

# **Choosing from an incomplete tournament**

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by

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# Abstract

By incomplete tournaments, we mean asymmetric binary relations over finite sets. Tournaments, which are complete and asymmetric binary relations, and tournament solutions are exhaustively investigated in the literature. We introduce the structure of incomplete tournaments, and we adapt three solution concepts -top cycle of Schwartz (1972), Miller (1977); uncovered set of Fishburn (1977), Miller (1977) and Miller (1980), Copeland solution of Copeland (1951)- established for tournaments to incomplete tournaments. We axiomatize top-cycle, and investigate the characterization of the uncovered set and the Copeland solution.

## Özet

Eksik turnuvalar sonlu kümeler üzerindeki asimetric ikili bağıntılardır. Tamamlanmış asimetric ikili bağıntı olan turnuvalar ve turnuva çözümleri literatürde kapsamlı bir şekilde incelenmiştir. Bu çalışmada eksik turnuvaların yapısı incelenmiş, ve üç önemli turnuva çözümü- Schwartz (1972), Miller (1977) tepe döngüsü; Fishburn (1977), Miller (1977), Miller (1980) kaplanmamış elemanlar kümesi; Copeland (1951) çözümü- eksik turnuvalara adapte edilmiştir. Tepe döngüsü karakterize edilmiş ve kaplanmamış elemanlar kümesi ile Copeland çözümünün karakterizasyonu incelenmiştir.

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## Preface

Since Arrow, in 1951, proved the impossibility of rational collective decision making, many propositions have been made to overcome the problem. Most of the tournament solutions are proposed for breaking the cyclical majorities by scientists in Economics and Voting Theory. Tournaments have also been a great interest in the field of Psychology, while researching non-transitive preferences of individuals (Tversky 1969, Ng 1989). In sports, ranking teams according to their wins and losses is another problem of choosing from a tournament. The design of the tournaments for sports competitions is another area of research where mathematicians and social choice theorists are involved.

Completeness and transitivity are traditionally defined as the postulates for rationality. A rational individual is assumed to reveal complete preferences, because any incomplete preference may lead to indecisiveness. However, it is very likely to observe incomplete preferences when we deal with the psychological preferences. Besides, an individual can possibly find it better to be indecisive over some alternatives. Aumann (1962, p.446) explains the reasons for such a behaviour as follows: "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. ... For example, certain decisions that an individual is asked to make might involve highly hypothetical situations, which he will never face in real life; he might feel that he can not reach an honest decision in such cases. Other decision problems might be

extremely complex, too complex for intuitive insight, and our individual might prefer to make no decision at all in these problems."

It is common to think that the revealed preferences must be complete. Eliaz and Ok (2006) argue that an agent can "reveal" indecisiveness between certain alternatives. It is also an interest to model the behaviour of an individual under imperfect information, and this situation can very well be represented by incomplete preferences.

So, the basis of incomplete preferences in individual choice is well-founded. However, tournaments are usually interpreted as the outcomes of pairwise majority voting. So, it is obtained "through" individual preferences. So, what can lead to incomplete tournament? Let's think about a social planner or a modeler, who has some missing information about individual's pairwise comparisons on some issues. The planner may also have some restrictions disabling comparison of some alternatives. In both cases the resulting binary relation may be written as an incomplete tournament. In many sports tournaments like tennis, players do not play with every other player, so, the resulting relation is not complete. The overall ranking of the players often depends on the results of their matches, and also the strength of the players they play with. All countries in Europe have football leagues. If our alternative set contains football teams in Europe, then it is very likely that many teams do not play with each other.

The solution concepts designed for choosing the best alternative(s) through a tournament are usually inspired by the voting rules based on the pairwise majority comparison of alternatives. In our work, we focus on the how to choose from asymmetric binary relations, as we call “incomplete tournaments”. So, we will propose ways to choose from possibly incomplete and possibly cyclic results of a majority voting.

This theses is organized as follows: We will first introduce some basic solution concepts for tournaments in Chapter 1. After we introduce the main structure of the incomplete tournaments in Chapter 2, we will go on with the solution concepts that we propose for incomplete tournaments. We finally axiomatize these solution concepts, and this will be the end of Chapter 2.

In the very beginning of his book "Tournament Solutions and Majority Voting", 1997; Jean François Laslier adressed the question "given a tournament, which are the best outcomes?" . Now, we will seek for an answer to the following question:

Given an incomplete tournament, which are the best outcomes?



# Chapter 1

## Tournaments

Tournaments are complete and asymmetric binary relations over a finite set. They have been exhaustively investigated in social choice theory since 1950s as well as in mathematics as connected graphs. In graph theory, tournaments are defined as complete and asymmetric directed graphs: for any vertex  $x$  and any vertex  $y$  with  $x \neq y$ , there exists exactly one of the two arcs,  $(x, y)$  or  $(y, x)$ . If we adapt this to voting theory, the arc  $(x, y)$  will mean  $x$  is preferred to  $y$  by a majority of voters.

When there is a team in a sports tournament that beats every other team, or, if there is a candidate that is preferred to any candidate by a majority of the voters, choosing this team or candidate as the winner is unquestioned. This element is the "Condorcet winner" of the tournament. In graph theory, when a vertex  $x$  is collectively preferred to other vertices, or in other words, all the arcs adjacent to  $x$  go from  $x$  towards the other vertices,  $x$  is the equivalent of Condorcet winner. The number of the arcs going from  $x$  to other vertices is called "the out-degree of  $x$ ". However, we know that a Condorcet winner does not always exist, and the problem of choosing from a tournament arises under this circumstance.

Many of great social choice theorists have proposed choice correspondences to determine the best outcomes of a tournament. These correspondences are generally called "tournament solutions".

We will emphasize the two very important tournament solutions, top-cycle and uncovered set, which are crucial for this work. We will then introduce some well-known tournament solutions.

## 1.1 Tournament Solutions

We will first introduce some basic notions which are only valid for this chapter of this work.

We will let  $A$  be a finite set of alternatives. We write  $\Theta$  for the set of complete and asymmetric binary relations over  $A$ . Any  $T \in \Theta$  is called a *tournament*. A *tournament solution* is a mapping  $f : \Theta \rightarrow 2^A$ . To any tournament  $T$ , a tournament solution associates a nonempty subset  $f(T)$  of "best" outcomes, called the choice set at  $T$ . For any  $B \subset A$ ,  $f(T|_B)$  is a restriction of  $T$  on  $B$ .

The following are the definitions of some appealing properties that a solution may satisfy. They are used to characterize or investigate the characterization of the tournament solutions.

*Condorcet Consistency*

$f : \Theta \longrightarrow 2^A$  satisfies *Condorcet Consistency* iff whenever  $xTy \forall y \in A$ , then  $f(T) = x$ .

### *Smith Consistency*

$f : \Theta \longrightarrow 2^A$  satisfies *Smith Consistency* iff  $y \notin f(T) \iff y$  is eliminated by some  $x \in f(T)$ .

### *Arrow's IIA*

$f : \Theta \longrightarrow 2^A$  satisfies Arrow's IIA iff whenever  $T \upharpoonright_B = T' \upharpoonright_B$ ,  $f(T \upharpoonright_B) = f(T' \upharpoonright_B)$  where  $B \subset A$ .

### *Neutrality*

We will define  $\sigma$  as a permutation of  $A$ . The binary relation  $T^\sigma$  is defined as  $aT^\sigma b \iff \sigma^{-1}(a)T\sigma^{-1}(b)$ . For all  $T \in \Theta$ ,  $f : \Theta \longrightarrow 2^A$  satisfies *neutrality* iff  $f(T^\sigma) = \sigma[f(T)]$ .

### *Expansion*

$f : \Theta \longrightarrow 2^A$  satisfies *expansion* iff  $f(T \upharpoonright_B) \cap f(T' \upharpoonright_B) \subset f(T \upharpoonright_{B \cup B'})$ .

### *Aizerman*

$f : \Theta \rightarrow 2^A$  satisfies *Aizerman* iff  $f(T \upharpoonright_{B'}) \subset B \subset B' \implies f(T \upharpoonright_B) \subset f(T' \upharpoonright_{B'})$ .

### *Idempotency*

$f : \Theta \rightarrow 2^A$  is idempotent iff  $f(f(T)) = f(T)$ .

#### **1.1.1 The Top-Cycle**

When a tournament  $T \in \Lambda$  is not connected, it is possible to decompose it into its strongly connected components. For any tournament  $T$ , there exists a strongly connected graph which is called the top-cycle. All the arcs in this tournament  $T$  are from the top-cycle to the out of it. Defining it in terms of candidates or teams, it assigns the set of the alternatives that beat every other alternative directly or indirectly.<sup>1</sup> Top-cycle always induces a strongly connected subtournament of the tournament.

Tournament solutions are characterized as satisfying some consistency axioms. The main consistency axiom in the literature is the "Condorcet Consistency", which requires to uniquely choose the Condorcet winner whenever it exists. The top-cycle satisfies Condorcet transitivity. Another consistency requirement, called "Smith Consistency", was introduced by Smith (1973) which was a weakening of the "Condorcet transitivity". Condorcet transitivity required that any element in the choice set beats every element outside. "Smith consistency" weakens this property by saying that if

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<sup>1</sup> An alternative  $x$  can either directly beat  $y$  ( $xTy$ ,  $T$  being a complete and asymmetric binary relation) or it beats  $y$  through a path: for instance  $x$  beats  $z$ ,  $z$  beats  $w$ ,  $w$  beats  $y$ . ( $xTzTwTy$ )

the elements of one subset of the alternative set beats every element outside, then the choice set must be from this set.

These consistency axioms both lead to the characterization of the top-cycle. Top-cycle is the smallest choice correspondence which satisfies Condorcet transitivity (Schwartz 1972), while it is the largest choice correspondence satisfying Smith consistency (Smith, 1973). The top-cycle choice correspondence was introduced by Schwartz (1972) and Miller (1977).

Although the top-cycle choice correspondence satisfies these indispensable axioms, it is shown in many examples that it is an undesirably large set. Besides, it leads to Pareto dominated outcomes in the choice set (Fishburn, 1977). This was an incentive for social choice theorists to seek for "better" choice correspondences. However, all of the choice correspondences proposed for tournaments assign sets which are subsets of the top-cycle. Choosing inside the top-cycle turned out to be a requirement for rationality of a choice correspondence. (Moon 1968, Schwartz 1972).

Schwartz (1986) defines the GETCHA (Generalized top-choice assumption) as the minimal set with respect to set inclusion where each element beats every other outside this set. GETCHA is defined for asymmetric binary relations. Let us have an asymmetric binary relation, say  $P$  on  $A$ , and Schwartz calls " $P$ -undominated subset of  $A$ " any set where no alternative outside this set beats any of the alternatives from this set. Whenever this set is minimal with respect to set inclusion, then it is called minimum " $P$ -undominated subset" of  $A$ . In case of missing relationships, we may

have more than one of " $P$ -undenominated subsets" GOCHA (Generalized optimal choice axiom) is defined as the union of some sets, in which  $P$ -undenominated subsets. Whenever we have a tournament  $T \in \Theta$  on  $A$ , both GETCHA and GOCHA coincide with the top-cycle. More detailed information on GOCHA will be given in the next chapter.

### 1.1.2 Uncovered Set

Fishburn (1977) investigated some Condorcet social choice functions that had been proposed before. He also introduces another Condorcet social choice correspondence which he names as "Fishburn's function". Fishburn induces complete binary relations through the simple majority voting over his alternatives. Fishburn's function is based on the notion that if everything that beats an alternative- say  $x$ - also beats  $y$  under simple majority, and if  $x$  beats or ties something that beats  $y$ , then  $x$  is better than  $y$  under simple majority comparisons. It is seen that Fishburn actually deals with weak tournament as Peris and Subiza (1999) will call later. Fishburn function is "superior" to Schwartz function in terms of some properties that Fishburn has defined. Please note that Schwartz function is equivalent to the top-cycle choice correspondence in Fishburn's work. One of these properties is the Pareto Optimality condition of Fishburn. This condition relates the choice function with the voters linear orders. It requires that if for an alternative - say  $y$ - there is at least one alternative -say  $x$ - which is ranked above in all individual profiles, then the choice correspondence should

not choose  $y$ . Schwartz function does not satisfy this property, while the Fishburn function does. Another appealing property of the Fishburn function compared to Schwartz function is its discriminability property. Discriminability is a condition related to "how small" the choice set is. Both Fishburn function and Schwartz function turn out to have low discriminability. However, Fishburn's function is more discriminating than Schwartz's Function since Fishburn shows that if we ignore ties, then the Schwartz's function will assign a superset of the Fishburn's functions.

After defining the Condorcet set (or minimal undominated set, which are equivalent to the top-cycle) in his 1977 work, Miller (1980) seeks a choice correspondence which is not as large as the Condorcet set, and which does not give Pareto dominated outcomes. He defines the covering relation for complete, asymmetric and irreflexive binary relations. What he does in this further work is to define the "Uncovered Set" through the covering relation. The uncovered set is the set of the alternatives that are not covered as well as it is the set of the alternatives that reach every other alternative at most in two steps. Consequently, he finds out that the uncovered set is the refinement of the Condorcet set. Miller (1980) also points out that if  $x$  covers  $y$ , then the Copeland score (dominion as Miller calls) of  $x$  must be larger than  $y$ 's. By pointing out this, he shows that Copeland winner is a subset of the uncovered set. He links the uncovered set with the sophisticated voting, cooperative voting and electoral competition.

Shepsle and Weingast (1984) gives a full characterization of the uncovered set as the equilibrium set of a sophisticated voting agenda. Mc Kelvey (1976) states in his theorem that from any initial point, there is an agenda that will lead sincere voters to any terminal point. Shepsle and Weingast show that from any initial point, there is an agenda that will lead *sophisticated* voters to any point *not covered* by the initial point. They use the term sophisticated voting as strategic voting in an institutional context. They define sophisticated agenda algorithm, and Banks (1985) finds that this algorithm ends up in the top-cycle of the tournament restricted on the uncovered set. In Shepsle and Weingast work, uncovered set is characterized through the two step principle as being the set of points which beat any other point by a path of length one or two. They also announce that the uncovered set is always a subset of the Pareto optimal outcomes.

The uncovered set is axiomatically characterized by Moulin (1986) based on the expansion axiom borrowed from rationalizable choice functions. The theorem of Moulin is the following:

Theorem (Moulin 1986): The uncovered set satisfies Neutrality, Arrow's IIA and Expansion. Conversely any mapping satisfying Neutrality, Arrow's IIA, Expansion and Condorcet consistency must contain the uncovered set

The theorem states that uncovered set is the smallest choice correspondence with respect to set inclusion satisfying Neutrality, Arrow's IIA and Expansion and Condorcet consistency.



### **1.1.3 Copeland Solution**

The Copeland score of an alternative is the number of alternatives beaten by that alternative. A Copeland winner of a tournament is an alternative with the highest Copeland score. It is proposed by Copeland (1951) and used in variety of fields, including biology as in Landau (1953); graph theory as in den Brink and Gilles (2003); economics as in Paul (1997); computer science as in Singh and Kurose (1991) and social choice theory as in Moulin (1986). Rubinstein (1980) characterizes the “Copeland welfare function” as a method to rank the participants of a tournament. This characterization is through neutrality, Arrow’s IIA and a type of strong monotonicity. Henriot (1985) extends this characterization to environments which allow ties between candidates. Moreover, he gives three characterizations of the “Copeland solution” which chooses among the participants of a tournament.

Copeland solution is always included in the uncovered set, and followingly, in the top-cycle.

### **1.1.4 Some other tournament solutions**

While choosing inside the top-cycle of a tournament has been a rationality requirement, most other solution concepts developed after uncovered set turned out to be the refinements of the uncovered set. One of them is the minimal covering set (Dutta, 1988), which was introduced as a Von-Neumann Morgenstern solution concept, satisfying both internal and external stability axioms. Minimal covering set is also defined

through the covering relation. It is included in the top-cycle and also in uncovered set.

Slater (1961) proposed the Slater Solution which is based on the idea of approximating a tournament by a linear order. One takes the usual distance between graphs and considers linear orders at minimal distance from the tournament, then the solution is by definition the set of outcomes which are top-element of one of these closest orders.

### **1.1.5 Weak Tournaments**

Peris and Subiza (1999) generalize some important tournament solutions to the context in which ties are possible. Any complete binary relation will be called a weak tournament. Two sports team may tie. Two candidates may obtain equal number of votes when mutually compared. In this work named "Condorcet choice correspondences for weak tournaments", top-cycle, uncovered set and minimal covering set solutions are generalized to weak tournaments context from a normative point of view. In all of these generalizations, whenever the binary relation of interest corresponds to a tournament, the extended weak tournament solution coincides with the tournament solution.

# Chapter 2

## Incomplete Tournaments

### 2.1 Basic Notions

Let  $A$  be a finite set of alternatives. We write  $\Theta$  for the set of asymmetric binary relations over  $A$ . Any  $T \in \Theta$  is called an *incomplete tournament*.

For any  $X \in 2^A$ , for any couple  $(x, y) \in X \times X$ , we say that “a path  $P(x, y)$ ” is the length of the shortest distance from  $x$  to  $y$ .  $P(x, y)$  is found through the sequence  $\{x^h\}_{h=1, \dots, H}$  in  $X$  such that  $x^1 = x$ ,  $x^H = y$  and  $x^h T x^{h+1}$  for all  $h = 1, \dots, H - 1$ . If there is no such sequence from  $x$  to  $y$ , then  $P(x, y) = \infty$ . We say that  $x$  reaches  $y$  (or  $y$  is reachable by  $x$ ) iff there is a path from  $x$  to  $y$ .

For any  $x, y \in X$  with neither  $xTy$  nor  $yTx$ , we write  $x \dashv y$ . We will denote by  $T|_X$  the restriction of  $T$  on  $X$ .  $T|_X$  is connected iff  $\nexists$  nonempty strict subset  $Y$  of  $X$  such that  $x \dashv y, \forall x \in X \setminus Y$  and  $\forall y \in Y$ . We say that  $T|_X$  is a strongly connected graph iff for any pair  $x, y \in X$ ,  $x$  is reachable by  $y$ . The maximal strongly connected subgraphs of a strongly connected graph are called “strongly connected components”. For any  $X \in 2^A$  we will denote by  $C^T(X)$  the set of strongly connected components on  $X$  induced by  $T \in \Theta$ .

## 2.2 The Structure of an Incomplete Tournament

Tournaments, which are complete and asymmetric binary relations, possibly admit cycles as we interpret them as pairwise majority voting outcomes. For a finite tournament, instead of linear order of the alternatives, we may have a linear order of “cycles”, in which it is possible to have a unique alternative. We call the maximal cycle of this linear order “top-cycle”. In light of what we know about the structure of tournaments, we investigate how this structure shapes under incompleteness. In case of incomplete tournaments, it is possible that an element of the sequence is repeated to complete the cycle. This type of cycle will be called "weak cycle". The elements belonging to a cycle in a tournament directly beat every element which is ranked in one of the below cycles. Similarly, we can decompose an incomplete tournament into its weak cycles in which the alternatives belonging to the weak cycle “are not beaten” by any alternatives in a weak cycle below.

This leads to the following definition:

**Definition 1** *Given  $T \in \Theta$ ,  $Y \subseteq X \in 2^A$  is an undominated set in  $X$  iff not  $xTy$  for all  $x \in X \setminus Y$  and all  $y \in Y$ .*

This definition is similar to Schwartz’s (1986)  $\rho$ -undominated subset definition. An undominated set might contain a subset which is also an undominated set. Note that  $X \in 2^A$  is also an undominated set of itself. We follow Schwartz’s tracks for

the following definition. This definition below will restrict us to an undominated set which does not contain any undominated set different than itself.

**Definition 2** (Schwartz 1986) *Given  $T \in \Theta$ ,  $Y \subseteq X$  is a minimal undominated set in  $X$  iff  $Y$  is an undominated set in  $X$  according to  $T$  which is minimal with respect to set inclusion.*

If  $Y$  is a minimal undominated set, then  $T|_Y$  is a strongly connected component.

We introduce a lemma for different minimal undominated sets.

**Lemma 1** *Given  $T \in \Theta$ , for the minimal undominated sets  $Y, Z \subseteq X$  of  $X$  with  $Y \neq Z, Y \cap Z = \emptyset$ .*

**Proof.** Let  $Y$  and  $Z$  are minimal undominated sets in  $X$ , and suppose for a contradiction that  $Y \cap Z \neq \emptyset$ . For any  $x \in Y \cap Z$ , either  $xTy$  or  $x \Vdash y$  for  $\forall y \in X \setminus Y$ , in particular  $\forall y \in Z \setminus Y$ . Similarly; for any  $x \in Y \cap Z$ , since  $x \in Z$  and  $Z$  is a minimal undominated set in  $X$ , either  $xTy$  or  $x \Vdash y$  for  $\forall y \in X \setminus Z$ , and in particular  $Y \setminus Z$ . These two results lead that  $Y \cap Z$  is an undominated set itself, and this contradicts that  $Y$  and  $Z$  are minimal with respect to set inclusion. ■

**Proposition 2** *Every incomplete tournament  $T \in \Theta$  admits the family of minimal undominated sets  $\{X_i\}_{i=1,\dots,k}$  in  $X$  with  $1 \leq k \leq n$  such that  $\forall X_i, X_j$  with  $i \neq j$ , the two sets are mutually disjoint.*

**Proof.** Take any  $T \in \Theta$  and  $X \in 2^A$ . Suppose that  $T$  admits no minimal undominated set. Since every minimal undominated set is an undominated set and  $X$  is finite, this leads that  $X$  admits no undominated set. This contradicts the fact that  $X$  is an undominated set of itself. It is easily seen that the number of the minimal undominated sets can be more than one but cannot exceed  $n$ , which is the cardinality of  $X$ .

To keep up with the proof, now we have to show that for all minimal undominated sets  $X_i, X_j$  with  $i \neq j$ ,  $x_i \Vdash x_j \forall x_i \in X_i, \forall x_j \in X_j$ . Since we already know from lemma 2.1, these sets are distinct. We will suppose for a contradiction and without loss of generality  $x_i T x_j$  for  $x_i \in X_i$  and  $x_j \in X_j$ . It immediately follows from the definition of minimal undominated set that this case is not impossible. ■

This result is also mentioned in Schwartz's work. Note that the family of the minimal undominated sets  $\{X_i\}_{i=1,\dots,k}$  for each  $X$  through  $T$  is uniquely defined.

### 2.3 Choosing from an Incomplete Tournament

An *incomplete tournament solution* is a mapping  $f : \Theta \times 2^A \longrightarrow 2^A$  such that  $f(T, X) \subseteq X \forall (T, X) \in \Theta \times 2^A$ .

We directly borrow a very crucial axiom, Condorcet consistency, defined for tournaments and we apply it to our world.

*Condorcet consistency*

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies *Condorcet consistency* iff  $xTy \forall y \in X \implies f(T, X) = x$

This axiom is very well known and it requires that the solution concept must choose the Condorcet winner whenever it exists.

Before we define a crucial axiom which is *Smith Consistency*, we will introduce a binary relation called "elimination".

**Definition 3** For any pair  $x, y \in X$ ,  $x$  eliminates  $y$  in  $X$  iff  $P(x, y) = k$  and  $P(y, x) = \infty$ .

Elimination is transitive and not complete.

*Smith Consistency*

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies *Smith Consistency* iff  $y \notin f(T, X) \iff y$  is eliminated by some  $x \in f(T, X)$ .

So, any nonchosen outcome is eliminated by some chosen outcome.

In tournaments, some other axioms are used to characterize the solution concepts. These are adapted into the world of incomplete tournaments.

*Arrow's IIA*

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies Arrow's IIA iff whenever  $T|_X = T'|_X$ ,  $f(T, X) = f(T', X)$ .

*Neutrality*

We will define  $\sigma$  as a permutation of  $A$ . The binary relation  $T^\sigma$  is defined as  $aT^\sigma b \iff \sigma^{-1}(a)T\sigma^{-1}(b)$ . For all  $T \in \Theta$  and all  $X \in 2^A$ ,  $f : \Theta \times 2^A \longrightarrow 2^A$  satisfies *neutrality* iff  $f(T^\sigma, X) = \sigma[f(T, X)]$

### *Expansion*

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies expansion iff  $f(T, X) \cap f(T, X') \subset f(T, X \cup X')$ .

### *Aizerman*

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies Aizerman iff  $f(T, X') \subset X \subset X' \implies f(T, X) \subset f(T, X')$ .

Finally, we introduce a very important consistency axiom for incomplete tournaments, which is monotonicity. We will define the following sets for  $x \in X$  :

$$D^+(T, x) = \{y \in X : xTy\}, \text{ and } D^-(T, x) = \{y \in X : yTx\}$$

### *Monotonicity*

Take  $T, T' \in \Theta$  such that  $T|_{X-\{x\}} = T'|_{X-\{x\}}$  for any  $x \in f(T, X)$ .

$f : \Theta \times 2^A \longrightarrow 2^A$  satisfies monotonicity iff  $[D^+(T, x) \subseteq D^+(T', x) \text{ and } D^-(T', x) \subseteq D^-(T, x)] \implies x \in f(T', X)$ .

## **2.3.1 Incomplete Tournament Solutions**



### Top-cycle

Using the structure of an incomplete tournament, and the properties of the top-cycle for tournaments, we are ready to introduce our top-cycle:

**Definition 4** *The top-cycle choice correspondence assigns the set  $TC(T, X) = \{x \in X : \nexists y \in X \text{ that eliminates } x\}$*

One can easily check that the top-cycle choice correspondence of  $X$  can also be defined as the union of minimal undominated sets that  $X$  admits. This set is equivalent to Schwartz's GOCHA set. Schwartz introduces a characterization of this set through the following conditions:

- nothing out of GOCHA beats anything in GOCHA.
- there is no subset, say  $B$ , of GOCHA, say  $C(A)$  such that something in  $B$  beats something in  $C(A) - B$ , and nothing in  $C(A) - B$  beats anything in  $B$ .
- if  $B$  is an undominated set of  $A$ , then some element of  $B$  belongs to GOCHA.

Through the Smith consistency axiom we defined for incomplete tournaments, we characterize our top-cycle as follows:

**Theorem 3** *The unique smallest (with respect to set inclusion) choice correspondence satisfying Smith Consistency is the top-cycle.*

**Proof.** Since top-cycle is the union of minimal undominated sets, it consists of elements which are not eliminated. So, it is obvious that top-cycle satisfies Smith consistency. Now, we will let  $f(T, X)$  be a choice correspondence satisfying Smith consistency, and suppose for a contradiction that  $f(T, X)$  is a strict subset of  $TC(T, X)$ . So, there is  $x \in TC(T, X) \setminus f(T, X)$ . Since  $f$  satisfies Smith consistency,  $x$  is eliminated by some  $z \in f(T, X)$ . So,  $P(x, z) = \infty$ , and  $P(z, x) = k$ . It immediately follows that this contradicts with  $x$  being in the top-cycle, establishing the result. ■

### The Uncovered Set

In tournaments, which are complete and asymmetric binary relations the uncovered set choice correspondence assigns the set of the elements which can beat every other alternative at most in 2 steps. The top-cycle is a superset of uncovered set since it contains the elements which can beat every other alternative in some steps. If we look for the alternatives that beat every other alternative at most in 2 steps in incomplete tournaments, we face the serious problem of not being well-defined as the following example illustrates:

**Example 1** *Let  $X = \{a, b, c, d\}$  and  $aTb, bTc, cTd, dTa$ . In this incomplete tournament, none of the alternatives can beat all others at most in 2 steps.*

However, if we change our definition of the uncovered set to “the set of the alternatives that beat every other alternative at most in 3 steps”, then the uncovered

set will be the whole set:  $\{a, b, c, d\}$ . Nonetheless, if we let  $X' = \{a, b, c, d, e\}$  and  $aTb, bTc, cTd, dTe$  and  $eTa$ ; this will not work

Take any  $T \in \Theta$ , and  $X \in 2^A$ . We know that  $X$  will admit the family of minimal undominated sets. For all  $x \in Y$  where  $Y$  is a minimal undominated set in  $X$ , we define the maximum attainable path as  $P(x) = \max\{P(x, y)_{y \in Y - \{x\}}\}$ .

**Definition 5** For any  $Y \in C^T(X)$ , the minimax choice correspondence  $M : Y \rightarrow 2^Y$  assigns the set  $M(T, Y) = \{x \in Y : P(x) \leq P(y) \forall y \in Y\}$ . The uncovered set  $UC : \Theta \times 2^A \rightarrow 2^A$  of an incomplete tournament  $T$  in  $X$  is  $UC(T, X) = \bigcup_{Y \in C^T(X)} M(T, Y)$ .

In case of tournaments, when we seek for the set of the alternatives that beat every other alternative at most in  $n - 1$  step, where  $n$  is the cardinality of  $X$ , we obtain the top-cycle. When we limit ourselves to 2 steps, the result is the uncovered set. It is obvious that 2 steps principle may not give any solution and is not well-defined in the case of incomplete tournaments. However, there is something in between top-cycle and uncovered set, which, for example, is the set of the alternatives that beat every other at most in 3 steps, when  $3 < n - 1$ . For incomplete tournaments, we seek for the minimum number of steps that will give us a well defined set for each minimal undominated set, and that gives us the minimax choice correspondence.. By definition of the uncovered set, it will always be inside the top-cycle.

An immediate result will link the uncovered set in tournaments with the sign uncovered set. Before we present it, we should introduce some very well known definitions from the world of tournaments. let us introduce a very well known lemma by Shepsle and Weingast (1982).

Uncovered set is known to be the smallest set that satisfies neutrality, Arrow's IIA, expansion and Condorcet consistency in the world of tournaments (Moulin 1986). We expect that our minimax choice correspondence satisfies the versions of these axioms in our world. However, it is not the case.

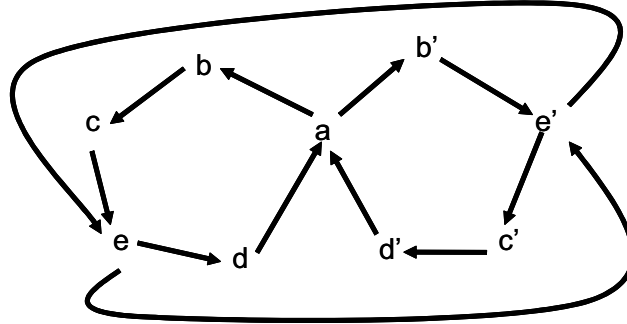
**Proposition 4** *The minimax choice correspondence satisfies Monotonicity, Arrow's IIA, and Neutrality.*

**Proof.** Arrow's IIA and neutrality are straightforward. To show that monotonicity is satisfied let  $T, T' \in \Theta$  and  $Y \in C^T(X)$ . Letting  $x \in M(T, Y)$ , suppose we have  $T \upharpoonright_{Y-\{x\}} = T' \upharpoonright_{Y-\{x\}}$ ,  $D^+(T, x) \subseteq D^+(T', x)$ , and  $D^-(T', x) \subseteq D^-(T, x)$ . Since  $x \in M(T, Y)$ ,  $P(x) \leq P(y) \forall y \in Y$ . This condition is still true for  $(T', Y)$  under these conditions. So,  $x \in M(T', Y)$  and  $M$  satisfies monotonicity. ■

**Proposition 5** *The minimax choice correspondence does not satisfy Expansion or Aizerman.*

**Proof.** We produce an example showing that Expansion or Aizerman are not satisfied.

Let  $Y = \{a, b, c, d, e, b', e', c', d'\}$ , and we have the following graph through  $T \in \Theta$ :



*Expansion:* Let  $Y_1 = \{a, b, c, d, e\}$  and  $Y_2 = \{a, b', c', d', e'\}$ . The corresponding minimax set are  $M(T, Y_1) = \{a, b, c, d, e\}$  and  $M(T, Y_2) = \{a, b', c', d', e'\}$ . Although  $a \in M(T, Y_1) \cap M(T, Y_2)$ , it is not in  $M(T, Y_1 \cup Y_2)$ .  $M(T, Y_1 \cup Y_2) = \{e\}$ , and expansion is violated.

*Aizerman:* For the same incomplete tournament, we have  $\{e\} \subset Y_1 \subset (Y_1 \cup Y_2)$ . However,  $M(T, Y_1)$  is not a subset of  $M(T, Y_1 \cup Y_2)$ . ■

Given a tournament, minimax choice correspondence coincides with the uncovered set.

In tournaments which are complete and asymmetric binary relations, there is another solution which coincides with the Uncovered set. Finding the minimum number of arrows to be reversed for each alternative to beat every other at most in two steps and choosing the alternative with the minimum number of necessary reversals would give us the Uncovered set. Now, we will adopt this to incomplete tournaments:

let  $D^+(x, T|_Y) = \{x' \in Y : xTx'\}$ . The completion score of  $x$  in  $T|_Y$  is defined as the minimal integer  $s(x)$  such that there exists  $T'$  on  $Y$  such that:

$$-D^+(x, T) \subseteq D^+(x, T')$$

$$-\forall x' \in Y, P(x, x') \leq 2$$

$$-|D^+(x, T') - D^+(x, T)| = s(x)$$

The completion score of  $x \in Y$  is the minimum number of additional points in  $Y$  that  $x$  must defeat in order to defeat every point at most two steps. It follows from definition that these additional points correspond to either an arrow reversal in  $T|_Y$  or an additional arrow in  $T|_Y$ .

**Definition 6** Let  $T \in \Theta$  and  $Y \in C^T(X)$ . For any  $x, x' \in Y$ ,  $x$  is said to dominate  $x'$  by min completion if  $s(x) < s(x')$ .  $Y^{mc}$  denotes the subset of undominated elements of any  $Y \in C^T(X)$ .

**Definition 7** The Min-completion uncovered set is defined by  $UC^{mc}(T, X) = \bigcup_{Y \in C^T(X)} Y^{mc}$ .

It follows from the well-known characterization of uncovered set that  $UC^{mc}(T) = UC(T)$  whenever  $T$  is complete.

### Copeland Solution

In a very recent work, Sanver et. al show that minimizing the number of steps from an alternative to the others gives us the Copeland solution in tournaments. The following definition originates from this result:

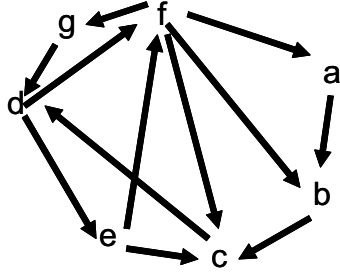
**Definition 8** Given  $T \in \Theta$ ,  $Y \in \mathcal{C}^T(X)$ , the sum score of  $y \in Y$  is  $sum(y) = \sum_{x \in Y} P(y, x)$ . The minisum choice correspondence  $MS : Y \rightarrow 2^Y$  assigns the set  $MS(T, Y)$  to each  $T \in \Theta$  the alternatives with the minimum sum scores:  $MS(T, Y) = \{y \in Y : sum(y) \leq sum(x) \forall x \in Y\}$  The Copeland solution  $C : \Theta \times 2^A \rightarrow 2^A$  of  $T$  in  $X$  is  $C(T, X) = \bigcup_{Y \in \mathcal{C}^T(X)} MS(T, Y)$ .

### A Set Theoretical Comparison

We have already shown that  $UC \subseteq TC$ . Now we will investigate the relationships between the other solution concepts.

Although the Copeland solution is included in the top-cycle for complete case, this is no longer true in incomplete tournaments. The following example shows that they may even be disjoint:

**Example 2** Let  $X = \{a, b, c, d, e, f, g\}$  and we have the following strongly connected component:



In this incomplete tournament  $sum(f) = 9$ , which is the minimum among the alternatives. However,  $f$  needs 3 steps to reach  $e$ , while  $e$  can reach all other alternatives at most in 2 steps. The sum score of  $e$  is  $sum(e) = 10$ . So,  $MS(T, X) = \{f\}$  while  $M(T, X) = \{e\}$ , showing that these two choice correspondences, and followingly the uncovered set and the Copeland solution can assign distinct sets in incomplete tournaments.

**Proposition 6** *There exist  $X$  and  $T \in \Theta$  such that  $UC^{mc}(T) \cap M(T) = \emptyset$ .*

Proof

Let  $X = \{a, b, c, d, e, f, g\} \cup \{b', c', d', e', f', g', h'\}$  and let  $T \in \Theta$  be defined as follows:

$$- aTbTcTdTeTfTgTa$$

$$- aTb'Tc'Td'Te'Tf'Tg'Th'Ta$$

$$- aTd$$

$$- dTc'$$

It is easily checked that,

$$- Max_{y \in Y} P(d, y) = 6$$



- $\text{Max}_{y \in Y} P(z, y) = 7$  for  $z = a, c, f', g', h'$
- $\text{Max}_{y \in Y} P(z, y) = 8$  for  $z = b, g, e'$
- $\text{Max}_{y \in Y} P(z, y) = 9$  for  $z = f, d'$
- $\text{Max}_{y \in Y} P(z, y) = 10$  for  $z = e, c'$
- $\text{Max}_{y \in Y} P(b', y) = 11$

Hence,  $M(T) = \{d\}$ . And,  $s(a) = 4$  and  $s(d) = 5$ . So,

- $d$  must defeat  $g$  in at most 2 steps  $\implies$  either  $dT'f$  or  $dT'g$
- $d$  must defeat  $c$  in at most 2 steps  $\implies$  either  $dT'b$  or  $dT'c$
- $d$  must defeat  $b$  in at most 2 steps  $\implies$  either  $dT'a$  or  $dT'b$
- $d$  must defeat  $b'$  in at most 2 steps  $\implies$  either  $dT'a$  or  $dT'b'$
- $d$  must defeat  $e'$  in at most 2 steps  $\implies$  either  $dT'd'$  or  $dT'e'$
- $d$  must defeat  $f'$  in at most 2 steps  $\implies$  either  $dT'e'$  or  $dT'f'$
- $d$  must defeat  $g'$  in at most 2 steps  $\implies$  either  $dT'f'$  or  $dT'g'$
- $d$  must defeat  $h'$  in at most 2 steps  $\implies$  either  $dT'g'$  or  $dT'h'$

A way to minimize the number of additional points defeated by  $d$  is to retain  $dT'f, a, b, e', g'$ , so that  $s(d) = 5$ . Furthermore,  $a$  defeats any other point in at most two steps if one adds up to the existing arrows the following:  $aT'f, d', f', h'$ , so that  $s(a) \leq 4$ . It is obviously seen that actually  $s(a) = 4$ . Thus,  $UC^{mc}(T) \cap M(T) = \emptyset$ , which concludes the proof.

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