Coalitional Stability and Efficiency of Partitions in Matching Markets

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Abstract

Özkal-Sanver (2005) studies stability and efficiency of partitions of agents in two-sided matching markets where agents are allowed to form partitions only by individual moves, and within each coalition of a partition a matching rule determines the matching. In this thesis, first we introduce some of the papers in the literature relating to this topic with their results. Then, we present Nizamoğulları and Özkal-Sanver (2007)'s work in which the relationship between stability and efficiency of partitions is analyzed for several matching rules and under various membership property rights codes, now allowing coalitional moves.

1 Introduction

This thesis consists of two parts. First, we present some of the papers in the literature with their results relating to this topic.

In the second part, after introducing our model we state the results with their proofs of Nizamoğulları and Özkal-Sanver (2007) where efficiency and coalitional stability of partitions under membership property codes for several matching rules are considered .

2 Literature Review

The stability (in our setting coalitional stability) of a partition depends on the existing membership property rights code which is the list of agents who have the right to object when an agent desires to exit from the coalition he belongs to and to enter another coalition. The idea of membership property rights was introduced by Sertel (1982) in his analysis of workers' enterprises where he proposes membership as a private property. For a more general treatment one can refer to Sertel (1992, 1998, 2003) and Eren (1993).

Özkal-Sanver (2005) studies stability and efficiency of partitions of agents in two-sided matching markets where agents are allowed to form partitions only by individual moves. In this work, she defines two different versions of stability and efficiency of partitions. For the strong versions, in a world where agents can freely exit from and enter to coalitions, for all Pareto optimal and individually rational rules φ the set of $\varphi-$efficient and $\varphi-$stable partitions coincide, there is only one; grand coalition. But, for the weaker versions for different codes she did not get similar results.

Aşan and Sanver (2003) state a stability-efficiency equivalence result when agents voluntarily contribute to the production of a pure public good under these membership property rights codes where agents are allowed to form partitions only by individual moves. Then, Aşan and Sanver (2007) study the relationship between stability and efficiency of partitions in public good production when there is a crowding effect and agents are allowed to form partitions also by coalitional moves. In fact, Aşan and Sanver (2007) analyze the relationhip between well-known Tiebout-equilibrium[1] and efficiency of jurisdiction structures under these membership property rights. Jurisdiction structure is just a partition in our setting. But our definition of coalitional stability differs from their definition of coalitional stable jurisdiction structures. They prove that coalitionaly stable jurisdiction structures are always efficient. Also they show that under a suitable membership property codes namely approved-entry and approved exist efficient jurisdiction structures are coalitionally stable jurisdiction structures. In other words,

[1] See Conley and Konishi (2000), Grenberg and Weber(1986).

under approved-entry and approved exist, efficient and coalitionally stable jurisdiction structures coincide.

And lastly, Nizamoğulları and Özkal-Sanver (2007) study efficiency and coalitional stability of partitions for both strong and weak versions. For strong versions, we find similar results as in Özkal-Sanver (2005). But for the weaker versions, while having Invisible Hand Theorem under some code we do not have Decentralization result under this code. Thus, for this version under none of the well-known codes, coalitional stable and efficient partitions coincide.

3 Coalitional Stability and Efficiency of Partitions in Matching Markets

In this part, we give results with their detailed proofs of Nizamoğulları and Özkal-Sanver (2007). In this work, we analyze stability and efficiency of partitions of agents in two-sided matching markets as well, now allowing agents to form partitions also by coalitional moves. Again, the coalitional stability of a partition depends on the existing membership property rights code, when we use the code Free Entry and Free Exit (FE-FX), our coalitional stability notion turns out to be the well-known Strong Tiebout equilibrium[2]. We can weaken coalitional stability by replacing FE-FX with the code Approved Entry and Approved Exit (AE-AX), where entrance to and exit from a coalition requires the consent of all members of that coalition to be entered to or exited from.

Our analysis depends on the matching rule in question which determines the mates of agents under any partition. Throughout the paper, we consider those which satisfy some well-known nice properties. The coalitional stability, as well as efficiency of a partition depends on the matching rule φ. We define two versions of coalitional stability. In the stronger versions, we ask the φ-coalitional stability (resp. φ-efficiency) of a partition for all preference profiles. The weaker versions of φ-coalitional stability and φ-efficiency are defined specifically for a given preference profile. We analyze the relationship between coalitional stability and efficiency (for each version) of a partition under several membership property rights axioms and for several matching rules.

First, we study the relationship between weaker versions, defined specifically for a preference profile, of coalitional stability under FE-FX (namely, the strong Tiebout equilibrium) and efficiency of a partition. We get an impossibility result: there is no stable rule φ, such that φ-efficiency of a

[2] See Conley and Konishi (2000), Grenberg and Weber(1986)

partition at a preference profile P would guarantee that this partition is coalitional $\varphi-$stable at this P under FE-FX. But we have Invisible Hand Theorem: Each coalitional φ-stable partition at some preference profile P is $\varphi-$efficient at this P under FE-FX. But for the code AE-AX we have the reverse situation, now Decentralization result holds but we do not have Invisible Hand Theorem. Namely each φ-efficient partition at some preference profile P is coalitional $\varphi-$stable. But for some stable rule φ, namely the men-optimal rule, there is a partition which is coalitional $\varphi-$stable but not $\varphi-$efficient.

For the stronger versions of coalitional $\varphi-$stability and $\varphi-$efficiency, defined for all preference profiles, there is only one partition, namely the grand coalition, that is strong coalitional $\varphi-$stable under FE-FX and $\varphi-$efficient for all stable rules. So under FE-FX we have an Invisible Hand Theorem and as well as a Decentralization Theorem. We cannot expand this result for any individually rational and Pareto optimal rule as in Özkal-Sanver (2005). Under some individually rational and Pareto optimal rule φ, for example under a modified version of the serial men-dictatorship rule imposing individual rationality, the grand coalition is not strong coalitional $\varphi-$stable, hence there exists no strong coalitional $\varphi-$stable partition. When we consider the code AE-AX, for all individually rational and Pareto optimal rules, strong coalitional $\varphi-$stable and strong $\varphi-$efficient partitions coincide, it is the grand coalition.

3.1 Our model

Let M and W be two disjoint universal sets. Let M be a nonempty and finite subset of M. Similarly, let W be a nonempty and finite subset of W. A society is a union of some $M \subset \mathsf{M}$ and some $W \subset \mathsf{W}$. Let $\mathcal{A} = \{M \cup W\}_{M \subset \mathsf{M}, W \subset \mathsf{W}}$ be the set of all possible societies. In the context of marriage, the set M stands for a set of men and the set W for a set of women. One can also interpret M as a set of firms and W as a set of workers.

For each agent $i \in A$ the set of potential mates of i, denoted by $A(i)$, is defined as

$$A(i) \equiv \{i\} \cup \begin{cases} W \text{ if } i \in M \\ M \text{ if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by P_i. Let $\mathcal{P}$ denote the set of all possible preference profiles $P \equiv (P_i)_A$.

A **matching** is a function $\mu : A \longrightarrow A$ such that for all $i \in A$, $\mu(i) \in A(i)$ and for all $j, k \in A$, $\mu(j) = k$ implies $\mu(k) = j$. Here, $\mu(i)$ is the mate of agent i under matching μ. Let $\mathcal{M}(A)$ denote the set of all matchings for A.

We extend each agent's preference over the agent's potential mates to the set of matchings in the following way: We say that agent i prefers μ to μ' if and only if agent i prefers his/her mate at μ to his/her mate at μ'. Slightly abusing notation we write this as $\mu \ P_i \ \mu'$. Similarly, agent i finds μ at least as desirable as μ' if and only if agent i finds his/her mate at μ at least as desirable as his/her mate at μ', we write this as $\mu \ R_i \ \mu'$. Finally, agent i is indifferent between μ to μ' if both $\mu \ R_i \ \mu'$and $\mu' \ R_i \ \mu$. Note that since preferences are strict, agent i can be indifferent between μ and μ' if and only if agent i is matched with the same mate at μ and μ'.

Let A be a set of agents and K be an index set. A **partition** of A is a finite family $\{S^k\}_{k\in K}$ of pairwise disjoint subsets of A such that $\cup_{k\in K} S^k = A$. Formally speaking, each S^k is non-empty. However, in our setting $\{S^k\}_{k\in K}$ and $\{S^k\}_{k\in K} \cup \{\emptyset\}$ are equivalent. Let $\Pi(A)$ be the set of all possible partitions of A.

For any nonempty subset $T \subseteq A$, $\Pi(T)$ denotes the set of all possible partitions of T.

A **matching problem** is a list $p \equiv (A, P, \pi)$, where A is the set of agents, P is the profile of their preferences over potential mates, and π is a partition of A. Let $\mathbf{P}$ denote the set of all matching problems. A **(matching) rule** is a function φ that associates with each matching problem $p \equiv (A, P, \pi)$ a matching $\mu \in \mathcal{M}(A, \pi)$

Given a set of agents A, a preference profile P, and a rule φ, each agent $i \in A$ has a complete and transitive preference relation over $\Pi(A)$. Keeping the set of agents A and preference profile P fix, we write $\pi \ R_i^{\varphi} \ \pi'$ if and only if $\varphi[A, P, \pi] \ R_i \ \varphi[A, P, \pi']$.

3.1.1 Stability and Efficiency of Matching Rules

Let $p \equiv (A, P, \pi) \in \mathbf{P}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A, \pi)$ is **individually rational** for p if and only if for all $i \in A$, $\mu(i) \ P_i \ i$ or $\mu(i) = i$. Let $\mathcal{M}_{IR}(p)$ denote the set of all such matchings. An individually rational rule φ associates with each $p \in \mathbf{P}$ an individually rational matching $\varphi[p] \in \mathcal{M}_{IR}(p)$. A pair of agents $(i, j) \in S^k \times S^k$ for some $k \in \{1, ..., K\}$ **blocks** a matching $\mu \in M(A, \pi)$ if and only if $j \ P_i \ \mu(i)$, $i \ P_j \ \mu(j)$. A matching $\mu \in \mathcal{M}(A, \pi)$ is **stable** for p if and only if it is individually rational for p and there is no pair $(i, j) \in S^k \times S^k$ for all $k \in \{1, ..., K\}$ that blocks μ. Let $\mathcal{M}^*(p)$ denote the set of all such matchings. A stable rule φ associates with each $p \in \mathbf{P}$ a stable matching $\varphi[p] \in \mathcal{M}^*(p)$. For each problem $p \equiv (A, P, \pi) \in \mathbf{P}$, there exists a matching $\mu \in \mathcal{M}(A, \pi)$ such that for all $\mu' \in \mathcal{M}^*(p)$ and all $i \in M$, we have $\mu \ R_i \ \mu'$.[3] (Gale and Shapley, 1962) Furthermore, this matching is unique. We call it the **men-optimal matching** for p. We denote μ_M the men-optimal matching for p. We obtain the men-optimal matching by applying for each coalition in π the Gale-Shapley deferred acceptance procedure in which men propose. The men-optimal rule φ_M associates with

[3] This result no longer holds when agents may be indifferent between possible mates.

each $p \in \mathbf{P}$ the men-optimal matching μ_M.

The women-optimal matching μ_W and the women-optimal matching rule φ_W are defined, similarly. A matching $\mu' \in \mathcal{M}(A, \pi)$ **dominates** a matching $\mu \in \mathcal{M}(A, \pi)$ at P if for all $i \in A$, $\mu' \ R_i \ \mu$ and for some $j \in A$, $\mu' \ P_j \ \mu$. A matching $\mu \in \mathcal{M}(A, \pi)$ is **Pareto optimal** for p if and only if there is no $\mu' \in \mathcal{M}(A, \pi)$ that dominates μ. Let $\mathcal{M}^E(p)$ denote the set of all such matchings. Finally, a Pareto optimal rule φ associates with each $p \in \mathbf{P}$ a Pareto optimal matching $\varphi[p] \in \mathcal{M}^E(p)$.

3.1.2 Stability and Efficiency of Partitions

Code Let $\gamma_{i,S^k,S^l} \subseteq A$ denote a set of agents who must agree to agent i leaving S^k and entering the (possibly empty) S^l. For convenience, we require $i \in \gamma_{i,S^k,S^l}$. We call the collection $\gamma \equiv \left\{\gamma_{i,S^k,S^l}\right\}_{i \in A \ and \ S^k \cap S^l = \emptyset}$ the membership property rights code, or simply the code, of the society.

We are especially interested in the following codes [4], where S^k means the group that agent i belongs to and S^l means the group that agent i enters.

[4] We are using the same terminology as in Sertel (1992, 1998, 2003). See also Greenberg (1977), Dreze and Greenberg (1980), and Bogomolnaia and Jackson (2002).

Free entry—free exit	(FE-FX):	$\gamma_{i,S^k,S^l} = \{i\}$
Approved entry—free exit	(AE-FX):	$\gamma_{i,S^k,S^l} = \{i\} \cup S^l$
Free entry—approved exit	(FE-AX):	$\gamma_{i,S^k,S^l} = S^k$
Approved entry—approved exit	(AE-AX):	$\gamma_{i,S^k,S^l} = S^k \cup S^l$

Individual Stability Let $i \in A$ be an arbitrary agent and $\pi \in \Pi(A)$ be an arbitrary partition. We define a function F_{i,S^k,S^l} from $\Pi(A)$ to $\Pi(A)$ as follows:

$$F_{i,S^k,S^l}(\pi) = \begin{cases} \pi \backslash \{S^k, S^l\} \cup \{S^k \backslash \{i\}, S^l \cup \{i\}\} & \text{if } i \in S^k \neq \{i\} \text{ and } S^l \in \pi. \\ \pi \backslash \{S^k, S^l\} \cup \{S^l \cup \{i\}\} & \text{if } i \in S^k = \{i\} \text{ and } S^l \in \pi. \\ \pi \backslash \{S^k\} \cup \{S^k \backslash \{i\}, \{i\}\} & \text{if } i \in S^k \neq \{i\} \text{ and } S^l = \emptyset \end{cases}$$

ie: By $F_{i,S^k,S^l}(\pi)$ we mean that, it is again a partition of A and agent i leaves the group S^k he/she belongs and enters to the S^l.

Definition 3.1 *: Let φ be a matching rule, P be a preference profile and γ be a code. A partition $\pi \in \Pi(A)$ is said to be individually φ-stable under γ at$P$:*

If there exists $k, l \in K$ and $i \in S^k$ with $F_{i,S^k,S^l}(\pi)\ P_i^{\varphi}\ \pi$ then for some $j \in \gamma_{i,S^k,S^l}$, $\pi\ P_j^{\varphi}\ F_{i,S^k,S^l}(\pi)$.

Remark 3.1 *:Individual stability of $\pi \in \Pi(A)$ under FE-FX is equivalent to the free mobility equilibrium defined in Conley and Konishi (2002).*

Coalitional Deviation Given any partition $\pi \in \Pi(A)$, any nonempty subset T of A and $\rho \in \Pi(T)$, $F_{T,\rho}(\pi) = \{S \backslash T : S \in \pi\} \cup \rho$ is called $\rho-$ move of the coalition T.

By $F_{T,\rho}(\pi)$ we mean that agents in T leave the groups they belong and form a new partition of T, and other people in the society stays as in π.

Definition 3.2 *: Let φ be a matching rule. We say that T forms a coalitional $\varphi-$deviation by $\rho \in \Pi(T)$ from π at the preference profile P iff for all $i \in T$, $F_{T,\rho}(\pi) P_i^{\varphi} \pi$*

Definition 3.3 *: A partition π is called coalitional φ-stable at P under the code γ iff*

i) π is individually $\varphi-$stable at P under γ.

ii) If there exists a coalition T that forms a coalitional deviation by $\rho \in \Pi(T)$, say $\rho = \{T^l\}_{l \in L}$, then there are $k \in K$, $l \in L$, $i \in S^k \cap T^l$ and $j \in \gamma_{i,S^k,T^l}$ such that $\pi P_j^{\varphi} F_{T,\rho}(\pi)$.

Let $C(\gamma, (P, \varphi)) \subset \Pi(A)$ denotes the set of all coalitional φ-stable partitions at P under the code γ for a society A.

Remark 3.2 *Coalitional $\varphi-$stability of a partition under FE-FX is equivalent to the strong Tiebout equilibrium defined in Conley and Konishi (2002). But this definition differs from the definition of coalitional stable jurisdiction structures which is defined in Aşan and Sanver (2007). Because in their definition of blocking $\rho-$move (what we say coalitional deviation), some agents in the coalition are allowed to be indifferent between before and after the deviation.*

We say that a partition $\pi' \in \Pi(A)$ **dominates** a partition $\pi \in \Pi(A)$ at P if for all $i \in A$ we have $\pi' \ R_i^{\varphi} \ \pi$ and for some $j \in A$, $\pi' \ P_j^{\varphi} \ \pi$.

Definition 3.4 *A partition π is φ-efficient at P iff there exists no partition π' that dominates π at P.*

Let $E(P, \varphi) \subset \Pi(A)$ denotes the the set of all $\varphi-$efficient partitions at P for a society A

3.2 Results

We study the relationship between coalitional φ-stability and $\varphi-$efficiency of a partition under several codes, and for individually rational and Pareto optimal rules.

3.2.1 Coalitional Stability and Efficiency of Partitions

Let A be a society and $\pi = \{S^k\}_{k \in K}$ be a partition of A. Throughout this section we fix this society and its partition.

First we consider the code FE-FX, and study whether Decentralization Theorem and/or Invisible Hand Theorem hold in this set up. Our answer for the former turns out to be negative and for the latter is positive. Namely, $\varphi-$coalitional stability implies $\varphi-$efficiency for any Pareto optimal and individually rational rule φ, but $\varphi-$efficieny of a partition does not guarantee its $\varphi-$coalitional stability. We state and prove these results below.

Proposition 3.1 *: For any Pareto optimal and individually rational rule φ, any coalitional $\varphi-$stable partition under FE-FX at P is $\varphi-$efficient at this P.*

Proof. Let φ be a Pareto optimal and individually rational rule, P be a preference profile and γ be FE-FX.

Take any $\pi \in C(\gamma, (P, \varphi))$. Suppose $\pi \notin E(P, \varphi)$. Then there exists $\pi^{'}$ such that for all $i \in A$ we have $\pi^{'} R_i^{\varphi} \pi$ and for some $j \in A$, $\pi' P_j^{\varphi} \pi$. Let $T = \{j \in A : \pi' P_j^{\varphi} \pi\}$, denote it by $T = \{j_1, ..., j_m\}$.

Claim: For any $j_k \in T$, we have $\varphi(A, P, \pi')(j_k) \in T$.

Proof: Take any $j_k \in T$, by definiton of T, $\varphi(A, P, \pi)(j_k) \neq \varphi(A, P, \pi^{'})(j_k) = l$. Hence $\varphi(A, P, \pi)(l) \neq \varphi(A, P, \pi^{'})(l)$. Since π' dominates π at P, for the agent l, we have $\pi' P_l^{\varphi} \pi$. Thus, for all $j_k \in T$, $\varphi(A, P, \pi^{'})(j_k) \in T$, proving our claim.

For each $j_k \in T$, define $T^k = \{j_k, \varphi(A, P, \pi^{'})(j_k)\}$. Then consider the following $\rho-$move: $\rho = \{T^k\}_{k=1,...,m} \in \Pi(T)$

Since φ is individually rational and Pareto optimal, $\varphi(A, P, F_{T,\rho}(\pi))(j_k) = \varphi(A, P, \pi^{'})(j_k)$ for all $j_k \in T$.

Then T forms a coalitional deviation by ρ from π. Also, for all $k \in K$, for all $l \in \{1, ..., m\}$ and for all $i \in S^k \cap T^l$ since $\gamma_{i,S^k,T^l} = \{i\}$, there exist no $j \in \gamma_{i,S^k,T^l}$ with $\pi P_j^{\varphi} F_{T,\rho}(\pi)$.

Thus, $\pi \notin C(\gamma, (P, \varphi))$ which gives a contradiction. Hence $\pi \in E(P, \varphi)$.

■

Proposition 3.2 *For any stable rule φ, there is a partition π which is $\varphi-$efficient but not coalitional $\varphi-$stable under FE-FX.*

Proof. Directly from the proposition 5.5 in Özkal-Sanver(2005), p.202. But for the completeness of the thesis, we give (some adaptation of) its proof below.

Let φ be a stable rule and γ be FE-FX.

Let π be a partition containing two coalitions S^1 and S^2 such that $S^1 = \{m_1, w_1\}, S^2 = \{m_2, w_2\}$. Let P be such that for $\{m_1, w_1, m_2, w_2\}$ and all $k \in A \diagdown \{m_1, m_{2,} w_{1,} w_2\}$

P_{m_1}	P_{w_1}	P_{m_2}	P_{w_2}	P_k
w_2	m_1	w_2	m_1	k
w_1	m_2	w_1	m_2	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

One can easily check that $\pi \in E(P, \varphi)$. Since $T = \{m_1, w_2\}$ forms a coalitional deviation by $\rho = \{\{m_1, w_2\}\}$ from π at P under γ, $\pi \notin C(\gamma, (P, \varphi))$.

■

Now, if we impose agents to get approval of the members of the coalitions which they exit from and enter to, there exists a Decentralization result for all rules. But under AE-AX there exists no Invisible hand theorem. Indeed, for an individually rational and Pareto optimal rule φ, there is a partition which is coalitional $\varphi-$stable but not $\varphi-$efficient under AE-AX.

Proposition 3.3 *For any rule φ, any $\varphi-$efficient partition at P is coalitional $\varphi-$stable under AE-AX at P.*

Proof. Let φ be a matching rule, P be a preference profile and γ be AE-AX.

Take any $\pi \in E(P, \varphi)$. Suppose $\pi \notin C(\gamma, (P, \varphi))$. There are two cases to consider:

Case 1: π is not individually $\varphi-$stable under γ. Then there are $k, l \in K$ and $i \in S^k$ such that $F_{i,S^k,S^l}(\pi) P_i^{\varphi} \pi$ and for all $j \in \gamma_{i,S^k,S^l}$, $F_{i,S^k,S^l}(\pi) R_j^{\varphi} \pi$. But then $F_{i,S^k,S^l}(\pi)$ dominates π at P, contradicting $\pi \in E(P, \varphi)$.

Case 2: There is a coalition T that forms a coalitional deviation by $\rho \in \Pi(T)$, say $\rho = \{T^l\}_{l \in L}$, and for all $k \in K$, $l \in L$, for all $i \in S^k \cap T^l$, and for all $j \in \gamma_{i,S^k,T^l}$, $F_{T,\rho}(\pi) R_j^{\varphi} \pi$. In this case $F_{T,\rho}(\pi)$ dominates π at P, contradicting $\pi \in E(P, \varphi)$.

Thus, $\pi \in C(\gamma, (P, \varphi))$. ■

Proposition 3.4 *Let φ be the men-optimal rule. Then there is a partition π which is coalitional $\varphi-$stable under AE-AX, but not $\varphi-$efficient.*

Proof. Let φ_M be the men-optimal rule and γ be AE-AX.

Let $M \equiv \{m_1, m_2, m_3, m_4\}$, $\widetilde{M} \equiv \{\widetilde{m_1}, \widetilde{m_2}, \widetilde{m_3}, \widetilde{m_4}\}$, $W \equiv \{w_1, w_2, w_3, w_4\}$, $\widetilde{W} \equiv \{\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}, \widetilde{w_4}\}$ and $A = M \cup W \cup \widetilde{M} \cup \widetilde{W}$.

And let P as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_2	w_2	$\widetilde{w}_3$	m_2	m_3	$\widetilde{m}_4$	m_1
w_4	w_4	....	w_4	m_4	m_2	m_4	m_2
....	w_1	....	w_1	m_1	....	....	m_3
....	w_3	....	w_3	m_3	....	....	m_4
....	....	....	w_2	....	....	....	....

$P_{\widetilde{m}_1}$	$P_{\widetilde{m}_2}$	$P_{\widetilde{m}_3}$	$P_{\widetilde{m}_4}$	$P_{\widetilde{w}_1}$	$P_{\widetilde{w}_2}$	$P_{\widetilde{w}_3}$	$P_{\widetilde{w}_4}$
$\widetilde{w}_1$	$\widetilde{w}_2$	$\widetilde{w}_2$	w_3	$\widetilde{m}_2$	$\widetilde{m}_3$	m_4	$\widetilde{m}_1$
$\widetilde{w}_4$	$\widetilde{w}_4$	....	$\widetilde{w}_4$	$\widetilde{m}_4$	$\widetilde{m}_2$	$\widetilde{m}_4$	$\widetilde{m}_2$
....	$\widetilde{w}_1$	....	$\widetilde{w}_1$	$\widetilde{m}_1$	....	....	$\widetilde{m}_3$
....	$\widetilde{w}_3$	....	$\widetilde{w}_3$	$\widetilde{m}_3$	....	....	$\widetilde{m}_4$
....	....	....	$\widetilde{w}_2$	....	....	....	....

Let $\pi = (M \cup W, \widetilde{M} \cup \widetilde{W})$.[5]

Note that $\varphi_M(A, P, \pi) = \{(m_1, w_4), (m_2, w_1), (m_3, w_2), (m_4, w_3), (\widetilde{m_1}, \widetilde{w_4}), (\widetilde{m_2}, \widetilde{w_1}), (\widetilde{m_3}, \widetilde{w_2}), (\widetilde{m_4}, \widetilde{w_3})\}$.

Then $\pi \notin E(P, \varphi_M)$, since $\pi' = \{\{m_1, w_4\}, \{m_2, w_1\}, \{m_3, w_2\}, \{m_4, \widetilde{w_3}\}, \{\widetilde{m_1}, \widetilde{w_4}\}, \{\widetilde{m_2}, \widetilde{w_1}\}, \{\widetilde{m_3}, \widetilde{w_2}\}, \{\widetilde{m_4}, w_3\}\}$ dominates π at P for φ_M.

[5] Note that, m_4 and $\widetilde{w_3}$, $\widetilde{m_4}$ and w_3 put eachother at the top . And the other agents in the society put the agents that are in the other coalition below the their own coalition's agents.

We need to show that $\pi \in C(\gamma, (P, \varphi))$.

The agents who may prefer to move other coalition or form coalitional deviations are $\{m_1, m_2, m_4, w_3, \widetilde{m_1}, \widetilde{m_2}, \widetilde{m_4}, \widetilde{w_3}\}$, denote this set by H. Because the remaining agents are matched with their top choices.

First, we will show individually $\varphi-$stability of π. Note that for any agent $i \in H$, we have $jP_i i$ for all $j \in H$. And m_1, m_2 do not want to enter $\widetilde{M} \cup \widetilde{W}$, since $w_1, w_2, w_4 \in M \cup W$. Similar situation holds for $\widetilde{m_1}$ and $\widetilde{m_2}$. Thus only $m_4, w_3, \widetilde{m_4}, \widetilde{w_3}$ may prefer to move the other coalition.

But whenever m_4 leaves $M \cup W$ and enters $\widetilde{M} \cup \widetilde{W}$, φ assigns m_4 to $\widetilde{w_3}$ and $\widetilde{m_4}$ becomes alone and worse off. So m_4 can not leave $M \cup W$ and enter $\widetilde{M} \cup \widetilde{W}$. Similarly whenever $\widetilde{m_4}$ leaves $\widetilde{M} \cup \widetilde{W}$, and enter $M \cup W$, m_4 will be worse off. When w_3 leaves $M \cup W$ and enters $\widetilde{M} \cup \widetilde{W}$, $\widetilde{w_3}$ becomes alone. And lastly when $\widetilde{w_3}$ leaves $\widetilde{M} \cup \widetilde{W}$, and enter $M \cup W$, then w_3 becomes alone. Therefore none of them can do these movings.

Next, we will consider any possible coalitional deviations. For m_1 and m_2, to form a coalitional deviation w_1 and w_2 or w_4 must be in this coalition but this agents are matched with their first choices and to have a coalitional deviation they should become strictly better. Similarly for $\widetilde{m_1}$ and $\widetilde{m_2}$.

So only $T = \{\ m_4, w_3, \widetilde{m_4}, \widetilde{w_3}\}$ can form coalitional deviations by five different moves:$\rho_1 = \{\{m_4, w_3\}\}$, $\rho_2 = \{\{m_4, \widetilde{w_3}\}\}$, $\rho_3 = \{\{\widetilde{m_4}, w_3\}\}, \rho_4 = \{\{\widetilde{m_4}, \widetilde{w_3}\}\}$ $\rho_5 = \{\{m_4, w_3, \widetilde{m_4}, \widetilde{w_3}\}\}$

One can easily check that by the moves ρ_1, ρ_4 none of the agents' mates are changed.

For ρ_2, $m_4 P_{w_3}\ \varphi(A, P, F_{T,\rho_2}(\pi))(w_3)$, so w_3 gets worse off. For ρ_3, $\widetilde{m_4} P_{\widetilde{w_3}}$ $\varphi(A, P, F_{T,\rho_3}(\pi))(\widetilde{w_3})$, so $\widetilde{w_3}$ gets worse off.

And finally for ρ_5, $\varphi(A, P, F_{T,\rho_5}(\pi))(m_1) = w_1$ and for w_1, we have $m_2 P_{w_1} m_1$. In this case w_1 gets worse off.

Thus, $\pi \in C(\gamma, (P, \varphi))$. ■

3.2.2 Stronger Versions of Coalitional Stability and Efficiency of Partitions

Since there is no equivalence between Decentralization Theorem and Invisible Hand Theorem to have this equivalence we define strong versions of coalitional stability and efficiency.

Let φ be a matching rule and γ be a code.

Definition 3.5 *A partition π is strong coalitional $\varphi-$stable under γ if $\pi \in C(\gamma, (P, \varphi))$ for all $P \in \mathcal{P}$.*

Let $C(\gamma, \varphi) \subset \Pi(A)$ denotes the set of all such partitions

Definition 3.6 *A partition π is strong $\varphi-$efficient if $\pi \in E(P, \varphi)$ for all $P \in \mathcal{P}$.*

Let $E(\varphi) \subset \Pi(A)$ denotes the set of all such partitions.

Since we use Proposition 5.2 of Özkal-Sanver(2005), p.200 to support one of our main result, we restate it below. Moreover for the completeness of the thesis we rewrite its proof (with some slight changes) below.

Proposition 3.5 *:For any Pareto optimal rule φ, the grand coalition A is the unique partition which is strong φ-efficient.*

Proof. *Let φ be a Pareto optimal rule. Let $\{A\}$ be the grand coalition. Suppose $\{A\} \notin E(\varphi)$. This means that there are $P \in \mathcal{P}$ and $\pi \in \Pi$ such that for all $i \in A$, $\pi \ R_i^{\varphi} \ \{A\}$ and for some $j \in A$, $\pi \ P_j^{\varphi} \ \{A\}$. Then, φ is not Pareto optimal.*

Now we want to show that there is no other $\pi \in E(\varphi)$. Let π be a partition containing at least two coalitions S^k and S^l such that $S^k \cap M \neq \emptyset$ and $S^l \cap W \neq \emptyset$. Let P be such that for some $i \in S^k$ and some $j \in S^l$, and all $k \in A \setminus \{i, j\}$

P_i	P_j	P_k
j	i	k
$\ldots$	$\ldots$	$\ldots$

Let $\pi' = \pi \setminus \{S^k, S^l\} \cup \{i, j\} \cup \{S^k \setminus \{i\}\} \cup \{S^l \setminus \{j\}\}$. At π', each agent in A is at least as well off as at π, and i and j are better off, showing that $\pi \notin E(\varphi)$. ■

Proposition 3.6 *For any stable rule φ, the grand coalition A is the unique partition which is strong coalitional $\varphi-$stable under the code FE-FX.*

Proof. Let φ be a stable rule and γ be FE-FX.

First we will show that $A \in C(\gamma, \varphi)$ then $\{A\} = C(\gamma, \varphi)$. Suppose $A \notin C(\gamma, \varphi)$. Then there exists P such that either A is not individually $\varphi-$stable under FE-FX at this P or there exists a coalition T that forms a coalitional deviation at P.

Case1: A is not individually $\varphi-$stable under FE-FX at this P, then there is $i \in A$ such that $F_{i,A,\emptyset}(A) P_i^{\varphi} A$. This means that agent i is better by leaving A and being alone. But this contradicts the fact that φ is a stable rule indeed its individual rationality.

Case2: There is a coalition T that forms a coalitional deviation at P by ρ from π .For any $i \in T$, define $j_i = \varphi(A, P, F_{T,\rho}(A))(i)$. Then for any $i \in T$ we have $j_i P_i \varphi(A, P, A)(i)$ and $i P_{j_i} \varphi(A, P, A)(j_i)$, which means that for each $i \in T$, $(i, \varphi(A, P, F_{T,\rho}(A))(i)) \in T \times T$ is blocking pair in the grand coalition.

To prove its uniqueness, we refer to Proposition 5.1 of Özkal-Sanver (2005),p.199, stating that "For all individually rational and Pareto optimal rules φ, the grand coalition is the unique partition which is φ-stable under FE-FX". But for the completeness we give its proof below.

Let π be a partition containing at least two coalitions S^k and S^l such that

$S^k \cap M \neq \emptyset$ and $S^l \cap W \neq \emptyset$. Let P be such that for some $i \in S^k$ and some $j \in S^l$, and all $k \in A \setminus \{i,j\}$

P_i	P_j	P_k
j	i	k
$\ldots$	$\ldots$	$\ldots$

Then, we have $F_{i,S^k,S^l}(\pi)\ P_i^{\varphi}\ \pi$ and there is no $j \in \gamma_{i,S^k,S^l}$ with $\pi\ P_j^{\varphi}\ F_{i,S^k,S^l}(\pi)$. But this contradicts individual $\varphi-$stability of π. Thus, $\{A\} = C(\gamma,\varphi)$. ■

Theorem 3.1 *For any stable matching rule φ, under FE-FX the set of strong coalitional $\varphi-$stable partitions equals to the set of strong $\varphi-$efficient partitions.*

Proof. As a corollary to Proposition 3.5 and Proposition 3.6. ■

Stability of a matching rule is crucial here. If we weaken the rule to Pareto optimal and individually rational ones, then theorem 3.1 is no more valid.

Example 3.1 *The grand coalition is not strong coalitional $\varphi-$stable under FE-FX for some Pareto optimal and individually rational rule φ.*

Let φ_D be individually rational serial men-dictatorship. Let $A = \{m_1, m_2, m_3, w_1, w_2, w_3\}$. And P as follows:

P_{m_1}	P_{m_2}	P_{m_3}
w_1	w_1	w_3
w_2	w_2	w_2
w_3	w_3	w_1

P_{w_1}	P_{w_2}	P_{w_3}
m_2	m_2	.....
m_1	w_2	.....
w_1	.....	w_3

Denote $\mu = \varphi_D(A, P, A)$.

Let men be placed in some order m_1, m_2, m_3. φ_D *matches* m_1 *to his first choice if his first choice prefer him to being alone(in other words, respecting individual stability),* m_2 *to his first choice of possible mates remaining after* $\mu(m_1)$ *removed from the society by respecting individual stability and* m_3 *to his first choice of possible mates remaining after* $\mu(m_2)$ *removed from the society again by respecting individual stability. At the end, the outcome will be* $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. *But* $\{m_2, w_1\}$ *form a coalitional deviation by* $\rho = \{\{m_2, w_1\}\}$. *Hence* $A \notin C(\gamma, \varphi_D)$.

Proposition 3.7 *For any rule* φ*, any strong* $\varphi-$*efficient partition* π *is strong coalitional* $\varphi-$*stable under AE-AX.*

Proof. Let φ be a matching rule and γ be AE-AX.

Take any $\pi \in E(\varphi)$. Suppose $\pi \notin C(\gamma, \varphi)$ Then there exists P such that $\pi \notin C(\gamma, (P, \varphi))$. But by proposition 3.3, $\pi \notin E(P, \varphi)$ contradicting $\pi \in E(\varphi)$.

■

Proposition 3.8 *For any Pareto optimal rule* φ*, the grand coalition* A *is* *strong coalitional $\varphi-$stable partition under AE-AX.*

Proof. Directly from Proposition 3.5 and Proposition 3.7. ■

Proposition 3.9 *For any Pareto optimal and individually rational rule φ, any strong coalitonal $\varphi-$stable partition π under AE-AX is strong φ-efficient.*

Proof. Let φ be a Pareto optimal and individually rational rule and γ be AE-AX.

Take any $\pi \in C(\gamma, \varphi)$. Suppose $\pi \notin E(\varphi)$. Then there exists P such that $\pi \notin E(P, \varphi)$. Hence there exists $\pi^{'}$ such that for all $i \in A$ we have $\pi' R_i^{\varphi} \pi$ and for some $j \in A$, $\pi' P_j^{\varphi} \pi$.Denote $\varphi(A, P, \pi)$ and $\varphi(A, P, \pi^{'})$ by μ and μ' respectively. Consider a new preference profile P^* such that for all $i \in A$,we have $\mu'(i) P^* j$ for all $j \in A(i)$.

Claim: There exists $j \in A$ such that $\mu(j) \in S^k$, $\mu'(j) \in S^l$ and $k \neq l$.

Proof. Suppose for a contradiction that for all $i \in A$, there is $k \in K$ such that $\mu(i), \mu'(i) \in S^k$. There exists $j \in A$ such that $\mu'(j) P_j \mu(j)$. Consider the coalition that j belongs say S^j. Then for all $i \in S^j$, $\mu(i)$ and $\mu'(i)$ are in S^j. Since π is not $\varphi-$efficient we have $\mu'(i) R_i \mu(i)$. Then in S^j, μ' dominates μ at P, contradicting that φ is Pareto optimal rule. ■

Let $T = \{i \in A : \mu(j) \in S^k, \mu'(j) \in S^l \text{and } k \neq l\}$ and denote $T = \{i_1, ..., i_s\}$.Then consider the following $\rho-$move:

$\rho = \{\{i_1, \mu'(i_s)\}, ..., \{i_s, \mu'(i_s)\}\} \in \Pi(T)$.

Since φ is Pareto optimal and individually rational, for all $j \in \{1, ..., s\}$ we have $\varphi(A, P, F_{T,\rho}(\pi))(i_j) = \mu'(i_j)$.

Then T forms a coalitional deviation by ρ from π at P^*.

Thus $\pi \notin C(\gamma, (P^*, \varphi))$, indeed $\pi \notin C(\gamma, \varphi)$. ■

Theorem 3.2 *For any Pareto optimal rule and individually rational rule φ, under AE-AX the set of strong coalitional $\varphi-$stable partitions equals to the set of strong $\varphi-$efficient partitions.*

Proof. As a corollary to Proposition 3.7 and Proposition 3.9. ■

4 Discussion and Conclusion.

The aim of this thesis is to redefine stability of a partition by allowing coalitional moves an then study under which axioms and for what rules we have Invisible Hand Theorem and Decentralization result.

First we define coalitional stability under many membership property rights. And then we analyze relationship between coalitional stability under these membership property rights and efficiency of partition. Under FE-FX, coalitional stability implies efficiency for all pareto optimal and individually rational rules. But for the converse we have a negative result. For the code AE-AX now we have Decentralization result but now Invisible Hand theorem no longer holds. More precisely, every efficient partition is coalitional stable under AE-AX for all rules. But coalitional stability of a partition under AE-AX at a preference profile does not imply its efficiency of a partition at this preference profile.

Since we do not have Invisible Hand Theorem and Decentralization result at the same time, we change definitions of efficiency and coalitional stability. We define strong versions. And our results are: for all stable rules , there is a unique partition, the grand coalition, that is strong coalitional stable under FE-FX and efficient. So we have an Invisible Hand Theorem and as well as a Decentralization Theorem. Stability is important for these results. Under some individually rational and Pareto optimal rule φ, for example under a modified version of the serial men-dictatorship rule imposing individual rationality, the grand coalition is not strong coalitional $\varphi-$stable, hence there exists no strong coalitional $\varphi-$stable partition.

Whenever we seek the approval of members of coalitions from the agent exits and which he/she desires to enter (AE-AX), when the rule is Pareto optimal and individually rational, again we have both these results.

The other way that we look the problem is to change our codes, in words axiom of mate approval when you leave a coalition and enter another coalition you have to get approval of your old and new mate.

For further research all these result can be extended to the world of many to one matching which can be seen as a workers to firms. To have Decentralization result and Invisible Hand Theorem we have to change the definition of coalitional stability.

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