ELECTING COMMITTEES BY APPROVAL BALLOTING

TUĞÇE CUHADAROĞLU

106622004

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Prof. JEAN LAINÉ

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Tuğçe Cuhadaroğlu
1066622004

Tez Danışmanının Adı Soyadı : Jean LAINÉ
Jüri Üyelerinin Adı Soyadı: Remzi SANVER
Jüri Üyelerinin Adı Soyadı: Gilbert LAFFOND

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Abstract

We study committee election models where voters' approvals over candidates are collected and the election outcome is determined according to a predetermined voting rule, in particular issue-wise majority rule. The properties of these models are investigated with a special attention to representativeness of the election outcomes, where representativeness relates to the way they are consistent with the voters' preferences over committees.

Özet

Bu çalışmada, seçmenlerin adaylar hakkındaki onaylarının toplandığı ve seçim sonucunun önceden belirlenmiş bir seçim kuralına, özelde çoğunluk kuralına gore belirlendiği komite seçimi modelleri incelenmektedir. Bu modellerin özellikleri, seçim sonuçlarının temsiliyet özelliklerine odaklanarak araştırılmaktadır. Temsiliyet, seçim sonuçlarının seçmenlerin komite tercihleriyle tutarlılığı ile ilişkili olarak tanımlanmıştır.

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Preface

Committee election problems are particular types of multidimensional choice problems where a wide range of approaches and models have been developed, solution rules have been proposed and underlying properties have been analyzed. What makes committee elections a significant voting problem is its frequency of being realized as a real-life case. In other words, rather than an abstract theory, committee elections embrace an applied nature and the work done in this field conveys ideas applicable to real life, either in state politics or more basic problems such as the election of a school committee.

Two main approaches to committee election problems should be distinguished. First, committees can be assessed in accordance with the outcomes they produce. From this perspective what the voters care about and decide accordingly is not the candidates or committees by themselves but the actions that would be performed by those committees. This approach rests upon an outcome function, which assigns to each possible combination of candidates one or several outcomes, e.g.; elements of some space of policy decision.

A second approach, that this study adopts, is regarding a committee election problem as the election of a subset from a set of candidates. Given a society of voters, and a set of candidates, a predetermined voting rule is applied, which necessarily consists of balloting and selection procedures in order to assign candidates as the committee members. This approach brings forth a large amount of variations in the design of the models. A committee may either be composed of hierarchic positions such as the chair, major members and secondary members, or it can simply consist of equivalent members such as in the case of assemblies. This symmetry or asymmetry of the committees will influence the voting rule to be applied. Another potential asymmetry may result from specific characteristics candidates may share or not. This suggests that committee members should be chosen from different sets of candidates. Gender restrictions or age restrictions can be given as examples. Furthermore, candidates may have positions, which will determine the votes they will get from the voters. The case of ideological positions where the voters vote according to the distance between their own position and the relative positions of the candidates is the best-known example for this interpretation of candidates. To concretize, candidates have positions on a [0,1] line, 0 denoting extreme left and 1 denoting extreme right and voters have their own positions in that line and vote in favor of the specified number of candidates that are closest to their own positions.

Apart from various candidate interpretations, those models may differentiate according to the balloting procedures. Voters may cast their ballots either candidate-wise or on entire committees. Eventually, moreover many voting rules may be considered. Plurality voting, simple majority voting and variations of majority voting, transferablevoting procedures, scoring systems are some of the voting rules that are utilized in both real-life elections and in the theory of committee elections.

The committee election model that will be studied in this paper focuses on approval balloting and most of the time, issue-wise majority voting. By approval balloting, we mean that voters cast their votes over candidates in a dichotomous way, that is they vote in favor of the committee that is composed of their approved candidates. The election outcome is determined according to the number of votes each candidate has collected. Two cases can be considered; either the committees without any size restriction or the case of fixed size committees. In the former case, the elected committee is the candidate-wise majority committee, that is, the committee including the majority approved candidates. In the case of committees of given size k, we distinguish between two procedures. In the first one, the number of approvals in each ballot is restricted to the same size k, and the winning committee is determined according to the issue-wise majority rule with an additional restriction over the permissible vote matrices. In the second procedure, which we call sequential approval balloting, voters are free to approve as many as candidates they wish and the elected committee involves the k candidates having collecting the highest number of approvals (with eventually some tie-breaking rule).

The main purpose of this paper is to study the properties of this committee election method. Special attention is paid to the representativeness properties of its outcomes, where representativeness relates to the way they are consistent with the voters' preferences over committees.

The paper is organized in the following way: The first chapter reviews the existing literature on multidimensional choice models with a special focus on the equilibrium conditions of majority voting. The committee election model based on approval balloting is introduced in the second chapter and the properties of two different voting rules over this model are analyzed. The third and fourth chapter focuses on the representativeness qualities of approval balloting in unrestricted commitee elections and fixed-size committee elections, respectively.

Chapter 1
Literature on satisfactory outcomes with majority voting

The search for a “satisfactory” outcome in multiperson decision-making mechanisms under the majority rule goes back to the old days that social choice theory began to emerge as a discipline. After Arrow’s classical work (1951), the major line of interest problematized the possibility for irrational choice through mechanisms which aim at respecting the majority will. With the characterization of May (1952), majority rule is promoted to be a satisfactory voting rule which fulfills some reasonable properties when the alternative space is dichotomous. However, an increase in the number of alternatives brings out the existence problem of an unbeaten alternative under majority rule, which had been the most discouraging property of it since Condorcet (1785). With Arrow, what became visible was that any social choice rule that takes into account not only the preferences of one predetermined person, namely the dictator, may result in socially "irrational" outcomes. Hence, a major line of research focused on the search for the conditions that would result in satisfactory outcomes without giving up the respect for majority will.

Although what is meant by being “satisfactory” changes throughout the literature, various stability concepts such as equilibrium, electing the Condorcet winner, or Condorcet consistent concepts such as Core, Top Cycle, Copeland winner, Uncovered Set or closeness to Condorcet winner are used to qualify those "satisfactory" outcomes. An equilibrium point is usually defined as a stable point in the sense that no majority of the voters would prefer to deviate from. The existence of an equilibrium is simply interpreted as the existence of an unbeaten alternative. In the cases that indifference relation between alternatives is allowed, the unbeaten alternative need not to be unique. The set of undefeated alternatives by a majority of voters is defined as the Core. Under the restriction of the admissible preferences to strict profiles and with odd numbers of voters, an unbeaten alternative will clearly be an alternative that majority defeats any other alternative, namely the Condorcet winner. Hence in the absence of indifference relation, Core necessarily consists of a unique element, the Condorcet winner.

The median-voter theorem suggests the first and the best-known model that guarantees the existence of an unbeaten alternative under majority voting. As established by Black (1948, 1958), whenever some particular restrictions upon the preferences of the voters and set of alternatives are sustained, an alternative cannot be beaten by any other alternative under majority voting if and only if it is the most preferred choice for the median-voter. The median-voter theorem requires (i) the existence of a linear ordering of the alternatives, (ii) strict-convexity of preference ordering of each individual over this set of alternatives. These two restrictions shape the set of preference orderings called as single-peaked preferences. To clarify, to have single-peaked preferences over the set of possible alternatives means that for each voter there is a best possible alternative in the linearly ordered alternative set and whenever s/he moves away from this alternative, either to the left or right, s/he will be less and less satisfied. Therefore, despite the quite restrictive assumptions, the median voter theorem set forth a way to escape from intransitive social choice and lead to a literature that investigates similar conditions to single-peakedness such as weak single-peakedness, single-cavedness, separability, value-restrictedness, extremal restriction, limited agreement and generalized exclusion. Major work in this line of research, as listed in the literature survey of Coughlin (1990), is due to Inada (1964, 1969), Ward (1965), Sen (1966, 1969), Sen and Pattanaik (1969), Pattanaik (1968, 1970a, 1970b) and Pattanaik and Sengupta (1974).

The non-existence of majority equilibrium in multidimensional choice space was first illustrated in Black and Newing (1951) and Black (1958). The median voter theorem is generalized to this setting by Tullock (1967a, 1967b). He showed that single-peaked indifference curves will ensure the existence of equilibrium outcome. Later on, this model is generalized by Grandmont (1978) by altering the shape of individual distributions over the set of preference relations. Kramer (1973) contributes to this literature by showing that the equilibrium conditions for majority rule equilibrium such as single-peakedness, weak single-peakedness, single-cavedness, separability, value-restrictedness, extremal restriction, limited agreement and generalized exclusion are extraordinarily restrictive for a voting equilibrium when applied to multi dimensional models because these conditions fail if there exists just three voters with contrasting preferences.

Resent research in this field is due to Holland and Le Breton (1996) and Vidu (1998, 1999, 2002). All these works consider the multidimensional choice spaces (either dichotomous or not) and show that under separable preferences which are single-peaked on each dimension, majority cycles remain in pair-wise relations over the outcome sets.

Once the search for stable majority outcomes goes beyond unidimensional models, following is a huge literature on spatial models of voting that establishes lack of equilibrium by changing the assumptions slightly and analyses the conditions that will result in majority equilibrium in those spatial models.

1.1 Search for equilibrium in spatial models of voting

Spatial models of voting consider voting on multiple issues where alternative social states are viewed as points in a convex policy space, such as E^n. Voters are assumed to have "positions" in the alternative space and their preferences on alternatives are shaped according to the distance to this position. The researchers in this wave were mostly in pursuit of majority equilibrium, defined as the majority undefeated point. The first representative of this line of research is Plott (1967). In his pioneering work, Plott builds necessary and sufficient conditions for equilibrium in multidimensional majority rule spatial voting games under the assumptions of (i) finite number of voters; (ii) differentiable individual utility functions that represent voters' preferences; (iii) one voter's ideal point coincides with the equilibrium social state. Under this premises, a local majority equilibrium exists if and only if a pair-wise symmetry condition is satisfied, which states that all of the voters except the one with the equilibrium as ideal point can be paired such that all nonzero utility gradients of voters in each pair point exactly opposite directions.

Following Plott's work, Davis and Hinich (1968) and Davis, DeGroot and Hinich (1972) deal with infinite population cases, where preferences can be represented by quadratic utility functions on E^n. Davis et al. (1972) prove that under the premises of Plott's model, a point x is a dominant point if and only if any hyperplane containing x divides the voter ideal points such that at least one half lie on either closed side of the hyperplane. This can obviously be read as a strong symmetry condition analogical to Plott's. The generalization of the spatial models of Plott and Davis et al. are studied to different extents by Sloss (1973), Wendell and Thorson (1974), Hoyer and Mayer (1975), McKelvey, Ordeshook, Ungar (1980). McKelvey and Wendell (1976) reviews the previous work of spatial voting models in multiple dimension, set some equivalence conditions among them and generalizes those models in a way to reach global equilibrium conditions not only under differentiable utility assumptions but also very general voter preference assumptions.

Thus, at the end of 70's the voting theorists in this wave have already agreed upon the rectricted nature of majority equilibrium conditions in multidimensional election problems. A more pessimistic result about the utilization of majority voting for decision making in real life politics is demonstrated by McKelvey (1976). McKelvey shows that when equilibrium collapses, it really collapses. More formally, in the absence of a Condorcet winning outcome, "the intransitivities extend to the whole policy space in such a way that all points are in the same cycle set." Thus, he destroys the more optimistic idea of selecting from the top-cycle, which is the set of alternatives that majority beats the ones outside, in the absence of an obviously winning outcome. He shows that any alternative, even a Pareto dominated one, is attainable through a specific sequence of votes, or to quote from himself "it is theoretically possible to design voting procedures which, starting from any given point, will end up at any other point in the space of alternatives, even at Pareto dominated ones." This discouraging result leads to a further literature of so-called "chaos theorems" (Riker 1980) and deepened and generalized by Bell (1978), Cohen (1979), Cohen and Matthews (1980) and Schofield (1978a, 1978b, 1983, 1985).

Feld and Grofman (1987) review this literature on the majority equilibrium conditions of spatial voting models and show that when they stick to the simple case that "all voters have an ideal point in the policy space and voters order the alternatives by how close they are to this ideal", most of the famous results done in this field (also reviewed here) can be stated in the course of the median voter theorem.

Following those spatial models of the first line of research on stable outcomes in multi-dimensional majority voting, voting theorists open up new lines of related research. One way to go forward is to investigate what kind of a majority can be sufficient to guarantee satisfactory results. A second idea is to weaken equilibrium requirements by defining choice functions that are Condorcet consistent, that is, which select the Condorcet winner when it exists. Making use of issue-wise majority rule and working in more particular settings that are inspired by real-life multiple issue voting problems are the extents that search for satisfactory outcomes keeps on.

1.2 d-majority equilibrium

Greenberg (1979) is one of the first to investigate what kind of majority rules give rise to equilibrium results other than simple majority rule, which can achieve this objective in excessively restricted conditions in multi-dimensional models. Hence, he sets the conditions that give rise to a d-majority equilibrium, defined as the choice of the alternative that no other alternative is preferred to this alternative by at least d individuals. Under a convex and a compact alternative set of dimension m, a d-majority equilibrium exists whenever d is greater than $(m/(m+1))n$, n being the number of voters. Greenberg's results trigger interest in d-majority equilibriums and many scholars including Slutsky (1979), Coughlin (1981), Nitzan and Paroush (1984), Greenberg and Weber (1985) search for equilibrium conditions under d-majority rules.

1.3 Weaker equilibrium concepts

Another idea that brought together huge diversification afterwards is to weaken the concept of equilibrium. In most of the previous models the search for equilibrium results has been the search for undominated outcomes and it has been established again and again that the absence of a majority undominated alternative is the norm rather than an exception. Hence choice functions like uncovered set, the Copeland winner or weaker criteria like Pareto-optimality become the focus of search in a wide range of models with a variety of assumptions on preferences (such as separability), space of alternatives (unidimensional, multidimensional, dichotomous, multichotomous) or voting behavior (sincere, sophisticated, simultaneous, sequential).

Uncovered set is the set of alternatives that a majority of the voters prefer to any other alternative either directly or at one move. To put formally, with P as the strict preference relation over alternatives, if x is in the uncovered set, then for all y, either xPy or there exists z such that zPy and xPz. Uncovered set is studied in Banks (1985), McKelvey (1986), Miller (1977, 1980, 1983), Shepsle and Weingast (1984).

A Copeland winner is an alternative defeated by the fewest number of alternatives (Copeland 1951). The Copeland winner coincides with the core if it exists. Glazer, Grofman, Noviello and Owen (1987) suggest Copeland as a stable solution concept in multidimensional spatial voting models especially because it always exists and in general uniquely exists; is included in the uncovered set and Pareto set, thus respects what most voters want. They analyze the characteristics of the Copeland winner (which they call as the strong point) in the context of legislative voting under the simple majority rule and characterize it in terms of a modification of Shapley value. Henriet (1984), Owen and Shapley (1985) and Straffin (1980) are some of the other researchers worked on Copeland winner as a stable outcome of majority voting.

Pareto-optimality is one of the oldest and mostly utilized criteria to qualify the minimum level of satisfactory outcomes. A Pareto-optimal outcome is an outcome which is not rejected under the unanimity rule. There does not exist any other alternative that all of the individuals in the society will be as happy as with that outcome and at least one will be happier. Formally, given a profile, a point x is said to be Pareto optimal, if there does not exist any other point $y,$ such that all of the voters are at least as good as with y compared to $x,$ and at least one voter prefers y to $x.$ Hence, it will not be too demanding to expect Pareto-optimal outcomes from a solution concept. All of the equilibrium concepts mentioned above; Core, Copeland winner, Uncovered Set and actually any refinement of the Uncovered Set will yield outcomes from the Pareto set, the set of Pareto-optimal points.

1.4 Issue-wise majority rule

In the multidimensional cases an alternative way to select an unbeaten alternative, if it exists, is the "division of the question". Kramer (1972) and Kadane (1972) consider issue-wise majority rule as a way to do so. Kramer shows that when the choice space is Euclidean, the Condorcet winner will be chosen under majority rule when votes are cast issue-wise by sophisticated voters. Kadane considers the case of multiple binary issues and shows that under separable preferences the issue-wise majority rule selects the Condorcet winner, whenever it exists. Separability of preferences is essential to this result. The intuition behind Kadane's proof is simple: Under the separability, the issue-wise majority platform will defeat either directly or indirectly any other platform.

This result is later extended by Schwartz (1977) to the context of vote trading and sophisticated voting. He shows that Kadane's result valid under either sophisticated or sincere or simultaneous or sequential voting. Therefore, Kadane's and Schwartz's results promote the use of issue-wise majority rule as a decisive tool in the settings that separation of issues is reasonably applicable such as committee elections or referendum voting. What is crucial to have satisfactory outcomes with issue-wise majority rule is the separability of preferences. As the voting rule takes into account only the majority will issue-wise, the outcome will be a good representative of voter preferences in cases individual will over issues depend on only the issue in consideration.

Outcomes of the issue-wise majority rule for multiple binary issues under non-separable preferences is deeply analyzed by Lacy and Niou (2000). They problematize the representativeness property of voting by referendum. In their own words, they search whether its outcomes give an answer to the question "what did the people want?". Thus, representativeness of a voting rule is interpreted in line with its ability to choose consistently with the preferences of voters. Issue-wise majority rule has been promoted to be representative of voters preferences if they are issue-wise independent. What Lacy and Niou show is that once preferences over separate issues become dependent to each other for some voters, issue-wise majority rule is not that much successfull to represent voters preferences. Consider the following simple example given in their work;

Given 3 voters and 2 issues, voters either accept or reject each issue by stating Yes or No on each issue. Following is the table of preference rankings of voters over possible outcomes.

Rank	**Voter 1**	**Voter 2**	**Voter 3**
1	YN	NY	NN
2	YY	YY	YY
3	NY	YN	NY
4	NN	NN	YN

It is easily seen that under sincere voting, the outcome of the issue-wise majority voting will be the rejection of both issues even if this is the majority defeated outcome. In addition, there is an obvious winner in this profile, which is the Condorcet winner YY, exactly the opposite of the issue-wise majority winner.

The election of a Condorcet loser is not, under sincere voting, the worst possible scenario, as illustrated by the following example;

Rank	Voter 1	Voter 2	Voter 3
1	YYN	YNY	NYY
2	YNY	NYY	YYN
3	NYY	YYN	YNY
4	NNY	NYN	YNN
5	YNN	YNN	NYN
6	NYN	NNY	NNY
7	NNN	NNN	NNN
8	YYY	YYY	YYY

Notice that the issue-wise majority winner is YYY which is the worst preference of all of the voters. Thus, in addition to electing a Condorcet loser in the existence of a Condorcet winner, under nonseparable preferences, sincere and simultaneous issue-wise majority voting may result in outcomes that are Pareto-dominated by any other alternative. What Lacy and Niou propose to the problem of nonseparable preferences is sophisticated sequential voting, which will ensure the election of the Condorcet winner, hence yield a stable and representative outcome.[1]

1.5 Search for stable outcomes in committee elections

Committee elections are particular types of multidimensional voting problems, where a given set of voters are faced with a given set of candidates and supposed to select a number of candidates from this set according to a predetermined voting rule. Certainly, the summarized literature can be read as the early literature of equilibrium conditions in committee elections where majority voting is used.

[1] The election of the Condorcet loser, or the election of a Pareto-dominated outcome are not the only examples of the representativness failure of the issue-wise majority rule. Another one is brought by the Paradox of Multiple Election (Brams, Kilgour and Zwicker, 1998; Scarsini 1998), which states that issue-wise majority winner may be shared as an ideal by a minimum number of voters.

A stable outcome in committee elections may refer to a representative committee for the given society of voters. A committee is qualified to be representative to the extent of its consistency with the preference profile of society. In other words, representativeness refers to the ability to reflect voters' preferences over committees. For instance, a Condorcet committee, being a stable outcome, is a good representative of voters committee preferences.

In committee election models, a Condorcet committee is defined in two different ways. In the first approach the premises are the voter preferences over committees. The Condorcet committee is defined as the winner of the pair-wise contests among committees (Fishburn (1981), Bock, Day and McMorris (1998)). The second approach focuses on the preferences over candidates and defines a Condorcet committee as a committee consisting of m members that would defeat every other candidate outside the committee in pair-wise contest. (Gehrlein (1985), Ratliff (2003)).

In line with the first approach, Fishburn (1981) shows that either with dichotomous or single-peaked preferences over candidates when these preferences over candidates are extended to preferences over committees in consistence with separability, those separable preference profiles over committees will have a Condorcet winner committee in single-member or all-but-one member cases. On the other hand, it is not possible to ensure this existence when the number of members is in between.

Following the second approach to Condorcet committees, Gehrlein (1985) investigates the probability of existence of a Condorcet committee. His work continues with the calculations of the likelihood that several voting rules will select the Condorcet winner.

Ratliff (2003) proposes electing the committee that is "closest" to being a Condorcet winner, when itself does not exist. Two different approaches are used to define this "closeness" to a Condorcet winner. Given the complete and transitive preference profiles of the voters, a k-size Dodgson's Committee consists of the k candidates that "requires the fewest adjacent switches in the voters' preferences to become the Condorcet winner", whereas, Kemeny's method considers all of the pair-wise contests and elects the committee that minimizes total margin of loss to be the Condorcet winner. Surprisingly, given a profile the Dodgson's committee and Kemeny's committee need not to coincide or even have common members. Ratliff's suggestion is an alternative to the utilization of majority rule as a decisive tool where the "the most stable" alternative according to majority rule does not exist.

Chapter 2
Approval balloting and committee elections

As a relatively young voting rule, approval voting has been introduced to literature at the end of 70's by Brams and Fishburn (1978). Since then, it became the focus of heated debates among voting theorists. Advocates of approval voting presented it as a practical and efficient voting rule as it is easy to conduct and it yields 'stable' outcomes under certain conditions. As it does not restrict the voter to vote in favor of a predetermined number of voters, it is claimed to promote sincere voting. And, most favorably, under dichotomous preferences, approval voting is proved to be the only single-ballot system with outcomes in the Core and with undominated strategies. (Brams and Fishburn (1978, 1981), Brams (1980)). A major opposition against approval voting was due to the fact that dichotomous preferences is essential for the nice results of approval voting to hold. Niemi (1985) showed that in the absence of dichotomous preferences, voters are inclined towards strategic voting, in addition, a Condorcet winner may not be selected both under sincere or sophisticated voting.

The idea of making use of approval balloting as a means of electing committees has been introduced by Brams, Kilgour and Sanver (2004, 2005, 2007). It should be noted here that what is proposed in their work and what will be studied in the sequel is not approval voting but approval balloting. Voters ballots indicating the approved candidates are collected and the elected committee is determined according to a specified procedure which is not necessarily majority voting. Brams et al. proposed two different methods to elect committees when voting ballots are approval ballots and the preferences of voters are extended to preferences over committees in a very specific way of extension. As will be analyzed in details in the following parts, approval balloting allows to consider majority will over candidates, which makes it a quite appealing election procedure as long as respecting the majority will is taken to be a representativeness criterion. Because, depending on the results of Kadane (1972) and Schwartz (1977), what is known is that the issue-wise majority winner of the approval ballots is necessarily the Condorcet winner, whenever it exists.

As long as there is no restriction upon the committee size, issue-wise majority voting on approval ballots works in the following way: Are elected all those candidates who are more often approved than disapproved.

Consider a society of voters $I = \{1, ..., n, ...N\}$ and a set of candidates $C = \{1, ..., q, ..., Q\}$. A committee is a subset of C. Each voter $n \in N$ casts an approval ballot in favor of the candidates s/he approves as committee member: the approval ballot of the voter n, defined as x_n is a vector of 0's and 1's, i.e.; $x_n = (x_n^q)_{q=1,\ldots,Q} \in \{0, 1\}^Q$ where $x_n^q = 1$ means that voter n approves candidate q as a committee member and $x_n^{q'} = 0$ denotes the disapproval of candidate q' by n. It should be clear that an approval ballot corresponds to a committee $x \in \{0, 1\}^Q$ with k members, where

$k = \mid \{x^q \in x : x^q = 1\} \mid$. Let $\Omega = \cup_{Q\geq 1}\{0,1\}^Q$ be the set of committees. Notice there exists 2^Q admissible approval ballots and hence 2^Q committees including the degenerate cases of no-member committee and all-member committee.

A (N, Q)**-ballot** is a vote matrix $X^{NQ} = [x_n^q]_{n=1,...,N}^{q=1,...,Q}$, where row n corresponds to voter n's approval ballot. Let $\mathcal{X} = \cup_{N,Q\geq 1} X^{NQ}$. The issue-wise majority committee of X^{NQ} will be the one that consists of the candidates whose number of approvals exceeds the number of disapprovals. To put formally;

Definition 1 *Let N be odd and let X^{NQ} be a (N, Q)-ballot, the majority committee $m(X^{NQ}) = (m^1, ..., m^Q) \in \{0,1\}^Q$ is defined by: $\forall q = 1, ..., Q$, $\mid \{n = 1, ..., N : x_n^q = m^q\} \mid > \frac{N}{2}$.*

This is the usual outcome of committee elections with approval balloting.

Example 1 *Consider the case $N = 5, Q = 3$ with the following approval ballots;*

$x_1 = (1, 1, 0)$

$x_2 = (0, 1, 0)$

$x_3 = (0, 1, 1)$

$x_4 = (1, 1, 1)$

$x_5 = (0, 0, 1)$

Obviously, the issue-wise majority winner will be $m(X^{NQ}) = (0, 1, 1)$, the committee excluding only the first candidate, as the first candidate is not supported by at least half of the voters.

Approval balloting provides an incomplete information about preferences: Candidates are separated into two groups; the ones who are approvable (hence, under sincere voting the ones liked by the voter) and the ones who are disliked. Thus, approval balloting works as if candidates within each group are assumed to be indifferent for the voter. In other words, approval balloting does not reveal individual's rankings over candidates. Consider the following examples how far this information loss can go in the case of a fixed size committee;

Example 2 *Let $Q = 4$ with $C = \{a, b, c, d\}$ and $N = 27$. Assume the voter preferences over candidates are as given in the following table;*

5	**8**	**8**	**4**	**1**	**1**
a	b	a	d	c	b
b	c	d	c	d	a
c	d	c	b	a	d
d	a	b	a	b	c

The numbers in the first row indicate the number of voters who have the preference order in the corresponding column, i.e.; exactly 5 voters rank the candidates from the most preferred to the least preferred as a,b, c, d, 8 voters have the ranking b, c, d, a, and etc.

In this example, there are 6 groups among 27 voters with identical preference orders within the group . One can check that a is the winner of pair-wise contest, namely the Condorcet winner.

Now, let us check what will be the composition of committees that will be chosen by approval voting under sincere voting. The assumption is that voters approve as *many candidates as the size of the committee that will be chosen and the candidates with the most number of approvals are elected.*

Hence, if a single candidate will be chosen, this will be the Condorcet winner a *with the* 13 *approvals.*

In a two-member committe election, the approval votes will be like:

number of voters	**approval ballot**
5	$(1,1,0,0)$
8	$(0,1,1,0)$
8	$(1,0,0,1)$
4	$(0,0,1,1)$
1	$(0,0,1,1)$
1	$(1,1,0,0)$
total	**approval voting winner**
27	$(1,1,0,0)$

Thus, Condorcet winner a *and another friend* b *are included in the committee with exactly the same number of votes* 14.

In case a committee of size 3 *will be elected, the approval ballots will be*

number of voters	**approval ballot**
5	$(1,1,1,0)$
8	$(0,1,1,1)$
8	$(1,0,1,1)$
4	$(0,1,1,1)$
1	$(1,0,1,1)$
1	$(1,1,0,1)$
total	**approval voting winner**
27	$(0,1,1,1)$

Notice that the only candidate that is not included in a committee of three is the remaining Condorcet winner!

Excluding the natural single winner is not the worst situation that can appear in this setting. Consider the next example where the elected committee is exactly the opposite of the winner of majority tournament among candidates, where majority tournament over the candidate set is defined as;

Definition 2 *Given a set of candidates* $C = \{1, ..., q, ..., Q\}$, *and a set of voters* $I = \{1, ..., n, ..., N\}$, *where* N *is odd, with individual preference orders over candidates denoted by* $\prec_n$, *the majority tournament* T *over the given preference profile is such that;* $\forall q, q' \in C, q\, T\, q' \Leftrightarrow \mid n \in I : q \succ_n q' \mid > \mid n \in I : q'_n \succ_n q \mid$

Observe that if the majority tournament among alternatives is well-defined, i.e., yields a transitive ranking of candidates, then the winner of the majority tournament will be precisely the Condorcet winner of the alternative set.

Now, let us consider the following example which points out how approval balloting may provide a committee which does not involve the first best candidates of the transitive majority tournament over candidates;

Example 3 *Let* $Q = 5$ *with* $C = \{a, b, c, d, e\}$ *and* $N = 29$. *Assume the voter preferences over candidates are as given in the following table;*

9	**8**	**6**	**6**
a	e	e	b
c	d	d	c
b	b	a	a
d	c	b	d
e	a	c	e

Similarly to the previous example, 9 *voters strictly order the candidates as* $a \succ c \succ b \succ d \succ e$, 8 *voters have the order* $e \succ d \succ b \succ c \succ a$ *and so on.*

The majority tournament over this profile is well defined and yield the following preorder: $a\ T\ b\ T\ c\ T\ d\ T\ e$. *Thus, a way to proceed to elect a committee of size* 2 *may be to select the most favored candidates* a *and* b [2].

On the other hand, consider the winning committee of size 2 *when approval ballots are utilized. The election method is assumed to be like in the former example.*

number of voters	**approval ballot**
9	$(1, 0, 1, 0, 0)$
8	$(0, 0, 0, 1, 1)$
6	$(0, 0, 0, 1, 1)$
6	$(0, 1, 1, 0, 0)$
total	**approval voting winners**
29	$(0, 0, 1, 0, 1)$ $(0, 0, 1, 1, 0)$

Again, the approval voting outcome, in both cases will exclude the two most favored candidates.

At first sight, it may seem a little bit surprising that in the previous examples the only candidates that is not included in the committes are the Condorcet winning candidates. However, once approval ballots from the specified preference orders are collected in this fashion, then the profiles under consideration is reduced or altered to the following ones respectively;.

For the first example;

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[2] In Laffond and Lainé (2008) this method of electing the k-first best issues in the majority tournament among alternatives is defined as Decision-wise procedure, whereas respecting the majority tournament over extended profiles is said to be Direct procedure. Laffond and Lainé shows that the winner of the decision-wise procedure, if it exists, can be defeated and even worse covered in the majority tournament among programs where rank-based, monotone and independent extension rules are used to extend preferences over committees to preferences over programs.

5	**8**	**8**	**4**	**1**	**1**
$a \sim b \sim c$	$b \sim c \sim d$	$a \sim d \sim c$	$d \sim c \sim b$	$c \sim d \sim a$	$b \sim a \sim d$
d	a	b	a	b	c

For the latter;

9	**8**	**6**	**6**
$a \sim c$	$e \sim d$	$e \sim d$	$c \sim b$
$d \sim b \sim e$	$b \sim c \sim a$	$b \sim c \sim a$	$e \sim d \sim a$

with $\sim$ as the indifference relation between candidates. Thus in the former, the original first group of 5 voters with the strict preference over a, b, c, d in the specified order, now are assumed to be indifferent among the candidates a, b, c and prefer those to the last one, d. As can be easily noticed, in this new profile, a is not the Condorcet winner but Condorcet loser, that is beaten by any other alternative in pair-wise contest.

Similarly, in the second example, the former Condorcet winner a and her closest friend b are now defeated by the remaining candidates c, d, e.

Thus, approval balloting works as if preferences over candidates are dichotomous ones.

Inada (1964) had proved that if each voter classifies all alternatives into two indifference groups, then there will be a majority undefeated alternative. Thus, dichotomous individual preference orderings imply the existence of a stable outcome. Later on, Brams and Fishburn (1978) showed that under dichotomous preferences, the approval voting outcome will be a majority undefeated alternative, which implies the election of Condorcet winner if it exists. But, all these results work when a single candidate is elected, thus do not provide insights about how to elect committees involving more than 1 member.

To comment on the representativeness of approval balloting in committee elections, we need to consider voter preferences over committees rather than preferences over candidates. With approval balloting, what can be observed from voters' ballots is only a profile of dichotomous preferences over candidates. Hence the domain of preferences over committees that are compatible with such approval ballots is quite large. What we know is that, even under the assumption of separable preferences, the committee chosen through approval balloting may fail to be representative of voters' preferences. Ozkal-Sanver and Sanver (2006) shows that, under the assumption of separable preferences, it is impossible to guarantee Pareto optimal outcomes through any kind of anonymous voting in multiple dichotomous choice models, in particular through issue-wise majority rule. Thus, as long as voting rules that does not discriminate among the voters are used, which is quite reasonable, approval balloting outcomes may be Pareto-dominated, which means that can not sustain even the minimum representativeness requirements. Further, when the commonly-used separability assumption is released, as illustrated by Lacy and Niou (2000) and Ratliff (2006) a committee which is the last preference of all of the voters may be the election outcome. To note, this non-separability of preferences would not be a meaningless assumption in committee election settings, even further it may be the actual case in most of real life election problems (Ratliff 2006). Nevertheless, ignoring potential spillover effects and restricting the profiles to separable ones does not help to ensure representable outcomes. Consider the following case where the preferences over candidates in Example 2 is used;

Example 4 $N = 27, C = \{a, b, c, d\}$

5	**8**	**8**	**4**	**1**	**1**
a	b	a	d	c	b
b	c	d	c	d	a
c	d	c	b	a	d
d	a	b	a	b	c

Again a committee of 3 will be selected and preferences over candidates are extended to preferences over 3-sized committees in the following separable and lexicographic-type way: $\forall n \in N, \forall k = 1, 2, 3, 4;$ *the* k^{th} *best ranked committee will be the one that excludes the* k^{th} *worst ranked candidate. This type of lexicographic extension leads the following profile over 3-sized committees;*

5	**8**	**8**	**4**	**1**	**1**
1110	*0111*	1011	*0111*	1011	1101
1101	1110	1101	1110	*0111*	1110
1011	1101	1110	1101	1110	*0111*
0111	1011	*0111*	1011	1101	1011

Under approval balloting, the elected committee is $(0, 1, 1, 1)$. *However, the committee* $(1, 1, 1, 0)$ *is the Condorcet winner of the profile and the remaining two committees* $(1, 0, 1, 1)$ *and* $(1, 1, 0, 1)$ *defeats the approval balloting outcome* $(0, 1, 1, 1)$ *in pair-wise contest! Thus, a majority defeated outcome is the winner of issue-wise majority voting over approval ballots.*

This example illustrates how poor approval balloting can behave to be representative of voter preferences deduced from the ballots. Now, consider another example where preferences over committees are extended again from the same profile over candidates, but with a different extension rule;

Example 5 $N = 27, C = \{a, b, c, d\}$. *A committee of size 2 will be elected, therefore voter preferences over 2-sized committees are considered. Voters rank the committees according to the inclusion of the best candidate and the second-best candidate in the following way;* $\forall n \in N$, *a committee with the most-preferred 2 candidates is the best. Then, the committees that include the best candidate but not the second-best candidate are ranked as the second-preferred committees, the third-ranked committees are the ones that include the second-best candidate but not the first-best one. And at last place, comes the committees that does not contain any of them. This extension leads the following profile:*

5	**8**	**8**	**4**	**1**	**1**
1100	0110	1001	0011	0011	*1100*
1010	*1100*	*1100*	1001	1010	0110
1001	0101	1010	0101	0110	0101
0110	1010	0101	1010	0101	1010
0101	0011	0011	0110	1001	1001
0011	1001	0110	*1100*	*1100*	0011

Notice that, the outcome of the issue-wise majority rule over approval ballots is $(1, 1, 0, 0)$, *which is the Condorcet winner of the profile.*

This is an example of the profiles where approval balloting is representative of the underlying preferences in the sense that it yields Condorcet consistent outcome.

The two different examples given here may give some clue about the range that the winning commitees will be representative of the voters preferences: it is possible to have counter-representative committees as well as representative committees depending on the admissible preference profiles. Thus, an intuitive path to follow is to stick to one particular type of preference profile and investigate the representative features of approval balloting over those profile. What Brams, Kilgour and Sanver (2004, 2005, 2007) did in successive papers is to introduce a very appealing definition of preferences over committees, preferences based on Hamming distance, that favors approval balloting as a representative method. Hamming extension rule orders committees according to the number of the members that coincide with the approval ballots with the additional assumption that approval ballots reflect true preferences of voters. Following Brams et al (2007), we study below the properties of approval balloting under the Hamming extension rule.

2.1 Hamming extension rule

As cited in the review of literature, at the beginning of the search for stable majority outcomes with multiple issues, multidimensional spatial models were used to represent voters' preferences consistent with a distance criteria in the Euclidean space. In those spatial models, voters are assumed to have a position, a best-preferred alternative in the predetermined space and order the other alternatives, positions according to the distance to their own position. How far an alternative to a best-preferred position, it is less and less preferred by the voter whose preferences are in consideration.

With a similar intuition, Brams, Kilgour and Sanver make use of a distance criterion to extend preferences from approval ballots to preorders over committees.

Before introducing Hamming extension rule, we should define what an extension rule is.

We denote by Γ^Q the set of all complete preorders on $\{0,1\}^Q$. Let $\mathcal{R} = \cup_{Q\geq 1}^{Q} \Gamma^Q$.

An extension rule is a function R from Ω to $\mathcal{R}$ which associates with each committee $x \in \{0,1\}^Q$ an element $R(x)$ of Γ^Q. The asymmetric counterpart of R is denoted by P and I stands for the indifference part.

Observe that as R is defined from the set of any size committees to the set of complete preorders over any size committees; e.i. voter preferences are defined over varying size committees.

An extension rule R associates with each (N,Q)-ballot X^{NQ} a preference profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$. To simplify notation, we will denote the extended preorder from voter n's approval ballot as R_n instead of $R(x_n)$.

Hamming distance between two committees is simply the number of candidates they differ about. Formally, the Hamming distance $d(x,y)$ between two committees x and y is defined as;

Definition 3 *$\forall Q$, $\forall x = (x^1, ..., x^Q)$, $y = (y^1, ..., y^Q) \in \{0,1\}^Q$, **Hamming distance** between x and y is defined as* $d(x,y) =\mid \{q = 1, ..., Q : x^q \neq y^q\} \mid$.

Obviously, Hamming distance criterion induces a very natural ordering of preferences over committees such that the closer to the ideal will be ranked higher. Formally;

Definition 4 *R^{Ham} is said to be **Hamming extension rule** if the following condition is satisfied:* $\forall Q$, $\forall x, y, z \in \{0,1\}^Q$, $d(x,y) < d(x,z) \Leftrightarrow y\ P_{(x)}^{Ham}\ z$ *and* $d(x,y) = d(x,z) \Leftrightarrow y\ I_{(x)}^{Ham}\ z$.

Obviously, R^{Ham} induces a preference order with non-singleton indifference classes. We denote as $I_x(d)$ the indifference class of committees that are at distance d to the committee x, i.e.; For each $x \in \{0,1\}^Q, \forall y \in \{0,1\}^Q$ such that $d(x,y) = d$, d being a non-negative integer, $y \in I_x(d)$.

Example 6 *Consider the Hamming extended profile of voter n with an ideal committee of* $(1,0,0)$.

$\mathbf{I(0)}$	$\mathbf{I(1)}$	$\mathbf{I(2)}$	$\mathbf{I(3)}$
100	$110, 101, 000$	$111, 001, 010$	011

In the table above, $I(d)$, $d \in \{0,1,2,3\}$ denotes the indifference classes induced by Hamming extension rule. The table shows the preorder of voter n, which is formally; $(\mathbf{1,0,0})\mathbf{P}_n(\mathbf{1,1,0})\mathbf{I}_n(\mathbf{1,0,1})\mathbf{I}_n(\mathbf{0,0,0})\mathbf{P}_n(\mathbf{1,1,1})\mathbf{I}_n(\mathbf{0,0,1})\mathbf{I}_n(\mathbf{0,1,0})$

$\mathbf{P}_n(\mathbf{0,1,1})$.

Notice that the three committees in the indifference class $I(1)$ are the ones which are at distance 1 to the ideal and similarly the committees at $I(2)$ have 2

disagreements with the ideal. In the last indifference class, exactly the opposite of the ideal stands.

Being based on a symmetric distance criterion, Hamming extension rule attributes equal importance to the election of favorable candidates and to the exclusion of the unfavorable candidates. This symmetry property can be interpreted from two aspects:

i. Hamming extended preferences does not discriminate among the candidates above approval line or similarly among the candidates below the approval line.

ii. Under Hamming extension rule, the exclusion of a favorable candidate will have the same effect with the inclusion of an unfavorable candidate.

The first point above shows that Hamming criteria creates a "reasonable" extension method under approval balloting. For instance, consider two candidates approved by a particular voter. The committees that exclude one of those two candidates and include the remaining one will be at the same indifference class for the voter, other members kept constant. Thus, Hamming extension rule has nothing to do with the rankings of favorable candidates, which makes it quite consistent to use with approval balloting in this committee elections setting.

To assume Hamming extended preferences over committees is at the same time to assume another condition that is frequently used to restrict preferences in this kind of multiple binary issues settings: Separability. Separability is roughly can be defined as the independence of the decision concerning a candidate (or a set of candidates) from the decisions about other candidates. In this varying size-committee setting, separability refers to the comparison of any two committees independent of their common decisions about candidates.

Before giving the formal definition of separability, we introduce a quick definition of a sub-committee, which will be used in some of the following definitions and proofs:

Definition 5 *Let $Q \geq 1$, $x \in \{0,1\}^Q$ and $B \subseteq \{1,...,Q\}$ be a subset of Q' candidates. The* ***sub-committee*** *x/B is the element of $\{0,1\}^{Q'}$ defined by:$\forall q \in B$, $(x/B)^q = x^q$, e.g.; x/B is the sub-committee of x that indicates the approval or disapproval of all candidates in B.*

Definition 6 *R is said to be* ***separable*** *(S) if the following condition holds; $\forall Q$, $\forall x, y, z \in \{0,1\}^Q$, $(y/Q^{y \neq z})\ R(x/Q^{y \neq z})\ (z/Q^{y \neq z}) \Rightarrow y\ R(x)\ z$, where $Q^{y \neq z} = \{q = 1, ..., Q : y^q \neq z^q\}$.*

Under separability axiom, if a voter approves a candidate as a committee member, then s/he will always prefer the committees including that candidate to the ones excluding that candidate, without any change in the other members.

Example 7 *Consider the Hamming preference order of individual n in the previous example;*

$\mathbf{I(0)}$	$\mathbf{I(1)}$	$\mathbf{I(2)}$	$\mathbf{I(3)}$
100	$110, 101, 000$	$111, 001, 010$	011

Consider the decision regarding the first candidate. Voter n prefers the candidate 1 to be in the committee, $(\mathbf{1},\mathbf{0},\mathbf{0})\mathbf{P}_n(\mathbf{0},\mathbf{0},\mathbf{0})$. Therefore, s/he prefers a committee with candidate 1 always to a committee without candidate 1, the other elements of the committee being kept constant; $(\mathbf{1},\mathbf{1},\mathbf{0})\mathbf{P}_n(\mathbf{0},\mathbf{1},\mathbf{0})$ and $(\mathbf{1},\mathbf{1},\mathbf{1})\mathbf{P}_n(\mathbf{0},\mathbf{1},\mathbf{1})$.

Lemma 1 *Any Hamming extension rule satisfies separability.*

Proof. Take any Hamming extension rule R. Assume, for a contradiction, R does not satisfy separability. Hence, there exists $n \in N$ and $x, y \in \{0,1\}^Q$ such that $(y/Q^{y\neq x})\ R_n(x/Q^{y\neq x})$ and $x\ P_n\ y$. Let $x_n \in \{0,1\}^Q$ denote the ideal of voter n. As $(y/Q^{y\neq x})\ R_n(x/Q^{y\neq x})$ and R is a Hamming extension rule, then $d((x_n/Q^{y\neq x}),(y/Q^{y\neq x}))$ $\leq d((x_n/Q^{y\neq x}),(x/Q^{y\neq x}))$, which in turn implies $d(x_n,y) \leq d(x_n,x)$. Therefore, y $R_n x$. ■

2.2 Minisum and minimax committees

Brams, Kilgour and Sanver (2004, 2005, 2007) in their successive studies propose two voting methods based on approval balloting to elect a representative committee. In those studies, the representativeness of the elected committee is assessed by means of a distance between committees and a vote matrix. The "closer" a committee to the approval ballots the more representative it is. This interpretation of representativeness is in line with what is said in this paper. A representative committee should allow a faithful definition of underlying voter preferences. As long as the committee rankings are compatible with a distance criteria, the closest committee will be consistent with the hidden preferences. "Closeness", here, is determined through two different methods. The former is the minimization of the sum of the distances to the vote matrix while the latter is the minimization of the maximum distance. To define the distance between two committees, they refer to the Hamming distance as defined above. A minisum outcome is defined to be the committee that minimizes the sum of Hamming distances to all approval ballots, while a minimax outcome minimizes the maximum distance to all ballots cast.

Brams, Kilgour and Sanver's setting is the one defined in details above. Voters' preferences are derived from their approval ballots according to Hamming distance criteria. From now on, let us assume that voters cast their ballots sincerely, thus their approval ballots coincide with their ideal committees. Actually, as proved by Brams et al. (2005) the minisum method does not necessitate this assumption as the voting procedure is not manipulable when the elected committee is the minisum outcome, while minimax outcome does not ensure sincere voting.

Note that while the balloting procedure is approval balloting, the election outcome is determined according to another procedure, which is not defined as the issue-wise majority rule.

However, Brams, Kilgour and Sanver (2004) prove that the minisum committee coincides with the issue-wise majority winner.

Proposition 2 ***(Brams et al. (2004) Proposition 4)*** *Given an* (N, Q)*-ballot* X^{NQ}*, the majority committee* $m(X^{NQ})$ *will be the committee that minimizes on* $\{0,1\}^Q$ *the total distance* $\Sigma_{n=1}^{N} d(y, x_n)$*, where* x_n *is defined to be the approval ballot of voter* $n \in N$*.*

The intuition under this proposition is quite simple. As the issue-wise majority winner $m(X^{NQ})$ is the committee that respects majority will on each candidate, hence minimizes the number of disagreeing voters on each candidate, it minimizes total number of disagreements either, which in turn coincides with minimizing sum of distances.

The second election method based on approval balloting proposed by Brams et. al is the election of minimax committee.

Definition 7 *Given any* X^{NQ}*,* $x \in \{0,1\}^Q$ *is said to be the* ***minimax*** *outcome if it minimizes on* $\{0,1\}^Q$ *the maximum distance* $Max_{n \in N}\{d(y, x_n)\}$*, where* x_n *is defined to be the approval ballot of voter* $n \in N$*.*

Brams et al. identify the minimax committee as a representative committee in the sense that "it does not antagonize any voter" too much.

Example 8 *The table below demonstrates the determination of minisum and minimax outcomes for the particular* X^{NQ} *given in the first example, with* 5 *voters,* 3 *issues and* $(0, 1, 1)$ *as* $m(X^{NQ})$*.*

In the first row the approval ballots of the voters take place. The first column shows all attainable committees with the specified number of candidates. Thus, minisum and minimax outcomes will be among the committees in the first column. Obviously, the integers under the ballots are the distances between the ballot and the corresponding committee. In the last two columns the total distances and the maximum distance of the corresponding committee to the approval ballots is denoted. As expected, the minisum outcome is $m(X^{NQ}) = (0, 1, 1)$ *with the minimum total distance of* 5 *and the minimax outcomes are* $(0, 1, 0)$ *and* $(0, 1, 1)$ *with maximum distances of* 2.

ballots	110	010	011	111	001	***total distance***	***max. distance***
000	2	2	2	3	1	10	3
100	1	2	3	2	2	10	3
010	1	0	1	2	2	6	**2**
001	3	2	1	2	0	8	3
110	0	1	2	1	3	7	3
101	2	3	2	1	1	9	3
011	2	1	0	1	1	**5**	**2**
111	1	2	1	0	2	6	3

Up to now, the size of the elected committees has been unrestricted. In other words, the number of the members involved in the elected committee is dependent on the approval ballots. However, most real-life elections require the election of a committee with a predetermined size, i.e.; a size restriction is imposed on the election outcome. Brams et al. show how their results can be adapted to such a case. In the case of a k-sized committee, the minisum outcome will be one that involves the k candidates collecting the highest number of approval votes (Brams, Kilgour and Sanver (2007)). Notice that in this case it is quite probable to have more than one minisum outcome. A minimax committee with a size restriction will be the one with the specified size and with the minimum maximum distance to the vote matrix. Consider the following example where a committee of size 3 is to be elected:

Example 9 *$N = 7, Q = 4, k = 2$ member committee will be elected.*

Example 10

$number\ of\ ballots$	$approval\ ballots$
2	0111
3	1100
2	0010

Observe that the 2 *sized committee involving the members with most approvals is* $(0, 1, 1, 0)$*, whose members respectively collect* $5, 4$ *approvals. The following table shows the distances of each admissible committee to the vote matrix.* $(0, 1, 1, 0)$ *is both the minimax and minisum outcome.*

approval ballots	**0111**	**1100**	**0010**	
number of approvals	**2**	**3**	**2**	**max.distance total distance**
1100	*3* *6*	*0* *0*	*3* *6*	*3* *12*
1010	*3* *6*	*2* *6*	*2* *4*	*3* *16*
1001	*3* *6*	*2* *6*	*3* *6*	*3* *18*
0110	*1* *2*	*2* *6*	*1* *2*	*2* *10*
0101	*1* *2*	*2* *6*	*3* *6*	*3* *14*
0011	*1* *2*	*4* *12*	*1* *2*	*4* *16*

Notice that in the examples given above, for both the restricted and unrestricted cases, the minisum outcome is also one of the minimax outcomes. But this does not have to be the case, as illustrated by Brams et. al (2007). Even, it is quite possible to have the minisum and the minimax outcomes as opposite committees. This triggers to search for the conditions under which minisum committee is included by the set of minimax outcomes. Although this question remains under investigation, the case of three candidates can be mentioned here:

Take any X^{NQ} with $(1,1,1)$ as the minisum outcome. This is not innocuous assumption, because through a relabelling of the issues, the same result can be reached for any case. The following table indicates all possible approval ballots, number of approvals, and the distances between each ballot.

approval ballots	**111**	**101**	**110**	**011**	**100**	**010**	**001**	**000**
number of approvals (weights)	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**
111	0	1	1	1	2	2	2	3
101	1	0	2	2	1	3	1	2
110	1	2	0	2	1	1	3	2
011	1	2	2	0	3	1	1	2
100	2	1	1	3	0	2	2	1
010	2	3	1	1	2	0	2	1
001	2	1	3	1	2	2	0	1
000	3	2	2	2	1	1	1	0

To have $(1,1,1)$ as the issue-wise majority winning committee, the following conditions have to hold;

$a+b+c+e>d+f+g+h$

$a+c+d+f>b+e+g+h$

$a+b+d+g>c+e+f+h$

Recall how minimax committee is calculated via the table: The maximum numbers in each row is found and the committee with the minimum maximum number is elected as the minimax committee. Let us call $w\in\{a,b,...,h\}$ as weights of the corresponding committees. Notice that if any of the $w \in \{a, b, ..., h\}$ is not equal to 0, that is if all the possible ballots are cast, $(1, 1, 1)$ will be one of the minimax committees as well as all the other ones. But, if any $w \in \{a, b, ..., h\}$ is equal to 0, meaning that the corresponding ballot is not cast, then the column of that committee can be removed from the table. In this case, the opposite committee of the removed committee will be the minimax winner as the maximum number in its row will be 2, while all the others are 3. Thus, the first condition to have $(1, 1, 1)$ as the minimax committee when at least one voter cast the ballot $(0, 0, 0)$ is to have all the other ballots cast as well.

Now, let us assume that $w(0, 0, 0) = 0$. Thus, we could remove the $(0, 0, 0)$ column from the table. In order to have $(1, 1, 1)$ as the minimax committee, there should not be any other committee without any 2 and 3 throughout its row. Hence, some particular weights should not be equal to 0 at the same time, that is each of the following conditions should not hold;

$f = d = c = 0$

$g = d = b = 0$

$b = c = e = 0$

$a = d = f = g = 0$

$a = b = e = g = 0$

$a = c = e = f = 0$

$a = b = c = d = 0$

Observe that the last four conditions are already inconsistent with the conditions that yield $(1,1,1)$ as the issue-wise majority outcome. Hence, we can drop them.

Notice that, each of the first three conditions are related to the ballots that agree with $(1,1,1)$ on one common issue. So, if the weight of at least one of $[(1,1,0),$ $(1,0,1), (1,0,0)]$, $[(1,1,0),(0,1,1),(0,1,0)]$ and $[(0,1,1),(0,0,1),(1,0,1)]$is different than 0, then $(1,1,1)$ will be the minimax outcome as well.

Formally;

Given $X^{N,3}$ and $m(X^{N,3})$ as the minisum committee, under each of the following conditions $m(X^{N,3})$ will be a minimax outcome:

(C1) If $w(m(X^{N,3})) = 0$, then for all distinct $x \in \{0,1\}^Q$, $w(x) \neq 0$.

(C2) Define as M_q, the set of committees that agree with $m(X^{N,3})$ on candidate q, i.e.; $x \in M_q \Leftrightarrow x^q = m^q(X^{N,3})$. If $w(m(X^{N,3})) \neq 0$,then $\forall q$, $\exists x \in M_q$ such that $w(x) \neq 0$.

The minisum and minimax committees are the outcomes of two different methods based on approval balloting. Given the approval ballots of the voters, with the assumption that voters vote for their most preferred outcomes, Hamming distance criteria is used to extend voters' preferences over committees. A quick observation is that the minisum outcome will be the utilitarian outcome if voters' preferences over committees is assumed to be represented by a utility function U inversely related with the distance to the ideal of the voter such that

Definition 8 *$\forall n \in I, \forall Q, \forall x, y \in \{0,1\}^Q$ function U over $\{0,1\}^Q$ is said to represent R if the following holds; $U_n(x) \geq U_n(y) \Leftrightarrow x \; R_n \; y$.*

Notice that, by definition of Hamming extension rule $R, U_n(x) \geq U_n(y) \Leftrightarrow d(x, x_n) \leq d(y, x_n)$ with $x_n \in \{0,1\}^Q$ as the ideal committee of voter $n,$ whose preference order is in consideration.

By this approach, the minisum outcome will definitely be the committee that maximizes total utility of the society in the sense that it solves on $\{0,1\}^Q$ the program: $Max_{x \in \{0,1\}^Q}\{\sum_{n \in \mathcal{N}} U_n(x)\}$.

On the other hand, the minimax outcome will be the one that maximizes the minimum utility in the society in the sense that it solves on $\{0,1\}^Q$ the program: $Max_{x \in \{0,1\}^Q}\{Min_{n \in \mathcal{N}} U_n(x)\}$.

Therefore, the minisum outcome represents the utilitarian approach, whereas the minimax committee can be interpreted as the egalitarian outcome. Thus, the representative qualities of these two committees will depend on the dominant values in the society, which is the subject matter of another discussion that will not be made here.

In the introduced methods of electing minisum and minimax committees, the number of the voters who are in favor of a particular committee does not influence the minimax outcome unlike the minisum outcome. Even if all voters but one cast the same approval ballot, these two committees, supported by the two types of voters, have the same influence in the determination of the minimax outcome. The minimax winner will be the one that lies in the middle of these two committees. Hence, extreme votes have "extreme" impact in the determination of the minimax committee. Although this feature of the minimax outcome is not inconsistent with the Rawlsian interpretation of minimax committee as the egalitarian outcome, Brams, Kilgour and Sanver (2005, 2005, 2007) propose two alternative weigthing system to overcome this vulnerability of the minimax outcome to extreme votes, respectively called the count weights and proximity weights.

Given a $X^{N,Q}$, a distinct vote matrix is a (M,Q) ballot, $X^{M,Q} = [x_m^q]_{m=1,\dots,M}^{q=1,\dots,Q}$ where M is the number of distinct ballots cast and the committee $x_m \in \{0,1\}^Q$ corresponds to the m^{th} row in $X^{M,Q}$. It is obvious that $M \leq N$. Weighted minisum committees are the ones that minimize the total weighted distance to the distinct vote matrix while weighted minimax committees minimize the maximum weighted distance to the distinct vote matrix. As expected, the weighted distance between two committees is the Hamming distance multiplied by a predetermined weight factor.

Definition 9 *Given $X^{N,Q}$, $x^* \in \{0,1\}^Q$ is a weighted minisum committee if it minimizes on $\{0,1\}^Q$, $\Sigma_{m=1}^{M}(d(x^*, x_m)w_m)$, where w_m denotes the weight given to committee x_m*

Definition 10 *Given $X^{N,Q}$, $x^{**} \in \{0,1\}^Q$ is a weighted minimax committee if it minimizes on $\{0,1\}^Q$, $Max_{m=1,\dots,M}\{d(x^{**}, x_m)w_m\}$, where w_m denotes the weight given to committee x_m.*

Let us consider a first weighting method, namely the count weight, in which approval ballots are weighted by the number of voters who cast them. This count weighting is already intrinsic to the usual minisum procedure, as pointed out by the next example;

Example 11 *$X^{N,Q} = \begin{bmatrix} (1,0,1) \\ (1,1,1) \\ (0,1,1) \\ (1,0,1) \end{bmatrix}$ can be written in terms of $X^{M,Q} = \begin{bmatrix} (1,0,1) \\ (1,1,1) \\ (0,1,1) \end{bmatrix}$ and a weight vector $w = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, each number in the weight vector shows how many times the corresponding ballot in the $X^{M,Q}$ is cast, namely the count weight w_m of committee x_m. Notice that the entries of the count weight vector sum up to the total number of voters, N. It is obvious that, the usual minisum outcome is exactly the count weighted minisum outcome, or in other words $x^* \in \{0,1\}^Q$ minimizes on $\{0,1\}^Q$, $\Sigma_{m=1}^{M}(d(x^*, x_m)w_m)$ if and only if $x^* = m(X^{N,Q})$ where w_m denotes the count weight of committee x_m.*

Using count weights in the determination of the minimax outcome will certainly reduce the significance of extreme ballots and increase the impact of "popular" ballots by giving more influence power to the committees approved by larger number of voters. At this point, Brams et al. propose a second weighting method that takes into account not only the counts but the proximity of ballots to each other. In the proximity weights method, weight of a committee x_m is defined as; $w_m = \frac{cw_m}{\Sigma_{l=1}^{M} cw_l d(x_m, x_l)}$, where cw_m denotes the count weight of committee x_m.

Consider the following example that illustrates the weighting methods mentioned;

Example 12

Ballot	***001***	***101***	***110***	$m(X^{N,Q})$	*101*
Count Weight	***2***	***2***	***1***		
Proximity Weight	***4***	***5***	***1***	*Maximum*	*Total*
000	*2* *4*	*4* *10*	*2* *2*	*4* *10*	*8* *16*
100	*4* *8*	*2* *5*	*1* *1*	*4* *8*	*7* *14*
010	*4* *8*	*6* *15*	*1* *1*	*6* *15*	*11* *24*
001	*0* *0*	*2* *5*	*3* *3*	*3* *5*	*5* *8*
110	*6* *12*	*4* *10*	*0* *0*	*6* *12*	*10* *22*
101	*2* *4*	*0* *0*	*2* *2*	***2*** ***4***	***4*** ***6***
011	*2* *4*	*4* *10*	*2* *2*	*4* *10*	*8* *16*
111	*4* *8*	*2* *5*	*1* *1*	*4* *8*	*7* *14*

$N = 5$, $Q = 3$ *and in the first two rows of the table below, the ballots cast and number of voters casting each distinct ballot is shown. When proximity weight of each ballot calculated accordingly, one will get* $2/5$, $2/4$ *and* $1/10$ *respectively. When working with weights what matters is not the exact amount of weight but the proportion of weights to each other. Hence, for simplification purposes, each proximity weight is multiplied by a common factor* 10, *and the weights in the third row is acquired. In the column of each approval ballot the distances between the corresponding committee is shown; first number is the count weighted distance, while second number indicates the proximity weighted distance. The maximum column de-*

notes the maximum distance of the each row, and the numbers in the total column indicates the total of the corresponding row. As expected, minimax winner is the one which has the smallest number in the maximum column, whereas minisum committee is the one with the smallest value in the total column.

In this example, the outcomes of both weighting procedures and both methods is the same committee, $(1,0,1)$. *For sure, this does not have to be the case. Consider the following example;*

Example 13 $N = 101, Q = 3$

Ballot	**100**	**010**	**101**	**011**	$m(X^{NQ}) = (1,0,1)$
Count Weight	**25**	**25**	**26**	**25**	
Proximity Weight	$\mathbf{25/151}$	$\mathbf{25/153}$	$\mathbf{26/150}$	$\mathbf{25/152}$	***Total***
000	$25/151$	$25/153$	$52/150$	$50/152$	$1,004575642$
100	0	$50/153$	$26/150$	$150/152$	$1,486972824$
010	$50/151$	0	$78/150$	$25/152$	$1,015599512$
001	$50/151$	$50/153$	$26/150$	$25/152$	$\mathbf{0,995730231}$
110	$25/151$	$25/153$	$52/150$	$50/152$	$1,004575642$
101	$50/151$	$75/153$	0	$50/152$	$1,150269275$
011	$75/151$	$25/153$	$52/150$	0	$1,006754101$
111	$50/151$	$50/153$	$26/150$	$25/152$	$\mathbf{0,995730231}$

Notice that, in this example there are two minisum committees with proximity weights, $(0,0,1)$, $(1,1,1)$ *and these two committees do not coincide with the issue-wise majority winner* $(1,0,1)$.

Consider the Hamming preference profiles of the same group of voters.

25	**25**	**26**	**25**
100	010	**101**	011
000	000	001	001
110	110	100	010
101	011	111	111
010	100	011	**101**
111	111	110	110
001	001	000	000
011	**101**	010	100

Notice that all three minisum committees are in the Pareto-set, while the issue-wise majority winner $(1, 0, 1)$ *defeats the proximity weighted minisum committees in pair-wise contest. Even further, the committee* $(1, 0, 1)$ *is the Condorcet winner of the profile.*

What this example implies is that once a weighting method that is not in direct proportion with count weights is applied, the election of a majority unbeaten committee is not ensured. Proximity weighting increases the influence of ballots that are close to each other, hence fosters voting for similar committees, at the cost of the possibility of dropping a majority winning alternative.

Chapter 3
Representativeness of approval balloting in unrestricted committee elections

Both minisum and minimax committees are to be seen as representative in that they maximize a social welfare function. Moreover, minisum committees coincide with issue-wise majority committees. This chapter investigates further the representativeness properties of minisum committees without a restriction on the elected committee size.

More precisely, it first addresses the following question; Does minisum committee fulfill the majority will regarding committees, when the latter is defined by using the Hamming distance criterion? We show that the answer is negative, more precisely a minisum committee can be majority defeated and even drop out of the Uncovered Set of committees. This suggests to study a weaker representativeness property by replacing the majority will with the unanimity will. Put differently, is it true that for any set of approval ballots the minisum committee is always Pareto-optimal? This is obviously true when the Hamming-distance prevails. We show that this is also true for a much larger domain of separable preferences over committee.

3.1 Minisum committees under majority will

We know that in multiple dichotomous choice models with separable voter preferences, if a committee that beats every other in pairwise context exists, then the issue-wise majority rule will select it (Kadane (1972), Schwartz (1977) and Kramer (1972)). Certainly this result applies for the domain of Hamming extension rule as shown by the following proposition;

Proposition 3 *Given any X^{NQ}, if a Condorcet winner exists under Hamming extension rule, the issue-wise majority rule will select it.*

Proof. *Take any X^{NQ}, and the extended Hamming preference profile from the ballots. Let $x_* \in \{0,1\}^Q$ be the Condorcet winner of the profile. Hence, for all distinct $y \in \{0,1\}^Q$ including the issue-wise majority winner $m(X^{NQ})$, $\mid n \in I : x_* P_n\, y \mid > N/2$.*

Suppose, w.l.g. $m(X^{NQ}) = (0,0,...0)$ and define $B = \{q \in C : m^q(X^{NQ}) \neq x_*^q\}$ with $\mid B \mid = Q' < Q$. Then, for all $q \in B, (m(X^{NQ})/B)^q = 0$, $(x_*/B)^q = 1$, $\mid n \in I : (m(X^{NQ})/B) P_n (x_*/B) \mid > N/2$ follows from the fact that $(m(X^{NQ})/B)$ is the issue-wise majority winner over $X^{NQ'}$, hence $\forall q \in B$, $\mid n \in I : x_n^q = 0 \mid > N/2$ which implies $\Sigma_{q=1}^{Q'} \Sigma_{n=1}^{N} x_n^q < NQ'/2$, suppose for a contradiction $\mid n \in I : (m(X^{NQ})/B) P_n (x_*/B) \mid \leq N/2$, then, $\mid n \in I : \Sigma_{q=1}^{Q'} x_n^q < Q'/2 \mid \leq N/2$, which implies $\Sigma_{n=1}^{N} \Sigma_{q=1}^{Q'} x_n^q \geq NQ'/2$, establishing the expected contradiction. Thus, as stated before, $\mid n \in I : (m(X^{NQ})/B) P_n (x_*/B) \mid > N/2$. Due to separability of individual preferences the common members will not alter the preferences and *hence,* $\mid n \in I : m(X^{NQ}) P_n \, x_* \mid > N/2$, *creating a direct contradiction with having* $x_* \neq m(X^{NQ})$ *as the Condorcet winner.* ■

This proposition promotes the minisum committee as the relevant one when satisfying the majority will is the goal. However, the problem of existence of the Condorcet winner remains unanswered. Consider the following example;

Example 14 $Q = 3, N = 5$ *with the following approval ballots;* $(1,1,0), (1,0,1), (0,1,1)$ *and two* $(0,0,0)$. *The extended Hamming profiles are given below:*

110	101	011	000	000
100, 010, 111	100, 001, 111	001, 010, 111	001, 010, 100	001, 010, 100
101, 000, 011	110, 011, 000	101, 000, 110	110, 101, 011	110, 101, 011
001	010	100	111	111

Observe that there is no Condorcet winner in this profile. Worse, the minisum outcome, i.e., the issue-wise majority winner is $m(X^{NQ}) = (0,0,0)$. *But as it can be easily noticed, the issue-wise majority is defeated by its opposite, which is* $(1,1,1)$ *beats* $(0,0,0)$.[3]

Condorcet consistent choice rules, the Copeland Set, the Top Cycle and the Uncovered Set, can be used to circumvent the existence of a Condorcet winner (recall that Condorcet consistency means the ability to uniquely select the Condorcet winner when it exists).

A Copeland winning committee will be one that beats the biggest number of committees. The Copeland Set induces a complete preorder over the set of commit-

[3] This condition is known as the Ostrogorski paradox, where the issue-wise majority winner is majority-defeated in the majority tournament defined over committees. For a detailed analysis, one can see Daudt and Rae (1976), Bezembinder and van Acker (1985), Deb and Kelsey (1987), Kelly (1989), Laffond and Laine (2006, 2008).

tees. For sure, Condorcet winner is the winner of the Copeland rule too, whenever it exists. But in the absence of a Condorcet winning committee, the Copeland ranking is shown to coincide with the ranking provided by the Kemeny ranking (Klamler, 2005). Thus, a Copeland winner can be said to be the "closest" to being a Condorcet winner. Think about the following example;

Example 15 $Q = 3$, $N = 5$ *and with the following profile;*

110	101	011	000	000
$100, 010, 111$	$100, 001, 111$	$001, 010, 111$	$001, 010, 100$	$001, 010, 100$
$101, 000, 011$	$110, 011, 000$	$101, 000, 110$	$110, 101, 011$	$110, 101, 011$
001	010	100	111	111

To identify the Copeland winner of this profile consider the table below:

$x \in \{0,1\}^3$	$\mid \{y \in \{0,1\}^3 : x \succ y\} \mid$	$\mid \{y \in \{0,1\}^3 : y \succ x\} \mid$
111	1	6
110	1	4
101	1	4
011	1	4
100	4	1
010	4	1
001	4	1
000	6	1

In the table, the committees are listed in the first column. The second column shows the number of alternatives that the corresponding committee beats, while the number of committees which beats that committee is given in the third column. Note that in cases where the sum of the second and the third column does not count to 7, *which is the number of pairwise contests in relation with a particular committee, indifference relation has taken place.*

The Copeland rule ranks the committees according to the difference between the second and the third columns.

As can be noticed, the issue-wise majority winner $m(X^{NQ})$ *is also the Copeland winner, which qualifies it as "closest" to being the Condorcet winner.*

This example may seem encouraging in the sense that in the absence of an obvious winner, the "closest" alternative is elected. However, this is not the general case when dealing with all possible profiles, i.e.; there exist profiles for which the issue-wise majority winner is not the Copeland winner. A recent result shows that under the assumptions made here, the issue-wise majority winner can be covered in the majority tournament among committees (Laffond and Lainé (2008)). Hence, Copeland winner being a refinement of the Uncovered Set leaves the stage.

A second result in the same work ensures the inclusion of the issue-wise majority winner to the Top-Cycle. But since McKelvey (1976), Top-Cycle is not considered to be a "reliable" stability concept as it can consist of all of the alternative set. For instance, in the previous example, all of the committees belong to the Top-Cycle of the majority tournament. Actually, this is always the case when the issue-wise majority winner is majority defeated by exactly the opposite committee (Laffond and Laine (2008)). In addition to poor-selectivity, the Top-Cycle may select Pareto-dominated outcomes. This suggests to investigate the Pareto-efficiency of issue-wise majority rule over approval ballots.

3.2 Pareto-efficiency of the minisum committees

As a reasonable minimum representativeness criterion, Pareto-optimality prohibits the election of an alternative while another alternative is preferred to it by every individual in the society. Thus, the benchmark becomes the unanimity rule instead of the majority rule. As mentioned before, issue-wise majority voting in multiple dichotomous choice models does not ensure the election of a Pareto-optimal outcome when any separable preference consistent with the ballot is assumed (Ozkal-Sanver and Sanver, 2006)[4]. Hence, an additional restriction on admissible preferences is required beyond separability in order to ensure Pareto-optimality of the elected committee.

Benoit and Kornhauser (1994) address this question in the different setting of numbered post election (which relates to asymmetric committee), where each issue refers to a specific position in an assembly or a specific post in a legislation, and they assume that the order of the importance of the issues is the same for all voters. For instance, the most important issue to be voted is the presidency position and the second important is the asistant president position, etc. They show that in a numbered post election with a common order of importance of the issues, if the preferences are separable and top-lexicographic then a Pareto-optimal assembly is always selected. This top-lexicographicity property can be roughly defined as such: Consider two assemblies with the most important position being the first issue for the voters and

[4] This result holds only when at least three candidates have to be appointed.

the second most important one being the second and so on. Preferences would be top-lexicographic if any voter prefers the first assembly to the second where the first issue that these two committees differ is occupied by the voter's best preference for this position in the first assembly. Notice that this top-lexicographicity property is designed for a very special type of voting procedure, which is the numbered post election with a common order of importance of the issues, and does not guarantee Pareto-optimality in more general settings.

It is obviously seen that a minisum committee is Pareto-optimal under Hamming extension rule; indeed we know that it maximizes the sum of utilities. Is it still the case for other preferences over committees? Indeed, the properties of the issue-wise majority rule are very sensitive to the choice of extension rule.

Now, we study this question for preference domains larger than the Hamming one. We prove that the preference domain that always insures a Pareto-optimal minisum committee is much larger than the Hamming extension rule. Next, we characterize a larger domain (for inclusion) for which this is true.

3.2.1 Pareto-efficiency under Hamming criteria

Let us begin with the formal definition of a Pareto-optimal committee:

Definition 11 *Given X^{NQ}, a committee $x \in \{0,1\}^Q$ is said to be **Pareto-optimal** if there is no $y \in \{0,1\}^Q$ such that $y \; R_n \; x$ for all $n \in N$ and $y \; P_{n^*} \; x$ for at least one $n^* \in N$.*

The next definition introduces the Pareto-efficiency property of the issue-wise majority rule:

Definition 12 *The issue-wise majority rule is said to be* ***Pareto-efficient*** *for the extension rule R if, for any N and Q, any (N,Q)-ballot $X^{NQ} = (x_1, ..., x_N)$ there is no $x \in \{0,1\}^Q$ such that $x\ R_n\ m(X^{NQ})$ for all $n \in N$ and $x\ P_{n^*}\ m(X^{NQ})$ for at least one $n^* \in N$.*

We first introduce a simple way to enlarge the Hamming preference domain by "cutting ties" within each indifference class:

Definition 13 *R^{HamC} is said to be a* ***Hamming-consistent extension rule*** *if and only if $\forall Q, \forall x, y, z \in \{0,1\}^Q$, $d(x,y) < d(x,z) \Rightarrow y\ P^{HamC}(x)\ z$.*

Hamming consistency means that any two committees that are not in the same indifference class (for the Hamming extension rule) remain compared according to the Hamming extension rule and any ranking (strict or not) of committees within the same class is allowed.

For instance consider the following example;

Example 16 *Consider $n \in N$ with the best preference $x_n = (1,1,1)$, a committee of three candidates. Hamming extension rule R^{Ham} yields the following profile;*

$(\mathbf{1,1,1})\ \mathbf{P}_n^{Ham}\ (\mathbf{1,1,0})\ \mathbf{I}_n^{Ham}\ (\mathbf{1,0,1})\ \mathbf{I}_n^{Ham}\ (\mathbf{0,1,1})\ \mathbf{P}_n^{Ham}\ (\mathbf{0,0,1})\ \mathbf{I}_n^{Ham}$ $(\mathbf{1,0,0})\ \mathbf{I}_n^{Ham}\ (\mathbf{0,1,0})\ \mathbf{P}_n^{Ham}\ (\mathbf{0,0,0})$

while the following belong to the set of Hamming-consistent extended preorders with the best-preference $(1,1,1)$;

$(\mathbf{1},\mathbf{1},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{1},\mathbf{1},\mathbf{0})\mathbf{P}_n^{HamC}\ (\mathbf{1},\mathbf{0},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{1},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{0},\mathbf{1})$ $\mathbf{I}_n^{HamC}\ (\mathbf{1},\mathbf{0},\mathbf{0})\ \mathbf{I}_n^{HamC}\ (\mathbf{0},\mathbf{1},\mathbf{0})\ \mathbf{P}_n^{HamC}(\mathbf{0},\mathbf{0},\mathbf{0})$

$(\mathbf{1},\mathbf{1},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{1},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{1},\mathbf{1},\mathbf{0})\ \mathbf{I}_n^{HamC}\ (\mathbf{1},\mathbf{0},\mathbf{1})\ \mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{0},\mathbf{1})$ $\mathbf{I}_n^{HamC}\ (\mathbf{1},\mathbf{0},\mathbf{0})\ \mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{1},\mathbf{0})\mathbf{P}_n^{HamC}\ (\mathbf{0},\mathbf{0},\mathbf{0})$

What is altered in the second and third profiles is exactly within the indifference classes of the first profile. One can extend 169 *Hamming-consistent preference profiles from the best preference* $(1,1,1)$, 36 *of them being strict orders.*

It should be clear that Hamming extension rule belongs to the class of Hamming-consistent extension rules. Notice that similar to the Hamming extension rule, Hamming-consistent extension rules satisfies separability and top-consistency. The following proposition shows the minisum committee is always Pareto-optimal for Hamming-consistent extension rules.

Proposition 4 *Any Hamming-consistent extension rule* R^{HamC} *is Pareto-efficient for the issue-wise majority rule.*

Proof. Given any X^{NQ} and any Hamming-consistent extension rule R^{HamC}, suppose for a contradiction that $m(X^{NQ})$ is Pareto dominated by a distinct $y \in \{0,1\}^Q$.

Formally, $y\ R_n^{HamC}\ m(X^{NQ})$ for all n and $y\ P_{n^*}^{HamC}\ m(X^{NQ})$ for at least one n^*. Due to Hamming-consistency, for all n, $d(x_n, y) \leq d(x_n, m(X^{NQ}))$, resulting in $\Sigma_{n\in N}d(x_n, y) \leq \Sigma_{n\in N}d(x_n, m(X^{NQ}))$. As proved by Brams, Kilgour and Sanver (2004), the committee that minimizes the total distance to the ideals is the issue-wise majority winner.

Hence, $\Sigma_{n\in N}d(x_n, y) = \Sigma_{n\in N}d(x_n, m(X^{NQ}))$ and together with Hamming-consistency for all n, $d(x_n, y) = d(x_n, m(X^{NQ}))$.

Let $B = \{q = 1, ..., Q : m^q(X^{NQ}) \neq y^q\}$ with $\mid B \mid = Q' < Q$. Q' must be an even integer. Suppose w.l.g. $m^q(X^{NQ}) = 0$ for all q. Then, for all $q \in B, (m(X^{NQ})/B)^q = 0$, $(y/B)^q = 1$ and for all n, $\mid q \in B : (x_n/B)^q = 0 \mid = \mid q \in B : (x_n/B)^q = 1 \mid = Q'/2$, implying for all n, $\Sigma_{q\in B}(x_n/B)^q = Q'/2 \Longrightarrow \Sigma_{n\in N}\Sigma_{q\in B}(x_n/B)^q = NQ'/2$ **(1)**

By definition of $m(X^{NQ})$ and by construction for all $q \in B$, $\mid n \in N : (x_n/B)^q = 1 \mid < N/2$. Hence, for all $q \in B$, $\Sigma_{n\in N}(x_n/B)^q < N/2 \Longrightarrow \Sigma_{q\in B}\Sigma_{n\in N}(x_n/B)^q < Q'N/2$ creating a direct contradiction with **(1).** ■

What this proposition shows is simply that the Pareto-efficiency of issue-wise majority voting remains valid as long as the extended preferences over committees are consistent with the partition into Hamming indifference classes.

3.2.2 A characterization result

The previous proposition suggests to specify the largest class of separable extension rules for which a minisum committee is Pareto-optimal for any set of approval ballots. We show that, under a mild additional restriction, this class is larger than the class of Hamming-consistent extension rules. It contains all *weak Hamming consistent* extension rules, which are formally defined as follows;

Definition 14 R^{WHam} *is said to be a* ***weak Hamming-consistent extension rule*** *if and only if* $\forall Q$, $\forall \alpha < \frac{Q}{2}$, $\forall x, y \in \{0,1\}^Q$, $d(x,y) \leq \alpha \Rightarrow y\ P(x)\ (-y)$,*where for any* $y \in \{0,1\}^Q$, $(-y)$ *denote the opposite committee of* y *such that:* $\forall q$, $(-y)^q \neq y^q$.

The weak Hamming-consistency property only determines the ranking of a committee relative to its opposite committee by comparing the distances to the ideal. If the ideal of an individual is closer to a committee than its opposite committee, any weakly Hamming consistent extension rule should rank this committee higher than its opposite. In other words, if a committee coincides with the ideal of a voter at more than half of the candidates, than this committee should be ranked higher than its opposite by the voter. Note that unlike Hamming-consistency, weak Hamming-consistency does not guarantee separability.

A neutral assumption is to consider that the admissible preferences are "minimally-consistent" with the observed approval ballots. This leads to the top-consistency property which states that ballots describe voters' ideals[5].

Definition 15 *An extension rule* R *is said to satisfy* ***top-consistency*** *if the following condition holds;* $\forall Q$, $\forall x \in \{0,1\}^Q$, $x\ P(x)\ y$ *for all* $y \in \{0,1\}^Q \backslash x$.

[5] Top-consistency is equivalent to sincere voting under the assumption of dichotomous preferences.

Top-consistency requires the single committee from which the preferences are extended to be the first-ranked committee in the extended preorder.

It should be clear that any Hamming consistent profile is also top-consistent, separable and weakly Hamming consistent.

How large is the set of weakly Hamming consistent extension rules in comparison to Hamming consistent ones? When the number of candidates $Q = 3$, the set of top-consistent, separable and weakly Hamming-consistent extension rules is exactly the same as Hamming-consistent ones. But, even if we increase the number of candidates by 1, with $Q = 4$, the number of attainable individual profiles under weak Hamming-consistency will be 2^{24} more than the number of attainable individual profiles under Hamming-consistency, for sure without harming neither separability nor top-consistency.

Example 17 *Consider $n \in N$ with the best preference $x_n = (1, 1, 1, 1)$, a committee of four candidates. The extended Hamming profile would be;*

$I(0)$	$I(1)$	$I(2)$	$I(3)$	$I(4)$
1111	1110 1101 1011 0111	1001 1100 1010 0011 0101 0110	1000 0100 0010 0001	0000

In the Hamming-consistent profile, the committees within the same indifference class can be ordered in any possible combination. That makes $4!6!4!$ different strict orders. Once indifference relation is taken into account, this number explodes.

In the separable and weakly Hamming-consistent profile, respecting indifference boundaries is not required as long as separability is preserved. In other words, a committee does not have to belong to the Hamming indifference class. Consider again the individual with the ideal $(1, 1, 1, 1)$ *and consider the indifference classes shown above. In a weakly Hamming consistent separable profile with* $(1, 1, 1, 1)$ *as the top-preference, it is admissible to have either* $(1, 0, 0, 1)$ $P_n(0, 1, 1, 1)$ *or* $(1, 0, 0, 1)$ $P_n(1, 0, 1, 1)$. *Actually, in a Hamming-consistent profile over committees with* 4 *candidates,* 24 *independent shifts in preferences will lead weakly Hamming-consistent, separable and top-consistent profiles.*

This example with 4 candidates illustrates how large is the set of weakly Hamming consistent profiles compared to Hamming-consistent ones. The next result characterizes the domain of weakly Hamming consistent extension rules as the largest domain of separable and top-consistent rules for which the issue-wise majority rule is Pareto-efficient *for any set of approval ballots*.

Theorem 5 *Let* D *be the domain of top-consistent and separable extension rules over committees. The issue-wise majority rule is Pareto efficient for* $R \in D$ *if and only if* R *is weakly Hamming consistent.*

Proof. *Proof of the sufficiency part:*

Let R be a weakly Hamming-consistent extension rule. Let Q, N be any non-zero integers and let X^{NQ} be a (N, Q)-ballot such that $m(X^{NQ})$ is Pareto-dominated by $x^* = (x^{*1}, ..., x^{*Q})$ in the profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$.

Step 1: $\forall Q, N$, $\exists n \in \{1, ..., N\}$ such that $d(x_n, m(X^{NQ})) < \frac{Q}{2}$.Indeed, it follows from the definition of $m(X^{NQ})$ that $\mid \{x_n^q : x_n^q = m^q(X^{NQ})\} \mid > \frac{NQ}{2}$(*). Suppose for a contradiction that for any n, $d(x_n, m(X^{NQ})) \geq \frac{Q}{2}$.Then, for any $n, \mid \{q = 1, ..., Q : x_i^q = m^q(X^{NQ})\} \mid \leq \frac{Q}{2}$, which immediate contradicts with (*). Thus, there exists n such that $\mid \{q = 1, ..., Q : x_i^q = m^q(X^{NQ})\} \mid > \frac{Q}{2}$, which means that $d(x_n, m(X^{NQ})) < \frac{Q}{2}$. Furthermore, it follows from the definition of weak-Hamming consistency that $x^* \neq m(X^{NQ})$.

Step 2: $m(X^{NQ})$ is not Pareto-dominated by $(-m(X^{NQ}))$.

This is an immediate consequence of Step 1 together with weak Hamming-consistency.

Step 3: $\forall Q, N$, $m(X^{NQ})$ is not Pareto-dominated.

Let $B = \{q = 1, ...Q : y^q \neq m^q(X^{NQ})\}$, with $\mid B \mid = Q' < Q$ (from Step 2). It follows from construction that $x^*/B = (-m(X^{NQ})/B)$. Moreover, it follows from Step 1 that $\exists n \in \{1, ..., N\}$ such that $d(x_n/B, m(X^{NQ})/B) \leq \frac{Q'}{2}$. Thus, weak Hamming-consistency ensures that $m(X^{NQ})/B \; P(x/B) \; x^*/B$. The conclusion follows from the separability property of R.

Proof of the necessary part:

Let R be an extension rule that is not weakly Hamming-consistent. Thus, $\exists Q$, $\exists \alpha \leq \frac{Q}{2}$ two integers such that $\exists x, y \in \{0,1\}^Q$ which verify $d(x,y) = \alpha$ and $(-y)$ $R(x)$ y. One may assume without loss of generality that $y^q = 1$ for all q (through a relevant relabelling of issues). We claim that there exists a (N,Q)-ballot X^{NQ} such that $m(X^{NQ})$ is Pareto-dominated.

Let $N = \beta + 1$, where $\beta = \binom{Q}{\alpha} = \frac{Q(Q-1)...(Q-\alpha+1)}{2.3.4....\alpha(\alpha-1)}$. The set of ideal committees for individuals in $\{1,...,\beta\}$, that is $\cup_{1 \leq n \leq \beta}\{x_n\}$, is $\{x \in \{0,1\}^Q :\mid \{q = 1,...,Q : x^q = 0\} \mid = \alpha\}$, while $x_N^q = 0$ for all $q = 1,...,Q$. Consider any candidate q, and let us compute the number $n(q)$ of approvals given to q. One get $n(q) = \binom{Q-1}{\alpha} = \frac{(Q-1)...(Q-\alpha)}{2.3.4....\alpha(\alpha-1)} = \frac{(Q-\alpha)}{Q}.\beta$.

SSuppose that Q is even. Thus, $\alpha \leq \frac{Q}{2} - 1 \Rightarrow \frac{(Q-\alpha)}{Q}.\beta > (\frac{1}{2} + \frac{1}{Q}).\beta$. Since $n(q)$ is an integer, then $n(q) > \frac{\beta+1}{2}$: indeed, if β is even, then $n(q) > (\frac{1}{2} + \frac{1}{Q}).\beta \Rightarrow n(q) > \frac{\beta}{2} + 1 > \frac{\beta+1}{2}$; if β is odd, then $n(q) > (\frac{1}{2} + \frac{1}{Q}).\beta \Rightarrow n(q) > \frac{\beta+1}{2} + \frac{\beta}{Q} > \frac{\beta+1}{2}$.

Suppose that Q is odd. Then $\alpha < \frac{Q}{2} \Rightarrow \alpha \leq \frac{Q-1}{2} \Rightarrow \frac{(Q-\alpha)}{Q}.\beta > (\frac{1}{2} + \frac{1}{2Q}).\beta$. Since $n(q)$ is an integer, then $n(q) > \frac{\beta+1}{2}$: indeed, if β is even, then $n(q) > (\frac{1}{2} + \frac{1}{2Q}).\beta \Rightarrow n(q) > \frac{\beta}{2} + 1 > \frac{\beta+1}{2}$; if β is odd, then $n(q) > (\frac{1}{2} + \frac{1}{Q}).\beta \Rightarrow n(q) > \frac{\beta+1}{2} + \frac{\beta}{2Q} > \frac{\beta+1}{2}$.

This proves that $y = m(X^{NQ})$. It follows from the definition of α, together with the lack of weak Hamming-consistency, that $(-y)$ R_n y for all $n = 1,...,\beta$. Moreover, it follows from top-consistency that $x_N = (-y)$ P_N y. Hence, $(-m(X^{NQ}))$ Pareto-dominates $m(X^{NQ})$, which completes the proof. ■

Chapter 4
Representativeness of approval balloting in fixed-size committee elections

In most of real life elections, the number of the committee members that will be elected is restricted. Upto now, while investigating the representativeness properties, we have not assumed any restriction over the size of the elected committee. In other words, the number of the members involved in the elected committee has been dependent on the approval ballots. However, most real-life elections require the election of a committee with a predetermined size, i.e.; a size restriction is imposed on the election outcome. Now, we will investigate the Pareto characteristics in fixed-size committee elections.

The problem under consideration is the election of a fixed number of candidates, namely k, from the set of Q candidates, $C = \{1, ..., q, ..., Q\}$. Same as the unrestricted model, all voters cast approval ballots indicating their approvals and disapprovals of candidates as committee members. Given any N and Q, the set of all admissible election outcomes is defined as $K_k = \{x \in \{0,1\}^Q :\mid \{q \in C : x^q = 1\} \mid= k\}$.

Two different election procedures are proposed to elect a fixed size committee by approval balloting: issue-wise majority rule and sequential approval balloting.

4.1 Issue-wise majority rule

In this procedure, each voter is asked to approve exactly k candidates and the winning committee is determined according to the issue-wise majority rule. In this case it is quite possible to have issue-wise majority winning committees with more or less than k members. Thus, an additional restriction over the vote matrix is required to ensure the election of k-member committees with issue-wise majority rule.

Formally, given a set of N voters, and Q candidates and a non-zero integer k, each voter casts an approval ballot $x_n \in K_k$. $\mathcal{X}^{N,Q}$ denotes the set of all approval ballots with Q candidates and N voters and $\mathcal{X}^{N,Q,k}$ is the subset of $\mathcal{X}^{N,Q}$ defined as: $X^{NQk} \in \mathcal{X}^{N,Q,k}$ if and only if $m(X^{NQk}) \in K_k$, where $m(X^{NQk})$ is the issue-wise majority winner. Note that $m(X^{NQk})$ is uniquely defined from the assumption that N is odd.

Voter preferences over committees are complete preorders over $\mathcal{K} = \cup_{k \geq 0, Q > k} K_k$ extended from approval ballots via an extension rule. We denote by $\Gamma^{Q,k}$ the set of all complete preorders over $\mathcal{K}$. An extension rule R defined from $\mathcal{K}$ to $\mathcal{R}^k = \cup_{Q \geq 1}^{Q} \Gamma^{Q,k}$ associates with each committee $x \in K_k$ an element $R(x)$ of $\Gamma^{Q,k}$.

Definition 16 *An extension rule R is said to satisfy* ***top-consistency*** *if the following condition holds;* $\forall Q$,$\forall k < Q$, $\forall x \in K_k$, $x \ P(x) \ y$ *for all* $y \in K_k \backslash \{x\}$.

Definition 17 *An extension rule R is said to satisfy* ***separability*** *if the following condition holds;* $\forall Q$,$\forall \ 0 < k \leq k' < Q$, $\forall x, y \in K_k$, $\forall z \in K_{k'-k}$, $x \ R \ y \Rightarrow$

$(x,z)R(y,z)$, where (x,z) refers to the k'-sized committee including the members of both x and z.

In this restricted setting, to characterize the largest domain of separable and top-consistent extension rules that yield out Pareto-optimal issue-wise majority winners with Hamming-consistency, we need to adapt some particular definitions:

The distance between two fixed-size committees is defined as the number of candidates that is excluded from one particular commitee while approved by the other, i.e; distance is calculated based on particular subsets of these two committees, which refer to the subset of approved candidates by one of the two.

Definition 18 $\forall Q, \forall k, \forall x, y \in K$, $d(x,y) =\mid \{q \in C : (y/A(x))^q = 0\} \mid$, *where* $A(x) = \{q \in C : x^q = 1\}$ *and* $y/A(x)$ *stands for the restriction of the committee* y *to the subset* $A(x)$.

Notice that this is a symmetric kind of distance in the sense that $d(x,y) = d(y,x)$, since x and y are of the same size.

Given any fixed size committee in $\mathcal{K}$, the exact opposite committee will not be a member of $\mathcal{K}$. Hence, we need to modify the definition of opposite committee of a fixed size committee as;

Definition 19 $\forall x \in \mathcal{K}$, $(-x)$ *is the subset of opposite committees of* x *defined by;* $y \in (-x) \Longleftrightarrow \mid \{q \in C : y^q = x^q\} \mid = Max\{0, [k - Q/2]\}$.

Notice that, if $k \leq Q/2$, $(-x)$ will include the committees that have no common member with x. But if $k > Q/2$, as it is not possible for any two committee to be disjoint, the committees that share the minimum possible common members with a committee will be opposites of it.

Definition 20 *R^{WHam} is said to be a* ***weak Hamming-consistent extension rule*** *if and only if $\forall Q$, $\forall \alpha < \frac{k}{2}$, $\forall x, y \in \mathcal{K}$, $\forall z \in (-y)$, $d(x,y) \leq \alpha \Rightarrow y\ P(x)\ z$.*

The next proposition shows that similar to the case of committee elections without a size restriction, the largest domain of separable and top-consistent extension rules over fixed size committees that yield Pareto-optimal issue-wise majority winners is the set of weak-Hamming consistent rules.

Proposition 6 *Let D^* be the domain of top-consistent and separable extension rules over fixed size committees. The issue-wise majority rule is Pareto efficient for $R \in D^*$ if and only if R is weakly Hamming consistent.*

Proof. *Sufficiency Part:*

Given any $X^{NQk} \in \mathcal{X}^{N,Q,k}$, let R be a weakly-Hamming consistent extension rule and suppose $m(X^{NQk})$ is Pareto-dominated by $x^* \in \mathcal{K}$ in the profile $R(X^{NQk}) = (R(x_1), ..., R(x_N))$.

We will first show that, $\exists n \in I$ such that $d(x_n, m(X^{NQk})) < \frac{k}{2}$. Indeed, it follows from the definition of $m(X^{NQk})$ that, for any elected member q, the number of approvals given to q, $n(q) > N/2$. Hence, total number of approvals given to the elected members of $m(X^{NQk})$, $n(m(X^{NQk})) > \frac{Nk}{2}$. Suppose for a contradiction that for any n, $d(x_n, m(X^{NQk})) \geq \frac{k}{2}$, which implies that $n(m(X^{NQk})) < \frac{Nk}{2}$, creating the desired contradiction. Thus, $\exists n \in I$ such that $d(x_n, m(X^{NQk})) < \frac{k}{2}$. Due to weak Hamming consistency, $m(X^{NQk}) P_n\, z, \forall z \in (-m(X^{NQk}))$, establishing that any $z \in (-m(X^{NQk}))$ cannot Pareto-dominate $m(X^{NQk})$. Let $B = \{q \in C : m^q(X^{NQk}) = 1$ and $x^{*,q} = 0\}$. As any $z \in (-m(X^{NQk}))$ cannot Pareto-dominate $m(X^{NQk})$, $| B |< Q$. Following the first claim, $\exists n \in I$ for who $d(x_n/B, (m(X^{NQk})/B)) < \frac{|B|}{2}$. Again, weak Hamming consistency ensures that $(m(X^{NQk})/B)\ P\ x^*/B$. The conclusion follows from the separability property of R.

Necessary Part:

Let R be an extension rule that is not weakly Hamming-consistent. We claim that $\exists\ X^{NQk}$ such that $m(X^{NQk})$ is Pareto dominated in the profile $R(X^{NQk}) = (R(x_1), ..., R(x_N))$.

Let $Q > 3$, $\alpha < k/2$, $N = \begin{pmatrix} k \\ \alpha \end{pmatrix} + 1$. Let $R(X^{NQk})$ be such that $\exists y \in K$ such that $\forall x_n \in \mathcal{K}$, $d(x_n, y) = \alpha$ and $zR_n y$ for one $z \in (-y)$, where ballots are given by;

- $\forall n, | A(x_n) - \{q : y^q = 1\} |= \alpha$

- $\forall n \neq n' < N, x_n \neq x_{n'}$

- $A(x_N) = z$

- $\forall q \notin \{1, ..., k\}$, $n(q) < N/2$ (note that Q can be chosen large enough to ensure this). It follows from construction that $m(X^{NQk}) = y$. Finally, $zR_n m(X^{NQk})$

for all n and due to top-consistency $zP_N\ m(X^{NQk})$. Hence, $m(X^{NQk})$ is not Pareto-optimal. ■

4.2 Sequential approval balloting

A second method to elect a committee of size k is sequential approval balloting, which is proposed by Brams, Kilgour and Sanver (2004) under the name of restricted size committee elections. Unlike to the former procedure, voters are free to approve as many candidates as they like. Exactly k number of candidates with the most approvals are elected as members of the winning committee.

Given a set of N voters, and Q candidates and a non-zero integer k, each voter casts an approval ballot $x_n \in \{0,1\}^Q$.Committee $c \in \{0,1\}^Q$ with $\mid q \in C : c^q = 1 \mid = k$ is elected if $\forall q$ with $c^q = 1$, $\nexists q'$ with $c^{q'} = 0$ such that $\Sigma_{n=1,\ldots,N}\ x_n^{q'} > \Sigma_{n=1,\ldots,N}\ x_n^q$. Thus, a selected committee involves k members among those having gathered as many approvals as possible. Given k, the set of eligible committtees of vote matrix X^{NQ} under this rule is denoted as $a_k(X^{NQ})$.

The distance between two committees is again calculated on a restricted area. Given any committee, the distance between this committee and any other one is defined as the number of candidates that is excluded from the second commitee while approved by the former.

Definition 21 *$\forall Q, \forall x, y \in \{0,1\}^Q$, $d(x,y) = \mid \{q \in C : (y/A(x))^q = 0\} \mid$, where $A(x) = \{q \in C : x^q = 1\}$ and $y/A(x)$ stands for the restriction of the committee y to the subset $A(x)$.*

Notice here that, unless x and y are committees of the same size, the distance between them is not a symmetric distance in the sense that $d(x,y) \neq d(y,x)$.

Preferences over committees are derived by the help of extension rules defined from Ω to $\mathcal{R}^k$, where $\Omega = \cup_{Q \geq 1}\{0,1\}^Q$ and $\mathcal{R}^k = \cup_{Q \geq 1}^{Q} \Gamma^{Q,k}$ with $\Gamma^{Q,k}$ being the set of all complete preorders on $\mathcal{K} = \cup_{k \geq 0, Q > k} K_k$

A slight change is required in the definition of top-consistency as;

Definition 22 *An extension rule R is said to satisfy **top-consistency** if: $\forall Q$,$\forall k < Q$,*

i. $\forall n \in I$ with $\mid q \in C : x_n^q = 1 \mid \leq k$, $x \; P_n \; y$ for all $x \in K_k$ with $d(x_n, x) = 0$ and $\forall y \in K_k$ with $d(x_n, y) > 0$

ii. $\forall n \in I$ with $\mid q \in C : x_n^q = 1 \mid > k$, $x \; P_n \; y$ for all $x \in K_k$ with $d(x, x_n) = 0$ and $\forall y \in K_k$ with $d(y, x_n) > 0$

Extension rule R satisfies top-consistency, if for the voters that approve at most k candidates, all the candidates approved are included as members in the first ranked committees and for the voters that approve more than k candidates, all the members of the first-ranked committee are approved in the ballots.

Preserving the modified definition of a distance, Hamming extension rule will rank the committees according to the distance to the ideal.

Definition 23 *R^{Ham} is said to be **Hamming extension rule** if the following condition is satisfied: $\forall Q$, $\forall k, \forall x \in \{0,1\}^Q, \forall y, z \in K^k, d(x,y) < d(x,z) \Leftrightarrow y\ P_{(x)}\ z$ and $d(x,y) = d(x,z) \Leftrightarrow y\ I_{(x)}\ z$.*

Definition 24 *R^{HamC} is said to be a **Hamming-consistent extension rule** if and only if $\forall Q, \forall k, \forall x \in \{0,1\}^Q, \forall y, z \in K^k, d(x,y) < d(x,z) \Leftrightarrow yP_{(x)}z$.*

With sequential approval balloting, the set of eligible committees $a_k(X^{NQ})$ does not necessarily consist of a single element regardless of N being odd or even. Thus, the search for the largest domain of extension rules which yield Pareto-optimal outcomes becomes a two-way problem: on one hand, we will search for the extensions for which any eligible committee is never Pareto-dominated; on the other hand, we will characterize the largest domain that an eligible committee is never Pareto-dominated by a non-eligible committee. In the former, the unique rule will be Hamming extension, while in the latter Hamming-consistent extension rules are made use of.

Proposition 7 *Let D^{**} be the domain of top-consistent and separable extension rules from set of committees to set of complete orders over fixed size committees. No eligible committee is Pareto dominated under sequential approval balloting for $R \in D^{**}$ if and only if R is Hamming extension rule.*

Proof. *Sufficiency Part:*

Given any X^{NQ}, let R be Hamming extension rule. Take any $c \in a_k(X^{NQ})$ and suppose c is Pareto-dominated by c^*in the profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$.

As c is an eligible committee, by definition, $n(q) \geq n(q') \; \forall q \in c, \forall q' \notin c$, where $n(q) =\mid \{n \in I : x_n^q = 1\} \mid$. Let, $N(c)$ denote the total number of approvals collected by the members of c.

Since, $\forall n,\ c^* R_n\ c$, due to Hamming consistency, $\forall n,\ d(x_n, c^*) \leq d(x_n, c)$, which implies $\forall n, \mid \{q \in C : (c^*/A_n)^q = 1\} \mid \geq \mid \{q \in C : (c/A_n)^q = 1\} \mid$ .Thus, $\Sigma_{n \in I}\ (c^*/A_n)^q \geq \Sigma_{n \in I}\ (c/A_n)^q$, yielding $N(c^*) \geq N(c)$. As, c is an eligible committee, $N(c^*) = N(c)$ and $\forall n, \mid \{q \in C : (c^*/A_n)^q = 1\} \mid = \mid \{q \in C : (c/A_n)^q = 1\} \mid$, which is $\forall n,\ d(x_{n,} c^*) = d(x_n, c)$.

As R is the Hamming rule, $\forall n,\ c^* I_n c$. Thus, an eligible committee is never Pareto-dominated.

Necessary Part:

Let R be an extension rule that is not Hamming rule. We claim that $\exists\ X^{NQ}$, $\exists k,\ \exists c \in a_k(X^{NQ})$ such that c is Pareto dominated in the profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$.

Consider the following X^{NQ} with $N = 6$, $k = 3$, $Q \geq 6$ and rows indicating the approval ballots.

1	**2**	**3**	**4**	**5**	**6**	**7**...
1	0	0	1	0	0	0
0	1	0	0	1	0	0
0	0	1	0	0	1	0
1	0	0	1	0	0	0
0	1	0	0	1	0	0
0	0	1	0	0	1	0

Both the committees $c_{1=}(1,1,1,0...)$ and $c_{2=}(0,0,0,1,1,1,0...)$ are elements of $a_k(X^{NQ})$. $\forall n$, $d(x_{n,}c_1) = d(x_n, c_2)$. As R is not Hamming rule, one can have the profile such that $\forall n$, $c_1 R_n c_2$ and $\exists n^*$, $c_1 P_{n^*} c_2$. ■

This proposition shows that with Hamming extension rule, an eligible profile under sequential approval voting is never Pareto-dominated and once another extension rule is used it is possible to find profiles that an eligible committee is dominated by another (not necessarily) eligible committee. Thus, it suggests to investigate the domain of extension rules that yield out profiles for which eligible committees can not be Pareto-dominated by any non-eligible committee. This domain will be the one that at least one eligible committee is not Pareto-dominated by any other committee. Obviously, this domain will be larger than Hamming extension rule.

The next proposition shows that the largest domain of extension rules at which at least one eligible committee is not Pareto-dominated is the class of Hamming consistent extension rules. The sufficiency part follows the previous proof closely till the last step.

Proposition 8 *Let D^{**} be the domain of top-consistent and separable extension rules from set of committees to set of complete orders over fixed size committees. At least one eligible committee is not Pareto dominated under sequential approval voting for $R \in D^{**}$ if and only if R is a Hamming consistent extension rule.*

Proof. *Sufficiency Part:*

Given any X^{NQ}, let R be a Hamming consistent extension rule. Take any $c \in a_k(X^{NQ})$ and suppose c is Pareto-dominated by $c^* \notin a_k(X^{NQ})$ in the profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$.

As c is an eligible committee, by definition, $n(q) \geq n(q') \; \forall q \in c$, $\forall q' \notin c$, where $n(q) =\mid \{n \in I : x_n^q = 1\} \mid$. Let, $N(c)$ denote the total number of approvals collected by the members of c.

Since, $\forall n$, $c^* R_n \; c$, due to Hamming consistency, $\forall n$, $d(x_n, c^*) \leq d(x_n, c)$, which implies $\forall n, \mid \{q \in C : (c^*/A_n)^q = 1\} \mid \geq \mid \{q \in C : (c/A_n)^q = 1\} \mid$ .Thus, $\Sigma_{n \in I} \; (c^*/A_n)^q \geq \Sigma_{n \in I} \; (c/A_n)^q$, yielding $N(c^*) \geq N(c)$. As, c is an eligible committee, $N(c^*) = N(c)$, which directly contradicts with $c^* \notin a_k(X^{NQ})$.

Thus, $c \in a_k(X^{NQ})$ can not be Pareto-dominated by any $c^* \notin a_k(X^{NQ})$.

Now, suppose $c^* \in a_k(X^{NQ})$ dominates $c \in a_k(X^{NQ})$. That is, $N(c^*) = N(c)$ and $\forall n$, $\mid \{q \in C : (c^*/A_n)^q = 1\} \mid = \mid \{q \in C : (c/A_n)^q = 1\} \mid$, which is $\forall n$, $d(x_n, c^*) = d(x_n, c)$. As R is a Hamming-consistent extension rule, one can find profiles that c^* Pareto-dominates c. Let X^{NQ} be such. Since Pareto-domination defines a transitive binary relation, $\exists \; \tilde{c} \in a_k(X^{NQ})$ that Pareto dominates all distinct $c \in a_k(X^{NQ})$ and is not Pareto-dominated by any of them.

Necessary Part:

Let R be an extension rule that is not Hamming consistent. We claim that $\exists$ X^{NQ}, $\exists k$, $\exists c \in a_k(X^{NQ})$, $\exists c^* \notin a_k(X^{NQ})$ such that c^* Pareto dominates c in the profile $R(X^{NQ}) = (R(x_1), ..., R(x_N))$.

Let $Q > 3$, $k \leq Q/2$, $N = \begin{pmatrix} k \\ k-1 \end{pmatrix} + 1$, the approval ballots;

-$\forall n < N, \forall q \in \{1, ..., k\}, \mid \{q : x_n^q = 1\} \mid = k - 1$

-$\forall n < N, \forall q \in \{(k+1), ..., 2k\}, \mid \{q : x_n^q = 1\} \mid = 1$

-$\forall n, n'$ with $n \neq n'$, $x_n \neq x_{n'}$

-$\forall n < N, \forall q > 2k$, the total number of approvals, $n(q) < \begin{pmatrix} k \\ k-1 \end{pmatrix} - 1.$

-For $n = N, \forall q \in \{(k+1), ..., 2k\}, x_N^q = 1$ and $\forall q \notin \{(k+1), ..., 2k\}$ $x_N^q = 0.$

It follows from construction that the unique sequential approval balloting outcome c is such that: $\forall q \in \{1, ..., k\}$, $c^q = 1$ and $\forall q \notin \{(k+1), ..., 2k\}$ $c^q = 0$. However, as R is not Hamming consistent, it may be the case that $\forall n$, x_N R c and from top-consistency $\exists n = N$ with x_N P_N c. One can check that this is compatible with separability as well. ■

Conclusion

A committee election rule is certainly promoted as long as the outcomes are representative of the underlying voter preferences. To represent underlying voter preferences faithfully, the outcomes produced by the voting rule should satisfy some requirements. Apart from Condorcet-consistency, maximizing some social welfare function or minimizing a particular distance, what we mainly focus in this study is a minimum level of requirement to secure representative election winners: Pareto-optimality. We have defined the largest class of separable and top-consistent preference extension rules that will yield out Pareto-optimal winners in both unrestricted and fixed-size committee elections based on approval balloting.

For further study, the investigation of the same problem under strategic voting would be complementary to this study. Here, we know that sincere voting is the dominant strategy with Hamming preferences as shown by Brams et al. (2004) but, it would not probably be the case with weakly-Hamming consistent preferences. Hence the representative properties of elections based on approval balloting where strategic voting is allowed stands as a problem to stick with.

Or to read the question other way round, specific properties of tournament solutions for alternative definitions of preferences over committees compatible with approval ballots may be investigated. Some results have been reached for tournament solutions for Hamming preferences (Laffond and Lainé, 2008) but, since the class of preferences compatible with approval ballots is a large class, there is still room to go on with.

A last proposition for further study would be fairness related questions. In addition to representativeness of an election rule, fairness of it is a crucial qualifier. In the model introduced, the minimax rule is assumed to be "fair" in the sense that it rests upon the egalitarian ideal of maximizing the minimum utility. However, fairness can be defined in different ways and related to different criteria. For instance, once we define a "fair" outcome in the sense that it coincides with all approval ballots for the same number of candidates, a voting rule that maximizes this would be a fair one. Hence, the search for fair voting rules would be an interesting question.

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