SOCIAL CHOICE WITHOUT THE PARETO PRINCIPLE UNDER WEAK INDEPENDENCE

by

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ABSTRACT

In this thesis, it is shown that the class of social welfare functions that satisfy a weak independence condition identified by Campbell (1976) and Baigent (1987) is fairly rich and freed of a power concentration on a single individual. This positive result prevails when a weak Pareto condition is imposed. Hence, the impossibility of Arrow (1951) can be overcame by simultaneously weakening the independence and Pareto conditions. Moreover, under weak independence, an impossibility of the Wilson (1972) type vanishes.

ÖZET

Bu tezde, Campbell (1976) ve Baigent (1987) tarafından tanımlanmış zayıf bağımsızlık koşulunu sağlayan sosyal refah fonksiyonları sınıfının oldukça zengin ve tek bir birey üzerindeki güç yoğunluğundan muaf olduğu gösterilmektedir. Bu pozitif sonuç, zayıf Pareto koşulu uygulandığında da geçerli olur. Sonuç olarak, bağımsızlık ve Pareto koşullarının eş zamanlı olarak zayıflatılmasıyla Arrow (1951) imkansızlığının üstesinden gelinebilir. Bunun yanı sıra, zayıf bağımsızlık altında, Wilson (1972) türü imkansızlık da kaybolur.

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CONTENTS

CHAPTER 1

Introduction

In this thesis, the preference aggregation problem in a society which confronts at least three alternatives is considered. A *Social Welfare Function* (SWF) is a mapping which assigns a social ranking to any logically possible profile of individual rankings. A SWF is *independent of irrelevant alternatives* (IIA) if the social ranking of any pair of alternatives depends only on individuals' preferences over that pair. Since the seminal work of Arrow (1951), it is known that IIA and Pareto optimality are incompatible, unless one is ready to admit dictatorial SWFs.

The Arrovian impossibility is remarkably robust against weakenings of IIA.[1] For example, letting k stand for the number of alternatives that the society confronts, Blau (1971) proposes the concept of *m-ary independence* for any integer between 2 and k. A SWF is *m-ary independent* if the social ranking of any set of alternatives with cardinality m depends only on individuals' preferences over that set. Clearly, when $m = 2$, m-ary independence coincides with IIA. Moreover, every SWF trivially satisfies m-ary independence when $m = k$. It is also straightforward to see that m-ary independence implies n-ary independence when $m < n$. Nevertheless, Blau (1971) shows that m-ary independence implies n-ary independence when $n < m < k$ as well. Thus, weakening IIA by imposing independence over

[1] In fact, it is robust against weakenings of other conditions as well: Wilson (1972) shows that the Arrovian impossibility essentially prevails when the Pareto condition is not used. Ozdemir and Sanver (2007) identify severely restricted domains which exhibit the Arrovian impossibility.

sets with cardinality more than two is not sufficient to escape from the Arrovian impossibility, unless independence is imposed over the whole set of alternatives - a condition which is satisfied by the definition of a SWF.

Campbell and Kelly (2000a, 2007) further weaken m-ary independence by requiring that the social preference over a pair of alternatives depends only on individuals' preferences over some proper subset of the set of available alternatives. This condition, which they call *independence of some alternatives* (ISA) is considerably weak. As a result, non-dictatorial SWF that satisfy Pareto optimality and ISA -such as the "gateau rules" identified by Campbell and Kelly (2000a)- do exist. On the other hand, "gateau rules" fail neutrality and as Campbell and Kelly (2007) later show, within the Arrovian framework, an extremely weaker version of ISA disallows both anonymity and neutrality.

Denicolo (1998) identifies a condition called *relational independent deciseveness* (RID). He shows that although IIA implies RID, the Arrovian impossibility prevails when IIA is replaced by RID.

Campbell (1976) proposes a weakening of IIA which requires that the social decision between a pair of alternatives cannot be reversed at two distinct preference profiles that admit the same individual preferences over that pair. We refer to this condition as *quasi IIA*.[2] Baigent (1987) shows that every Pareto optimal and quasi IIA SWF must be dictatorial in a sense which is close to the Arrovian meaning of the concept - hence a version of the Arrovian impossibility.[3]

[2] See Campbell (1976) for a discussion of the computational advantages of quasi IIA. Note that when social indifference is not allowed, IIA and quasi IIA are equivalent.

[3] Baigent (1987) claims this impossibility in an environment with at least three alternatives.

Results corresponding to the effects of weakening IIA on the Arrovian impossibility is presented as a negative nature in the literature. In order to contribute a positive result, this thesis is conducted. Under the weakening proposed by Baigent (1987), it is shown that the Arrovian impossibility can be surpassed by avoiding the Pareto condition: The class of quasi IIA SWFs is described and shown as being a fairly large class which is not restricted to SWFs where the decision power is concentrated on one given individual. Actually, SWFs included in this class are both anonymous and neutral. In case of imposing a weak version of the Pareto condition, this positive result holds.

According to the findings of this thesis, it is established that the tension between quasi IIA and the transitivity of the social outcome does not exist. Hence, Wilson (1972) and Barberà (2003)'s results which states that the Pareto condition has little impact on the Arrovian impossibility which is essentially a tension between IIA and the range restriction imposed over SWFs depart from our result.

Chapter 2 presents the basic notions.

Chapter 3 reviews the literature.

Chapter 4 states our results.

Chapter 5 makes some concluding remarks.

Nevertheless, Campbell and Kelly (2000b) show the existence of Pareto optimal and quasi IIA SWF when there are precisely three alternatives. They also show that the impossibility announced by Baigent (1987) prevails when there are at least four alternatives and even under restricted domains.

CHAPTER 2

Basic Notions

We consider a finite set of individuals N with $\#N \geq 2$, confronting a finite set of alternatives A with $\#A \geq 3$.

Definition 1 *An aggregation rule is a mapping* $f : \Pi^N \to \Theta$ *where* Π *is the set of complete, transitive and antisymmetric binary relations over* A *while* Θ *is the set of complete binary relations over* A.

We conceive $P_i \in \Pi$ as the *preference* of $i \in N$ over A.[1] We write $P = (P_1, ..., P_{\#N}) \in \Pi^N$ for a *preference profile* and $f(P) \in \Theta$ reflects the social preference obtained by the aggregation of P through f. Note that $f(P)$ need not be transitive. Moreover, as $f(P)$ need not be antisymmetric, we write $f^*(P)$ for its strict counterpart.[2]

Definition 2 *An aggregation rule* f *is independent of irrelevant alternatives (IIA) iff given any distinct* $x, y \in A$ *and any* $P, P' \in \Pi^N$ *with* $x \ P_i \ y \Longleftrightarrow x \ P_i' \ y \ \forall i \in N$, *we have* $x \ f(P) \ y \Longleftrightarrow x \ f(P') \ y$.

We write Φ for the set of aggregation rules which satisfy IIA. For any distinct $x, y \in A$, let $\{\substack{x \\ y}, \substack{y \\ x}, xy\}$ be the set of possible preferences over $\{x, y\}$.[3]

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[1] As usual, for any distinct $x, y \in A$, we intepret $x \ P_i \ y$ as x being preferred to y in view of i.

[2] So for any distinct $x, y \in A$, we have $x \ f^*(P) \ y$ whenever $x \ f(P) \ y$ and not $y \ f(P) \ x$.

[3] We interpret $\substack{x \\ y}$ as x being preferred to y; $\substack{y \\ x}$ as y being preferred to x; and xy as indifference between x and y.

Definition 3 *An elementary aggregation rule is a mapping* $f_{\{x,y\}} : \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^N \rightarrow \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}, xy\}$.

Any family $f = \{f_{\{x,y\}}\}$ of elementary aggregation rules indexed over all possible distinct pairs $x, y \in A$ induces an aggregation rule as follows: For each $P \in \Pi^N$ and each $x, y \in A$, let $x\ f(P)\ y \iff f_{\{x,y\}}(P^{\{x,y\}}) \in \{\genfrac{}{}{0pt}{}{x}{y}, xy\}$ where $P^{\{x,y\}} \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^N$ is the restriction of $P \in \Pi^N$ over $\{x, y\}$.[4] Note that $f = \{f_{\{x,y\}}\} \in \Phi$. Moreover, any $f \in \Phi$ can be expressed in terms of a family $\{f_{\{x,y\}}\} = f$ of elementary aggregation rules.

Let $\Re$ be the set of complete and transitive binary relations over A. A *Social Welfare Function* (SWF) is an aggregation rule whose range is restricted to $\Re$.

Definition 4 *A SWF* $\alpha : \Pi^N \rightarrow \Re$ *is Pareto optimal iff given any distinct* $x, y \in A$ *and any* $P \in \Pi^N$ *with* $x\ P_i\ y\ \forall i \in N$, *we have* $x\ \alpha^*(P)\ y$.

Definition 5 *A SWF* $\alpha : \Pi^N \rightarrow \Re$ *is dictatorial iff* $\exists i \in N$ *such that* $x\ P_i\ y$ *implies* $x\ \alpha^*(P)\ y\ \forall P \in \Pi^N, \forall x, y \in A$.

The Arrovian impossibility, as we consider, announces that a SWF $\alpha : \Pi^N \rightarrow \Re$ is Pareto optimal and IIA if and only if is α dictatorial.

[4] So for any $i \in N$, we have $P_i^{\{x,y\}} = \genfrac{}{}{0pt}{}{x}{y} \iff x\ P_i\ y$.

CHAPTER 3

Literature Review

3.1 Arrovian Impossibility Theorem with Weak Independence

The first attempt for weakening the IIA condition comes from Blau (1971) where the consequences of weakening IIA is stated, i.e whether it is still inconsistent with the other conditions in Arrow's Theorem or not. We will present here precise definitions and theorems that Blau (1971) states and consider the results that is reached.

Definition 6 *A SWF a is m-ary independent if* $\forall X \subseteq A$ *with* $\#X = m$ *where* $m < \#A$ *and* $\forall R, Q \in \Re^N$ *with* $R^X = Q^X$*, we have* $\alpha(R)^X = \alpha(Q)^X$*.*

As it is clear from the definition, when $m = 2$, it is the usual IIA condition. Also, Blau (1971) calls it as *binary*. Similarly, when $m = 3$, it is called as *ternary*. Here is the first theorem that Blau (1971) states;

Theorem 7 *Let* $\#A = 4$*. Then, ternary implies binary.*

Proof. Take any SWF a which satisfies ternary. Let $A = \{a, b, c, d\}$. Take any two profiles $R, Q \in \Re^N$ with $R^{\{a,b\}} = Q^{\{a,b\}}$. If there is an other alternative, say c, with $R^{\{a,b,c\}} = Q^{\{a,b,c\}}$, then $\alpha(R)^{\{a,b,c\}} = \alpha(Q)^{\{a,b,c\}}$ since a satisfies ternary. Then, $\alpha(R)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$ by deleting c. Hence, α satisfies binary. However, it is not always the case of an existing alternative like c. Therefore, we need to find a third profile S such that $R^{\{a,b,c\}} = S^{\{a,b,c\}}$ and $Q^{\{a,b,d\}} = S^{\{a,b,d\}}$.

Claim: Let R be an ordering on $\{a, b, c\}$ and Q be an ordering on $\{a, b, d\}$ with $R^{\{a,b\}} = Q^{\{a,b\}}$. Then there exist an ordering S on $\{a, b, c, d\}$ such that $R^{\{a,b,c\}} = S^{\{a,b,c\}}$ and $Q^{\{a,b,d\}} = S^{\{a,b,d\}}$.

Proof: Construct S as follows; order $\{a, b, c\}$ same as on R. Then insert d in a way that ordering R is same with the ordering Q on $\{a, b, d\}$. Since this is always possible, the proof is done.

Then, $\alpha(R)^{\{a,b,c\}} = \alpha(S)^{\{a,b,c\}}$ and $\alpha(Q)^{\{a,b,d\}} = \alpha(S)^{\{a,b,d\}}$ since α satisfies ternary. By deleting c, we have $\alpha(R)^{\{a,b\}} = \alpha(S)^{\{a,b\}}$ and similarly, by deleting d, we have $\alpha(Q)^{\{a,b\}} = \alpha(S)^{\{a,b\}}$ which means that $\alpha(R)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$. Hence, α satisfies binary. ■

Theorem 8 *Let $\#A \geq 4$. Then, ternary implies binary.*

Proof. Take any SWF α which satisfies ternary and take any $X \subseteq A$ with $\#X = 4$. Also for any $\{a, b\} \in X$, take any two profiles $R, Q \in \Re^N$ with $R^{\{a,b\}} = Q^{\{a,b\}}$.

Claim1: α satisfies quaternary.

Proof: Let $X = \{a, b, c, d\}$. Also take any $R, Q \in \Re^N$ with $R^{\{a,b,c,d\}} = Q^{\{a,b,c,d\}}$. Since X has 4 elements, then there are 6 doubletons. Take any doubleton, say $\{a, b\}$. Then, $R^{\{a,b,c\}} = Q^{\{a,b,c\}}$. Since α satisfies ternary, then $\alpha(R)^{\{a,b,c\}} = \alpha(Q)^{\{a,b,c\}}$. Hence, by deleting c, we get $\alpha(R)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$. Therefore, $\alpha(R)^{\{a,b,c,d\}} = \alpha(Q)^{\{a,b,c,d\}}$ which means that α satisfies quarternary.

After showing that α satisfies quarternary, we are sure that α induces a SWF α' on X. Now, we want to show that α' satisfies ternary.

Claim2: α' satisfies ternary.

Proof: Let T be a triple of alternatives in X. Take any $R, Q \in \Re^N$ restricted to X with $R^T = Q^T$. Since α is ternary, $\alpha(R)^T = \alpha(Q)^T$. Moreover, since $\alpha(R)^T = \alpha'(R^X)^T$ and $\alpha(Q)^T = \alpha'(Q^X)^T$, we have $\alpha'(R^X)^T = \alpha'(Q^X)^T$. Hence, α' satisfies ternary.

Next step is to show that α' satisfies binary.

Claim3: α' satisfies binary.

Proof: By using Theorem 7, α' satisfies binary.

Now, we get $\alpha'(R^X)^{\{a,b\}} = \alpha'(Q^X)^{\{a,b\}}$ since α' satisfies binary. Also, by definition of α', we have $\alpha(R)^X = \alpha'(R^X)$ and $\alpha(Q)^X = \alpha'(Q^X)$. If we restrict these orderings on $\{a, b\}$, we get $\alpha(R)^{\{a,b\}} = \alpha'(R^X)^{\{a,b\}}$ and $\alpha(Q)^{\{a,b\}} = \alpha'(Q^X)^{\{a,b\}}$. Since $\alpha'(R^X)^{\{a,b\}} = \alpha'(Q^X)^{\{a,b\}}$, we have $\alpha(R)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$. Therefore, α satisfies binary. ■

Next theorem is a generalization of Theorem 7.

Theorem 9 *Let $\#A = m + 1$ where $m \geq 3$. Then, m-ary implies $(m - 1)$-ary.*

Proof. Take any SWF α which satisfies m-ary. Proof of theorem 7 is the case where $m = 3$. So, suppose $m > 3$. Take any $X \subseteq A$ with $\#X = m - 1$ and take any two profiles $R, Q \in \Re^N$ with $R^X = Q^X$. There are only two elements in $A - X$. Let's denote them as c and d. Let $C = X \cup \{c\}$ and $D = X \cup \{d\}$. Now, we need to find a profile S such that $R^C = S^C$ and $Q^D = S^D$.

Claim: Let R be an ordering on C and Q be an ordering on D with $R^X = Q^X$. Then there exists an ordering S on A such that $R^C = S^C$ and $Q^D = S^D$.

Proof: Construct S as follows; order C same as on R. Then insert d in a way that ordering R is same with the ordering Q on D. Since this is always possible, the proof is done.

Then, $\alpha(R)^C = \alpha(S)^C$ and $\alpha(Q)^D = \alpha(S)^D$ since α satisfies m-ary. By deleting c, we have $\alpha(R)^X = \alpha(S)^X$ and similarly, by deleting d, we have $\alpha(Q)^X = \alpha(S)^X$ which means that $\alpha(R)^X = \alpha(Q)^X$. Hence, α satisfies $(m-1)$-ary. ■

Similarly, next theorem is a generalization of Theorem 8.

Theorem 10 *Let $\#A \geq m+1$ where $m \geq 3$. Then, m-ary implies $(m-1)$-ary.*

Proof. Take any SWF α which satisfies m-ary. Proof of theorem 8 is the case where $m = 3$. So, suppose $m > 3$. Take any $X \subseteq A$ with $\#X = m-1$ and take any two profiles $R, Q \in \Re^N$ with $R^X = Q^X$. Also take any two distinct elements $c, d \in A - X$. Let $K = X \cup \{c, d\}$. As we have shown in proof of theorem 8, ternary implies quaternary, similarly, m-ary implies $(m+1)$-ary. Thus, α induces a SWF α' on K. Also, α' preserves the m-ary property of α. So, by theorem3, α' satisfies $(m-1)$-ary. Also, by definition of α', we have $\alpha(R)^K = \alpha'(R^K)$ and $\alpha(Q)^K = \alpha'(Q^K)$. If we restrict these orderings on X, we get $\alpha(R)^X = \alpha'(R^K)^X$ and $\alpha(Q)^X = \alpha'(Q^K)^X$. Since $\alpha'(R^K)^X = \alpha'(Q^K)^X$, we have $\alpha(R)^X = \alpha(Q)^X$. Therefore, α satisfies $(m-1)$-ary. ■

Now, we are ready to present the main result of Blau (1971).

Theorem 11 *Let $\#A > m$ and $1 < n < m$. Then, m-ary implies n-ary.*

Proof of the theorem is an easy consequence of the preceeding theorems.

3.2 Contribution of Campbell and Kelly

Campbell and Kelly (2000a, 2007) follow a similar path as Blau (1971). As we already know, m-ary independence cannot avoid the impossibility. What Campbell and Kelly (2000a, 2007) did is to weaken the m-ary independence. Here are the precise definitions and theorems that is stated.

Definition 12 *A SWF α satisfies weak unanimity if for any $P \in \Pi^N$ with $x\ P_i\ y$ $\forall i \in N, \forall y \in A - \{x\}$, we have $x\ \alpha^*(P)\ y, \forall y \in A - \{x\}$.*

Weak unanimity is also a weaker version of Pareto optimality.

Definition 13 *A SWF α is anonymous if given any permutation σ of N, and given any $P \in \Pi^N$, $\sigma(P)$ is also in Π^N and $\alpha(\sigma(P)) = \alpha(P)$.*

Definition 14 *A SWF α is neutral if given any permutation τ of A, and given any $P \in \Pi^N$, $\tau(P)$ is also in Π^N and $\alpha(\tau(P)) = \alpha(P)$.*

Definition 15 *A SWF α satisfies independence of some alternatives (ISA) if given any pair of alternatives x and y in A, there exist a proper subset $X \subset A$ such that $\forall P, Q \in \Pi^N$ with $P^X = Q^X$, we have $\alpha(P)^{\{x,y\}} = \alpha(Q)^{\{x,y\}}$.*

Definition 16 *A SWF α satisfies weakest independence if for at least one pair of alternatives x and y in A, there exists a proper subset $X \subset A$ such that $\forall P, Q \in \Pi^N$ with $P^X = Q^X$, we have $\alpha(P)^{\{x,y\}} = \alpha(Q)^{\{x,y\}}$.*

Definition 17 *A subset $X \subseteq A$ is said to be sufficient for $\{x, y\}$ if $\forall P, Q \in \Pi^N$ with $P^X = Q^X$, we have $\alpha(P)^{\{x,y\}} = \alpha(Q)^{\{x,y\}}$.*

Theorem 18 *Let A is finite and $\#A \geq 3$. Then, there does not exist a SWF α satisfying weak unanimity, nondictatorship, weakest independence and neutrality.*

In order to prove theorem 18, we need to prove two lemmas.

Lemma 19 *Let A is finite and $\#A \geq 3$. If a SWF α satisfies weakest independence and neutrality, then it also satisfies ISA.*

Proof. Take any SWF α which satisfies weakest independence and neutrality. Since α satisfies weakest independence, $\exists a, b \in A$ and $\exists B \subset A$ such that $\forall P, Q \in \Pi^N$ with $P^B = Q^B$, we have $\alpha(P)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$.

Claim: $\exists\ c \in A$ such that $A - \{c\}$ is sufficient for $\{a, b\}$.

Proof: Since B is a proper subset of A, then there is at least one alternative in A which does not belong to B. So let's denote it as c. Clearly, $B \subseteq A - \{c\}$ and $A - \{c\}$ is sufficient for $\{a, b\}$ since B is sufficient for $\{a, b\}$.

Given any pair $x, y \in A$, let τ be a permutation with $\tau(a) = x$, $\tau(b) = y$ and $\tau(c) = z$.

Claim2: $A - \{z\}$ is sufficient for $\{x, y\}$.

Proof: Suppose not. Then, there exist profiles $P, Q \in \Pi^N$ with $P^{A-\{z\}} = Q^{A-\{z\}}$ such that $x\alpha(P)^{\{x,y\}}y$ and $y\alpha^*(Q)^{\{x,y\}}x$. Then consider the profiles $\tau^{-1}(P)$ and $\tau^{-1}(Q)$. Since $P^{A-\{z\}} = Q^{A-\{z\}}$, then $\tau^{-1}(P)^{A-\{c\}} = \tau^{-1}(Q)^{A-\{c\}}$. By neutrality, $a\alpha(\tau^{-1}(P))^{\{a,b\}}b$ and $b\alpha^*(\tau^{-1}(Q))^{\{a,b\}}a$ which violates the sufficiency of $A - \{c\}$ for $\{a, b\}$.

Therefore, α satisfies ISA. ■

Lemma 20 *Let A is finite and $\#A \geq 3$. If a SWF α satisfies weak unanimity, weakest independence and neutrality, then it also satisfies IIA.*

Proof. Take any SWF α which satisfies weak unanimity, weakest independence and neutrality. Since α satisfies weakest independence and neutrality, then it also satisfies ISA. Similarly, $\exists\ c \in A$ such that $A - \{c\}$ is sufficient for $\{a, b\}$. Also we know by Campbell and Kelly (2000a) that intersection of two sufficient sets for $\{a, b\}$ is also sufficient for $\{a, b\}$. then, by finiteness of A, there is a smallest set which is sufficient for $\{a, b\}$ and it is denoted by $\varphi(\{a, b\})$. So, $\varphi(\{a, b\}) \subseteq A - \{c\}$.

Claim: $\{a, b\} \subseteq \varphi(\{a, b\})$.

Proof: Suppose not. Then, there are two cases;

case1: Both a and b is not in $\varphi(\{a, b\})$.

Then, consider the profile $P \in \Pi^N$ with $aP_ibP_ic, \forall i \in N, \forall c \in A - \{a, b\}$. Let $Q \in \Pi^N$ be another profile obtained from P by interchanging a and b in each individual profile. So, we have $P^{A-\{a,b\}} = Q^{A-\{a,b\}}$. Since $\varphi(\{a, b\}) \subseteq A - \{a, b\}$, then $A - \{a, b\}$ is sufficient for $\{a, b\}$. Hence, $\alpha(P)^{\{a,b\}} = \alpha(Q)^{\{a,b\}}$. However, by weak unanimity, $a\ \alpha(P)^{\{a,b\}}b$ and $b\ \alpha^*(Q)^{\{a,b\}}a$ which leads to a contradiction. Therefore case1 does not hold.

case2: Either a or b is not in $\varphi(\{a, b\})$.

If $a \in \varphi(\{a, b\})$ and $b \notin \varphi(\{a, b\})$, then by neutrality it leads to a contradiction. Hence, case2 does not hold as well.

Therefore, we have $\{a, b\} \subseteq \varphi(\{a, b\})$. So, c cannot be a or b. Then, by neutrality, for every $c \in A - \{a, b\}$, $A - \{c\}$ is sufficient for $\{a, b\}$. By finiteness of A and repeated application of the intersection principle,

$$\{a, b\} = \bigcap_{c \in A - \{a,b\}} A - \{c\}$$

is sufficient for $\{a, b\}$. Therefore, α satisfies IIA. ■

Proof. of Theorem 18. By Lemma 20, α satisfies IIA. By IIA and weak unanimity, α satisfies Pareto Optimality. But, by Arrow's Theorem, α cannot satisfy nondictatorship. ■

3.3 Independent Decisiveness and the Arrow Theorem

Denicolo (1998) shows that Arrovian impossibility still remains even if a weaker condition is imposed instead of IIA. Here are the precise definitions and the theorem.

Definition 21 *A subset $K \subseteq N$ is said to be locally decisive over the ordered pair (x, y) if for any profile $R \in \Re^N$ with $xP_j y$, $\forall j \in K$, we have $x\ \alpha^*(R)\ y$.*

Definition 22 *A subset $K \subseteq N$ is said to be decisive if it is locally decisive over every ordered pair (x, y).*

The following definition is stated by Baigent (1996).

Definition 23 *A subset $K \subseteq N$ can enforce x against y if for any profile $R \in \Re^N$ with $xP_j y$, $\forall j \in K$, there exists a profile $R' \in \Re^N$ with $R^{\{x,y\}} = R'^{\{x,y\}}$ such that $x\ \alpha^*(R')\ y$.*

Definition 24 *A SWF a satisfies relational independent decisiveness if $\forall x, y \in A$, $K \subseteq N$ can enforce x against y, we have K is locally decisive over $\{x, y\}$.*

Theorem 25 *Let A is finite and $\#A \geq 3$. Then, there does not exist a SWF α satisfying relational independent decisiveness, weak Pareto principle and nondictatorship.*

Lemma 26 *For any $K \subseteq N$, if there exist $a, b \in A$ such that K is locally decisive over $\{a, b\}$, then K is decisive.*

Proof. Take any SWF α which satisfies relational independent decisiveness and weak Pareto principle. Also take any $K \subseteq N$. Suppose $\exists a, b \in A$ such that K is locally decisive over $\{a, b\}$. Then, take any profile $R \in \Re^N$ with aP_jb and bP_jc $\forall j \in K$ and bP_ic, $\forall i \in N - K$ where c is any different alternative from a and b in A. By local decisiveness of K over $\{a, b\}$, we have $a\ \alpha^*(R)\ b$ and by the weak Pareto principle, we have $b\ \alpha^*(R)\ c$. Then, by transitivity of $\alpha(R)$, we have a $\alpha^*(R)\ c$. Hence, K can enforce a against c. By relational independent decisiveness, K is locally decisive over $\{a, c\}$. Therefore, K is decisive. ■

Lemma 27 *Let K is decisive and $\#K > 1$. Then, there exists a proper subset of K which is also decisive.*

Proof. Partition K into K_1 and K_2. Take any profile $R \in \Re^N$ with aP_ib and aP_ic $\forall i \in K_1$ and aP_ib and cP_ib $\forall i \in K_2$. Since K is decisive, $a\ \alpha^*(R)\ b$. Now, suppose $a\ \alpha^*(R)\ c$. Then, K_1 can enforce a against c. Otherwise, $\exists\ R' \in \Re^N$ with $aP_i'c$ $\forall i \in K_1$ such that for every $R'' \in \Re^N$ with $R'^{\{a,c\}} = R''^{\{a,c\}}$, we have $c\ \alpha(R')$ a. Since $R^{\{a,c\}} = R'^{\{a,c\}}$ for a suitable choice of R, we have $c\ \alpha(R)\ a$ which leads to a contradiction. By relational independent decisiveness and lemma 26, K_1 is decisive.

Now, consider the case $c\ \alpha(R)\ a$. By transitivity, $c\ \alpha^*(R)\ b$. Then, by the same reasoning, K_2 can enforce c against a. By relational independent decisiveness and lemma 26, K_2 is decisive. Hence, either K_1 or K_2 is decisive. ■

Proof. of the Theorem 25 By the weak Pareto principle, N is decisive. Since it is finite, there exists an individual that must be decisive by iterated application of Lemma 27. ■

3.4 Contribution of Baigent

Another way of weakening IIA is the following; Given any two profiles0 where the individual orderings are the same for two alternatives, social ordering for these alternatives cannot be reversed at those profiles. It is first used by Campbell (1976) and further that Baigent (1987) replaces IIA with these weaker version in Arrow's theorem and reaches a weaker version of Dictatoriality. However, Campbell and Kelly (2000b) state that Baigent (1987) result fails when there are three alternatives and they show that the result holds for at least four alternatives. Here are the precise definitions and the theorem.

Definition 28 *An individual $i \in N$ is decisive over $\{x, y\}$ if for any profile $R \in \Re^N$ with xP_iy, we have $x\ \alpha^*(R)\ y$.*

Definition 29 *An individual $i \in N$ is semi decisive over $\{x, y\}$ if for any profile $R \in \Re^N$ with xP_iy, we have $x\ \alpha(R)\ y$.*

Definition 30 *A SWF α is weakly IIA iff given any distinct $x, y \in A$ and any $R, R' \in \Re^N$ with $x\ R_i\ y \Longleftrightarrow x\ R'_i\ y\ \forall i \in N$, we have $x\ \alpha^*(R)\ y \Rightarrow x\ \alpha(R')\ y$.*

Definition 31 *A SWF α is weakly dictatorial iff $\exists i \in N$ such that $x\ P_i\ y$ implies $x\ \alpha(R)\ y\ \forall R \in \Re^N, \forall x, y \in A$.*

Theorem 32 *Let A is finite and $\#A \geq 4$. If a SWF α satisfies weak Pareto optimality and weakly IIA, then it is weakly dictatorial.*

Lemma 33 *For any SWF α that satisfies weak Pareto optimality and weakly IIA, if $K \subseteq N$ is semi decisive over $\{x, y\}$, then it is also semi decisive over all pairs of alternatives.*

Proof. Take any $K \subseteq N$ and any profile $R \in \Re^N$ with xP_iy and yP_iz $\forall i \in K$ and yP_jz, $\forall j \in N - K$ where z is any different alternative from x and y in A. Suppose K is semi decisive over $\{x, y\}$. Then, $x\ \alpha(R)\ y$. By weak Pareto optimality, $y\ \alpha^*(R)\ z$ and by transitivity of $\alpha(R)$, $x\ \alpha^*(R)\ z$. Then, for any other profile $R' \in \Re^N$ with $R^{\{x,z\}} = R'^{\{x,z\}}$, we have $x\ \alpha(R)\ z$ by weakly IIA. Hence, K is semi decisive over $\{x, z\}$. Similarly, if K is semi decisive over $\{x, z\}$, then K is semi decisive over $\{x, y\}$. In general, semi decisiveness over $\{x, y\}$ can be extended to all pairs of alternatives. ■

Proof. of Theorem 32. By weak Pareto optimality, there exists a decisive subset of N and therefore it is a semi decisive subset. Since N is finite, there exist a smallest semi decisive subset $K' \subseteq N$. Suppose $\#K' > 1$. Take any $K \subseteq K'$ and any profile $R \in \Re^N$ with xP_iy and xP_iz $\forall i \in K$ and xP_jy and zP_jy, $\forall j \in K' - K$ and xP_ky, $\forall k \in N - K'$. By weak Pareto optimality, $x\alpha^*(R)y$. If $z\alpha^*(R)y$ then $K' - K$ is semi decisive over $\{y, z\}$ by weak IIA condition. Then, by lemma 33, $K' - K$ is semi decisive over all pairs which leads to a contradiction. Therefore, $y\alpha(R)z$. Then, by transitivity, $x\alpha^*(R)z$. But, by weak IIA, K is semi decisive over $\{x, z\}$ and by lemma 33, K is semi decisive over all pairs which leads to a contradiction. Hence, $\#K' = 1$. Therefore, there is a weak dictator. ■

CHAPTER 4

Social Choice without the Pareto Principle under Weak Independence

4.1 Introduction

In brief, the literature which explores the effects of weakening IIA on the Arrovian impossibility presents results of a negative nature. We revisit this literature in order to be contribute a positive result. We show that under the weakening proposed by Baigent (1987), the Arrovian impossibility can be surpassed by dropping the Pareto condition: We characterize the class of quasi IIA SWFs and show that this is a fairly large class which is not restricted to SWFs where the decision power is concentrated on one given individual. In fact, this class contains SWFs that are both anonymous and neutral. This positive result prevails when a weak version of the Pareto condition is imposed.

Our findings pave the way to surpass the impossibility of Arrow (1951). Moreover, we establish that there is no tension between quasi IIA and the transitivity of the social outcome. Thus, we also contrast the results of Wilson (1972) and Barberà (2003) who show that the Pareto condition has little impact on the Arrovian impossibility which is essentially a tension between IIA and the range restriction imposed over SWFs.

4.2 Results

Baigent (1987) proves a version of the Arrovian impossibility where IIA and dictatoriality are replaced by their following weaker versions:

Definition 34 *A SWF α is quasi IIA iff given any distinct $x, y \in A$ and any $P, P' \in \Pi^N$ with $x\ P_i\ y \Longleftrightarrow x\ P'_i\ y\ \forall i \in N$, we have $x\ \alpha^*(P)\ y \Rightarrow x\ \alpha(P')\ y$.*

Definition 35 *A SWF α is weakly dictatorial iff $\exists i \in N$ such that $x\ P_i\ y$ implies $x\ \alpha(P)\ y\ \forall P \in \Pi^N, \forall x, y \in A$.*

Baigent (1987) establishes that every Pareto optimal and quasi IIA SWF is a weak dictatorship. Nevertheless, we remark that, unlike the original version of the Arrovian impossibility, the converse statement is not true: Although every weak dictatorship is quasi IIA, there exist weak dictatorships that are not Pareto optimal.[1] Following this remark, we allow ourselves to the state a slight generalization of this theorem of Baigent (1987), corrected by Campbell and Kelly (2000b)[2]:

Theorem 36 *Let $\#A \geq 4$. Within the family of Pareto optimal SWFs, a SWF $\alpha : \Pi^N \rightarrow \Re$ is quasi IIA iff α is weakly dictatorial.*

We now explore the effect of being confined to the class of Pareto optimal SWFs. The strict counterpart of $T \in \Theta$ is denoted T^*. Let $\rho : \Theta \longrightarrow 2^{\Re}$ stand

[1] For example the SWF α where $x\ \alpha(P)\ y\ \forall x, y \in A$ and $\forall P \in \Pi^N$ is a weak dictatorship but not Pareto optimal.

[2] Baigent (1987) claims this impossibility in an environment with at least three alternatives. Nevertheless, Campbell and Kelly (2000b) show the existence of Pareto optimal and quasi IIA SWF when there are precisely three alternatives. They also show that the impossibility announced by Baigent (1987) prevails when there are at least four alternatives and even under restricted domains.

for the correspondence which transforms each $T \in \Theta$ over A into a non-empty subset of $\Re$ such that $\rho(T) = \{R \in \Re : xTy \Longrightarrow xRy, \forall x, y \in A\}$. To have a clearer understanding of ρ, we recall that every $T \in \Theta$ induces an ordered list of "cycles".[3] A set $Y \in 2^A \backslash \{\emptyset\}$ is a *cycle* (with respect to $T \in \Theta$) iff Y can be written as $Y = \{y_1, ..., y_{\#Y}\}$ such that $y_i \ T \ y_{i+1} \ \forall i \in \{1, ..., \#Y - 1\}$ and $y_{\#Y} \ T \ y_1$. The *top-cycle* of $X \in 2^A \backslash \{\emptyset\}$ with respect to $T \in \Theta$ is a cycle $C(X,T) \subseteq X$ such that $y \ T^* x \ \forall y \in C(X,T)$, $\forall x \in X \backslash C(X,T)$.[4] Now let $A_1 = C(A,T)$ and recursively define $A_i = C(A \backslash \bigcup\limits_{k=1}^{i-1} A_k, T), \forall i \geq 2$. Given the finiteness of A, there exists an integer k such that $A_{k+1} = \emptyset$. So every $T \in \Theta$ induces a unique ordered partition $(A_1, A_2, ..., A_k)$ of A. It follows from the definition of the top-cycle that whenever $i < j$, we have $xT^*y \ \forall x \in A_i$, $\forall y \in A_j$.

Lemma 37 *Take any $T \in \Theta$ which induces the ordered partition $(A_1, A_2, ..., A_k)$. Given any A_i with no indifferences among alternatives and any $x, y \in A_i$, we have $x \ R \ y$ and $y \ R \ x$, $\forall R \in \rho(T)$.*

Proof. Take any $T \in \Theta$ which induces the ordered partition $(A_1, A_2, ..., A_k)$. Take any A_i, any $x, y \in A_i$ and any $R \in \rho(T)$.If $\#A_i = 1$, then $x \ R \ y$ and $y \ R \ x$ holds by the completeness of R. As $\#A_i = 2$ cannot hold we complete the proof by considering the case $\#A_i = k \geq 3$. Let $A_i = \{x_1, x_2, ..., x_k\}$. Suppose, without loss of generality, $x_1 R \ x_2$ and *not* $x_2 \ R \ x_1$. This implies $x_1 \ T^* \ x_2$, as $R \in \rho(T)$. Moreover, as A_i is a cycle with no indifferences, $\exists x \in A_i$ such that $x_2 \ T^* x$. Let,

[3] We use the definition of "cycle" as stated by Peris and Subiza (1999).

[4] The top-cycle, introduced by Good (1971) and Schwartz (1972), has been explored in detail. Moreover, Peris and Subiza (1999) extend this concept to weak tournaments. In their setting, as $C(X,T)$ is a cycle, $\nexists Y \subset C(X,T)$ with $y \ T^* \ x \ \forall y \in Y$, $\forall x \in C(X,T) \backslash Y$.

without loss of generality, $x_2\ T^*\ x_3$. Thus, $x_2\ R\ x_3$ by definition of ρ which implies $x_1 R\ x_3$ and *not* $x_3\ R\ x_1$ by the transitivity of R. Again by the definition of ρ, we have $x_1\ T^* x_3$. As A_i is a cycle, $\exists j \in \{4, ..., k-1\}$ such that $x_3\ T^*\ x_j$. Suppose, without loss of generality, $j = 4$. So $x_3\ T^*\ x_4$, hence $x_3\ R\ x_4$, implying $x_1 R\ x_4$ and *not* $x_4\ R\ x_1$, which in turn implies $x_1 T^*\ x_4$. So, iteratively, $\forall i \in \{4, ..., k-1\}$, we have $x_i\ T^*\ x_{i+1}$,which implies $x_i\ R\ x_{i+1}$ and moreover $x_1 R\ x_{i+1}$ and *not* $x_{i+1}\ R\ x_1$. Hence, $x_1\ T^*\ x_{i+1}$. As A_i is a cycle, $x_k\ T^*\ x_1$. So, $x_k\ R\ x_1$ by the definition of ρ. Then, $x_i\ R\ x_{i+1}, \forall i \in \{2, 3, ..., k-1\}$ and $x_k\ R\ x_1$ implies by transitivity of R, $x_2\ R\ x_1$ which leads to a contradiction. Therefore, $x\ R\ y$ and $y\ R\ x$ for all $x, y \in A_i, \forall R \in \rho(T)$. ■

Thus for any $T \in \Theta$ which induces the ordered partition $(A_1, A_2, ..., A_k)$ and any $R \in \Re$, we have $R \in \rho(T)$ if and only if for any $x, y \in A$

(i) $x, y \in A_i$ for some $A_i \Longrightarrow xRy$ and yRx

and

(ii) $x \in A_i$ and $y \in A_j$ for some A_i, A_j with $i < j \Longrightarrow xRy$.

We now proceed towards characterizing the family of quasi IIA SWFs. Take any aggregation rule $f \in \Phi$ which satisfies IIA. By composing f with ρ, we get a *social welfare correspondence* $\rho \circ f : \Pi^N \longrightarrow 2^{\Re}$ which assigns to each $P \in \Pi^N$ a non-empty subset $\rho(f(P))$ of $\Re$. Clearly, every singleton-valued selection of $\rho \circ f$ is a SWF.[5] Let $\Sigma^f = \{\alpha : \Pi^N \rightarrow \Re \mid \alpha$ is a singleton-valued selection of $\rho \circ f\ \}$. We write $\Sigma = \cup_{f \in \Phi} \Sigma^f$. Interestingly, the class of quasi IIA SWFs coincides with Σ.

[5] We say that $\alpha : \Pi^N \rightarrow \Re$ is a singleton-valued selection of $\rho \circ f$ iff $\alpha(P) \in \rho \circ f(P)\ \forall P \in \Pi^N$.

Theorem 38 *A SWF* $\alpha : \Pi^N \to \Re$ *is quasi IIA iff* $\alpha \in \Sigma$.

Proof. To establish the "only if" part, let $\alpha : \Pi^N \to \Re$ be a quasi IIA SWF. For any distinct $x, y \in A$, we define $f_{\{x,y\}} : \{\begin{smallmatrix} x \\ y \end{smallmatrix}, \begin{smallmatrix} y \\ x \end{smallmatrix}\}^N \to \{\begin{smallmatrix} x \\ y \end{smallmatrix}, \begin{smallmatrix} y \\ x \end{smallmatrix}, xy\}$ as follows: For any $r \in \{\begin{smallmatrix} x \\ y \end{smallmatrix}, \begin{smallmatrix} y \\ x \end{smallmatrix}\}^N$,

$$f_{\{x,y\}}(r) = \begin{cases} \begin{smallmatrix} x \\ y \end{smallmatrix} & \text{if} \quad x\ \alpha^*(P)\ y \text{ for some } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \\ \begin{smallmatrix} y \\ x \end{smallmatrix} & \text{if} \quad y\ \alpha^*(P)\ x \text{ for some } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \\ xy & \text{if } x\ \alpha(P)\ y \text{ and } y\ \alpha(P)\ x \text{ for all } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \end{cases}.$$

As α is quasi IIA, $f_{\{x,y\}}$is well-defined. Thus $f = \{f_{\{x,y\}}\} \in \Phi$. We now show $\alpha(P) \in \rho(f(P))$ $\forall P \in \Pi^N$. Take any $P \in \Pi^N$ and any distinct $x, y \in A$. First let x $f^*(P)$ y. So $f_{\{x,y\}}(P^{\{x,y\}}) = \begin{smallmatrix} x \\ y \end{smallmatrix}$. By definition of $f_{\{x,y\}}$, we have $x\ \alpha^*(Q)\ y$ for some $Q \in \Pi^N$ with $Q^{\{x,y\}} = P^{\{x,y\}}$ which implies $x\ \alpha(P)\ y$ as α is quasi IIA. If y $f^*(P)$ x, then one can similarly $y\ \alpha(P)\ x$. Now, let $x\ f(P)\ y$ and $y\ f(P)\ x$. So, $f_{\{x,y\}}(P^{\{x,y\}}) = xy$ which, by definition of $f_{\{x,y\}}$, implies $x\ \alpha(Q)\ y$ and $y\ \alpha(Q)\ x$ for all $Q \in \Pi^N$ with $Q^{\{x,y\}} = P^{\{x,y\}}$, hence $x\ \alpha(P)\ y$ and $y\ \alpha(P)\ x$. Thus, $x\ f(P)$ $y \Longrightarrow x\ \alpha(P)\ y$ for any $x, y \in A$, establishing $\alpha(P) \in \rho(f(P))$.

To establish the "if" part, take any $\alpha \in \Sigma$. So there exists $f \in \Phi$ such that $\alpha(P) \in \rho(f(P))$ $\forall P \in \Pi^N$. Suppose α is not quasi IIA. So, $\exists x, y \in A$ and $\exists P, Q \in \Pi^N$ with $P^{\{x,y\}} = Q^{\{x,y\}}$such that $x\ \alpha^*(P)\ y$ and $y\ \alpha^*(Q)\ x$. By the definition of ρ we have $x\ f^*(P)\ y$ and $y\ f^*(Q)\ x$ which implies $f_{\{x,y\}}(P^{\{x,y\}}) = \begin{smallmatrix} x \\ y \end{smallmatrix}$ and $f_{\{x,y\}}(Q^{\{x,y\}}) = \begin{smallmatrix} y \\ x \end{smallmatrix}$, giving a contradiction as $P^{\{x,y\}} = Q^{\{x,y\}}$, thus showing that α is quasi IIA. ■

By juxtaposing Theorems 36 and 38, one can conclude that removing the Pareto condition has a dramatic impact, as the class Σ of quasi IIA SWFs is fairly large and allows those where the decision power is not concentrated on a single individual. This positive result prevails when the following weak Pareto condition is imposed:

Definition 39 *A SWF α is weakly Pareto optimal iff given any distinct $x, y \in A$ and any $P \in \Pi^N$ with $x\ P_i\ y\ \forall i \in N$, we have $x\ \alpha(P)\ y$.*

Definition 40 *An aggregation rule $f \in \Phi$ is weakly Pareto optimal iff for any $x, y \in A$ and any $r \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^N$ with $r_i = \genfrac{}{}{0pt}{}{x}{y}$ $\forall i \in N$, we have $f_{\{x,y\}}(r) \in \{\genfrac{}{}{0pt}{}{x}{y}, xy\}$.*

Let Φ^* stand for the set of weakly Pareto optimal and IIA aggregation rules and $\Sigma^* = \cup_{f \in \Phi^*} \Sigma^f$.

Theorem 41 *A SWF $\alpha : \Pi^N \to \Re$ is weakly Pareto optimal and quasi IIA iff α $\in \Sigma^*$.*

Proof. To show the "only if" part, take any SWF $\alpha : \Pi^N \to \Re$ which is weakly Pareto optimal and quasi IIA. For any distinct $x, y \in A$, we define $f_{\{x,y\}} : \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^N \to \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}, xy\}$ as follows: For any $r \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^N$,

$$f_{\{x,y\}}(r) = \begin{cases} \genfrac{}{}{0pt}{}{x}{y} & \text{if} \quad x\ \alpha^*(P)\ y \text{ for some } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \\ \genfrac{}{}{0pt}{}{y}{x} & \text{if} \quad y\ \alpha^*(P)\ x \text{ for some } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \\ xy & \text{if } x\ \alpha(P)\ y \text{ and } y\ \alpha(P)\ x \text{ for all } P \in \Pi^N \text{ with } P^{\{x,y\}} = r \end{cases}.$$

As α is quasi IIA, $f_{\{x,y\}}$is well-defined. Thus $f = \{f_{\{x,y\}}\} \in \Phi$. Suppose, f is not weakly Pareto optimal. So, $\exists x, y \in A$ and $\exists P \in \Pi^N$ with $x \; P_i \; y \; \forall i \in N$ such that $y \; f^*(P) \; x$, implying $f_{\{x,y\}}(P^{\{x,y\}}) = \begin{array}{c} y \\ x \end{array}$. By definition of $f_{\{x,y\}}$, we have $y \; \alpha^*(Q) \; x$ for some $Q \in \Pi^N$ with $Q^{\{x,y\}} = P^{\{x,y\}}$, contradicting that α is weakly Pareto optimal, which establishes $f = \{f_{\{x,y\}}\} \in \Phi^*$. We now show $\alpha(P) \in \rho(f(P))$ $\forall P \in \Pi^N$. Take any $P \in \Pi^N$ and any distinct $x, y \in A$. First let $x \; f^*(P) \; y$. So $f_{\{x,y\}}(P^{\{x,y\}}) = \begin{array}{c} x \\ y \end{array}$. By definition of $f_{\{x,y\}}$, we have $x \; \alpha^*(Q) \; y$ for some $Q \in \Pi^N$ with $Q^{\{x,y\}} = P^{\{x,y\}}$ which implies $x \; \alpha(P) \; y$ as α is quasi IIA. If $y \; f^*(P) \; x$, then one can similarly $y \; \alpha(P) \; x$. Now, let $x \; f(P) \; y$ and $y \; f(P) \; x$. So, $f_{\{x,y\}}(P^{\{x,y\}}) = xy$ which, by definition of $f_{\{x,y\}}$, implies $x \; \alpha(Q) \; y$ and $y \; \alpha(Q) \; x$ for all $Q \in \Pi^N$ with $Q^{\{x,y\}} = P^{\{x,y\}}$, hence $x \; \alpha(P) \; y$ and $y \; \alpha(P) \; x$. Thus, $x \; f(P) \; y \Longrightarrow x \; \alpha(P) \; y$ for any $x, y \in A$, establishing $\alpha(P) \in \rho(f(P))$.

To show the "if" part, take any $\alpha \in \Sigma^*$. So there exists $f \in \Phi^*$ such that $\alpha(P) \in \rho(f(P)) \; \forall P \in \Pi^N$. Take any distinct $x, y \in A$ and any $P \in \Pi^N$ with $x \; P_i \; y$ $\forall i \in N$. By the weak Pareto optimality of f, we have $f_{\{x,y\}}(P^{\{x,y\}}) \in \{\begin{array}{c} x \\ y \end{array}, xy\}$, hence $x \; f(P) \; y$, which implies $x \; \alpha(P) \; y$ by the definition of ρ. Thus, α is weakly Pareto optimal. The "if"part of Theorem 38 establishes that α is quasi IIA, completing the proof. ■

CHAPTER 5

Conclusion

Within the scope of the preference aggregation problem, we contribute to the understanding of the well-known tension between requiring the pairwise independence of the aggregation rule and the transitivity of the social preference. As Wilson (1972) shows, a SWF $\alpha : \Pi^N \to \Re$ is *non-imposed*[1] and IIA if and only if α is dictatorial or *antidictatorial*[2] or *null*[3]. Thus, aside from these, any aggregation rule which is IIA allows intransitive social outcomes. In case these outcomes are rendered transitive according to one of the prescriptions made by ρ, we attain a SWF which fails IIA but satisfies quasi IIA. In fact, as Theorem 38 states, the class of quasi IIA SWFs coincides with those which can be attained through a selection made out of the social welfare correspondence obtained by the composition of a SWF that is IIA with ρ. This can be interpreted as a positive result, as the class of quasi IIA SWFs is fairly rich and not restricted to those where the decision power is concentrated on one individual. In fact, this class contains SWFs that are both anonymous and neutral.[4] Moreover, as Theorem 41 states, this positive result prevails when a weaker version of the Pareto condition is imposed. Thus, we can conclude that the transitivity of the social outcome can be achieved at a

[1] $\alpha : \Pi^N \to \Re$ is *non-imposed* iff for any $x, y \in A$, there exists $P \in \Pi^N$ with $x\ \alpha(P)\ y$.

[2] α is *anti-dictatorial* iff $\exists i \in N$ such that $x\ P_i\ y$ implies $y\ \alpha^*(P)\ x\ \forall P \in \Pi^N, \forall x, y \in A$.

[3] $\alpha : \Pi^N \to \Re$ is *null* iff $x\ \alpha(P)\ y\ \forall x, y \in A$ and $\forall P \in \Pi^N$.

[4] For instance, the SWF in Example 2 of Campbell and Kelly (2000b), which shows the failure of Theorem 36 for $\#A = 3$, belongs to this class.

cost of reducing IIA to quasi IIA and compromising of the strenght of the Pareto condition - hence an escape from an impossibility of both the Arrow (1951) and Wilson (1972) type.

This escape imposes indifference in social preference, as quasi IIA and IIA coincide otherwise. One can ask for minimizing this imposition. It is straightforward to see that given an aggregation rule $f \in \Phi$, there exists a unique selection of $\rho \circ f$ which minimizes the imposed indifferences in the social decision: Writing $(A_1, A_2, ..., A_k)$ for the ordered partition induced by $f(P) \in \Theta$ at $P \in \Pi^N$, take $\alpha(P) \in \rho(f(P))$ where $x \; \alpha^*(P) \; y \; \forall x \in A_i$ and $\forall y \in A_j$ with $i < j$. On the other hand, an open question of interest is the choice of the (non-dictatorial) f that minimizes the imposed social indifference. We conjecture, by relying on Dasgupta and Maskin (2008), that this will be the pairwise majority rule.

CHAPTER 6

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