# KÜME ÜZERİNDEKİ SIRALAMALARIN ALTKÜMELERİNE GENİŞLETİLMESİ

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İSTANBUL BİLGİ ÜNİVERSİTESİ
SOSYAL BİLİMLER ENSTİTÜSÜ
EKONOMİ YÜKSEK LİSANS PROGRAMI

PROF. DR. M. REMZİ SANVER
2007

# Abstract

This thesis investigates the problem of extending a (complete) order over a set to its power set. We interpret the set under consideration as a set of alternatives and we conceive orders as individual preferences. The elements of the power sets are the non-resolute outcomes. To determine how an individual with a given preference over alternatives is required to rank certain sets, we need a concept of *extension axioms*.

In the first part, the final outcome is determined by an "(external) chooser" which is a resolute choice function. The individual whose preference is under consideration confronts a set of resolute choice functions which reflects the possible behaviors of the chooser. Every such set naturally induces an extension axiom (i.e., a rule that determines how an individual with a given preference over alternatives is required to rank certain sets). Our model allows to revisit various extension axioms of the literature. Interestingly, the Gärdenfors (1976) and Kelly (1977) principles are singled-out as the only two extension axioms compatible with the non-resolute outcome interpretation.

In the second part, the extension axioms we consider generate orderings over sets according to their expected utilities induced by some assignment of utilities over alternatives and probability distributions over sets. The model we propose gives a general and unified exposition of expected utility consistent extensions while it allows to emphasize various subtleties, the effects of which seem to be underestimated - particularly in the literature on strategy-proof social choice correspondences.

# Özet

Bu tezde, kümeler üzerindeki sıralamalardan bu kümelerin altkümeleri üzerindeki sıralamaları oluşturma problemini ele alıyoruz. İncelediğimiz kümeleri seçenekler kümesi olarak, sıralamaları bireysel tercihler olarak, altkümeleri de sosyal seçim kurallarının kesin olmayan sonuçları olarak değerlendiriyoruz. Bireysel tercihlerden sosyal seçim kurallarının çözülmemiş sonuçları arasında ilişki kurabilmek için *genişletme aksiyomları*'ndan yararlanıyoruz.

İlk bölümde, seçmenlerin seçenekler üzerindeki tercihlerinin sosyal seçim kurallarının kesin olmayan sonuçlarını incelemekte nasıl kullanılabileceğini araştırırken kesin sonucun yetkili bir seçici tarafından belirleneceği genel bir model kuruyoruz. Bu model çerçevesinde yetkili seçicinin olası tercihlerinin belirsizliği altında ortaya çıkacak stratejik seçmenlerin toplumsal sonuçları ne şekilde etkileyeceğini inceliyoruz. Araştırmamız sonucunda, bu alandaki zengin literatür içerisinden Gärdenfors (1976) ve Kelly (1977)'deki genişletme aksiyomlarının yetkili seçicilerin ne şekilde tercihte bulunacaklarının öngörülmesinde kullanılabileceği ortaya çıkmaktadır.

İkinci bölümde, kullandığımız genişletme aksiyomlarından, seçeneklere atanan belirli değerler ve kümeler üzerindeki olasılık dağılımlarıyla belirlenen "*beklenen fayda*"larına göre sıralamalar oluşturuyoruz. Burada önerdiğimiz model, bu alandaki literatüre hem daha genel ve toparlayıcı bir bakış açısı sağlamakta, hem de nispeten muğlak olan kısımlara netlik kazandırmaktadır.

# Acknowledgements

Among so many people I owe to thank , I will of course start by Remzi Sanver. Everything which is written under this cover is thanks to him. He enlarged my my perspective not only in Economics but also in many aspects of life. He was with me whenever I needed him. I thank him wholeheartedly.

I owe special thanks to Nick Baigent. He kindly agreed to be in my thesis commitee. It has been a big plessure and honour for me to meet him. I benefited a lot from his friendly comments and critiques.

I would like to thank Göksel Aşan for his interest in my study. He corrected several points and gave me useful comments.

I gratefully acknowledge the support of TÜBİTAK (Scientific and Technical Research Council of Turkey) for the first part of this study which is also the outcome of a project (#106K380).

I would also thank my family who always supported my studies.

I would like to dedicate this study to my fiancee, Filiz Tumel, who has been with me at all stages of this study. I thank her also for her sympathy and warmly support during this study.

Of course, all remaining errors are mine.

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# 1 Introduction

In this thesis, we consider the problem of extending a (complete) order over a set to its power set. We interpret the set under consideration as a set of alternatives and we conceive orders as individual preferences.

It is quite typical that collective decision problems are resolved through the initial choice of a non-resolute set of outcomes which is followed by the final decision of an "external chooser". This two-stage structure is sometimes an explicit part of the social choice rule -hence the external chooser truly exists.[1] But even without an explicit reference to the "external chooser", a two-stage structure is implicit in the nature of the social choice problem. For, the impossibility of making a resolute choice under desirable axioms is well-known. In fact, as one can see in Moulin (1983), every anonymous and neutral social choice rule must exhibit non-resoluteness, thus leaving the final choice to an "external chooser" - who does not necessarily exist in flesh and bone.

This two-stage nature of collective decision problems raises the question of extending a preference over a set to its power set. This question is typically answered through an *extension axiom* which is a rule that determines how an individual with a given preference over alternatives is required to rank certain sets. Moreovover, given an extension axiom, we need a condition of the *compatibility* of a preference over sets with a preference over alternatives which is the obedience of the extended order to the requirements of the extension axiom.[2] As Barberà, Bossert and Pattanaik (2004) beautifully survey, there is a vast literature on extending an order over a set to its power set. To be sure, this literature contains various interpretations of a set, such as being a list of mutually incompatible outcomes[3] or a list of

[1]Such social choice rules are analyzed by Barberà and Coelho (2004) who call them "rules of k names".

[2]To be more formal, given an extension axiom $\pi$, a complete order $R$ over sets is compatible with an order $\rho$ over alternatives if and only if $R$ is a completion of the partial order $\pi(\rho)$ that $\pi$ assigns to $\rho$.

[3]e.g., Gärdenfors (1976), Barberà (1977), Kelly (1977), Feldman (1979), Duggan and Schwarz (2000), Barberà, Dutta and Sen (2001), Benoit (2002), Ching and Zhou (2002), Ozyurt and Sanver (2006).

mutually compatible outcomes[4] or a menu from which the individual whose preference under consideration makes a choice[5] or a collection states[6]. All these interpretations have their own axioms. Throughout this thesis our consideration is limited to an interpretation where a set is conceived as an initial non-resolute refinement of outcomes from which a final choice will be made. We propose a model that underlies this conception of a set. As social choice correspondences are typically social choice rules which give non-resolute outcomes, the problem we consider is connected to the analysis of strategy-proof social choice correspondences.[7]

First part of this thesis is mainly composed of the results of Erdamar and Sanver(2007). In this paper, we admit a resolute choice function[8] to be an "(external) chooser" who makes the final decision from any non-resolute outcome. Hence a (non-empty) set $\mathcal{D}$ of resolute choice functions is the list of admissible behaviors that choosers may exhibit. In principle, $\mathcal{D}$ can be anything, ranging from a singleton set to the set of all choice functions. In particular, $\mathcal{D}$ may be determined by well-established axioms of choice theory, such as the weak axiom of reveal preference. After all, any given $\mathcal{D}$ induces an extension axiom in the following natural way: For each possible ordering $\rho$ of alternatives, a set $X$ is required to be ranked above a set $Y$ if and only if the final decision made from $X$ is preferred (according to $\rho$) to the final decision made from $Y$, for any chooser belonging to $\mathcal{D}$.

Our model allows to revisit the existing extension axioms of the literature. Among these, two prevalent ones, namely the Gärdenfors (1976) and Kelly (1977) principles are singled out. For, every "regular" axiom of choice theory determines a domain of admissible chosers which induces either the Gärdenfors (1976) or the Kelly (1977) principle.

[4]e.g., Barberà, Sonnenschein and Zhou (1991), Ozyurt and Sanver (2007).

[5]e.g., Kreps (1979), Dutta and Sen (1996), Dekel et al. (2001), Gul and Pesendorfer (2001).

[6]e.g., Lainé et al. (1986), Weymark (1997).

[7]The literature on strategy-proof social choice correspondences contains Fishburn (1972), Pattanaik (1973), Gärdenfors (1976), Barberà (1977), Kelly (1977), Feldman (1980), Duggan and Schwarz (2000), Barberà, Dutta and Sen (2001), Benoit (2002), Ching and Zhou (2002), Ozyurt and Sanver (2006). This list is certainly non-exhaustive. One can see Taylor (2005) for an excellent account of the literature.

[8]A resolute choice function assigns to each non-empty set $X$ a single element of $X$.

In the second part of this thesis, one can find the results of Can, Erdamar and Sanver(2007). In this paper our focus is on extension axioms that order sets according to their expected utilities induced by some assignment of utilities over alternatives and probability distributions over sets. This approach leads to what is generally called the *expected utility consistent extension of a preference*. Nevertheless, the idea needs to be made more precise by determining which utility functions and probability distributions are admissible. Moreover, the order generated by expected utilities is complete or partial also matters. In fact, completing a generated partial order and directly generating a complete order may lead to different admissible orderings. The literature seems to be missing a unified exposition of these subtleties - a treatment of which is one of our aims.

We set our framework for the first part in Section 3.1 and Section 3.2 and then state our results in Section 3.3 . We also consider, in Section 3.4, a probabilistic variant of our model where we allow randomizations over $\mathcal{D}$. However, our findings remain essentially unaltered by this variation.

In Section 4.1 we introduce the basic notions for Part II. In this section, we adjust the definitions of an "*extension axiom*" and a *"prior"* to set a framework for expected utility consistent extensions of preferences. Throughout the second part, an extension axiom is a mapping which assigns to an order over the set of alternatives a strict partial order over the subsets of the alternatives. Moreover, we define a *"prior"* as a vector that collects a probability distribution over each element of the power set of the alternatives. We devote Section 4.2 to give an account of expected utility consistent extensions in our unified framework. In Section 4.3 , we note that different admissible orderings are obtained when we complete a generated partial order and directly generate a complete order. In Section 4.4, we discuss the effects of our findings to definitions of strategy-proofness. Moreover, we are able to remark that not all the finesses of expected utility consistent extensions are incorporated into the literature on strategy-proof social choice correspondences. Section 5 concludes.

## 2 Basic Notions

Consider a finite non-empty set of alternatives $A$ and let $\underline{A} = 2^A \backslash \{\emptyset\}$. We let $\#A \geq 3$ and write $\Pi$ for the set of complete, transitive and antisymmetric binary relations over $A$ and $\Re$ for the set of complete and transitive binary relations over $\underline{A}$.[9] We write $\rho \in \Pi$ and $R \in \Re$ for typical orders over $A$ and $\underline{A}$, respectively. We let $\rho^*$ and $P$ stand for the for the strict counterparts of $\rho \in \Pi$ and $R \in \Re$, respectively.[10]

## 3 PART I: Choosers as Extension Axioms

### 3.1 Extension Axioms

An *extension axiom* is a mapping $\varepsilon$ which assigns to each $\rho \in \Pi$ a transitive binary relation $\varepsilon(\rho)$ over $\underline{A}$ such that $x\ \rho^*\ y \Leftrightarrow \{x\}\ \varepsilon(\rho)\ \{y\}\ \forall x, y \in A$. We interpret $(X, Y) \in \varepsilon(\rho)$ as the requirement of ranking the set $X$ at least as good as the set $Y$ when the ranking of alternatives is $\rho$. Note that our definition of an extension axiom, perhaps untypically, does not require the antisymmetry of $\varepsilon(\rho)$. Nevertheless, most of the extension axioms we consider turn out to induce antisymmetric binary relations.

We define below three principal extension axioms that we consider:

- The extension axiom used by Kelly (1977) in his analysis of strategy-proof social choice correspondences, is defined for each $\rho \in \Pi$ as $\varepsilon^{KELLY}(\rho)$ $= \{(X, Y) \in \underline{A} \times \underline{A} \backslash \{X\} : x\ \rho\ y\ \forall\ x \in X\ \forall\ y \in Y\}$. We refer to $\varepsilon^{KELLY}$ as the **Kelly principle**.
- The extension axiom used by Gärdenfors (1976) in his analysis of strategy-proof social choice correspondences, is defined for each $\rho \in \Pi$

[9] So for any $\rho \in \Pi$ and any $x, y \in A$, by completeness, we have $x\ \rho\ y$ or $y\ \rho\ x$. This implies reflexivity, i.e., $x\ \rho\ x\ \ \forall x \in A$. Note that by antisymmetry, $x\ \rho\ y \Longrightarrow$ not $y\ \rho\ x$ when $x$ and $y$ are distinct. Finally, transitivity ensures $x\ \rho\ y$ and $y\ \rho\ z \Longrightarrow x\ \rho\ z$ $\forall x, y, z \in A$.

[10] So for any $\rho \in \Pi$ and any $x, y \in A$, we have $x\ \rho^*\ y$ whenever $x\ \rho\ y$ holds and $y\ \rho\ x$ fails. Similarly, for any $X, Y \in \underline{A}$, we have $X\ P\ Y$ whenever $X\ R\ Y$ holds but $Y\ R\ X$ does not. As $\rho$ is antisymmetric, when $x$ and $y$ are distinct, we have either $x\ \rho^*\ y$ or $y\ \rho^*\ x$.

as $\varepsilon^{GF}(\rho) = \{(X,Y) \in \underline{A} \times \underline{A}\backslash\{X\} : (x\ \rho^*\ y\ \forall\ x \in X\backslash Y\ \forall\ y \in Y)$ and $(x\ \rho^*\ y\ \forall\ x \in X\ \forall\ y \in Y\backslash X)\}$. We refer to $\varepsilon^{GF}$ as the **Gärdenfors principle.**

- The extension axiom $\varepsilon^{SE}$, to which we refer as the **separability principle**, is defined for each $\rho \in \Pi$ as $\varepsilon^{SE} = \{(X \cup \{x\}, X \cup \{y\}) : X \in 2^A$ and $x\ \rho^*\ y$ for distinct $x, y \in A\backslash X\ \}$.[11]

The Gärdenfors principle is stronger than the Kelly principle, i.e., $\varepsilon^{KELLY}(\rho) \subsetneq \varepsilon^{GF}(\rho)\ \forall \rho \in \Pi$. On the other hand, the separability principle is logically independent of both the Kelly and the Gärdenfors principles. Note that all three extension axioms induce antisymmetric binary relations.

## 3.2 Choice Functions

A *(resolute) choice function* is a mapping $C : \underline{A} \rightarrow A$ such that $C(X) \in X$, $\forall X \in \underline{A}$. We write $\mathcal{C}$ for the set of all choice functions and $\mathcal{D} \subseteq \mathcal{C}$ stands for any non-empty subclass of choice functions. We consider axiomatic restrictions over $\mathcal{C}$. The definitions below are quoted from Aizerman and Aleskerov (1995):

- A choice function $C$ satisfies the *Weak Axiom of Revealed Prefence (WARP)* iff $C(Y) \in X$ and $C(X) \in Y \implies C(X) = C(Y)$ $\forall X, Y \in \underline{A}$.[12] We write $\mathcal{C}^{WARP}$ for the set of (resolute) choice functions that satisfy WARP.[13] It is to be noted that, defining at each

[11] The separability principle, which is a modified version of the monotonicity axiom of Kannai and Peleg (1984), is used by Roth and Sotomayor (1990) in their manipulation analysis of many-to-one matching rules.

[12] For resolute choice functions, the version of WARP we use and the definition given by Aizerman and Aleskerov (1995) are equivalent.

[13] Note that a variety of conditions which differ from WARP over the class of choice correspondences turn out to be equivalent to WARP over the class of resolute choice functions. Among these, we have

($i$) *postulate 4* of Chernoff (1954) (called *axiom C2* by Arrow (1959), *condition alpha* by Sen (1974), *upper semi-fidelity* by Sertel and van der Bellen (1979), *heredity* by Aizerman and Aleskerov (1995));

($ii$) the independence of irrelevant alternatives condition of Nash (1950) (called *postulate 5** by Chernoff (1954), *axiom 2* by Sen (1974), *outcast* by Aizerman and Aleskerov (1995)

$\tau \in \Pi$, the choice function $C_\tau(X)\ \tau\ x\ \forall x \in X$, $\forall X \in \underline{A}$, we have $\mathcal{C}^{WARP} = \{C_\tau\}_{\tau \in \Pi}$.[14]

- A choice function $C$ satisfies *Concordance* iff $C(X) = C(Y) \implies C(X) = C(X \cup Y)\ \forall X, Y \in \underline{A}$. We write $\mathcal{C}^{CONC}$ for the set of (resolute) choice functions that satisfy concordance.
- A choice function $C$ satisfies *direct Condorcet* iff $x \in C(X) \implies x \in \bigcap_{y \in X} C(\{x, y\})\ \forall X \in \underline{A}$, $\forall x \in A$. We write $\mathcal{C}^{DC}$ for the set of (resolute) choice functions that satisfy direct Condorcet.

**Remark 3.2.1** *As one can see in Aizerman and Aleskerov (1995), we have* $\mathcal{C}^{WARP} \subsetneq \mathcal{C}^{CONC} \subsetneq \mathcal{C}^{DC} \subsetneq \mathcal{C}$.

## 3.3 Inducing Extension Axioms through Choice Functions

Any non-empty $\mathcal{D} \subseteq \mathcal{C}$ induces an extension axiom $\varepsilon^{\mathcal{D}}$ as follows: At each $\rho \in \Pi$, for all distinct $X, Y \in \underline{A}$, we have $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho) \iff C(X)\ \rho\ C(Y)\ \forall C \in \mathcal{D}$. Note that $\varepsilon^{\mathcal{D}}(\rho)$ is antisymmetric if and only if $\mathcal{D}$ satisfies the following **richness condition**: Given any distinct $X, Y \in \underline{A}$, there exists $C \in \mathcal{D}$ such that $C(X) \neq C(Y)$. The definition of $\varepsilon^{\mathcal{D}}$ conjoined with Remark 3.2.1 leads to the following proposition:

**Proposition 3.3.1** $\varepsilon^{\mathcal{C}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{DC}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{CONC}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)\ \forall \rho \in \Pi$.

Although the set inclusions stated by Remark 3.2.1 are proper, those announced by Proposition 3.3.1 need not be so, as we show soon.

We first establish the equivalence between the Kelly principle and the extension axiom induced by allowing all logically possible choice functions.

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and *absorbance* by Sertel and van der Bellen (1979));

(*iii*) *postulate 6* of Chernoff (1954) (called *axiom C4* by Arrow (1959) and *constancy* by Aizerman and Aleskerov (1995));

(*iv*) The *inverse condorcet* condition of Aizerman and Aleskerov (1995).

[14] What we note follows from many results of the literature, e.g., Theorem 2.10 of Aizerman and Aleskerov (1995).

**Theorem 3.3.1** $\varepsilon^{\mathcal{C}}(\rho) = \varepsilon^{KELLY}(\rho)\ \forall\rho \in \Pi$.

**Proof.** Take any $\rho \in \Pi$. To see $\varepsilon^{\mathcal{C}}(\rho) \subseteq \varepsilon^{KELLY}(\rho)$, pick any $(X,Y) \in \varepsilon^{\mathcal{C}}(\rho)$. So, $C(X)\ \rho\ C(Y)\ \forall C \in \mathcal{C}$. Now, consider a choice function $C_0$ with $x\ \rho\ C_0(X)$ $\forall x \in X$ and $C_0(Y)\ \rho\ y\ \forall y \in Y$. Clearly, $C_0 \in \mathcal{C}$. Thus, $C_0(X)\ \rho\ C_0(Y)$ which, by the choice of $C_0$, implies $x\ \rho\ y\ \forall x \in X$, $\forall y \in Y$, hence establishing $(X,Y)$ $\in \varepsilon^{KELLY}(\rho)$. To see $\varepsilon^{KELLY}(\rho) \subseteq \varepsilon^{\mathcal{C}}(\rho)$, pick any $(X,Y) \in \varepsilon^{KELLY}(\rho)$. Let $x_o \in X$ be such that $x\ \rho\ x_o\ \forall x \in X$ and $y_0 \in Y$ be such that $y_0\ \rho\ y\ \forall y \in Y$. As $(X,Y) \in \varepsilon^{KELLY}(\rho)$, we have $x_0\ \rho\ y_0$. Now, take any $C \in \mathcal{C}$. By the choice of $x_0$ and $y_0$, we have $C(X)\ \rho\ x_0$ and $y_0\ \rho\ C(Y)$ which implies $C(X)\ \rho\ C(Y)$, establishing $(X,Y) \in \varepsilon^{\mathcal{C}}(\rho)$. ■

**Remark 3.3.1** *The antisymmetry of $\varepsilon^{\mathcal{C}}$ follows from the antisymmetry of $\varepsilon^{KELLY}$ as well as from the richness of $\mathcal{C}$.*

**Remark 3.3.2** *For any $\mathcal{D}$, we have $\varepsilon^{KELLY}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)\ \forall\rho \in \Pi$. In other words, the Kelly principle is the weakest extension axiom that can be conceived in our environment.*

We now show that restricting the set of admissible choice functions to those which satisfy the concordance axiom does not induce an extension axiom stronger than the Kelly principle.

**Theorem 3.3.2** $\varepsilon^{\mathcal{C}^{CONC}}(\rho) = \varepsilon^{KELLY}(\rho)\ \forall\rho \in \Pi$.

**Proof.** Take any $\rho \in \Pi$. The inclusion $\varepsilon^{KELLY}(\rho) \subseteq \varepsilon^{\mathcal{C}^{CONC}}(\rho)$ follows from Remark 3.3.2. To see $\varepsilon^{\mathcal{C}^{CONC}}(\rho) \subseteq \varepsilon^{KELLY}(\rho)$, pick some $(X,Y) \notin \varepsilon^{KELLY}(\rho)$. So, $\exists\overline{y} \in Y$ and $\exists\overline{x} \in X\backslash\{\overline{y}\}$ such that $\overline{y}\ \rho^*\ \overline{x}$. First, consider the first case where $\overline{y} \notin X$. Pick some $\tau \in \Pi$ with $\overline{y}\ \tau\ \ \overline{x}\ \tau\ z\ \forall z \in A\backslash\{\overline{x},\overline{y}\}$. Note that $C_\tau \in \mathcal{C}^{WARP} \subsetneq \mathcal{C}^{CONC}$. As $\overline{y} \notin X$, we have $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, thus $C_\tau(X)\ \rho\ C_\tau(Y)$ fails, establishing $(X,Y) \notin \varepsilon^{\mathcal{C}^{CONC}}(\rho)$ . Next, consider the case where $\overline{x} \notin Y$. Pick some $\tau \in \Pi$ with $\overline{x}\ \ \tau\ \ \overline{y}\ \tau\ z\ \ \forall z \in A\backslash\{\overline{x},\overline{y}\}$. Note that $C_\tau \in \mathcal{C}^{CONC}$. As $\overline{x} \notin Y$, we have $C_\tau(Y) = \overline{y}$ and $C_\tau(X) = \overline{x}$, thus $C_\tau(X)\ \rho\ C_\tau(Y)$ fails, establishing $(X,Y) \notin \varepsilon^{\mathcal{C}^{CONC}}(\rho)$. Finally, consider the case where $\overline{y} \in X$ and $\overline{x} \in Y$. Pick some $\tau \in \Pi$ with $z\ \tau\ \overline{x}\ \tau\ \ \overline{y}\ \forall z \in$

$A\backslash\{\overline{x},\overline{y}\}$. Consider the choice function $\overline{C}$ defined as $\overline{C}(X)=\overline{x}$, $\overline{C}(Y)=\overline{y}$ and $\overline{C}(Z)=C_{\tau}(Z)$ $\forall Z\in \underline{A}\backslash\{X,Y\}$. Note that $\overline{C}(X)$ $\rho$ $\overline{C}(Y)$ fails. So we complete the proof by showing $\overline{C}\in\mathcal{C}^{CONC}$. To see this, take any distinct $S,T\in\underline{A}$ with $\overline{C}(S)=\overline{C}(T)$. Note that $S,T\in\{X,Y\}$ cannot hold, by construction of $\overline{C}$. Now, consider the following three exhaustive cases:

Case 1: $X\in\{S,T\}$, say $S=X$ without loss of generality. So $\overline{C}(T)=\overline{x}$, which implies $T\in\{\{\overline{x},\overline{y}\},\{\overline{x}\}\}$ which in turn implies $S\cup T=S$, establishing $\overline{C}(S\cup T)=\overline{C}(S)$.

Case 2: $Y\in\{S,T\}$, say $S=Y$ without loss of generality. So $\overline{C}(T)=\overline{y}$, which implies $T=\{\overline{y}\}$, which in turn implies $S\cup T=S$, establishing $\overline{C}(S\cup T)=\overline{C}(S)$.

Case 3: $X,Y\notin\{S,T\}$. Let $z=\overline{C}(S)=\overline{C}(T)$. So $z\ \tau\ s$ $\forall s\in S$ and $z\ \tau\ t$ $\forall t\in T$, thus $z\ \tau\ u$ $\forall u\in S\cup T$, implying $z=\overline{C}(S\cup T)$.

Therefore, $\overline{C}\in\mathcal{C}^{CONC}$, hence $(X,Y)\notin\varepsilon^{\mathcal{C}^{CONC}}(\rho)$. ■

**Remark 3.3.3** *The antisymmetry of* $\varepsilon^{\mathcal{C}^{CONC}}$ *follows from the antisymmetry of* $\varepsilon^{KELLY}$ *as well as from the richness of* $\mathcal{C}^{CONC}$.

The following result is a corollary to Theorem 3.3.1 and Theorem 3.3.2.

**Theorem 3.3.3** *Given any* $\mathcal{D}\supseteq\mathcal{C}^{CONC}$ *we have* $\varepsilon^{\mathcal{D}}(\rho)=\varepsilon^{KELLY}(\rho)$ $\forall\rho\in\Pi$.

Note that Theorem 3.3.3 covers the particular case where $\mathcal{D}=\mathcal{C}^{DC}$. Our next result shows that by further restricting the set of admissible choice functions through WARP, we fall into the Gärdenfors principle.[15]

**Theorem 3.3.4** $\varepsilon^{\mathcal{C}^{WARP}}(\rho)=\varepsilon^{GF}(\rho)$ $\forall\rho\in\Pi$.

[15] Sanver and Zwicker (2007) consider various monotonicity and manipulability properties of irresolute social choice rules. Among other things, they show that certain monotonicity conditions turn out to be equivalent, independent of whether the irresolute social choice rule is refined through a total order or preferences over alternatives are extended over sets through the Gärdenfors principle. In fact, it is the result announced by Theorem 3.3.4 which underlies this equivalence.

**Proof.** Take any $\rho \in \Pi$. To see $\varepsilon^{\mathcal{C}^{WARP}}(\rho) \subseteq \varepsilon^{GF}(\rho)$, pick some $(X,Y) \notin \varepsilon^{GF}(\rho)$. So $\exists \overline{y} \in Y$, $\exists \overline{x} \in X \backslash Y$ with $\overline{y}\ \rho^*\ \overline{x}$ or $\exists \overline{y} \in Y \backslash X$, $\exists \overline{x} \in X$ with $\overline{y}\ \rho^*\ \overline{x}$. In the former case, pick some $\tau \in \Pi$ with $\overline{x}\ \tau\ \overline{y}\ \tau\ z\ \ \forall z \in A \backslash \{\overline{x}, \overline{y}\}$, thus $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, implying the failure of $C_\tau(X)\ \rho\ C_\tau(Y)$ while $C_\tau \in \mathcal{C}^{WARP}$, hence establishing $(X,Y) \notin \varepsilon^{\mathcal{C}^{WARP}}(\rho)$. In the latter case, pick some $\tau \in \Pi$ with $\overline{y}\ \tau\ \overline{x}\ \tau\ z\ \forall z \in A \backslash \{\overline{x}, \overline{y}\}$, thus $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, implying the faliure of $C_\tau(X)\ \rho\ C_\tau(Y)$ while $C_\tau \in \mathcal{C}^{WARP}$, hence establishing $(X,Y) \notin \varepsilon^{\mathcal{C}^{WARP}}(\rho)$.

To see $\varepsilon^{GF}(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)$, pick any $(X,Y) \in \varepsilon^{GF}(\rho)$. So we have $(x\ \rho^*\ y\ \forall x \in X \backslash Y,\ \forall y \in Y)$ and $(x\ \rho^*\ y\ \forall x \in X,\ \forall y \in Y \backslash X)\}$. In particular, $C(X \backslash Y)\ \rho^*\ C(Y)\ \forall C \in \mathcal{C}$ whenever $X \backslash Y \neq \emptyset$ and $C(X)\ \rho^*\ C(Y \backslash X)\ \forall C \in \mathcal{C}$ whenever $Y \backslash X \neq \emptyset$. Note that $X$ and $Y$ are distinct, thus $X \backslash Y$ and $Y \backslash X$ cannot be both empty. Let, without loss of generality, $X \backslash Y \neq \emptyset$. Take any $C \in \mathcal{C}^{WARP}$. First, consider the case where $C(X) \in X \backslash Y$. Since $X \backslash Y \subseteq X$ and $C \in \mathcal{C}^{WARP}$, we have $C(X) = C(X \backslash Y)$. Thus, $C(X)\ \rho^*\ C(Y)$. Now, consider the case where $C(X) \notin X \backslash Y$. So $C(X) \in X \cap Y$. Since $X \cap Y \subseteq X$ and $C \in \mathcal{C}^{WARP}$, we have $C(X) = C(X \cap Y)$. If $C(Y) \in X \cap Y$ then $C(Y) = C(X \cap Y)$ follows by $C \in \mathcal{C}^{WARP}$, establishing $C(X)\ \rho^*\ C(Y)$. If $C(Y) \notin X \cap Y$, then $C(Y) \in Y \backslash X$, and we get $C(Y) = C(Y \backslash X)$ by $C \in \mathcal{C}^{WARP}$, implying $C(X)\ \rho^*\ C(Y)$. Thus $(X,Y) \in \varepsilon^{\mathcal{C}^{WARP}}(\rho)$ and $\varepsilon^{GF}(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)$. ■

**Remark 3.3.4** *The antisymmetry of $\varepsilon^{\mathcal{C}^{WARP}}$ follows from the antisymmetry of $\varepsilon^{GF}$ as well as from the richness of $\mathcal{C}^{WARP}$.*

We summarize below our findings upto now.

**Corollary 3.3.1** $\varepsilon^{KELLY}(\rho) = \varepsilon^{\mathcal{C}}(\rho) = \varepsilon^{\mathcal{C}^{DC}}(\rho) = \varepsilon^{\mathcal{C}^{CONC}}(\rho) \subsetneq \varepsilon^{\mathcal{C}^{WARP}}(\rho) = \varepsilon^{GF}(\rho)\ \forall \rho \in \Pi$.

Remark that a rich variety of choice axioms[16] single out the Kelly and Gärdenfors principles. As an interesting observation, the separability principle has not been induced by any of the choice axioms we considered. In fact,

[16]see footnote 13

as we show below, there exists no class of admissible choice functions that induces the separability principle. Before proving this, we state a lemma.

**Lemma 3.3.1** *Let* $\mathcal{D} \subseteq \mathcal{C}$ *ensure* $\varepsilon^{SE} \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$. *Given any* $C \in \mathcal{D}$ *and any* $X, Y \in \underline{A}$ *with* $\#X = \#Y = 2$ *and* $\#(X \cap Y) = 1$, *we have* $C(X) = X \cap Y \implies C(Y) = X \cap Y$.

**Proof.** Let $\mathcal{D}$ be as in the statement of the lemma. Take any $C \in \mathcal{D}$. Let $X = \{x, y\}$ and $Y = \{x, z\}$ for some distinct $x, y, z \in A$. Take any $\rho \in \Pi$ with $y\ \rho^*\ z\ \rho^*\ x$. Suppose $C(X) = x$ and $C(Y) = z$. So $C(X)\ \rho\ C(Y)$ fails, hence $(X, Y) \notin \varepsilon^{\mathcal{D}}(\rho)$ while $(X, Y) \in \sigma(\rho)$, contradicting the choice of $\mathcal{D}$. ■

**Theorem 3.3.5** $\nexists \mathcal{D} \subseteq \mathcal{C}$ *which ensures* $\varepsilon^{SE} \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$.

**Proof.** Let, for a contradiction, $\mathcal{D} \subseteq \mathcal{C}$ ensure $\varepsilon^{SE} \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$. Take any $C \in \mathcal{D}$ and any distinct $x, y, z \in A$. Let, without loss of generality, $C(\{x, y\}) = x$. By Lemma 3.3.1, we have $C(\{x, z\}) = x$ and $C(\{y, z\}) = z$. However, again by Lemma 3.3.1, $C(\{x, z\}) = x$ implies $C(\{y, z\}) = y$, giving the desired contradiction. ■

The impossibility announced by Theorem 3.3.5 prevails for any variant of Kannai and Peleg (1984) monotonicity which is stronger than separability.

We close the section by a remark regarding the strenghts of the extension axioms that are conceivable in our environment. As noted by Remark 3.3.2, the Kelly principle is the weakest among all conceivable extension axioms. On the other hand, although the Gärdenfors principle is the strongest extension axiom we encountered, we cannot claim it to be the strongest among all conceivable extension axioms. For, although WARP is a fairly demanding condition, the set of admissible choice functions can be further reduced. In fact, at the extreme, $\mathcal{D}$ can be assumed to contain only one choice function. Actually, the strongest conceivable extension axioms will be those which are induced by singleton sets of admissible choice functions. In fact, any $\mathcal{D} = \{C\}$ with $C \in \mathcal{C}$ induces a complete and transitive binary relation

$\varepsilon^{\mathcal{D}}(\rho) = \{(X,Y) \in \underline{A} \times \underline{A} : C(X)\ \rho\ C(Y)\}$ at each $\rho \in \Pi$.[17] Nevertheless, as we note below, it is not possible to speak about "the strongest" extension axiom.

**Proposition 3.3.2** *Given any $\mathcal{D} = \{C\}$ and $\mathcal{D}' = \{C'\}$ with distinct $C, C' \in \mathcal{C}$, both $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ and $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ fail at every $\rho \in \Pi$.*

**Proof.** Take any $\mathcal{D} = \{C\}$ and $\mathcal{D}' = \{C'\}$ with distinct $C, C' \in \mathcal{C}$. So, there exists $X \in \underline{A}$ such that $C(X) \neq C'(X)$. Note that $\#X \geq 2$. Take any $\rho \in \Pi$. Consider the first case where $C'(X)\ \rho^*\ C(X)$. Note that $(\{C(X)\}, X) \in \varepsilon^{\mathcal{D}}(\rho)$ but $(\{C(X)\}, X) \notin \varepsilon^{\mathcal{D}'}(\rho)$. Moreover $(X, \{C'(X)\}) \in \varepsilon^{\mathcal{D}'}(\rho)$ but $(X, \{C'(X)\}) \notin \varepsilon^{\mathcal{D}}(\rho)$. Hence, neither $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ nor $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ holds. Now, consider the case where $C(X)\ \rho^*\ C'(X)$. Note that $(X, \{C(X)\}) \in \varepsilon^{\mathcal{D}}(\rho)$ but $(X, \{C(X)\}) \notin \varepsilon^{\mathcal{D}'}(\rho)$. Moreover $(\{C'(X)\}, X) \in \varepsilon^{\mathcal{D}'}(\rho)$ but $(\{C'(X)\}, X) \notin \varepsilon^{\mathcal{D}}(\rho)$. Hence, neither $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ nor $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ holds. ■

As a case of particular interest, we have $\mathcal{D} = \{C\}$ for $C \in \mathcal{C}^{WARP}$. Let $\beta_\rho(X) \in X$ denote the best element of $X \in \underline{A}$ at $\rho \in \Pi$, i.e., $\beta_\rho(X)\ \rho\ x$ $\forall x \in X$. The *leximax extension* is the extension axiom $\lambda^+$ defined for each $\rho \in \Pi$ as $\lambda^+(\rho) = \{(X,Y) \in \underline{A} \times \underline{A} \backslash \{X\} : \beta_\rho(X)\ \rho\ \beta_\rho(Y)\}$. Similarly, let $\omega_\rho(X) \in X$ satisfy $x\ \rho\ \omega_\rho(X)\ \forall x \in X$. The *leximin extension* is the extension axiom $\lambda^-$ defined for each $\rho \in \Pi$ as $\lambda^-(\rho) = \{(X,Y) \in \underline{A} \times \underline{A} \backslash \{X\} : \omega_\rho(X)$ $\rho\ \omega_\rho(Y)\}$.[18]

**Proposition 3.3.3** *Given any $\mathcal{D}$ and any $\rho \in \Pi$, we have*

*(i) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$ if and only if $\mathcal{D} = \{C_\rho\}$.*

*(ii) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$ if and only if $\mathcal{D} = \{C_\tau\}$ for $\tau \in \Pi$ with $x\ \tau\ y \Longleftrightarrow y$ $\rho\ x\ \forall x, y \in A$.*

**Proof.** Take any $\mathcal{D}$ and any $\rho \in \Pi$.

[17] Remark that no $\mathcal{D} = \{C\}$ is rich hence the corresponding complete preorder $\varepsilon^{\mathcal{D}}(\rho)$ is not antisymmetric.

[18] Pattanaik and Peleg (1984), Bossert (1995), Campbell and Kelly (2002), Kaymak and Sanver (2003), Dogan and Sanver (2007) explore lexicographic extensions under a variety of definitions.

We prove $(i)$. To establish the "if" part, let $\mathcal{D} = \{C_\rho\}$. To see $\varepsilon^{\mathcal{D}}(\rho) \subseteq \lambda^+(\rho)$, take some $(X,Y) \in \varepsilon^{\mathcal{D}}(\rho)$. So $C_\rho(X)\ \rho\ C_\rho(Y)$. Moreover, by the definition of $C_\rho$, we have $C_\rho(X) = \beta_\rho(X)$ and $C_\rho(X) = \beta_\rho(Y)$, thus, $\beta_\rho(X)\ \rho\ \beta_\rho(X)$, showing $(X,Y) \in \lambda^+(\rho)$. To see $\lambda^+(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$, pick some $(X,Y) \in \lambda^+(\rho)$. So $\beta_\rho(X)\ \rho\ \beta_\rho(Y)$, thus $C_\rho(X)\ \rho\ C_\rho(Y)$, showing $(X,Y) \in \varepsilon^{\mathcal{D}}(\rho)$. To establish the "only if" part, assume $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$ and suppose $\exists C \in \mathcal{D}$ with $C \neq C_\rho$. So, $C(X) \neq C_\rho(X)$ for some $X \in \underline{A}$. Check that $(X, \{C_\rho(X)\}) \in \lambda^+(\rho)$ but $(X, \{C_\rho(X)\}) \notin \varepsilon^{\mathcal{D}}(\rho)$, contradicting $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$.

We prove $(ii)$. To establish the "if" part, let $\mathcal{D} = \{C_\tau\}$ for $\tau \in \Pi$ with $x\ \tau\ y \iff y\ \rho\ x\ \forall x,y \in A$. To see $\varepsilon^{\mathcal{D}}(\rho) \subseteq \lambda^-(\rho)$, take some $(X,Y) \in \varepsilon^{\mathcal{D}}(\rho)$. So $C_\tau(X)\ \rho\ C_\tau(Y)$. Moreover, by the choice of $\tau$, we have $C_\tau(X) = \omega_\rho(X)$ and $C_\tau(Y) = \omega_\rho(Y)$, thus $\omega_\rho(X)\ \rho\ \omega_\rho(Y)$, showing $(X,Y) \in \lambda^-(\rho)$. To see $\lambda^-(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$, pick some $(X,Y) \in \lambda^-(\rho)$. So $\omega_\rho(X)\ \rho\ \omega_\rho(Y)$, thus $C_\tau(X)\ \rho\ C_\tau(Y)$, showing $(X,Y) \in \varepsilon^{\mathcal{D}}(\rho)$. To establish the "only if" part, assume $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$ and suppose $\exists C \in \mathcal{D}$ with $C \neq C_\tau$. So, $C(X) \neq C_\tau(X)$ for some $X \in \underline{A}$. Check that $(\{C_\tau(X)\}, X) \in \lambda^-(\rho)$ but $(\{C_\tau(X)\}, X) \notin \varepsilon^{\mathcal{D}}(\rho)$, contradicting $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$. ■

So at a given $\rho$ the leximax ordering $\lambda^+(\rho)$ is induced if and only if $\mathcal{D} = \{C_\rho\}$. Similarly, at a given $\rho$ the leximin ordering $\lambda^-(\rho)$ is induced if and only if $\mathcal{D} = \{C_\tau\}$ such that $\tau$ is the opposite ranking of $\rho$. As a corollary which we state below, there exist no $\mathcal{D}$ which induces leximax (or leximin) orderings at every $\rho$.

**Theorem 3.3.6** *There exists no* $\mathcal{D}$ *such that*

$(i)$ $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)\ \forall \rho \in \Pi$

*or*

$(ii)$ $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)\ \forall \rho \in \Pi$.

## 3.4 A Probabilistic Variant of the Model

We now consider a probabilistic variant of our model by allowing randomizations over the set of admissible choice functions $\mathcal{D}$. A *prior* over $\mathcal{D}$ is a mapping $\theta : \mathcal{D} \to (0,1]$ such that $\sum_{C \in \mathcal{D}} \theta(C) = 1$. We write $\Theta_{\mathcal{D}}$ for some arbitrary (non-empty) set of priors over $\mathcal{D}$. Let $U_\rho$ stand for the set of all (real-valued) utility functions over $A$ that represent $\rho \in \Pi$.[19] Any $\mathcal{D}$ and $\Theta_\mathcal{D}$ induce an extension axiom $\varepsilon^{\Theta_\mathcal{D}}$ as follows: At each $\rho \in \Pi$, for all distinct $X, Y \in \underline{A}$, we have $(X,Y) \in \varepsilon^{\Theta_\mathcal{D}}(\rho) \iff \sum_{C \in \mathcal{D}} \theta(C)u(C(X)) \geq \sum_{C \in \mathcal{D}} \theta(C)u(C(Y))\ \forall u \in U_\rho$, $\forall \theta \in \Theta_\mathcal{D}$.

**Theorem 3.4.1** *Given any $\mathcal{D}$ and $\Theta_\mathcal{D}$, we have $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\Theta_\mathcal{D}}(\rho)\ \forall \rho \in \Pi$.*

**Proof.** Take any $\rho \in \Pi$ and any $(X,Y) \notin \varepsilon^{\Theta_\mathcal{D}}(\rho)$. So there exists $u \in U_\rho$ and $\theta \in \Theta_\mathcal{D}$ such that $\sum_{C \in \mathcal{D}} \theta(C).u(C(Y)) > \sum_{C \in \mathcal{D}} \theta(C).u(C(X))$. Since $\theta(C) > 0$ for each $C \in \mathcal{D}$, the inequality holds only if there exists $C' \in \mathcal{D}$ with $u(C'(Y)) > u(C'(X))$, hence $C'(Y)\ \rho^*\ C'(X)$, establishing $(X,Y) \notin \varepsilon^{\mathcal{D}}(\rho)$. ■

Whether the set inclusion announced by Theorem 3.4.1 is proper or not depends on the richness of the set of admissible priors $\Theta_\mathcal{D}$. For example, as we show below, when $\Theta_\mathcal{D}$ allows all priors over $\mathcal{D}$, Theorem 3.4.1 holds as an equality.

**Theorem 3.4.2** *Take any $\mathcal{D}$ and let $\Theta_\mathcal{D}$ be the set of all priors over $\mathcal{D}$. We have $\varepsilon^{\mathcal{D}}(\rho) = \varepsilon^{\Theta_\mathcal{D}}(\rho)\ \forall \rho \in \Pi$.*

**Proof.** Take any $\rho \in \Pi$. The inclusion $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\Theta_\mathcal{D}}(\rho)$ is already established by Theorem 3.4.1. To see $\varepsilon^{\Theta_\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$, pick some $(X,Y) \in \varepsilon^{\Theta_\mathcal{D}}$. Take any $C \in \mathcal{D}$ and consider some prior $\theta^* \in \Theta_\mathcal{D}$ with $\theta^*(C') = \varepsilon\ \forall C' \in \mathcal{D}\backslash\{C\}$ and $\theta^*(C) = 1 - (\#\mathcal{D} - 1).\varepsilon$ where $\varepsilon \in (0, \frac{1}{\#\mathcal{D}-\infty})$. As $(X,Y) \in \varepsilon^{\Theta_\mathcal{D}}(\rho)$, we have $(1-(\#\mathcal{D}-1).\varepsilon).u(C(X)) + \sum_{C' \in \mathcal{D}\backslash\{C\}} \varepsilon.u(C'(X)) \geq (1-(\#\mathcal{D}-1).\varepsilon).u(C(Y)) + \sum_{C' \in \mathcal{D}\backslash\{C\}} \varepsilon.u(C'(Y))$. Picking $\varepsilon$ arbitrarily small, we get $u(C(X)) \geq u(C(Y))$, hence $C(X)\ \rho\ C(Y)$, establishing $(X,Y) \in \varepsilon^{\mathcal{D}}(\rho)$. ■

Nevertheless, there are restricted choices of $\Theta_\mathcal{D}$ which render the set inclusion of Theorem 3.4.1 proper. To see this, let $\mathcal{D} = \mathcal{C}^{WARP}$ and $\Theta_\mathcal{D} = \{\overline{\theta}\}$ where $\overline{\theta}(C) = \frac{1}{\#\mathcal{D}}\ \forall C \in \mathcal{D}$. Take any distinct $x$, $y$, $z \in A$ and any $\rho \in \Pi$

[19] A utility function $u$ over $A$ represents $\rho \in \Pi$ iff $u(x) \geq u(y) \Leftrightarrow x\ \rho\ y\ \forall x, y \in A$.

with $x\rho\ y\rho\ z$. Note that $(\{x,y\},\{x,z\}) \in \varepsilon^{\Theta_{\mathcal{D}}}(\rho)$ since $\frac{u(x)+u(y)}{2} \geq \frac{u(x)+u(z)}{2}$ $\forall u \in U_\rho$ while $(\{x,y\},\{x,z\}) \notin \varepsilon^{\mathcal{D}}(\rho)$.

# 4 PART II : Expected Utility Consistent Extensions

## 4.1 Extension Axioms and Priors Revisited

In this part, by an *extension axiom*, we mean a mapping $\pi$ which assigns to each $\rho \in \Pi$ a strict partial order[20] $\pi(\rho)$ of $\underline{A}$ such that for all distinct $x, y \in A$ we have $x\ \rho\ y \Longleftrightarrow \{x\}\ \pi(\rho)\ \{y\}$. Given any extension axiom $\pi$ and any $\rho \in \Pi$, we write $D^\pi(\rho) = \{R \in \Re : X\ \pi(\rho)\ Y \Rightarrow X\ P\ Y$ for all distinct $X, Y \in \underline{A}\}$ for the set of complete and transitive binary relations over $\underline{A}$ which are compatible with $\pi(\rho)$. [21]

Let $\Omega_X$ be the set of all non-degenerate probability distributions over $X \in \underline{A}$, i.e., each $\omega_X \in \Omega_X$ is a probability distribution $\{\omega_X(x)\}_{x\in X}$ over $X$ where $\omega_X(x) \in (0,1]$ is interpreted as the (positive) probability that $x \in X$ will be chosen from $X$.[22] We call $\Omega = \prod_{X\in\underline{A}} \Omega_X$ the set of *priors* over $\underline{A}$. So, in this part a prior $\omega = (\omega_X)_{X\in\underline{A}} \in \Omega$ is a vector which collects a probability distribution over each element of $\underline{A}$. Any given non-empty set $\Gamma \subseteq \Omega$ of admissible priors over $\underline{A}$ induces an extension axiom $\pi^\Gamma$ which assigns to each $\rho \in \Pi$ a binary relation $\pi^\Gamma(\rho)$ over $\underline{A}$ as follows: For all distinct $X, Y \in \underline{A}$, we have $X\ \pi^\Gamma(\rho)\ Y$ if and only if $\sum_{x\in X} \omega_X(x).u(x) > \sum_{y\in Y} \omega_Y(y).u(y)$ $\forall\ u \in U_\rho$, $\forall\ \omega \in \Gamma$.[23] So $D^{\pi^\Gamma}(\rho)$ is the set of orderings which are completions of the partial order $\pi^\Gamma$ that the set of admissible priors $\Gamma$ induces. We call $D^{\pi^\Gamma}(\rho)$ the set of orderings over $\underline{A}$ which are *expected utility consistent with* $\rho$ *(* under

[20] A strict partial order is a transitive and antisymmetric (but not necessarily complete) binary relation.

[21] So every $R \in D^\pi(\rho)$ is a completion of the strict partial order $\pi(\rho)$ and $D^\pi(\rho)$ is non-empty by Spilrajn's Theorem.

[22] So we have $\sum_{x\in X} \omega_X(x) = 1$ for all $X \in \underline{A}$.

[23] One can immediately check that $\pi^\Gamma$ is an extension axiom, i.e., $\pi^\Gamma(\rho)$ is transitive and antisymmetric, while $x\ \rho\ y \Longleftrightarrow \{x\}\ \pi^\Gamma(\rho)\ \{y\}$ for all distinct $x, y \in A$.

the set of admissible priors $\Gamma$). Note that for any $\rho \in \Pi$ and any $R \in \Re$ we have $R \in D^{\pi^\Gamma}(\rho) \iff \forall\ X, Y \in \underline{A}$ with $X\ R\ Y$, there exists $(u, \omega) \in U_\rho \times \Gamma$ such that $\sum_{x\in X} \omega_X(x).u(x) \geq \sum_{y\in Y} \omega_Y(y).u(y)$. One could impose a stronger expected utility consistency requirement by reversing the order of the quantifiers. In other words, one could say that $R \in \Re$ is *strongly expected utility consistent with* $\rho \in \Pi$ (*under the set of admissible priors* $\Gamma$) iff there exists $(u, \omega) \in U_\rho \times \Gamma$ such that $X\ R\ Y \iff \sum_{x\in X} \omega_X(x).u(x) \geq \sum_{y\in Y} \omega_Y(y).u(y)$ for all $X, Y \in \underline{A}$. We write $D^\Gamma(\rho)$ for the set of orderings over $\underline{A}$ which are *strongly expected utility consistent with* $\rho \in \Pi$. In what follows, we say that a triple $(\rho, u, \omega) \in \Pi \times U_\rho \times \Gamma$ *directly generates* $R \in \Re$ iff $X\ R\ Y \iff \sum_{x\in X} \omega_X(x).u(x) \geq \sum_{y\in Y} \omega_Y(y).u(y)$ for all $X, Y \in \underline{A}$. So $D^\Gamma(\rho)$ is the set of orderings over $\underline{A}$ which are directly generated by some $(\rho, u, \omega) \in \Pi \times U_\rho \times \Gamma$. Note that $D^\Gamma(\rho) \subseteq D^{\pi^\Gamma}(\rho)\ \forall \rho \in \Pi$ follows from the definitions. On the other hand, as we show in Section 4.3, the properness of the set inclusion depends on the choice of admissible priors $\Gamma$.

## 4.2 The choice of admissible priors

The precise meaning of the "expected utility consistency" of an extension depends on the set of admissible priors and the set of admissible utility functions. Given a preference $\rho \in \Pi$ over alternatives, we let any $u \in U_\rho$ to be admissible. On the other hand, we allow the set of admissible priors $\Gamma$ to vary. The literature exhibits three choices of $\Gamma$:

### 4.2.1 General Expected Utility Consistency (GEUC)

Any prior is allowed, i.e., $\Gamma = \Omega$. As one can also deduce from Theorem 4.4.1 in Taylor (2005), the extension axiom $\pi^\Omega$ induced by GEUC is equivalent to the extension axiom introduced by Kelly (1977):

**Theorem 4.2.1** $\pi^\Omega(\rho) = \pi^{KELLY}(\rho)\ \forall \rho \in \Pi$.

**Proof.** Take any $\rho \in \Pi$. To see $\pi^{KELLY}(\rho) \subseteq \pi^\Omega(\rho)$, pick some $(X, Y) \in \pi^{KELLY}(\rho)$. Let $x_o \in X$ be such that $x\ \rho\ x_o\ \forall\ x \in X$ and $y_0 \in Y$ be such that $y_0 \; \rho \; y \; \forall y \in Y$ . As $(X,Y) \in \pi^{KELLY}(\rho)$ we have $x_0 \; \rho \; y_0$. Thus, for any $u \in U_\rho$, any $\omega_X \in \Omega_X$ and any $\omega_Y \in \Omega_Y$, we have $\sum_{x \in X} \omega_x(x).u(x) \geqslant u(x_0) \geqslant u(y_0) \geqslant \sum_{y \in Y} \omega_y(y).u(y)$ . If $X \cap Y = \varnothing$, then $u(x_0) > u(y_0)$, implying $\sum_{x \in X} \omega_x(x).u(x) > \sum_{y \in Y} \omega_y(y).u(y)$ . If $X \cap Y \neq \varnothing$, then at least one of $X$ and $Y$ is not a singleton as otherwise $X$ and $Y$ would coincide. In case $X$ is not a singleton we have $\sum_{x \in X} \omega_x(x).u(x) > u(x_0)$ and in case $Y$ is not a singleton we have $u(y_0) > \sum_{y \in Y} \omega_y(y).u(y)$, both of which implies $\sum_{x \in X} \omega_x(x).u(x) > \sum_{y \in Y} \omega_y(y).u(y)$, showing that $(X,Y) \in \pi^{\Omega}(\rho)$.

To see $\pi^{\Omega}(\rho) \subseteq \pi^{KELLY}(\rho)$, pick some $(X,Y) \notin \pi^{KELLY}(\rho)$ . So there exist $y_0 \in Y$ and $x_0 \in X \setminus \{y_0\}$ with $y_0 \; \rho \; x_0$.Now, let $x_1 \in X$ be such that $x_1 \; \rho \; x \quad \forall x \in X$. Take any $u \in U_\rho$ and any $r \in (0,1)$ which satisfies $r.u(x_1) + (1-r) \; .[u(x_0) - u(y_0)] < 0$. So $r.u(x_1) < (1-r) \; .[u(y_0) - u(x_0)]$. Let $\omega_X(x_o) \; = \omega_Y(y_o) \; = 1 - r$. So we have

$$\begin{aligned}
\sum_{x \in X} \omega_x(x).u(x) &\leq \omega_X(x_o).u(x_0) + (1 - \omega_X(x_o)).u(x_1) \\
&= (1-r) \; .u(x_0) + r.u(x_1) \\
&< (1-r) \; .u(x_0) + (1-r) \; .[u(y_0) - u(x_0)] \\
&= (1-r) \; .u(y_0) = \omega_Y(y_o). \; u(y_0) \\
&\leq \sum_{y \in Y} \omega_y(y).u(y)
\end{aligned}$$

which implies $(X,Y) \notin \pi^{\Omega}(\rho)$. ■

### 4.2.2 Bayesian Expected Utility Consistency (BEUC)

This is a restriction of GEUC that Barberà, Dutta and Sen (2001) and Ching and Zhou (2002) use in their analysis of strategy-proof social choice correspondences.[24] The set of admissible priors is defined as $\Gamma^{BEUC} = \{\omega \in \Omega : \omega_X(x) = \frac{\omega_A(x)}{\sum_{y \in X} \omega_A(y)}$ for all $X \in \underline{A} \setminus \{A\}$ and for all $x \in X\}$. As one can also deduce from Lemma 1 of Ching and Zhou (2002), the extension axiom $\pi^{\Gamma^{BEUC}}$ induced by BEUC is equivalent to the extension axiom introduced by Gärdenfors (1976):

[24] Barberà, Dutta and Sen (2001) call it Conditional Expected Utility Consistency.

The proof of the equivalence theorem we will state benefits from the following two lemmata.

**Lemma 4.2.1** *For all* $\rho \in \Pi$ *and all* $(X,Y) \in \pi^{GF}(\rho)$ *with* $X \cap Y \neq \emptyset$ *and* $X \backslash Y \neq \emptyset$, *we have* $(X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

**Proof.** Take any $\rho \in \Pi$ and let $(X,Y)$ be as in the statement of the lemma. As $(X,Y) \in \pi^{GF}(\rho)$, we have $x\ \rho\ y\ \forall\ x \in X\backslash Y\ \forall\ y \in Y$, thus $x\ \rho\ y\ \forall\ x \in X\backslash Y\ \forall\ y \in X \cap Y$. Therefore, given any $u \in U_\rho$ and any $\omega \in \Gamma^{BEUC}$, we have $\sum\limits_{x \in X\backslash Y} \omega_{X\backslash Y}(x)u(x) > \sum\limits_{x \in X \cap Y} \omega_{X\cap Y}(x)u(x)$, which implies $\frac{1}{\sum\limits_{x\in X\backslash Y}\omega_A(x)} \sum\limits_{x \in X\backslash Y} \omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)$. Multiplying both sides by $\frac{\sum\limits_{x\in X\backslash Y}\omega_A(x)}{\sum\limits_{x\in X}\omega_A(x)}$ gives

$$\frac{1}{\sum\limits_{x\in X}\omega_A(x)} \sum\limits_{x \in X\backslash Y} \omega_A(x)u(x) > \left(\frac{\sum\limits_{x\in X\backslash Y}\omega_A(x)}{\sum\limits_{x\in X}\omega_A(x)}\right) \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)$$
$$\Rightarrow \frac{1}{\sum\limits_{x\in X}\omega_A(x)} \sum\limits_{x \in X\backslash Y} \omega_A(x)u(x) > \frac{\sum\limits_{x\in X}\omega_A(x) - \sum\limits_{x\in X\cap Y}\omega_A(x)}{\sum\limits_{x\in X}\omega_A(x)\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)$$
$$\Rightarrow \frac{1}{\sum\limits_{x\in X}\omega_A(x)} \left(\sum\limits_{x \in X\backslash Y} \omega_A(x)u(x) + \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)\right) > \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)$$
$$\Rightarrow \frac{1}{\sum\limits_{x\in X}\omega_A(x)} \sum\limits_{x \in X} \omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x)$$
$$\Rightarrow \sum\limits_{x \in X} \omega_X(x)u(x) > \sum\limits_{x \in X\cap Y} \omega_{X\cap Y}(x)u(x)$$
$$\Rightarrow (X, X\cap Y) \in \pi^{\Gamma^{BEUC}}(\rho). \blacksquare$$

**Lemma 4.2.2** *For all* $\rho \in \Pi$ *and all* $(X,Y) \in \pi^{GF}(\rho)$ *with* $X \cap Y \neq \emptyset$ *and* $Y \backslash X \neq \emptyset$ *we have* $(X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

**Proof.** Take any $\rho \in \Pi$ and let $(X,Y)$ be as in the statement of the lemma. As $(X,Y) \in \pi^{GF}(\rho)$, we have $x\ \rho\ y\ \forall\ x \in X\ \forall\ y \in Y\backslash X$, thus $x\ \rho\ y\ \forall\ x \in X \cap Y\ \forall\ y \in Y\backslash X$. Therefore, given any $u \in U_\rho$ and any $\omega \in \Gamma^{BEUC}$, we have $\sum\limits_{x \in X\cap Y} \omega_{X\cap Y}(x)u(x) > \sum\limits_{x \in Y\backslash X} \omega_{Y\backslash X}(x)u(x)$, which implies $\frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)} \sum\limits_{x \in X\cap Y} \omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in Y\backslash X}\omega_A(x)} \sum\limits_{x \in Y\backslash X} \omega_A(x)u(x)$. Multiplying both sides by $\frac{\sum\limits_{x\in Y\backslash X}\omega_A(x)}{\sum\limits_{x\in Y}\omega_A(x)}$ gives

$$\frac{\sum\limits_{x\in Y\backslash X}\omega_A(x)}{\sum\limits_{x\in Y}\omega_A(x)\sum\limits_{x\in X\cap Y}\omega_A(x)}\sum_{x\in X\cap Y}\omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in Y}\omega_A(x)}\sum_{x\in Y\backslash X}\omega_A(x)u(x)$$
$$\Rightarrow \frac{\sum\limits_{x\in Y}\omega_A(x) - \sum\limits_{x\in X\cap Y}\omega_A(x)}{\sum\limits_{x\in Y}\omega_A(x)\sum\limits_{x\in X\cap Y}\omega_A(x)}\sum_{x\in X\cap Y}\omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in Y}\omega_A(x)}\sum_{x\in Y\backslash X}\omega_A(x)u(x)$$
$$\Rightarrow \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)}\sum_{x\in X\cap Y}\omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in Y}\omega_A(x)}\left(\sum_{x\in Y\backslash X}\omega_A(x)u(x) + \sum_{x\in X\cap Y}\omega_A(x)u(x)\right)$$
$$\Rightarrow \frac{1}{\sum\limits_{x\in X\cap Y}\omega_A(x)}\sum_{x\in X\cap Y}\omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in Y}\omega_A(x)}\sum_{x\in Y}\omega_A(x)u(x)$$
$$\Rightarrow (X\cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho). \blacksquare$$

**Theorem 4.2.2** $\pi^{\Gamma^{BEUC}}(\rho) = \pi^{GF}(\rho)\ \forall\rho\in\Pi$.

**Proof.** Take any $\rho\in\Pi$. We first show $\pi^{GF}(\rho) \subseteq \pi^{\Gamma^{BEUC}}(\rho)$. Take any $(X,Y)\in\pi^{GF}(\rho)$. Consider the following 4 exhaustive cases:

CASE 1: $X\cap Y\neq\emptyset$, $X\backslash Y\neq\emptyset, Y\backslash X=\emptyset$. So $Y=(X\cap Y)\subset X$ and by Lemma 3.1, we have $(X, X\cap Y)\in\pi^{\Gamma^{BEUC}}(\rho)$, thus $(X,Y)\in\pi^{\Gamma^{BEUC}}(\rho)$.
CASE 2: $X\cap Y\neq\emptyset$, $Y\backslash X\neq\emptyset, X\backslash Y=\emptyset$. So $X=(X\cap Y)\subset Y$ and by Lemma 3.2, we have $(X\cap Y, Y)\in\pi^{\Gamma^{BEUC}}(\rho)$, thus $(X,Y)\in\pi^{\Gamma^{BEUC}}(\rho)$.

CASE 3: $X\cap Y\neq\emptyset$, $Y\backslash X\neq\emptyset, X\backslash Y\neq\emptyset$. The conjunction of Lemma 3.1 and Lemma 3.2 implies $(X, X\cap Y)\in\pi^{\Gamma^{BEUC}}(\rho)$ and $(X\cap Y, Y)\in\pi^{\Gamma^{BEUC}}(\rho)$ while by transitivity we have $(X,Y)\in\pi^{\Gamma^{BEUC}}(\rho)$.

CASE 4: $X\cap Y=\emptyset$. As $(X,Y)\in\pi^{GF}(\rho)$, we have $x\ \rho\ y\ \forall x\in X, \forall y\in Y$. So $\sum\limits_{x\in X}\omega_X(x)u(x) > \sum\limits_{y\in Y}\omega_Y(y)u(y)$ holds for all $u\in U_\rho$ and all $\omega\in\Gamma^{BEUC}$, showing $(X,Y)\in\pi^{\Gamma^{BEUC}}(\rho)$.

We now show $\pi^{\Gamma^{BEUC}}(\rho)\subseteq\pi^{GF}(\rho)$. Take some $(X,Y)\in\underline{A}\times\underline{A}\backslash\{X\}$ with $(X,Y)\notin\pi^{GF}(\rho)$. So at least one of the following two conditions holds:

$(i)$ $\exists x\in X\backslash Y$, $\exists y\in Y$ such that $y\ \rho\ x$

$(ii)$ $\exists x\in X$, $\exists y\in Y\backslash X$ such that $y\ \rho\ x$

First let $(i)$ hold. Let $a\in X\backslash Y$ be such that $x\ \rho\ a\ \forall x\in X\backslash Y$ and $b\in Y$ be such that $b\ \rho\ y\ \forall y\in Y$. As $(i)$ holds, we have $b\ \rho\ a$. Now fix some $u\in U_\rho$. Take some $\epsilon\in(0,1)$ and consider the prior $\omega\in\Gamma^{BEUC}$ where $\omega_A(a)=\omega_A(b)=\dfrac{1-\epsilon}{2}$ and $\omega_A(x)=\dfrac{\epsilon}{\#A-2}\ \forall x\in A\backslash\{a,b\}$. Consider first the case where $b \in X$. We have

$$\sum_{x \in X} \omega_X(x)u(x) = \frac{1}{\sum_{x \in X} \omega_A(x)} \left( \frac{1-\epsilon}{2}u(a) + \frac{1-\epsilon}{2}u(b) + (\#X - 2) \frac{\epsilon}{\#A-2} \sum_{x \in X \backslash \{a,b\}} u(x) \right)$$

and $\sum_{y \in Y} \omega_Y(y)u(y) = \frac{1}{\sum_{y \in Y} \omega_A(y)} \left( \frac{1-\epsilon}{2}u(b) + (\#Y - 1) \frac{\epsilon}{\#A-2} \sum_{y \in Y \backslash \{b\}} u(y) \right)$. So when $\epsilon$ is picked arbitrarily small, $\sum_{x \in X} \omega_X(x)u(x)$ approaches to $\frac{u(a)+u(b)}{2}$ while $\sum_{y \in Y} \omega_Y(y)u(y)$ approaches to $u(b)$ and as $u(b) > u(a)$, this allows $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing that $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. Now consider the case where $b \notin X$. We have $\sum_{x \in X} \omega_X(x)u(x) = \frac{1}{\sum_{x \in X} \omega_A(x)}$ $\left( \frac{1-\epsilon}{2}u(a) + (\#X - 1) \frac{\epsilon}{\#A-2} \sum_{x \in X \backslash \{a,b\}} u(x) \right)$ and $\sum_{y \in Y} \omega_Y(y)u(y) = \frac{1}{\sum_{y \in Y} \omega_A(y)}$ $\left( \frac{1-\epsilon}{2}u(b) + (\#Y - 1) \frac{\epsilon}{\#A-2} \sum_{y \in Y \backslash \{b\}} u(y) \right)$. So when $\epsilon$ is picked arbitrarily small, $\sum_{x \in X} \omega_X(x)u(x)$ approaches to $u(a)$ while $\sum_{y \in Y} \omega_Y(y)u(y)$ approaches to $u(b)$ and as $u(b) > u(a)$, this allows $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. Now let $(ii)$ hold. Let $a \in X$ be such that $x \; \rho \; a$ $\forall x \in X$ and $b \in Y \backslash X$ be such that $b \; \rho \; y \; \forall y \in Y \backslash X$. As $(ii)$ holds, we have $b \; \rho \; a$. Fixing some $u \in U_\rho$, taking some $\epsilon \in (0, 1)$ and considering a prior $\omega \in \Gamma^{BEUC}$ as above, one can obtain $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. ■

### 4.2.3 Equal-Probability Expected Utility Consistency (EEUC)

This is a restriction of BEUC (hence of GEUC) that Feldman (1980) and Barberà, Dutta and Sen (2001) use in their analysis of strategy-proof social choice correspondences.[25] Letting $\omega^{\approx}$ be defined for each $X \in \underline{A}$ as $\omega^{\approx}_X(x) =$

[25] Barberà, Dutta and Sen (2001) call it Conditional Expected Utility Consistency With Equal Probabilities.

$\frac{1}{\#X}$ for all $x \in X$, we have $\Gamma^{EEUC} = \{\omega^{\approx}\}$. We characterize $\Gamma^{EEUC}$ in terms of an axiom that we call *componentwise dominance*. We define two equivalent versions of it.

**The Componentwise Dominance Principle 1:** For any real number $r$, we write $\lceil r \rceil$ for the lowest integer no less than $r$. Let $N$ stand for the set of natural numbers. Picking any two $m, n \in N$, we introduce a mapping $f_{mn} : N \longrightarrow N$ defined for each $i \in N$ as $f_{mn}(i) = \lceil \frac{1+n.(i-1)}{m} \rceil$. Note that $f_{mn}$ is an increasing function on $N$. Now take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{A}$. Let, without loss of generality,$X = \{x_1, .., x_{\#X}\}$ with $x_i \rho x_{i+1} \forall i \in \{1, .., \#X - 1\}$ and $Y = \{y_1, .., y_{\#Y}\}$ with $y_j \rho y_{j+1} \forall j \in \{1, .., \#Y - 1\}$. The *componentwise dominance principle 1* is defined through the strict partial order $\pi^{CD1}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \backslash \{X\} : \ x_i \rho \ y_{f_{\#X\#Y}(i)} \ \ \forall \ i \in \{1, .., \#X\}\}$.[26]

**The Componentwise Dominance Principle 2:** Take any $\rho \in \Pi$ and any $X = \{x_1, .., x_{\#X}\} \in \underline{A}$ with $x_i \rho \ x_{i+1} \ \forall i \in \{1, .., \#X - 1\}$. Given any $t \in N$, we define a $t.\#X$ dimensional vector $\vec{X}^t$ such that given any $i \in \{1, .., t.\#X\}$, we have $\vec{X}^t_i = x_{\lceil \frac{i}{t} \rceil}$.[27] In other words, we can write $\vec{X}^t = (x_1, ..., x_1, ..., x_{\#X}, .., x_{\#X})$ where each $x \in X$ appears $t$ times while given any $x_i, x_j \in X$ with $i < j$, $x_i$ appears at the left of $x_j$. Take also $Y = \{y_1, .., y_{\#Y}\} \in \underline{A} \backslash \{X\}$ with $y_i \rho \ y_{i+1} \ \forall i \in \{1, .., \#Y - 1\}$ and define $\vec{Y}^t$ similarly. The *componentwise dominance principle 2* is defined through the strict partial order $\pi^{CD2}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \backslash \{X\} : \vec{X}^{\#Y}_i \rho \, \vec{Y}^{\#X}_i \ \forall i \in \{1, .., \#X.\#Y\}\}$.[28]

**Lemma 4.2.3** *For all $\rho \in \Pi$, we have* $\pi^{CD1}(\rho) = \pi^{CD2}(\rho)$.

**Proof.** Take any $\rho \in \Pi$.To see $\pi^{CD1}(\rho) \subseteq \pi^{CD2}(\rho)$, pick some $(X, Y) \in \pi^{CD1}(\rho)$. Now take any $k \in \{1, .., \#X.\#Y\}$. We have $\vec{X}^{\#Y}_k = x_{\lceil \frac{k}{\#Y} \rceil}$ and $\vec{Y}^{\#X}_k = y_{\lceil \frac{k}{\#X} \rceil}$. As $(X, Y) \in \pi^{CD1}(\rho)$, we have $x_{\lceil \frac{k}{\#Y} \rceil} \ \rho \ y_{f_{\#X\#Y}(\lceil \frac{k}{\#Y} \rceil)}$. Now check that $f_{\#X\#Y}(\lceil \frac{t}{\#Y} \rceil) \leq \lceil \frac{t}{\#X} \rceil$ for all $t \in \{1, .., \#X.\#Y\}$. As

[26] The fact that $\varepsilon^{CD1}(\rho)$ is a strict partial order may not be visible at the first glance and we discuss the matter at the end of the section.

[27] As usual, $\vec{X}^t_i$ is the $i^{th}$ entry of $\vec{X}^t$.

[28] The fact that $\varepsilon^{CD2}(\rho)$ is a strict partial order may not be visible at the first glance and we discuss the matter at the end of the section.

a result, $y_{f_{\#X\#Y}(\lceil \frac{k}{\#Y} \rceil)}$ $\rho$ $y_{\lceil \frac{k}{\#Y} \rceil}$, which implies $x_{\lceil \frac{k}{\#Y} \rceil}$ $\rho$ $y_{\lceil \frac{k}{\#Y} \rceil}$, showing that $(X,Y) \in \pi^{CD2}(\rho)$.

To see $\pi^{CD2}(\rho) \subseteq \pi^{CD1}(\rho)$, pick some $(X,Y) \in \pi^{CD2}(\rho)$. So $\vec{X}_i^{\#Y} \rho\, \vec{Y}_i^{\#X}$ $\forall i \in \{1,..,\#X.\#Y\}$. Suppose, for a contradiction, that $(X,Y) \notin \pi^{CD1}(\rho)$. So there exists $i \in \{1,..,\#X\}$ such that $x_i$ $\rho$ $y_{f_{\#X\ \#Y}(i)}$ fails. Thus, if $x_i$ $\rho$ $y_j$ for some $y_j \in Y$ then $j \geq f_{\#X\ \#Y}(i)+1$. This, combined with the fact that $\vec{X}_i^{\#Y} \rho\ \vec{Y}_i^{\#X}$ for each $i \in \{1,..,\#X.\#Y\}$, implies $(i-1).\#Y \geq f_{\#X\ \#Y}(i).\#X$, which in turn implies $f_{\#X\ \#Y}(i) \leq (i-1).\frac{\#Y}{\#X}$, contradicting the definition of $f_{\#X\ \#Y}$, hence showing $\pi^{CD2}(\rho) \subseteq \pi^{CD1}(\rho)$. ■

So, for each $\rho \in \Pi$, we write $\pi^{CD}(\rho) = \pi^{CD1}(\rho) = \pi^{CD2}(\rho)$.

**Theorem 4.2.3** $\pi^{CD}(\rho) = \pi^{\Gamma^{EEUC}}(\rho)$ $\forall \rho \in \Pi$.

**Proof.** Take any $\rho \in \Pi$. To see $\pi^{CD}(\rho) \subseteq \pi^{\Gamma^{EEUC}}(\rho)$, pick some $(X,Y) \in \pi^{CD}(\rho)$. So $\vec{X}_i^{\#Y} \rho\ \vec{Y}_i^{\#X}$ $\forall i \in \{1,..,\#X.\#Y\}$. Thus, for any $u \in U_\rho$, we have $\sum\limits_{i=1}^{\sharp X.\sharp Y} u(\vec{X}_i^{\#Y}) > \sum\limits_{i=1}^{\sharp X.\sharp Y} u(\vec{Y}_i^{\#X})$ , the inequality being strict due to the fact that $X$ and $Y$ are distinct. This inequality can be rewritten as $\sum\limits_{i=1}^{\sharp X} \sharp Y.u(x_i) > \sum\limits_{j=1}^{\sharp Y} \sharp X.u(y_i)$, which implies $\frac{\sum_{i=1}^{\sharp X} u(x_i)}{\sharp X} > \frac{\sum_{j=1}^{\sharp Y} u(y_j)}{\sharp Y}$, thus showing $(X,Y) \in \Pi^{\Gamma^{EEUC}}(\rho)$.

To see $\Pi^{\Gamma^{EEUC}}(\rho) \subseteq \Pi^{CD}(\rho)$, pick some $(X,Y) \notin \Pi^{CD}(\rho)$. So there exists $j \in \{1,\ldots,\sharp X\}$ such that $x_j$ $\rho$ $y_{f_{\#X\ \#Y}(j)}$ fails, hence $u(x_j) < u(y_{f_{\#X\ \#Y}(j)})$ for any $u \in U_\rho$. Now, let $X \cup Y = Z = \{z_1,..,z_{\#Z}\}$ with $z_i \rho\ z_{i+1}$ $\forall i \in \{1,..,\#Z-1\}$ and take some $\epsilon > 0$ and some $M > 0$. Let $z_k \in Z$ coincide with $x_j$. Consider the following $u \in U_\rho$ defined as $u(z_{\#Z}) = 0$, $u(z_i) - u(z_{i+1}) = \epsilon$ for all $i \in \{k,...,\#Z-1\}$, $u(z_{k-1}) - u(z_k) = M$, and $u(z_i) - u(z_{i+1}) = \epsilon$ for all $i \in \{1,...,k-2\}$. Picking $M$ arbitrarily large and $\epsilon$ arbitrarily close to 0, we have $\frac{\sum_{j=1}^{\sharp Y} u(y_j)}{\sharp Y} > \frac{\sum_{i=1}^{\sharp X} u(x_i)}{\sharp X}$, showing that $(X,Y) \notin \Pi^{\Gamma^{EEUC}}(\rho)$. ■

We close by noting the straightforwardness of checking that $\pi^{\Gamma^{EEUC}}(\rho)$ is a strict partial order, thus answering the issue raised by Footnotes 26 and 28.

## 4.3 Completing partial orders versus direct generation of complete orderings

Whether an ordering over sets is obtained by completing a partial order generated through expected utilities (i.e., expected utility consistency) or is directly generated with reference to expected utilities (i.e., strong expected utility consistency) matters. In other words, given a set $\Gamma$ of admissible priors, the extension axiom $\pi^{\Gamma}$ induced by $\Gamma$ and a preference $\rho \in \Pi$, the sets $D^{\Gamma}(\rho)$ and $D^{\pi^{\Gamma}}(\rho)$ need not coincide. In fact, as we note in the beginning of Part II, $D^{\Gamma}(\rho)$ being a subset of $D^{\pi^{\Gamma}}(\rho)$ follows from the definitions. A formal statement of this logical relationship is given by the following theorem.

**Theorem 4.3.1** *Given any set $\Gamma$ of admissible priors over $\underline{A}$, we have $D^{\Gamma}(\rho) \subseteq D^{\pi^{\Gamma}}(\rho)$ $\forall \rho \in \Pi$.*

**Proof.** Take any set $\Gamma$ of admissible priors over $\underline{A}$, any $\rho \in \Pi$ and any $R^* \in \Re \backslash D^{\pi^{\Gamma}}(\rho)$. So there exist distinct $X, Y \in \underline{A}$ with $Y \ R^* \ X$ while $\sum_{x \in X} \omega_X(x).u(x) > \sum_{y \in Y} \omega_Y(y).u(y)$ $\forall\ u \in U_{\rho}$, $\forall\ \omega \in \Gamma$. Thus, there exists no $(\rho, u, \omega) \in \Pi \times U_{\rho} \times \Gamma$ that directly generates $R^*$, showing $R^* \notin D^{\Gamma}(\rho)$. ■

Whether the set inclusion announced by Theorem 4.3.1 is proper or not depends on the choice of admissible priors $\Gamma$. To explore this, we define the strong leximax extension $\Lambda^{+}(\rho) \in \Re$ and the strong leximin extension $\Lambda^{-}(\rho) \in \Re$ of $\rho \in \Pi$.[29] Under the strong leximax extension, sets are ordered according to their best elements. If these are the same, then the ordering is made according to the second best elements, etc. The elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the smaller set is preferred.[30] To speak formally, given any $\rho \in \Pi$, the *strong leximax extension* $\Lambda^{+}(\rho) \in \Re$ is defined as follows: Take any distinct $X, Y \in \underline{A}$. First consider the case

[29] Kaymak and Sanver (2003) show that at each $\rho \in \Pi$, the leximax and leximin extensions determine unique orderings $\Lambda^{+}(\rho)$ and $\Lambda^{-}(\rho)$ over $\underline{A}$ which are complete, transitive and antisymmetric.

[30] This is exactly how words are ordered in a dictionary. For example, given three alternatives $a$, $b$ and $c$, the leximax extension of the ordering $a\ \rho\ b\ \rho\ c$ is $\{a\}\ \Lambda^{+}(\rho)\ \{a,b\}\ \Lambda^{+}(\rho)\ \{a,b,c\}\ \Lambda^{+}(\rho)\ \{a,c\}\ \Lambda^{+}(\rho)\ \{b\}\ \Lambda^{+}(\rho)\ \{b,c\}\ \Lambda^{+}(\rho)\ \{c\}$.

where $\#X = \#Y = k$ for some $k \in \{1, ..., \#A - 1\}$. Let, without loss of generality, $X = \{x_1, ..., x_k\}$ and $Y = \{y_1, ..., y_k\}$ such that $x_j \ \rho \ x_{j+1}$ and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1, ..., k - 1\}$. We have $X \ \Lambda^+(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest $h \in \{1, ..., k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1, ..., x_{\#X}\}$ and $Y = \{y_1, ..., y_{\#Y}\}$such that $x_j \ \rho \ x_{j+1}$ for all $j \in \{1, ..., \#X - 1\}$and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1, ..., \#Y - 1\}$. We have either $x_h = y_h$ for all $h \in \{1, ..., \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1, ..., \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case, $X \ \Lambda^+(\rho) \ Y$ if and only if $\#X < \#Y$. For the second case, $X \ \Lambda^+(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest $h \in \{1, ..., \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

The concept of a leximin extension is similarly defined while it is based on ordering two sets according to a lexicographic comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the larger set is preferred.[31] So given given any $\rho \in \Pi$, the *strong leximin extension* $\Lambda^-(\rho) \in \Re$ is defined as follows: Take any distinct $X, Y \in \underline{A}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1, ..., \#A - 1\}$. Let, without loss of generality, $X = \{x_1, ..., x_k\}$ and $Y = \{y_1, ..., y_k\}$ such that $x_j \ \rho \ x_{j+1}$ and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1, ..., k - 1\}$. We have $X \ \Lambda^-(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the greatest $h \in \{1, ..., k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1, ..., x_{\#X}\}$ and $Y = \{y_1, ..., y_{\#Y}\}$such that $x_j \ \rho \ x_{j+1}$ for all $j \in \{1, ..., \#X - 1\}$and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1, ..., \#Y - 1\}$. We have either $x_h = y_h$ for all $h \in \{1, ..., \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1, ..., \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case, $X \ \Lambda^-(\rho) \ Y$ if and only if $\#X > \#Y$. For the second case, $X \ \Lambda^-(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest $h \in \{1, ..., \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

The first application of Theorem 4.3.1 is for GEUC, when $\Omega$ is taken as the set of admissible priors. In this case, Theorem 4.3.1 holds as an equality. Before establishing this, we state a lemma.

[31] For example, the leximin extension of the ordering $a \ \rho \ b \ \rho \ c$ is $\{a\} \ \Lambda^-(\rho) \ \{a, b\} \ \Lambda^-(\rho) \ \{b\} \ \Lambda^-(\rho) \ \{a, c\} \ \Lambda^-(\rho) \ \{a, b, c\} \ \Lambda^-(\rho) \ \{b, c\} \ \Lambda^-(\rho) \ \{c\}$.

**Lemma 4.3.1** *Take any one-to-one and real-valued function $u$ defined over $A$ and any $X \in \underline{A}$ with $\#X > 1$. Given any real number $r \in (\min_{x \in X} u(x), \max_{x \in X} u(x))$, there exists $w_X \in \Omega_X$ such that $\sum_{x \in X} w_X(x).u(x) = r$.*

**Proof.** Let $u$, $X$ and $r$ be as in the statement of the lemma. Let $x^+, x^- \in X$ be such that $x^+ \rho\ x\ \forall x \in X$ and $x\ \rho\ x^-\ \forall x \in X$. We define $X^+ = \{x \in X : u(x) \geq r\}$ and $X^- = \{x \in X : u(x) < r\}$. Both $X^+$ and $X^-$ are non-empty, as $x^+ \in X^+$ and $x^- \in X^-$. Take any $\omega_{X^+} \in \Omega_{X^+}$ and any $\omega_{X^-} \in \Omega_{X^-}$. Let $q^+ = \sum_{x \in X^+} \omega_{X^{+\prime}}(x).u(x)$ and $q^- = \sum_{x \in X^-} \omega_{X^-}(x).u(x)$. Note that $q^- < r < q^+$. Let $\Lambda = \frac{q^+ - r}{q^+ - q^-} \in (0, 1)$. Now define the following function $\omega_X$ over $X$: For each $x \in X$, we have $\omega_X(x) = (1 - \Lambda)\omega_{X^+}(x)$ if $x \in X^+$ and $\omega_X(x) = \Lambda\omega_{X^-}(x)$ if $x \in X^-$. It is clear that $\omega_X(x) \in (0, 1)$ for all $x \in X$. Moreover, $\sum_{x \in X} \omega_X(x) = (1 - \Lambda) \sum_{x \in X^+} \omega_{X^+}(x) + \Lambda \sum_{x \in X^-} \omega_{X^-}(x) = (1 - \Lambda) + \Lambda = 1$.Thus $\omega_X \in \Omega_X$. Finally, $\sum_{x \in X} \omega_X(x).u(x) = (1 - \Lambda) \sum_{x \in X^+} \omega_{X^+}(x).u(x) + \Lambda \sum_{x \in X^-} \omega_{X^-}(x).u(x) = (1 - \Lambda).q^+ + \Lambda q^-$ which, by the choice of $\Lambda$, equals to $r$. ■

**Theorem 4.3.2** $D^{\Omega}(\rho) = D^{\pi^{\Omega}}(\rho)\ \forall\ \rho \in \Pi$.

**Proof.** Take any $\rho \in \Pi$. The inclusion $D^{\Omega}(\rho) \subseteq D^{\pi^{\Omega}}(\rho)$ follows from Theorem 4.3.1. We now show $D^{\pi^{\Omega}}(\rho) \subseteq D^{\Omega}(\rho)$ or by Theorem 4.2.1 equivalenty $D^{\pi^{KELLY}}(\rho) \subseteq D^{\Omega}(\rho)$. Let $A = \{a_1, ..., a_m\}$ for some integer $m \geq 2$ and assume, without loss of generality, that $a_i\ \rho\ a_{i+1}$ for each $i \in \{1, ..., m\}$. Take any $R \in D^{\pi^{KELLY}}(\rho)$. Let $C_1 = \{X \in \underline{A} : X\ R\ Y\ \forall\ Y \in \underline{A}\}$ and define recursively $C_i = \{X \in \underline{A} : X\ R\ Y\ \forall\ Y \in \underline{A} \setminus \bigcup_{j=1}^{i-1} C_j\}$. So we express $R$ in terms of a family $\{C_1, ..., C_k\}$ of equivalence classes where $k$ is some integer that cannot exceed $2^m - 1$. Note that for all $X\ , Y \in \underline{A}$, we have $X\ R\ Y$ if and only if given any $X \in C_i$ and $Y \in C_j$ for some $i, j \in \{1, ..., k\}$ with $i < j$. As $R \in D^{\pi^{KELLY}}(\rho)$, $C_1 = \{\{a_1\}\}$ and $C_k = \{\{a_m\}\}$. Consider the function $f : \{1, ..., m\} \to \{1, ..., k\}$ where for each $i \in \{1, ..., m\}$ we have $\{a_i\} \in C_{f(i)}$. So $f(1) = 1$ and $f(m) = k$. Moreover, as $R \in D^{\pi^{KELLY}}(\rho)$, for any $i, j \in \{1, ..., m\}$ with $i < j$, we have $f(i) < f(j)$. Now we define a real valued utility function $u$ over $A$ as $u(a_i) = k - f(i) + 1$ for each

$i \in \{1, ..., m\}$. We complete the proof by showing the existence of some $\{\omega_X\}_{X \in \underline{A}} \in \Omega$ such that for each $j \in \{1, ..., k\}$ and for each $X \in C_j$ we have $\sum_{x \in X} \omega_X(x).u(x) = k - j + 1$, as this ensures that the triple $(\rho, u, \{\omega_X\}_{X \in \underline{A}})$ directly generates $R$. So take any $j \in \{1, ..., k\}$ and any $X \in C_j$. Consider first the case where $\{a_i\} \in C_j$ for some $a_i \in A$. If $X = \{a_i\}$, then $\sum_{x \in X} \omega_X(x).u(x) = u(a_i) = k - j + 1$. If $X$ and $\{a_i\}$ are distinct, then, as $R \in D^{\pi^{KELLY}}(\rho)$, there exist $x, y \in X \setminus \{a_i\}$ such that $x \; \rho \; a_i$ and $a_i \; \rho \; y$. So $\min_{z \in X} u(z) < u(a_i) < \max_{z \in X} u(z)$ and by Lemma 4.3.1, there exists $\omega_X \in \Omega_X$ such that $\sum_{x \in X} \omega_X(x).u(x) = u(a_i) = k - j + 1$. Now consider the case where $\{x\} \in C_j$ for no $x \in A$. Let $i \in \{1, ..., m\}$ be such that $\{a_d\} \; P \; X$ for all $i \in \{1, ..., i\}$ and $X \; P \; \{a_d\}$ for all $d \in \{i+1, ..., m\}$. As $R \in D^{\pi^{KELLY}}(\rho)$, there exists $x \in X \setminus \{a_i\}$ such that $a_i \; \rho \; x$ and there exists $y \in X \setminus \{a_{i+1}\}$ such that $y \; \rho \; a_{i+1}$. Thus, $\min_{z \in X} u(z) \leq u(a_{i+1}) = k - f(i+1) + 1$ and $\max_{z \in X} u(z) \geq u(a_i) = k - f(i) + 1$. Moreover, $f(i) < j < f(i+1)$ implying $\min_{z \in X} u(z) < k - j + 1 < \max_{z \in X} u(z)$ which, by Lemma 4.3.1, implies the existence of $\omega_X \in \Omega_X$ such that $\sum_{x \in X} \omega_X(x).u(x) = k - j + 1$. ■

**Remark 4.3.1** *For each $\rho \in \Pi$, we have $\Lambda^+(\rho), \Lambda^-(\rho) \in D^{\pi^{KELLY}}(\rho)$, hence by Theorem 4.2.1, $\Lambda^+(\rho), \Lambda^-(\rho) \in D^{\Omega}(\rho)$.*

The next application of Theorem 4.3.1 is for BEUC and EEUC, which is a case in point to show that the converse of the inclusion expressed by Theorem 4.3.1 need not hold.

**Theorem 4.3.3** *$D^{\Gamma^{BEUC}}(\rho) \subsetneq D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subsetneq D^{\pi^{\Gamma^{EEUC}}}(\rho)$ $\forall$ $\rho \in \Pi$.*

**Proof.** Take any $\rho \in \Pi$. By Theorem 4.3.1, we have $D^{\Gamma^{BEUC}}(\rho) \subseteq D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subseteq D^{\pi^{\Gamma^{EEUC}}}(\rho)$. To see that both inclusions are strict, we check that $\Lambda^+(\rho) \in D^{\pi^{\Gamma^{BEUC}}}(\rho) \cap D^{\pi^{\Gamma^{EEUC}}}(\rho)$ while $\Lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho) \cup D^{\Gamma^{EEUC}}(\rho)$. As $D^{\pi^{\Gamma^{EEUC}}}(\rho) \subset D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subset D^{\Gamma^{BEUC}}(\rho)$, it suffices to check that $\Lambda^+(\rho) \in D^{\pi^{\Gamma^{EEUC}}}(\rho)$ and $\Lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho)$. We recall that by Theorem 3.3 $D^{\pi^{\Gamma^{EEUC}}}(\rho) = D^{\pi^{CD}}(\rho)$ and leave checking

$\Lambda^+(\rho) \in D^{\pi^{CD}}(\rho)$ as an exercice to the reader. To see $\Lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho)$, suppose there exists a triple $(\rho, u, \omega) \in \Pi \times U_\rho \times \Omega$ that directly generates $\Lambda^+(\rho)$. Take any distinct $a, b, c \in A$ with $a \ \rho \ b \ \rho \ c$. Note that by definition of the strong leximax extension, we have $\{a, b, c\} \ \Lambda^+(\rho) \ \{a, c\} \ \Lambda^+(\rho) \ \{b\}$. Therefore, $\frac{1}{\sum\limits_{x\in\{a,b,c\}} \omega_A(x)} \sum\limits_{x\in\{a,b,c\}} \omega_A(x)u(x) > \frac{1}{\sum\limits_{x\in\{a,c\}} \omega_A(x)} \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x)$

$\Rightarrow \frac{1}{\sum\limits_{x\in\{a,b,c\}} \omega_A(x)} \left( \omega_A(b)u(b) + \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x) \right) > \frac{1}{\sum\limits_{x\in\{a,c\}} \omega_A(x)} \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x)$

$\Rightarrow \frac{\omega_A(b)u(b)}{\sum\limits_{x\in\{a,b,c\}} \omega_A(x)} > \left( \frac{1}{\sum\limits_{x\in\{a,c\}} \omega_A(x)} - \frac{1}{\sum\limits_{x\in\{a,b,c\}} \omega_A(x)} \right) \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x)$

$\Rightarrow \frac{\omega_A(b)u(b)}{\sum\limits_{x\in\{a,b,c\}} \omega_A(x)} > \frac{\omega_A(b)}{\sum\limits_{x\in\{a,c\}} \omega_A(x) \sum\limits_{x\in\{a,b,c\}} \omega_A(x)} \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x)$

$\Rightarrow u(b) > \frac{1}{\sum\limits_{x\in\{a,c\}} \omega_A(x)} \sum\limits_{x\in\{a,c\}} \omega_A(x)u(x)$, contradicting that $Y \ \Lambda^+(\rho) \ Z$, thus that $(\rho, u, \omega)$ directly generates $\Lambda^+(\rho)$. ■

As one can see from the proof of Theorem 4.3.3, lexicographic extensions may or may not be expected utility consistent, depending on whether a partial order is completed or complete orderings are directly generated.

## 4.4 A Remark on Strategy-Proof Social Choice Correspondences

The "strategy-proofness" of a social choice correspondence depends on how preferences over alternatives is extended over sets. If this extension is made through expected utility consistency, then the subtleties discussed in the previous section affect the definition of strategy-proofness.

To argue this formally, let $\underline{\rho} = (\rho_1, ..., \rho_n) \in \Pi^N$ stand for a *preference profile* over $A$ where $\rho_i$ is the preference of $i \in N$. A *social choice correspondence* (SCC) is a mapping $f : \Pi^N \longrightarrow \underline{A}$. Consider a set of admissible priors $\Gamma$ inducing the extension axiom $\pi^\Gamma$. We say that a SCC $f : \Pi^N \longrightarrow \underline{A}$ is

- *strategy-proof under* $\Gamma$ iff given any $i \in N$ and any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \ \forall \ j \in N \backslash \{i\}$, we have $f(\underline{\rho}) \ R \ f(\underline{\rho}')$ for all $R \in D^\Gamma(\rho_i)$.

- *strongly strategy-proof under* $\Gamma$ iff given any $i \in N$ and any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \ \forall \ j \in N \backslash \{i\}$, we have $f(\underline{\rho}) \ R \ f(\underline{\rho}')$ for all $R \in D^{\pi^\Gamma}(\rho_i)$.

At a first glance, the second definition deserves to be qualified as "strong", because, by Theorem 4.3.1, we have $D^{\Gamma}(\rho) \subseteq D^{\pi^{\Gamma}}(\rho)$ for all $\rho \in \Pi$. Nevertheless, the two definitions coincide, as the following theorem announces:

**Theorem 4.4.1** *Take any non-empty* $\Gamma \subseteq \Omega$ *inducing the extension axiom* $\pi^{\Gamma}$*. A SCC* $f : \Pi^N \longrightarrow \underline{A}$ *strategy-proof under* $\Gamma$ *if and only if* $f$ *is strongly strategy-proof under* $\Gamma$*.*

**Proof.** Take any non-empty $\Gamma \subseteq \Omega$.The "if" part follows from Theorem 4.3.1. To show the "only if" part, consider a SCC $f : \Pi^N \longrightarrow \underline{A}$ which fails to be strongly strategy-proof. So there exist $i \in N$ and $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \ \forall \ j \in N \backslash \{i\}$ such that $f(\underline{\rho}') \ P \ f(\underline{\rho})$ for some $R \in D^{\pi^{\Gamma}}(\rho_i)$. Thus $(f(\underline{\rho}), f(\underline{\rho}')) \notin \pi^{\Gamma}(\rho_i)$, implying the existence of some $\widetilde{u} \in U_{\rho_i}$ and some $\widetilde{\omega} \in \Gamma$ such that $\sum\limits_{x \in f(\underline{\rho}')} \widetilde{\omega}_{f(\underline{\rho}')}(x).\widetilde{u}(x) > \sum\limits_{x \in f(\underline{\rho})} \widetilde{\omega}_{f(\underline{\rho})}(x).\widetilde{u}(x)$. Therefore, letting $\widetilde{R} \in \Re$ be directly generated by $(\rho_i, \widetilde{u}, \widetilde{\omega})$, there exist $i \in N$ and $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \ \forall \ j \in N \backslash \{i\}$ such that $f(\underline{\rho}') \ \widetilde{P} \ f(\underline{\rho})$ for $\widetilde{R} \in D^{\Gamma}(\rho_i)$, showing that $f$ fails to be strategy-proof. ■

Thus, in analyzing the strategy-proofness of SCCs, it does not matter whether orderings over sets are obtained by completing a partial order generated through expected utilities or are directly generated with reference to expected utilities. The literature on strategy-proof SCCs exhibits both definitions of strategy-proofness. For example, Ching and Zhou (2002) use strong strategy-proofness while Barberà, Dutta and Sen (2001) adopt the "weaker" version. We know by Theorem 4.4.1 that this choice, everything else being equal, does not affect the analysis.[32]

On the other hand, it would be no surprise that the choice of the set of admissible priors $\Gamma$ matters. In fact, it immediately follows from the definitions that expanding $\Gamma$ can only strenghten strategy-proofness. As a case in point, we have Barberà, Dutta and Sen (2001) who consider strategy-proofness under $\Gamma^{EEUC}$ and $\Gamma^{BEUC}$. They show that under $\Gamma^{EEUC}$ strategy-proof SCCs

[32] It is worth noting that the analysis of Barberà, Dutta and Sen (2001) is for social choice rules that map preference profiles over sets into sets. These being more general than standard social choice correspondences, their impossibility under $\Gamma^{BEUC}$ implies the impossibility that Ching and Zhou (2002) establish under $\Gamma^{BEUC}$.

are either dictatorial or bidictatorial[33] while $\Gamma^{BEUC}$ admits only dictatorial rules. Hence the fact that $\Gamma^{EEUC} \subset \Gamma^{BEUC}$ matters and strategy-proofness under $\Gamma^{BEUC}$ is effectively stronger than it is under $\Gamma^{EEUC}$. On the other hand, Ozyurt and Sanver (2006) pick $\Gamma^{GEUC}$ as the set of admissible priors and show the equivalence between strategy-proofness and dictatoriality. Thus expanding $\Gamma^{EEUC}$ to $\Gamma^{GEUC}$ leaves the definition of strategy-proofness intact.

[33] A SCC $f : \Pi^N \longrightarrow \underline{A}$ is *dictatorial* iff $\exists i \in N$ such that $f(\underline{\rho}) = \{\arg\max \rho_i\}$ $\forall \underline{\rho} \in \Pi^N$. A SCC $f : \Pi^N \longrightarrow \underline{A}$ is *bidictatorial* iff $\exists i, j \in N$ such that $f(\underline{\rho}) = \{\arg\max \rho_i, \arg\max \rho_j\}$ $\forall \underline{\rho} \in \Pi^N$.

# 5 Conclusion

As Barberà et al. (2004) eloquently survey, the literature on extending an order a set to its power set admits a plethora of extension axioms. Nevertheless, the appropriateness of an extension axiom depends on how elements of the power set are interpreted. We propose a model which incorporates the "non-resolute outcome" interpretation. In the first part, we show that among the plethora of extension axioms of literature, two of them −namely the Gärdenfors (1976) and Kelly (1977) principles− arise as the appropriate ones. This observation does not necessarily exclude the use of extension axioms based on "expected utility consistency", as these are essentially equivalent to either the Gärdenfors (1976) or the Kelly (1977) principle, depending on the precise meaning attributed to "expected utility consistency".[34] On the other hand, Theorem 3.3.5 sets an obstacle in using the separability principle when sets are conceived as non-resolute outcomes.[35]

In the second part, we explore the problem of extending a complete order over a set to its power set by the assignment of utilities over alternatives and probability distributions over sets - hence the idea of expected utility consistent extensions. We express three well-known expected utility consistent extensions of the literature as a function of admissible priors and we characterize them in terms of extension axioms which do not refer to the concept of expected utility. Moreover, we display that

- assigning utilities and probabilities which end-up ordering sets according to their expected utilities

and

- completing the partial order determined by the pairs of sets whose ordering is independent of the utility and probability assignment

[34] One can see Can et al. (2007) for a detailed exploration of this matter.

[35] To be sure, this does not criticize Roth and Sotomayor (1990) who use separability in their manipulation analysis of many-to-one matching rules, as their environments conceives sets as lists of mutually compatible outcomes.

are different approaches. This difference has an immediate reflection to the analysis of strategy-proof social choice correspondences which we also discuss and clarify. In brief, we present a framework which allows a general and unified exposition of expected utility consistent extensions while it allows to emphasize various subtleties, the effects of which seem to be underestimated - particularly in the literature on strategy-proof social choice correspondences.

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