M = (W, R, V)$ be the model for β. By the truth lemma, we have $M \models \neg\phi$, contradicting the validity of ϕ. □

4.7 Variants

In this section, we extend our systems with adding new tableau rules which are Sym (for symmetric models), Ref (for reflexive models) and Den (for dense models). We give sound and complete tableaux based decision procedure for some variants of contact logics.

4.7.1 Symmetric models

A model $M = (W, R, V)$ is symmetric if for all $x, y \in W$, xRy if and only if yRx. Let us consider the class of all symmetric models of generalized contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Sym)} \qquad \frac{x \triangle y}{y \triangle x} \; (x \text{ and } y \text{ already occur in the branch})$$

Obviously, (Sym) preserves the property of interpretability in symmetric models. Thus, if ϕ is satisfiable in a symmetric model, then all semantic tableaux computed from the initial tableau of ϕ are open. Seeing that there is no need to apply (Sym) twice to the same pair (x, y), termination of our extended tableau system easily follows. Thus, after a finite number of steps from an initial tableau, no rules can be applied. Finally, within the context of our extended system, the model defined in the previous section is clearly symmetric. The proof of the Truth Lemma being repeated as such, we therefore obtain the following theorem:

Theorem 4.7.1. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules augmented with (Sym). If ϕ is valid in the class of all symmetric models then t is closed.*

Proof. Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Sym). If ϕ is valid in the class of all symmetric models then t is closed. □

Again, a careful analysis of the tableau rules augmented with (Sym) immediately leads us to the conclusion that the depth of a tableau computed from a given formula ϕ is linear in the number of symbols in ϕ. Consequently, the problem of determining if a given formula is satisfiable in the class of all symmetric models is NP-complete.

4.7.2 Reflexive models

A model $M = (W, R, V)$ is reflexive if for all $x \in W$, xRx. Let us consider the class of all reflexive models of generalized contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Ref)} \qquad \frac{\cdot}{x \triangle x} \qquad (x \text{ already occurs in the branch})$$

The same line of reasoning as the one considered in the case of the class of all models would easily allow us to obtain the soundness, termination, completeness of the extended tableau system.

4.7.3 Dense models

A model $\mathcal{M} = (W, R, V)$ is said to be dense if for all $x, y \in W$, if xRy then there exists $z \in W$ such that xRz and zRy. Let us consider the class of all dense models of generalized contact logics. In order to decide satisfiability with respect to this class, we should add the following rule to our system:

$$\text{(Den)} \qquad \frac{x \triangle y}{\begin{array}{c} x \triangle z \\ z \triangle y \end{array}} \qquad \begin{array}{l} (x \text{ and } y \text{ already occur in the branch} \\ \text{and } z \text{ is new in the branch)} \end{array}$$

Obviously, (Den) is sound with respect to dense models, i.e. it preserves the interpret ability property of tableaux. Nevertheless, as in the case of (Ser), it may lead to infinite computations. Thus, we have to define a strategy that will guarantee the soundness, the completeness and the termination of our tableaux-based system.

Let β be a branch and the pair (x, y) be a symbol occurring in β. We will say the pair (x, y) is intermediate-free if $x\Delta y$ occurs in β and for all symbols z in β, either $x\Delta z$ does not occur in β, or $z\Delta y$ does not occur in β. Let β be a branch, x be a symbol occurring in β. let $term(x, \beta) = \{a : \; x : a \text{ occurs in } \beta\}$. We will say the pair (x, y) of symbols occurring in β is said to be twin-free if $x\Delta y$ occurs in β and for all symbols z_1, z_2, z_3 occurring in β, if $term(x, \beta) = term(z_1, \beta)$ and $term(y, \beta) = term(z_3, \beta)$ then either $z_1\Delta z_2$ does not occur in β, or $z_1\Delta z_3$ does not occur in β, or $z_2\Delta z_3$ does not occur in β.

Our strategy is the following:

(i) Apply the formula rules and the term rules as much as possible,

(ii) Choose an intermediate-free twin-free pair (x, y) of symbols occurring in a branch β; Apply the rule (Den) to (x, y) and go to (i) otherwise go to (iii),

(iii) Halt.

In order to show that our strategy terminate, it suffices to follow an argument similar to the one developed in the previous section. Let us be more precise. Firstly, remark that in any branch β of a tableau constructed from the initial formula ϕ and for any x occurring in β, $term(x, \beta)$ only contains sub-terms or negation of sub-terms from ϕ. Seeing that there exists finitely many sub-terms from ϕ, at some point of the computation, each intermediate-free pair (x, y) of symbols occurring in a branch β is not twin-free. Therefore, our strategy terminates. Secondly, let us prove the soundness and the completeness of our tableau system extended with (Den) and following the above strategy. Obviously, every tableau constructed, by following the above strategy, from the initial tableau of a formula ϕ satisfiable in a dense model will be open. Conversely, suppose β is an open branch obtained, by means of our strategy, at the end of the tableau computation from an initial formula ϕ. Let W be the set of all x, y, etc occurring in β. As expected, we define on W the valuation V such that for all Boolean variables p, $V(p) = \{x \in W\colon\ x : p$ occurs in $\beta\}$. Now, for the accessibility relation R on W, it is defined as follows: for all $x, y \in W$, xRy iff $x\Delta y$ occurs in β. Defining $\mathcal{M} = (W, R, V)$, as in the cases of the previous classes of models that we have considered we obtain the following Truth Lemma which proof is similar to the proof of Lemma 4.6.2.

Lemma 4.7.2. *Let t be a tableau, β be a branch in t and M be a model for β. We have the following:*

(i) *If β contains $x : A$ and $x \in W$, then $x \in \bar{V}(A)$.*

(ii) *If β contains a formula ϕ, then $\mathcal{M} \models \phi$.*

Theorem 4.7.3. *Let ϕ be a formula and t a tableau obtained from the initial tableau of $\neg\phi$ by applying the tableau rules augmented with (Den) and following the above strategy. If ϕ is valid in the class of all dense models then t is closed.*

Proof. Let ϕ be a formula and t be a tableau obtained from the initial tableau of $\neg\phi$ by means of the tableau rules augmented with (Den). If ϕ is valid in the class of all dense models then t is closed. □

Thus, the problem of the satisfiability of formulas with respect to the class of all dense models is decidable. Unfortunately, we do not know its exact computational complexity. Note that this decidability result is new; it does not seem that it can be easily obtained by means of an argument based on the filtration method, seeing that the filtration construction does not preserve the elementary property of density.

Chapter 5

TABLEAUX APPROACHES FOR CONTACT LOGICS INTERPRETED OVER INTERVALS

In this chapter, we focus our attention on tableau methods for contact logics interpreted over intervals on the reals. We give sound and complete tableaux-based decision procedures for contact logics.

5.1 Syntax and Semantics

5.1.1 Syntax

Let BV be a countably infinite set of Boolean variables (with members denoted by p, q, etc). The set of all Boolean terms based on BV (with members denoted by a, b, etc) is defined as follows:

- $a := p \mid 0 \mid -a \mid (a \cup b)$.

We obtain the Boolean constructs 1 and $a \cap b$ as an abbreviation 1 for -0 and

$a \cap b$ for $-(-a \cup -b)$. We will sometimes write p^0 for $-p$ and p^1 for p.

The set of all formulas based on BV (with members denoted by ϕ, ψ, etc) is defined as follows:

- $\phi := a \equiv b \mid \bot \mid \neg\phi \mid (\phi \vee \psi)$.

The other Boolean constructs for formulas ($\top$, $\wedge$, $\rightarrow$) are defined as usual.

5.1.2 Semantics

An interpretation is a function associating a finite union $f(p)$ of regular closed intervals of $\mathbb{R}$ to each propositional variable p. A regular closed intervals on $\mathbb{R}$ is a closed interval of the form $[x; y] = \{z \in \mathbb{R} : x \leq z \leq y\}$ where $x < y$. An extension of f to atomic terms, $\bar{f}$ is defined as follows:

- $\bar{f}(p) = f(p)$,
- $\bar{f}(0) = \emptyset$,
- $\bar{f}(-a) = Cl(\mathbb{R} \setminus \bar{f}(a))$,
- $\bar{f}(a \cup b) = \bar{f}(a) \cup \bar{f}(b)$.

Let f be an interpretation. Satisfiability of formulas on f is defined as follows:

- $f \models a \equiv b$ iff $\bar{f}(a) = \bar{f}(b)$,
- $f \not\models \bot$
- $f \models \neg\phi$ iff $f \not\models \phi$,
- $f \models \phi \vee \psi$ iff $f \models \phi$ or $f \models \psi$.

We shall say that ϕ is valid iff for all interpretations f, $f \models \phi$.

Disjunction Rule

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

Conjunction Rule

$$\frac{\neg(\phi \vee \psi)}{\begin{array}{c}\neg\phi \\ \neg\psi\end{array}}$$

Negation Rule

$$\frac{\neg\neg\phi}{\phi}$$

$\equiv$ Rule

$$\frac{a \equiv b}{\begin{array}{c|c} x : a & x : -a \\ x : b & x : -b \end{array}} \quad (x = x_0 \text{ or } x \text{ is old in the branch})$$

$\not\equiv$ Rule

$$\frac{a \not\equiv b}{\begin{array}{c|c} x : a & x : -a \\ x : -b & x : b \end{array}} \quad (x \text{ is new in the branch})$$

5.2 Tableau rules

In this section we present the tableau rules for contact logics interpreted over intervals of $\mathbb{R}$. We will base the tableau rules on the connective of equality $\equiv$.

Definition 5.2.1. Tableau nodes will be labeled by the following types of expressions:

(i) formulas,

(ii) $x \in a$,

(iii) $x \notin a$

where x is a variable, and a is a Boolean term.

Given a formula ϕ, its initial tableau is the labeled tree consisting of exactly one node (called root) labeled with ϕ.

T Negation Rule

$$\frac{x \in -a}{x \notin a}$$

F Negation Rule

$$\frac{x \notin -a}{x \in a}$$

T Union Rule

$$\frac{x \in a \cup b}{x \in a \;\big|\; x \in b}$$

F Union Rule

$$\frac{x \notin a \cup b}{\begin{array}{c} x \notin a \\ x \notin b \end{array}}$$

The tableau rules are given in two parts: formula-rules (Page 48) and Boolean-rules (this page). Rules are applied in a standard way by extending branches of constructed trees.

A branch is said to be closed if and only if one of the following conditions holds:

(i) it contains a node labeled with $\perp$;

(ii) it contains a node labeled with $x \in \emptyset$;

(iii) it contains two nodes respectively labeled with $x \in a$, $x \notin a$.

where x is a variable and a is a Boolean term. Given a formula ϕ, its initial tableau is the labeled tree consisting of exactly one node (called root) labeled with ϕ. A tableau is closed when all its branches are closed. Let us consider the formula $(a \cup b) \not\equiv a \wedge (a \cap b) \equiv 1$ and show how the rules apply. The tableau obtained for this formula by applying our rules has three closed branches.

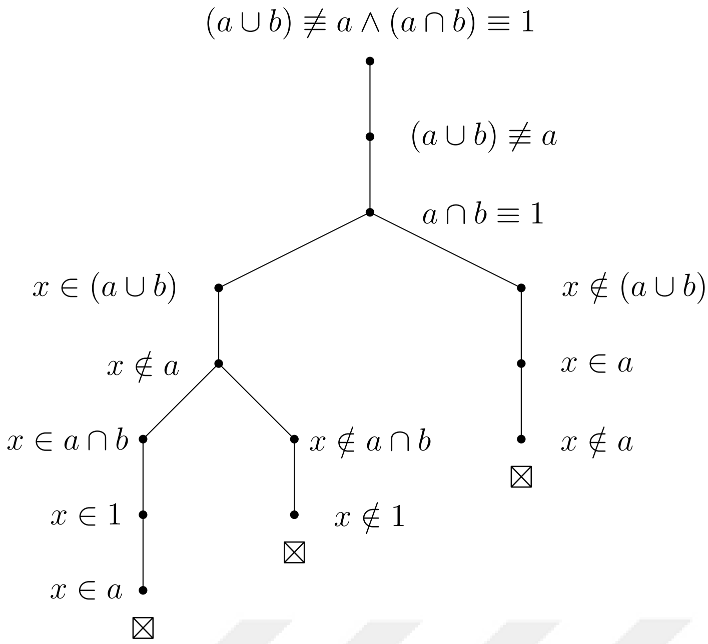

Figure 5.1 Closed tableau

5.3 Soundness

In this section, we show the soundness of the tableau rules for contact logics. In order to prove that the tableaux of satisfiable formulas cannot be closed, we introduce the concept of interpretability of a branch in an interpretation.

Let β be a branch. Let $x_0, x_1, ..., x_n$ be a list of the variables occurring in β. Note that we will always consider that x_0 occurs in β, even if x_0 does not explicitly occurs in β.

Definition 5.3.1. Let f be an interpretation. We say β is interpretable in f, if there exists $\bar{x}_0, \bar{x}_1, ..., \bar{x}_n$ in $\mathbb{R}$ such that the following conditions holds:

- for all labels "ϕ" occuring in β, $f \models \phi$,
- for all labels "$x_i \in a$" occuring in β, $\bar{x}_i \in \bar{f}(a)$,
- for all labels "$x_i \notin a$" occuring in β, $\bar{x}_i \notin \bar{f}(a)$.

We say that the variable x_i is regularly interpretable, if for all propositional variables p, $\bar{x}_i \in Int(f(p))$ or $\bar{x}_i \notin f(p)$.

Lemma 5.3.2. *Let f be an interpretation and x_i be a regularly interpreted variable, then for all terms a, $\bar{x}_i \in Int\bar{f}(a)$ or $\bar{x}_i \notin \bar{f}(a)$.*

Proof. By induction on term a, we only consider the case of $a = -b$. Suppose $\bar{x}_i \in Int\bar{f}(b)$ or $\bar{x}_i \notin \bar{f}(b)$, we demonstrate that $\bar{x}_i \in Int\bar{f}(-b)$ or $\bar{x}_i \notin \bar{f}(-b)$. By contradiction, suppose $\bar{x}_i \notin Int\bar{f}(-b)$ and $\bar{x}_i \in \bar{f}(-b)$. Since $\bar{x}_i \notin Int\bar{f}(-b)$, therefore $\bar{x}_i \in \mathbb{R} \setminus Int\bar{f}(-b)$. Hence, $\bar{x}_i \in Cl(\mathbb{R} \setminus \bar{f}(-b))$. Thus, $\bar{x}_i \in Cl(\mathbb{R} \setminus Cl(\mathbb{R} \setminus \bar{f}(b)))$. Consequently, $\bar{x}_i \in Cl(Int\bar{f}(b))$. Since $\bar{f}(b)$ is a finite union of regular closed closed intervals, therefore $Cl(Int\bar{f}(b)) = \bar{f}(b)$. Since $\bar{x}_i \in Cl(Int\bar{f}(b))$, therefore $\bar{x}_i \in \bar{f}(b)$. Since $\bar{x}_i \in Int\bar{f}(b)$ or $\bar{x}_i \notin \bar{f}(b)$, therefore $\bar{x}_i \in Int\bar{f}(b)$. Since $\bar{x}_i \in \bar{f}(-b)$, therefore $\bar{x}_i \in Cl(\mathbb{R} \setminus \bar{f}(b))$. Hence, $\bar{x}_i \notin Int\bar{f}(b)$: a contradiction.

□

The tuple $(\bar{x}_0, \bar{x}_1, ..., \bar{x}_n)$ is called interpretation of β in f. It is regular for the variable x_i, if x_i is regularly interpreted in it.

We say a tableau is interpretable in f, if one of its branches is interpretable in f.

Lemma 5.3.3. *Let ϕ be a formula. If ϕ is satisfiable then the initial tableau for ϕ is interpretable by an interpretation $(\bar{x}_0)$ in some interpretation f. Moreover, x_0 is regularly interpreted in it.*

Proof. Let f be an interpretation such that $f \models \phi$. Let $p_1, ..., p_k$ be the propositional variables occuring in ϕ. Let $\epsilon_1, .., \epsilon_k \in \{0, 1\}$ be such that $\bar{f}(p_1^{\epsilon_1} \cap ... \cap p_k^{\epsilon_k})$ is a nonempty finite union of regular closed intervals. Let $\bar{x}_0$ be an element in the interior of this finite union. Obviously, $(\bar{x}_0)$ is a regular interpretation. □

Lemma 5.3.4. *A closed branch cannot be interpreted.*

Proof. Let β be a branch and f be an interpretation. Suppose β is interpretable in f and $x_0, x_1, ..., x_n$ occurs in β. There exists $\bar{x}_0, \bar{x}_1, ..., \bar{x}_n \in \mathbb{R}$ which satisfy compatibility conditions. Since β is closed, β contains β contains $\perp$, "$x \in 0$" or "$x \in a$" and "$x \notin a$".

Case 1, β contains $\perp$. Therefore $f \models \perp$, it is a contradiction.

Case 2, β contains "$x \in 0$", since β interpretable in f, $\bar{x}_i \in \bar{f}(0)$ which means $\bar{x}_i \in \emptyset$. It is a contradiction.

Case 3, β contains "$x_i \in a$" and "$x_i \notin a$". Since β interpretable in M, $\bar{x}_i \in \bar{f}(a)$ and $\bar{x}_i \notin \bar{f}(a)$. It is a contradiction. □

Lemma 5.3.5. *If a tableau t is regularly interpretable then all tableaux obtained from t by applying one of the tableau rules in regularly interpretable.*

Proof. Let $x_0, ..., x_n$ be a list of the variables occurring in a branch β of t that is regularly interpretable in an interpretation of f. Let $\bar{x}_0, ..., \bar{x}_n$ in $\mathbb{R}$ be the associated elements. Suppose t' is extension of t after applying one of the tableau rules. We want to show that t' is regularly interpretable in f.

Let us consider $\not\equiv$ rule. Suppose $a \not\equiv b$ occurs in t. We want to show that t' is regularly interpretable. This rule extend the current branch β in two new branches β' and β'' by adding (for β'), $y' \in a$" and "$y' \notin b$" and adding (for β") "$y' \notin a$" and "$y' \in b$", y' being a new variable. We have to show that either the branch β' can be regularly interpreted or the branch β'' can be regularly interpreted. Since the branch β is regularly interpreted, it means that $f \models a \not\equiv b$. This implies that $\bar{f}(a) \neq \bar{f}(b)$. Since $\bar{f}(a)$ and $\bar{f}(b)$ are finite unions of closed intervals, this implies that there is a real number r which either belongs to $Int\bar{f}(a)$ and does not belong to $\bar{f}(b)$, or belongs to $Int\bar{f}(b)$ but does not belong to $\bar{f}(a)$. The former case corresponds to the branch β' whereas the latter case corresponds to the branch β'. In the branch β', we should extend the interpretation f by the interpretation f' which is like f on W and which is such that $f'(y') = r$. In the branch β'', it is the same: we should extend the interpretation f by the interpretation f'' which is like f on W and which is such that $f''(y') = r$. Then, clearly, in the former case, f' is a regular interpretation of the branch β' whereas in the latter case, f'' is a regular interpretation of the branch β''.

□

Theorem 5.3.6. *Let ϕ be a formula. If ϕ is satisfiable then all tableaux computed from the initial tableau for ϕ are regularly interpretable and, therefore, open.*

Proof. By Lemmas 5.3.3 and 5.3.5, since ϕ is satisfiable, then ϕ is regularly interpretable in some interpretation f. All tableaux from the initial tableau for ϕ are regularly interpretable and also open. □

Theorem 5.3.7. *Let ϕ be a formula. After finitely many steps, from the initial tableau of ϕ, no tableau rules can be applied.*

Proof. Let t be a tableau computed from the initial tableau for ϕ. We have to show that after a finite number of steps, no rule can be applied. Suppose the contrary. Thus there exists an infinite sequence $t_0, t_1, \ldots$ of tableaux such that $t_0 = t$ and t_{n+1}, is obtained from t_n by applying a rule. Note that each node occurring in these tableaux has at most two successors. As a result, from the sequence $t_0, t_1, \ldots$, we can extract an infinite branch β. The branch β contains information of the form: ψ, $x \in a$ and $x \notin a$ where ψ is a subformula or the negation of a subformula of ϕ and a is a subterm or the complement of a subterm of ϕ. Note that the variables $x_0, x_1 \ldots, x_n$ occurring in β have been introduced by the $\not\equiv$ rule. Note also that the application of this rule are triggered by the occurrence of atomic formula of the form $a \not\equiv b$. These atomic formulas being subformulas or negations of subformulas of ϕ, $\not\equiv$ rule will be applied finitely many time only. This means that, all in all, the infinite sequence $t_0, t_1, \ldots$ does not exist. □

5.4 Completeness

In this section, we prove the completeness of the tableau method, i.e. for all valid formulas ϕ, after finitely many steps one can obtain a closed tableau from the initial tableau of $\neg\phi$. In this respect, the concept of saturation is essential. A branch β in some tableau is called saturated, if the following conditions holds for all nodes $n \in \beta$:

- if n is labeled with $\neg\neg\phi$, then β contains a node labeled with ϕ,
- if n is labeled with $\phi \vee \psi$, then β contains a node labeled with ϕ or ψ,
- if n is labeled with $\neg(\phi \vee \psi)$, then β contains nodes labeled with $\neg\phi$ and $\neg\psi$,
- if n is labeled with $a \equiv b$ and β contains the variables x, then β contains nodes labeled with $x \in a$ and $x \in b$, or β contains nodes labeled with $x \in -a$ and $x \in -b$,
- if n is labeled with $a \not\equiv b$, then for some variable x either β contains nodes labeled with $x \in a$ and $x \in -b$, or β contains nodes labeled with $x \in -a$ and $x \in b$,
- if n is labeled with $x \in -a$, then β contains a node labeled with $x \notin a$,
- if n is labeled with $x \notin -a$, then β contains a node labeled with $x \in a$,
- if n is labeled with $x \in a \cup b$, then β contains a node labeled with $x \in a$ or $x \in b$,
- if n is labeled with $x \notin a \cup b$, then β contains nodes labeled with $x \in -a$ and $x \in -b$.

A tableau is said to be saturated if all its branches are saturated.

Let β be a saturated open branch and $x_0, x_1, ..., x_n$ be a list of the variables occurring in β. Let $I_0, I_1, ..., I_n$ be the following finite unions of regular intervals (where $0 \leq \epsilon \leq \frac{1}{2}$):

$\cdot\ I_0 =]-\infty, 1-\epsilon] \cup [1+\epsilon, 2-\epsilon] \cup ... \cup [n-1+\epsilon, n-\epsilon] \cup [n+\epsilon, +\infty[$,

$\cdot\ I_1 = [1-\epsilon, 1+\epsilon]$,

...

$\cdot\ I_n = [n-\epsilon, n+\epsilon]$.

Let β be a saturated open branch. Let $x_0, x_1, ..., x_n$ be a list of the variables occurring in β. Let $f : p \to f(p) \subseteq \mathbb{R}$ be the function defined as follows:

$$f(p) = \cup\{I_k : 0 \leq k \leq n \text{ and "} x_k \text{ in } p\text{" occurs in } \beta\}.$$

Remark 5.4.1. For all propositional variables p, $f(p)$ is a finite union of regular closed intervals.

Remark 5.4.2. Let $\bar{x}_k = k$ for each $k \in \mathbb{N}$ such that $k \leq n$. For all $k \in \mathbb{N}$, $k \leq n$, and for all propositional variables p, $\bar{x}_k \in Int(f(p))$ or $\bar{x}_k \notin f(p)$.

The following lemma is important for proving the completeness of our method.

Lemma 5.4.3. *Let t be an open saturated tableau, β be an open branch in t and f be an interpretation for β defined as above. Let p be a Boolean variable and $x_1, x_2, ..., x_n$ be variables which occur in β and for all $k, n \in \mathbb{N}$, $0 \leq k \leq n$. The following conditions are hold:*

(i) If $x_k \in a$ occurs in β, then $I_k \subseteq \bar{f}(a)$.

(ii) If $x_k \notin a$ occurs in β, then $Int(l_k) \cap \bar{f}(a) = \emptyset$.

Proof. The proof is done by induction on the Boolean term a. The case "p" follows by definition of f(p). Proof by induction on terms.

Let us consider $x_k \in -a$ occurs in β. We want to show that $I_k \subseteq \bar{f}(-a)$. Since $x_k \in -a$ occurs in β,then $x_k \notin a$ occurs in β. By induction hypothesis $Int(I_k) \cap \bar{f}(a) = \emptyset$. Therefore, $Int(I_k) \subseteq (\mathbb{R} \setminus \bar{f}(a)) \subseteq Cl(\mathbb{R} \setminus \bar{f}(a))$. By the definition of $\bar{f}$, $Int(I_k) \subseteq \bar{f}(-a)$. Consequently, $I_k \subseteq \bar{f}(-a)$.

Let us consider $x_k \in a \cup b$. We want to show that $I_k \subseteq \bar{f}(a \cup b)$. Since β is saturated, $x_k \in a$ occurs in β or $x_k \in b$ occurs in β. By induction hypothesis

$I_k \subseteq \bar{f}(a)$ or $I_k \subseteq \bar{f}(b)$. Therefore, $I_k \subseteq \bar{f}(a) \cup \bar{f}(b)$. By the definition of $\bar{f}$, $I_i \subseteq \bar{f}(a \cup b)$.

Let us consider "$x_k \notin -a$" occurs in β. We want to show that $Int(I_k) \cap \bar{f}(-a) = \emptyset$. Obviously, "$x_k \in a$" occurs in β. By induction hypothesis, $I_k \subseteq \bar{f}(a)$. Therefore $Int(I_k) \cap Cl(\mathbb{R} \setminus \bar{f}(a)) = \emptyset$. So, $Int(I_k) \cap \bar{f}(-a) = \emptyset$.

Let us consider $x_k \notin a \cup b$. We want to show that $Int(I_k) \cap \bar{f}(a \cup b) = \emptyset$. Since β is saturated, $x_k \notin a$ occurs in β and $x_k \notin b$ occurs in β. By the induction hypothesis, $Int(I_k) \cap \bar{f}(a) = \emptyset$ and $Int(I_k) \cap \bar{f}(b) = \emptyset$. By the definition of $\bar{f}$, $Int(I_k) \cap \bar{f}(a \cup b) = \emptyset$.

This completes the induction. □

Theorem 5.4.4. *If $\models \phi$, then there is a closed tableau computed from $\neg\phi$.*

Proof. By contraposition. Suppose t is a saturated and open tableau computed from $\neg\phi$ and β is an open branch in it. Since t is saturated, therefore β is saturated too. Let f be the interpretation for ϕ determined by β. By the truth lemma, we have $f \not\models \phi$, contradicting the validity of ϕ.

□

CONCLUSION

We have given sound and complete tableaux-based decision procedures for region based theories of space. Developing such tableaux procedures, we have obtained decidability/complexity results concerning the satisfiability problem with respect to several classes of models (class of all models, class of all symmetric models, class of all reflexive models, class of all serial models, class of all dense models). In Chapter 3 and Chapter 4, the decidability/complexity results concerning the class of all serial models and the class of all dense models are new. They have been obtained in this thesis by means of the tableau approach. Much remains to be done. For example, the exact computational complexity of the problem of the satisfiability of formulas with respect to the class of all dense models is unknown. As for Chapter 5, we can extend the language with predicates of the form $a < b$, $convex(a)$, $meets(a, b)$, etc. For instance, $a < b$ is true in a model if all real numbers in a's interpretation precede all real numbers in b's interpretation, $convex(a)$ is true in a model if a's interpretation consists of a regular closed interval, $meets(a, b)$ is true in a model if the intersection of a's interpretation with b's interpretation is a singleton.

We believe that specific tableau approaches for such or such fragment of a modal languages deserves to be developed.

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